L9. Machine Learning
(Classification)

Contents

- **Non-Bayesian approaches**
  - Discriminative model
    - ✓ Logistic regression
    - ✓ Neural Network
  - Generative model
    - ✓ Gaussian Discriminative Analysis
    - ✓ Naïve Bayes classification
- **Full Bayesian approach for classification**
  - Bayesian Logistic regression
  - Bayesian Neural Network

Non-Bayesian vs Bayesian

- **Non-Bayesian approaches**

✓ *discriminative* probabilistic classification

$$p(y|x) = f(w^T x)$$

Directly model posterior $p(y|x)$ using parametric form

✓ *Generative* probabilistic classification

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)} = \frac{P(x|y)P(y)}{\sum_{y \in Y} P(x|y)P(y)}$$

Model $P(x|y)$ and $P(y)$ and combined them in Bayes' rule

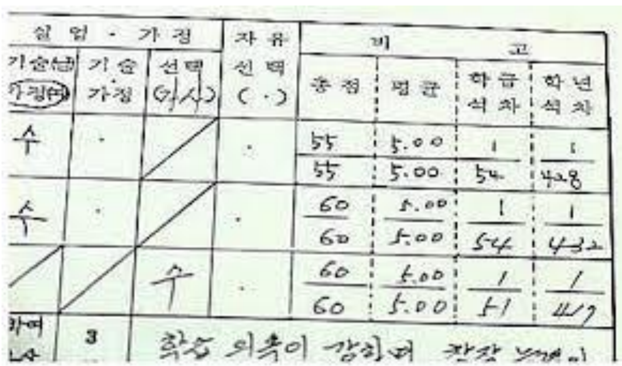
$$\hat{y} = \underset{y \in Y}{\operatorname{argmax}}\, p(y|x)$$

- **Full Bayesian approach for classification**

1. Construct prior $p(w)$
2. Construct likelihood $p(D|w)$, where $D = \{(x_i, y_i)\}_{i=1}^{m}$
3. Construct posterior $p(w|D) = \frac{p(D|w)p(w)}{p(D)}$
4. Posterior predictive distribution $p(y_*|x_*, D) = \int_w p(y_*|x_*, w)p(w|D)dw$

University admission committee

High school grades



National Exam score

〈2016학년도 대학수학능력시험 성적통지표(예시)〉

수험번호	성 명	생년월일	성별	출신고교 (반 또는 졸업년도)		
12345678	홍 길 동	97.09.05.	남	한국고등학교 (9)		
구 분	국어 영역	수학 영역	영어 영역	사회탐구 영역		제2외국어/한문 영역
	B형	A형		생활과 윤리	사회 · 문화	일본어 I
표준점수	131	137	141	53	64	69
백 분 위	93	95	97	75	93	95
등 급	2	2	1	4	2	2

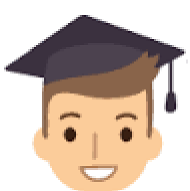
2015. 12. 2.

한국교육과정평가원장

Rejected

Student 1

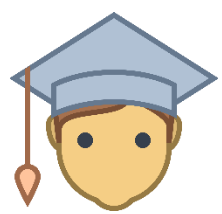
- Exam: 3/10
- Grades: 4/10



?

Student 2

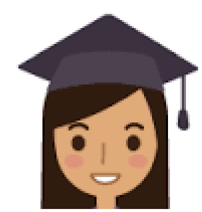
- Exam: 7/10
- Grades: 6/10



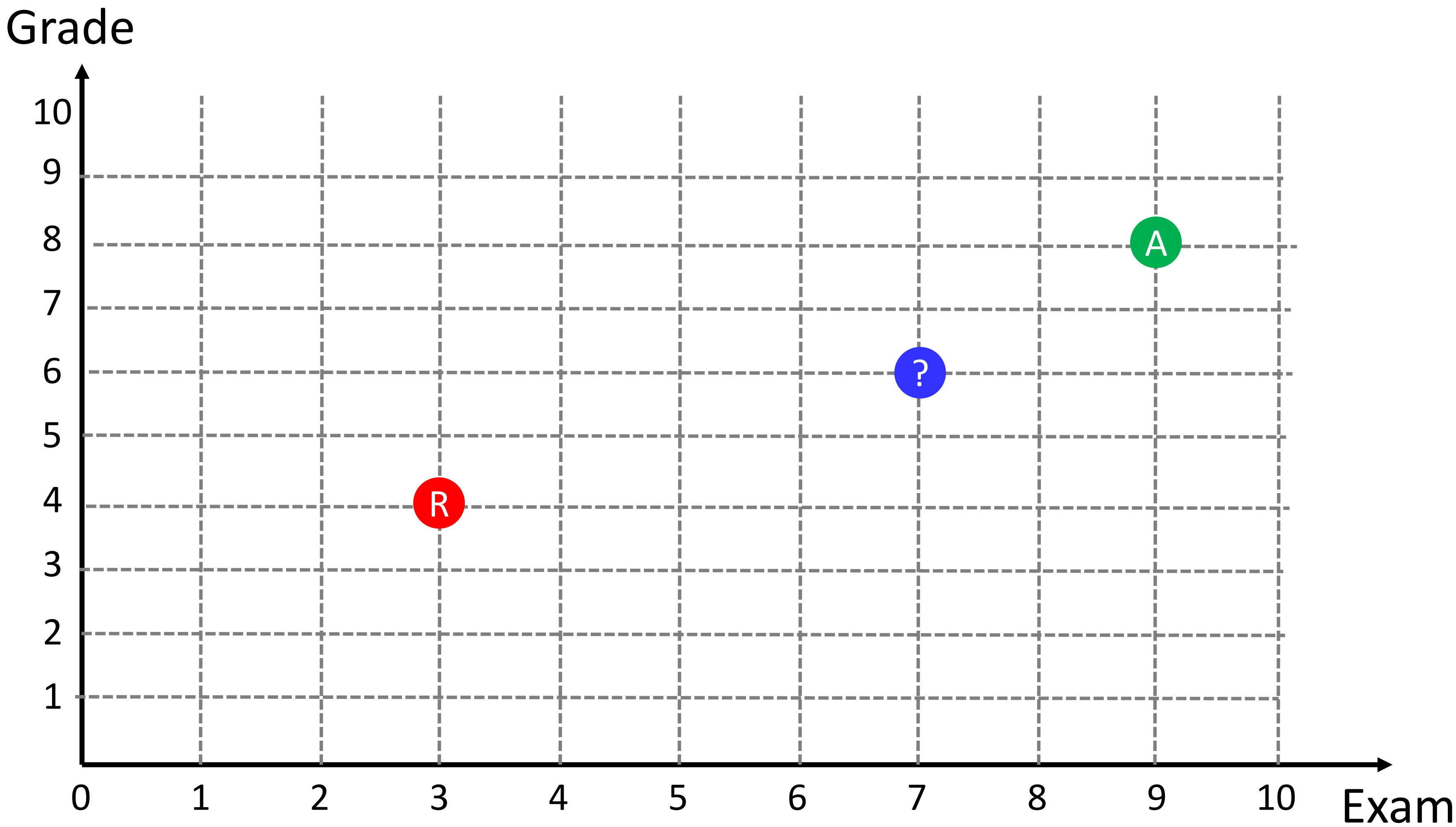
Accepted

Student 3

- Exam: 9/10
- Grades: 8/10

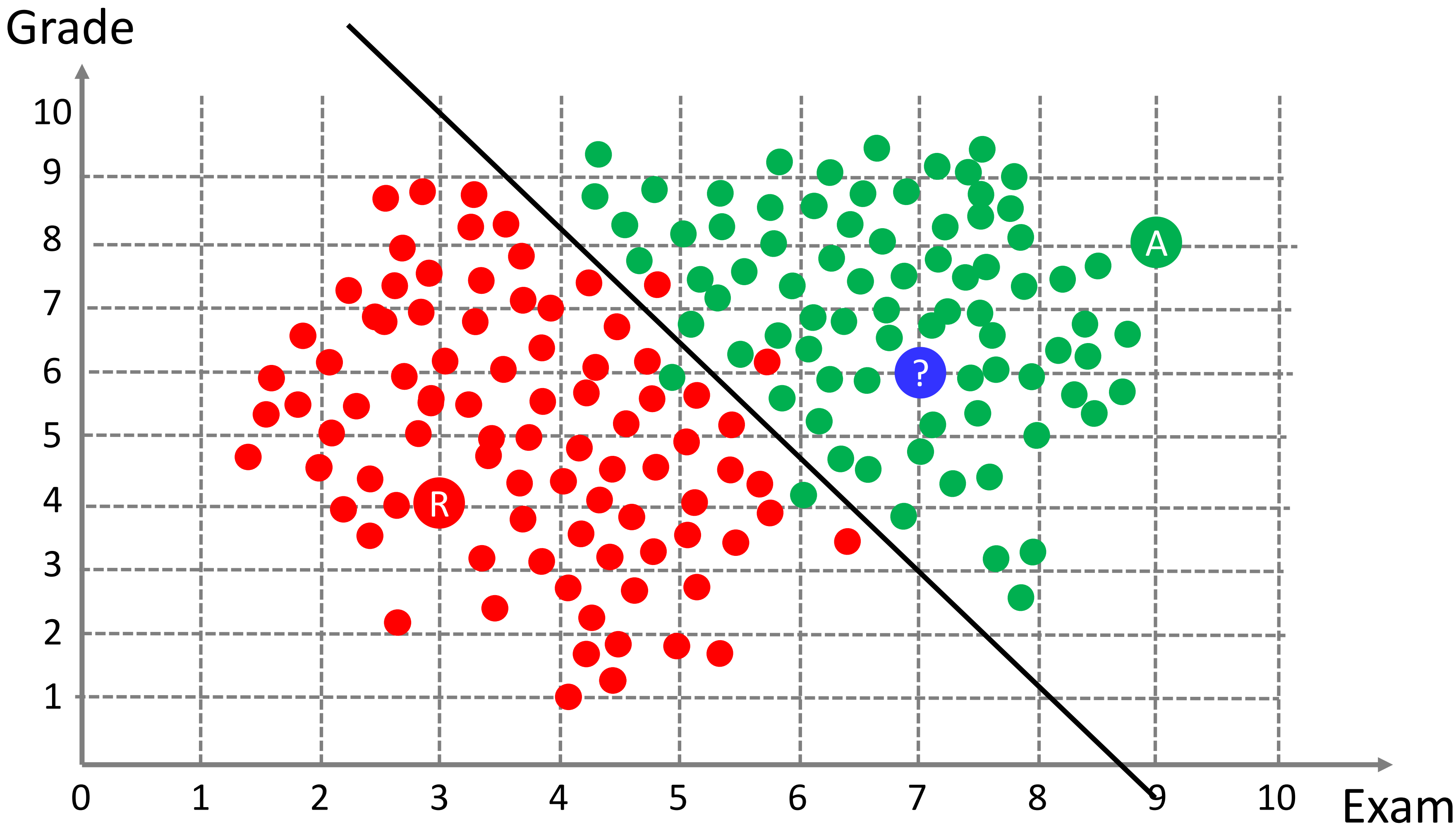


University admission committee



University admission committee

Look at the **historical data** on the admission results



Logistic regression

- Logistic regression is *discriminative* probabilistic linear classification : $p(y|x) = g(w^T x)$

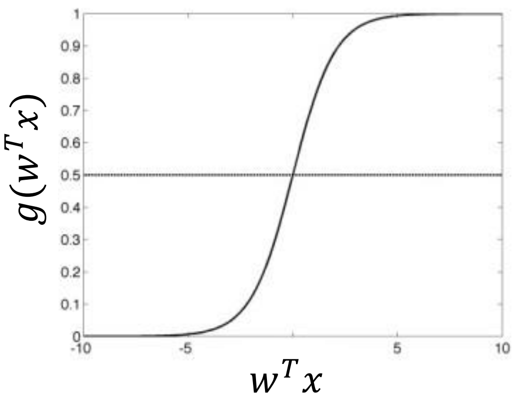
Let's denote p a probability of having $y = 1$

$$\text{logit}(p) = \log\left(\frac{p}{1-p}\right) = w^T x$$

$$\frac{p}{1-p} = \exp(w^T x)$$

$$p = \frac{\exp(w^T x)}{1+\exp(w^T x)} = \frac{1}{1+\exp(-w^T x)} = g(w^T x)$$

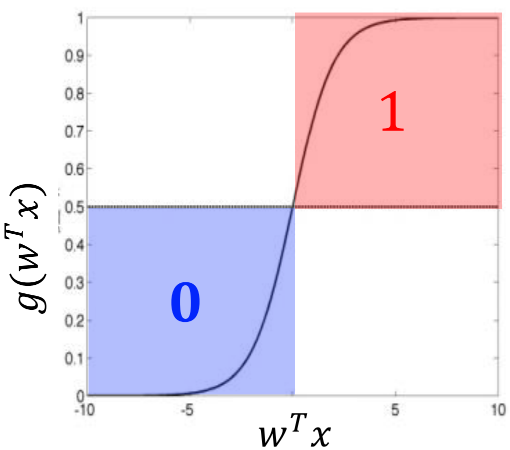
- Larger $w^T x$ →lareger→ $g(w^T x)$ → higher p for $y = 1$
- Smaller $w^T x$ →smaller→ $g(w^T x)$ → lower p for $y = 1$



$$g(z) = \frac{1}{(1+\exp(-w^T x))}$$

- Classification rule:

$$y = \begin{cases} 0, & \text{if } p(Y=1|x) = g(w^T x) < 0.5 \Leftrightarrow w^T x < 0 \\ 1, & \text{if } p(Y=1|x) = g(w^T x) \geq 0.5 \Leftrightarrow w^T x \geq 0 \end{cases}$$

