

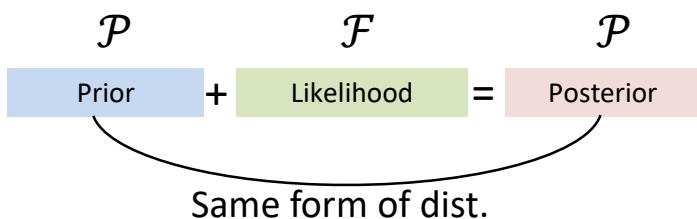
## **L3. Conjugate Models**

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int_{\theta} p(y|\theta)p(\theta)d\theta}$$

1. Non-informative prior (objective prior, vague prior, reference prior...)
2. Weakly informative prior
3. Informative prior (subjective prior)

## Conjugacy

From the Bayes rule,  $p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int_{\theta} p(y|\theta)p(\theta)d\theta}$



$p(\theta|y) \in \mathcal{P}$  for all  $p(\cdot|\theta) \in \mathcal{F}$  and  $p(\cdot) \in \mathcal{P}$

$\rightarrow \mathcal{P}$  is conjugate for  $\mathcal{F}$

If the **posterior** is a distribution that is of the same family as our **prior**  
 $\rightarrow$  the prior is conjugate to the **likelihood**.

## Advantages

- Due to the analytical form available, **we can carry out the integration**

$$p(y) = \int_{\theta} p(y|\theta)p(\theta)d\theta$$

- By using conjugate prior, we know the form of the resultant posterior (don't need math!)
  - ✓ we can easily summarize the results using mean, mode, variance, etc.
- Easy to understand the meaning of the prior used in the analysis (insight).
  - ✓ For example, Beta prior is just adding pseudo counts to the data.

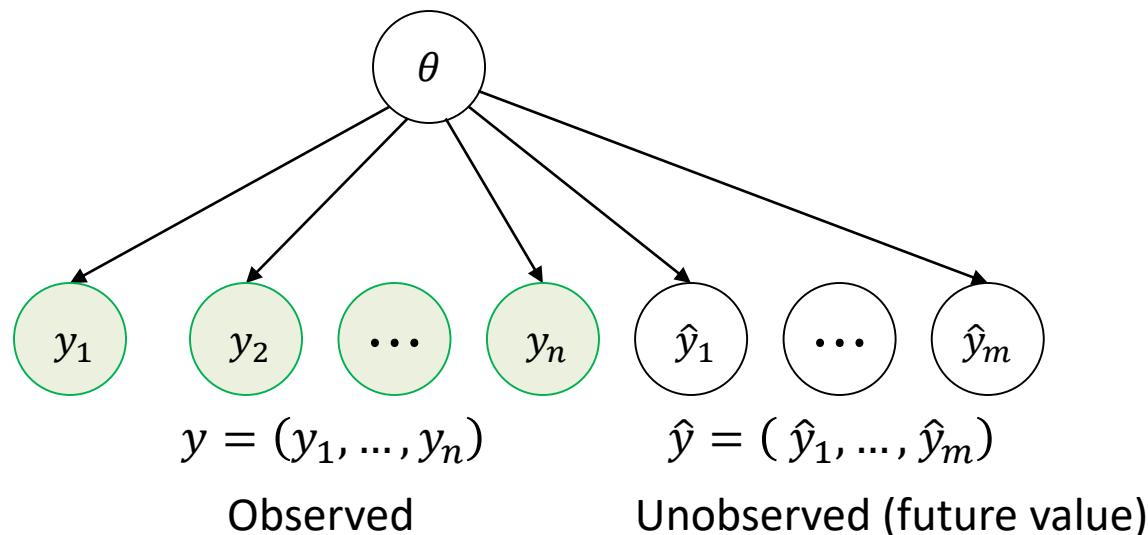
## Conjugate pairs

	Likelihood	Prior	Posterior
Single parameter model ←	Binomial	Beta	Beta
	Negative Binomial	Beta	Beta
	Geometric	Beta	Beta
	Poisson	Gamma	Gamma
	Exponential	Gamma	Gamma
	Normal (mean unknown)	Normal	Normal
Multi parameters model ←	Normal (variance unknown)	Inverse Gamma	Inverse Gamma
	Normal (mean and variance unknown)	Normal-Gamma	Normal-Gamma
	Multinomial	Dirichlet	Dirichlet

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Multi parameters model ←	Normal (mean and variance unknown)	Normal-Gamma	Normal-Gamma
	Multinomial	Dirichlet	Dirichlet

## Bayesian Inference Problems



### Objectives

- Prior predictive distribution

$$p(\hat{y}) = \int_{\theta} p(\hat{y}, \theta) d\theta = \int_{\theta} p(\hat{y}|\theta) p(\theta) d\theta$$

- Posterior distribution

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int_{\theta} p(y|\theta)p(\theta)d\theta} \propto p(y|\theta)p(\theta)$$

- Posterior predictive distribution

$$p(\hat{y}|y) = \int_{\theta} p(\hat{y}, \theta|y) d\theta = \int_{\theta} p(\hat{y}|\theta, y) p(\theta|y) d\theta = \int_{\theta} p(\hat{y}|\theta) p(\theta|y) d\theta$$

$$p(\hat{y}|\theta, y) = p(\hat{y}|\theta) \text{ because } \hat{y} \perp y | \theta$$

## Binomial Likelihood and Beta Prior Distribution (Recap)

## Binomial Likelihood and Beta Prior Distribution (Recap)

Likelihood :

$$Y \sim \text{Bin}(n, \theta) \rightarrow p(y|\theta) = \binom{n}{y} \theta^y (1 - \theta)^{n-y}$$

Prior:

$$\theta \sim \text{Beta}(\alpha, \beta) \rightarrow p(\theta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$

