JLI Maths Workbook Solutions

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1 Review of Algebra

1.

$$C(q) = 4 + 2q + \frac{1}{2}q^2$$

$$C(4) = 20$$

$$C(1) = \frac{13}{2}$$

$$C(0) = 4$$

2.

$$(-2)^3 \cdot (-10 + 7) = -8 \cdot -3 = 24$$

3.

$$\text{a) } 3x(x + y - 5) \quad \text{b) } 2\left(z^3 + z - 6\right)$$

4.

$$\text{a) } \tfrac{6a^4b \cdot 4b}{8ab^3c} = \tfrac{3a^3}{bc} \quad \text{b) } \sqrt{\tfrac{3x^3y}{27xy}} = \sqrt{\tfrac{x^2}{9}} = \pm\tfrac{x}{3} \quad \text{c) } \left(2x^3\right)^3 \cdot \left(xz^2\right)^4 = 8x^9 \cdot x^4z^8 = 8x^{13}z^8$$

5.

$$\text{a) } \tfrac{2y}{3x} + \tfrac{4y}{5x} = \tfrac{22y}{15x} \quad \text{b) } \tfrac{x+1}{4} - \tfrac{2x-1}{3} = \tfrac{7-5x}{12}$$

6.

$$\text{a) } x^2 - 7x + 12 = (x-3)(x-4)$$
$$\text{b) } 16y^2 - 25 = (4y+5)(4y-5)$$
$$\text{c) } 3z^2 - 10z - 8 = (3z+2)(z-4)$$

7.

$$\text{a) } 4^{\frac{3}{2}} = 8 \quad \text{b) } \log_{10} 100 = 2 \quad \text{c) } \log_5 125 = 3$$

8.

$$\text{a) } 2\log_a(3x) + \log_a x^2 = \log_a 9x^4 \quad \text{b) } \log_a y - 3\log_a z = \log_a\left(\tfrac{y}{z^3}\right)$$

9.

$$\text{a) } 5(2x-9) = 2(5-3x) \implies 16x = 55 \implies x = \tfrac{55}{16}$$
$$\text{b) } 1 + \tfrac{6}{y-8} = -1 \implies y - 8 + 6 = 8 - y \implies y = 5$$
$$\text{c) } z^{\frac{2}{5}} = 7 \implies z = 7^{\frac{5}{2}}$$
$$\text{d) } 3^{2t-1} = 4 \implies (2t-1)\ln 3 = \ln 4 \implies 2t - 1 = \tfrac{\ln 4}{\ln 3} \implies t = \tfrac{1}{2}\left(1 + \tfrac{\ln 4}{\ln 3}\right)$$

10.

$$\text{a) } ax - 7a = 1 \implies x = \tfrac{1+7a}{a}$$
$$\text{b) } 5x - a = \tfrac{x}{a} \implies x\left(5 - \tfrac{1}{a} = a\right) \implies x = \tfrac{a^2}{5a-1}$$
$$\text{c) } \log_a 2x + 5 = 2 \implies 2x + 5 = a^2 \implies x = \tfrac{a^2-5}{2}$$

11.

$$P = \sqrt{\frac{a}{Q^2+b}} \implies P^2 = \frac{a}{Q^2+b} \implies Q^2 = \frac{a}{P^2} - b \implies Q = \sqrt{\frac{a}{P^2} - b}$$

12.

$$\text{a) } 2x^2+5x-7=0 \implies (2x+7)(x-1)=0 \implies x=-\tfrac{7}{2} \text{ or } x=1$$
$$\text{b) } y^2+3y-\tfrac{1}{2}=0 \implies \left(y+\tfrac{3}{2}\right)^2=\tfrac{11}{4} \implies y=\tfrac{-3\pm\sqrt{11}}{2}$$

13.
a)

$$\begin{array}{c} 2x-y=4 \\ 5x-4y=13 \end{array} \implies \begin{array}{c} 8x-4y=16 \\ 5x-4y=13 \end{array} \implies 13x=29 \implies x=\frac{29}{13},\ y=\frac{6}{13}$$

b)

$$y=\frac{3x+4}{2}=x^2+1 \implies 2x^2-3x-2=0 \implies (2x+1)(x-2)=0 \implies x=-\frac{1}{2} \text{ or } x=2$$

$$x=-\frac{1}{2} \iff y=\frac{5}{4}$$

$$x=2 \iff y=5$$

14.

