# JLI Maths Workbook Solutions

J.J. Winterburn

Comments and corrections to jjw79@cam.ac.uk

# 1 Review of Algebra

1.

$$C(q) = 4 + 2q + \frac{1}{2}q^2$$

$$C(4) = 20$$

$$C(1) = \frac{13}{2}$$

$$C(0) = 4$$

2.

$$(-2)^3 \cdot (-10+7) = -8 \cdot -3 = 24$$

3.

$$\text{a) } 3x(x+y-5) \quad \text{b) } 2\left(z^3+z-6\right)$$

4.

$$\text{a) } \tfrac{6a^4b \cdot 4b}{8ab^3c} = \tfrac{3a^3}{bc} \quad \text{b) } \sqrt{\tfrac{3x^3y}{27xy}} = \sqrt{\tfrac{x^2}{9}} = \pm\tfrac{x}{3} \quad \text{c) } \left(2x^3\right)^3 \cdot \left(xz^2\right)^4 = 8x^9 \cdot x^4z^8 = 8x^{13}z^8$$

5.

$$\text{a) } \tfrac{2y}{3x} + \tfrac{4y}{5x} = \tfrac{22y}{15x} \quad \text{b) } \tfrac{x+1}{4} - \tfrac{2x-1}{3} = \tfrac{7-5x}{12}$$

6.

$$\text{a) } x^2 - 7x + 12 = (x-3)(x-4)$$
$$\text{b) } 16y^2 - 25 = (4y+5)(4y-5)$$
$$\text{c) } 3z^2 - 10z - 8 = (3z+2)(z-4)$$

7.

$$\text{a) } 4^{\frac{3}{2}} = 8 \quad \text{b) } \log_{10} 100 = 2 \quad \text{c) } \log_5 125 = 3$$

8.

$$\text{a) } 2\log_a(3x) + \log_a x^2 = \log_a 9x^4 \quad \text{b) } \log_a y - 3\log_a z = \log_a\left(\tfrac{y}{z^3}\right)$$

9.

$$\text{a) } 5(2x-9) = 2(5-3x) \implies 16x = 55 \implies x = \tfrac{55}{16}$$
$$\text{b) } 1 + \tfrac{6}{y-8} = -1 \implies y - 8 + 6 = 8 - y \implies y = 5$$
$$\text{c) } z^{\frac{2}{5}} = 7 \implies z = 7^{\frac{5}{2}}$$
$$\text{d) } 3^{2t-1} = 4 \implies (2t-1)\ln 3 = \ln 4 \implies 2t - 1 = \tfrac{\ln 4}{\ln 3} \implies t = \tfrac{1}{2}\left(1 + \tfrac{\ln 4}{\ln 3}\right)$$

10.

$$\text{a) } ax - 7a = 1 \implies x = \tfrac{1+7a}{a}$$
$$\text{b) } 5x - a = \tfrac{x}{a} \implies x\left(5 - \tfrac{1}{a} = a\right) \implies x = \tfrac{a^2}{5a-1}$$
$$\text{c) } \log_a 2x + 5 = 2 \implies 2x + 5 = a^2 \implies x = \tfrac{a^2-5}{2}$$

11.

$$P = \sqrt{\frac{a}{Q^2 + b}} \implies P^2 = \frac{a}{Q^2 + b} \implies Q^2 = \frac{a}{P^2} - b \implies Q = \sqrt{\frac{a}{P^2} - b}$$

12.

a) $2x^2 + 5x - 7 = 0 \implies (2x+7)(x-1) = 0 \implies x = -\frac{7}{2}$ or $x = 1$

b) $y^2 + 3y - \frac{1}{2} = 0 \implies \left(y + \frac{3}{2}\right)^2 = \frac{11}{4} \implies y = \frac{-3 \pm \sqrt{11}}{2}$

13.

a)

$$\begin{array}{c} 2x - y = 4 \\ 5x - 4y = 13 \end{array} \implies \begin{array}{c} 8x - 4y = 16 \\ 5x - 4y = 13 \end{array} \implies 13x = 29 \implies x = \frac{29}{13},\ y = \frac{6}{13}$$

b)

$$y = \frac{3x+4}{2} = x^2 + 1 \implies 2x^2 - 3x - 2 = 0 \implies (2x+1)(x-2) = 0 \implies x = -\frac{1}{2} \text{ or } x = 2$$

$$x = -\frac{1}{2} \iff y = \frac{5}{4}$$

$$x = 2 \iff y = 5$$

14.

a) $2y - 7 \leq 3 \implies y \leq 5$

b) $3 - z > 4 + 2z \implies -1 > 3z \implies -\frac{1}{3} > z$

c) $3x^2 - 5x - 2 < 0 \implies (3x+1)(x-2) > 0 \implies -\frac{1}{3} < x < 2$

# 2 Lines and Graphs

# 3 Sequences, Series and Limits; The Economics of Finance

## Quick Questions

1.

$$\text{a) } u_n = 20 - 5n$$
$$\text{b) } u_n = n^3$$
$$\text{c) } u_n = 4u_{n-1} = 4^2 u_{n-2} ... = 4^n u_0 = \frac{2}{10} 4^n = \frac{2^{2n+1}}{10}$$

2.

$$1 + 1 + 9 + 25 + ... + (3n-5)^2 + (2n-3)^2$$

3.

a) $\sum_{i=3}^{n-1} i = \frac{1}{2} \cdot 21 \cdot 20 - 2 - 1 = 207$ b) $\sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i = \frac{1-2^{n-1}}{\frac{1}{2}} = 2 - 2^n$ c) $\sum_{i=0}^{n} 5 \cdot 2^i = \frac{5(1-2^n)}{1-2} = 5(2^n - 1)$

4.

$$\sum_{i=0}^{n} (4i + 3)$$

5.

$$\text{a) } £500 \cdot (1+i)^4 = £562.75$$
$$\text{b) } £500 \cdot \left(1 + \frac{i}{12}\right)^{4 \cdot 12} = £563.66;\ \text{AER} = \left(1 + \frac{i}{12}\right)^{4 \cdot 12} - 1 = 3.04\%$$
$$\text{c) } £500 \cdot e^{4i} = £563.75$$

If paid annually, savings will exceed £600 when $£500 \cdot (1+i)^\tau > £600 \implies (1+i)^\tau > \frac{6}{5} \implies \tau > \frac{\ln \frac{6}{5}}{\ln 1+i} = 6.16$ years

6.

$$\text{a) } \frac{A}{1+i} + \frac{A}{(1+i)^2} + ... + \frac{1}{(1+i)^N} = \frac{A\left\{1 - \left(\frac{A}{1+i}\right)^N\right\}}{i} = £1246.22$$
$$\text{b) } \frac{A}{1+i} + \frac{A}{(1+i)^2} + ... + \frac{A}{(1+i)^\infty} = \frac{A}{i} = £2000$$

