L6. Bayesian Decision Analysis

Bayesian Regression

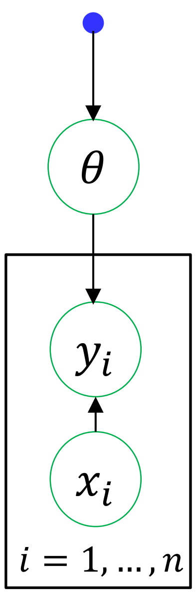
Linear regression:

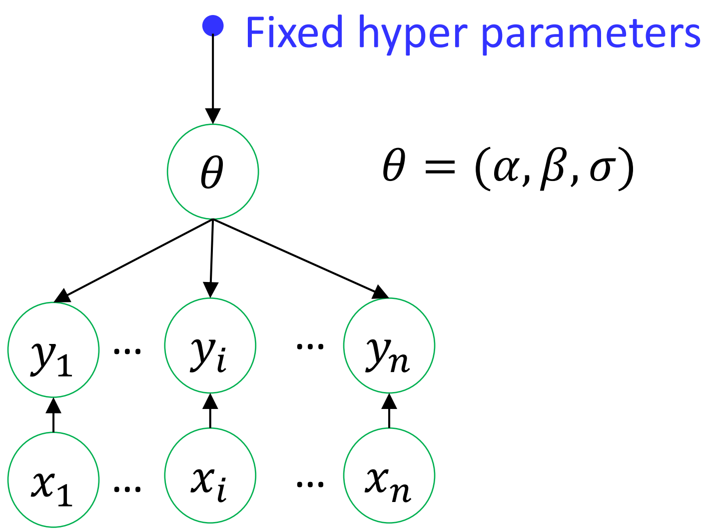
$$y_i = \alpha + \beta x_i + \epsilon$$

Probabilistic reformulation:

$$y_i \sim N(\alpha + \beta x_i, \sigma^2)$$

$$\epsilon \sim N(0, \sigma^2)$$





$$\theta = (\alpha, \beta, \sigma)$$

$$\alpha \sim N(\mu_\alpha = 0, \sigma_\alpha = 20^2)$$

$$\beta \sim N(\mu_\beta = 0, \sigma_\beta = 20^2)$$

$$\sigma \sim U(0,20)$$

Bayesian Regression

Jupyter Demo Simulation

Hierarchical Bayesian Regression

Group 1 $\left(x_{1,1}, y_{1,1}\right), \left(x_{2,1}, y_{2,1}\right), \ldots, \left(x_{i,1}, y_{i,1}\right), \ldots, \left(x_{n_1,1}, y_{n_1,1}\right)$

Group 2 $\left(x_{1,2}, y_{1,2}\right), \left(x_{2,1}, y_{2,2}\right), \ldots, \left(x_{i,2}, y_{i,2}\right), \ldots, \left(x_{n_2,2}, y_{n_2,2}\right)$

$\vdots$

Group j $\left(x_{1,j}, y_{1,j}\right), \left(x_{2,j}, y_{2,j}\right), \ldots, \left(x_{i,j}, y_{i,j}\right), \ldots, \left(x_{n_j,j}, y_{n_j,j}\right)$

$\vdots$

Group J $\left(x_{1,J}, y_{1,J}\right), \left(x_{2,J}, y_{2,J}\right), \ldots, \left(x_{i,J}, y_{i,J}\right), \ldots, \left(x_{n_J,J}, y_{n_J,J}\right)$

Hierarchical Bayesian Regression

Linear regression:

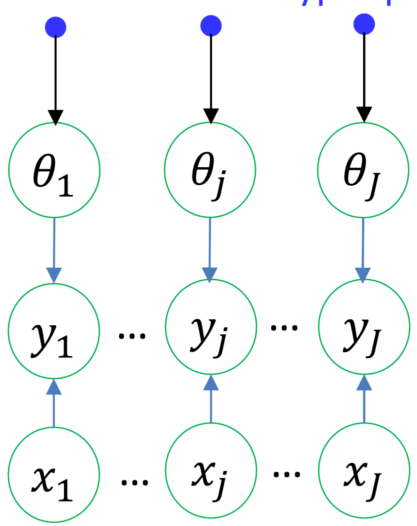
$$y_{i,j} = \alpha_j + \beta_j x_{i,j} + \epsilon$$

Probabilistic reformulation:

$$y_{i,j} \sim N\left(\alpha_j + \beta_j x_{i,j}, \sigma_j^2\right)$$

$$\epsilon \sim N(0, \sigma_j^2)$$

Fixed hyper parameters



$$\theta_j = (\alpha_j, \beta_j, \sigma_j)$$

$$\alpha_j \sim N(\mu_\alpha = 0, \sigma_\alpha = 20^2)$$

$$\beta_j \sim N(\mu_\beta = 0, \sigma_\beta = 20^2)$$

$$\sigma_j \sim U(0{,}20)$$

$$y_j = (y_{1,j}, y_{2,j}, \dots, y_{nj,j})$$

$$x_j = (x_{1,j}, x_{2,j}, \dots, x_{nj,j})$$

Linear regression:

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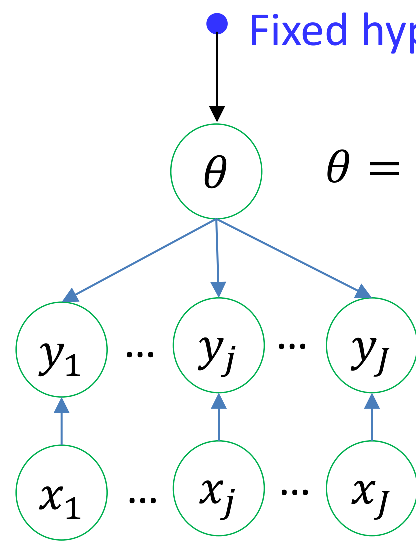

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Linear regression:

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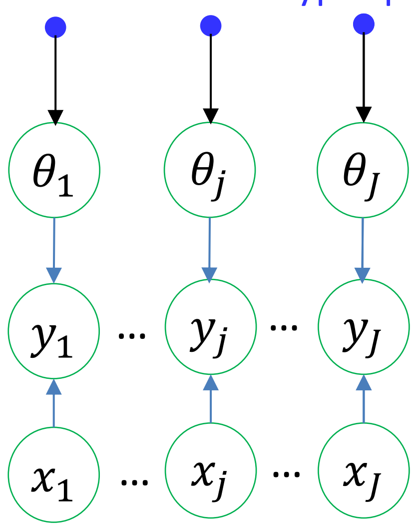

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$$y_j = (y_{1,j}, y_{2,j}, \dots, y_{nj,j})$$
$$x_j = (x_{1,j}, x_{2,j}, \dots, x_{nj,j})$$

Hierarchical Bayesian Regression

Linear regression:

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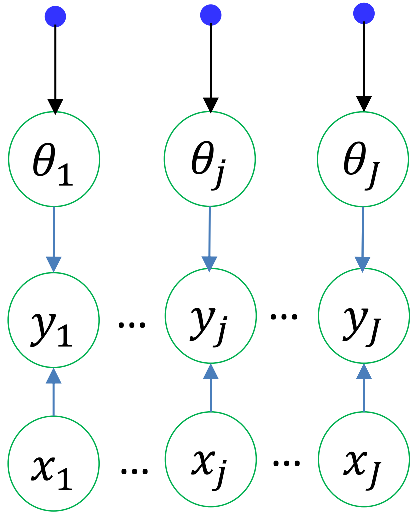
Fixed hyper parameters



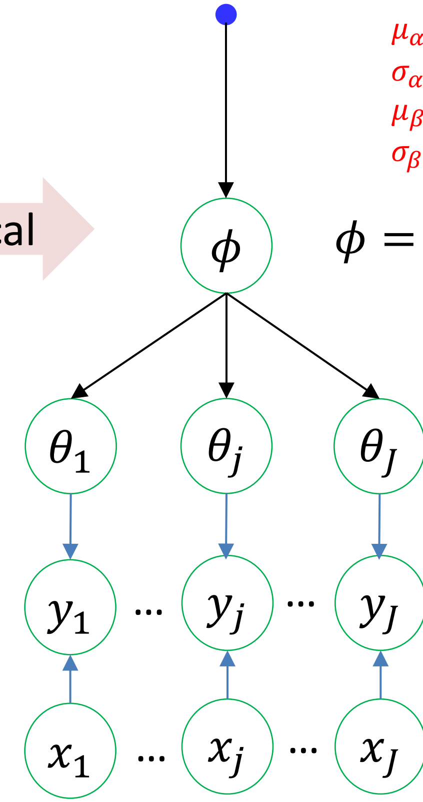
$$\theta_j = (\alpha_j, \beta_j, \sigma_j)$$
$$\alpha_j \sim N(\mu_\alpha = 0, \sigma_\alpha = 20^2)$$
$$\beta_j \sim N(\mu_\beta = 0, \sigma_\beta = 20^2)$$
$$\sigma_j \sim U(0,20)$$

$$y_j = (y_{1,j}, y_{2,j}, \ldots, y_{nj,j})$$
$$x_j = (x_{1,j}, x_{2,j}, \ldots, x_{nj,j})$$

Hierarchical

Fixed hyper parameters

$$\mu_\alpha \sim N(0, 20^2)$$
$$\sigma_\alpha \sim U(0,10)$$
$$\mu_\beta \sim N(0, 20^2)$$
$$\sigma_\beta \sim U(0,10)$$



$$\phi = \left(\mu_\alpha, \sigma_\alpha, \mu_\beta, \sigma_\beta\right)$$

Shared hyper parameters

$$\theta_j = (\alpha_j, \beta_j, \sigma_j)$$
$$\alpha_j \sim N(\mu_\alpha, \sigma_\alpha)$$
$$\beta_j \sim N(\mu_\beta, \sigma_\beta)$$
$$\sigma_j \sim U(0,20)$$

$$y_j = (y_{1,j}, y_{2,j}, \ldots, y_{nj,j})$$
$$x_j = (x_{1,j}, x_{2,j}, \ldots, x_{nj,j})$$

