CHAPTER 6
Functions of Random Variables

Motivation

- All quantities used to estimate population parameters or to make decisions about a population are functions of the n random observations that appear in a sample.

- For example, we draw a random sample of n observations, $y_1, y_2, \ldots, y_n$, from the population and employ the sample mean

$$\bar{y} = \frac{y_1 + y_2 + \cdots + y_n}{n} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

  as an estimate for μ

- How good is this estimate?
  - Depends on the behavior of the random variables $Y_1, Y_2, \ldots, Y_n$ which affect on the distribution on

$$\bar{Y} = \frac{Y_1 + Y_2 + \cdots + Y_n}{n} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

Motivation

- A measure of the goodness of an estimate is the error of estimation, the difference between the estimate and the parameter estimated (for our example, the difference between y and μ).
- If we can determine the probability distribution of the estimator $\bar{Y}$

$$\bar{Y} = \frac{Y_1 + Y_2 + \cdots + Y_n}{n} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

  - which is a function of random variables.
  - this probability distribution can be used to determine the probability that the error of estimation is less than or equal to B.
- To determine the probability distribution of n random variables, $Y_1 + Y_2 + \cdots + Y_n$, we must find the joint probability distribution for the random variables.
- We will assume that the random variables obtained through a random sample
  - $p(y_1, y_2, \ldots, y_n) = p(y_1)p(y_2)\cdots p(y_n)$.
  - $f(y_1, y_2, \ldots, y_n) = f(y_1)f(y_2)\cdots f(y_n)$.

Motivation

- Consider random variables $Y_1, Y_2, \dots, Y_n$ and a function $U(Y_1, Y_2, \dots, Y_n)$, denoted simply as U.
- Then three of the methods for finding the probability distribution of U are as follows:
  - The method of distribution functions:
  - The method of transformations
  - The method of moment-generating functions

Procedure

Distribution Function Method

Let U be a function of the random variables $Y_1, Y_2, \dots, Y_n$.

1. Find the region $U = u$ in the $(y_1, y_2, \dots, y_n)$ space.
2. Find the region $U \leq u$.
3. Find $F_U(u) = P(U \leq u)$ by integrating $f(y_1, y_2, \dots, y_n)$ over the region $U \leq u$.
4. Find the density function $f_U(u)$ by differentiating $F_U(u)$. Thus, $f_U(u) = dF_U(u)/du$.

The Method of Distribution Functions : Example

EXAMPLE 6.1

A process for redefining sugar yields up to 1 ton of pure sugar per day, but the actual amount produced, Y, is a random variable because of machine breakdowns and other slowdowns. Suppose that Y has density function given by

$$f(y) = \begin{cases} 2y, & 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}.$$

The company is paid at the rate of \$300 per ton for the refined sugar, but it also has a fixed overhead cost of \$100 per day. Thus the daily profit, in hundreds of dollars, is $U = 3Y - 1$. Find the probability density function for U.

The Method of Distribution Functions : Example

SOLUTION 6.1

$$F_U(u) = P(U \le u) = P(3Y - 1 \le u) = P\left(Y \le \frac{u+1}{3}\right)$$

- If $u < -1, F_U(u) = P(Y \le 0) = 0$. Also, if $u > 2$, then $F_U(u) = P(Y \le 1) = 1$.
- If $-1 \le u \le 2$,

$$P\left(Y \le \frac{u+1}{3}\right) = \int_{-\infty}^{(u+1)/3} f(y)dy = \int_{-\infty}^{(u+1)/3} 2y dy = \left(\frac{u+1}{3}\right)^2.$$

Thus, the distribution function of the random variable U is given by

$$F_U(u) = \begin{cases} 0 & u < -1 \\ \left(\frac{u+1}{3}\right)^2 & -1 \le u \le 2 \\ 1 & u > 2 \end{cases}$$

and the density function for U is

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} (2/9)(u+1), & -1 \le u < 2 \\ 0, & elsewhere \end{cases}$$

The Method of Distribution Functions : Example

EXAMPLE 6.2

In Example 5.4, we considered the random variables Y_1 (the proportional amount of gasoline stocked at the beginning of a week) and Y_2 (the proportional amount of gasoline sold during the week). The joint density function of Y_1 and Y_2 is given by

$$f(y) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the probability density function for $U = Y_1 - Y_2$, the proportional amount of gasoline remaining at the end of the week. Use the density function of U to find $E(U)$.