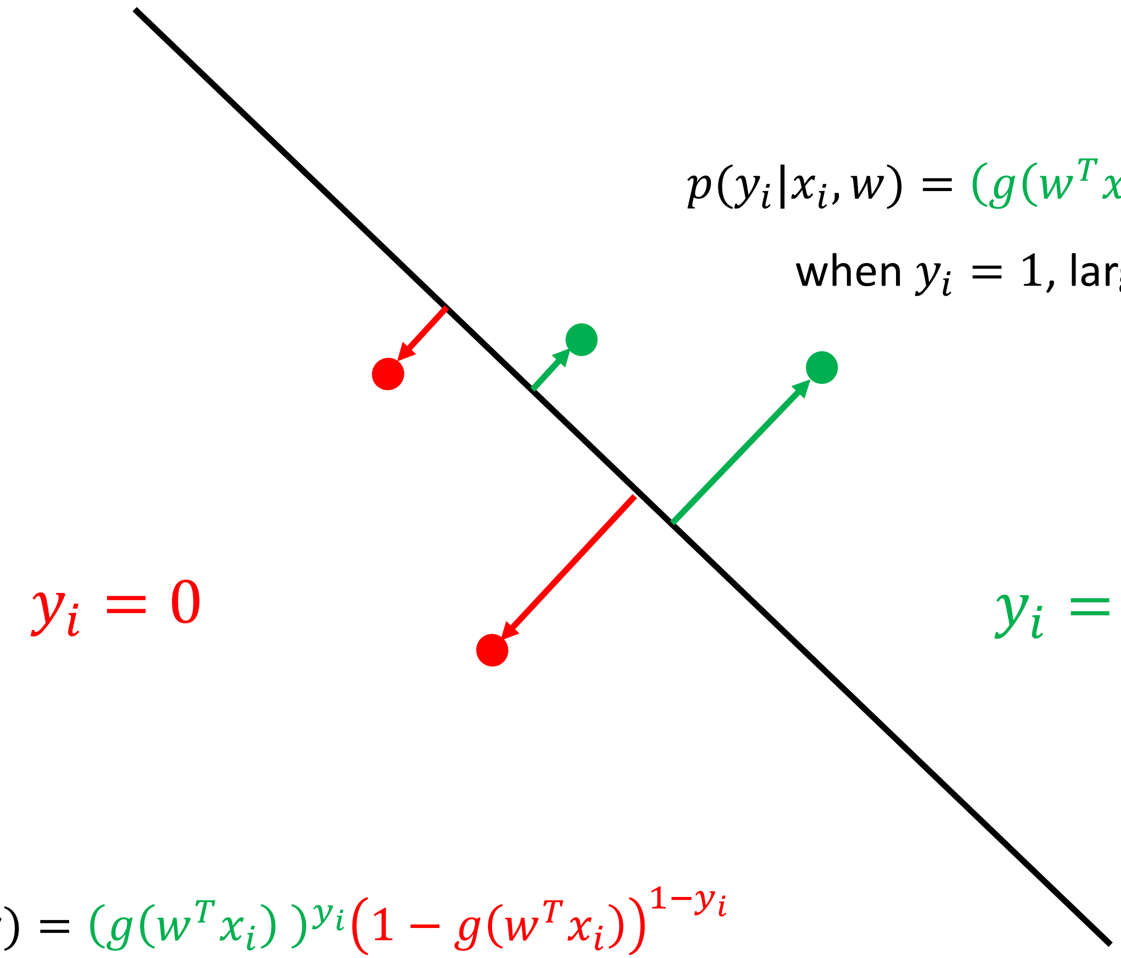


University admission committee

How to draw **a separating line** ?

$$p(y_i|x_i,w) = (g(w^T x_i))^{y_i} \left(1 - g(w^T x_i)\right)^{1-y_i}$$

when $y_i = 1$, larger $g(w^T x_i)$ is better

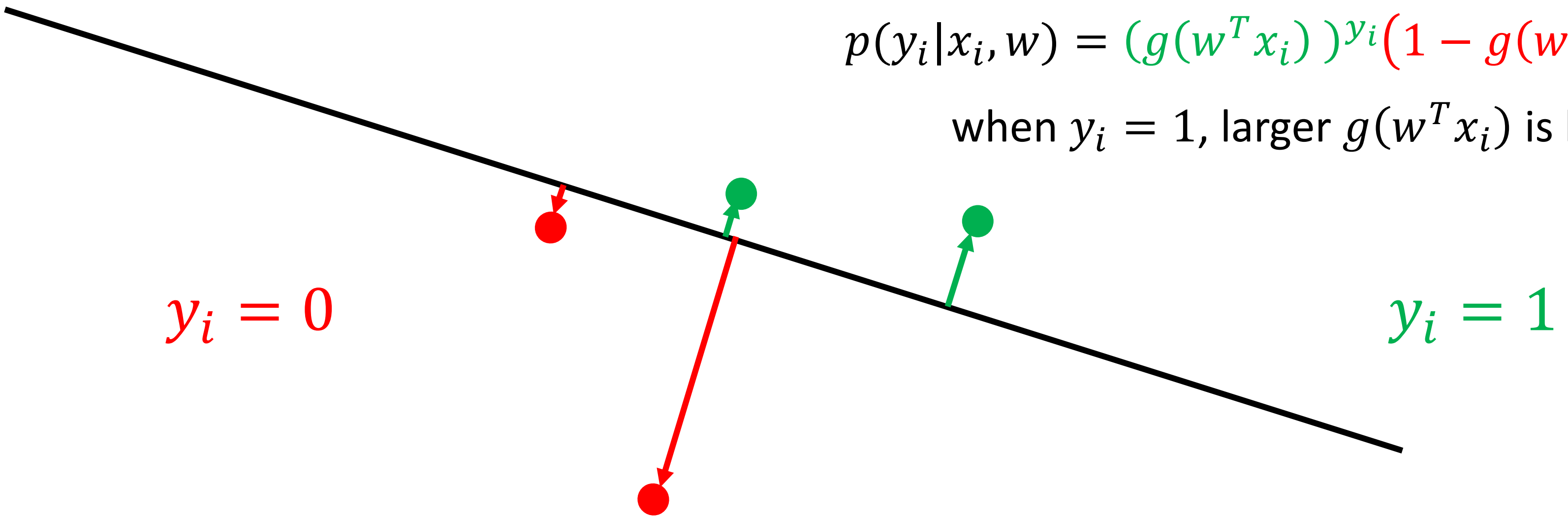


$$p(y_i|x_i,w) = (g(w^T x_i))^{y_i} \left(1 - g(w^T x_i)\right)^{1-y_i}$$

When $y_i = 0$, smaller $g(w^T x_i)$ is better

University admission committee

How to draw **a separating line** ?



$$p(y_i|x_i, w) = (g(w^T x_i))^{y_i} \left(1 - g(w^T x_i)\right)^{1-y_i}$$

when $y_i = 1$, larger $g(w^T x_i)$ is better

$$p(y_i|x_i, w) = (g(w^T x_i))^{y_i} \left(1 - g(w^T x_i)\right)^{1-y_i}$$

When $y_i = 0$, smaller $g(w^T x_i)$ is better

Logistic regression – objective function

- Likelihood for **a single point** (x_i, y_i) can be specified as

$$p(y_i|x_i,w) = (g(w^T x_i))^{y_i}\big(1 - g(w^T x_i)\big)^{1-y_i}$$

- Likelihood for **whole training data** (X, y) can be specified as

$$p(y|X,w) = \prod_{i}^{m} p(y_i|x_i,w) = \prod_{i=1}^{m} (g(w^T x_i))^{y_i}\big(1 - g(w^T x_i)\big)^{1-y_i}$$

Note that this is similar to the likelihood of Binomial dist.

- **Log**-likelihood

$$L(w) = \log \prod_{i}^{m} p(y_i|x_i,w) = \sum_{i=1}^{m} y_i \log g(w^T x_i) + (1 - y_i)\log\big(1 - g(w^T x_i)\big)$$

Logistic regression – learning (optimization)

- **Log**-likelihood

$$L(w) = \log \prod_{i}^{m} p(y_i | x_i, w) = \sum_{i=1}^{m} y_i \log g(w^T x_i) + (1 - y_i) \log\left(1 - g(w^T x_i)\right)$$

- We can find the parameters that maximizes the log-likelihood function

$$w^* = \text{argmax}_w \ L(w)$$

- **Gradient ascent** algorithm

Repeat until convergence{

$w_j := w_j + \alpha \frac{\partial}{\partial w_j} L(w)$ (for every j)

}

α : learning rate

$$\frac{\partial}{\partial w_j} L(w) = \sum_{i=1}^{m} \left(y_i - g(w^T x_i)\right) x_{ij}$$

Logistic regression – learning (optimization)

- **Log**-likelihood

$$L(w) = \log \prod_{i}^{m} p(y_i|x_i, w) = \sum_{i=1}^{m} y_i \log g(w^T x_i) + (1 - y_i) \log\big(1 - g(w^T x_i)\big)$$

- We can find the parameters that maximizes the log-likelihood function

$$w^* = \text{argmax}_w \ L(w)$$

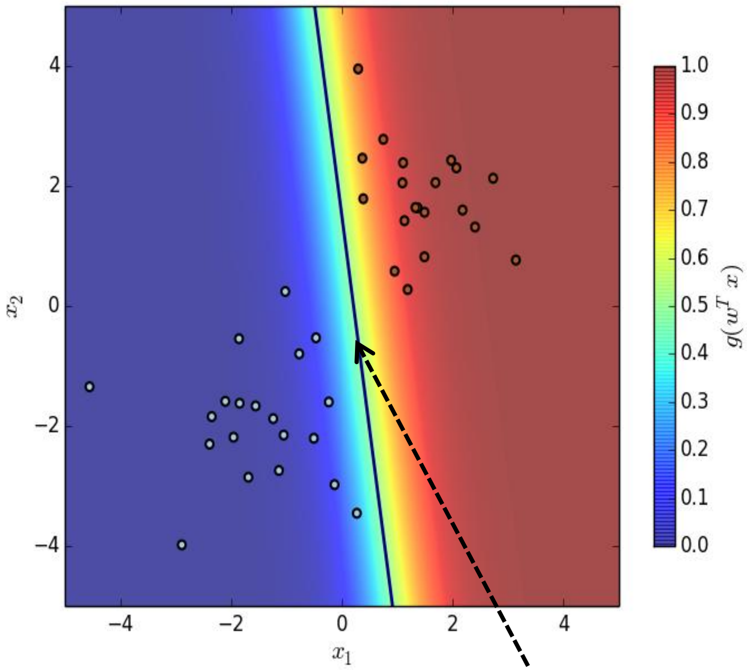
- **Stochastic gradient ascent** algorithm

Repeat until convergence{
  for $i = 1, \dots, m$ {
    $w_j := w_j + \alpha\big(y_i - g(w^T x_i)\big)x_{ij}$ (for every j)
  }
}