Hierarchical Bayesian Regression

Bayesian Regression

$$y_{i,j} = \alpha + \beta x_{i,j} + \epsilon$$
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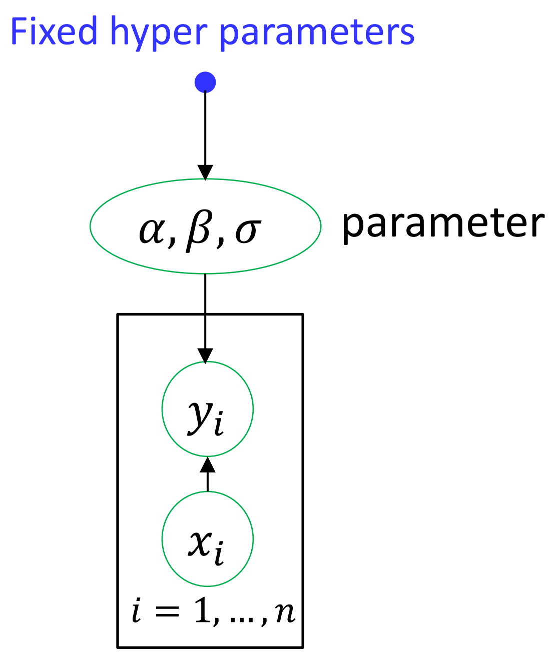

Hierarchical Bayesian Regression

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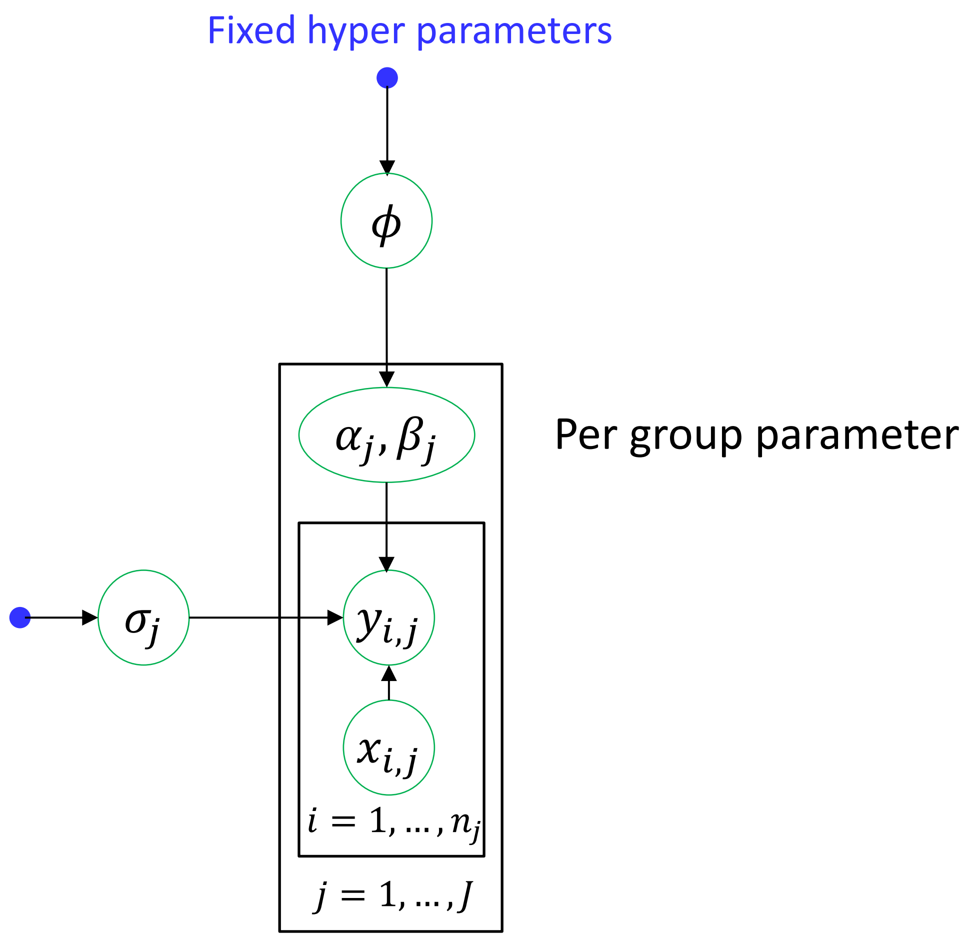


Hierarchical Bayesian Regression

Jupyter Demo Simulation

http://pymc-devs.github.io/pymc3/notebooks/GLM-hierarchical.html

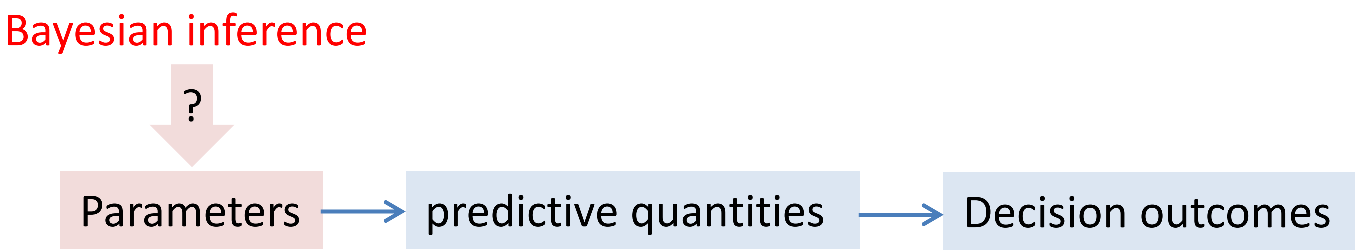
Bayesian decision theory

- ✓ Model has been built
- ✓ The inferences are conducted
- ✓ Model has been checked

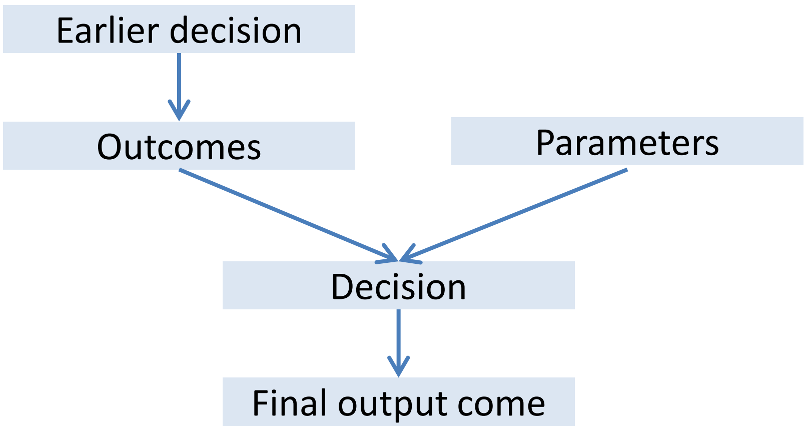
What else?

- Two ways of using *Bayesian inference in decision making*

(1) Measure uncertainties regarding predictive quantiles



(2) Within decision analysis (Multi-stages decision problems)



Elements of Bayesian decision analysis

Decision analysis is inherently more complicated than statistical inference because it involves *optimization over decision* as well *averaging over uncertainties*

Elements of Bayesian decision analysis

1. Enumerate the space of all possible decisions (actions) a and outcome o

   -The vector of outcomes o can include observables (predicted values $\tilde{y}$) and parameters θ

2. Determine $p(o|a)$, a *conditional posterior probability distribution* of outcomes o for each decision option a

   -Outcome o is random variable, while decision a is deterministic (set by user)

3. Define a utility function $U(o)$ mapping outcomes o onto the real numbers

   - If multiple attributes (vector) o is considered, the utility function must trade off different goods

4. Compute the $\mathrm{E} =$ expected utility $\mathrm{E}(U(o)|a)$ as a function of the decision a, and choose the decision with highest expected utility

Examples that will discussed in the lecture

1. Survey incentives
   - ✓ Conduct only step 1 and 2 to estimate the expected effect depending on the option that can be taken

2. Medical test decision
   - ✓ Conduct only step 1 and 2 in a sequential decision making and computed value of information

3. Risk analysis of Radon exposure and making prevention strategies
   - ✓ Conduct the full Bayesian analysis

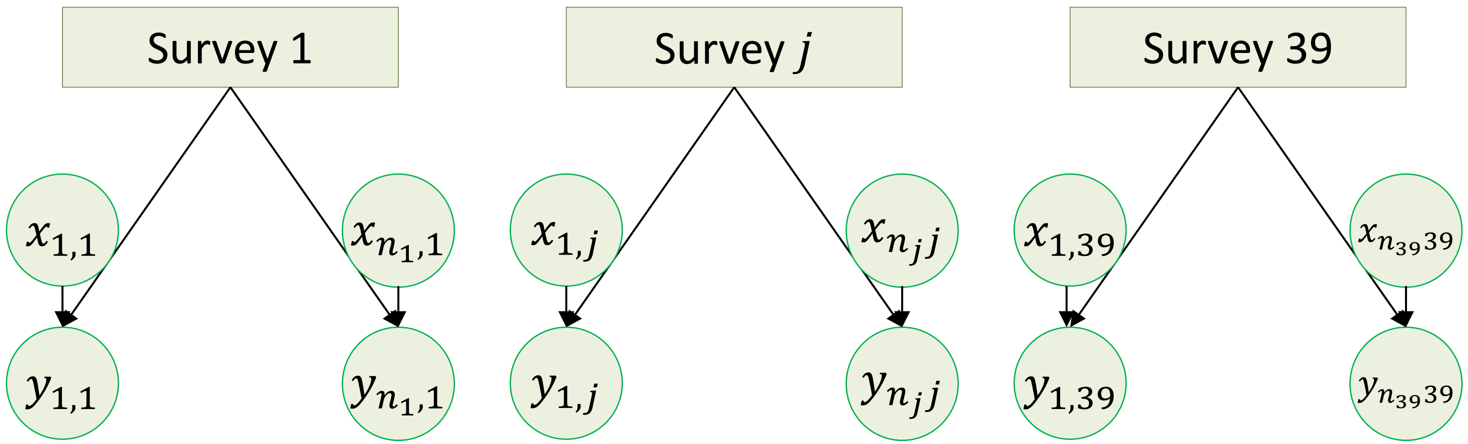
Example 1: Survey incentives

- Do the benefits of incentives outweigh the cost?
- If an incentive is given,
  - ✓ how and when should it be offered
  - ✓ whom should it be offered to
  - ✓ what form should it take
  - ✓ how large should its value be?