Prior predictive distribution :

$$p(y) = \int_{\theta} p(y, \theta) d\theta = \int_{\infty} p(y|\theta) p(\theta) d\theta$$

$$\begin{aligned} p(y) &= \int_0^1 p(y, \theta) d\theta \\ &= \int_0^1 p(y|\theta) p(\theta) d\theta \quad P(y|\theta) = \text{Bin}(y|n, \theta), \quad p(\theta) = \text{Beta}(\alpha, \beta) \\ &= \int_0^1 \binom{n}{y} \theta^y (1 - \theta)^{n-y} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1} d\theta \quad \binom{n}{y} = \frac{n!}{y!(n-y)!} = \frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)} \\ &= \frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^y (1 - \theta)^{n-y} \theta^{\alpha-1} (1 - \theta)^{\beta-1} d\theta \\ &= \frac{\Gamma(n+1)\Gamma(\alpha + \beta)}{\Gamma(y+1)\Gamma(n-y+1)\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^{y+\alpha-1} (1 - \theta)^{n-y+\beta-1} d\theta \\ &= \frac{\Gamma(n+1)\Gamma(\alpha + \beta) \Gamma(y + \alpha) \Gamma(n - y + \beta)}{\Gamma(y+1)\Gamma(n-y+1)\Gamma(\alpha)\Gamma(\beta) \Gamma(n + \alpha + \beta)} \int_0^1 \frac{\Gamma(n + \alpha + \beta)}{\Gamma(y + \alpha) \Gamma(n - y + \beta)} \theta^{y+\alpha-1} (1 - \theta)^{n-y+\beta-1} d\theta \\ &= \frac{\Gamma(n+1)\Gamma(\alpha + \beta) \Gamma(y + \alpha) \Gamma(n - y + \beta)}{\Gamma(y+1)\Gamma(n-y+1)\Gamma(\alpha)\Gamma(\beta) \Gamma(n + \alpha + \beta)} \int_0^1 \text{Beta}(\theta|y + \alpha, n - y + \beta) d\theta \\ &= \frac{\Gamma(n+1)\Gamma(\alpha + \beta) \Gamma(y + \alpha) \Gamma(n - y + \beta)}{\Gamma(y+1)\Gamma(n-y+1)\Gamma(\alpha)\Gamma(\beta) \Gamma(n + \alpha + \beta)} 1 \\ &= \text{Beta-Binomial}(y|n, \alpha, \beta) \end{aligned}$$

The beta-binomial distribution is the binomial distribution in which the probability of success at each trial is not fixed but random and follows the beta distribution

$$\text{Beta-bin}(y|n, \alpha, \beta) = \int \text{Bin}(y|n, \theta) \text{Beta}(\theta|\alpha, \beta) d\theta$$



## Binomial Likelihood and Beta Prior Distribution (Recap)

Likelihood :

$Y \sim \text{Bin}(n, \theta) \rightarrow p(y|\theta) = \binom{n}{y} \theta^y (1 - \theta)^{n-y}$   $Y$  is the number of successes among  $n$  Bernoulli trials

Prior:

$$\theta \sim \text{Beta}(\alpha, \beta) \rightarrow p(\theta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$

Posterior :

$$P(\theta|y) \propto p(y|\theta)p(\theta)$$

$$= \binom{n}{y} \theta^y (1 - \theta)^{n-y} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$

$$\propto \theta^y (1 - \theta)^{n-y} \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$

$$= \theta^{\alpha+y-1} (1 - \theta)^{\beta+n-y-1}$$

$$= \text{Beta}(\theta|\alpha + y, \beta + n - y)$$

( $\alpha$  is a pseudo-count for the success, while  $\beta$  is a pseudo-count for the failure)

## Binomial Likelihood and Beta Prior Distribution (Recap)

Likelihood :

$$Y \sim \text{Bin}(n, \theta) \rightarrow p(y|\theta) = \binom{n}{y} \theta^y (1 - \theta)^{n-y}$$

$Y$  is the number of success among  $n$  Bernoulli trial

Prior:

$$\theta \sim \text{Beta}(\alpha, \beta) \rightarrow p(\theta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$

Posterior :

$$P(\theta|y) = \text{Beta}(\theta|\alpha + y, \beta + n - y)$$

Posterior predictive distribution :

$$p(\hat{y}|y) = \int_0^1 p(\hat{y}, \theta|y) d\theta = \int_0^1 p(\hat{y}|\theta) p(\theta|y) d\theta$$

$$p(y) = \int_0^1 p(y, \theta) d\theta = \int_0^1 p(y|\theta) p(\theta) d\theta$$

$$= \text{Beta-Binomial}(y|n, \alpha, \beta)$$

Using this result, the posterior predictive distribution is

$$p(\hat{y}|y) = \int_0^1 p(\hat{y}, \theta|y) d\theta = \int_0^1 p(\hat{y}|\theta) p(\theta|y) d\theta$$

$$= \text{Beta-Binomial}(\hat{y}|n, \alpha + y, \beta + n - y)$$

$$p(\theta) = \text{Beta}(\alpha, \beta)$$



$$p(\theta|y) = \text{Beta}(\alpha + y, \beta + n - y)$$

## Poisson Likelihood-Gamma Prior

## Poisson Likelihood-Gamma Prior

Likelihood :

$$Y_i \sim \text{Poisson}(\lambda) \rightarrow p(y_i|\lambda) = \frac{\lambda^{y_i} e^{-\lambda}}{y_i!}$$

Prior:

$$\lambda \sim \text{Gamma}(\alpha, \beta) \rightarrow p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