The Method of Distribution Functions : Example

SOLUTION 6.2

- Consider $y_1 - y_2 = u$.
- If $u < 0$, the line $y_1 - y_2 = u$ has intercept $-u < 0$ and $F_U(u) = P(Y_1 - Y_2 \le u) = 0$.
- When $u > 1$, the line $y_1 - y_2 = u$ has intercept $-u < -1$ and $F_U(u) = 1$.
- For $0 \le u \le 1$,

$$F_U(u) = P(U \le u) = 1 - P(U \ge u) = 1 - \int_u^1 \int_0^{y_1 - u} 3y_1 dy_2 dy_1 = 1 - \int_u^1 3y_1(y_1 - u)dy_1$$

$$= 1 - 3\left(\frac{y_1^3}{3} - \frac{uy_1^2}{2}\right)\Bigg]_u^1 = 1 - \left[1 - \frac{3}{2}(u) + \frac{u^3}{2}\right] = \frac{1}{2}(3u - u^3).$$

- Thus, $F_U(u) = \begin{cases} 0 & u < 0 \\ (3u - u^3)/2 & 0 \le u \le 1. \\ 1 & u > 1 \end{cases}$
- It follows that $f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} 3(1 - u^2)/2, & 0 \le u < 1 \\ 0, & \text{elsewhere} \end{cases}$.
- $E(U) = \int_0^1 u\left(\frac{3}{2}\right)(1 - u^2)du = \frac{3}{2}\left(\frac{u^2}{2} - \frac{u^4}{4}\right)\Big]_0^1 = \frac{3}{8}$.

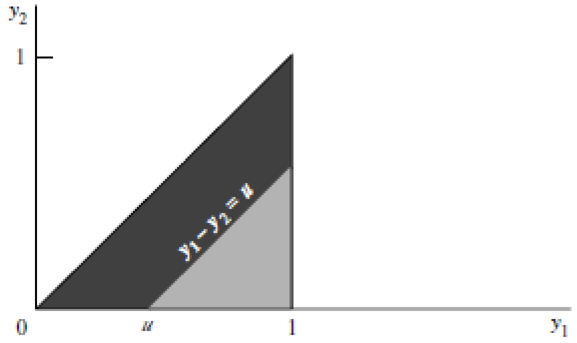


FIGURE 6.1 Region over which $f(y_1, y_2)$ is positive, Example 6.2

The Method of Distribution Functions : Example

EXAMPLE 6.3

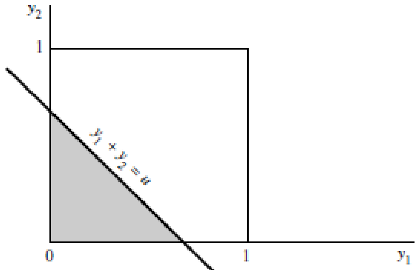
Let (Y_1, Y_2) denote a random sample of size $n = 2$ from the uniform distribution on the interval $(0,1)$. Find the probability density function for $U = Y_1 + Y_2$.

The Method of Distribution Functions : Example

SOLUTION 6.3

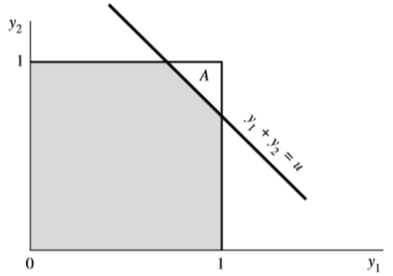
The density function for each Y_i is

$$f(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$



Because Y_1 and Y_2 are independent,

$$f(y_1, y_2) = f(y_1)f(y_2) = \begin{cases} 1 & 0 \le y_1 \le 1, 0 \le y_2 \le 1 \\ 0 & elsewhere \end{cases}$$



Consider $y_1 + y_2 = u$.

- If $u < 0$, $F_U(u) = P(U \le u) = 0$, and for $u > 2$, $F_U(u) = P(U \le u) = 1$.
- If $0 \le u \le 1$, we have

$$F_U(u) = \int_0^u \int_0^{u-y_2} (1) dy_1\, dy_2 = \int_0^u (u - y_2) dy_2 = \left(uy_2 - \frac{y_2^2}{2} \right) \Big]_0^u = u^2 - \frac{u^2}{2} = \frac{u^2}{2}.$$

- For $1 \le u \le 2$,

$$F_{U(u)} = 1 - \int_{u-1}^1 \int_{u-y_2}^1 (1) dy_1 dy_2 = 1 - \int_{u-1}^1 (y_1]_{u-y_2}^1) dy_2 = 1 - \int_{u-1}^1 (1 - u + y_2) dy_2$$
$$= (-u^2/2) + 2u - 1.$$