Example 1: Survey incentives

Data descriptions:

The New York City Social Indicators Survey, a telephone study conduced every two years that has had a response rate below 50%



- In total 39 surveys including 101 experiment conditions are conducted:
- y_i is the observed response rate for observation $i = 1, \dots, 101$
- All experiments has different conditions x_i on:

  -The value of the incentive (in tens of 1999dollars)
  -The timing of the incentive payment (given before the survey or after)
  -The form of the incentive (cash or gift)
  -The mode of the survey (face-to-face or telephone)
  -The burden or effort, required to survey respondents (high burden or low burden)

- Use the differences, $z_i = y_i - y_i^0$,
  where y_i^0 corresponds to the lowest valued incentive condition

A simple linear regression approach

Response rate : $y_i = X_i\beta + \beta_0$

$$y_i = \frac{n_i}{N_i} = \frac{repondents}{Trial}$$

Predictor variables : $X_i = (X_{1,i}, X_{2,i}, X_{3,i}, X_{4,i}, X_{5,i})$

$X_{1,i}$:The value of the incentive (in tens of 1999dollars)
$X_{2,i}$:The timing of the incentive payment (given before the survey or after)
$X_{3,i}$:The form of the incentive (cash or gift)
$X_{4,i}$:The mode of the survey (face-to-face or telephone)
$X_{5,i}$:The burden or effort, required to survey respondents (high burden or low burden)

Limitations in employing a classical regression model relating y_i to the predictor variables:

- Unable to model interactions
- Unable to reflect the hierarchical structure of the data
- Difficulty in dealing with unequal sample size

Example 1: Survey incentives

- **Step1** : perform a meta-analysis to estimate the effects of incentives on response rate, as a function of the amount of the incentive and the way it is implemented

$$\tau \sim \mathrm{U}(0,1)$$

$$\sigma \sim \mathrm{U}(0,1) \qquad \alpha_j \sim \mathrm{N}(0, \tau^2) \qquad \beta \sim \mathrm{U}(0,1)$$

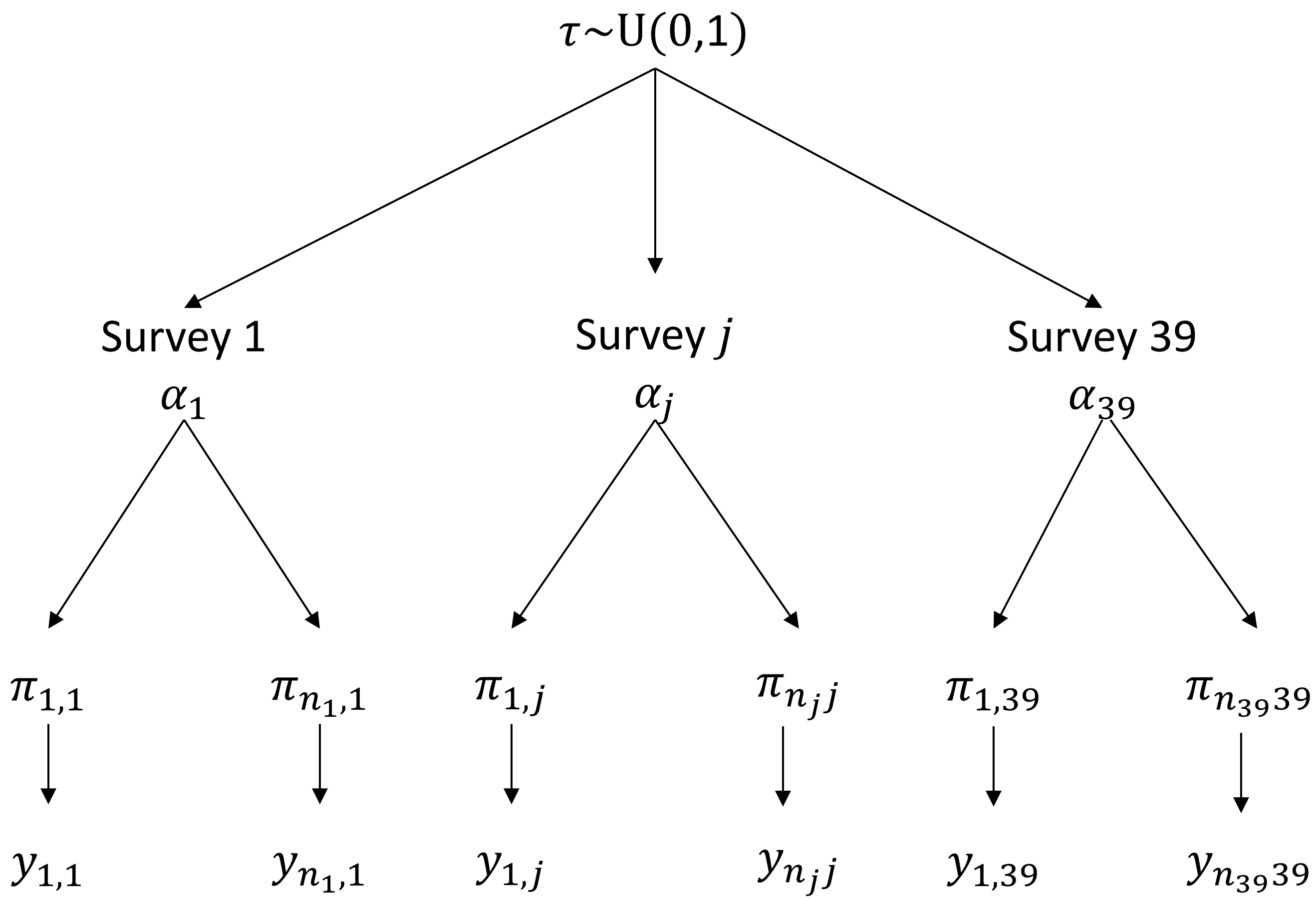
$$\pi_i \sim \mathrm{N}(X_i\beta + \alpha_{j(i)}, \sigma^2)$$

$$n_i \sim \mathrm{Bin}(N_i, \pi_i)$$

$\alpha_{j(i)}$: random effect for the survey $j = 1, \dots, 39$
(address the hierarchical structure)
$X_i\beta$: linear predictor for conditioned on data point i
π_i : population response probability $i = 1, \dots, 101$
N_i : number of persons contacted
n_i : number of respondents ($y_i = n_i / N_i$)

Example 1: Survey incentives

- **Step1** : perform a meta-analysis to estimate the effects of incentives on response rate, as a function of the amount of the incentive and the way it is implemented



$\alpha_j \sim \mathrm{N}(0, \tau^2)$

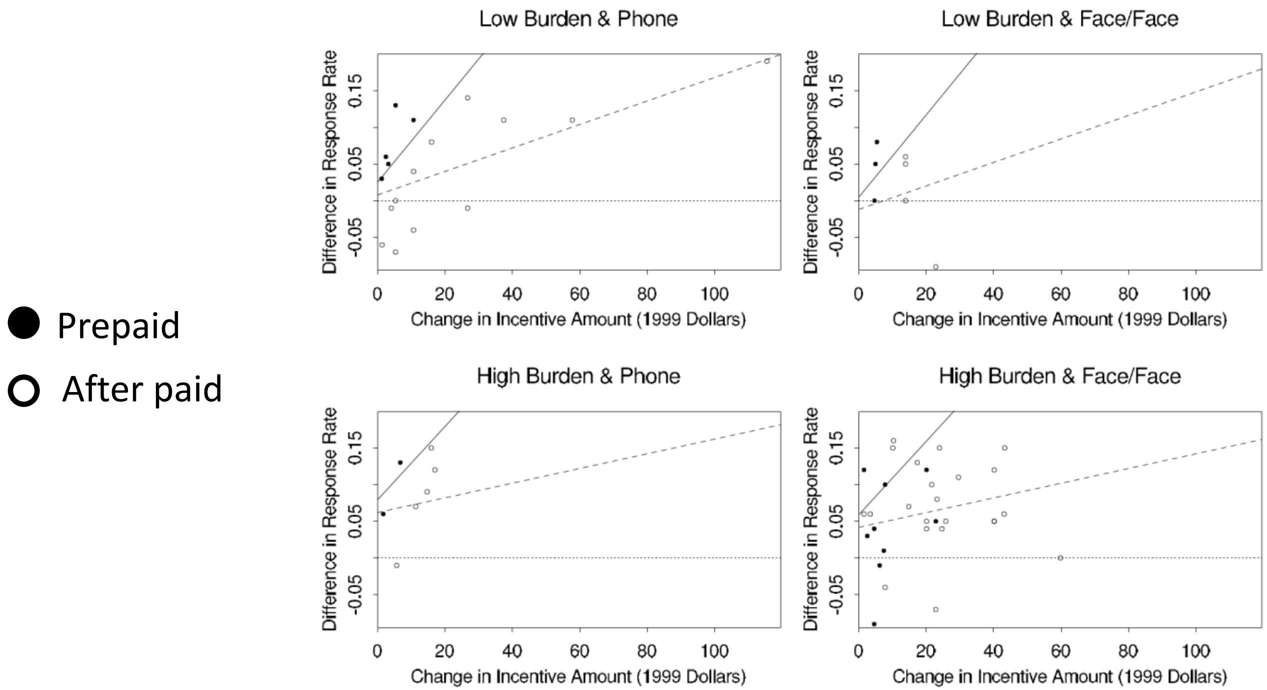
$\pi_{i,j} \sim \mathrm{N}(X_i\beta + \alpha_{j(i)}, \sigma^2)$