7.

$$\text{a) } \lim_{n\to\infty} 3\left(1 + \frac{1}{5^n}\right) = 3$$
$$\text{b) } \lim_{n\to\infty} \frac{5n^2+4n+3}{2n^2+1} = \frac{5 + \frac{4}{n} + \frac{3}{n^2}}{2 + \frac{1}{n^2}} = \frac{5}{2}$$
$$\text{c) } \sum_{i=1}^{\infty} (0.75)^i = 3$$

# Long Questions

1.

a)

i)

$$£30000 \cdot (1+i)^3 = £30909.03$$

ii)

$$£30000 \cdot (1+i)^n$$

iii)

$$£30000\left\{(1+i)+(1+i)^2+...+(1+i)^{40}\right\} = £30000\sum_{n=1}^{40}(1+i)^n = £1481257£30000\frac{(1+i)\left\{(1+i)^{40}-1\right\}}{i} = £2536795$$

b)

i)

$$£20000\frac{(1+i)\left\{(1+i)^{40}-1\right\}}{i}$$

ii)

$$£30000(1+i_A)^\tau < £20000(1+i_B)^\tau \implies \frac{3}{2} < \left(\frac{1+i_B}{1+1_A}\right)^\tau \implies \tau > \frac{\ln\frac{3}{2}}{\ln\frac{1+i_B}{1+i_A}} = 10.4 \therefore \text{ 11th year.}$$

c)

As an acrobat, earnings increase by 1% each year.

i)

$$PV\Big|_{\text{1st year}} = \frac{£30000 \cdot 1.01}{1+i}$$

ii)

$$PV\Big|_{\text{nth year}} = \frac{£30000 \cdot (1.01)^n}{(1+i)^n}$$

iii)

$$PV\Big|_T = \frac{£30000 \cdot 1.01}{1+i} + \frac{£30000 \cdot (1.01)^2}{(1+i)^2} + ... + \frac{£30000 \cdot (1.01)^{40}}{(1+i)^{40}}$$

$$= £30000\sum_{n=1}^{40}\left(\frac{1.01}{1+i}^n\right) = £30000\frac{\frac{1.01}{1+i}\left[1-\left(\frac{1.01}{1+i}\right)^{40}\right]}{1-\frac{1.01}{1+i}}$$

d)

Using the above formula with the respective annual earning increase and base salary, the Present Value of both careers at interest rates of 3% and 15% are:

$$PV_{A,3\%} = £82352.63$$

$$PV_{B,3\%} = £1216064.53$$

$$PV_{A,15\%} = £215225.60$$

$$PV_{B,15\%} = £204480.78$$

2.

Assuming that the computer must be repaired each year:
a)

$$PV/£ = 1000 + \left[40 + \frac{35}{1+i} + \frac{30}{(1+i)^2} + \frac{25}{(1+i)^3}\right]$$

$$+\frac{50}{1+i} + \frac{50\cdot 1.5}{(1+i)^2} + \frac{50\cdot 1.5^2}{(1+i)^3} + \frac{50\cdot 1.5^3}{(1+i)^4}$$

$-\frac{100}{(1+i)^4} - 360 - \frac{360}{1+i} - \frac{360}{(1+i)^2} - \frac{360}{(1+i)^3} = +£99.04$(at $i = 10\%$).Hence the employer should not agree as there is a cost to the em

Denoting the price of the laptop as $x$, we require:

$$x - \frac{£100}{1.1^4} - £832.65 \leq 0 \implies x \leq £832.65 + \frac{£100}{1.1^4} \implies x \leq £900.95$$

b)

$$PV/£ = 1000 + 40 + \frac{35}{1+i} + \frac{50}{1+i} + \frac{75}{(1+i)^2} + \frac{1000}{(1+i)^2} + \frac{40}{(1+i)^2} + \frac{35}{(1+i)^3} + \frac{50}{(1+i)^3} + \frac{75}{(1+i)^4}$$

$$-360 - \frac{360}{1+i} - \frac{360}{(1+i)^2} - \frac{360}{(1+i)^3} - \frac{600}{(1+i)^2} - \frac{600}{(1+i)^6} = £64.03 \text{ (at } i = 10\%)$$

# 4 Functions

## Quick Questions

1.
a)

$$h\left(\frac{1}{3}\right) = \frac{3}{4}$$

$$g\left(h\left(2\right)\right) = g\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

b)

$$\frac{1}{1+x} = \frac{3}{4} \implies 1+x = \frac{4}{3} \implies x = \frac{1}{3}$$

c)

$$h\left(f\left(x\right)\right) = \frac{1}{1+2x-5} = \frac{1}{2x-4}$$

$$f^{-1}\left(x\right) = \frac{x+5}{2}$$

$$h^{-1}\left(x\right) = \frac{1}{x} - 1$$

$$f\left(g\left(x\right)\right) = 6x^2 - 5$$

2.
a)

$$Y(0) = Y_0$$

b&c)

$$e^{g\tau} = 2 \implies \tau = \frac{\ln 2}{g} =$$

3.
a)

$$g(0) = 0, \quad g(1) = 1 - e^{-1}, \quad g(2) = 1 - e^{-2}$$

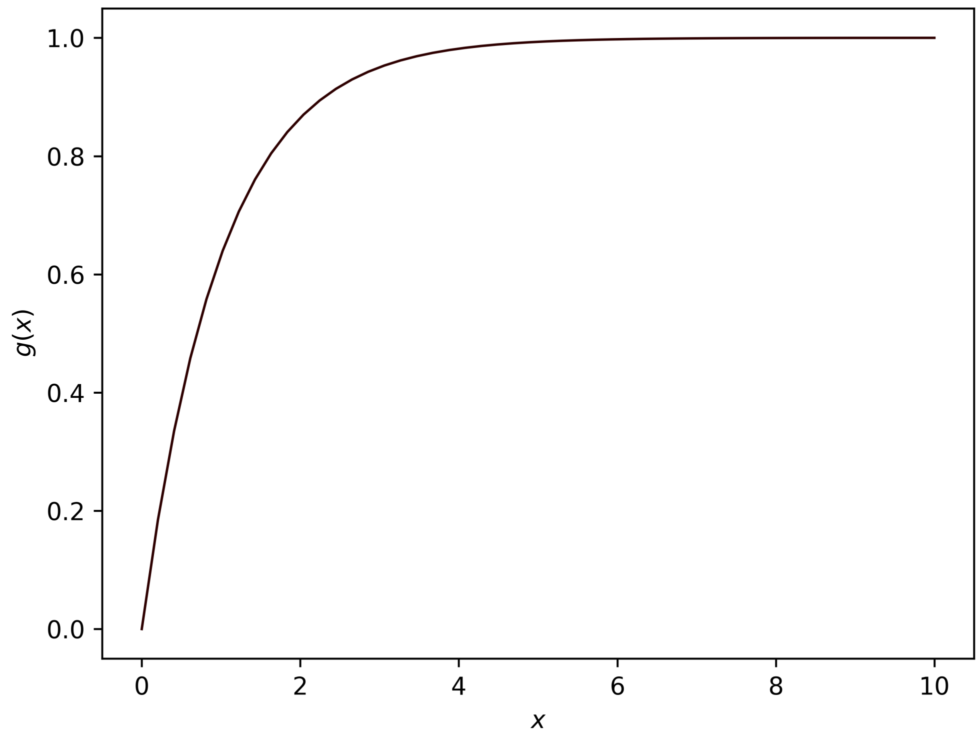
b)