**Prior predictive distribution :**

$$p(y) = \int_{\lambda} p(y, \lambda) d\lambda = \int_{\lambda} p(y|\lambda) p(\lambda) d\lambda$$

$$\begin{aligned} p(y) &= \int_0^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda \\ &= \frac{\beta^\alpha}{y! \Gamma(\alpha)} \int_0^{\infty} \lambda^y e^{-\lambda} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda \\ &= \frac{\beta^\alpha}{y! \Gamma(\alpha)} \int_0^{\infty} \lambda^{y+\alpha-1} e^{-(\beta+1)\lambda} d\lambda \\ &= \frac{\beta^\alpha}{y! \Gamma(\alpha)} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}} \int_0^{\infty} \frac{(\beta+1)^{y+\alpha}}{\Gamma(y+\alpha)} \lambda^{y+\alpha-1} e^{-(\beta+1)\lambda} d\lambda \\ &= \frac{\beta^\alpha}{y! \Gamma(\alpha)} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}} \text{Gamma}(\lambda|\alpha+y, \beta+1) \\ &= \binom{y+\alpha-1}{y} \left( \frac{1}{\beta+1} \right)^y \left( \frac{\beta}{\beta+1} \right)^\alpha = \text{Neg-bin}(y|\alpha, \beta) \end{aligned}$$

**Negative Binomial distribution (Neg – bin)** is a discrete probability distribution of the number of successes  $y$  in a sequence of independent and identically distributed Bernoulli trials before a specified (non-random) number of failures (denoted  $\alpha$ ) occurs.

$$\text{Neg-bin}(y|\alpha, \beta) = \int \text{Poisson}(y|\lambda) \text{Gamma}(\lambda|\alpha, \beta) d\lambda$$

## Poisson Likelihood-Gamma Prior

Likelihood :

$$Y_i \sim \text{Poisson}(\lambda) \rightarrow p(y_i|\lambda) = \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} \quad E(Y_i) = \lambda$$

Prior:

$$\lambda \sim \text{Gamma}(\alpha, \beta) \rightarrow p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \quad E(\lambda) = \frac{\alpha}{\beta}$$

Posterior :

$P(\lambda|y) \propto P(y|\lambda)p(\lambda)$   $y = (y_1, \dots, y_n)$  is a sequence of i.i.d. observation

$$= \left( \prod_{i=1}^n P(y_i|\lambda) \right) \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$= \left( \prod_{i=1}^n \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} \right) \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$= \frac{\lambda^{\sum_i y_i} e^{-n\lambda}}{\prod_i y_i!} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$\propto \lambda^{n\bar{y}} e^{-n\lambda} \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$\propto \lambda^{\alpha+n\bar{y}-1} e^{-(\beta+n)\lambda}$$

$$= \text{Gamma}(\alpha + n\bar{y}, \beta + n)$$

$$\because \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$

$$E(\lambda|y) = \frac{\alpha + n\bar{y}}{\beta + n} = \frac{\alpha + \sum_i y_i}{\beta + n}$$

( $\alpha$  is a pseudo-count for # of events, while  $\beta$  is a pseudo count for # of observations)

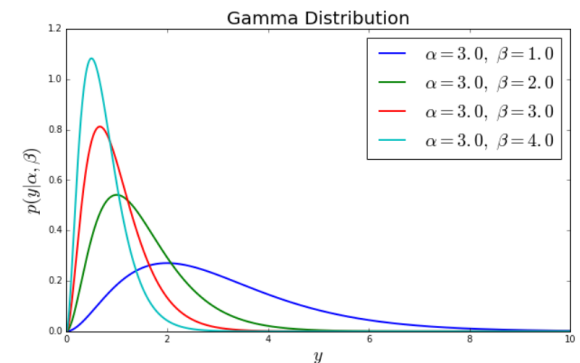
## Poisson Likelihood-Gamma Prior

The posterior mean is

$$\begin{aligned} E(\lambda|y) &= \frac{\alpha + \sum_i y_i}{\beta + n} \\ &= \frac{\alpha}{\beta + n} + \frac{\sum_i y_i}{\beta + n} \\ &= \frac{\beta}{\beta + n} \left( \frac{\alpha}{\beta} \right) + \frac{n}{\beta + n} \left( \frac{\sum_i y_i}{n} \right) \\ &= \frac{\beta}{\beta + n} E(\lambda) + \frac{n}{\beta + n} \hat{\theta}_{ML} \end{aligned}$$

Again, the data get weighted more heavily as  $n \rightarrow \infty$

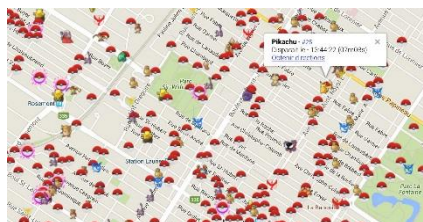
$\beta$  control strength of prior



## Example

Counts the numbers of Pokemons in each district of SF. The numbers are

14 13 7 10 15 15 2 13 13 11 10 13 5 13 9 12 9 12 8 7



- It is assumed that the numbers are independent and drawn from a Poisson distribution with mean  $\lambda$

$$Y_i \sim \text{Poisson}(\lambda) \rightarrow p(y_i|\lambda) = \frac{\lambda^{y_i} e^{-\lambda}}{y_i!}$$

- The prior distribution for  $\lambda$  is a Gamma distribution with mean 20 and standard deviation 10

$$\lambda \sim \text{Gamma}(\alpha, \beta) \rightarrow p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$E(\lambda) = \frac{\alpha}{\beta} = 20, \text{Var}(\lambda) = \frac{\alpha}{\beta^2} = 10^2 \rightarrow \alpha = 4, \beta = 0.2$$

- The posterior is

$$P(\lambda|y) = \text{Gamma}(4 + 211, 0.2 + 20) \quad P(\lambda|y) = \text{Gamma}(\alpha + n\bar{y}, \beta + n)$$

$$E(\lambda|y) = \frac{215}{20.2} = 10.64, \text{Var}(\lambda) = \frac{215}{20.2^2} = 0.5269$$