The Method of Distribution Functions : Example

SOLUTION 6.3

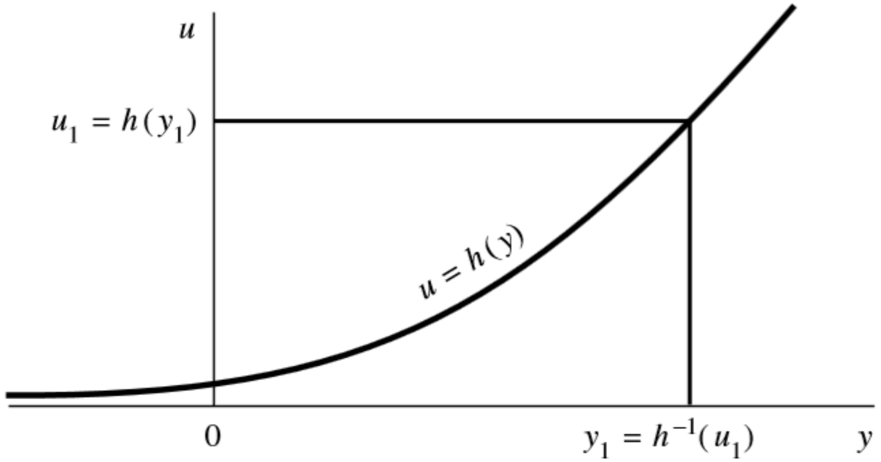
Thus, $F_U(u) = \begin{cases} 0 & u < 0 \\ u^2/2 & 0 \le u \le 1 \\ (-u^2/2) + 2u - 1 & 1 < u \le 2 \\ 1 & u > 2 \end{cases}.$

The density function $f_U(u)$ is given by

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} \frac{d}{du}(u^2/2) = u & 0 \le u \le 1 \\ \frac{d}{du}[(-u^2/2) + 2u - 1] = 2 - u & 1 < u \le 2 \\ 0 & otherwise \end{cases}.$$

Motivation

- Through the distribution function approach, we can arrive at a simple method of writing down the density function of $U = h(Y)$, provided that $h(y)$ is either decreasing or increasing.



$$P(U \le u) = P[h(Y) \le u] = P\{h^{-1}[h(Y)] \le h^{-1}(u)\} = P[Y \le h^{-1}(u)]$$

$$F_U(u) = F_Y[h^{-1}(u)].$$

- Then differentiating with respect to u, we have

$$f_U(u) = \frac{dF_U(u)}{du} = \frac{dF_Y[h^{-1}(u)]}{du} = f_Y\big(h^{-1}(u)\big)\frac{d[h^{-1}(u)]}{du}$$

Procedure

Summary of the Transformation Method

Let $U = h(Y)$, where $h(y)$ is either an increasing or decreasing function of y for all y such that $f_Y(y) > 0$.

1. Find the inverse function, $y = h^{-1}(u)$.
2. Evaluate $\frac{dh^{-1}}{du} = \frac{d[h^{-1}(u)]}{du}$.
3. Find $f_U(u)$ by

$$f_U(u) = f_Y[h^{-1}(u)]\left|\frac{dh^{-1}}{du}\right|.$$

The Method of Transformations : Example

EXAMPLE 6.6

In Example 6.1, we worked with a random variable Y (amount of sugar produced) with a density function given by

$$f(y) = \begin{cases} 2y, & 0 \leq y \leq 1 \\ 0, & elsewhere \end{cases}.$$

We were interested in a new random variable (profit) given by $U = 3Y - 1$. Find the probability density function for U by the transformation method.

The Method of Transformations : Example

SOLUTION 6.6

The function of interest here is $h(y) = 3y - 1$, which is increasing in y. If $u = 3y - 1$, then

$$y = h^{-1}(u) = \frac{u+1}{3} \quad \text{and} \quad \frac{dh^{-1}}{du} = \frac{1}{3}.$$

Thus,

$$f_U(u) = f_Y[h^{-1}(u)]\frac{dh^{-1}}{du} = \begin{cases} 2[h^{-1}(u)]\dfrac{dh^{-1}}{du} = 2\left(\dfrac{u+1}{3}\right)\left(\dfrac{1}{3}\right) & 0 \le \dfrac{u+1}{3} \le 1 \\ 0 & elsewhere \end{cases},$$

or,

$$f_U(u) = \begin{cases} 2(u+1)/9 & -1 \le u \le 2 \\ 0 & elsewhere \end{cases}.$$

The Method of Transformations : Example

EXAMPLE 6.7

Let Y have the probability density function given by

$$f(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & elsewhere \end{cases}.$$

Find the density function of $U = -4Y + 3$.