$y_{i,j} \sim \mathrm{Bin}(\pi_{i,j}, N_{i,j})$

$y_{1,1} = \frac{n_{1,1}}{N_{i,i}}$ Response rate

Example 1: Survey incentives

- **Step2** : use the inference to estimate the costs and benefits of incentives



$$y_i = X_i\hat{\beta} + \beta_0$$

Some of the findings are :

- an extra $10 in incentive is expected to increase the response rate by 3–4 percentage points
- cash incentives increase the response rate by about 1 percentage point relative to noncash
- prepaid incentives increase the response rate by 1–2 percentage points relative to postpaid
- incentives have a bigger impact (by about 5 percentage points) on high-burden surveys compared to low-burden surveys.

These inferences can be used to design a new survey (cost vs. response rate)

Example 2: Multistage decision making for medical screening

95-year-old man with an apparently malignant tumor in the lung

<Multi-stages decision problems>

Earlier decision — **Test or not?**

↓

Outcomes   Parameters

↘ ↙

Decision — **Treatment type**

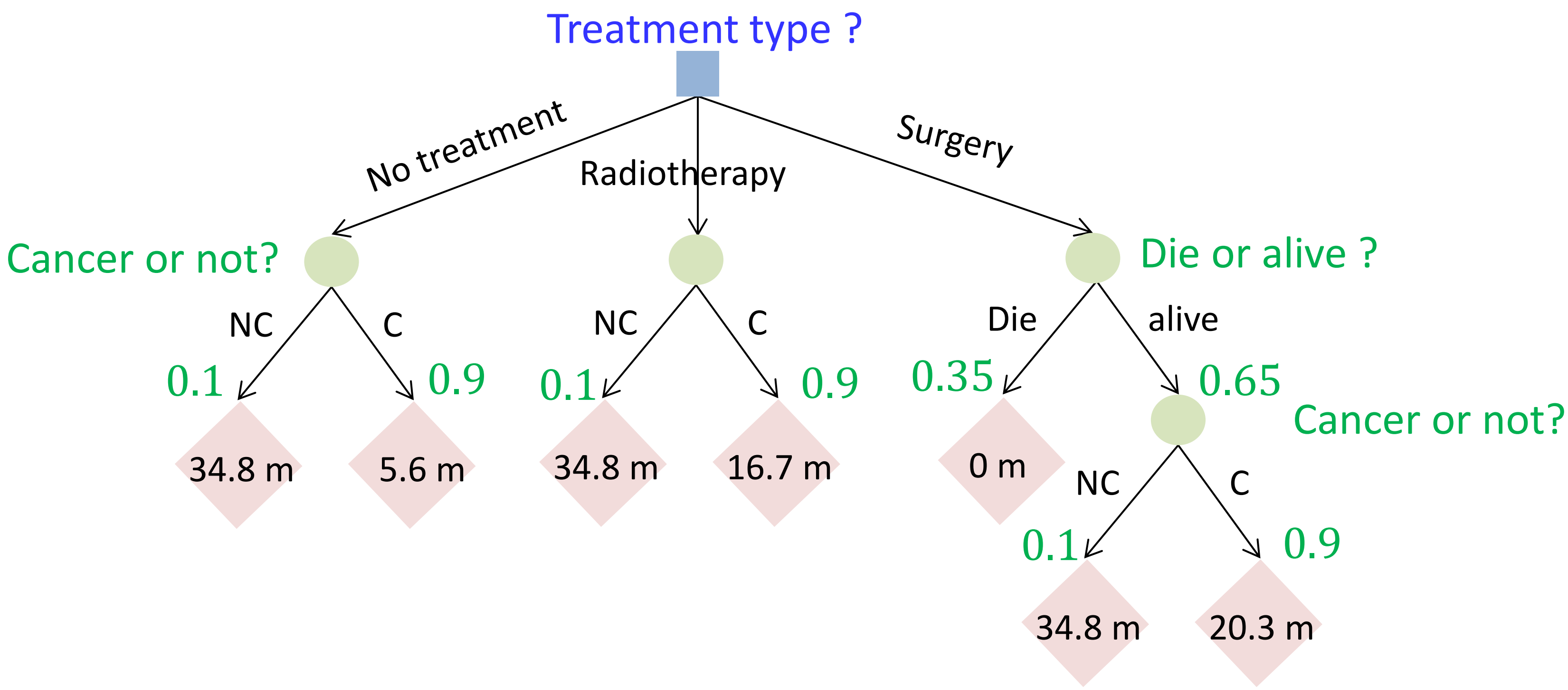
↓

Final output come

Bayesian inference is particularly useful in updating the state of knowledge with the information gained at each step.

Example 2: Multistage decision making for medical screening

Single decision point (without test)

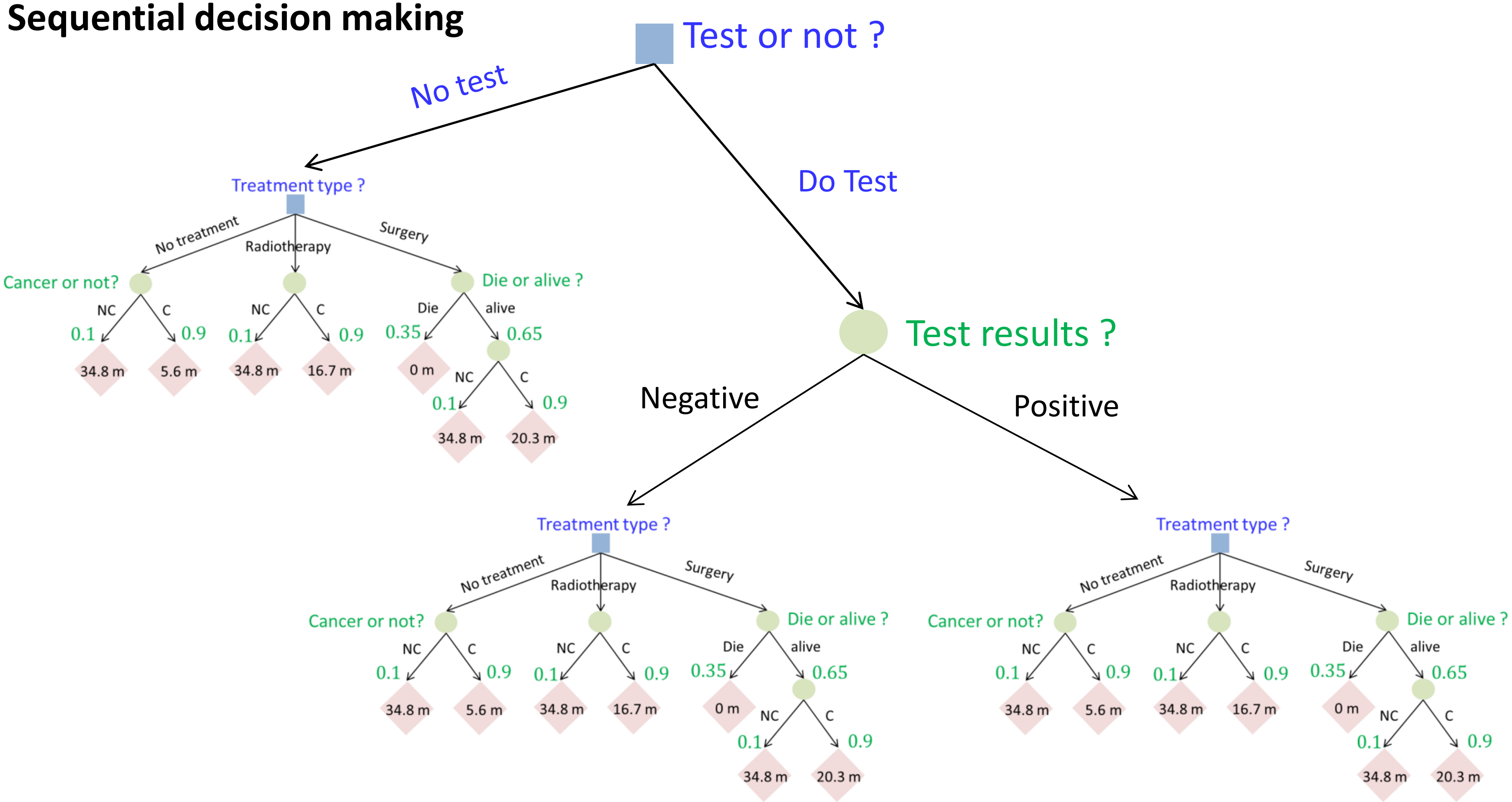


Quality-adjusted life expectancy under each treatment:

