A method for computing acoustic fields based on the principle of wave superposition

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A method for computing the acoustic fields of arbitrarily shaped radiators is described that uses the principle of wave superposition. The superposition integral, which is shown to be equivalent to the Helmholtz integral, is based on the idea that the combined fields of an array of sources interior to a radiator can be made to reproduce a velocity prescribed on the surface of the radiator. The strengths of the sources that produce this condition can, in turn, be used to compute the corresponding surface pressures. The results of several numerical experiments are presented that demonstrate the simplicity of the method. Also, the advantages that the superposition method has over the more commonly used boundary-element methods are discussed. These include simplicity of generating the matrix elements used in the numerical formulation and improved accuracy and speed, the latter two being due to the avoidance of uniqueness and singularity problems inherent in the boundary-element formulation.

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INTRODUCTION

During the past decade, computational methods in acoustics using both finite elements and boundary elements have progressed to the point where commercial software is now available for modeling complex acoustic fields. Of the two, finite-element acoustics is at a more mature stage, and numerical experiments involving internal and external radiated fields coupled with the structural response of the radiators are becoming more commonplace. Boundary elements are emerging in a complementary mode to the finite elements and are better suited to problems of infinite extent, e.g., free-field radiation. Several researchers[1–8] have laid the groundwork to help make the boundary-element formulation compatible with that of the finite element, allowing both methods to share, for example, isoparametric, quadrilateral elements, and, hence, grid geometries, graphic presentation, etc. While the boundary-element method will undoubtably continue to play an important role in computational acoustics, it is still hampered with difficulties that arise in approximating the Helmholtz integral in numerical form. While the uniqueness problem along with that of the singularity of the Green function can be overcome with some resourceful mathematics, the end result is always to increase the complexity, and thus the extent, of the computations. For example, in computing acoustic fields via the CHIEF formulation[3,9] of the boundary-element method using isoparametric, quadratic elements, the time required to compute the matrix coefficients is a substantial portion of the overall computing time.

The search to find a simpler, more straightforward computational method that circumvents the above complications was the motivation behind this study. The idea for a superposition method emerged as an offspring of the calibration procedure that is often used in boundary-element studies to provide benchmark data for cases where no analytical solutions exist. In this procedure, a simple source is surrounded by a fictitious surface having the same geometry as that of the acoustic radiator for which the boundary element is being applied. Next, a set of nodal points corresponding to those used in the boundary-element formulation is identified. The field of the enclosed point source is then evaluated at each of the node points for both pressure and velocity (normal to the surface). Although the surface pressure and velocity are dependent on the location of the interior point source, they do provide an exact pair of calibration values at each point for a given source location. With the grid geometry already prescribed, the boundary-element formulation is calibrated simply by taking the computed velocities on the fictitious surface as inputs and computing the corresponding surface pressures. A comparison of the "exact" and computed pressures at each nodal point indicates the extent of validity of the boundary-element formulation, e.g., whether or not the element size to wavelength ratio is adequate, the outward normals have been properly defined, and so on.

It was the calibration procedure that triggered the idea that superposition could be used to determine the acoustic fields of complex radiators. Since the field of a single source located within a surface boundary was used to provide the exact surface pressure and velocity distribution of a radiator for calibration purposes, by superimposing the fields of an array of sources (of the same number of node points on the surface), within the prescribed surface, could values for the magnitude and phase for the strength of each source be determined to give the correct surface pressure and normal velocity at each node point independently of the source locations? That this is possible will be demonstrated in the sections to follow. A series of computational experiments were conducted that provided guidelines for the development of a corresponding theory. Interestingly, the mathematical steps leading to the theory that validated the above hypothesis are closely aligned to the Helmholtz integral. In the final analysis, the theory behind superposition is shown to be equivalent to the Helmholtz-integral formulation. However, by locating sources internal to the radiating surface, the problems of uniqueness and singular kernels are circumvented altogether. Thus the method of superposition represents an enormous simplification computationally. From computations involving simple geometries, given the same density of nodes or computational points on an arbitrary surface, it appears that the superposition method is more accurate than the boundary-element formulation, especially at values of high wavenumbers. This feature together with its increased computational speed should make it an attractive alternative to the boundary-element method.