The Method of Transformations : Example

SOLUTION 6.7

The function of interest here is $h(y) = -4y + 3$, which is decreasing in y. If $u = -4y + 3$, then

$$y = h^{-1}(u) = \frac{3-u}{4} \quad \text{and} \quad \frac{dh^{-1}}{du} = -\frac{1}{4}.$$

Thus,

$$f_U(u) = f_Y[h^{-1}(u)]\left|\frac{dh^{-1}}{du}\right| = \begin{cases} 2\left(\frac{3-u}{4}\right)\left|-\frac{1}{4}\right| & 0 \leq \frac{3-u}{4} \leq 1 \\ 0 & elsewhere \end{cases},$$

or,

$$f_U(u) = \begin{cases} \frac{3-u}{8} & -1 \leq u \leq 3 \\ 0 & elsewhere \end{cases}.$$

The Method of Transformations : Example

EXAMPLE 6.8

Let Y_1 and Y_2 have a joint density function given by

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)} & 0 \leq y_1, 0 \leq y_2 \\ 0, & elsewhere \end{cases}.$$

Find the density function for $U = Y_1 + Y_2$.

The Method of Transformations : Example

SOLUTION 6.8

Let Y_1 be fixed at a value $y_1 \geq 0$. Then $U = h(Y_2) = y_1 + Y_2$, and $y_2 = u - y_1 = h^{-1}(u)$. Letting $g(y_1, u)$ denote the joint density of Y_1 and U gives

$$g(y_1, u) = \begin{cases} f[y_1, h^{-1}(u)] \left|\dfrac{dh^{-1}}{du}\right| = e^{-(y_1+u-y_1)}(1) & 0 \leq y_1, 0 \leq u - y_1, \\ 0 & elsewhere \end{cases}$$

Simplifying, we obtain

$$g(y_1, u) = \begin{cases} e^{-u} & 0 \leq y_1 \leq u \\ 0 & elsewhere \end{cases}.$$

The marginal density of U is then given by

$$f_U(u) = \int_{-\infty}^{\infty} g(y_1, u) dy_1 = \begin{cases} \displaystyle\int_0^u e^{-u} dy_1 = ue^{-u} & 0 \leq u \\ 0 & elsewhere \end{cases}$$

Motivation

The roles of Moments:

- Moments can be used as numerical descriptive measures to describe the data that we obtain in an experiment

- Moments can be used in a theoretical sense to prove that a random variable possesses a particular probability distribution
  - If two random variables Y and Z possess identical moment-generating functions, then Y and Z possess identical probability distributions.

Procedure

Summary of the Moment-Generating Functions Method

Let U be a function of the random variables $Y_1, Y_2, \dots, Y_n$.

1. Find the moment-generating function for U, $m_U(t)$.
2. Compare $m_U(t)$ with other well-known moment-generating functions. If $m_U(t) = m_V(t)$ for all values of t, Theorem 6.1 implies that U and V have identical distributions.

Uniqueness of Moment-Generating Functions

THEOREM 6.1

Let $m_X(t)$ and $m_Y(t)$ denote the moment-generating functions of random variables X and Y, respectively. If both moment-generating functions exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

Proof:

Far beyond the scope of this course.

The Method of Moment-Generating Functions : Example

EXAMPLE 6.10

Suppose that Y is a normally distributed with mean μ and variance σ^2. Show that

$$Z = \frac{Y - \mu}{\sigma}$$

has a *standard normal* distribution, a normal distribution with mean 0 and variance 1.

The Method of Moment-Generating Functions : Example

SOLUTION 6.10

In Example 4.16, $Y-\mu$ has moment-generating function $e^{t^2\sigma^2/2}$.
Hence,

$$m_Z(t)=E(e^{tZ})=E\left[e^{(t/\sigma)(Y-\mu)}\right]=m_{(Y-\mu)}\left(\frac{t}{\sigma}\right)=e^{(t/\sigma)^2(\sigma^2/2)}=e^{t^2/2}.$$

On comparing $m_Z(t)$ with the moment-generating function of a normal random variable, we see that Z must be normally distributed with mean $E(Z)=0$ and $V(Z)=1$.