- With no treatment : $0.1 \times 34.8 + 0.9 \times 5.6 = 8.5\ m$
- With radiotherapy : $0.1 \times 34.8 + 0.9 \times 16.7 - 1 = 17.5\ m$ Best option
- With surgery : $0.35 \times 0 + 0.65 \times (0.1 \times 34.8 + 0.9 \times 20.3 - 1) = 13.5\ m$

Example 2: Multistage decision making for medical screening

Sequential decision making



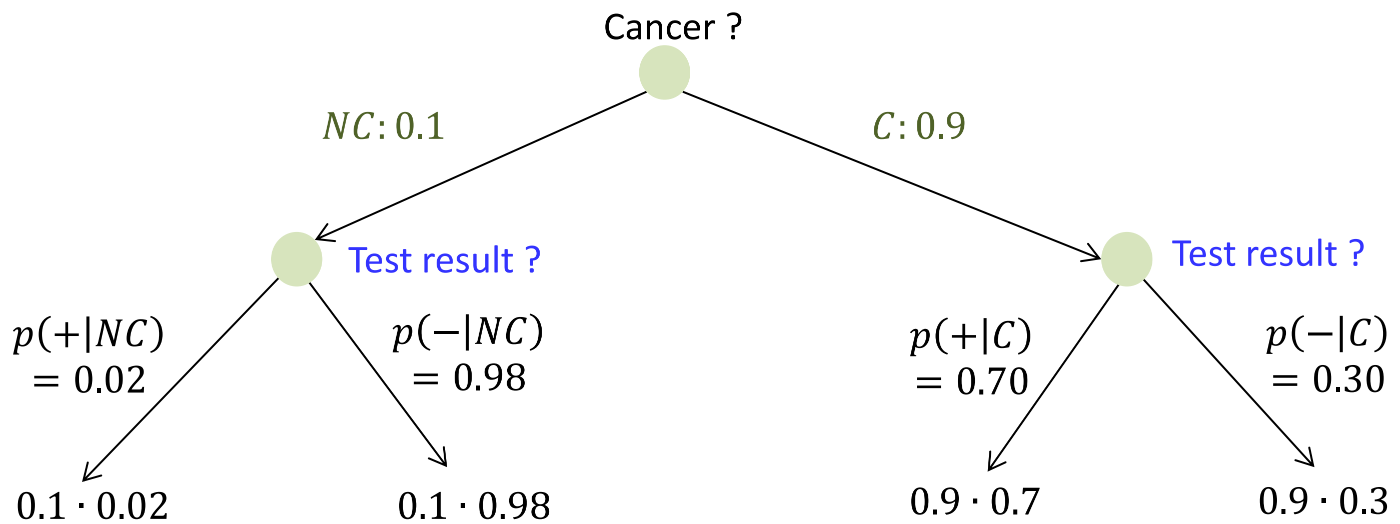
- Based on the test result, the probability of cancer can be updated using Bayes rule:

$$Pr(C|T) = \frac{\Pr(C)\, p(T|C)}{\Pr(C)\, p(T|C) + \Pr(NC)\, p(T|NC)}$$

- Updated information can be used for sequential decision making

Example 2: Multistage decision making for medical screening

Update information based on the test results



$$\Pr(T=+) = \Pr(NC)\,p(T=+|NC) + \Pr(C)\,p(T=+|C)$$
$$= 0.1 \cdot 0.02 + 0.9 \cdot 0.7 = 0.632$$
$$\Pr(T=-) = \Pr(NC)\,p(T=-|NC) + \Pr(C)\,p(T=-|C)$$
$$= 0.1 \cdot 0.98 + 0.9 \cdot 0.3 = 0.368$$

$$Pr(C|T) = \frac{\Pr(C)\,p(T|C)}{\Pr(T)}$$

$$Pr(C|T=+) = \frac{0.9 \cdot 0.7}{0.1 \cdot 0.02 + 0.9 \cdot 0.7} = 0.997$$

$$Pr(C|T=-) = \frac{0.9 \cdot 0.3}{0.1 \cdot 0.98 + 0.9 \cdot 0.3} = 0.734$$

$$Pr(NC|T) = \frac{\Pr(NC)\,p(T|NC)}{\Pr(T)}$$

$$Pr(NC|T=+) = \frac{0.1 \cdot 0.7}{0.1 \cdot 0.02 + 0.9 \cdot 0.7} = 0.003$$

$$Pr(NC|T=-) = \frac{0.1 \cdot 0.3}{0.1 \cdot 0.98 + 0.9 \cdot 0.3} = 0.266$$

Example 2: Multistage decision making for medical screening

Choose the best action at the second stage using the updated information

$$Pr(cancer|T=+) = \frac{0.9 \cdot 0.7}{0.9 \cdot 0.7 + 0.1 \cdot 0.02} = 0.997$$

Quality-adjusted life expectancy under each treatment:

- With no treatment : $0.997 \cdot 5.6 + 0.003 \cdot 34.8 = 5.7$ months
- With radiotherapy : $0.997 \cdot 16.7 + 0.003 \cdot 34.8 - 1 = 15.8$ months ⟵ Best action
- With surgery : $0.35 \cdot 0.65(0.997 \cdot 20.3 + 0.003 \cdot 34.8 - 1) = 12.6$ months

$$Pr(cancer|T=-) = \frac{0.9 \cdot 0.3}{0.9 \cdot 0.3 + 0.1 \cdot 0.98} = 0.734$$

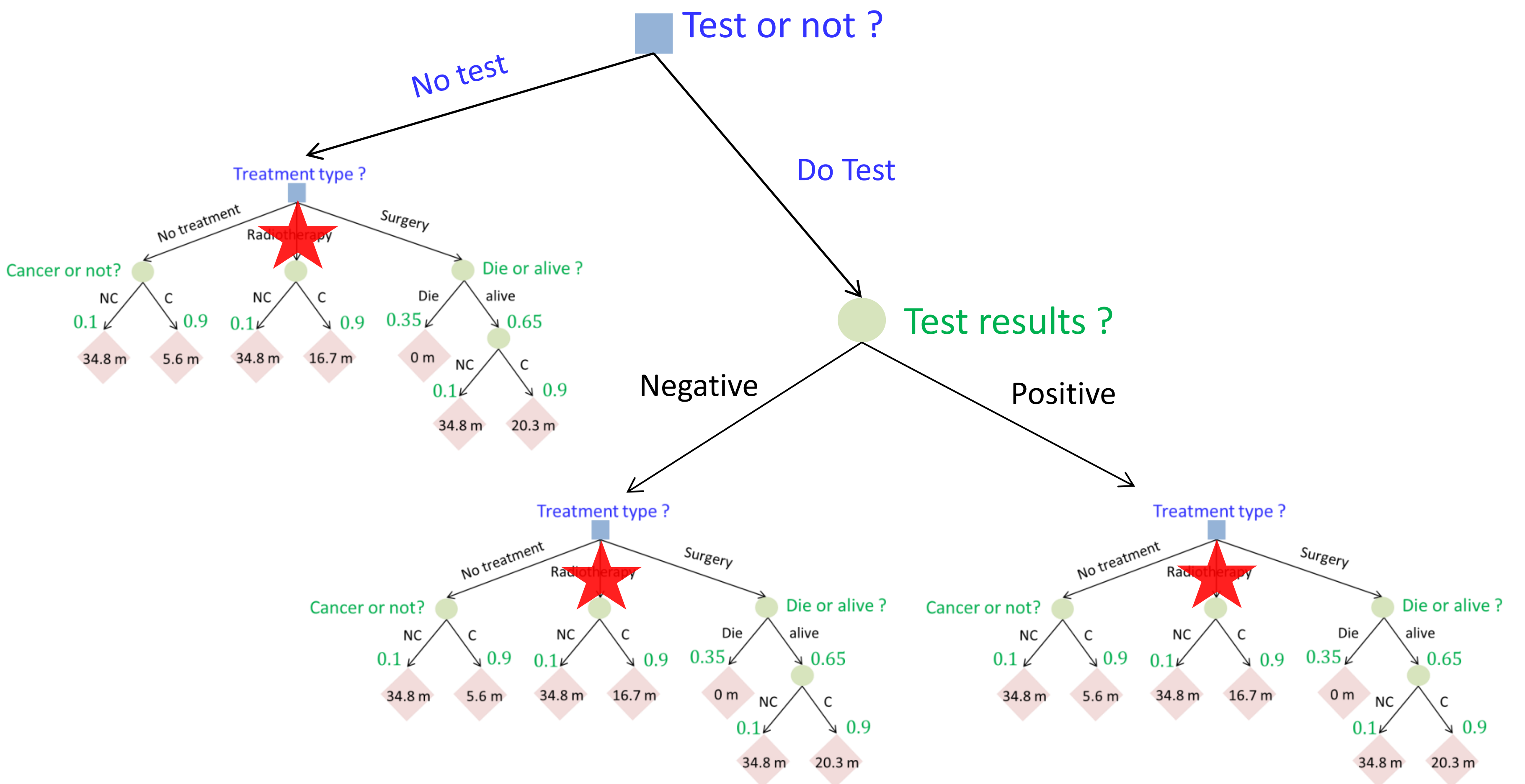
Quality-adjusted life expectancy under each treatment:

- With no treatment : $0.734 \cdot 5.6 + 0.266 \cdot 34.8 = 13.4$ months
- With radiotherapy : $0.734 \cdot 16.7 + 0.266 \cdot 34.8 - 1 = 20.5$ months ⟵ Best action
- With surgery : $0.35 \cdot 0.65(0.734 \cdot 20.3 + 0.266 \cdot 34.8 - 1) = 15.1$ months

It is clear conducting a test is not a good idea since no change in the selected option!