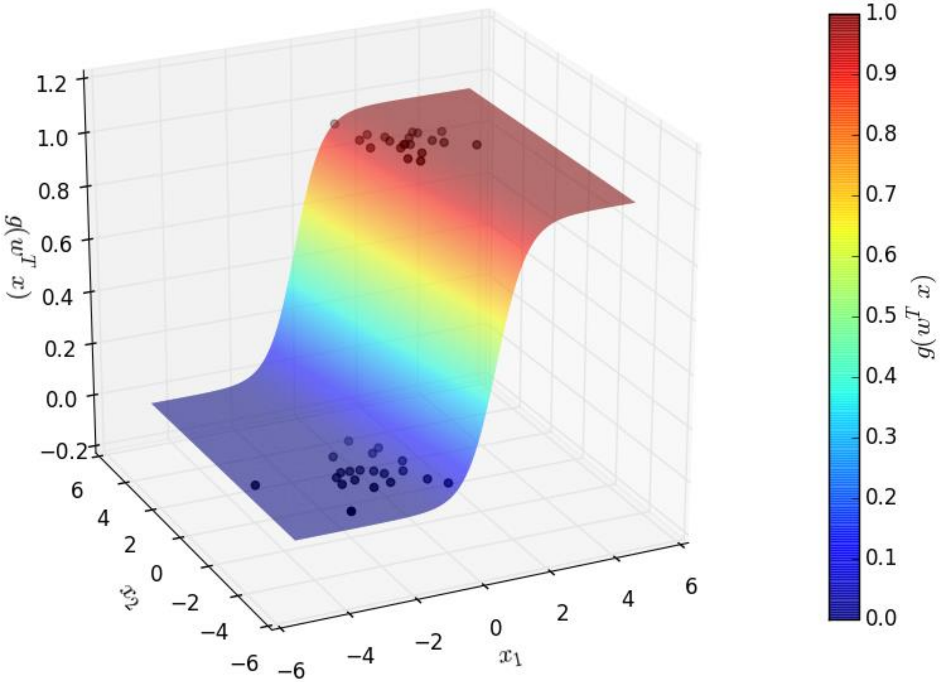
α : learning rate

$$\frac{\partial}{\partial w_j} L(w) = \sum_{i=1}^{m} \big(y_i - g(w^T x_i)\big)x_{ij} \sim \big(y_i - g(w^T x_i)\big)x_{ij}$$

Logistic regression



Classification line $\boldsymbol{w}^T\boldsymbol{x} = \boldsymbol{0}$



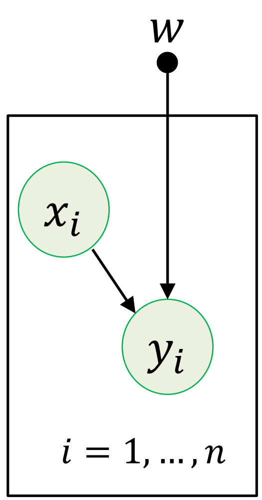
Logistic regression

Jupyter Demo Simulation

Bayesian Logistic Regression

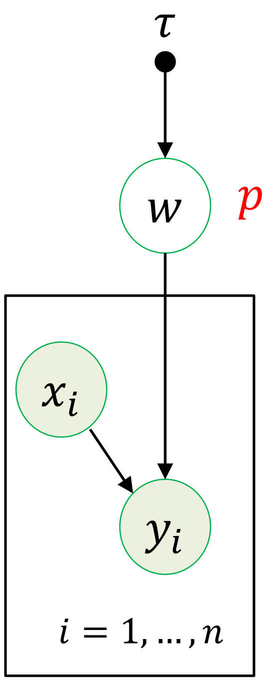
Logistic Regression

Fixed parameter
(to be determined)



Bayesian Logistic Regression

Fixed hyper-parameter



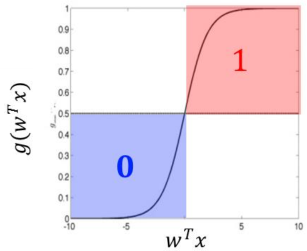
$p(w) = N(w|\mathbf{0}, \tau^2\mathbf{I})$

$$y_i = \begin{cases} 0, \text{if } g(w^T x_i) < 0.5 \Leftrightarrow w^T x_i < 0 \\ 1 \ \text{ if } g(w^T x_i) \geq 0.5 \Leftrightarrow w^T x_i \geq 0 \end{cases}$$

Bayesian Logistic Regression with Gaussian Prior (Ridge Logistic Regression)

- We have a logistic regression model :

$$p(Y=1|x) = g(w^Tx) = \frac{1}{(1+\exp(-w^Tx))}$$
$$p(Y=0|x) = 1 - g(w^Tx)$$



- **Likelihood** can be specified as

$$p(y_i|x_i,w) = (g(w^Tx_i)\,)^{y_i}\left(1-g(w^Tx_i)\right)^{1-y_i}$$

for $y = (y_1, \dots, y_m)$

$$p(y|X,w) = \prod_{i}^{m} p(y_i|x_i,w) = \prod_{i=1}^{m} (g(w^Tx_i)\,)^{y_i}\left(1-g(w^Tx_i)\right)^{1-y_i}$$

- **Prior** on parameter w can be specified as

$$p(w_j) = N(w_j|0,\tau_i^2) = \frac{1}{\sqrt{2\pi\tau_j^2}}\exp\left(-\frac{w_j^2}{2\tau_j^2}\right)$$

for $w = (w_1, \dots, w_n)$

$$p(w) = \prod_{i=1}^{n} N(w_j|0,\tau_i^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\tau_j^2}}\exp\left(-\frac{w_j^2}{2\tau_j^2}\right)$$

✓ With the fixed variance τ_j^2 quantifying our belief that w_j is close to 0.

✓ For simple case, $\tau_j^2 = \tau^2$ for $j = 1, \dots, n$

Bayesian Logistic Regression with Gaussian Prior (Ridge Logistic Regression)

- We need to compute **the posterior**:

$$p(w|X,y) = p(y|X,w)p(w)$$
$$= \prod_{i=1}^{m} (g(w^T x_i))^{y_i} (1 - g(w^T x_i))^{1-y_i} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{w_j^2}{2\tau_j^2}\right)$$

$$\log p(w|X,y) = \sum_{i=1}^{m} y_i \log g(w^T x_i) + (1-y_i)\log(1 - g(w^T x_i)) + n\log\left(\frac{1}{\sqrt{2\pi\tau^2}}\right) - \sum_{j=1}^{n} \frac{w_j^2}{2\tau_j^2}$$

- The **MAP** estimate of w is then simply

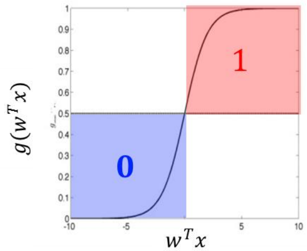
$$\hat{w} = \underset{w}{\operatorname{argmax}}\, p(w|X,y)$$
$$= \underset{w}{\operatorname{argmax}} \log p(w|X,y)$$
$$= \underset{w}{\operatorname{argmax}} \underbrace{\sum_{i=1}^{m} y_i \log g(w^T x_i) + (1-y_i)\log(1 - g(w^T x_i))}_{\text{Data fitness}} \underbrace{- \lambda \|w\|_2^2}_{\text{complexity}}$$

Bayesian Logistic Regression with Laplace Prior (Lasso Logistic Regression)

- We have a logistic regression model :

$$p(Y=1|x) = g(w^T x) = \frac{1}{(1+\exp(-w^T x))}$$

$$p(Y=0|x) = 1 - g(w^T x)$$



- **Likelihood** can be specified as

$$p(y_i|x_i, w) = (g(w^T x_i))^{y_i} \left(1 - g(w^T x_i)\right)^{1-y_i}$$

for $y = (y_1, \dots, y_m)$

$$p(y|X, w) = \prod_i^m p(y_i|x_i, w) = \prod_{i=1}^m (g(w^T x_i))^{y_i} \left(1 - g(w^T x_i)\right)^{1-y_i}$$

- **Prior** on parameter w can be specified using Laplacian as

$$p(w_j) = \frac{\lambda_j}{2} \exp(-\lambda_j |w_j|)$$

for $w = (w_1, \dots, w_n)$

$$p(w) = \prod_{j=1}^n \frac{\lambda_j}{2} \exp(-\lambda_j |w_j|)$$

✓ With the fixed variance λ_j quantifying our belief that w_j is close to 0.

✓ For simple case, $\lambda_j = \lambda$ for $j = 1, \dots, n$

Bayesian Logistic Regression with Laplace Prior (Lasso Logistic Regression)

- We need to compute **the posterior**:

$$p(w|X,y) = p(y|X,w)p(w)$$

$$= \prod_{i=1}^{m} (g(w^T x_i))^{y_i} (1 - g(w^T x_i))^{1-y_i} \prod_{j=1}^{n} \frac{\lambda_j}{2} \exp(-\lambda_j |w_j|)$$

$$\log p(w|X,y) = \sum_{i=1}^{m} y_i \log g(w^T x_i) + (1-y_i) \log(1 - g(w^T x_i)) + n\log\left(\frac{\lambda}{2}\right) - \lambda \sum_{j=1}^{n} |w_j|$$

- The **MAP** estimate of w is then simply

$$\hat{w} = \underset{w}{\operatorname{argmax}}\, p(w|X,y)$$

$$= \underset{w}{\operatorname{argmax}} \log p(w|X,y)$$

$$= \underset{w}{\operatorname{argmax}} \underbrace{\sum_{i=1}^{m} y_i \log g(w^T x_i) + (1-y_i) \log(1 - g(w^T x_i))}_{\text{Data fitness}} \underbrace{- \lambda \sum_{j=1}^{n} |w_j|}_{\text{Complexity (sparsity)}}$$

Bayesian Logistic Regression with Gaussian Prior (Ridge Logistic Regression)

Jupyter Demo Simulation

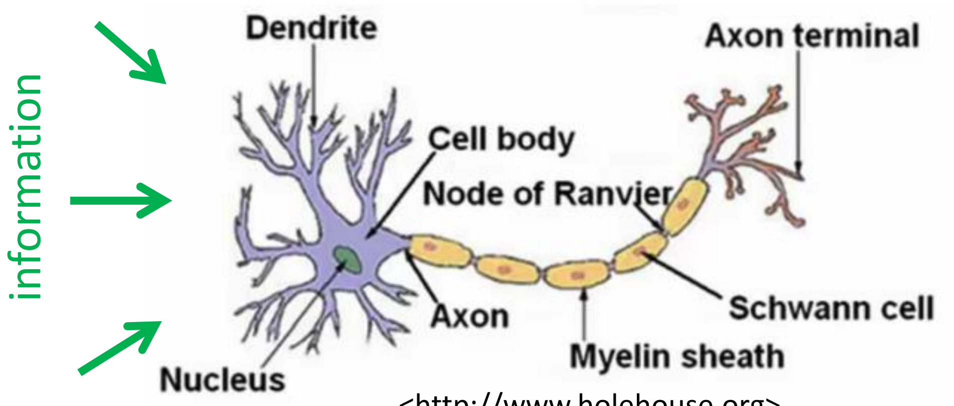
Fully Bayesian Logistic Regression with MCMC algorithm

Jupyter Demo Simulation (working on)

Neural Network

Concept

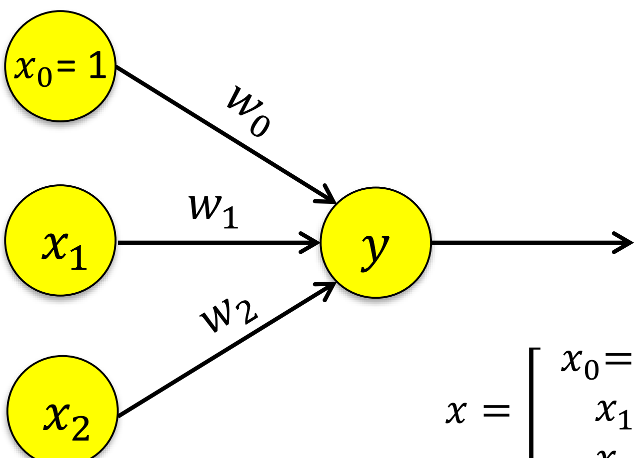
Neuron



<http://www.holehouse.org>

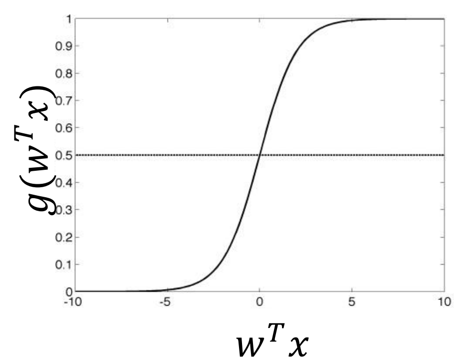
Dendrite: receive signal from multiple neurons
Cell body: Process signal
Axon: Send signal to other neuron

Logistic regression mimics the functionality of a single neuron



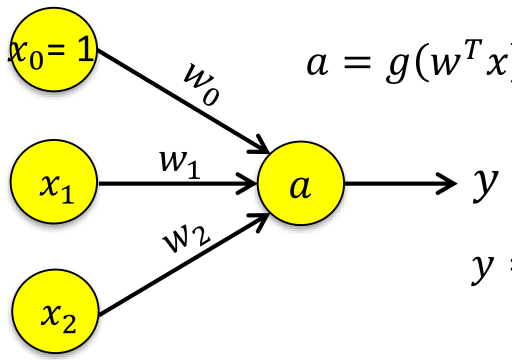
$$y = g(w^T x) = \frac{1}{(1 + \exp(-w^T x))}$$

$$x = \begin{bmatrix} x_0 = 1 \\ x_1 \\ x_2 \end{bmatrix}, \quad w = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$