The Method of Moment-Generating Functions : Example

EXAMPLE 6.11

Let Z be a normally distributed random variable with mean 0 and variance 1. Use the method of moment-generating functions to find the probability distribution of Z^2.

The Method of Moment-Generating Functions : Example

SOLUTION 6.11

$$m_{Z^2}(t) = E\left(e^{tZ^2}\right) = \int_{-\infty}^{\infty} e^{tz^2} f(z)dz = \int_{-\infty}^{\infty} e^{tz^2}\frac{e^{-z^2}}{\sqrt{2\pi}}dz = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(z^2/2)(1-2t)}dz$$

$$= \frac{1}{(1-2t)^{1/2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}(1-2t)^{-1/2}}e^{-(z^2/2)(1-2t)}dz = \frac{1}{(1-2t)^{1/2}}\ (t < 1/2)$$

Note that the integration is equal to 1 by definition of normal density function. A comparison of $m_{Z^2}(t)$ with the moment-generating function for the gamma-distributed random variable shows that Z^2 is a gamma random variable with $\alpha = 1/2$ and $\beta = 2$. Thus, using Definition 4.10, Z^2 has a χ^2 distribution with $\nu = 1$ degree of freedom.

Moment-Generating Functions of Sum of Random Variables

THEOREM 6.2

Let $Y_1, Y_2, \dots, Y_n$ be independent random variables with moment-generating function $m_{Y_1}(t), m_{Y_2}(t), \dots, m_{Y_n}(t)$, respectively. If $U = Y_1 + Y_2 + \cdots + Y_n$, then

$$m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) \times \cdots \times m_{Y_n}(t).$$

Proof:

Moment-Generating Functions of Sum of Random Variables

THEOREM 6.2

Let $Y_1, Y_2, \dots, Y_n$ be independent random variables with moment-generating function $m_{Y_1}(t), m_{Y_2}(t), \dots, m_{Y_n}(t)$, respectively. If $U = Y_1 + Y_2 + \cdots + Y_n$, then

$$m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) \times \cdots \times m_{Y_n}(t).$$

Proof:

Because the random variables $Y_1, \dots, Y_n$ are independent,

$$\begin{aligned} m_U(u) &= E\left[e^{t(Y_1+\cdots+Y_n)}\right] \\ &= E\left(e^{tY_1} e^{tY_2} \dots e^{tY_n}\right) \\ &= E\left(e^{tY_1}\right) \times E\left(e^{tY_2}\right) \times \cdots \times E\left(e^{tY_n}\right) \\ &= m_{Y_1}(t) \times m_{Y_2}(t) \times \cdots \times m_{Y_n}(t) \end{aligned}$$

The Method of Moment-Generating Functions : Example

EXAMPLE 6.12

The number of customer arrivals at a checkout counter in a given interval of time possesses approximately a Poisson probability distribution (see Section 3.8). If Y_1 denotes the time until the first arrival, Y_2 denotes the time between the first and second arrival, . . . , and Y_n denotes the time between the $(n - 1)$st and nth arrival, then it can be shown that $Y_1, Y_2, \ldots, Y_n$ are independent random variables, with the density function for Y_i given by

$$f_{Y_i}(y_i) = \begin{cases} \frac{1}{\theta} e^{-y_i/\theta}, & y_i > 0 \\ 0, & otherewise \end{cases}.$$

[Because the Y_i , for $i = 1, 2, \ldots, n$, are exponentially distributed, it follows that $E(Yi) = \theta$; that is, θ is the average time between arrivals.] Find the probability density function for the waiting time from the opening of the counter until the nth customer arrives. (If $Y_1, Y_2, \ldots$ denote successive interarrival times, we want the density function of $U = Y_1 + Y_2 + \cdots + Y_n$.)

The Method of Moment-Generating Functions : Example

SOLUTION 6.12

Because each of the Y_i's is exponentially distributed with mean θ, $m_{Y_i}(t) = (1 - \theta t)^{-1}$ and, by Theorem 6.2,

$$m_U(t) = \prod_{i=1}^{n} m_{Y_i}(t) = (1 - \theta t)^{-n}.$$

This is the moment-generating function of a gamma-distributed random variable with $\alpha = n$ and $\beta = \theta$. Theorem 6.1 implies that U actually has this gamma distribution and therefore that

$$f_U(u) = \begin{cases} \dfrac{1}{\Gamma(n)\theta^n}\left(u^{n-1}e^{-u/\theta}\right), & u > 0, \\ 0, & elsewhere \end{cases}.$$