$$g'(x) = e^x > 0\,\forall x \therefore \text{increasing}$$

c)

$$\lim_{x\to\infty} g(x) = 1$$

d)




4.
Equilibrium acheived when $P^s = P^d$:

$$1 + Q = a - bQ \implies Q(1+b) = a - 1 \implies Q = \frac{a-1}{1+b}$$

To be positive, either both numerator and denominator are positive, or both are negative:

$$(a > 1 \wedge b > -1) \vee (a < 1 \wedge b < -1)$$

5.
a)

$$g(\lambda z, \lambda t) = 2\lambda^3 t^2 z = \lambda^3 g(z,t) \therefore \text{homogeneous of degree 3.}$$

b)

$$h(\lambda a, \lambda b) = \left(\lambda^2 a^2 + \lambda^2 b^2\right)^{\frac{1}{3}} = \lambda^{\frac{2}{3}} \left(a^2 + b^2\right)^{\frac{1}{3}} = \lambda^{\frac{2}{3}} h(a,b) \therefore \text{homogeneous of degree } \frac{2}{3}.$$

# Long Questions

1.

a)

$$q^d(5) = 100\left(\frac{12}{5} - 1\right) = 140$$

b)

$$q^d = 0 \implies \frac{12}{p} = 1 \implies p = 12$$

c)

$$q^s = q^d \implies 50p = 100\left(\frac{12}{p} - 1\right) \implies p^2 + 2p - 24 = 0 \implies (p+6)(p-4) = 0 \implies p = 4$$

$$p = 4 \implies q^s(4) = q^d(4) = 200$$

d)

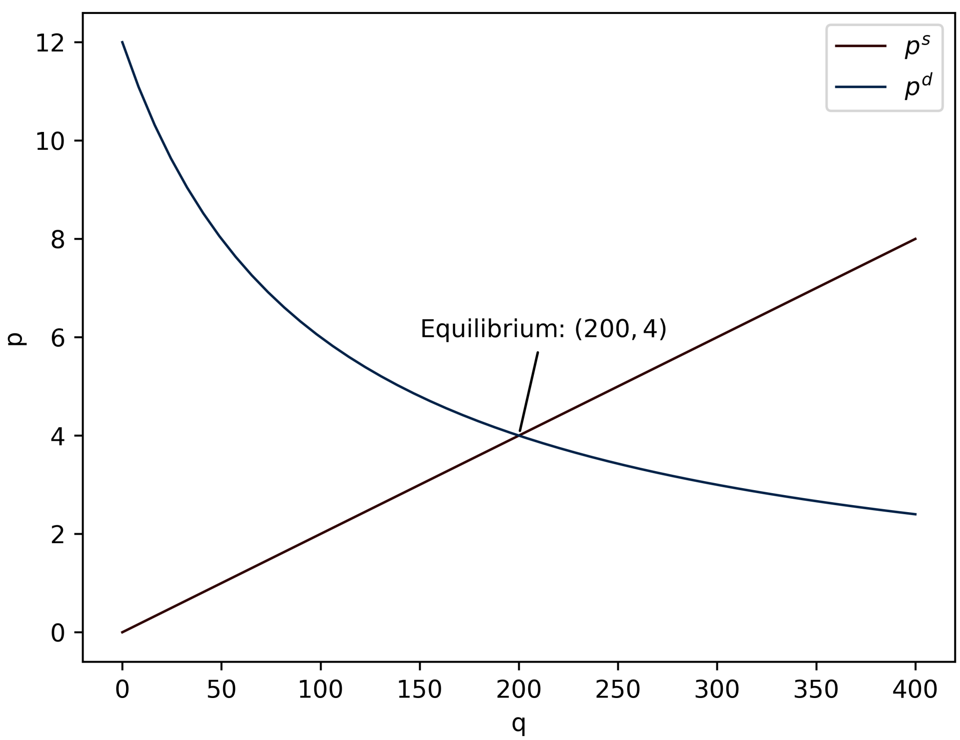
$$p^s(q) = \frac{q}{50}$$

$$p^d(q): \quad \frac{12}{p} - 1 = \frac{q}{100} \implies \frac{12}{p} = \frac{q}{100} + 1 \implies p^d(q) = \frac{1200}{q+100}$$

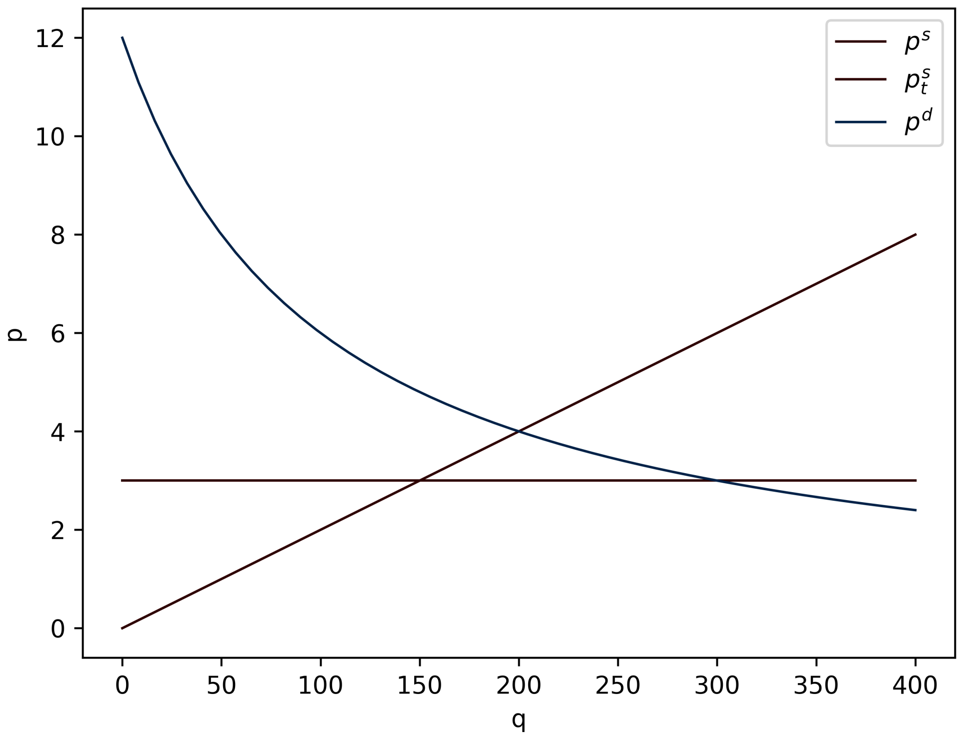
e)

$$\lim_{q\to\infty} p^d(q) = 0$$

f)




g)