This paper presents the results of some computational experiments on simple geometries using the superposition method along with some theoretical considerations.

I. METHOD OF WAVE SUPERPOSITION

A. Analytical formulation

The superposition method is based on the idea that the acoustic field of a complex radiator can be constructed as a superposition of fields generated by an array of simple sources enclosed within the radiator. The equivalency of the superposition method to the Helmholtz-integral formulation, and hence its validity, can be developed in the following way. By letting the enclosed sources be distributed continuously inside the radiator as shown in Fig. 1, the acoustic pressure at a field point $\mathbf{r}$ is the integral of the contribution from all sources:

$$p(\mathbf{r}) = j\rho_0\omega \int_V q(\mathbf{r}_0) g(|\mathbf{r}-\mathbf{r}_0|) dV(\mathbf{r}_0), \tag{1}$$

where ρ_0 is the mean density of the medium, ω is the angular frequency of the harmonic vibration of the surface S enclosing the volume V of the radiator, $q(\mathbf{r}_0)$ is the strength of the simple source distribution evaluated at $\mathbf{r}_0$ inside V. The free-space Green's function is defined as

$$g(|\mathbf{r}-\mathbf{r}_0|) = -e^{jk|\mathbf{r}-\mathbf{r}_0|}/4\pi|\mathbf{r}-\mathbf{r}_0|, \tag{2}$$

which satisfies

$$(\nabla^2 + k^2) g(|\mathbf{r}-\mathbf{r}_0|) = \delta(\mathbf{r}-\mathbf{r}_0), \tag{3}$$

where k is the wavenumber and δ is the Dirac delta function.

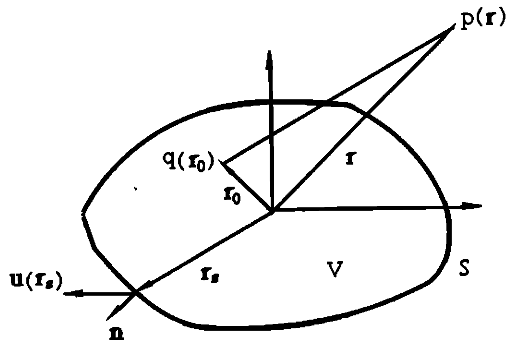

FIG. 1. The diagram for formulating the superposition integral.

Equation (1) is the equation based on which the superposition method will be formulated.

If the method is a valid formulation, the Eq. (1) should be equivalent to the Helmholtz-integral equation. To prove this, begin by assuming that the medium interior and exterior to the radiator is identical. This assumption is valid since the radiation field is independent of the property of the interior medium. Applying the law of mass conservation to the medium contained within the volume V leads to the equation

$$\frac{\partial \rho(\mathbf{r}_0)}{\partial t} + \nabla \cdot [\rho(\mathbf{r}_0)\mathbf{u}(\mathbf{r}_0)] = \rho(\mathbf{r}_0) q(\mathbf{r}_0), \tag{4}$$

where ρ is the total density of the medium. Then, neglecting the nonlinear terms and using the familiar expression for the sound speed c_0, Eq. (4) reduces to

$$-j\omega p(\mathbf{r}_0) + \rho_0 c_0^2 \nabla \cdot \mathbf{u}(\mathbf{r}_0) = \rho_0 c_0^2 q(\mathbf{r}_0). \tag{5}$$

Then, by substituting the above relation for $q(\mathbf{r}_0)$ in Eq. (1), the pressure field becomes

$$p(\mathbf{r}) = \int_V [k^2 p(\mathbf{r}_0) + j\omega\rho_0 \nabla \cdot \mathbf{u}(\mathbf{r}_0)] \times g(|\mathbf{r}-\mathbf{r}_0|) dV(\mathbf{r}_0). \tag{6}$$