Moment-Generating Functions of Sum of Random Variables

THEOREM 6.3

Let $Y_1, Y_2, \ldots, Y_n$ be independent normally distributed random variables with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$, for all $i = 1,2, \ldots, n$, and let $a_1, a_2, \ldots, a_n$ be constants. If

$$U = \sum_{i=1}^{n} a_i Y_i = a_1 Y_1 + a_2 Y_2 + \cdots + a_n Y_n,$$

then U is a normally distributed random variable with

$$E(U) = \sum_{i=1}^{n} a_i \mu_i = a_1 \mu_1 + \cdots + a_n \mu_n$$

and

$$V(U) = \sum_{i=1}^{n} a_i^2 \sigma_i^2 = a_1^2 \sigma_1^2 + \cdots + a_n^2 \sigma_n^2.$$

Moment-Generating Functions of Sum of Random Variables

Proof:

Since Y_i is normally distributed with mean μ_i and variance σ_i^2,

$$m_{Y_i}(t) = \exp\left(\mu_i t + \frac{\sigma_i^2 t^2}{2}\right), \text{ and}$$

Then,

$$m_{a_i Y_i}(t) = E\left(e^{t a_i Y_i}\right) = m_{Y_i}(a_i t) = \exp\left(a_i \mu_i t + \frac{a_i^2 \sigma_i^2 t^2}{2}\right).$$

By Theorem 6.2,

$$m_U(t) = \prod_{i=1}^{n} m_{a_i Y_i}(t) = \prod_{i=1}^{n} \exp\left(a_i \mu_i t + \frac{a_i^2 \sigma_i^2 t^2}{2}\right) = \exp\left(t \sum_{i=1}^{n} a_i \mu_i + \frac{t^2}{2} \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

Thus, U has a normal distribution with mean $\sum_{i=1}^{n} a_i \mu_i$ and variance $\sum_{i=1}^{n} a_i^2 \sigma_i^2$.

Moment-Generating Functions of Sum of Random Variables

THEOREM 6.4

Let $Y_1, Y_2, \ldots, Y_n$ be independent normally distributed random variables with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$, for all $i = 1,2, \ldots, n$, and define Z_i by

$$Z_i = \frac{Y_i - \mu_i}{\sigma_i}, \qquad i = 1,2, \ldots, n.$$

Then $\sum_{i=1}^{n} Z_i^2$ has a χ^2 distribution with n degrees of freedom.

Proof:

Moment-Generating Functions of Sum of Random Variables

Proof:

Note that Z_i is normally distributed with mean 0 and variance 1 by Example 6.10.

We have Z_i^2 is a χ^2-distributed random variable with 1 degree of freedom. Thus,

$$m_{Z_i^2}(t) = (1-2t)^{-1/2},$$

and from Theorem 6.2, with $V = \sum_{i=1}^{n} Z_i^2$,

$$m_V(t) = \prod_{i=1}^{n} m_{Z_i^2}(t) = (1-2t)^{-n/2}.$$

Because moment-generating functions are unique, V has a χ^2 distribution with n degrees of freedom.

The Bivariate Transformation Method

The Bivariate Transformation Method

Suppose that Y_1 and Y_2 are continuous random variables with joint density function $f_{Y_1,Y_2}(y_1, y_2)$ and that for all (y_1, y_2), such that $f_{Y_1,Y_2}(y_1, y_2) > 0$,

$$u_1 = h_1(y_1, y_2) \quad \text{and} \quad u_2 = h_2(y_1, y_2)$$

is a one-to-one transformation from (y_1, y_2) to (u_1, u_2) with inverse

$$y_1 = h_1^{-1}(u_1, u_2) \quad \text{and} \quad y_2 = h_2^{-1}(u_1, u_2).$$

If $h_1^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to u_1 and u_2 and *Jacobian*

$$J = \det\begin{bmatrix} \dfrac{\partial h_1^{-1}}{\partial u_1} & \dfrac{\partial h_1^{-1}}{\partial u_2} \\ \dfrac{\partial h_2^{-1}}{\partial u_1} & \dfrac{\partial h_2^{-1}}{\partial u_2} \end{bmatrix} = \frac{\partial h_1^{-1}}{\partial u_1}\frac{\partial h_2^{-1}}{\partial u_2} - \frac{\partial h_2^{-1}}{\partial u_1}\frac{\partial h_1^{-1}}{\partial u_2} \neq 0,$$

then the joint density of U_1 and U_2 is

$$f_{U_1,U_2}(u_1, u_2) = f_{Y_1,Y_2}\left(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2)\right)|J|,$$

where $|J|$ is the absolute value of the Jacobian.