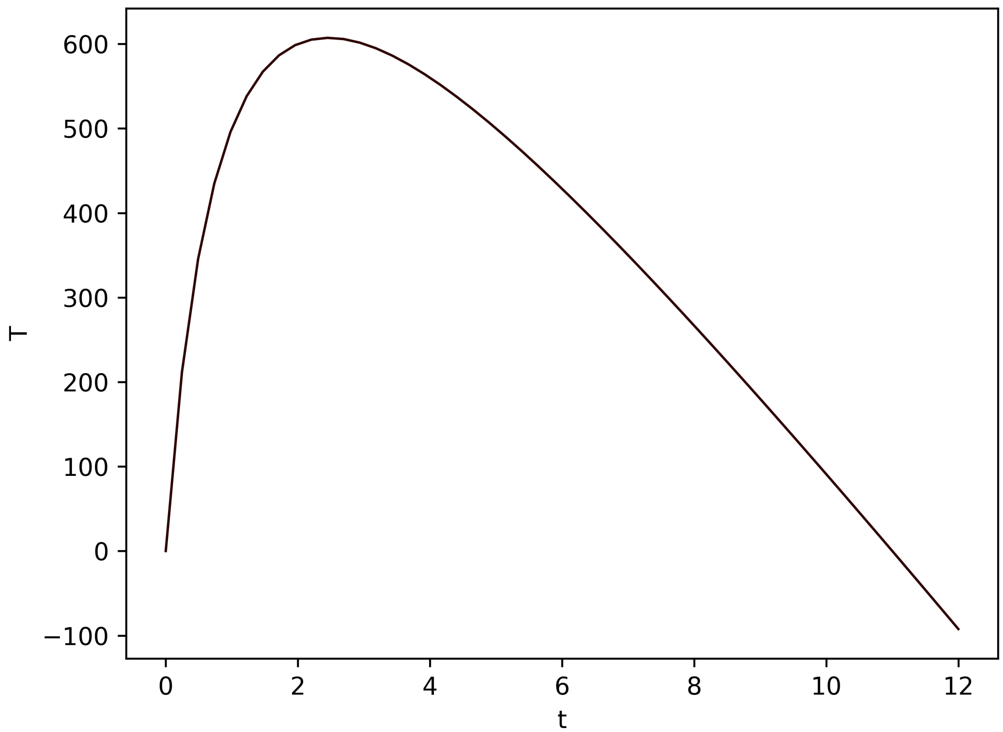

h)

$$1 + t = \frac{1200}{q_e + 100} \implies q_e + 100 = \frac{1200}{1+t} \implies q_e = \frac{1200}{1+t} - 100$$

$$p(q_e) = 1 + t$$

i)
Total taxed raised is $T = qt$ :

$$T = qt = \frac{1200t}{1+t} - 100t$$




2.

a)

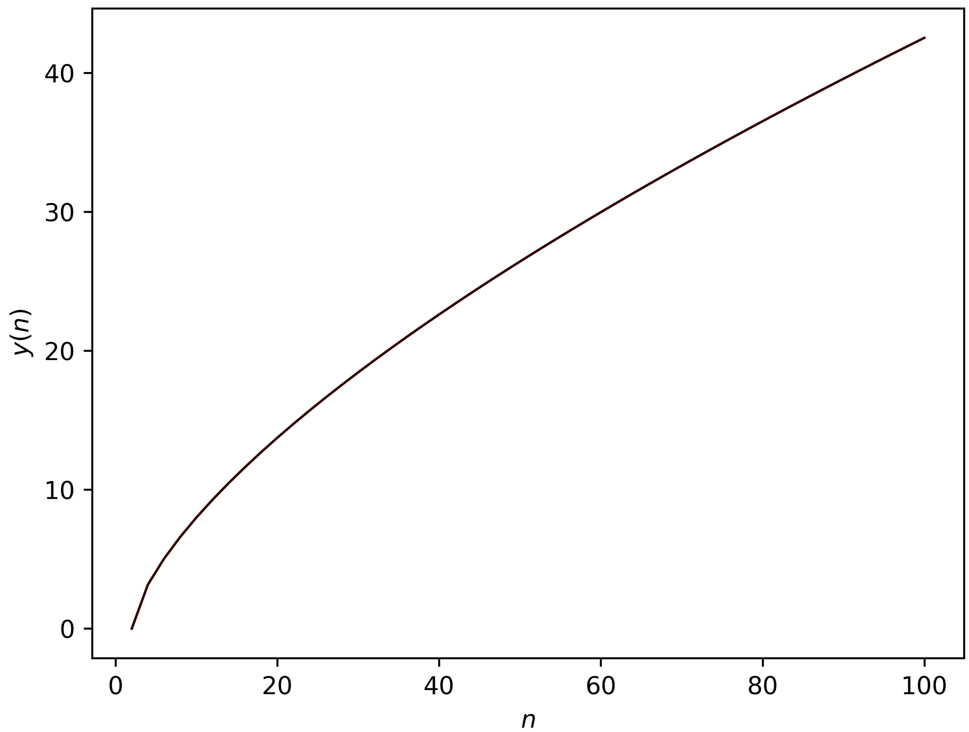
$$Y(\lambda n, \lambda m) = \left(\lambda n - \frac{\lambda m}{4}\right)^{\frac{2}{3}} \lambda^{\frac{1}{3}} m^{\frac{1}{3}} = \lambda^{\frac{2}{3}} \lambda^{\frac{1}{3}} \left(n - \frac{m}{4}\right)^{\frac{2}{3}} m^{\frac{1}{3}} = \lambda Y(n, m) \therefore \text{ homogenous of degree 1 - constant returns to scale.}$$

b)

i)