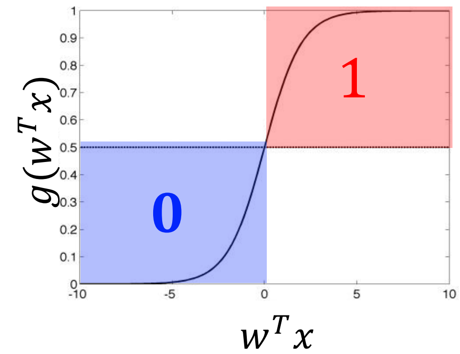
Concept

Classification with logistic regression

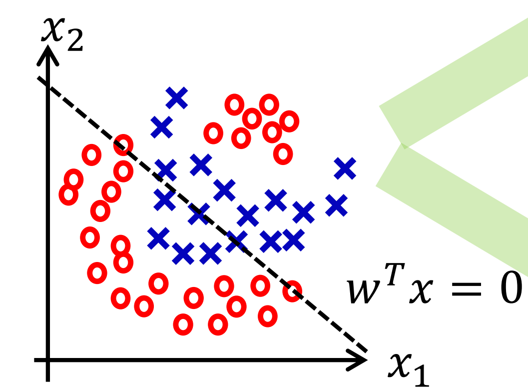


$$a = g(w^T x) = \frac{1}{(1 + \exp(-w^T x))}$$

$$y = h_w(x) = \begin{cases} 1, \text{if } a \ \geq 0.5 \Rightarrow w^T x \geq 0 \\ 0, \text{if } a \ < 0.5 \Rightarrow w^T x < 0 \end{cases}$$

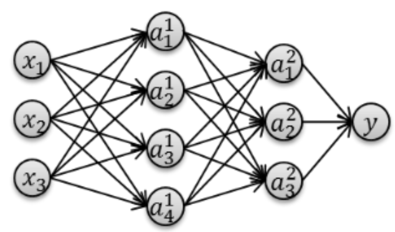


How to obtain non-linear decision boundaries?

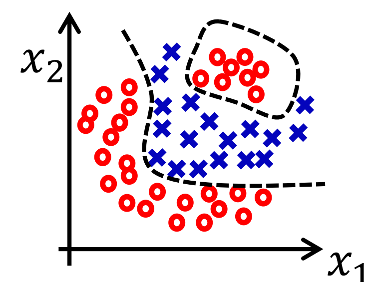


Linear decision boundary by single logistic regression

Use multilayer neural networks



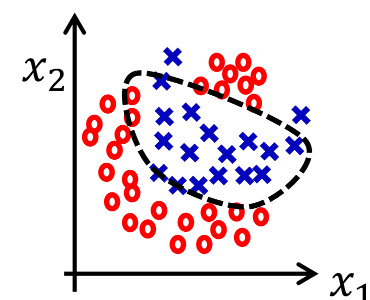
Any continuous function can be approximated well with a growing number of hidden unis.



Use non-linear feature mapping

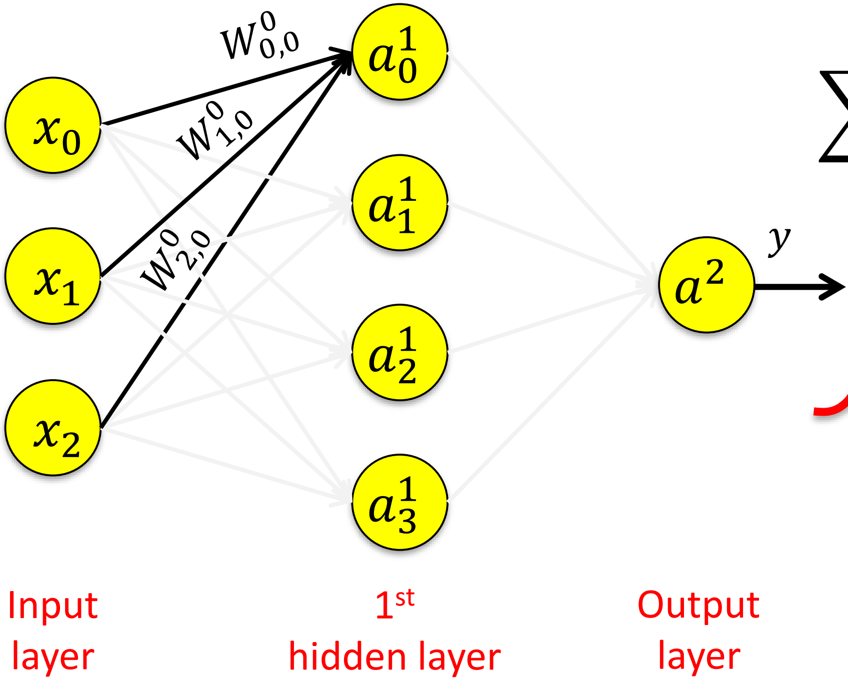
$$\phi: \chi \mapsto \mathcal{H}$$

e.g., $\phi(x) = (x_1, x_2, \sqrt{2}x_1x_2, x_1^2, x_2^2)$



Use kernel method (simplify the computation)

Concept



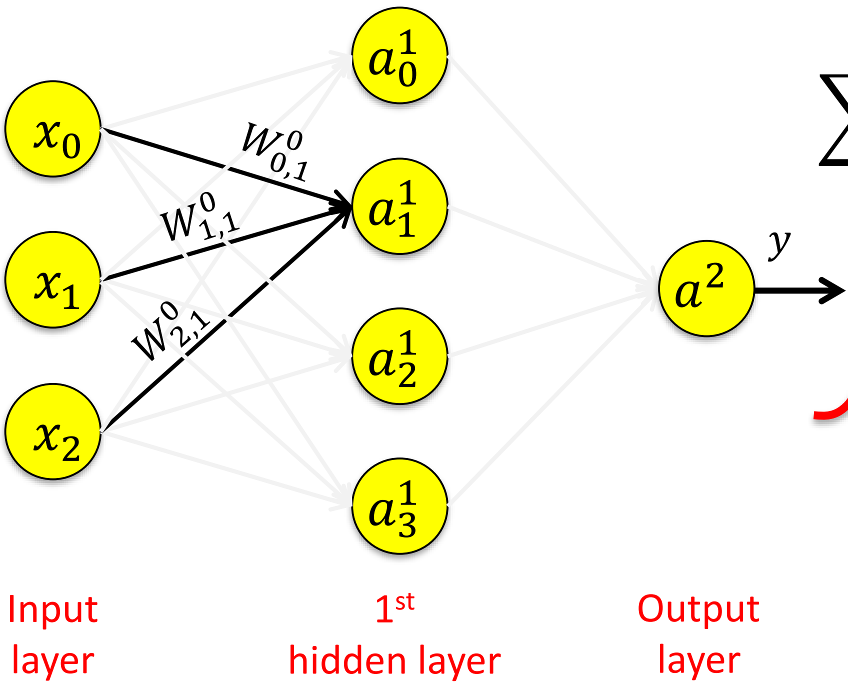
$\sum$ Weighted sum of the previous node values:

$$z_0^1 = W_{0,0}^{0} x_0 + W_{1,0}^{0} x_1 + W_{2,0}^{0} x_2$$

Logistic transform:

$$a_0^1 = g(z_0^1) = g\left(W_{0,0}^{0} x_0 + W_{1,0}^{0} x_1 + W_{2,0}^{0} x_2\right)$$

Concept



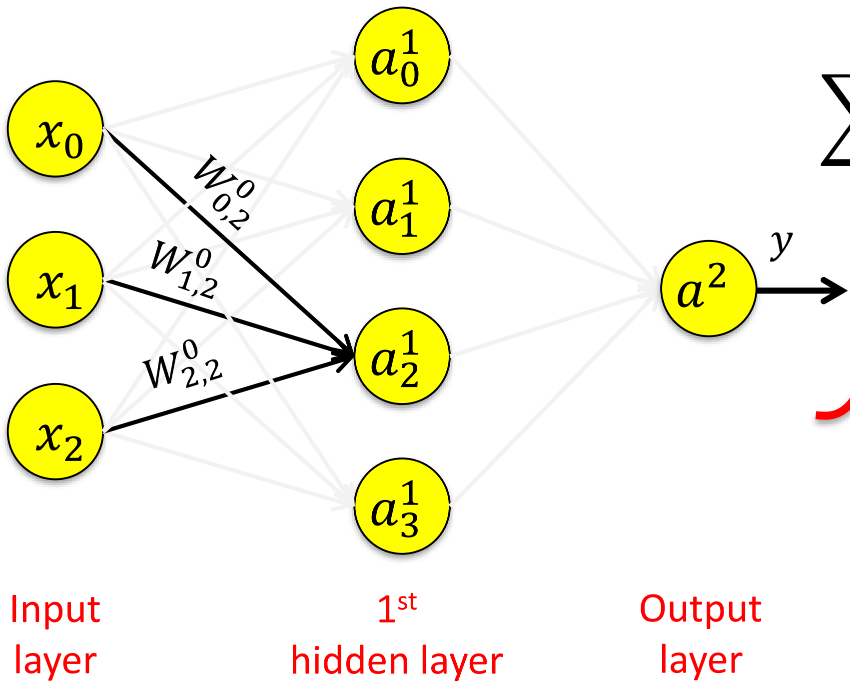
$\sum$ Weighted sum of the previous node values:

$$z^1_1 = W^0_{0,1}x_0 + W^0_{1,1}x_1 + W^0_{2,1}x_2$$

Logistic transform:

$$a^1_1 = g(z^1_1) = g\left(W^0_{0,1}x_0 + W^0_{1,1}x_1 + W^0_{2,1}x_2\right)$$

Concept



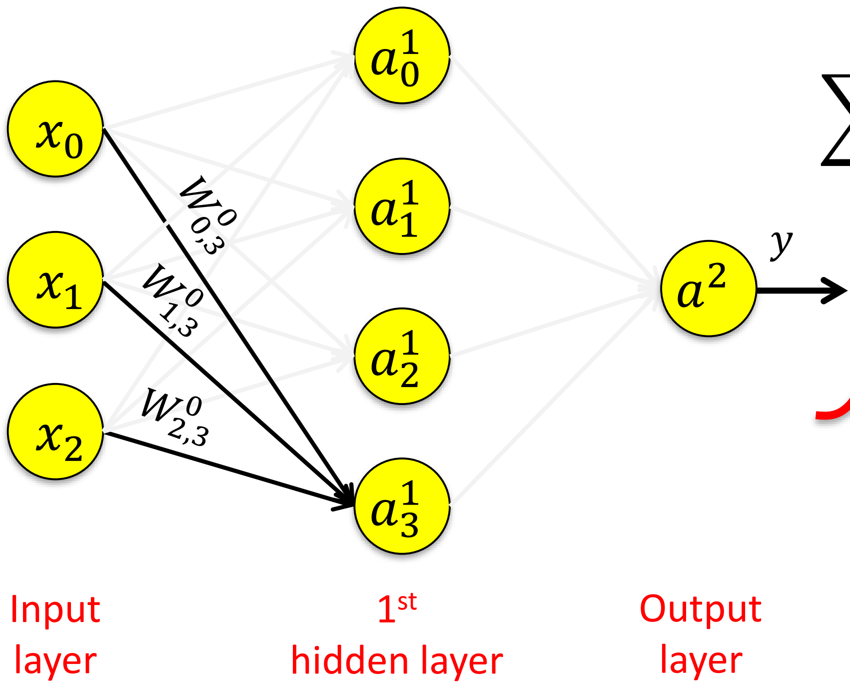
$\sum$ Weighted sum of the previous node values:

$$z^1_2 = W^0_{0,2}x_0 + W^0_{1,2}x_1 + W^0_{2,2}x_2$$

Logistic transform:

$$a^1_2 = g(z^1_2) = g\left(W^0_{0,2}x_0 + W^0_{1,2}x_1 + W^0_{2,2}x_2\right)$$

Concept



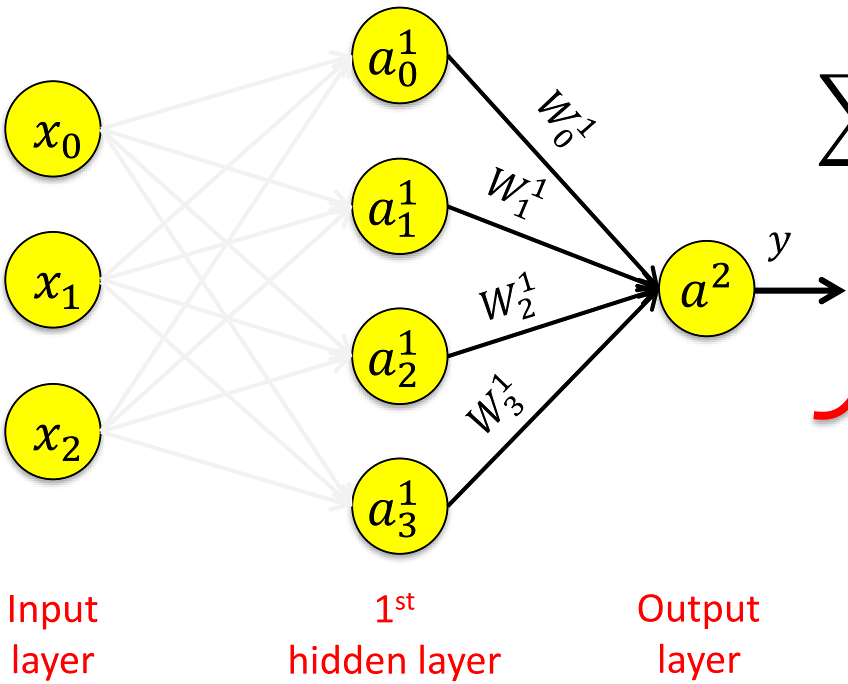
$\sum$ Weighted sum of the previous node values:

$$z_3^1 = W_{0,3}^0 x_0 + W_{1,3}^0 x_1 + W_{2,3}^0 x_2$$

Logistic transform:

$$a_3^1 = g(z_3^1) = g\left(W_{0,3}^0 x_0 + W_{1,3}^0 x_1 + W_{2,3}^0 x_2\right)$$

