Expectation Maximization and Total Variation Based Model for Computed Tomography Reconstruction from Undersampled Data

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Abstract

Computerized tomography (CT) plays an important role in medical imaging, especially for diagnosis and therapy. However, higher radiation dose from CT will result in increasing of radiation exposure in the population. Therefore, the reduction of radiation from CT is an essential issue. Expectation maximization is an iterative method used for CT image reconstruction that maximizes the likelihood function under Poisson noise assumption. Total variation regularization is a technique used frequently in image processing to preserve edges, given the assumption that most images are piecewise constant. Here, we propose a method combining expectation maximization and total variation regularization, called EM+TV. This method can reconstruct a better image using fewer views, thus reducing the overall dose of radiation. The numerical results show the efficiency of the EM+TV method by comparison with the result obtained by filtered back projection.

1 Introduction and Description of Purpose

As a group of methods for reconstructing two dimensional and three dimensional images from the projections of the object, iterative reconstruction has many applications such as in computerized tomography (CT), positron emission tomography (PET), and magnetic resonance imaging (MRI). This technique is quite different from the filtered back projection (FBP) method [7], which is the most commonly used algorithm in practice by manufacturers. The main advantages of iterative reconstruction technique over FBP are insensitivity to noise and flexibility [3]. The data can be collected over any set of lines, the projections do not have to be distributed uniformly in angle, and the projections can be even incomplete.

There are many available algorithms for iterative reconstruction. Most of these algorithms are based on the system of linear equations

$$Ax = b, \tag{1}$$

where $x = (x_1, \cdots, x_N)^T \in \mathbf{R}^N$ is the original unknown image represented as a vector, b is the given measurement with $b = (b_1, \cdots, b_M)^T \in \mathbf{R}^M$, and A is a $M \times N$ matrix describing the direct transformation from the original image to the measurements, which is different for different purposes. For example, in CT, A is the discrete Radon transform, with each row describing an integral along one straight line and all the elements are nonnegative.

One example of iterative reconstruction algorithm is expectation maximization (EM) [2, 8]. This is based on the assumption that the noise in b is Poisson noise. If x is given, the probability of obtaining b is

$$P(b|Ax) = \Pi_{i=1}^{M} \frac{e^{-(Ax)_i}((Ax)_i)^{b_i}}{b_i!}.$$

Therefore, given b and A, the objective is to find x such that the above probability is maximized. However, instead of maximizing the probability, we can minimize $-\log P(b|Ax) = (Ax)_i - b_i \log((Ax)_i) + C$, with C being a constant. Then the EM iteration is as follows:

$$x_j^{n+1} = \frac{\sum\limits_{i=1}^{M} (a_{ji}(\frac{b_i}{(Ax^n)_i}))}{\sum\limits_{i=1}^{M} a_{ji}} x_j^n. \tag{2}$$

Total-variation regularization method was proposed by Rudin, Osher and Fatemi [6] to remove noise in an image, while preserving edges. This technique is widely used in image processing and amounts to minimize an energy functional of the form

$$\min_x \int_\Omega |\nabla x| + \alpha \int_\Omega F(Ax, b),$$

where here x is viewed as a two or three-dimensional image with spatial domain Ω, A is usually a blurring operator, b is the given noisy-blurry image, and $F(Ax, b)$ is a data-fidelity term. For example, for Gaussian noise, $F(Ax, b) = \|Ax - b\|_2^2$. Here we combine the EM method with the total variation TV regularization. The assumption is that the reconstructed image can not have too large total-variation (thus noise and reconstruction artifacts are removed). For related relevant work, we refer to the Compressive Sensing Resources [11].