$$y(n) = Y(n, 8) = 2(n-2)^{\frac{2}{3}}$$

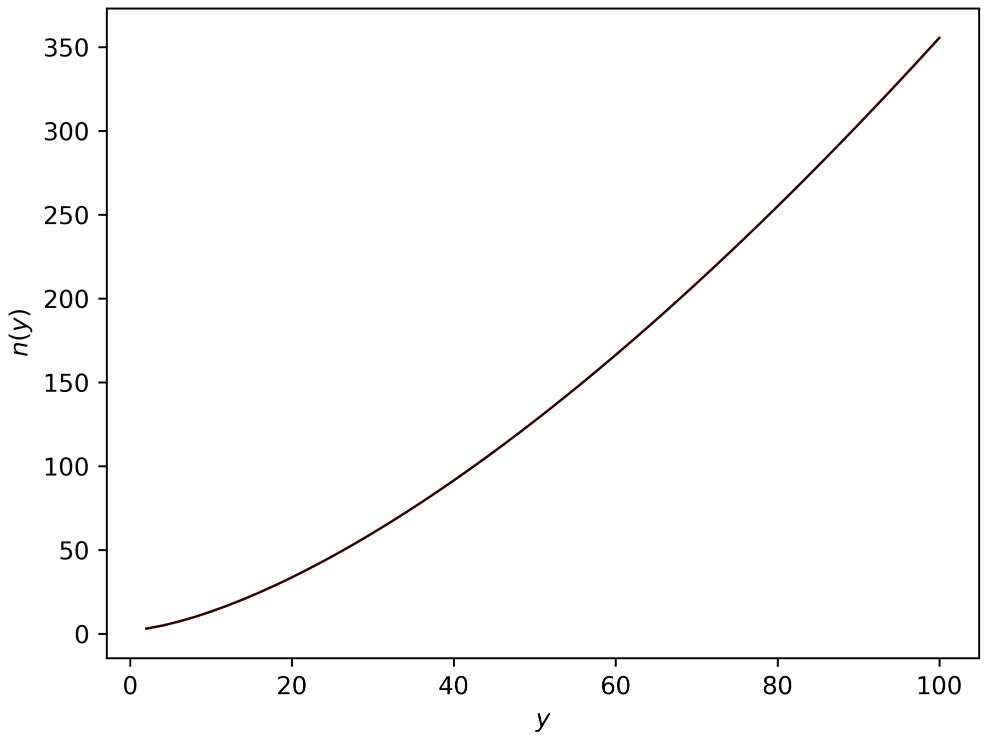
ii)




Decreasing returns to labour.

iii)

$$\frac{y}{2} = (n-2)^{\frac{2}{3}} \implies n = 2 + \left(\frac{y}{2}\right)^{\frac{3}{2}}$$

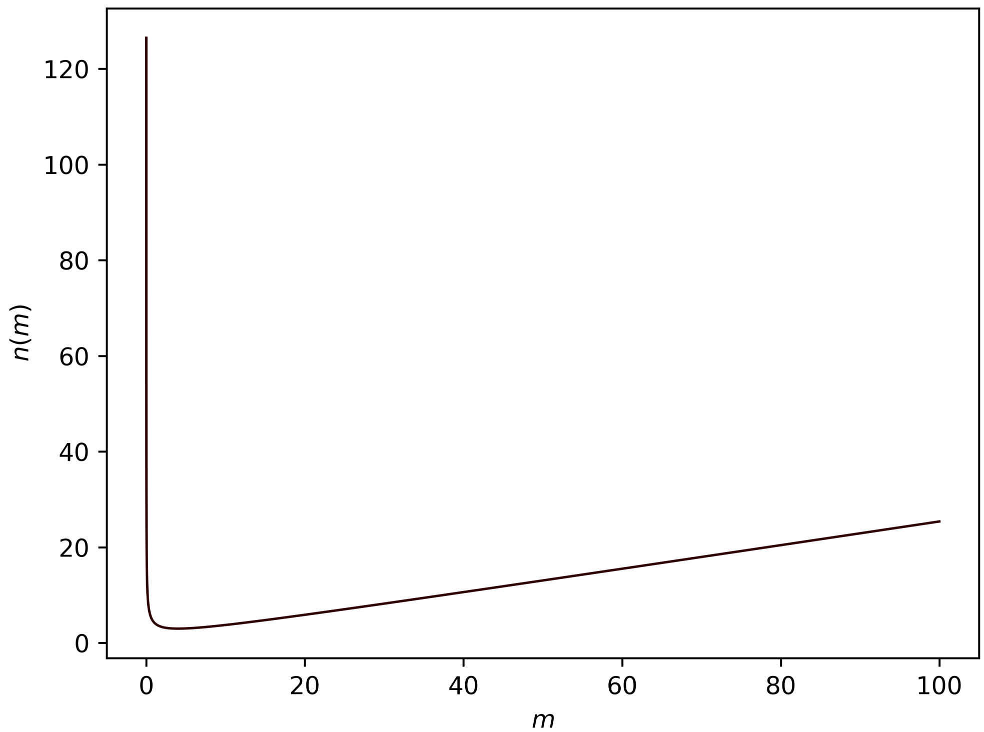



$$n(16) = 2 + 8^{\frac{3}{2}}$$

c)

i)

$$\left(n - \frac{m}{4}\right)^{\frac{2}{3}} m^{\frac{1}{3}} = 16 \implies \left(n - \frac{m}{4}\right)^2 = \frac{16}{m} \implies n = \frac{m}{4} + \sqrt{\frac{16}{m}}$$




As m increases, n must also increase in order to maintain the same output. Hence it would not be sensible to invest in a large number of machines, as the same output could be acheived with fewer machines, and fewer workers.

# 5 Differentiation

## Quick Questions

1.

$$\text{a) } \tfrac{\mathrm{d}y}{\mathrm{d}x} = 27x^2 - 14x$$
$$\text{b) } \tfrac{\mathrm{d}f}{\mathrm{d}x} = -\tfrac{3}{2x^3}$$
$$\text{c) } \tfrac{\mathrm{d}Y}{\mathrm{d}t} = 130t^{0.3}$$
$$\text{d) } \tfrac{\mathrm{d}P}{\mathrm{d}Q} = 2(Q - Q^{-\frac{1}{2}})$$

2.

$$\text{a) } y''(x) = 10 \geq 0 \quad \forall x \;\therefore\; \text{convex}$$
$$\text{b) } C''(y) = -\tfrac{1}{2y^{\frac{1}{2}}} \leq 0 \quad \forall y \geq 0 \;\therefore\; \text{concave}$$
$$\text{c) } P''(q) = 2 + \tfrac{1}{q^{\frac{3}{2}}} \geq 0 \quad \forall q \geq 0 \;\therefore\; \text{convex}$$
$$\text{d) } k''(x) = 2 - 6x \;\therefore\; \text{neither}$$

3.