## Poisson Likelihood-Gamma Prior

Likelihood :

$$Y_i \sim \text{Poisson}(\lambda) \rightarrow p(y_i | \lambda) = \frac{\lambda^{y_i} e^{-\lambda}}{y_i!}$$

Prior:

$$\lambda \sim \text{Gamma}(\alpha, \beta) \rightarrow p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

Posterior :

$$P(\lambda | y) = \text{Gamma}(\alpha + n\bar{y}, \beta + n) \quad y = (y_1, \dots, y_n) \text{ is a sequence of i.i.d. observations}$$

Posterior predictive distribution (Using Bayes' rule)

$$p(\hat{y} | y) = \int_{\lambda} p(\hat{y}, \lambda | y) d\lambda = \int_{\lambda} p(\hat{y} | \lambda) p(\lambda | y) d\lambda$$

$$\begin{aligned} p(y) &= \int_0^\infty p(y, \lambda) d\lambda = \int_0^\infty p(y | \lambda) p(\lambda) d\lambda \\ &= NB(y | \alpha, \beta) = \binom{y + \alpha - 1}{y} \left( \frac{1}{\beta + 1} \right)^y \left( \frac{\beta}{\beta + 1} \right)^\alpha \end{aligned}$$

Using this result, the posterior predictive distribution is

$$\begin{aligned} p(\hat{y} | y) &= \int_0^\infty p(\hat{y}, \theta | y) d\lambda = \int_0^\infty p(\hat{y} | \theta) p(\theta | y) d\lambda \\ &= NB(\hat{y} | \alpha + n\bar{y}, \beta + n) = \binom{\hat{y} + \alpha + n\bar{y} - 1}{\hat{y}} \left( \frac{1}{\beta + n + 1} \right)^{\hat{y}} \left( \frac{\beta + n}{\beta + n + 1} \right)^{\alpha + n\bar{y}} \end{aligned}$$

$$p(\theta) = \text{Gamma}(\alpha, \beta)$$



$$p(\theta | y) = \text{Gamma}(\alpha + n\bar{y}, \beta + n)$$



## Poisson model parameterized in terms of rate and exposure

Likelihood :

$$Y_i \sim \text{Poisson}(x_i \lambda) \rightarrow p(y_i | \lambda, x_i) = \frac{(x_i \lambda)^{y_i} e^{-(x_i \lambda)}}{(x_i \lambda)!}$$

where the value  $x_i$  is called the exposure of the  $i$ th unit

$$p(y | \lambda, x_i) = \prod_{i=1}^n \frac{(x_i \lambda)^{y_i} e^{-(x_i \lambda)}}{(x_i \lambda)!} \quad y = (y_1, \dots, y_n) \text{ is a sequence of i.i.d. observations}$$

$$\propto \lambda^{(\sum_{i=1}^n y_i)} e^{-(\sum_{i=1}^n x_i) \lambda}$$

It is more flexible model in that we can control the unit time or area

Prior:

$$\lambda \sim \text{Gamma}(\alpha, \beta) \rightarrow p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta \lambda}$$

Posterior :

$$P(\lambda | y) = \text{Gamma} \left( \alpha + \sum_{i=1}^n y_i, \beta + \sum_{i=1}^n x_i \right)$$

## Estimating a rate from Poisson data: an idealized example

- A Poisson sampling model is often used for epidemiological data.
- 3 people, out of a population of 200,000 died of asthma  
→ 1.5 cases per 100,000 people per year : exposure  $x = 2.0$
- Under the Poisson model, the sampling distribution of  $y$ , the number of deaths in a city of 200,000 in one year, can be expressed as

$$y \sim \text{Poisson}(2.0\lambda)$$

Where  $\lambda$  represents the true underlying long-term asthma mortality rate in our city (measured in cases per 100,000 people per year)

- We can use knowledge about asthma mortality rates around the world to construct a prior distribution for  $\lambda$  and then combine the datum  $y = 3$  to obtain a posterior distribution

## Estimating a rate from Poisson data: an idealized example

Prior:

$$\lambda \sim \text{Gamma}(\alpha = 3, \beta = 5)$$

$\lambda = \text{\#death per 100,000}$

Posterior :

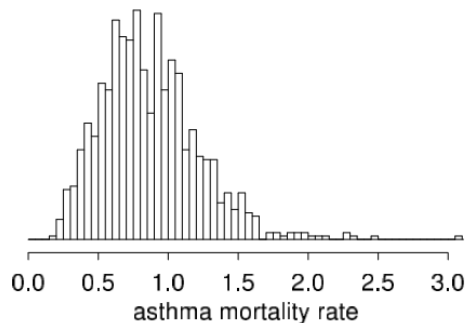
$$P(\lambda|y) = \text{Gamma}\left(\alpha + \sum_{i=1}^n y_i, \beta + \sum_{i=1}^n x_i\right)$$

**Case 1:** 3 persons, out of a population of 200,000 died of asthma for years

$$\sum_{i=1}^n x_i = 2 : \text{exposure}$$

$$\sum_{i=1}^n y_i = 3 \text{ (average)}$$

$$P(\lambda|y) = \text{Gamma}(3 + 3, 5 + 2)$$

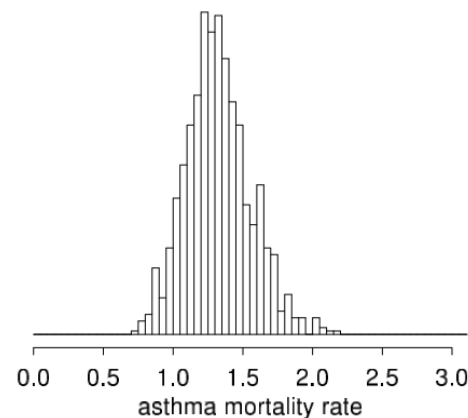


**Case 1:** 30 persons, out of a population of 200,000 died of asthma for 10 years

$$\sum_{i=1}^{n=10} x_i = 20 : \text{exposure}$$

$$\sum_{i=1}^n y_i = 30 \text{ (average)}$$

$$P(\lambda|y) = \text{Gamma}(3 + 30, 5 + 20)$$



## Exponential Likelihood-Gamma Prior

## Exponential Likelihood-Gamma Prior

Likelihood :

$$Y_i \sim \text{Exp}(\lambda) \rightarrow p(y_i | \lambda) = \lambda e^{-\lambda y_i}$$

$$E(Y_i) = \frac{1}{\lambda}$$

Prior:

$$\lambda \sim \text{Gamma}(\alpha, \beta) \rightarrow p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta \lambda}$$

