Problem 1 (§1.4, 24). The textbook just draws a picture of stereographic projection, so before we get started, let's derive the equations that describe this projection. Note that it's not even clear from the picture on page 55 what sphere we should be projecting from; there are a couple choices here, which yield different equations, but everything works the same way no matter which we use. I'll go with the most popular choice and suppose that we're projecting from the sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - 1/2)^2 = 1/4\} \subset \mathbb{R}^3$, that is, a sphere with the south pole at $(0, 0, 0)$ and north pole at $(0, 0, 1)$.

Given a point $w = (x, y, z)$ on the sphere, to find its image under the stereographic projection $S : S^2 \setminus \{\infty\} \to \mathbb{R}^2$, we draw a line from the north pole $(0, 0, 1)$ through w and define $S(w)$ to be its intersection with the xy-plane. This line has the equation $\gamma(t) = (0, 0, 1) + (x, y, z - 1)t$, so $\gamma(0) = (0, 0, 1)$ and $\gamma(1) = (x, y, z)$. This intersects the plane at $t = 1/(1 - z)$, so

$$S((x, y, z)) = \gamma(1/(1 - z)) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right).$$

From this, we see that S is continuous on the sphere without the North Pole (with respect to the chordal metric ρ). It is the restriction to a subset of $\mathbb{R}^3$ (the sphere) of a map $\mathbb{R}^3 \to \mathbb{R}^2$ which is continuous whereever $z \neq 1$. All points on $S^2 \setminus (0, 0, 1)$ have $z \neq 1$, so stereographic projection is continuous here as a map $\mathbb{R}^3 \to \mathbb{R}^2$. This means exactly that given any ϵ, there is δ such that $|w_1 - w_2|_{\mathbb{R}^3} < \delta$ implies that $|S(w_1) - S(w_2)|_{\mathbb{R}^2} < \epsilon$. But the norm $|\cdot|_{\mathbb{R}^3}$ restricted to the sphere is exactly our chordal metric, and $|\cdot|_{\mathbb{R}^2}$ is the usual metric on $\mathbb{C}$. So continuity of S away from the north pole means exactly that given any ϵ, there is δ such that $\rho(w_1, w_2) < \delta$ implies $|S(w_1) - S(w_2)| < \epsilon$.

We can similarly derive an equation for the inverse T of the stereographic projection. Given $(x, y) \in \mathbb{C} \cong \mathbb{R}^2$, define a line from $(x, y, 0)$ to the north pole by $\gamma(t) = (0, 0, 1) + (x, y, -1)t$, so that $\gamma(0) = (0, 0, 1)$ and $\gamma(1) = (x, y, 0)$. This intersects the sphere when

$$\begin{aligned}\frac{1}{4} &= (xt)^2 + (yt)^2 + \left((1 - t) - \frac{1}{2}\right)^2 \\ &= (x^2 + y^2 + 1)t^2 - t + \frac{1}{4}.\end{aligned}$$

This has two solutions, at $t = 0$ and $t = 1/(x^2 + y^2 + 1)$. It's the latter of these that we're interested in; the first just corresponds to the intersection of the line with the north pole. So

$$T((x, y)) = \left(\frac{x}{x^2 + y^2 + 1}, \frac{y}{x^2 + y^2 + 1}, \frac{x^2 + y^2}{1 + x^2 + y^2}\right).$$

This is a continuous map from $\mathbb{R}^2 \to \mathbb{R}^3$. Using the definition of continuity, and again noting that the chordal metric is equivalent to the usual metric on $\mathbb{R}^3$, this means that given any ϵ, there is δ such that $|z_1 - z_2| < \delta$ implies that $\rho(T(z_1), T(z_2)) < \epsilon$.

a) We're now in position to approach the problem. We claim first that $z_n \to z$ in $\mathbb{C}$ if and only if $\rho(z_n, z) \to 0$. First suppose that $z_n \to z$, and fix $\epsilon > 0$; we want to find N such that $n > N$ implies that $\rho(z_n, z) < \epsilon$ for $n > N$.

By the preceding discussion, there exists ϵ' such that $|z_1 - z_2| < \epsilon'$ implies that $\rho(T(z_1), T(z_2)) < \epsilon$. Since $z_n \to z$, there is N such that for $n > N$, $|z - z_n| < \epsilon'$, and so $\rho(z, z_n) < \epsilon$. This establishes the first direction. Suppose conversely that $\rho(z, z_n) \to 0$. There is ϵ' such that if $\rho(z, z_n) < \epsilon'$ we have $|z - z_n| < \epsilon$. But there exists N such that for $n > N$ we have $\rho(z, z_n) < \epsilon'$, and thus $|z - z_n| < \epsilon$, so $z \to z_n$, as desired.