$$F'(L) = 100 + \frac{400}{3}L^{-\frac{1}{3}}$$

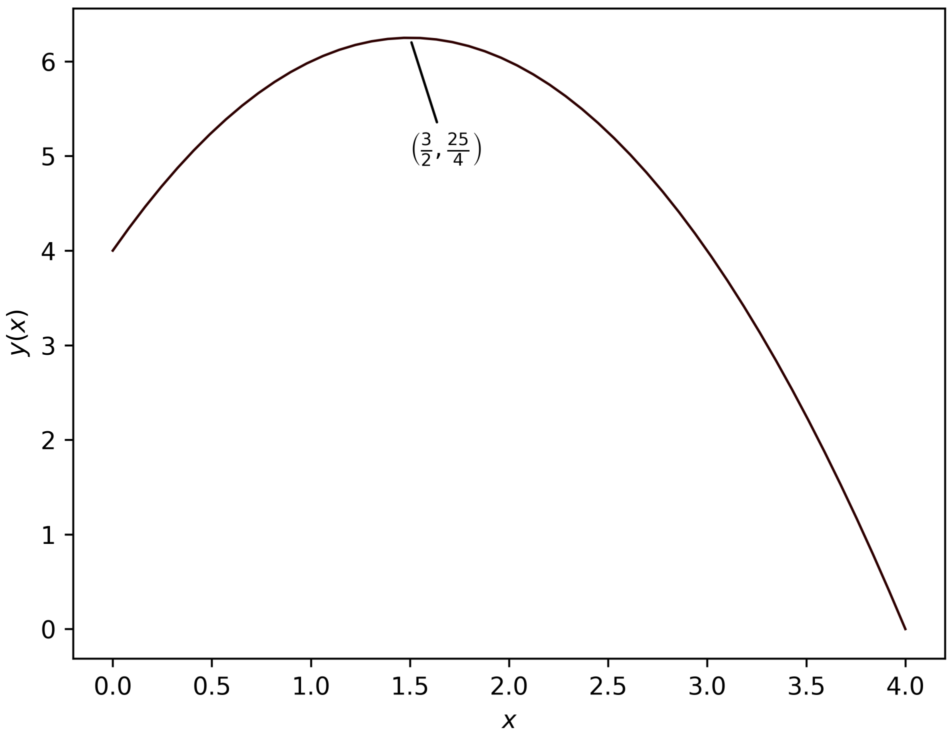
$$F''(L) = -\frac{400}{9}L^{-\frac{4}{3}}$$

$$F''(L) \leq 0 \quad \forall L \geq 0 \;\therefore\; \text{diminishing returns to labour.}$$

4.
a)

$$y = 3x - x^2 + 4 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = 3 - 2x \overset{!}{=} 0 \implies x = \frac{3}{2},\ y\left(\frac{3}{2}\right) = \frac{25}{4}$$

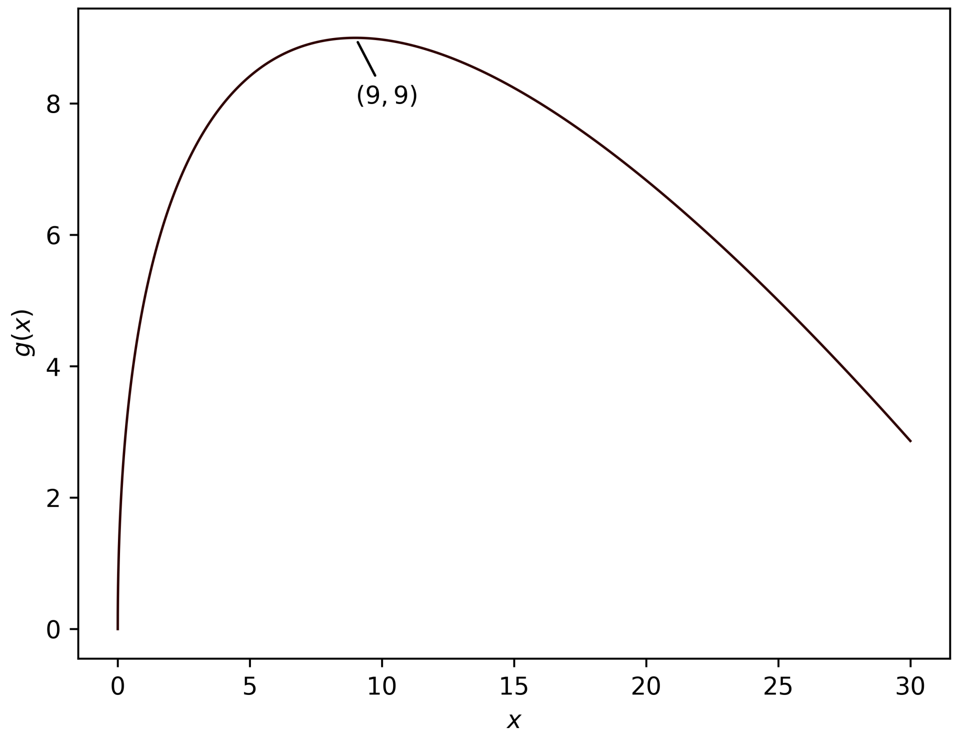
$$y''(x) = -2 < 0 \;\therefore\; \text{maximum}$$




b)

$$g = 6x^{\frac{1}{2}} - x \implies \frac{\mathrm{d}g}{\mathrm{d}x} = 3x^{-\frac{1}{3}} - 1 \stackrel{!}{=} 0 \implies x^{\frac{1}{2}} = 3 \implies x = 9,\ g(9) = 9$$

$$g''(9) = -\frac{3}{2}9^{-\frac{3}{2}} = -\frac{3}{2} \cdot 3^{-3} < 0 \ \therefore \text{ maximum}$$