2 Method (EM+TV)

The objective is to reconstruct an image with both minimal total-variation and maximal probability. So we can consider finding a Pareto optimal point by solving a scalarization of these two objective functions and the problem is to solve

$$\begin{cases} \underset{x}{\text{minimize}} & \int_\Omega |\nabla x| + \alpha \sum\limits_{i=1}^M \left((Ax)_i - b_i \log(Ax)_i\right), \\ \text{subject to} & x_j \geq 0, \quad j = 1, \cdots, N. \end{cases} \tag{3}$$

This is a convex constraint problem and we can find the optimal solution by solving the Karush-Kuhn-Tucker (KKT) conditions [5, 4]:

$$\begin{aligned} -\text{div}\Big(\frac{\nabla x}{|\nabla x|}\Big)_j + \alpha \sum_{i=1}^M \left(a_{ji}(1 - \frac{b_i}{(Ax)_i})\right) - y_j &= 0, \qquad j = 1, \cdots, N, \\ y_j \geq 0, \quad x_j &\geq 0, \qquad j = 1, \cdots, N, \\ y^T x &= 0. \end{aligned}$$

By positivity of $\{x_j\}$, $\{y_j\}$ and the complementary slackness condition $y^T x = 0$, we have $x_j y_j = 0$ for every $j = 1, \cdots, N$. Thus we obtain

$$-x_j \text{div}\Big(\frac{\nabla x}{|\nabla x|}\Big)_j + \alpha \sum_{i=1}^M \left(a_{ji}(1 - \frac{b_i}{(Ax)_i})\right) x_j = 0, \qquad j = 1, \cdots, N,$$

or equivalently

$$-\frac{x_j}{\sum\limits_{i=1}^M a_{ji}} \text{div}\Big(\frac{\nabla x}{|\nabla x|}\Big)_j + \alpha x_j - \alpha \frac{\sum\limits_{i=1}^M \left(a_{ji}(\frac{b_i}{(Ax)_i})\right)}{\sum\limits_{i=1}^M a_{ji}} x_j = 0, \qquad j = 1, \cdots, N.$$

After plugging the EM step into the KKT condition, we obtain

$$-\frac{x_j}{\sum\limits_{i=1}^M a_{ji}} \text{div}\Big(\frac{\nabla x}{|\nabla x|}\Big)_j + \alpha x_j - \alpha x_j^{EM} = 0, \qquad j = 1, \cdots, N, \tag{4}$$

which is the optimality for the following TV minimization problem

$$\underset{x}{\text{minimize}} \int_\Omega |\nabla x| + \alpha \sum_{j=1}^N \sum_{i=1}^M a_{ji} \left((x)_j - x_j^{EM} \log x_j\right). \tag{5}$$

To solve the above EM+TV minimization problem, we can use semi-implicit iteration for several steps or follow the lines of Chambolle's projection method initially proposed for removing Gaussian noise [1]. Each iteration is called a TV step. Thus the algorithm is as follows:

Input: $x^0 = 1$;
for *Out=1:1:IterMax* **do**
  $x^{0,0} = x^{Out-1}$;
  for k = *1:1:K* **do**
    $x^{k,0} = EM(x^{k-1,0})$;
  end
  for l = *1:1:L* **do**
    $x^{K,l} = TV(x^{K,l-1})$;
  end
  $x^{Out} = x^{K,L}$;
end

Algorithm 1: Proposed EM+TV algorithm.

3 Numerical Results

In this section, we will compare the reconstruction results obtained by EM+TV with those obtained by filtered back projection. For the numerical experiments, we choose the two dimensional Shepp-Logan phantom, and we obtain the projections using Siddon's algorithm [9, 10]. We consider both the noise-free and noise cases. With the FBP method, we present results using 36 views (every 10 degrees; for each view there are 301 measurements), 180 views, and 360 views. In order to show that we can reduce the number of views by using EM+TV, we only use 36 views for the proposed method. The results are shown in Figure 1. We notice the much improved results obtained with EM+TV using only 36 views, by comparison with FBP using 36, 180 or even 360 views.

4 Conclusion

We proposed a method combining EM and TV for CT image reconstruction. This method provides comparable results when using fewer views, comparing to filtered back projection with many more views. As illustrated in Figure 1, we can see that the result of EM+TV with 36 views is still better than that of filtered back projection with 360 views. It shows that this method needs much fewer measurements to obtain a comparable image, which results in decreasing of radiation dose. The method is easily extended to three dimensions. Future work includes faster implementation using the advantage of graphics processing units (GPUs), parallel computing, and applications to real data.

