Definition

EXAMPLE 9.1

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the uniform distribution on the interval $(0, \theta)$. Two unbiased estimators for θ are $\hat{\theta}_1 = 2\overline{Y}$ and $\hat{\theta}_2 = \left(\frac{n+1}{n}\right) Y_{(n)}$, where $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$. Find the efficiency of $\hat{\theta}_1$relative to $\hat{\theta}_2$.

Definition

SOLUTION 9.1

Each Y_i has a uniform distribution on the interval $(0,\theta)$. Thus, $\mu = E(Y_i) = \theta/2$ and $\sigma^2 = V(Y_i) = \theta^2/12$. Therefore,
$E(\hat{\theta}_1) = E(2\bar{Y}) = 2(\mu) = 2\left(\frac{\theta}{2}\right) = \theta$ (unbiased), $V(\hat{\theta}_1) = V(2\bar{Y}) = 4V(\bar{Y}) = 4[\frac{V(Y_i)}{n}] = \left(\frac{4}{n}\right)\left(\frac{\theta^2}{12}\right) = \frac{\theta^2}{3n}$.

To find the mean and variance of $\hat{\theta}_2$, recall (see Exercise 6.74) that the density function of $Y_{(n)}$ is given by

$$g_{(n)}(y) = n[F_Y(y)]^{n-1} f_Y(y) = \begin{cases} n\left(\frac{y}{\theta}\right)^{n-1}\left(\frac{1}{\theta}\right), 0 \le y \le \theta, \\ 0, \qquad\qquad elsewhere \end{cases}$$

Thus, $E(Y_{(n)}) = \frac{n}{\theta^n}\int_0^\theta y^n\,dy = \frac{n}{n+1}\theta \;\rightarrow\; E\{\left[\frac{n+1}{n}\right] Y_{(n)}\} = \theta$; that is, $\hat{\theta}_2$ is an unbiased estimator for θ.

Because $E(Y_{(n)}^2) = \frac{n}{\theta^n}\int_0^\theta y^{n+1}dy = (\frac{n}{n+2})\theta^2$,
we obtain

$$V(Y_{(n)}) = E(Y_{(n)}^2) - [E(Y_{(n)})]^2 = \left[\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2\right]\theta^2$$

$$V(\hat{\theta}_2) = V\left[\left(\frac{n+1}{n}\right)Y_{(n)}\right] = \left(\frac{n+1}{n}\right)^2 V(Y_{(n)}) = \left[\frac{(n+1)^2}{n(n+2)} - 1\right]\theta^2 = \frac{\theta^2}{n(n+2)}.$$

Therefore, eff $(\hat{\theta}_1, \hat{\theta}_2) = V(\hat{\theta}_2)/V(\hat{\theta}_1) = \frac{\theta^2/n(n+2)}{\theta^2/3n} = \frac{3}{n+2}$.
This efficiency is less than 1 if $n > 1$.
That is, if $n > 1$, $\hat{\theta}_2$has a smaller variance than $\hat{\theta}_1$, and therefore $\hat{\theta}_2$ is generally preferable to $\hat{\theta}_1$ as an estimator of θ.

Overview

EXAMPLE 9.3

Suppose that $Y_1, Y_2, \ldots, Y_n$ represent a random sample such that $E(Y_i) = \mu$, $E(Y_i^2) = \mu_2'$ and $E(Y_i^4) = \mu_4'$ are all finite. Show that

$$S_n^2 = \frac{1}{(n-1)} \Sigma_{i=1}^{n} (Y_i - \bar{Y}_n)^2$$

is a consistent estimator of $\sigma^2 = V(Y_i)$.

(*Note:* We use subscript n on both S^2 and Y to explicitly convey their dependence on the value of the sample size n.)

Overview

SOLUTION 9.3

$$S_n^2 = \frac{1}{(n-1)}\Sigma_{i=1}^{n}(Y_i - \bar{Y}_n)^2 = \frac{1}{(n-1)}\left(\Sigma_{i=1}^{n}Y_i^2 - n\bar{Y}_n^2\right) = \frac{n}{(n-1)}\left(\frac{1}{n}\Sigma_{i=1}^{n}Y_i^2 - \bar{Y}_n^2\right)$$

- By the law of large numbers, $\frac{1}{n}\Sigma_{i=1}^{n}Y_i^2$ converges in probability to μ_2'.
- Also, $\bar{Y}_n$ converges in probability to μ. Because
  - ✓ $\bar{Y}_n$ is an unbiased estimator
  - ✓ $\lim_{n\to\infty} V(\bar{Y}_n) = \lim_{n\to\infty} \sigma^2/n = 0.$
- Because the function $g(x) = x^2$ is continuous for all finite values of x, $\bar{Y}_n^2$ converges in probability to μ^2.
- Then, $\frac{1}{n}\Sigma_{i=1}^{n}Y_i^2 - \bar{Y}_n^2$ converges in probability to $\mu_2' - \mu^2 = \sigma^2$.

➢ S_n^2, the sample variance, is a consistent estimator for σ^2, the population variance.

Example

EXAMPLE 9.4

Suppose that $Y_1, Y_2, \ldots, Y_n$ is a random sample of size *n* from a distribution with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$. Define S_n^2 as $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2$.

Show that the distribution function of $\sqrt{n}\left(\frac{\bar{Y}_n - \mu}{S_n}\right)$ converges to a standard normal distribution function.

Overview

SOLUTION 9.4

In Example 9.3, we showed that S_n^2 converges in probability to σ^2. Notice that $g(x) = \sqrt{x/c}$ is a continuous function of x if both x and c are positive. Hence, it follows from Theorem 9.2(d) that $S_n/\sigma = \sqrt{S_n^2/\sigma^2}$ converges in probability to 1.

We also know from the central limit theorem (Theorem 7.4) that the distribution function of $U_n = \sqrt{n}\left(\frac{\bar{Y}_n-\mu}{\sigma}\right)$ converges to a standard normal distribution function. Therefore, Theorem 9.3 implies that the distribution function of $\sqrt{n}\left(\frac{\bar{Y}_n-\mu}{\sigma}\right) / \frac{S_n}{\sigma} = \sqrt{n}\left(\frac{\bar{Y}_n-\mu}{S_n}\right)$ converges to a standard normal distribution function.