b) Now we show that $\rho(z_n, \infty) \to 0$ if and only if $|z_n| \to \infty$. I claim that given ϵ, there is a constant R_ϵ such that $|z| = R_\epsilon$ precisely if $\rho(\infty, z) = \epsilon$. Indeed, from the preceding equation, equation, computing the distance from $T(z)$ to ∞ we obtain

$$\rho(z, \infty) = \frac{1}{\sqrt{|z|^2 + 1}}.$$

Now, suppose $\rho(z_n, \infty) \to 0$. For fixed R, there is N such that $\rho(z_n, \infty) < \epsilon_R$ for $n > N$. But this means exactly that $|z_n| > R$, as required. Conversely, suppose $|z_n| \to \infty$. Given ϵ, there is N such that for $n > N$ we have $|z_n| > R_\epsilon$, so $\rho(z_n, \infty) < \epsilon$, as required.

For those who have seen some topology, we can interpret this result in topological terms. We have two ways to put a topology on $\widehat{\mathbb{C}}$: one is as the one-point compactification of $\mathbb{C}$, and the other is the subspace topology inherited from $\mathbb{R}^3$ (with the usual euclidean metric). We've checked that the two topologies are in fact the same; part (a) shows that neighborhoods of points other than ∞ are the same, and part (b) shows that neighborhoods of ∞ are also the same.

c) Consider the function $f(z) = \frac{az+b}{cz+d}$. We're asked to show that it's continuous at ∞. This doesn't make any sense as stated, since it isn't even defined there. What we need to do is define $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as a function on the entire sphere. We do this in the expected way; if $c \neq 0$, then define

$$f(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq -d/c, \infty, \\ \infty & \text{if } z = -d/c \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}$$

If $c = 0$, we set

$$f(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty, \\ \infty & \text{if } z = \infty. \end{cases}$$

We'll prove continuity at ∞ (considering these two cases seperately). Continuity has the usual $\epsilon-\delta$ definition, but with respect to the metric ρ. We want to show that given ϵ, there is δ such that $\rho(z, \infty) < \delta$ implies that $\rho(f(z), f(\infty)) < \epsilon$.

Consider the $c \neq 0$ case first, so $f(\infty) = a/c$. There is an ϵ' such that $|f(z), a/r| < \epsilon'$ implies $\rho(f(z), f(\infty)) < \epsilon$, and given R, there is R_δ such that $|z| > R_\delta \leftrightarrow \rho(z, \infty) < \delta$. So it suffices to find R so $|z| > R$ implies $|f(z) - a/c| < \epsilon'$.

We have

$$\left| \frac{a}{c} - \frac{az+b}{cz+d} \right| = \left| \frac{acz+ad}{c^2z+cd} - \frac{acz+bc}{c^z+cd} \right| = \left| \frac{ad-bc}{c^2z+cd} \right| \leq \frac{|ad-bc|}{|c|^2\,|z| - |cd|}.$$

But this is less than ϵ' as long as

$$|z| > \frac{1}{|c|^2} \left(\frac{|ad-bc|}{\epsilon'} - |cd| \right) = M,$$

Set δ to be ϵ_M in the preceding notation. We have continuity at ∞, since $\rho(z, \infty) < \delta \leftrightarrow |z| > M \to |f(z) - f(\infty)| < \epsilon' \to \rho(f(z), f(\infty)) < \epsilon$, as needed.

Now consider the case $c = 0$. Suppose ϵ is given; we want to find δ so $\rho(z, \infty) < \delta$ implies $\rho(f(z), f(\infty)) = \rho(f(z), \infty) < \epsilon$. We have $f(z) = (az+b)/d$, and then

$$\left| \frac{az+b}{d} \right| \geq \left| \frac{a}{d} \right| |z| - \left| \frac{b}{d} \right|,$$

which exceeds R_ϵ as long as $|z| > \left|\frac{d}{a}\right| \left(R_\epsilon + \left|\frac{b}{d}\right|\right) = M$. But taking $\delta = \epsilon_M$, we have $\rho(z,\infty) < \delta \to |z| > M \to f(z) > R_\epsilon \to \rho(f(z),\infty) < \epsilon$, as needed.

The function $f = (az+b)/(cz+d)$ is called a *Möbius transformation* or *fractional linear transformation*. This function wasn't just devised for this problem; such functions are important and will appear again in this course. One may check that the Möbius transformations are in fact bijections, and thus automorphisms of $\widehat{\mathbb{C}}$; it turns out that in fact every automorphism of the sphere is of this form.