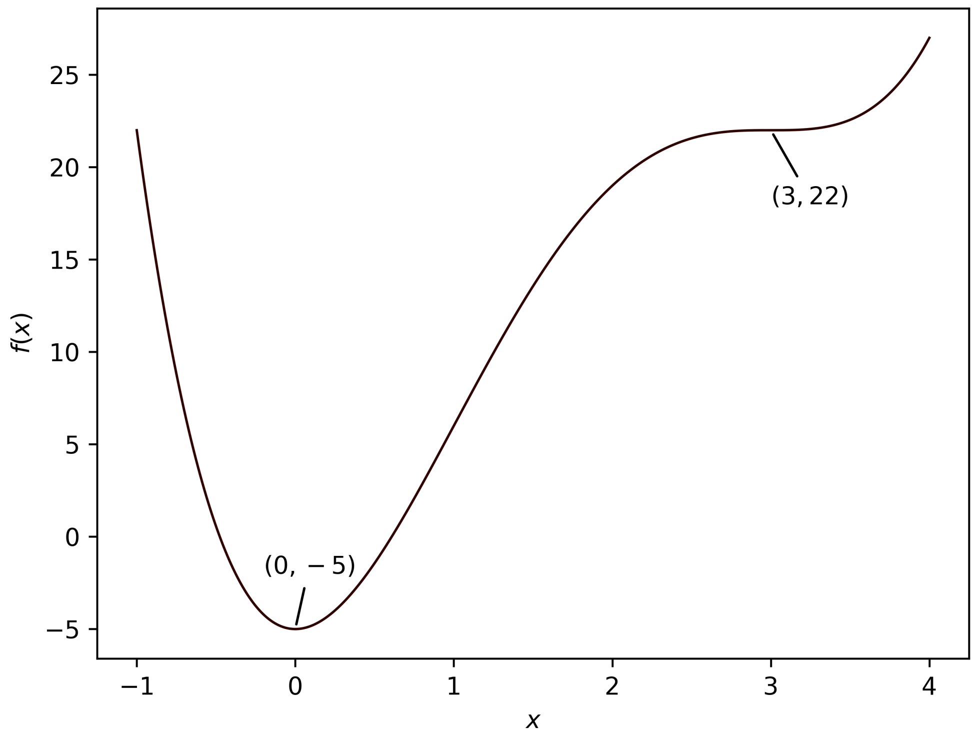



c)

$$f = x^4 - 8x^3 + 18x^2 - 5 \implies \frac{\mathrm{d}f}{\mathrm{d}x} = 4x^3 - 24x^2 + 36x \overset{!}{=} 0 \implies x(x-3)^2 = 0 \implies x = 0,\ x = 3$$

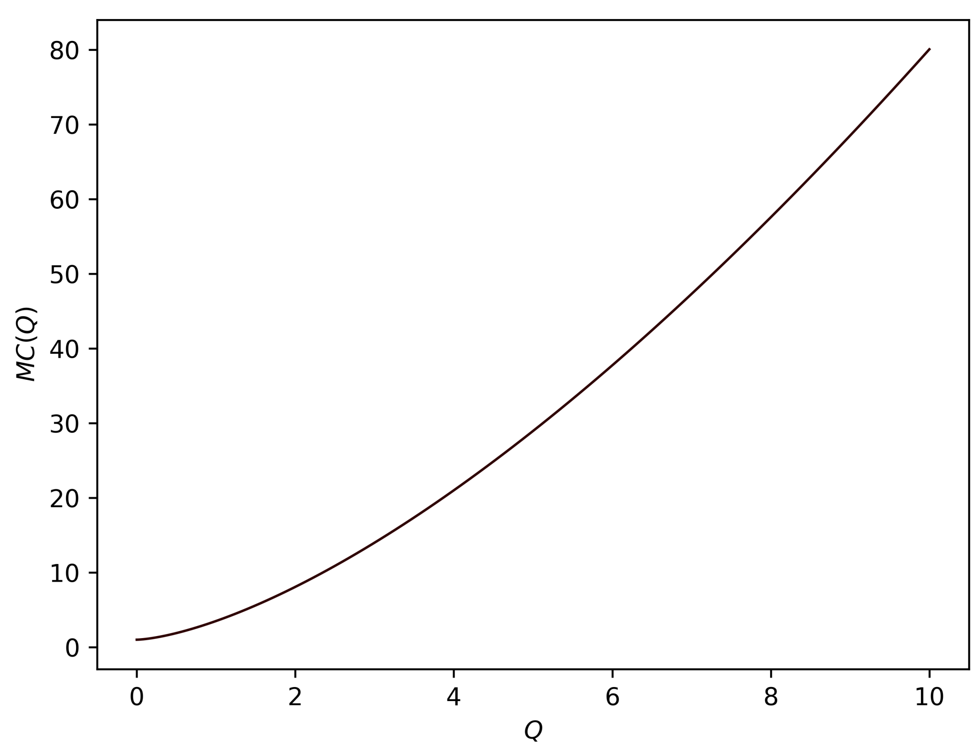
$$x = 0,\ f(0) = -5,\ f''(0) = 36\ \therefore\ \text{minimum}$$

$$x = 3,\ f(3) = 22,\ f''(3) = 0\ \therefore\ \text{inflection}$$




5.

$$MC(Q) = \frac{\mathrm{d}C}{\mathrm{d}Q} = a\left[b + \frac{5}{2}Q^{\frac{3}{2}}\right]$$



$$C''(Q) = a \cdot \frac{15}{4} Q^{\frac{1}{2}} \geq 0 \quad \forall Q \geq 0 \therefore \text{ convex.}$$

## Long Questions

1.
a)

$$MPL = \frac{\mathrm{d}y}{\mathrm{d}n} = 60 - \frac{3}{5}n^2$$

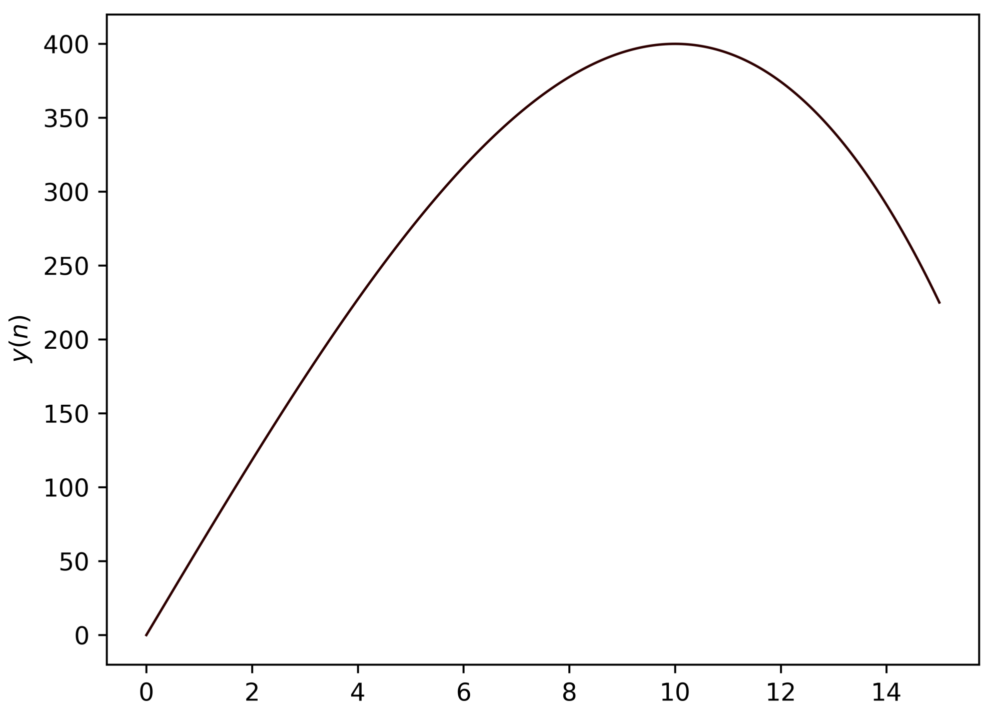
b)

$$y''(n) = -\frac{6}{5}n \leq 0 \quad \forall n \geq 0 \therefore \text{ decreasing returns to labour.}$$

c)

| $n$ | 1 | 5 | 10 | 15 |
|---|---|---|---|---|
| $y(n)$ | 59.8 | 275 | 400 | 225 |
| $A(n) = \frac{y(n)}{n}$ | 59.8 | 55 | 40 | 15 |
| $MPL(n) = \frac{\mathrm{d}y(n)}{\mathrm{d}n}$ | 59.4 | 45 | 0 | -75 |

d)
"Too many cooks spoil the broth."
e)