Posterior :

$$P(\lambda | y) \propto P(y | \lambda) p(\lambda) \quad y = (y_1, \dots, y_n) \text{ is a sequence of i.i.d. observations}$$

$$= \left( \prod_{i=1}^n P(y_i | \lambda) \right) p(\lambda)$$

$$= \left( \prod_{i=1}^n \lambda e^{-\lambda y_i} \right) \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta \lambda}$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^n y_i} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta \lambda} \quad \because \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$

$$\propto \lambda^n e^{-\lambda n \bar{y}} \lambda^{\alpha-1} e^{-\beta \lambda}$$

$$= \lambda^{\alpha+n-1} e^{-(\beta+n\bar{y})\lambda}$$

$$= \text{Gamma}(\alpha + n, \beta + n\bar{y})$$

## Exponential Likelihood-Gamma Prior (Example)

A machine continuously produces nylon filament. From time to time the filament snaps. Suppose that the time intervals, in minutes, between snaps are random, independent and have an exponential distribution.

- Time interval between two successive failures :  $Y_i \sim \text{Exp}(\lambda) \rightarrow p(y_i|\lambda) = \lambda e^{-\lambda y_i}$   
The mean time =  $E(Y_i) = \frac{1}{\lambda}$
- Prior  $\lambda \sim \text{Gamma}(\alpha = 6, \beta = 1800)$   
 $E(\lambda) = \frac{6}{1800} = 0.0033, \text{Var}(\lambda) = \frac{6}{1800^2} = 1.85 \times 10^{-6}$
- The mean time  $\frac{1}{\lambda}$  follows the *invers-Gamma* distribution, since its inverse follows Gamma

$$E\left(\frac{1}{\lambda}\right) = \int_0^{\infty} \frac{1}{\lambda} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda \quad \lambda \sim \text{Gamma}(\alpha, \beta) \rightarrow p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$
$$= \frac{\beta^\alpha \Gamma(\alpha-1)}{\beta^{\alpha-1} \Gamma(\alpha)} \int_0^{\infty} \frac{\beta^{\alpha-1}}{\Gamma(\alpha-1)} \lambda^{\alpha-1-1} e^{-\beta\lambda} d\lambda = \frac{\beta}{\alpha-1} \quad \alpha > 1$$

$$E\left(\frac{1}{\lambda^2}\right) = \frac{\beta^2}{(\alpha-1)(\alpha-2)}$$

$$\text{Var}\left(\frac{1}{\lambda}\right) = \frac{\beta^2}{(\alpha-1)(\alpha-2)} - \left(\frac{\beta}{\alpha-1}\right)^2 = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \quad \alpha > 2$$

## Exponential Likelihood-Gamma Prior (Example)

$$Y_i \sim \text{Exp}(\lambda) \rightarrow p(y_i | \lambda) = \lambda e^{-\lambda y_i}$$

$$\lambda \sim \text{Gamma}(\alpha = 6, \beta = 1800)$$

- The prior mean for the time interval :  $E\left(\frac{1}{\lambda}\right) = \frac{\beta}{\alpha-1} = \frac{1800}{6-1} = 360(6 \text{ hours})$
- The prior variance for the time interval :  $\text{var}\left(\frac{1}{\lambda}\right) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} = \frac{1800^2}{5^2 \times 4} = 32,400$
- The prior std. for the time interval :  $\text{std}\left(\frac{1}{\lambda}\right) = \sqrt{32,400} = 180(3 \text{ hours})$

### Observations:

55 30 231 592 141 139 695 56 803 642 1890 208 246 183 38 486 264 1091 368 222 662 150  
2 133 417 418 743 216 138 306 201 145 804 193 66 577 773 268 388 861

$$P(\lambda | y) = \text{Gamma}(\alpha + n, \beta + \sum_i y_i) = \text{Gamma}(6 + 40, 1800 + 15,841)$$

- The posterior mean for the time interval :  $E\left(\frac{1}{\lambda} | y\right) = \frac{\beta}{\alpha-1} = \frac{1800+15,841}{46-1} = 392.0(\text{minutes})$
- The posterior variance for the time interval :  $\text{var}\left(\frac{1}{\lambda} | y\right) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} = \frac{17641^2}{45^2 \times 44} = 3492.76$
- The posterior std. for the time interval :  $\text{std}\left(\frac{1}{\lambda} | y\right) = \sqrt{3492.76} = 59.1(\sim 1 \text{ hours})$

**Normal Likelihood-Normal Prior (unknown mean and known variance)**



## Normal Likelihood-Normal Prior (unknown mean and known variance)

Likelihood :

$$Y_i \sim N(\theta, \sigma_Y^2) \rightarrow p(y_i | \theta, \sigma_Y^2) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(y_i - \theta)^2}{2\sigma_Y^2}\right)$$

Note that  $\theta = \mu_Y$ , ( $\sigma_Y^2$  is known)

Prior:

$$\theta \sim N(\mu_0, \tau_0^2) \rightarrow p(\theta) = \frac{1}{\sqrt{2\pi\tau_0^2}} \exp\left(-\frac{(\theta - \mu_0)^2}{2\tau_0^2}\right)$$