Interestingly, letting $\mathcal{M}$ denote the group of Möbius transformations (with composition as the group operation), one may check that the map $\phi : \mathcal{M} \to \mathrm{GL}_2(\mathbb{C})$ sending $\frac{az+b}{cz+d} \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a group homomorphism, so that composition of Möbius transformations works like multiplication of matrices. Moreover, note that multiplying a, b, c, d all by some constant λ doesn't change the transformation, so we in fact have a map $\phi : \mathcal{M} \to \mathrm{PGL}_2(\mathbb{C})$, the projectivization; this map turns out to be an isomorphism between $\mathcal{M}$ and $\mathrm{PGL}_2(\mathbb{C})$, and so $\mathrm{Aut}(\widehat{\mathbb{C}}) \cong \mathcal{M} \cong \mathrm{PGL}_2(\mathbb{C})$. (Don't worry about this stuff if you haven't taken 122. You really don't need to worry about it even if you have.)

Problem 2 (§1.5, 9). We will verify the Cauchy-Riemann equations for $f(z) = z^2 + 3z + 2$. Writing $z = x + iy$, we then expand

$$f(x+iy) = (x+iy)^2 + 3(x+iy) + 2 = x^2 + 2xyi - y^2 + 3x + 3iy + 2 = (x^2 - y^2 + 3x + 2) + i(2xy + 3y).$$

So $f = u + iv$, where

$$u(x,y) = x^2 - y^2 + 3x + 2, \qquad v(x,y) = 2xy + 3y.$$

Then we compute

$$\frac{\partial u}{\partial x} = 2x + 3 \qquad \frac{\partial u}{\partial y} = -2y$$
$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x + 3.$$

We see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, so the Cauchy-Riemann equations are satisfied.

Problem 3 (§1.5, 10). Again set $z = x + iy$. Consider the function

$$f(z) = |z| = \sqrt{x^2 + y^2}.$$

The book's vague about where we're supposed to show this function isn't analytic; I'll check that it's not analytic at any point in $\mathbb{C}$. First, consider any point $z_0 = (x_0, y_0) \neq (0,0)$ which is not the origin. Note that the real part $u(x+iy) = \sqrt{x^2+y^2}$ and $v(x+iy) = 0$. Computing the partial derivatives,

$$\frac{\partial u}{\partial x} = \frac{1}{2}\sqrt{x^2+y^2}(2x) = \frac{x}{x^2+y^2} \qquad \frac{\partial v}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} = \frac{1}{2}\sqrt{x^2+y^2}(2y) = \frac{y}{x^2+y^2} \qquad \frac{\partial v}{\partial y} = 0$$

At z_0, either x_0 or y_0 is nonzero, so either $\frac{\partial u}{\partial x}(z_0) \neq 0 = \frac{\partial v}{\partial y}$ or $\frac{\partial u}{\partial y} \neq 0 = -\frac{\partial v}{\partial x}$, and the Cauchy-Riemann equations are not satisfied so the function is not analytic. We can check also that the function is not differentiable at $(0,0)$ by computing the limit in two different directions. If z is a positive real, then

$$\lim_{\substack{x\to 0^+\\ y=0}} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0^+\\ y=0}} \frac{|z|}{z} = \lim_{\substack{x\to 0^+\\ y=0}} \frac{x}{x} = 1.$$

If z is a negative real,

$$\lim_{\substack{x\to 0^-\\ y=0}} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0^-\\ y=0}} \frac{|z|}{z} = \lim_{\substack{x\to 0^-\\ y=0}} \frac{-x}{x} = -1.$$

These are different, so the function is not analytic here either.

Problem 4 (§1.5, 11). We claim that in polar coordinates the Cauchy-Riemann equations have the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

First we give the equations for switching between polar and rectangular coordinates. Writing $x = r\cos\theta$ and $y = r\sin\theta$, the chain rule then yields

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}, &\qquad \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y},\\
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial v}{\partial x} + \sin\theta\frac{\partial v}{\partial y}, &\qquad \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial v}{\partial x} + r\cos\theta\frac{\partial v}{\partial y}.
\end{aligned}$$

Solving these for the partials in rectangular coordinates, for which we know the Cauchy-Riemann equations, we obtain

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}, &\qquad \frac{\partial u}{\partial y} &= \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta},\\
\frac{\partial v}{\partial x} &= \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}, &\qquad \frac{\partial v}{\partial y} &= \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}.
\end{aligned}$$

By the usual Cauchy-Riemann equations, we can equate $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, and so

$$\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}, \qquad \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} = -\cos\theta\frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}.$$

By multiplying the first of these by $\cos\theta$ and the second by $\sin\theta$, then adding together, we can use the identity $\sin^2\theta + \cos^2\theta = 1$ to obtain

$$\begin{gathered}
\cos^2\theta\frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial u}{\partial \theta} + \sin^2\theta\frac{\partial u}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial u}{\partial \theta} = \sin\theta\cos\theta\frac{\partial v}{\partial r} + \frac{\cos^2\theta}{r}\frac{\partial v}{\partial \theta} - \sin\theta\cos\theta\frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial v}{\partial \theta}\\
\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.
\end{gathered}$$