Motivation

- Many functions of random variables of interest in practice depend on the relative magnitudes of the observed variables.

- For instance, we may be interested in
  - the fastest time in an automobile race or
  - the heaviest mouse among those fed on a certain diet.

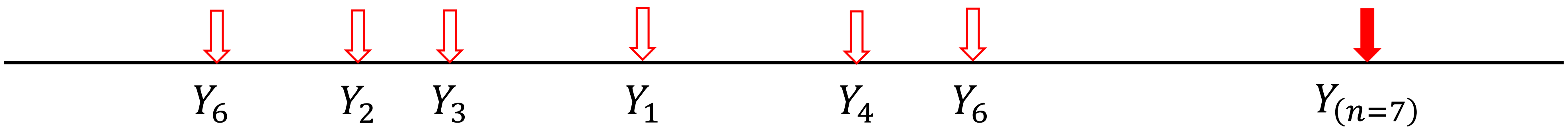
- Thus, we often order observed random variables according to their magnitudes. The resulting ordered variables are called order statistics.

Derivations

- Formally, let $Y_1, Y_2, \dots, Y_n$ denote independent continuous random variables with distribution function $F(y)$ and density function $f(y)$.

- We denote the ordered random variables Y_i by $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$, where $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$.
  - $Y_{(1)} = \min(Y_1, Y_2, \dots, Y_n)$
  - $Y_{(n)} = \max(Y_1, Y_2, \dots, Y_n)$

Derivations



- The event $(Y_{(n)} \leq y)$ will occur if and only if the events $(Y_i \leq y)$ occur for every $i = 1,2,\dots,n$. That is,

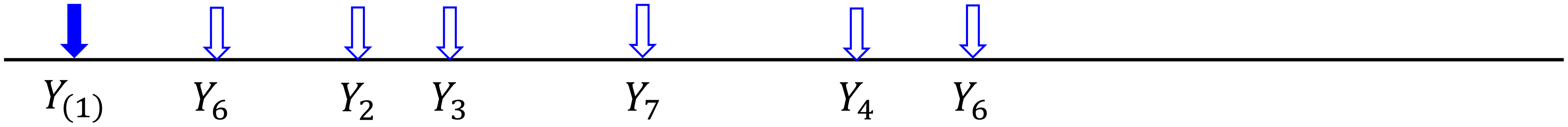
$$P\left(Y_{(n)} \leq y\right) = P(Y_1 \leq y, Y_2 \leq y, \dots, Y_n \leq y)$$

$$F_{Y(n)}(y) = P(Y(n) \leq y) = P(Y_1 \leq y)P(Y_2 \leq y) \cdots P(Y_n \leq y) = [F(y)]^n.$$

- Letting $g_{(n)}(y)$ denote the density function of $Y_{(n)}$, we see that, on taking derivatives of both sides

$$g_{(n)}(y) = n[F(y)]^{n-1}f(y)$$

Derivations



- The distribution function of $Y_{(1)}$ is

$$P\left(Y_{(1)} \leq y\right) = 1 - P\left(Y_{(1)} > y\right)$$

- Because $Y_{(1)}$ is the minimum of $Y_1, Y_2, \ldots, Y_n$, it follows that the event $\left(Y_{(1)} > y\right)$ occurs if and only if the events $(Y_i > y)$ occur for $i = 1, 2, \ldots, n$.
- Because the Y_i are independent and $P(Y_i > y) = 1 - F(y)$ for $i = 1, 2, \ldots, n$, we see that

$$\begin{aligned} F_{Y_{(1)}}(y) &= P(Y_{(1)} \leq y) = 1 - P(Y_{(1)} > y) \\ &= 1 - P(Y_1 > y, Y_2 > y, \ldots, Y_n > y) \\ &= 1 - [P(Y_1 > y)P(Y_2 > y) \cdots P(Y_n > y)] \\ &= 1 - [1 - F(y)]^n. \end{aligned}$$

$$g_{(1)}(y) = n[1 - F(y)]^{n-1} f(y).$$

Order Statistics : Example

EXAMPLE 6.16

Electronic components of a certain type have a length of life Y, with probability density given by

$$f(y) = \begin{cases} (1/100)e^{-y/100}, & y > 0 \\ 0, & elsewhere \end{cases}.$$

(Length of life is measured in hours.) Suppose that two such components operate independently and in series in a certain system (hence, the system fails when either component fails). Find the density function for X, the length of life of the system.

