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Exercise sheet 4: Solutions

Caveat emptor: *These are merely extended hints, rather than complete solutions.*

1. Prove that $\kappa(G) \le \kappa'(G) \le \delta(G)$ for any graph G.

   Solution. *The second inequality is trivial, since removal of the $\delta(G)$ edges incident to a vertex of degree $\delta(G)$ disconnects the graph.*
   The first inequality is less obvious. Suppose $\kappa'(G) = k$ and that we can disconnect G by deleting some k edges $e_1, \ldots, e_k$. After the deletion G splits into two subgraphs G_1 and G_2 that cannot reach each other. If there is a vertex in one of the parts G_1 or G_2 that is not incident to any of the deleted edges, we are done, since then we can remove from G all the vertices from that part that are incident to one of the edges $e_1, \ldots, e_k$: we will have at most k deleted vertices and G will be disconnected.
   Otherwise, we have that all vertices in G are covered by $e_1, \ldots, e_k$. If all vertices are incident to at least two of the k edges, then $n \le k$ and consequently $\kappa(G) < k$. So we may suppose that a vertex v is incident to just one edge e_i. Let H be the component of $G - \{e_1, \ldots, e_k\}$ where v lies. Note that H contains at most k vertices (since all the vertices are covered by the k edges and none of them runs within H). Therefore, $d(v) \le k$, which implies that $\kappa(G) \le k$.

2. Prove that a 3-regular 3-edge-connected graph is 3-connected.

   Solution. *Suppose that $\{u, v\} \subset V(G)$ separates the graph into two parts G_1 and G_2 such that there is no path from G_1 to G_2 except via $\{u, v\}$. Since $d(u) = d(v) = 3$ we have only a few cases to consider concerning how the edges from u and v to G_1 and G_2 are distributed (uv can also be an edge). All the cases lead to a contradiction, e.g. if there are two edges between u and G_1, two edges between v and G_2, one edge between u and G_2, and one edge between v and G_1, then by deleting the edge between u and G_2 and the edge between v and G_1 we disconnect the graph (contradicting 3-edge-connectedness).*

3. Construct a 100-edge-connected graph, which is 2-connected but not 3-connected.

Solution. *Consider the graph* $G = K_2 + (K_{10^{10}} \cup K_{10^{10}})$ *.*

4. Prove that a critically 2-connected graph has a vertex of degree 2.

   Solution. *We know that any 2-connected graph is obtained from a cycle by adding repeatedly new paths between two already existing vertices. If our graph is a cycle, we are done. Otherwise, it's obtained from a cycle in a path-adding process. If the last added path is of length at least 2, we have a vertex of degree 2 and we are done. If not, then the last path is an edge whose removal leaves a 2-connected graph. The latter can't happen as we deal with a* critical *2-connected graph.*

5. Prove that if G is a critically k-connected graph and H is a k-connected subgraph of G then H is also critically k-connected.

   Solution. *Suppose the contrary, i.e. that* H *is not critical. Thus* $H - uv$ *is still* k*-connected for some edge* $uv \in E(H)$*. However,* $G - uv$ *is not* k*-connected, so* $G - uv - \{v_1, \ldots, v_{k-1}\}$ *is disconnected for some vertices* $v_1, \ldots, v_{k-1}$*. By assumption* $H - uv - \{v_1, \ldots, v_{k-1}\}$ *is still connected, hence for some connected component* C *of* $G - uv - \{v_1, \ldots, v_{k-1}\}$*, we have that* H *lies in* $G[V(C) \cup \{v_1, \ldots, v_{k-1}\}]$*. That means in particular that* u *and* v *lie in* $V(C) \cup \{v_1, \ldots, v_{k-1}\}$*, which implies that* $G - \{v_1, \ldots, v_{k-1}\}$ *is disconnected. Contradiction.*

6. In a 2-connected graph call two edges *equivalent* if they are the same OR by removing them we can disconnect the graph. Show that this relation is in fact an equivalence relation.

   Solution. *Since* G *is 2-connected, for every edge* e *we have that* $G - e$ *is connected, i.e.* e *lies on a cycle. For every edge* e *denote by* $C(e)$ *the set of cycles that contain* e*. Then we have that* $e_1 \sim e_2$ *iff* $C(e_1) = C(e_2)$*. Now it is clear that we have an equivalence relation.*

If you spot any mistakes on this sheet, please drop an email to filip.moric@epfl.ch.