Concept



$\sum$ Weighted sum of the previous node values:

$$z^2 = W_0^1 a_0^1 + W_1^1 a_1^1 + W_2^1 a_2^1 + W_3^1 a_3^1$$

Logistic transform:

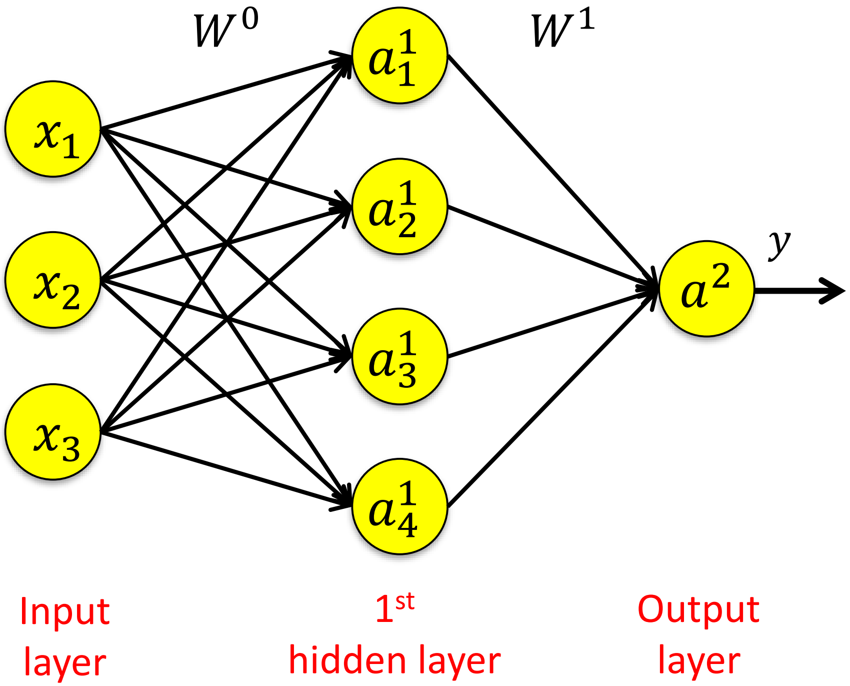
$$a^2 = g(z^2)$$

$$= g(W_0^1 a_0^1 + W_1^1 a_1^1 + W_2^1 a_2^1 + W_3^1 a_3^1)$$

Output node:

$$y = \begin{cases} 1, \text{if } a^2 \geq 0.5 \\ 0, \text{if } a^2 < 0.5 \end{cases}$$

Concept



$\sum$

In every layer, two computations, weighted sum and the evaluations of sigmoid functions, are conducted to find the values at the next layer

Input layer → 1st hidden layer

Linear combination

$$\begin{bmatrix} z_0^1 \\ z_1^1 \\ z_2^1 \\ z_3^1 \end{bmatrix} = \begin{bmatrix} W_{0,0}^0 W_{1,0}^0 W_{2,0}^0 \\ W_{0,1}^0 W_{1,1}^0 W_{2,1}^0 \\ W_{0,2}^0 W_{1,2}^0 W_{2,2}^0 \\ W_{0,3}^0 W_{1,3}^0 W_{2.3}^0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$z^1 = W^1 x$$

Sigmoid

$$\begin{bmatrix} a_0^1 \\ a_1^1 \\ a_2^1 \\ a_3^1 \end{bmatrix} = g\left(\begin{bmatrix} z_0^1 \\ z_1^1 \\ z_2^1 \\ z_3^1 \end{bmatrix}\right)$$

$$a^1 = g(z^1)$$

1st hidden layer → output layer

Linear combination

$$[z^2] = [W_0^1 W_1^1 W_2^1 W_3^1] \begin{bmatrix} a_0^1 \\ a_1^1 \\ a_2^1 \\ a_3^1 \end{bmatrix}$$

$$z^2 = W^2 a^1$$

Sigmoid

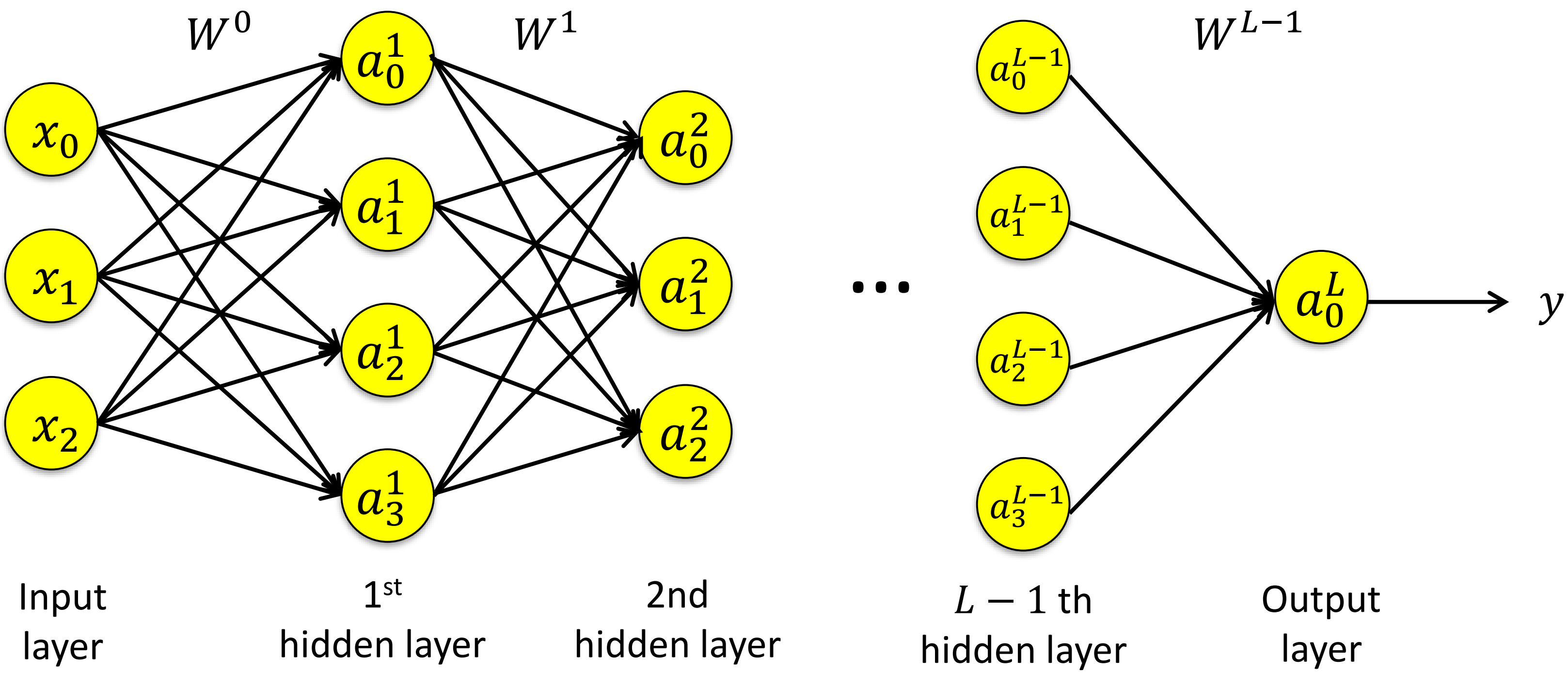
$$[\,a^2] = g([z^2])$$

$$a^2 = g(z^2)$$

Output layer

$$y = h_W(x) = \begin{cases} 1 \text{ if } a^2 \geq 0.5 \\ 1 \text{ if } a^2 < 0.5 \end{cases}$$

Prediction

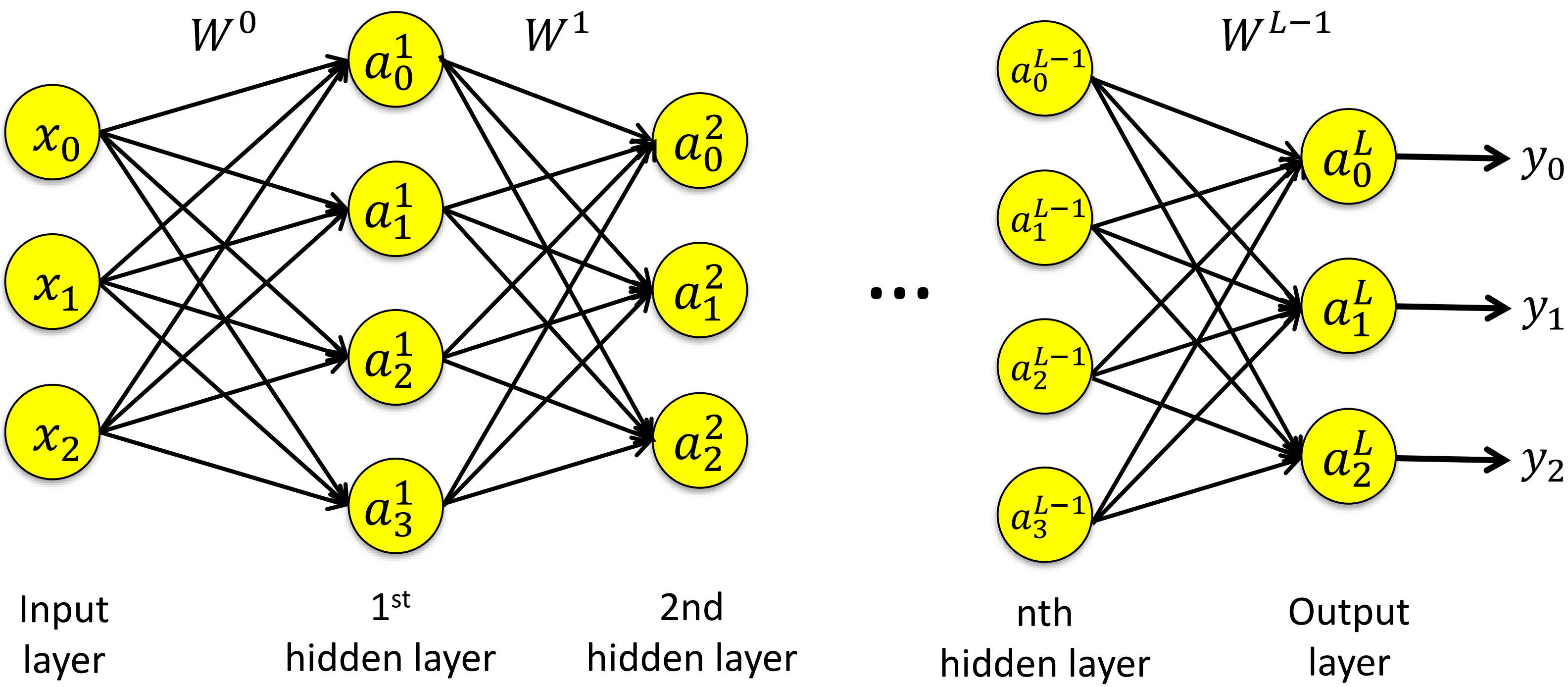


Prediction (Forward propagation)

Input layer	$z^1 = W^1 x,\ a^1 = g(z^1)$
Hidden layer	$\begin{cases} \text{for } l = 1, \dots, L \\ \quad z^l = W^{l-1} a^{l-1}, a^l = g(z^l) \end{cases}$
Output layer	$y = h_W(x) = \begin{cases} 1 \text{ if } a_0^L \geq 0.5 \\ 1 \text{ if } a_0^L < 0.5 \end{cases}$