Next, by using the vector identities

$$\nabla \cdot (\mathbf{u}g) = g\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla g,$$
$$\nabla \cdot (p\nabla g) = \nabla p \nabla g + p\nabla^2 g = \nabla p \nabla g + p[\delta(\mathbf{r}-\mathbf{r}_0) - k^2 g], \tag{7}$$

and the linearized Euler's equation

$$j\omega\rho_0 \mathbf{u}(\mathbf{r}_0) = \nabla p(\mathbf{r}_0), \tag{8}$$

Eq. (6) can be rewritten as

$$p(\mathbf{r}) = \int_V p(\mathbf{r}_0)\delta(\mathbf{r}-\mathbf{r}_0) dV(\mathbf{r}_0) - \int_V \nabla \cdot [p(\mathbf{r}_0)\nabla g(|\mathbf{r}-\mathbf{r}_0|) - j\omega\rho_0 \mathbf{u}(\mathbf{r}_0) g(|\mathbf{r}-\mathbf{r}_0|)] dV(\mathbf{r}_0). \tag{9}$$

Finally, applying Gauss' theorem to the second term on the right side of the above equation gives

$$p(\mathbf{r}) = \int_V p(\mathbf{r}_0)\delta(\mathbf{r}-\mathbf{r}_0) dV(\mathbf{r}_0) - \oint_s [p(\mathbf{r}_0)\nabla g(|\mathbf{r}-\mathbf{r}_0|) - j\omega\rho_0 \mathbf{u}(\mathbf{r}_0) g(|\mathbf{r}-\mathbf{r}_0|)] \cdot \mathbf{n}\, dS(\mathbf{r}_0), \tag{10}$$

where $\mathbf{n}$ is the outward normal on the enclosed surface S. Equation (10) is the well-known Helmholtz-integral equation for exterior radiation problems.

In the preceding analysis, the acoustic fields produced by the superposition integral [Eq. (1)] and the Helmholtz integral [Eq. (10)] are identical, thus proving that the superposition method is a valid formulation.

B. Numerical formulation

To reduce the superposition integral to a numerical form, the following steps are necessary. From the linearized Euler equation [Eq. (8)] and the superposition integral [Eq. (1)], the acoustic velocity at $\mathbf{r}$ can be calculated by

$$\mathbf{u}(\mathbf{r}) = \int_V q(\mathbf{r}_0) \nabla g(|\mathbf{r}-\mathbf{r}_0|) dV(\mathbf{r}_0). \tag{11}$$

The normal velocity on the surface of the radiator is given by

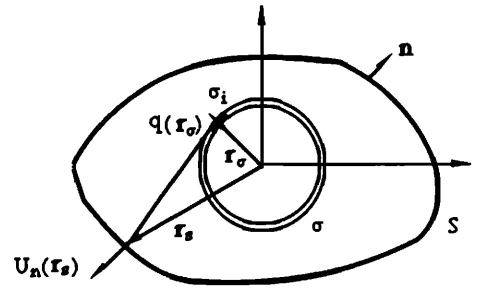

FIG. 2. The diagram of the fictitious source sphere inside of a radiator.

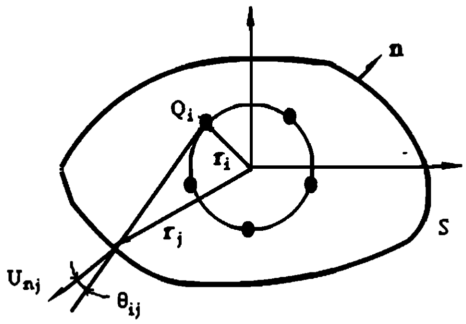

FIG. 3. The diagram of the simple source array.