### Prior predictive distribution :

Before the data  $y$  are considered, the distribution of the unknown but observable  $y$  is

$$p(y) = \int_{\theta} p(y, \theta) d\theta = \int_{-\infty}^{\infty} p(y|\theta)p(\theta)d\theta$$

(Without integration)

$$E(y) = E[E(y|\theta)] = E[\theta] = \mu_0$$

$$\because E(y|\theta) = \theta$$

$$\begin{aligned} \text{var}(y) &= E[\text{var}(y|\theta)] + \text{var}(E(y|\theta)) \\ &= E(\sigma_Y^2) + \text{var}(\theta) \\ &= \sigma_Y^2 + \tau_0^2 \end{aligned}$$

$$\because \text{var}(y|\theta) = \sigma_Y^2$$

$$p(y) = N(y|\mu_0, \sigma_Y^2 + \tau_0^2)$$

## Normal Likelihood-Normal Prior (unknown mean and known variance)

Likelihood :

$$Y_i \sim N(\theta, \sigma_Y^2) \rightarrow p(y_i | \theta, \sigma_Y^2) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(y_i - \theta)^2}{2\sigma_Y^2}\right)$$

Note that  $\theta = \mu_Y$ , ( $\sigma_Y^2$  is known)

Prior:

$$\theta \sim N(\mu_0, \tau_0^2) \rightarrow p(\theta) = \frac{1}{\sqrt{2\pi\tau_0^2}} \exp\left(-\frac{(\theta - \mu_0)^2}{2\tau_0^2}\right)$$

Posterior :

$$P(\theta|y) \propto P(y|\theta, \sigma_Y^2)p(\theta) \quad y = (y_1, \dots, y_n) \text{ is a sequence of i.i.d. observations}$$

$$= \left( \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(y_i - \theta)^2}{2\sigma_Y^2}\right) \right) \frac{1}{\sqrt{2\pi\tau_0^2}} \exp\left(-\frac{(\theta - \mu_0)^2}{2\tau_0^2}\right)$$

$$\propto \exp\left(-\sum_{i=1}^n \frac{(y_i - \theta)^2}{2\sigma_Y^2} + \frac{(\theta - \mu_0)^2}{2\tau_0^2}\right)$$

$$= \exp\left[-\frac{1}{2}\left(\sum_{i=1}^n \frac{(y_i - \theta)^2}{\sigma_Y^2} + \frac{(\theta - \mu_0)^2}{\tau_0^2}\right)\right]$$

$$= \exp\left[-\frac{1}{2\sigma_Y^2\tau_0^2}\left(\tau_0^2 \sum_{i=1}^n (y_i - \theta)^2 + \sigma_Y^2(\theta - \mu_0)^2\right)\right]$$

$$= \exp\left[-\frac{1}{2\sigma_Y^2\tau_0^2}\left(\tau_0^2 \sum_{i=1}^n (y_i^2 - 2\theta y_i + \theta^2) + \sigma_Y^2(\theta^2 - 2\theta\mu_0 + \mu_0^2)\right)\right]$$

## Normal Likelihood-Normal Prior (unknown mean and known variance)

Posterior :

$$P(\theta|y) \propto \exp \left[ -\frac{1}{2\sigma_Y^2\tau_0^2} \left( \tau_0^2 \sum_{i=1}^n (y_i^2 - 2\theta y_i + \theta^2) + \sigma_Y^2(\theta^2 - 2\theta\mu_0 + \mu_0^2) \right) \right]$$

$$= \exp \left[ -\frac{1}{2\sigma_Y^2\tau_0^2} \left( \tau_0^2 \sum_{i=1}^n y_i^2 - 2\tau_0^2\theta n\bar{y} + \tau_0^2 n\theta^2 + \sigma_Y^2\theta^2 - 2\sigma_Y^2\theta\mu_0 + \sigma_Y^2\mu_0^2 \right) \right]$$

$$= \exp \left[ -\frac{1}{2\sigma_Y^2\tau_0^2} \left( \tau_0^2 \sum_{i=1}^n y_i^2 - 2\tau_0^2\theta n\bar{y} + \tau_0^2 n\theta^2 + \sigma_Y^2\theta^2 - 2\sigma_Y^2\theta\mu_0 + \sigma_Y^2\mu_0^2 \right) \right]$$

$$\propto \exp \left[ -\frac{1}{2\sigma_Y^2\tau_0^2} (\theta^2(\sigma_Y^2 + n\tau_0^2) - 2\theta(\mu_0\sigma_Y^2 + n\bar{y}\tau_0^2) + \text{const}) \right]$$

$$= \exp \left[ -\frac{1}{2} \left( \theta^2 \left( \frac{\sigma_Y^2 + n\tau_0^2}{\sigma_Y^2\tau_0^2} \right) - 2\theta \left( \frac{\mu_0\sigma_Y^2 + n\bar{y}\tau_0^2}{\sigma_Y^2\tau_0^2} \right) + \text{const}' \right) \right]$$

$$= \exp \left[ -\frac{1}{2} \left( \theta^2 \left( \frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2} \right) - 2\theta \left( \frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma_Y^2} \right) + \text{const}' \right) \right]$$

$$= \exp \left[ -\frac{1}{2} \left( \frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2} \right) \left( \theta^2 - 2\theta \left( \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma_Y^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}} \right) + \text{const}' \right) \right]$$

$$= \exp \left[ -\frac{1}{2} \left( \frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2} \right) \left( \theta - \left( \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma_Y^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}} \right) \right)^2 \right]$$

$$\left( \because \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i \right)$$