- By adding more hidden layers, more complex features can be constructed.
- Prediction is conducted by sequence of matrix multiplication and the evaluations of logistic function

Multi-labels Classification



$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = h_W(x) = \text{one of} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Multiple binary classifications are executed for a certain class and the rest class. (one vs. all method)

Cost Functions For Neural Network

- **Log-likelihood of a logistic regression**

$$\sum_{i=1}^{m} y_i \log g(w^T x_i) + (1 - y_i)\log\big(1 - g(w^T x_i)\big) - \lambda \sum_{i=1}^{n} w_i^2$$

Penalizing parameters

$$g(w^T x) = \frac{1}{(1 + \exp(-w^T x))}$$

- **Log-likelihood of a Neural Network**

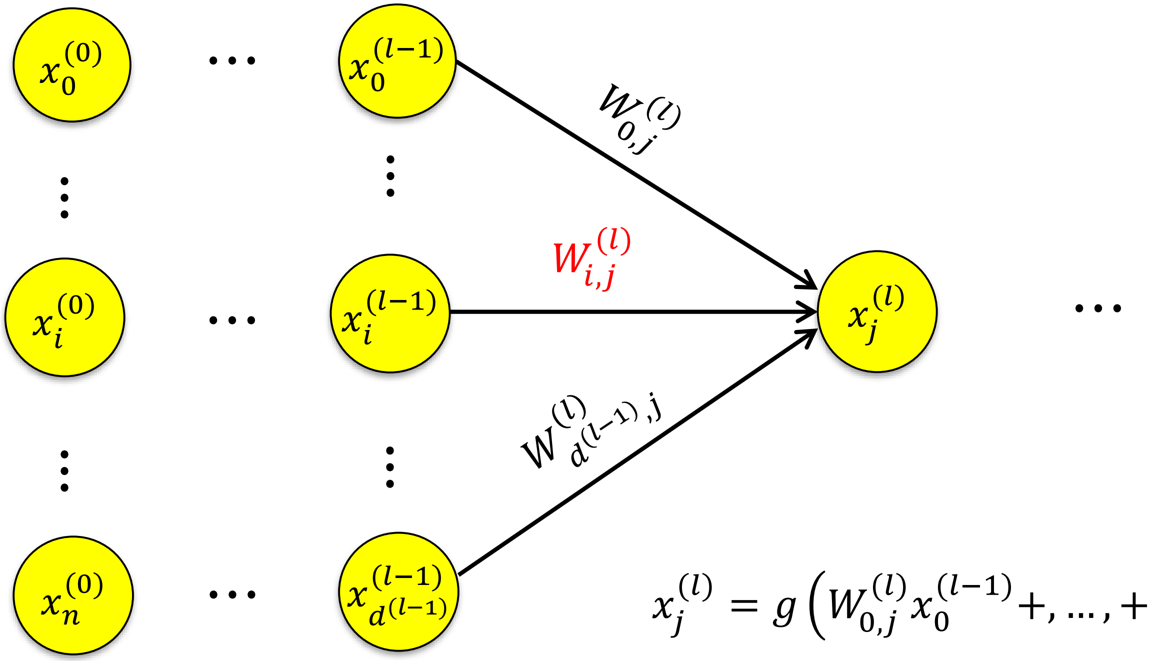
$$\sum_{i=1}^{m} y_i \log G_W(x_i) + (1 - y_i)\big(1 - \log\big(G_W(x_i)\big)\big) - \lambda \sum_{l}^{L} \sum_{i} \sum_{j} \left(W_{i,j}^{l}\right)^2$$

Penalizing parameters

$G_W(x) = g(W_3^T g(W_2^T g(W_1^T x)))$ is a nested function of sigmoid functions

→ **Training is difficult!!**

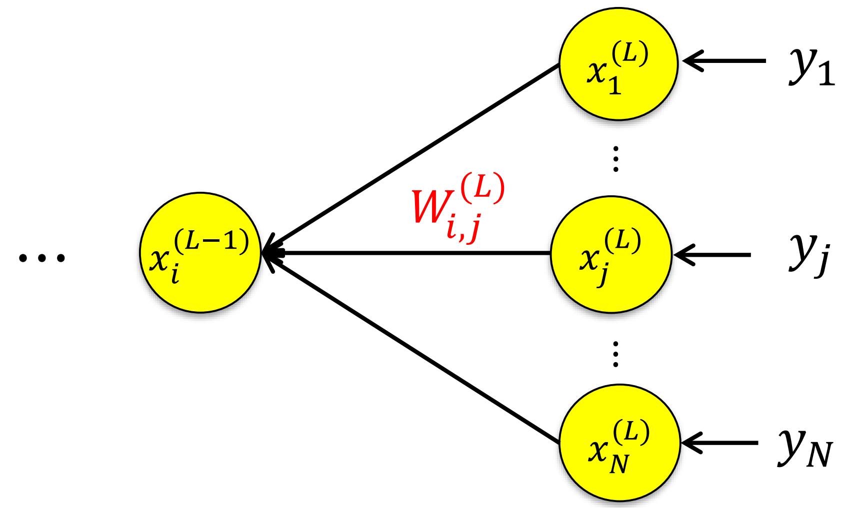
Forward Propagation



$$
\begin{aligned}
x_j^{(l)} &= g\left(W_{0,j}^{(l)} x_0^{(l-1)} +, \dots, + W_{i,j}^{(l)} x_i^{(l-1)} +, \dots +, W_{d^{(l-1)},j}^{(l)} x_{d^{(l-1)}}^{(l-1)}\right) \\
&= g\left(\sum_{k=0}^{d^{(l-1)}} W_{k,j}^{(l)} x_k^{(l-1)}\right) \\
&= g\left(\left(W_{\cdot,j}^{(l)}\right)^T \boldsymbol{x}^{(l-1)}\right) \\
&= h_{W_{\cdot,j}^{(l)}}\left(\boldsymbol{x}^{(l-1)}\right)
\end{aligned}
$$

The value for each node is determined by logistic regression for a single layer

Backward Propagation



$$E = \frac{1}{2}\sum_{k=1}^{N}\left(x_k^{(L)} - y_k\right)^2$$

At output node

$$E = \frac{1}{2}\sum_{k=1}^{N}\left(x_k^{(L)} - y_k\right)^2$$

Define the summed variable for simplicity

$$e_j^{(L)} = x_j^{(L)} - y_j \qquad z_j^{(L)} = \sum_{k=0}^{d^{(L-1)}} W_{k,j}^{(L)} x_k^{(L-1)}, \text{then } x_j^{(L)} = g\left(z_j^{(L)}\right)$$

$$\frac{\partial E}{\partial W_{i,j}^{(L)}} = \left(x_j^{(L)} - y_j\right)\frac{\partial}{\partial W_{i,j}^{(L)}} x_j^{(L)}$$

$$= e_j^{(L)} \frac{\partial}{\partial W_{i,j}^{(L)}} g\left(z_j^{(L)}\right)$$

$$= e_j^{(L)} g'\left(z_j^{(L)}\right) \frac{\partial}{\partial W_{i,j}^{(L)}} z_j^{(L)}$$

$$= e_j^{(L)} g'\left(z_j^{(L)}\right) x_i^{(L-1)}$$

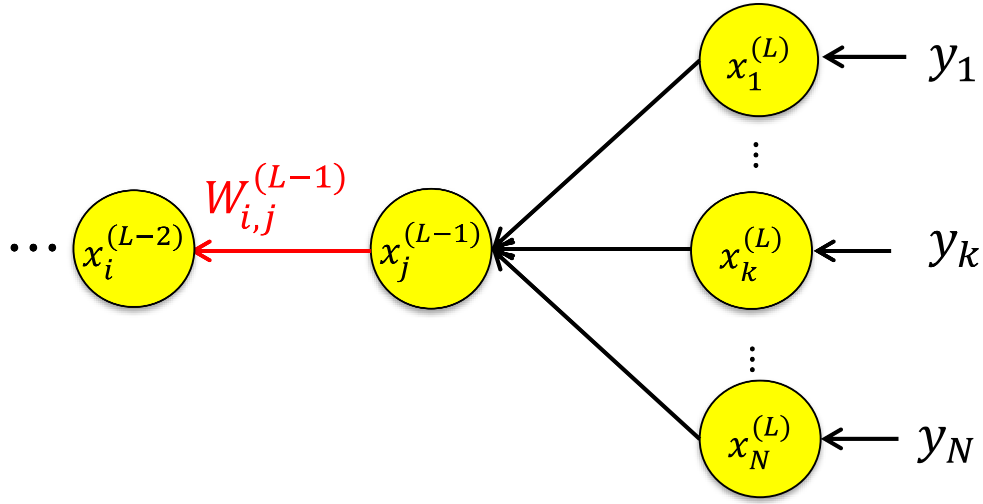
$$= x_i^{(L-1)} \delta_j^{(L)}$$

where $\delta_j^{(L)} = e_j^{(L)} g'\left(z_j^{(L)}\right)$

Backward Propagation

At hidden layer

$$E = \frac{1}{2}\sum_{k=1}^{N}\left(x_k^{(L)} - y_k\right)^2$$



$$E = \frac{1}{2}\sum_{k=1}^{N}\left(x_k^{(L)} - y_k\right)^2$$

$$\frac{\partial E}{\partial W_{i,j}^{(L-1)}} = \sum_{k=1}^{N}\left(x_k^{(L)} - y_k\right)\frac{\partial}{\partial W_{i,j}^{(L-1)}} x_k^{(L)}$$

$$= \sum_{k=1}^{N} e_k^{(L)} \frac{\partial}{\partial W_{i,j}^{(L-1)}} g\left(z_k^{(L)}\right)$$

$$= \sum_{k=1}^{N} e_k^{(L)} g'\left(z_k^{(L)}\right)\frac{\partial}{\partial W_{i,j}^{(L-1)}} z_k^{(L)}$$

$$e_k^{(L)} = x_k^{(L)} - y_k$$

$$= \sum_{k=1}^{N} e_k^{(L)} g'\left(z_k^{(L)}\right)\frac{\partial}{\partial W_{i,j}^{(L-1)}} W_{j,k}^{(L)} x_j^{(L-1)}$$

$$z_k^{(L)} = \sum_{r=0}^{d^{(L-1)}} W_{r,k}^{(L)} x_r^{(L-1)}$$

$$= \sum_{k=1}^{N} e_k^{(L)} g'\left(z_k^{(L)}\right)\frac{\partial W_{j,k}^{(L)} x_j^{(L-1)}}{\partial x_j^{(L-1)}}\frac{\partial x_j^{(L-1)}}{\partial W_{i,j}^{(L-1)}}$$

$$= \sum_{k=1}^{N} e_k^{(L)} g'\left(z_k^{(L)}\right) W_{j,k}^{(L)} g'\left(z_j^{(L-1)}\right) x_i^{(L-2)}$$

$\frac{\partial x_j^{(L-1)}}{\partial W_{i,j}^{(L-1)}} = g'\left(z_j^{(L-1)}\right) x_i^{(L-2)}$ from previous slide

$$= x_i^{(L-2)} g'\left(z_j^{(L-1)}\right)\sum_{k=1}^{N} e_k^{(L)} g'\left(z_k^{(L)}\right) W_{j,k}^{(L)}$$

$$= x_i^{(L-2)}\, \delta_j^{(L-1)}$$

where $\delta_j^{(L-1)} = g'\left(z_j^{(L-1)}\right)\sum_{k=1}^{N} e_k^{(L)} g'\left(z_k^{(L)}\right) W_{j,k}^{(L)} = g'\left(z_j^{(L-1)}\right)\sum_{k=1}^{N} \delta_j^{(L)} W_{j,k}^{(L)}$