Instead multiplying the first term by $-\sin\theta$ and the second by $\cos\theta$ and adding, we obtain

$$-\sin\theta\cos\theta\frac{\partial u}{\partial r}+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial\theta}+\sin\theta\cos\theta\frac{\partial u}{\partial r}+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial\theta}=-\sin^2\theta\frac{\partial v}{\partial r}-\frac{\sin\theta\cos\theta}{r}\frac{\partial v}{\partial\theta}-\cos^2\theta\frac{\partial v}{\partial r}+\frac{\sin\theta\cos\theta}{r}\frac{\partial v}{\partial\theta}$$
$$\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta}.$$

We conclude as desired that

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta},\qquad\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta}.$$

Problem 5 (§1.6, 2). a) Since the maps $f(z)=z\log 3$ and $g(z)=e^z$ are entire, $g\circ f(z)=e^{z\log 3}=3^z$ is entire as well. Thus, its region of analyticity is $\mathbb{C}$ and its derivative is $\log 3\cdot e^{z\log 3}$.

b) Since $\log z$ is analytic on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$, $\log(z+1)$ is analytic on $\mathbb{C}\setminus\mathbb{R}_{\leq -1}$. Its derivative is $1/(z+1)$.

c) Since z^{1+i} has the form z^a for $a\in\mathbb{C}\setminus\mathbb{Z}$, it is analytic on $\mathbb{C}\setminus\mathbb{R}_{<0}$. Its derivative is $(1+i)z^i$.

d) $\sqrt{z}$ also has the form z^a for $a\in\mathbb{C}\setminus\mathbb{Z}$, and so it is analytic on $\mathbb{C}\setminus\mathbb{R}_{<0}$. It derivative is $1/(2\sqrt{z})$.

e) $\sqrt[3]{z}$ also has the form z^a for $a\in\mathbb{C}\setminus\mathbb{Z}$, and so it is analytic on $\mathbb{C}\setminus\mathbb{R}_{<0}$. It derivative is $1/(3z^{2/3})$.

Problem 6 (§1.6,3). a) Since e^z is an entire function, we know that

$$\lim_{z\to 0}\frac{e^z-1}{z}=\lim_{z\to 0}\frac{e^z-e^0}{z-0}=\frac{d}{dz}\left(e^z\right)$$

exists and equals $e^0=1$.

b) This limit does not exist. Observe, by L'Hopital's rule, that

$$\lim_{z\to 0^+}\frac{\sin|z|}{z}=\lim_{z\to 0^+}\frac{\cos z}{1}=1,$$

while

$$\lim_{z\to 0^-}\frac{\sin|z|}{z}=\lim_{z\to 0^+}\frac{\sin|-z|}{-z}=\lim_{z\to 0^+}\frac{\cos z}{-1}=-1.$$

Problem 7 (§1.6, 8). We can see by inspection that $u^2(x,y)-v^2(x,y)$ and $2u(x,y)v(x,y)$ are the real and imaginary parts of the function $(u(x,y)+iv(x,y))^2$. Since u and v satisfy the Cauchy-Riemann equations, and are presumably continuous with continuous partial derivatives, $u(x,y)+iv(x,y)$ is a holomorphic function. This isn't stated explicitly in the problem. If the partials aren't continuous, the result is still true, but we have to check it by a direct computation. Thus, since the function $g(z)=z^2$ is holomorphic, $(u(x,y)+iv(x,y))^2=g(u(x,y)+iv(x,y))$ is holomorphic as well, which means that its real and imaginary parts satisfy the Cauchy-Riemann equations. Similarly, $e^{u(x,y)}\cos v(x,y)+ie^{u(x,y)}\sin v(x,y)=\exp(u(x,y)+iv(x,y))$, and since e^z is holomorphic, $e^{u(x,y)}\cos v(x,y)+ie^{u(x,y)}\sin v(x,y)$ is holomorphic, and so its real and imaginary parts satisfy the Cauchy-Riemann equations.

Problem 8 (§1.6, 9). a) We proved on problem set 2 that the only two complex roots of unity are 1 and -1. Therefore, since a rational function is analytic on $\mathbb{C}$ minus the zero set of its denominator, $z/(z^2-1)$ is analytic on $\mathbb{C} \setminus \{1, -1\}$ and has derivative $(-z^2-1)/(z^4-2z^2+1)$.

b) $e^{z+(1/z)}$ is analytic on $\mathbb{C}^*$ and has derivative $(1-1/z^2)e^{z+1/z}$.