## Normal Likelihood-Normal Prior (unknown mean and known variance)

Likelihood :

$$Y_i \sim N(\theta, \sigma_Y^2) \rightarrow p(y_i | \theta, \sigma_Y^2) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(y_i - \theta)^2}{2\sigma_Y^2}\right)$$

Prior:

$$\theta \sim N(\mu_0, \tau_0^2) \rightarrow p(\theta) = \frac{1}{\sqrt{2\pi\tau_0^2}} \exp\left(-\frac{(\theta - \mu_0)^2}{2\tau_0^2}\right)$$

Posterior on  $\theta = \mu_Y$

$$P(\theta | y) = N\left(\theta \left| \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma_Y^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}}, \left(\frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}\right)^{-1} \right.\right)$$

• Posterior mean  $\mu_1$  :

$$\mu_1 = \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma_Y^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}} = \frac{\sigma_Y^2 \mu_0}{\sigma_Y^2 + n\tau_0^2} + \frac{n\tau_0^2 \bar{y}}{\sigma_Y^2 + n\tau_0^2}$$

$\mu_0$ : Prior mean  
 $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$ : Data mean

- $\tau_0^2 \downarrow \Rightarrow$  Prior mean  $\mu_0$  becomes accurate and influence more to  $\mu_1$
- $\sigma_Y^2 \downarrow \Rightarrow$  the data become precise, making  $\bar{y}$  stronger
- $n \uparrow \Rightarrow \bar{y}$  stronger

• Posterior variance  $\tau_1^2$  :

$$\tau_1^2 = \left(\frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}\right)^{-1}$$

• Posterior precision :

$$\frac{1}{\tau_1^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}$$

$\frac{1}{\tau_0^2}$ : Prior precision  
 $\frac{n}{\sigma_Y^2}$ : Data precision

## Normal Likelihood-Normal Prior (unknown mean and known variance)

Likelihood :

$$Y_i \sim N(\theta, \sigma_Y^2) \rightarrow p(y_i | \theta, \sigma_Y^2) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(y_i - \theta)^2}{2\sigma_Y^2}\right)$$

Note that  $\theta = \mu_Y$ , ( $\sigma_Y^2$  is known)

Posterior :  $\theta = \mu_Y$

$$P(\theta|y) = N(\theta | \mu_1, \tau_1^2)$$

$$\mu_1 = \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma_Y^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}} \quad \tau_1^2 = \left(\frac{1}{\tau_0^2} + \frac{n}{\sigma_Y^2}\right)^{-1}$$

$$E[\theta|y] = \mu_1$$

$$\text{var}(\theta|y) = \tau_1^2$$

Posterior predictive distribution

$$p(\hat{y}|y) = \int_{\theta} p(\hat{y}, \theta|y) d\theta = \int_{\theta} p(\hat{y}|\theta, y) p(\theta|y) d\theta = \int_{\theta} p(\hat{y}|\theta) p(\theta|y) d\theta$$

(Without integration)

$$E(\hat{y}|y) = E[E(\hat{y}|\theta, y)|y] = E[\theta|y] = \mu_1$$

$$\because E(\hat{y}|\theta, y) = E(\hat{y}|\theta) = \theta$$

$$\text{var}(\hat{y}|y) = E[\text{var}(\hat{y}|\theta, y)|y] + \text{var}(E(\hat{y}|\theta, y)|y)$$

$$\because \text{var}(\hat{y}|\theta, y) = \sigma_Y^2$$

$$= E(\sigma_Y^2|y) + \text{var}(\theta|y)$$

$$= \sigma_Y^2 + \tau_1^2$$

$$p(\hat{y}|y) = N(\hat{y} | \mu_1, \sigma_Y^2 + \tau_1^2)$$

## Normal Likelihood-Normal Prior (unknown mean and known variance)

$$E(y) = E[E(y|\theta)] = E[\theta] = \mu_0$$

$$\because E(y|\theta) = \theta$$

$$\begin{aligned} \text{var}(y) &= E[\text{var}(y|\theta)] + \text{var}(E(y|\theta)) \\ &= E(\sigma_Y^2) + \text{var}(\theta) \\ &= \sigma_Y^2 + \tau_0^2 \end{aligned}$$

$$\because \text{var}(y|\theta) = \sigma_Y^2$$

$$p(y) = N(y|\mu_0, \sigma_Y^2 + \tau_0^2)$$

$$E(\hat{y}|y) = E[E(\hat{y}|\theta, y)|y] = E[\theta|y] = \mu_1$$

$$\because E(\hat{y}|\theta, y) = E(\hat{y}|\theta) = \theta$$

$$\begin{aligned} \text{var}(\hat{y}|y) &= E[\text{var}(\hat{y}|\theta, y)|y] + \text{var}(E(\hat{y}|\theta, y)|y) \\ &= E(\sigma_Y^2|y) + \text{var}(\theta|y) \\ &= \sigma_Y^2 + \tau_1^2 \end{aligned}$$

$$\because \text{var}(\hat{y}|\theta, y) = \sigma_Y^2$$

$$p(\hat{y}|y) = N(\hat{y}|\mu_1, \sigma_Y^2 + \tau_1^2)$$

**Normal Likelihood-Inverse-Gamma Prior (known mean and unknown variance)**

## Normal Likelihood-Inverse-Gamma Prior (known mean and unknown variance)

Likelihood :

$$Y_i \sim N(\mu, \theta) \rightarrow p(y_i | \mu, \theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(y_i - \mu)^2}{2\theta}\right)$$

Note that  $\theta = \sigma^2$ , ( $\mu$  is known)

Prior:

$$\theta \sim \text{Inv - Gamma}(\theta | \alpha_0, \beta_0) \rightarrow p(\theta) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \theta^{-(\alpha_0+1)} \exp\left(-\frac{\beta_0}{\theta}\right)$$