2.
a)

$$q(p) = q(p-1) - 10$$

$$= q(p-2) - 20$$

$$... = q(0) - 10p$$

$$q(10) = q(0) - 100 = 100 \implies q(0) = 200$$

$$\therefore q(p) = 200 - 10p$$

b)

$$C(q) = 5q$$

c)

$$p(q) = 20 - \frac{q}{10}$$

d)

$$\Pi(q) = q \cdot p(q) - C(q) = 15q - \frac{1}{10}q^2$$

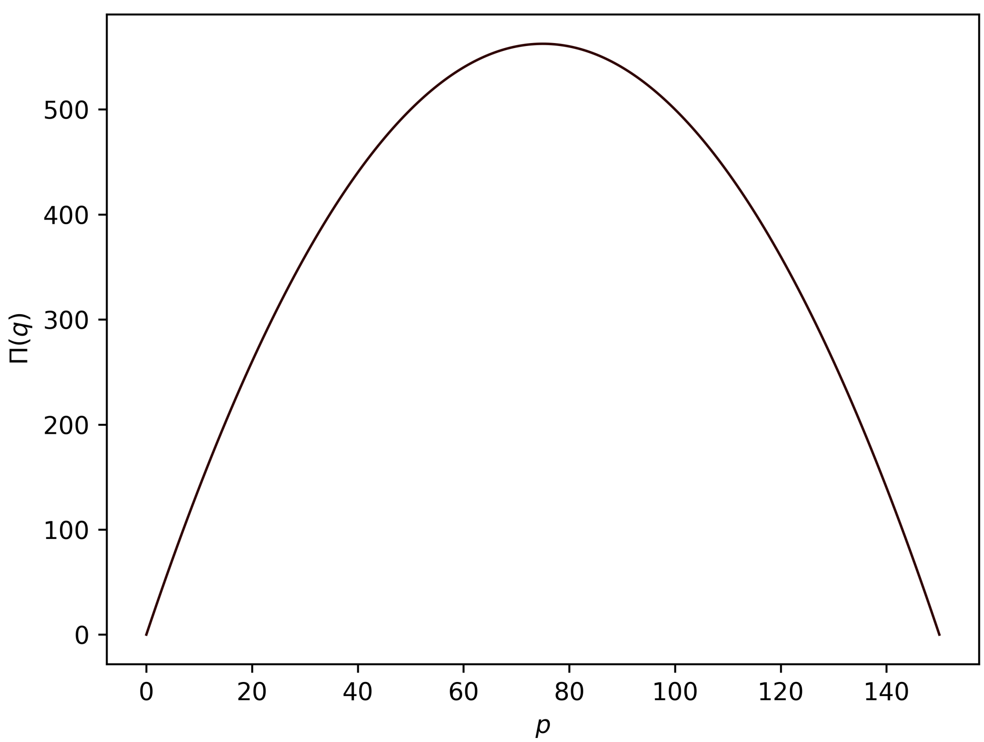
e)

$$\frac{\mathrm{d}\Pi}{\mathrm{d}q} = 15 - \frac{1}{5}q \stackrel{!}{=} 0 \implies q = 75$$

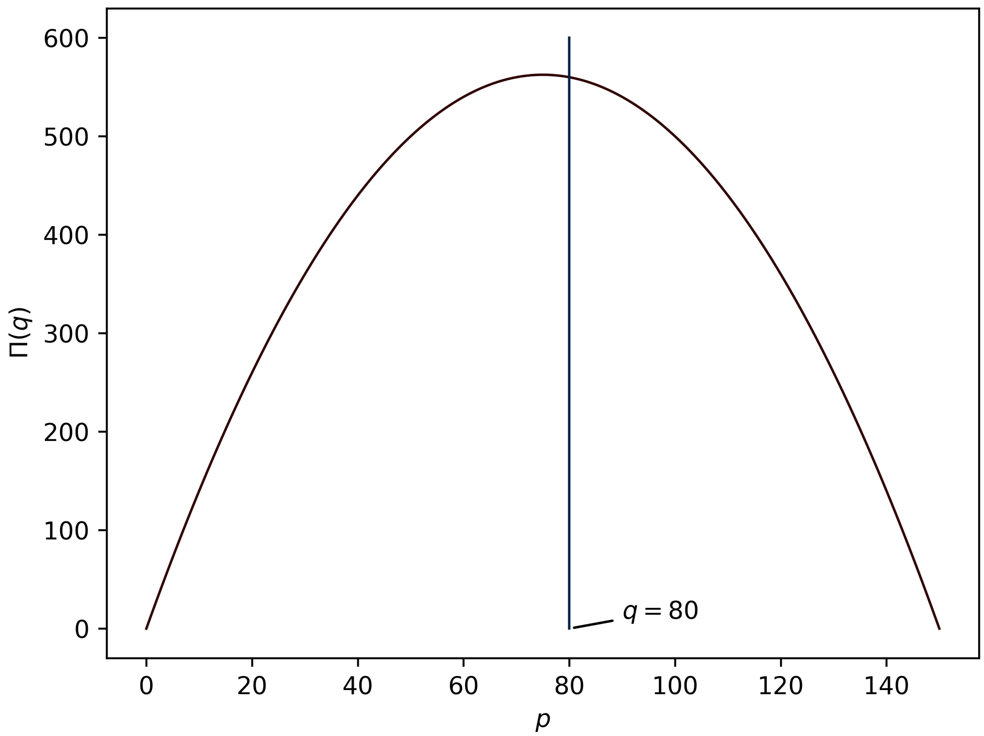
$$p(75) = £13.5$$

$$\Pi(75) = £562.50$$

f)




g)




To maximise profits subject to the new law, we must find the maximal point on the profit curve in the region $q \geq 80$. This is clearly at $q = 80$, so the new price is $p(80) = £12$.