$$\text{a) } 2y-7\le 3 \implies y\le 5$$
$$\text{b) } 3-z>4+2z \implies -1>3z \implies -\tfrac{1}{3}>z$$
$$\text{c) } 3x^2-5x-2<0 \implies (3x+1)(x-2)>0 \implies -\tfrac{1}{3}<x<2$$

2 Lines and Graphs

3 Sequences, Series and Limits; The Economics of Finance

Quick Questions

1.

$$\text{a) } u_n = 20 - 5n$$
$$\text{b) } u_n = n^3$$
$$\text{c) } u_n = 4u_{n-1} = 4^2u_{n-2}... = 4^n u_0 = \frac{2}{10}4^n = \frac{2^{2n+1}}{10}$$

2.

$$1 + 1 + 9 + 25 + ... + (4n-5)^2 + (2n-3)^2$$

3.

a) $\sum_{i=3}^{n-1} i = \frac{1}{2} \cdot 20 \cdot 19 - 2 - 1 = 187$ b) $\sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i = \frac{1-2^{1-n}}{\frac{1}{2}} = 2 - 2^{2-n}$ c) $\sum_{i=0}^{n} 5 \cdot 2^i = \frac{5(1-2^n)}{1-2} = 5(2^n - 1)$

4.

$$\sum_{i=0}^{n}(4i+3)$$

5.

$$\text{a) } £500 \cdot (1+i)^4 =$$
$$\text{b) } £500 \cdot \left(1 + \tfrac{i}{12}\right)^{4\cdot 12} =; \text{ AER} = \left(1 + \tfrac{i}{12}\right)^{4\cdot 12} - 1 =$$
$$\text{c) } £500 \cdot e^{4i} =$$

If paid annually, savings will exceed £600 when $£500 \cdot (1+i)^\tau > £600 \implies (1+i)^\tau > \frac{6}{5} \implies \tau > \frac{\ln\frac{6}{5}}{\ln 1+i} =$

6.

$$\text{a) } \frac{A}{1+i} + \frac{A}{(1+i)^2} + ... + \frac{A}{(1+i)^N} = \frac{A\left\{1-\left(\frac{A}{1+i}\right)^N\right\}}{i} =$$
$$\text{b) } \frac{A}{1+i} + \frac{A}{(1+i)^2} + ... + \frac{A}{(1+i)^\infty} = \frac{A}{i} =$$

7.

$$\text{a) } \lim_{n\to\infty} 3\left(1 + \tfrac{1}{5^n}\right) = 3$$
$$\text{b) } \lim_{n\to\infty} \frac{5n^2+4n+3}{2n^2+1} = \frac{5+\frac{4}{n}+\frac{3}{n^2}}{2+\frac{1}{n^2}} = \frac{5}{2}$$

Long Questions

1.

a)

i)

$$£30000 \cdot (1+i)^3 =$$

ii)

$$£30000 \cdot (1+i)^n$$

iii)

$$£30000 \left\{(1+i) + (1+i)^2 + ... + (1+i)^{40}\right\} = £30000 \sum_{n=1}^{40} (1+i)^n = £30000 \frac{(1+i)\left\{(1+i)^{40} - 1\right\}}{i} =$$

b)

i)

$$£20000 \frac{(1+i)\left\{(1+i)^{40} - 1\right\}}{i} =$$

ii)

$$£30000(1+i_A)^\tau < £20000(1+i_B)^\tau \implies \frac{3}{2} < \left(\frac{1+i_B}{1+1_A}\right)^\tau \implies \tau > \frac{\ln\frac{3}{2}}{\ln\frac{1+i_B}{1+i_A}} =$$

c)

As an acrobat, earnings increase by 1% each year.

