Functions

$f: A \longrightarrow B$

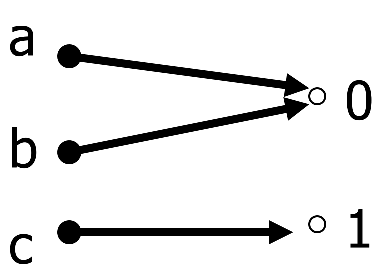
A - domain of the function f

B - codomain of f

To each element $a \in A$, the function assigns an element of B denoted f(a), the image of a

Example

$A=\{a,b,c\}$ $B=\{0,1\}$



f_1

x	$f_1(x)$
a	0
b	0
c	1

$f_1=(0,0,1)$

$f_2=(1,0,1)$

$f_3=(1,1,1)$

How many functions from A to B are there?

B^A the set of all functions from A to B

Theorem. $$|B^A| = |B|^{|A|}$$

Proof.

$$
\begin{array}{cccccc}
A=\{a_1, & a_2, & \ldots, & a_n\} & & \\
\downarrow & \downarrow & & \downarrow & & \\
|B| & |B| & \cdots & |B| & = & |B|^{|A|}
\end{array}
$$

∎

Surjection

Definition. A function $f: A \rightarrow B$ is a surjection if for each element $b \in B$ there is an $a \in A$ such that $f(a)=b$

Which of the following functions (with $B=\{0,1\}$) are surjections?

$f_1=(0,0,1)$ $f_2=(1,0,1)$ $f_3=(1,1,1)$

For $A=\{a,b\}$ and $B=\{0,1,2\}$, is $f=(2,1)$ a surjection? No

because for any surjection $f: A \rightarrow B$, $|A| \geq |B|$

Surjection

Definition. A function f: A→B is a surjection if for each element b∈B there is an a∈A such that f(a)=b

The number of all functions from A to B is $|B|^{|A|}$

How many of them are surjections?

The number of surjections

Theorem. The number of surjections from a set of n elements to a set of k elements is

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

Proof. f: A={1,2,…,n} $\longrightarrow$ B={1,2,…,k}

A_i the set of functions from A to B that never take on the value $i \in B$

The number of surjections from A to B is $k^n - |A_1 \cup \ldots \cup A_k|$

$$S_i = \sum_{j_1<j_2<\cdots<j_i} |A_{j_1} \cap A_{j_2} \cap \ldots \cap A_{j_i}|$$

$$\sum_{i=0}^{k}(-1)^i S_i = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^k S_k$$

The number of surjections

Theorem. The number of surjections from a set of n elements to a set of k elements is

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

Proof.

$$|A_{j_1}\cap A_{j_2}\cap\ldots\cap A_{j_i}|=$$ the number of functions from A to B-$\{j_1,j_2,\ldots,j_i\}$ $=(k-i)^n$

$$S_i=\sum_{j_1<j_2<\ldots<j_i}|A_{j_1}\cap A_{j_2}\cap\ldots\cap A_{j_i}|=\binom{k}{i}(k-i)^n$$

$$\sum_{i=0}^{k}(-1)^i S_i=\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

■

Partitions

1. How many partitions of an n-set are there?

   B_n Bell number

2. How many ways to partition an n-set into k subsets are there?

   $\left\{ {n \atop k} \right\}$ Stirling number of the second kind

3. How many ordered partitions of an n-set into k subsets are there?

   A partition $(A_1,\ldots,A_k)$ is *ordered* if the order of the subsets matters

4. How many ordered partitions of an n-set into k subsets of cardinalities $n_1,\ldots,n_k$ are there?

$$\frac{n!}{n_1!n_2!\cdots n_k!} = \binom{n}{n_1,\ldots,n_k}$$

The number of ordered partitions

Theorem. The number of ordered partitions of a set of n elements into k non-empty subsets is

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

Proof.

Let A be a set of n elements and B={1,2,…,k}.
Each surjection f from A to B defines an ordered partition of A into k non-empty subsets $A_1,\ldots,A_k$ as follows: $A_i=\{a\in A \mid f(a)=i\}$.

Conversely, each ordered partition of A into k non-empty subsets defines a surjection f: A→ B

Therefore, the number of ordered partitions of A coincides with the number of surjections from A to B. ■

The number of partitions

Theorem. The number of partitions of a set of n elements into k non-empty subsets is

$$\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

Injection

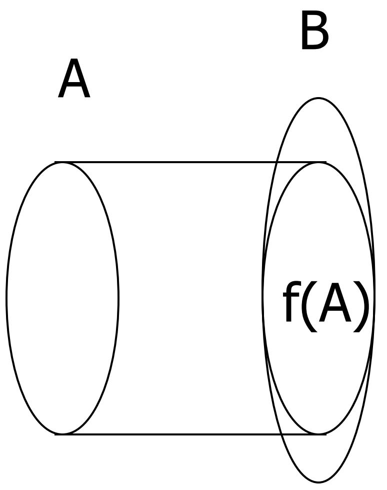
Definition. A function f: $A \to B$ is a injection if for any $a,b \in A$, $a \neq b$ implies $f(a) \neq f(b)$

For $A=\{a,b,c\}$ and $B=\{0,1\}$, is $f=(0,0,1)$ injection? No

because for any injection f: $A \to B$, $|B| \geq |A|$

How many injections from A to B are there?

$$\binom{|B|}{|A|}|A|! = \frac{|B|!}{(|B|-|A|)!}$$



Bijection

Definition. A function $f: A \rightarrow B$ is a bijection if f is an injection and surjection

Example: For $A=\{a,b,c\}$ and $B=\{0,1,2\}$, $f=(2,1,0)$ is a bijection

For any injection $f: A \rightarrow B$, $|B| \geq |A|$

For any surjection $f: A \rightarrow B$, $|A| \geq |B|$

For any bijection $f: A \rightarrow B$, $|A|=|B|$ Does $|A|=|B|$ imply that f is bijection?

How many bijections from A to B are there? $|A|!$

Bijection

For any function $f: A \rightarrow B$, any two of the following three statements imply the remaining one

1. f is surjection
2. f is injection
3. $|A|=|B|$

Proof.

(1,3 $\rightarrow$ 2) By contradiction, assume $f(a)=f(b)$ for some $a \neq b$. Then the number of elements of B that are images of some elements of A is strictly less than $|B|=|A|$, contradicting 1.

(2,3$\rightarrow$1) Analogously