Forward Backward Propagation algorithm

Forward propagation

$$x_j^{(l)} = g\left(\sum_{k=0}^{d^{(l-1)}} W_{k,j}^{(l)} x_k^{(l-1)}\right)$$

$$g(z) = \frac{1}{(1 + \exp(-z))}$$

Backward propagation

$$\delta_j^{(l)} = \begin{cases} e_j^{(l)} g'\left(z_j^{(l)}\right) & \text{If } l = L \text{ (output layer)} \\ g'\left(z_j^{(l)}\right) \sum_{k=1}^{N} \delta_j^{(l+1)} W_{j,k}^{(l+1)} & \text{If } l < L \text{ (hidden layer)} \end{cases}$$

$$g'\left(z_j^{(l)}\right) = x_j^{(l)}\left(1 - x_j^{(l)}\right)$$

Forward Backward Propagation algorithm

1 Initialize all weights $W_{i,j}^{(l)}$ at random

2 For $t = 0, 1, 2, ..$ do

3 Pick a single data point in $\boldsymbol{D} = \{(\boldsymbol{x}^{(i)}, y^{(i)}); i = 1, \dots, m\}$

4 **Forward propagation** : compute all $x_j^{(l)}$

5 **Backward propagation** : compute all $\delta_j^{(l)}$

6 Update the weights $W_{i,j}^{(l)} \leftarrow W_{i,j}^{(l)} - \alpha x_i^{(l-1)} \delta_j^{(l)}$

7 Iterate until $W_{i,j}^{(l)}$ converges

8 Return the final weights $W_{i,j}^{(l)}$

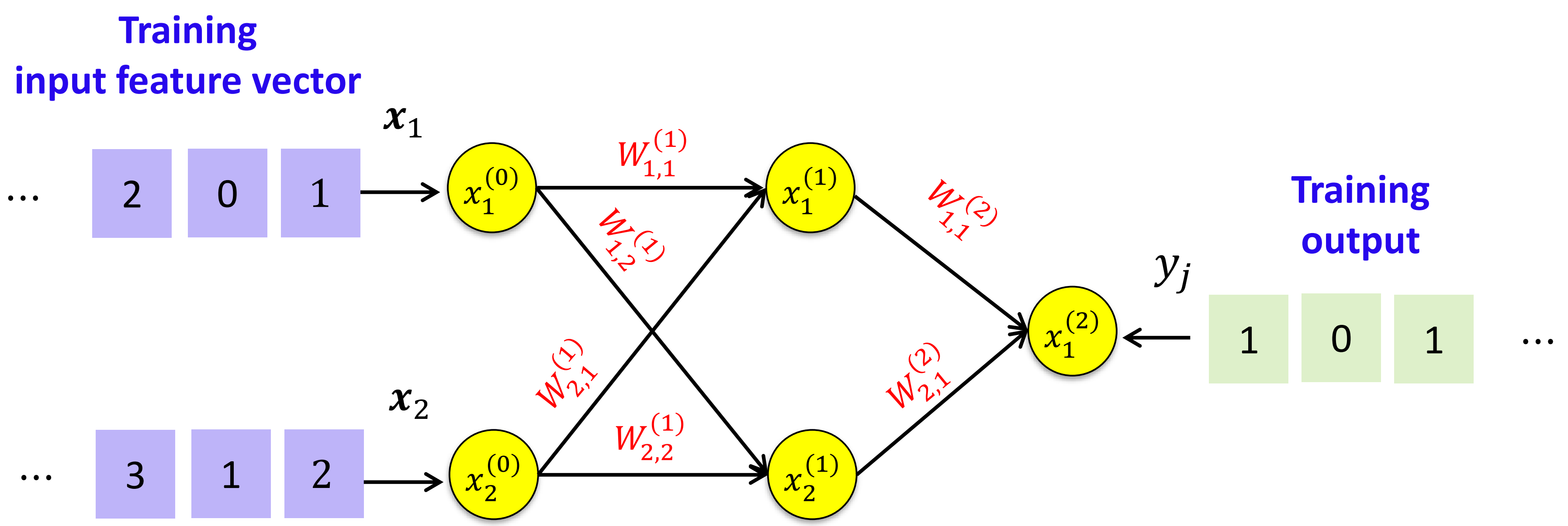
Applying gradient decent

$$W_{i,j}^{(l)} = W_{i,j}^{(l)} - \alpha \frac{\partial E}{\partial W_{i,j}^{(l)}}$$

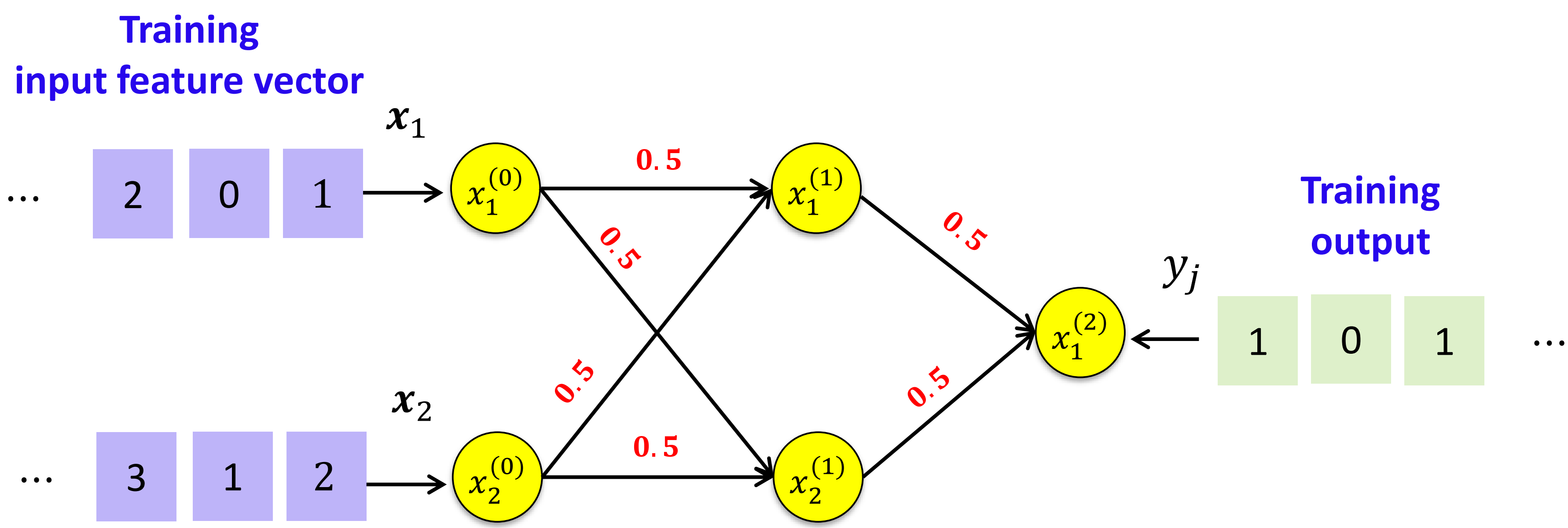
$$\frac{\partial E}{\partial W_{i,j}^{(l)}} = x_i^{(l-1)} \delta_j^{(l)}$$

α is learning rate

Illustrations

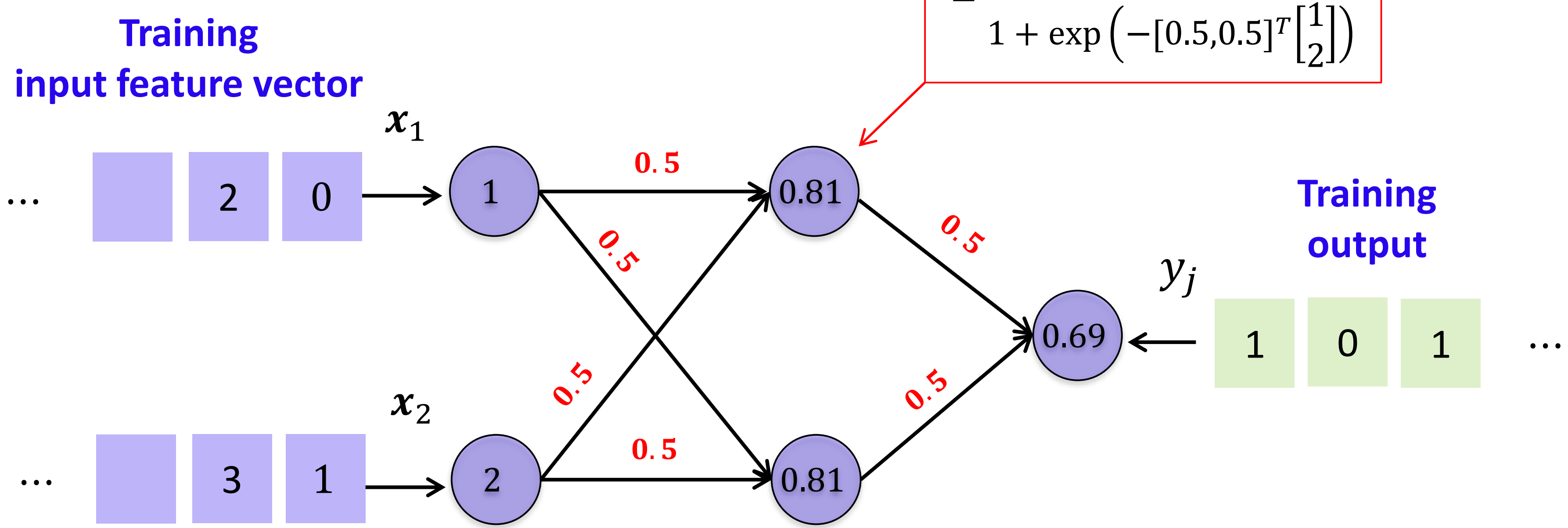


Initialize weight parameters $w_{i,j}^{(L-1)}$ (iteration = 0)



Forward Propagation (iteration = 1)

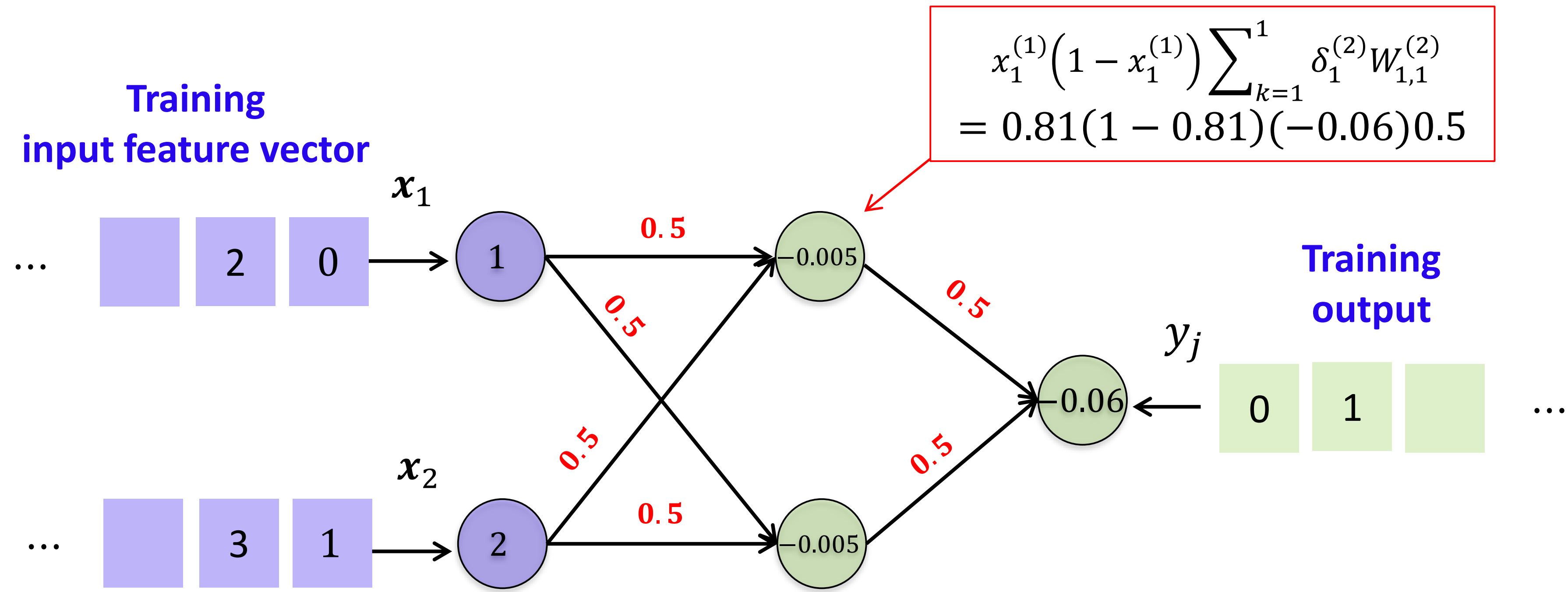
$$x_1^{(1)} = g\left(\sum_{k=1}^{2} W_{k,1}^{(1)} x_k^{(0)}\right) = \frac{1}{1 + \exp\left(-[0.5,0.5]^T \begin{bmatrix}1\\2\end{bmatrix}\right)}$$



Forward propagation

$$x_j^{(l)} = g\left(\sum_{k=0}^{d^{(l-1)}} W_{k,j}^{(l)} x_k^{(l-1)}\right)$$

Backward Propagation (iteration = 1)