$$U_n(\mathbf{r}_s) = \int_V q(\mathbf{r}_0)\nabla_n g(|\mathbf{r}_s - \mathbf{r}_0|)dV(\mathbf{r}_0), \qquad (12)$$

where $\mathbf{r}_s$ is the location vector of a point on S. However, there is no restriction on the location of $q(\mathbf{r}_0)$. Therefore, $q(\mathbf{r}_0)$ can be placed anywhere inside of S. For convenience, the simple sources are assumed to distribute over on the surface of a fictitious spherical shell with a thin thickness $\delta\tau$, as shown in Fig. 2. The spherical shell is called the source sphere. Thus Eq. (12) becomes

$$U_n(\mathbf{r}_s) = \delta\tau \oint_\sigma q(\mathbf{r}_\sigma)\nabla_n g(|\mathbf{r}_s - \mathbf{r}_\sigma|)d\sigma(\mathbf{r}_\sigma), \qquad (13)$$

where σ is the surface of the source sphere and $\mathbf{r}_\sigma$ is the location vector of a simple source on σ. Since $\mathbf{r}_\sigma$ can be made always to be smaller than $\mathbf{r}_s$, there is no singularity in Eq. (13). Consequently, the singularity problems common to the boundary-element formulation are avoided altogether.

Next, the surface σ is divided into N segments and the area of each segment is represented by σ_i. Equation (13) can be rewritten as

$$U_n(\mathbf{r}_s) = \delta\tau \sum_{i=1}^{N} \int_{\sigma_i} q(\mathbf{r}_\sigma)\nabla_n g(|\mathbf{r}_s - \mathbf{r}_\sigma|)d\sigma(\mathbf{r}_\sigma). \qquad (14)$$

No approximations have been made up to this point. If σ_i for all segments is made to be sufficiently small, the integrand in Eq. (14) can be taken as a constant. By doing so, the surface velocity is approximated by

$$U_n(\mathbf{r}_s) \approx \sum_{i=1}^{N} Q_i \nabla_n g(|\mathbf{r}_s - \mathbf{r}_{\sigma_i}|), \qquad (15)$$

where Q_i is the volume velocity of the simple source at the location $\mathbf{r}_{\sigma_i}$ on the source sphere. Equation (15) gives the normal velocity of the surface of the radiator, which is generated by N simple sources distributed on the source sphere. Since $U_n(\mathbf{r}_s)$ is known, Eq. (15) can be used to evaluate Q_i.

Suppose that the normal velocity values are prescribed at N points on the surface S. Applying Eq. (15) to each point gives N equations for N unknown Q_i's. In matrix form, the source vector $\mathbf{Q}$ is related with the normal velocity $\mathbf{U}_n$ by

$$\mathbf{Q} = [D]^{-1}\mathbf{U}_n, \qquad (16)$$

where D is an $N \times N$ coefficient matrix called a dipole matrix. Each element of the dipole matrix is determined by the following equation:

$$D_{ij} = -\frac{1}{4\pi}\frac{jk|\mathbf{r}_j - \mathbf{r}_i| - 1}{|\mathbf{r}_j - \mathbf{r}_i|^2} e^{jk|\mathbf{r}_j - \mathbf{r}_i|}\cos\theta_{ij}, \qquad (17)$$

where $\mathbf{r}_i$, $\mathbf{r}_j$, and θ_{ij} are shown in Fig. 3. The superposition method has a useful feature to quantify how well the discretized velocities U_{nj}'s represent the continuous velocity distribution on the surface S. By using the Q_i's to compute velocities at points other than at the U_{nj} points and comparing these with the specified values, the degree of the approximation of the velocity discretization can be determined.

After finding the source strength vector $\mathbf{Q}$, the pressure field is a straightforward calculation from the equation

$$p(\mathbf{r}) = \sum_{i=1}^{N} M(|\mathbf{r} - \mathbf{r}_{\sigma_i}|)Q_i, \qquad (18)$$

where the coefficient M is defined as

$$M(|\mathbf{r} - \mathbf{r}_{\sigma_i}|) = j\omega\rho_0 g(|\mathbf{r} - \mathbf{r}_{\sigma_i}|). \qquad (19)$$

On the surface of the radiator, the pressure vector $\mathbf{P}$ corresponding to the velocity vector $\mathbf{U}_n$ is simply given by

$$\mathbf{P} = [M][D]^{-1}\mathbf{U}_n, \qquad (20)$$

where the element of the monopole matrix M is evaluated by replacing $\mathbf{r}$ in Eq. (19) with $\mathbf{r}_j$.

