About Probability

IE241 Engineering Statistics I – Session 1

What is probability? The textbook says, "The term *probability* is a measure of one's belief in the occurrence of a future event."

Example 1.1) Consider a balanced die. What is the probability of getting 1 in the die toss?

Solution)

Note that we can think of the probability of an event as a relative frequency. This brings us a natural definition of the probability.

Definition 1.1) Let A be an event of interest. Then, the probability of A is given by

$$P(A) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} 1_{\{A\ occurs\ in\ ith\ trial\}}}{n} .$$

According to Definition 1.1, the statement "The probability of getting 1 from a die toss is $\frac{1}{6}$." means that "As we continue to toss a die, one-sixth of total tosses results in 1." But sometimes, it is impossible to think of identical and repeatable experiments. What is the probability that you have your lover in the next 5 years? (In case you have one already, think of the probability that you are still in love with him/her.) What is the probability that you get A+ in this course? Unless there are parallel universes, it is impossible to find the relative frequency of the events above, since the experiments associated with them happen once and only once. In such cases, we have to enforce our own degree of belief to the probability value, which is called subjective probability. For example, if you are so sure of something happens, say you graduate from KAIST in the next 10 years, you may assign the value that is very close to 1 to the event. It purely depends on your own belief. Then a question arises; Is every belief can work as a probability?

Example 1.2) Consider an experiment that you toss a fair coin until you get a head. You believe that every simple event is equally likely. What is the probability of getting a head in the first trial?

Solution)

As we see in the example above, not all beliefs can function as a probability. To be a probability associated with an experiment, certain conditions should be met. These are called the axioms of probability.

Definition 1.2) Suppose S is a sample space associated with an experiment. For an event $A \subset S$, $P(A)$ is called the probability of A if the following axioms holds:

i) $P(A) \geq 0$.

ii) $P(S) = 1$.

iii) $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$,

provided that A_i's are disjoint.

Note that the probability value is defined on events, which are subsets of a sample space. That is, the probability is a function whose argument is a set. Consider a die tossing experiment, and any probability function that satisfies the axioms of probability. The sample space is $S = \{1,2,3,4,5,6\}$. So $P(\{1\}), P(\{1,3,5\})$ are well-defined, while $P(1), P(1,2,5)$ are NOT defined, since 1 or $(1,3,5)$ are sample points, which are elements of the sample space, not sets.

Any belief that satisfies the axioms of probability can work as probability. Several facts that you are familiar with can be derived by using Definition 1.2. as corollaries. For the following theorems, assume that $A, B \subset \mathrm{S}$.

Theorem 1.1) $P(\phi) = 0$.

Proof)

Theorem 1.2) If $A \subset B$, then $P(A) \leq P(B)$.

Proof)

Theorem 1.3) For any event A, $P(A) \leq 1$.

Proof)

Theorem 1.4) For any event A, $P(A^C) = 1 - P(A)$.

Proof)

Often, a simple belief of a situation is expanded to a complete probability space by the conditions illustrated by the axioms of probability. Therefore, using them allows us to calculate the probability of events that are seemed not to be specified yet. Here are some examples with discrete sample space, though the logics behind the problems are all the same when it comes to continuous sample space. (For continuous case, refer to the problems at the end of the note.)

Example 1.3) 3 students, including Jinkyoo, Joonho, and Seung-hoon stand in a row. What is the probability that Jinkyoo stands in front of Joonho?

Solution)

Example 1.4) Two representatives are randomly picked out of 6 men and 4 women. What is the probability that one man and one woman are picked?

Solution)

Probabilities can be calculated using relations of known events, by means of multiplicative law, additive law, and conditional probability and independence concepts.

Before solving the next example, we prove the following useful property regarding independence of the events.

Theorem 1.5) If the events A, B are independent, so are A and B^C, and A^C, B^C.

Proof)

Example 1.5) Three radar sets, operating independently, are set to detect any aircraft flying through a certain area. Each set has a probability of 0.02 of failing to detect a plane in its area. If an aircraft enters the area, what is the probability that it goes undetected? What is the probability that it is detected by all three radar sets?

Solution)

Example 1.6) Suppose that the probability of exposure to the flu during an epidemic is 0.6. Experience has shown that a serum is 80% successful in preventing an inoculated person from acquiring the flu, if exposed to it. A person not inoculated faces a probability of 0.90 of acquiring the flu if exposed to it. Two persons, one inoculated and one not, perform a highly specialized task in a business. Assume that they are not at the same location, are not in contact with the same people, and cannot expose each other to the flu. What is the probability that at least one will get the flu?

Solution)

Example 1.7) Three prisoners, A, B and C, are in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the others who is going to be executed. "If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C." The warden tells A that B is to be executed. Prisoner A is pleased because he believes that his probability of surviving has gone up from $\frac{1}{3}$ to $\frac{1}{2}$, as it is now between him and C. Do you really think prisoner A's probability of survival increased?

Solution)

For those who are actively interested in the probability concept, here are some problems worth trying.

1. Suppose you roll two dice at the same time, and record the sum of the numbers that showed up. Your belief is that a total of 11 outcomes, namely from 2 to 12, are all equally likely. Assume that each die has its own probability of the result, and let p_i, q_i denote the probability of getting i on the first die and on the second die, respectively. Prove that you cannot construct a probability based on your belief.
2. Prove that $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$ where A_i's are events.
3. The number of *repeated combinations* of n objects taken r at a time is the number of subsets, each of size r, that can be formed from the distinct n object allowing repetitions. This number is denoted by H_r^n. Show that $H_r^n = C_r^{n+r-1}$.
4. How many ways are there to paint each side of the cube with 6 different colors?
5. Jinkyoo and Joonho are supposed to meet at KAIST gate. Unfortunately, they all forgot when to meet. They independently decided to go to the place sometime between 15:00 and 16:00, and wait for his companion for the next 15 minutes. If his companion does not show up, he leaves. What is the probability that they can meet up at the gate?
6. Joonho has two children, one of which is a boy born on Tuesday. What is the probability that the other child is a boy as well?
7. Can an event be independent of itself?
8. Show that there are no two events with positive probability that both mutually exclusive and independent.

9. A biased coin (with probability of obtaining a head equal to $p > 0$) is tossed repeatedly and independently until the first head is observed. Compute the probability that the first head appears at an even numbered toss.