i)

$$PV\Big|_{\text{1st year}} = \frac{£30000 \cdot 1.01}{1+i}$$

ii)

$$PV\Big|_{\text{nth year}} = \frac{£30000 \cdot (1.01)^n}{(1+i)^n}$$

iii)

$$PV\Big|_T = \frac{£30000 \cdot 1.01}{1+i} + \frac{£30000 \cdot (1.01)^2}{(1+i)^2} + ... + \frac{£30000 \cdot (1.01)^{40}}{(1+i)^{40}}$$

$$= £30000 \sum_{n=1}^{40} \left(\frac{1.01}{1+i}^n\right) = £30000 \frac{\frac{1.01}{1+i}\left[1 - \left(\frac{1.01}{1+i}\right)^{40}\right]}{1 - \frac{1.01}{1+i}}$$

d)

Using the above formula with the respective annual earning increase and base salary, the Present Value of both careers at interest rates of 3 and 15% are:

$$PV_{A,3\%} =$$

$$PV_{B,3\%} =$$

$$PV_{A,15\%} =$$

$$PV_{B,15\%} =$$

2.

Assuming that the computer must be repaired each year:

a)

$$PV/£ = 1000 + \left[40 + \frac{35}{1+i} + \frac{30}{(1+i)^2} + \frac{25}{(1+i)^3}\right]$$

$$+\frac{50}{1+i} + \frac{50 \cdot 1.5}{(1+i)^2} + \frac{50 \cdot 1.5^2}{(1+i)^3} + \frac{50 \cdot 1.5^3}{(1+i)^4}$$

$$-100 - 360 - \frac{360}{1+i} - \frac{360}{(1+i)^2} - \frac{360}{(1+i)^3} = (\text{at } i = 10\%)$$

b)

$$PV/£ = 2\left[1000 + 40 + \frac{35}{1+i} + \frac{50}{1+i} + \frac{50 \cdot 1.5}{(1+i)^2}\right]$$

$$-360 - \frac{360}{1+i} - \frac{360}{(1+i)^2} - \frac{360}{(1+i)^3} - 600 = (\text{at } i = 10\%)$$

4 Functions

Quick Questions

1.
a)

$$h\left(\frac{1}{3}\right) = \frac{3}{4}$$

$$g\left(h\left(2\right)\right) = g\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

b)

$$\frac{1}{1+x} = \frac{3}{4} \implies 1 + x = \frac{4}{3} \implies x = \frac{1}{3}$$

c)

$$h\left(f\left(x\right)\right) = \frac{1}{1+2x-5} = \frac{1}{2x-4}$$

$$f^{-1}\left(x\right) = \frac{x+5}{2}$$

$$h^{-1}\left(x\right) = \frac{1}{x} - 1$$

$$f\left(g\left(x\right)\right) = 6x^2 - 5$$

2.
a)

$$Y(0) = Y_0$$

b&c)

$$e^{g\tau} = 2 \implies \tau = \frac{\ln 2}{g} =$$

3.
a)

$$g(0) = 0, \quad g(1) = 1 - e^{-1}, \quad g(2) = 1 - e^{-2}$$

b)

$$g'(x) = e^x > 0 \, \forall x \therefore \text{increasing}$$

c)

$$\lim_{x\to\infty} g(x) = 1$$

d)
[Insert Sketch]
4.
Equilibrium acheived when $P^s = P^d$:

$$1 + Q = a - bQ \implies Q(1+b) = a - 1 \implies Q = \frac{a-1}{1+b}$$

To be positive, either both numerator and denominator are positive, or both are negative:

$$(a > 1 \wedge b > -1) \vee (a < 1 \wedge b < -1)$$

5.

a)

$$g(\lambda z, \lambda t) = 2\lambda^3 t^2 z = \lambda^3 g(z,t) \therefore \text{homogeneous of degree } 3.$$

b)

$$h(\lambda a, \lambda b) = \left(\lambda^2 a^2 + \lambda^2 b^2\right)^{\frac{1}{3}} = \lambda^{\frac{2}{3}} \left(a^2 + b^2\right)^{\frac{1}{3}} = \lambda^{\frac{2}{3}} h(a,b) \therefore \text{homogeneous of degree } \frac{2}{3}.$$

Long Questions

1.