C. Numerical accuracy

Two approximations have been introduced into the formulation of the superposition method. The first one is that the integration on the right side of Eq. (14) is replaced by the finite summation in Eq. (15), and the second one is that a finite number of the normal velocity values on the surface of the radiator are used to determine the source strengths. Obviously, increasing the number of the simple sources N will reduce the errors caused by both approximations.

For a fixed N, however, the locations of the simple sources inside the radiator can be adjusted to improve the accuracy of the method. As the sources are located farther from the surface S or as the radius of the source sphere becomes smaller, the area of each segment, σ_i, becomes smaller, and the variation of the integrand in Eq. (14), as measured by $\delta\theta_{ij}$, becomes smaller. This is illustrated by Fig. 4. As a result of compacting the sources, the error introduced by the second approximation becomes smaller. This effect will be demonstrated by the numerical experiments that follow. Physically, the compactness of the simple sources increases the relative contribution of each source to each discrete surface.

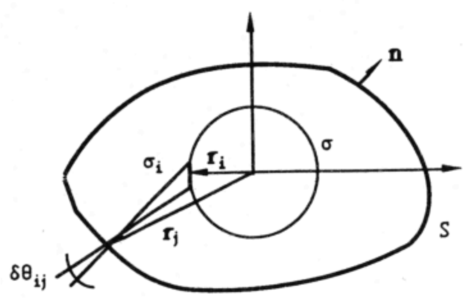

FIG. 4. The measure of the degree of variation of the integrand in Eq. (14).

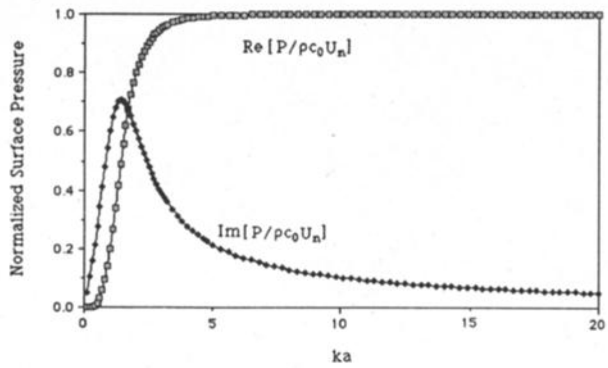

FIG. 6. Variation of normalized surface pressure with nondimensional wavenumber *ka* for a dipole, where *a* is the radius of the dipole sphere.

II. NUMERICAL EXPERIMENTS

The following numerical experiments were performed on two simple geometries. The first was a spherical radiator having 20 surface nodes given both monopole and dipole velocity distributions. The second was a cube radiator with nine nodes evenly distributed on each surface and with the velocity nodes given values corresponding to the field of an interior spherical radiator evaluated at each node point. The pressures at the surface nodes were calculated by the superposition method and compared with the exact values.

Figure 5 gives the normalized real and imaginary pressure on the surface of a monopole as a function of nondimensional wavenumber. For the range of *ka*'s presented, the pressures computed with the superposition method for all practical purposes agreed exactly with the theory, and thus, to avoid confusion, values for the corresponding theoretical points are omitted from the graph. Interestingly, the agreement with theory continued well beyond the range shown to exceedingly high wavenumbers, e.g., above 2000.

For the dipole with the same radius as that of the above monopole, the matrices D and M are the same. Thus Eq. (20) was evaluated simply by replacing U_n by a dipole velocity vector to give the corresponding pressure vector. Generally, the same procedure can be used to determine the surface pressure for any arbitrary velocity distribution on the surface of a radiator once the matrices D and M are available. The results for the dipole are shown in Fig. 6. The accuracy in this case is nearly identical to that in the monopole case. An agreement between theory and this numerical experiment at much higher wavenumbers reinforces the observation that the superposition method seems to perform substantially better at high wavenumbers than the boundary-element formulation (see, for example, Ref. 7).