Example 2: Multistage decision making for medical screening

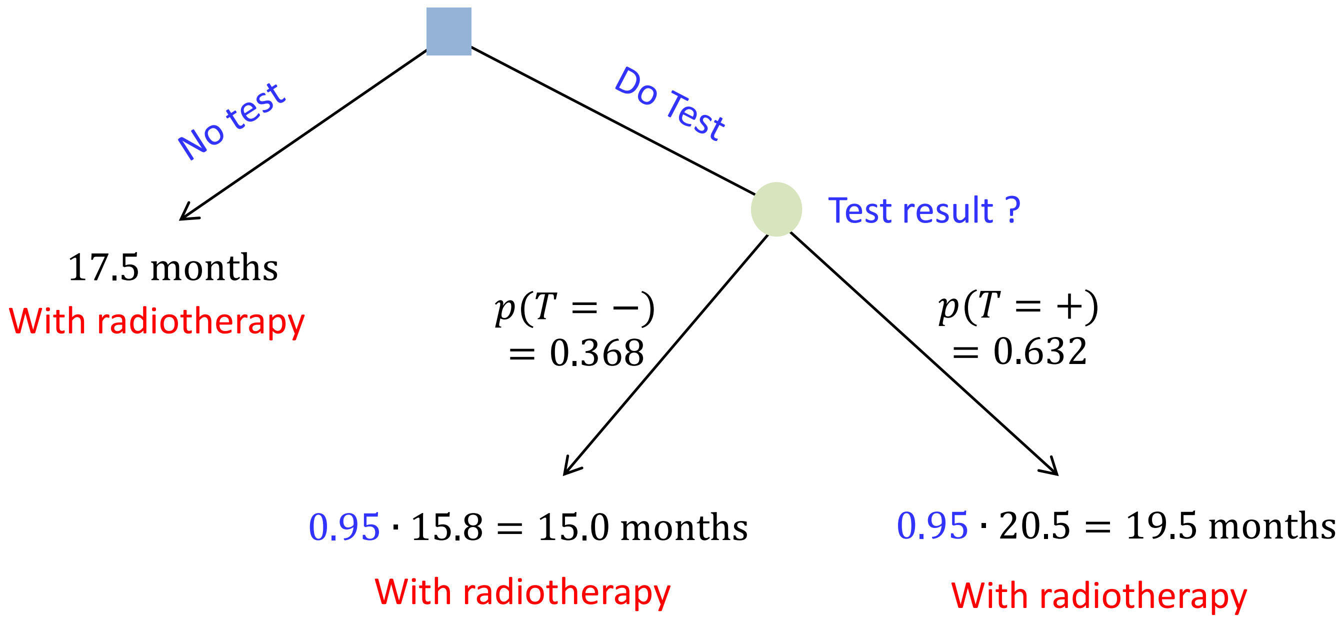
Sequential decision making



When selecting a decision at the first state, we should assume that all the decisions at the subsequent stages will be made optimum!! (key idea of sequential decision making: Control, MDP)

Example 2: Multistage decision making for medical screening

Decision analysis for performing test



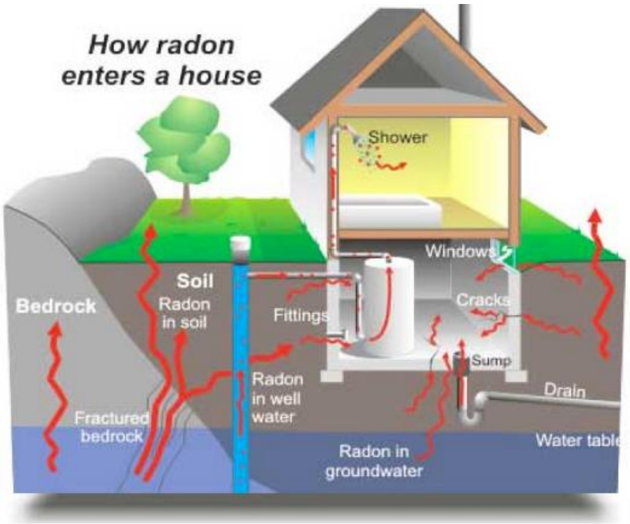
(0.95 is accounting for the 5% chance that the test can be fatal)

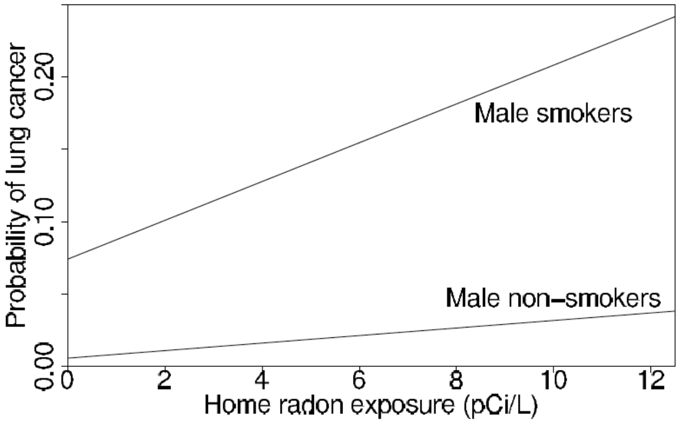
- No test : 17.5 months is the best
- Test : $0.368 \cdot 19.5 + 0.632 \cdot 15.0 = 16.6\ months$

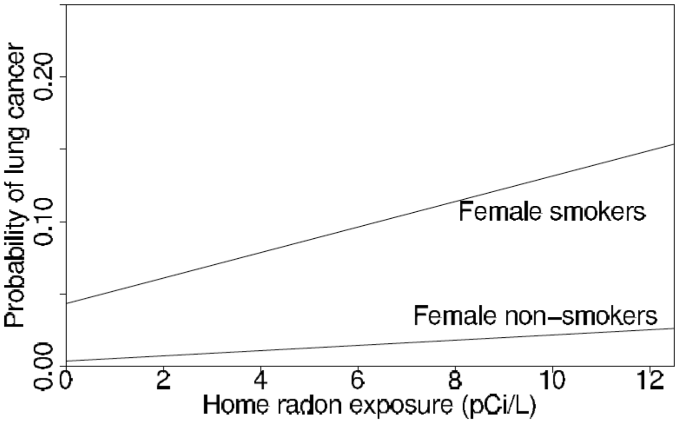
No change in the selected option and no improvement in the expected cost

→ Not conducting the test is the best option

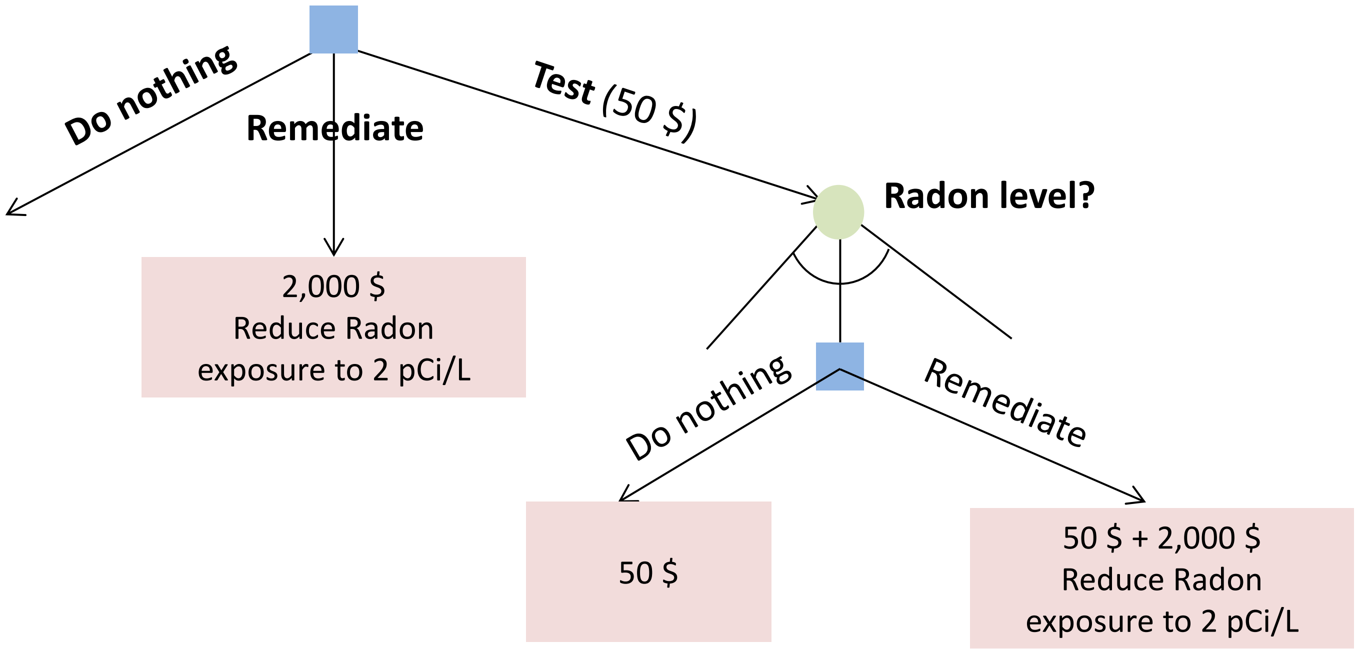
Example 3: Hierarchical decision analysis for home radon







Example 3: Hierarchical decision analysis for home radon



Two tasks:

- Given prior information about Radon concentration, decide whether to conduct a measurement test (first decision)
- Given the measurement of Radon concertation (we have a posterior of Radon con.), decide whether to remediate (second decision)

Performing the decision analysis requires estimating for the risks, which is done by using hierarchical Bayesian model

Example 3: Hierarchical decision analysis for home radon

Define utility : trades off dollars and lives

- D_d, the dollar value associated with a reduction of 10^{-6} in probability of death for lung cancer (the money that you are willing to pay to reduce 10^{-6} in probability of death for lung cancer)
- D_r, the dollar value associated with a reduction of 1 pCi/L in home radon level for 30-year period
  - depends on the number of lives saved by a drop in the Radon level
  - affected by variety of factors including gender, smoking status…
  - $D_r = 4800D_d$ (typical U.S. household)
- R_{action}, the home radon level above which you should remediate if your radon level is known
  - depends on the dollar value of radon reduction and the benefits of remediation

Action level R_{action} is determined as the value at which the benefit of remediation

$$\$D_r(R_{action} - R_{remed}) = \text{remediation cost of } \$2{,}000$$

$$R_{action} = \frac{\$2{,}000}{D_r} + R_{remedy}$$

$R_{action} = 4$ pCi/L is used as exemplary value, which can vary depending on financial resources, general risk tolerance, attitude toward risk, smoking status, etc.