Posterior :

$$P(\theta | y) \propto p(y | \theta) p(\theta) \quad y = (y_1, \dots, y_n) \text{ is a sequence of i.i.d. observations}$$

$$= \left( \prod_{i=1}^n \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(y_i - \mu)^2}{2\theta}\right) \right) \times \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \theta^{-(\alpha_0+1)} \exp\left(-\frac{\beta_0}{\theta}\right)$$

$$\propto \left( \prod_{i=1}^n \theta^{-\frac{1}{2}} \exp\left(-\frac{(y_i - \mu)^2}{2\theta}\right) \right) \times \theta^{-(\alpha_0+1)} \exp\left(-\frac{\beta_0}{\theta}\right)$$

$$= \theta^{-(\alpha_0+1+\frac{n}{2})} \exp\left(-\left(\frac{\beta_0}{\theta} + \frac{\sum_{i=1}^n (y_i - \mu)^2}{2\theta}\right)\right)$$

$$= \theta^{-(\alpha_0+\frac{n}{2}+1)} \exp\left(-\left(\frac{\beta_0 + \frac{1}{2} \sum_{i=1}^n (y_i - \mu)^2}{\theta}\right)\right)$$

$$= \text{Inv - Gamma}\left(\theta \middle| \alpha_0 + \frac{n}{2}, \beta_0 + \frac{1}{2} \sum_{i=1}^n (y_i - \mu)^2\right)$$



## Multi-Parameters Model

## Multinomial Likelihood-Dirichlet Prior

Likelihood :

$$p(y|\theta) = \text{Multin}(y|n, \theta_1, \dots, \theta_k) = \binom{n}{y_1 \ y_2 \ \dots \ y_k} \theta_1^{y_1} \dots \theta_k^{y_k}$$
$$= \frac{\Gamma(n+1)}{\prod_{j=1}^k \Gamma(y_j+1)} \prod_{j=1}^k \theta_j^{y_j}$$

$y = (y_1, \dots, y_j, \dots, y_k)$   
 $y_j \in \{0, 1, \dots, n\}, \sum_{j=1}^k y_j = n$

Prior:

$$\theta \sim \text{Dirichlet}(\alpha_1, \dots, \alpha_k) \rightarrow p(\theta) = \frac{\Gamma(\alpha_1 + \dots + \alpha_k)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_k)} \theta_1^{\alpha_1-1} \dots \theta_k^{\alpha_k-1}$$
$$= \frac{\Gamma(\sum_{j=1}^k \alpha_j)}{\prod_{j=1}^k \Gamma(\alpha_j)} \theta_1^{\alpha_1-1} \dots \theta_k^{\alpha_k-1}$$

$\sum_{j=1}^k \theta_j = 1$

Posterior :

$$P(\theta|y) \propto P(y|\theta) p(\theta)$$
$$= \frac{\Gamma(n+1)}{\prod_{j=1}^k \Gamma(y_j+1)} \prod_{j=1}^k \theta_j^{y_j} \frac{\Gamma(\sum_{j=1}^k \alpha_j)}{\prod_{j=1}^k \Gamma(\alpha_j)} \theta_1^{\alpha_1-1} \dots \theta_k^{\alpha_k-1}$$
$$\propto \prod_{j=1}^k \theta_j^{y_j} \prod_{j=1}^k \theta_j^{\alpha_j-1}$$
$$\propto \prod_{j=1}^k \theta_j^{\alpha_j+y_j-1}$$
$$= \text{Dirichlet}(\alpha_1 + y_1, \dots, \alpha_k + y_k)$$

### Posterior predictive distribution

$$p(\hat{y}|y) = \int_{\theta} p(\hat{y}, \theta|y) d\theta = \int_{\theta} p(\hat{y}|\theta, y) p(\theta|y) d\theta = \int_{\theta} p(\hat{y}|\theta) p(\theta|y) d\theta$$

$$\begin{aligned} p(\hat{y}|y) &= \int_{\theta} p(\hat{y}|\theta_1, \dots, \theta_k) p(\theta_1, \dots, \theta_k|y) d\theta \\ &= \int_{\theta} \frac{\Gamma(n+1)}{\prod_{j=1}^k \Gamma(y_j+1)} \prod_{j=1}^k \theta_j^{y_j} \frac{\Gamma(\sum_{j=1}^k \alpha_j)}{\prod_{j=1}^k \Gamma(\alpha_j)} \prod_{j=1}^k \theta_j^{\alpha_j-1} d\theta \\ &= \frac{\Gamma(n+1)}{\prod_{j=1}^K \Gamma(y_j+1)} \frac{\Gamma(\sum_{j=1}^K \alpha_j)}{\prod_{j=1}^K \Gamma(\alpha_j)} \int_{\theta} \prod_{j=1}^K \theta_j^{y_j+\alpha_j-1} d\theta \\ &= \frac{\Gamma(n+1)}{\prod_{j=1}^K \Gamma(y_j+1)} \frac{\Gamma(\sum_{j=1}^K \alpha_j)}{\prod_{j=1}^K \Gamma(\alpha_j)} \frac{\prod_{j=1}^K \Gamma(y_j+\alpha_j)}{\Gamma(n+\sum_{j=1}^K \alpha_j)} \int_{\theta} \frac{\Gamma(n+\sum_{j=1}^K \alpha_j)}{\prod_{j=1}^K \Gamma(y_j+\alpha_j)} \prod_{j=1}^K \theta_j^{y_j+\alpha_j-1} d\theta \\ &= \frac{\Gamma(n+1)}{\prod_{j=1}^K \Gamma(y_j+1)} \frac{\Gamma(\sum_{j=1}^K \alpha_j)}{\prod_{j=1}^K \Gamma(\alpha_j)} \frac{\prod_{j=1}^K \Gamma(y_j+\alpha_j)}{\Gamma(n+\sum_{j=1}^K \alpha_j)} \end{aligned}$$