The next set of tests was conducted to examine how the relative position of the inner source surface to that of the radiator influences the computational accuracy. For the cases of the monopole and dipole, the radius of the inner spherical surface was varied from nearly zero to nearly one times the radius of the radiator surface. The results are shown in Fig. 7, which give the normalized pressures as a function of the ratio of the radius of the inner source sphere to that of the radiator. As expected, the accuracy improves as this ratio becomes smaller. For the wavenumber shown ($ka = 1$), the numerically computed pressures are nearly the same as predicted by theory when the ratio is less than approximately 0.4. As the wavenumber increases, for a given accuracy, this preferred ratio should decrease.

The results of the superposition study on the cube radiator are shown in Fig. 8, where the normalized pressures at the center node of a cube radiator are given as a function of *ka* (*a* is defined as half of the length of one side of the cube). In this case, the reference surface velocity is a complex quantity, since it is determined by evaluating the field of a spherical source on the surfaces of a cube. As expected with this type of a reference source, the dependence of the pressure on

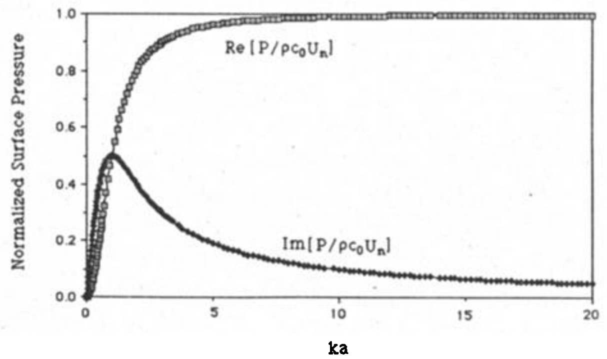

FIG. 5. Variation of normalized surface pressure with nondimensional wavenumber *ka* for a monopole, where *a* is the radius of the monople sphere.

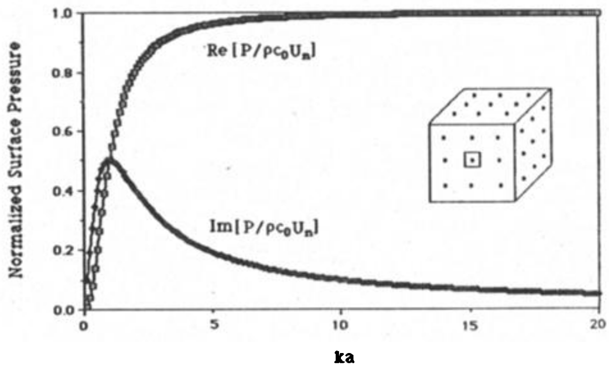

FIG. 7. Variation of normalized surface pressure with nondimensional wavenumber *ka* for a cubic radiator, where *a* is the half-length of one side of the cube.

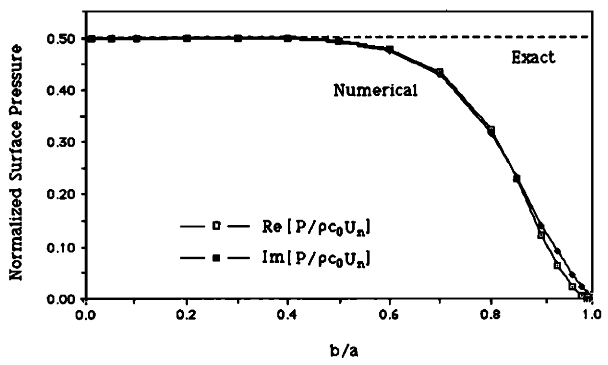

FIG. 8. Variation of normalized surface pressure with relative size of the inner source sphere for the monople and dipole cases, where b is the radius of the inner sphere.