Example 3: Hierarchical decision analysis for home radon

Bayesian inference for country radon level

Two data sets:

- Long-term measurements from approximately 5,000 houses, selected as a cluster sample from 125 randomly selected countries
- Short-term measurements from about 80,000 houses, sampled at random from all the counties in the U.S.

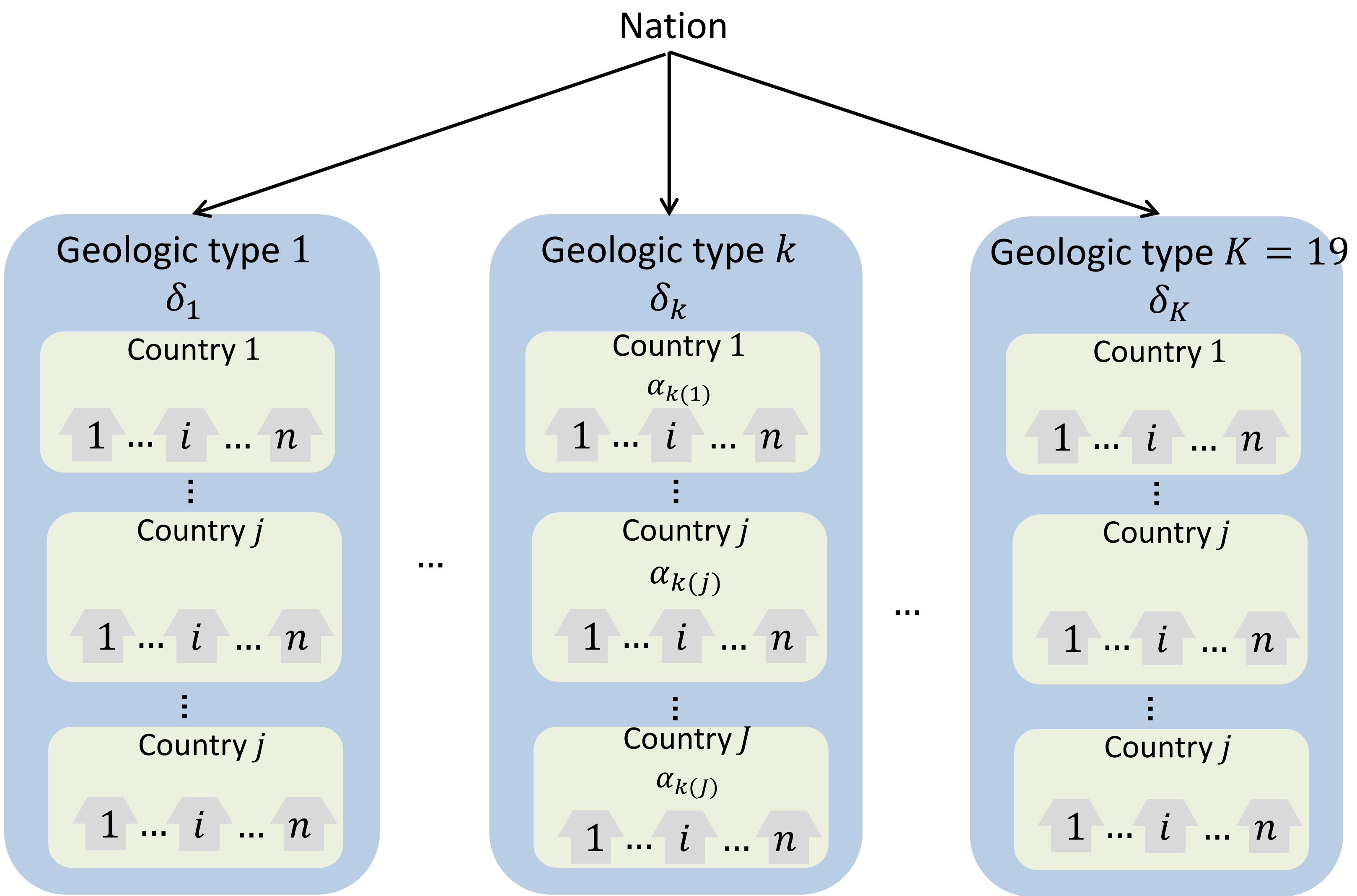
a relatively small amount of accurate data VS. a large amount of biased and imprecise data

Challenges:

Use the good data to calibrate the bad data, so that inference can be made about the entire country, not merely the 125 countries in the sample of long-term measurements

Example 3: Hierarchical decision analysis for home radon

Hierarchical modeling for individual house radon level



Example 3: Hierarchical decision analysis for home radon

Hierarchical modeling for individual house radon level

η

δ_1 Geological type k: $\delta_k \sim N(0, \eta^2)$ δ_K

α_1 Country j: $\alpha_j \sim N(W_j\gamma + \delta_{k(j)}, \tau^2)$ α_J

y_1 House i: $y_i \sim N(X_i\beta + \alpha_{j(i)}, \sigma_i^2)$ y_n

W_j: county-level predictors including climate data and a measure of the uranium level in the soil

X_i: household-level predictors including indicators for whether the house has a basement and **measurement type (**σ_i^2 varies depending on measurement type (long-term or short term**)**

Example 3: Hierarchical decision analysis for home radon

Bayesian inference for individual home level

$$R_i = \text{radon concntration in house } i$$
$$\theta_i = \log(R_i)$$

$$\theta_i = N(M_i, S_i^2)$$

- The mean $M_i = X_i\hat{B} + \hat{\alpha}_{j(i)}$ is computed from the posterior simulations of the model estimation.

  $\hat{B}$ and $\hat{\alpha}$ are the posterior means from the analysis in the appropriate region of the country. (computed from country level measurement data)

- The variance S_i^2 is computed from the posterior simulations of the model estimation taking into account the posterior uncertainty in the coefficients α, β and also hierarchical variance components η^2 and τ^2
- Serves as a **prior distribution** for the homeowner in that the distribution is constructed solely based on the basement information X_i and the county level parameter $\hat{\alpha}_{j(i)}$

Using $p(\theta_i)$ Decision whether to perform measurement test or not can be made

Example 3: Hierarchical decision analysis for home radon

Bayesian inference for individual home level

Likelihood :

$$Y_i \sim N(\theta, \sigma_Y^2) \rightarrow p(y_i|\theta, \sigma_Y^2) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(y_i - \theta)^2}{2\sigma_Y^2}\right)$$

Prior:

$$\theta \sim N(\mu_0, \tau_0^2) \rightarrow p(\theta) = \frac{1}{\sqrt{2\pi\tau_o^2}} \exp\left(-\frac{(\theta - \mu_0)^2}{2\tau_o^2}\right)$$

Posterior on $\theta = \mu_Y$

$$P(\theta|y) = N\left(\theta \left| \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma_Y^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}}, \left(\frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}\right)^{-1} \right.\right)$$

R_i = radon concntration in house i

$\theta_i = \log(R_i)$

Assume we measure Radon concentration $y_i \sim N(\theta_i, \sigma_i^2)$

Prior (from previous slide) :

$$p(\theta_i) = N(M_i, S_i^2)$$

Likelihood

$$y_i \sim N(\theta_i, \sigma_i^2)$$

The posterior distribution :

$$\theta_i | M_i, y_i \sim N(\Lambda_i, V_i) \qquad \Lambda_i = \frac{\frac{M_i}{S_i^2} + \frac{y_i}{\sigma_i^2}}{\frac{1}{S_i^2} + \frac{1}{\sigma_i^2}} \qquad \mathrm{V}_i = \frac{1}{\frac{1}{S_i^2} + \frac{1}{\sigma_i^2}}$$