References

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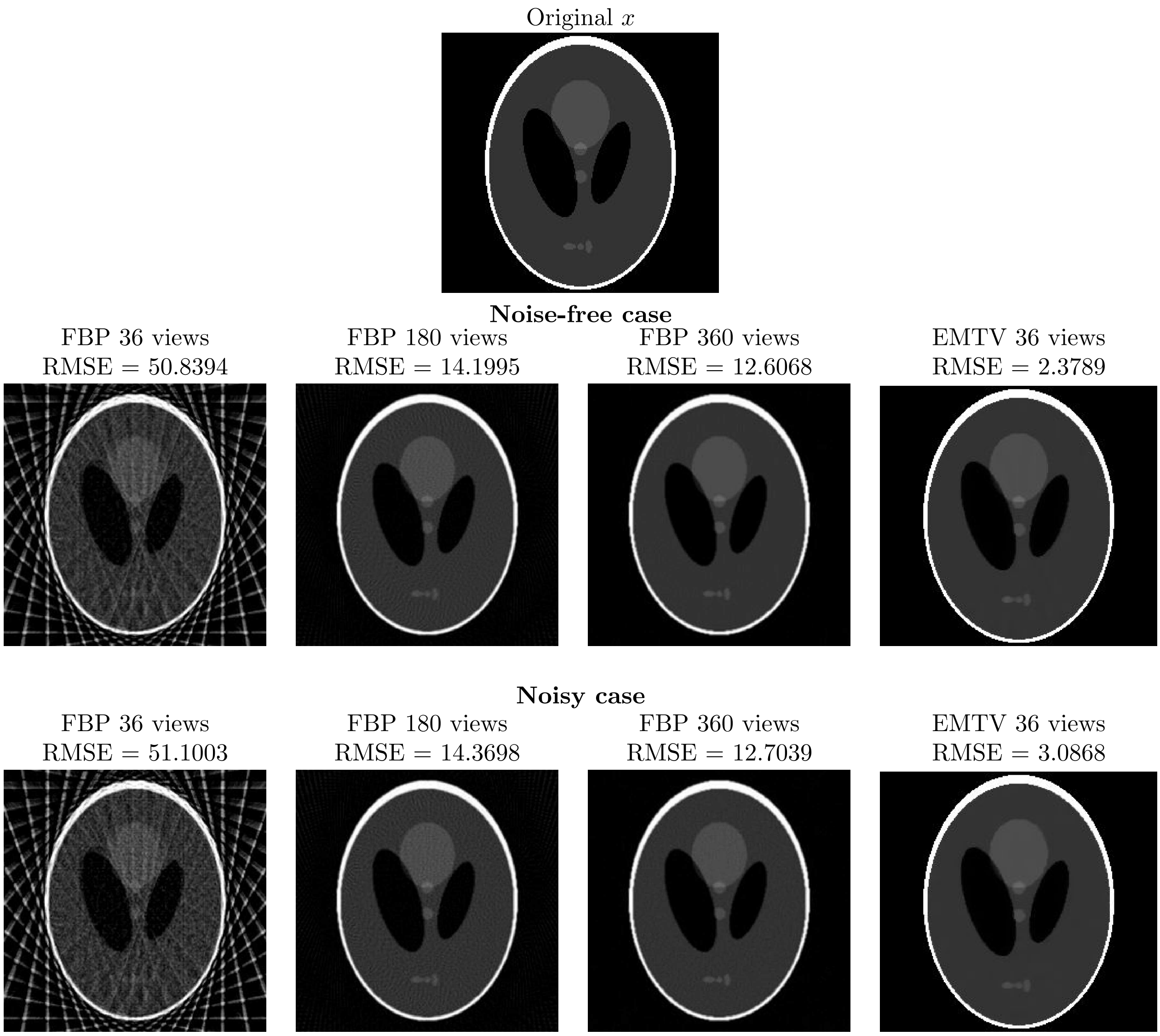

Figure 1: Top: original image (the Shepp-Logan phantom). Middle from left to right: reconstruction result in the noise-free case using FBP with 36, 180 and 360 views, and result using EM+TV with 36 views. Bottom from left to right: reconstruction result in the noisy case using FBP with 36, 180 and 360 views, and result using EM+TV with 36 views. The root mean square errors are also given.

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