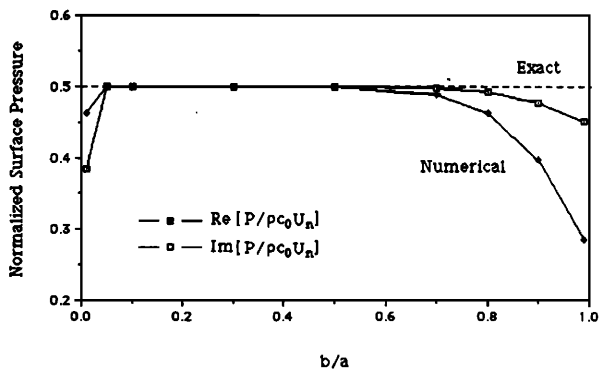

FIG. 10. Variation of normalized surface pressure with relative size of the inner source cube for the cubic radiator, where b is the half length of one side of the inner cube.

ka is similar to that of a spherical radiator. Using the reference velocities as input to the superposition equation (16) at 54 nodal points, the corresponding Q_i's were computed to give the strengths of the 54 sources located within the cube. Their combined fields were then computed via Eq. (20) to give the surface pressures at each of the 54 surface nodes. As in the monopole and dipole studies, the agreement between the exact pressures computed with a single reference source and those computed with the superposition method was excellent, e.g., in this later case approximately to within 0.02%.

Two types of internal source geometries were used in the cubic radiator study: an inner cube and an inner sphere. Both source arrangements predicted accurate surface pressures as long as the ratios of their characteristic lengths (the half-length of one side for the inner cube and the radius for the inner sphere) to the half-length of one side of the radiator were between 0.05 and 0.5, as shown in Figs. 9 and 10. When this ratio is below 0.05, both dipole and monopole matrices become ill-conditioned since the matrix elements all approach nearly the same value. Above 0.5, it appears that the matrices become too diagonally dominant as the singularities in the Green's function and its derivative are approached. As this ratio approaches unity, the pressures approach zero as expected, since the singularity in the dipole matrix D is of a higher order than that in the monopole matrix M in Eq. (20). While this study indicates that the geometry of the inner source has little effect on the computation accuracy, additional studies are needed to provide working guidelines for cases where more complicated geometries are to be treated.

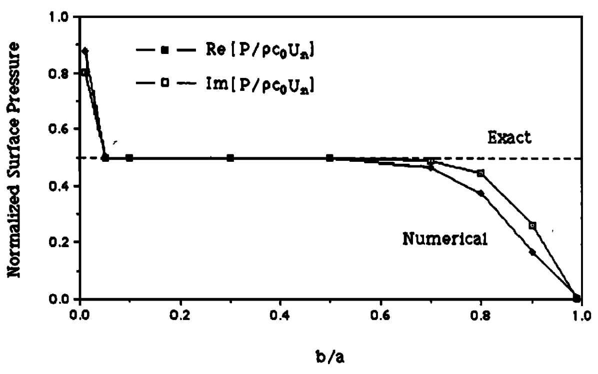

FIG. 9. Variation of normalized surface pressure with relative size of the inner source sphere for the cubic radiator case, where b is the radius of the inner sphere.

III. CONCLUSIONS

It has been shown that a method using the superposition of acoustic fields generated by an array of simple sources can be used to predict the pressures on the surface of a radiator in terms of the surface velocities. This is achieved by enclosing an array of sources within the radiator and computing the source strengths necessary to give the specified velocities on the surface of the radiator. The source strengths are then used to compute the corresponding surface pressures. Should a new surface-velocity distribution be specified, the corresponding change in surface pressure can be computed easily without having to perform a new matrix operation.

The acoustic field generated via the superposition method has been shown to be mathematically equivalent to the Helmholtz formulation, thus proving that the superposition method has a valid formulation.

In numerical experiments involving spherical and cubical radiators, it has been demonstrated that the superposition method has the following advantages over the more commonly used boundary-element method. First, the computations are immensely simplified since the method has neither uniquesness problems nor singularities to deal with. Second, the procedure to generate the matrix elements is simpler since nodes and not elements are the basis for the formulation. Consequently, calculations that require several mathematical operations in the boundary-element method (e.g., Gauss integration routine) are reduced to a single operation in the superposition method. Third, for a wide range of the inner source locations, the accuracy of the superposition method represents at least an order of magnitude improvement over a similar boundary-element computation using the density of nodes.

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