Example 3: Hierarchical decision analysis for home radon

Decision analysis for individual homeowners

$$R_i = \text{radon concntration in house } i$$
$$\theta_i = \log(R_i)$$

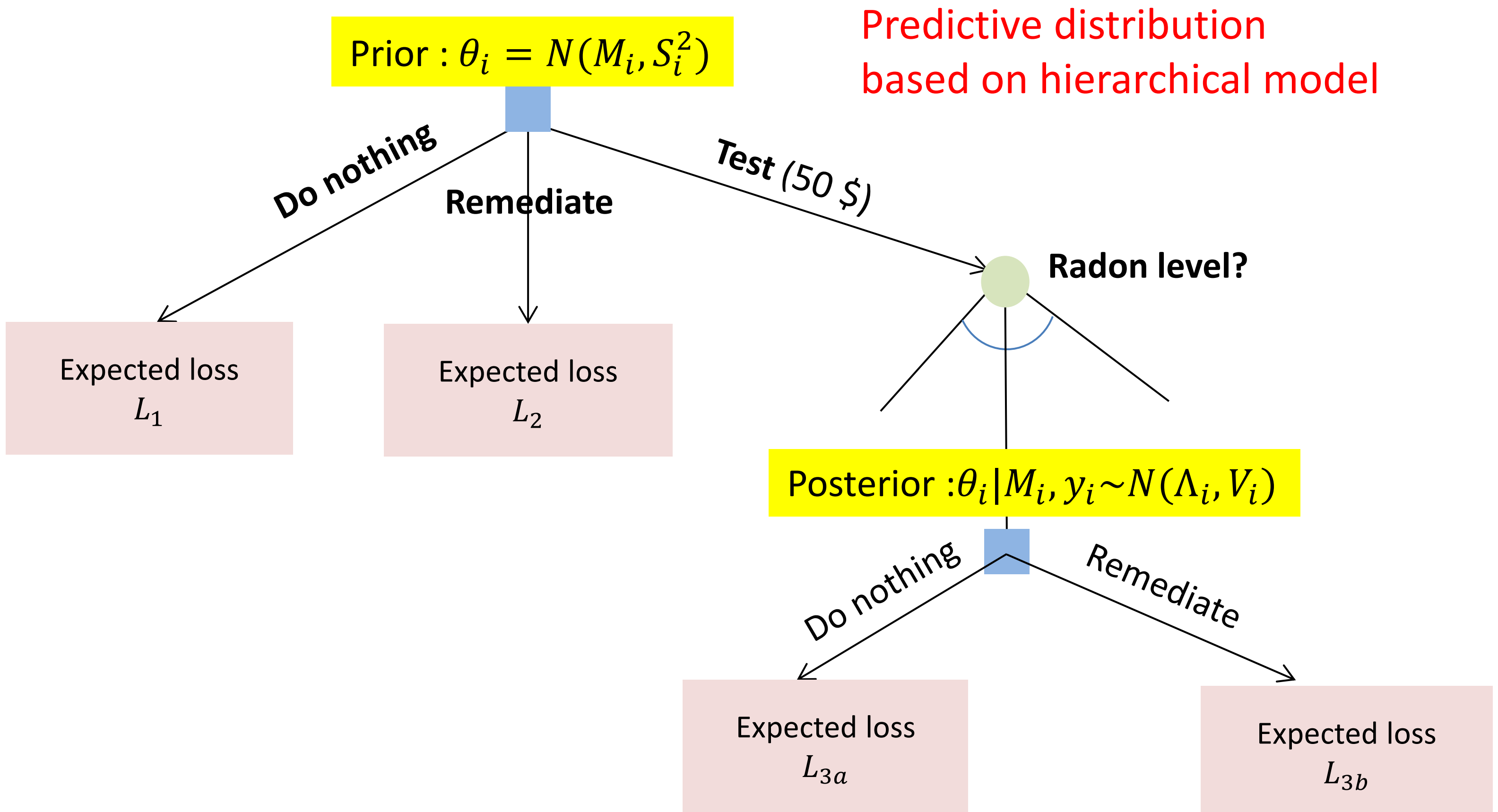
Prior : $\theta_i = N(M_i, S_i^2)$ Predictive distribution based on hierarchical model

Decision whether to perform measurement test or not

Posterior :$\theta_i | M_i, y_i \sim N(\Lambda_i, V_i)$

Decision whether to remediate or not

Example 3: Hierarchical decision analysis for home radon



Expected loss = cost + $D_r E(R)$

Example 3: Hierarchical decision analysis for home radon

Decision analysis for individual homeowners

R_i = radon concntration in house i
$\theta_i = \log(R_i)$

$$R = e^{\theta}, \theta \sim N(M, S^2)$$

$$E[R] = E\left[e^{\theta}\right] = e^{M+\frac{1}{2}S^2}$$

$$E[R|\theta > a] = E\left[e^{\theta}|\theta > a\right] = e^{M+\frac{1}{2}S^2}\left(1 - \Phi\left(\frac{M + S^2 - a}{S}\right)\right)$$

Do nothing

$$L_1 = D_r E(R) = D_r e^{M+\frac{1}{2}S^2}$$

Remediate without test

$$L_2 = \$2000 + D_r \mathrm{E}(\min(R, R_{remed}))$$

$$= \$2000 + D_r[R_{remed} \Pr(R \geq R_{remed}) + E(R|R < R_{remed}) \Pr(R < R_{remed})]$$

$$= \$2000 + D_r\left[R_{remed}\Phi\left(\frac{M - \log(R_{remed})}{S}\right) + e^{M+\frac{1}{2}S^2}\left(1 - \Phi\left(\frac{M + S^2 - \log(R_{remed})}{S}\right)\right)\right]$$

Example 3: Hierarchical decision analysis for home radon

Decision analysis for individual homeowners

Remediate without test

$$
\begin{aligned}
L_2 &= \$2000 + D_r \mathrm{E}(\min(R, R_{remed})) \\
&= \$2000 + D_r [R_{remed} \Pr(R \geq R_{remed}) + E(R|R < R_{remed}) \Pr(R < R_{remed})] \\
&= \$2000 + D_r \left[R_{remed} \Phi\left(\frac{M - \log(R_{remed})}{S}\right) + e^{M+\frac{1}{2}S^2}\left(1 - \Phi\left(\frac{M + S^2 - \log(R_{remed})}{S}\right)\right)\right]
\end{aligned}
$$

Prior

(M, S^2)

Perform measurement test

(Λ, V)

Posterior

- Do nothing

$$
L_{3a} = \$50 + D_r \frac{1}{30} e^{\mathrm{M}+\frac{1}{2}S^2} + D_r e^{\Lambda+\frac{1}{2}V}
$$

Exposure of 1 year

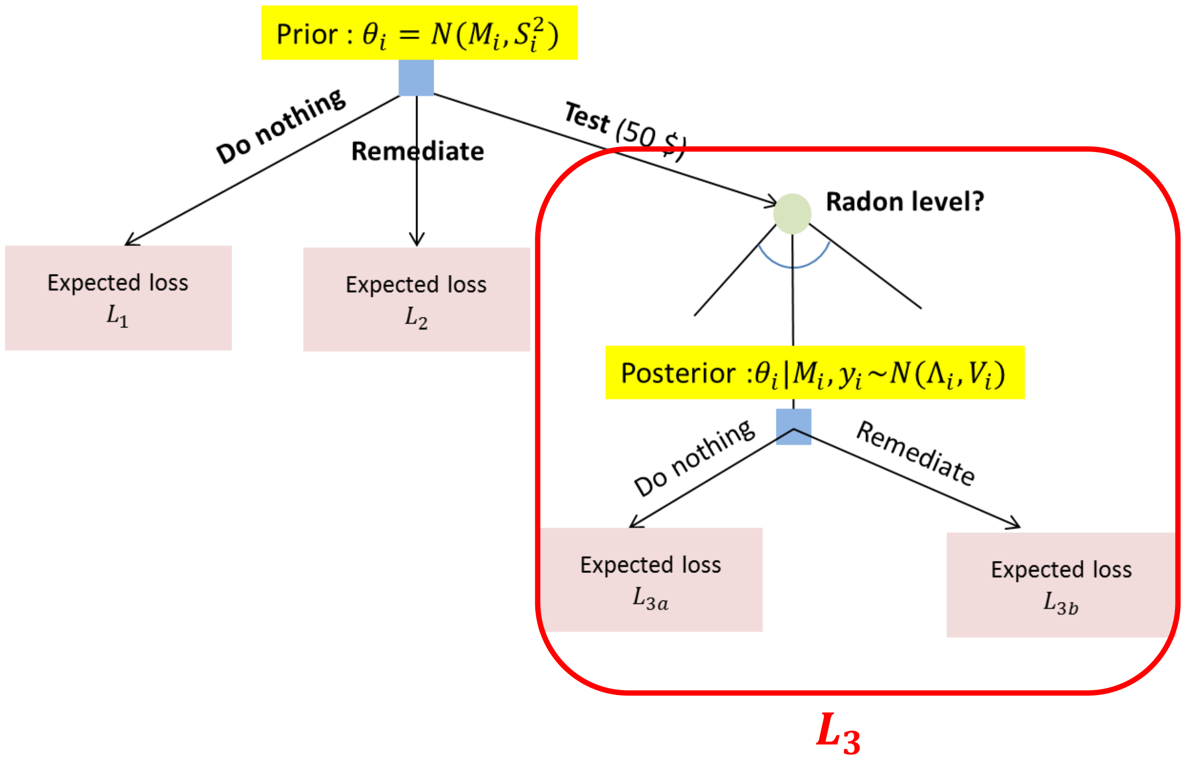
29 years of exposure after test is done

- Remediate

$$
\begin{aligned}
L_{3b} = \$50 + D_r \frac{1}{30} e^{\mathrm{M}+\frac{1}{2}S^2} + \$2000 \\
+ D_r \left[R_{remed} \Phi\left(\frac{\Lambda - \log(R_{remed})}{\sqrt{V}}\right) + e^{M+\frac{1}{2}S^2}\left(1 - \Phi\left(\frac{\Lambda + V - \log(R_{remed})}{\sqrt{V}}\right)\right)\right]
\end{aligned}
$$

Example 3: Hierarchical decision analysis for home radon

Decision analysis for individual homeowners



Expected loss for performing test : $L^3 = \mathrm{E}(\min(L_{3a}, L_{3b}))$:
(optimum decision in the second state is imbedded)

1. Simulate 5000 draws of $y \sim \mathrm{N}(\mathrm{M}, \mathrm{S}^2 + \sigma^2)$ (Considering uncertainty of y)
2. For each draw of y, compute $\min(L_{3a}, L_{3b})$
3. Estimate L_3 as the average of these 5000 values

Decision analysis for individual homeowners