Order Statistics : Example

SOLUTION 6.16

Because the system fails at the first component failure, $X = \min(Y_1, Y_2)$, where Y_1 and Y_2 are independent random variables with the given density. Then, because $F(y) = 1 - e^{-y/100}$, for $y \geq 0$,

$$f_X(y) = g_{(1)}(y) = n[1 - F(y)]^{n-1} f(y) = \begin{cases} 2e^{-y/100}(1/100)e^{-y/100}, & y > 0 \\ 0, & elsewhere \end{cases}$$

and it follows that

$$f_X(y) = \begin{cases} (1/50)e^{-y/50}, & y > 0 \\ 0, & elsewhere \end{cases}.$$

Thus, the minimum of two exponentially distributed random variables has an exponential distribution.

Order Statistics : Example

EXAMPLE 6.17

Suppose that the components in Example 6.16 operate in parallel (hence, the system does not fail until both components fail). Find the density function for X, the length of life of the system.

Order Statistics : Example

SOLUTION 6.17

Now, $X = \max(Y_1, Y_2)$, and

$$f_X(y) = g_{(2)}(y) = n[F(y)]^{n-1}f(y) = \begin{cases} 2(1 - e^{-y/100})(1/100)e^{-y/100}, & y > 0 \\ 0, & elsewhere \end{cases}$$

and, therefore

$$f_X(y) = \begin{cases} (1/50)\left(e^{-y/100} - e^{-y/50}\right), & y > 0 \\ 0, & elsewhere \end{cases}.$$

We see here that the maximum of two exponential random variables is not an exponential random variable.

Density of Order Statistics

THEOREM 6.5

Let $Y_1, Y_2, \dots, Y_n$ be independent identically distributed continuous random variables with common distribution function $F(y)$ and common density function $f(y)$. If $Y_{(k)}$ denotes the kth-order statistics, then the density function of $Y_{(k)}$ is given by

$$g_{(k)}(y_k) = \frac{n!}{(k-1)!\,(n-k)!}[F(y_k)]^{k-1}[1-F(y_k)]^{n-k}f(y_k), \qquad y_k \in (-\infty, \infty).$$

If j and k are two integers such that $1 \le j < k \le n$, the joint density of $Y_{(j)}$ and $Y_{(k)}$ is given by

$$g_{(j)(k)}(y_j, y_k) = \frac{n!}{(j-1)!\,(k-1-j)!\,(n-k)!}[F(y_j)]^{j-1}[F(y_k)-F(y_j)]^{k-1-j}[1-F(y_k)]^{n-k}f(y_j)f(y_k),$$

$-\infty < y_j < y_k < \infty$.

Proof: Omitted.

Order Statistics : Example

EXAMPLE 6.18

Suppose that $Y_1, Y_2, \dots, Y_5$ denotes a random sample from a uniform distribution defined on the interval $(0,1)$. That is,

$$f(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & elsewhere \end{cases}.$$

Find the density function for the second-order statistic. Also, give the joint density function for the second- and fourth- order statistics.

Order Statistics : Example

SOLUTION 6.18

The distribution function associated with each of the Y's is

$$F(y) = \begin{cases} 0, & y < 0 \\ y, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}.$$

From Theorem 6.5, with $n = 5, k = 2,$

$$g_{(2)}(y_2) = \frac{5!}{(2-1)!\,(5-2)!}[F(y_2)]^{2-1}[1 - F(y_2)]^{5-2} f(y_2), \qquad -\infty < y_2 < \infty$$

$$= \begin{cases} 20y_2(1-y_2)^3, & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

The preceding density is a beta density with $\alpha = 2$ and $\beta = 4$.

Order Statistics : Example

SOLUTION 6.18

The joint density of the second- and fourth- order statistics is readily obtained from Theorem 6.5, with $j = 2, k = 4$, and $n = 5$;

$$g_{(2)(4)}(y_2, y_4)$$

$$= \frac{5!}{(2-1)!\,(4-2-1)!\,(5-4)!}[F(y_2)]^{2-1}[F(y_4) - F(y_2)]^{4-2-1}[1 - F(y_4)]^{5-4} f(y_2) f(y_4),$$

$$-\infty < y_2 < y_4 < \infty$$

$$= \begin{cases} 5!\, y_2(y_4 - y_2)(1 - y_4), & 0 \le y_2 < y_4 \le 1 \\ 0, & elsewhere \end{cases}$$