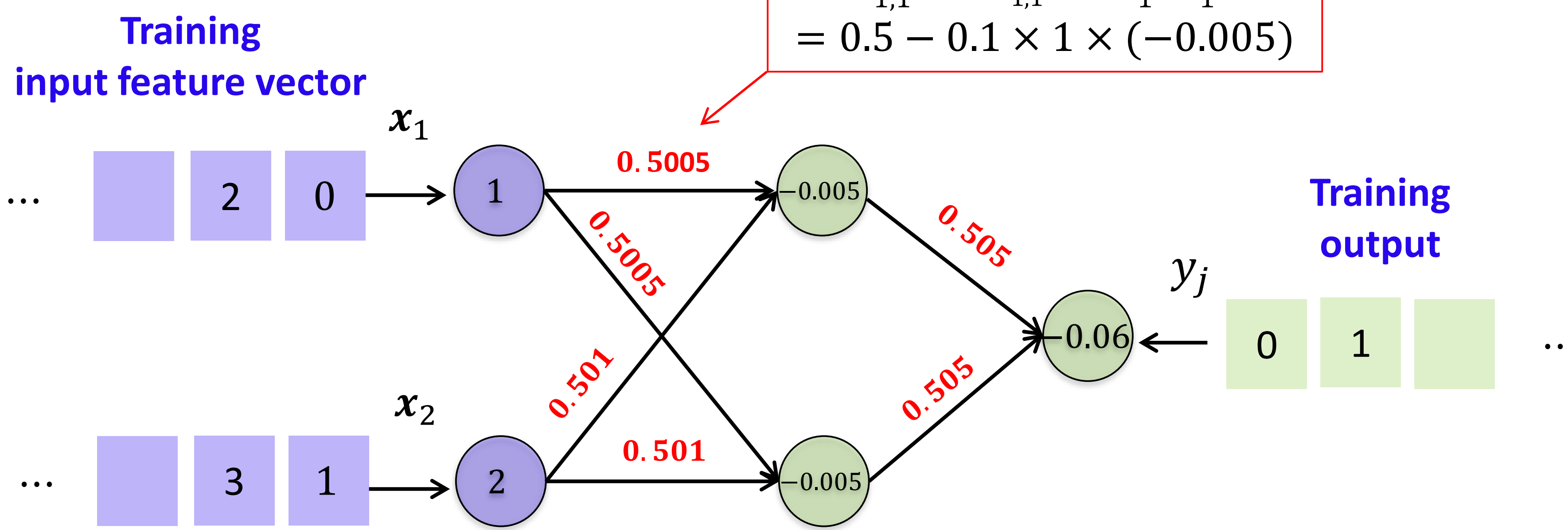
Backward propagation

$$\delta_j^{(l)} = \begin{cases} e_j^{(l)} g'\left(z_j^{(l)}\right) \\ g'\left(z_j^{(l)}\right) \sum_{k=1}^{N} \delta_j^{(l+1)} W_{j,k}^{(l+1)} \end{cases}$$

$$g'\left(z_j^{(l)}\right) = x_j^{(l)}\left(1 - x_j^{(l)}\right)$$

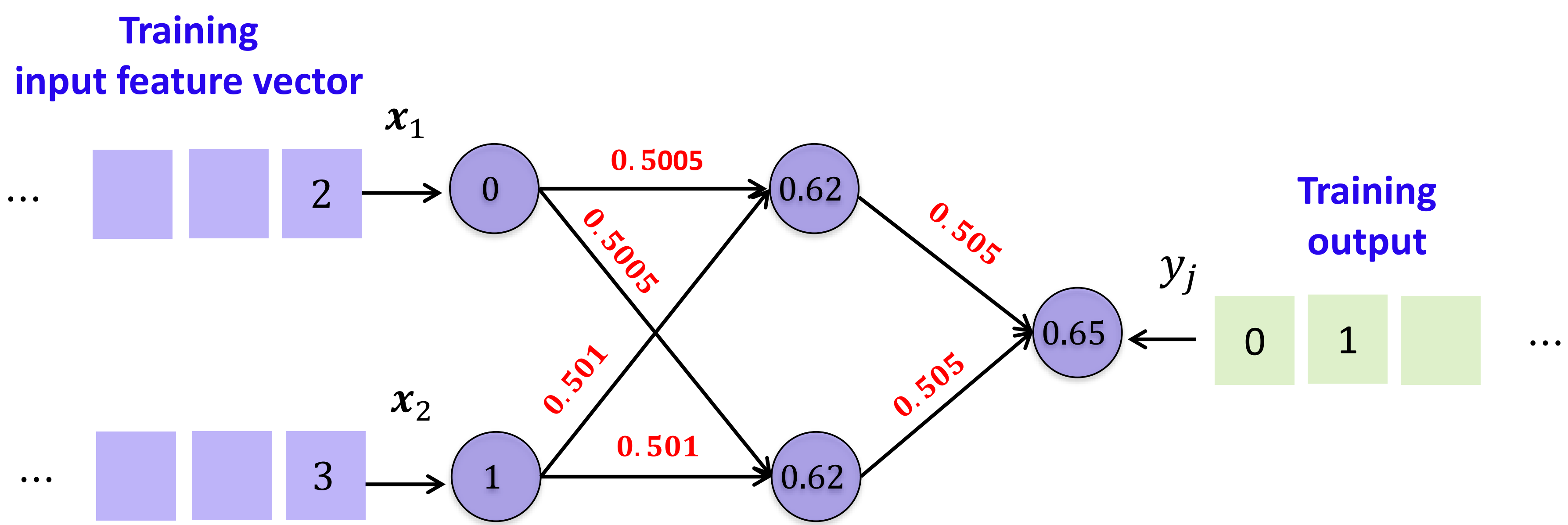
Parameters updating (iteration = 1)

$$W_{1,1}^{(1)} \leftarrow W_{1,1}^{(1)} - \alpha x_1^{(0)} \delta_1^{(1)}$$
$$= 0.5 - 0.1 \times 1 \times (-0.005)$$



Update the weights $W_{i,j}^{(l)} \leftarrow W_{i,j}^{(l)} - \alpha x_i^{(l-1)} \delta_j^{(l)}$

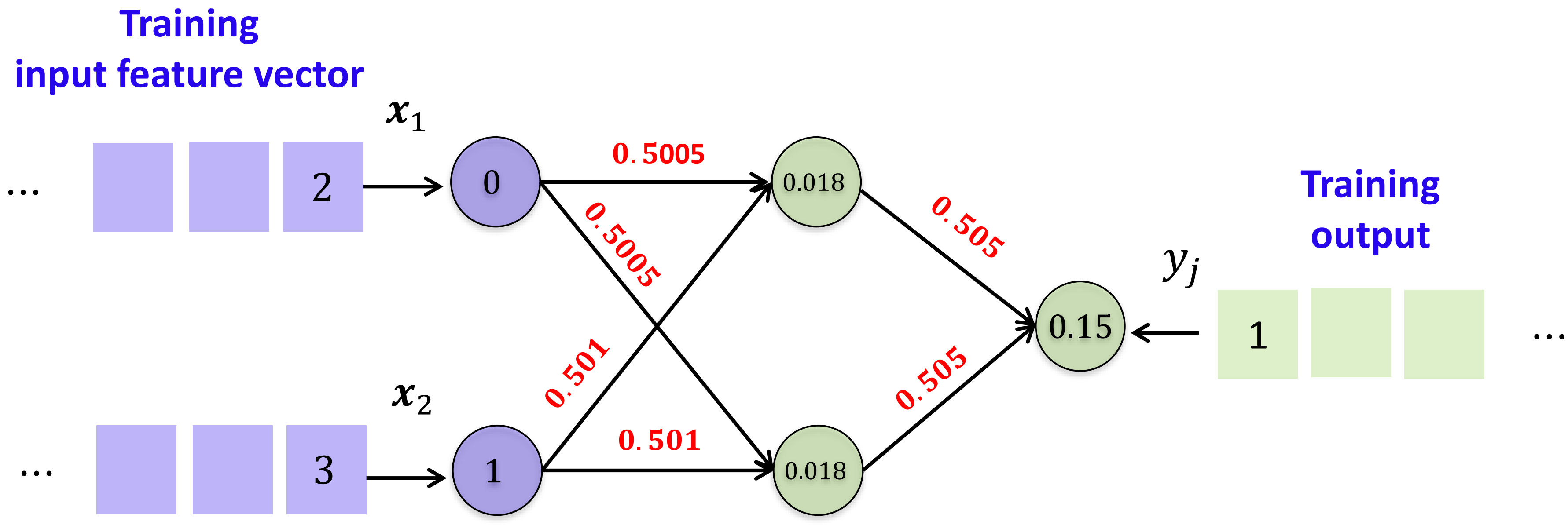
Forward Propagation (iteration = 2)



Forward propagation

$$x_j^{(l)} = g\left(\sum_{k=0}^{d^{(l-1)}} W_{k,j}^{(l)} x_k^{(l-1)}\right)$$

Backward Propagation (iteration = 2)

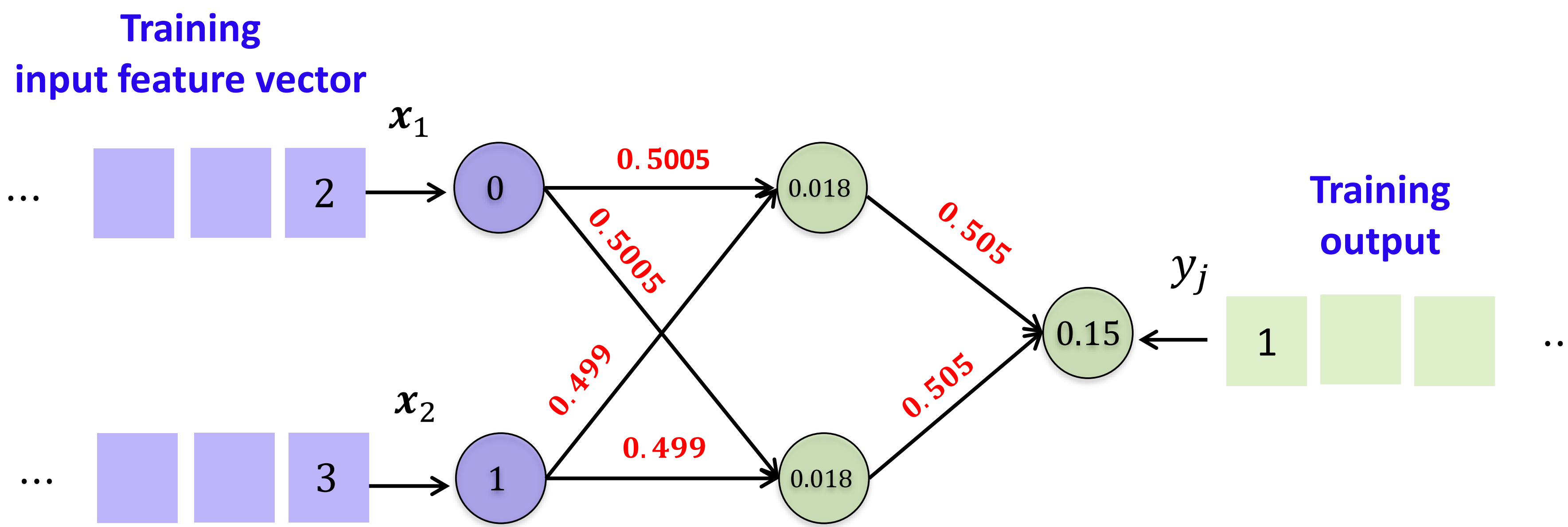


Backward propagation

$$\delta_j^{(l)} = \begin{cases} e_j^{(l)} g'\left(z_j^{(l)}\right) \\ g'\left(z_j^{(l)}\right) \sum_{k=1}^{N} \delta_j^{(l+1)} W_{j,k}^{(l+1)} \end{cases}$$

$$g'\left(z_j^{(l)}\right) = x_j^{(l)}\left(1 - x_j^{(l)}\right)$$

Backward Propagation (iteration = 2)



Update the weights $W_{i,j}^{(l)} \leftarrow W_{i,j}^{(l)} - \alpha x_i^{(l-1)} \delta_j^{(l)}$

Generative model

Generative model

1. Define Class prior $p(y)$ and likelihood $P(x|y)$

2. Learn the parameters of the models, $P(y)$ and $P(x|y)$

3. Express posterior distribution on class y given the input vector x

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)} = \frac{P(x|y)P(y)}{\sum_{y \in Y} P(x|y)P(y)}$$

4. Prediction step: any new input feature vector x_{new} can be classified according to the maximum a posteriori detection principle (MAP)

$$\hat{y} = \underset{y}{\operatorname{argmax}} P(y|x_{new}) = \underset{y}{\operatorname{argmax}} \frac{P(x_{new}|y)P(y)}{\sum_y P(x_{new}|y)P(y)}$$

$$= \underset{y}{\operatorname{argmax}} P(x_{new}|y)P(y)$$

Generative model

1. Define prior and likelihood

$$p(x|y = \text{dog}), p(y = \text{dog})$$
$$p(x|y = \text{cat}),\ p(y = \text{cat})$$

2. Learn the parameters for the models

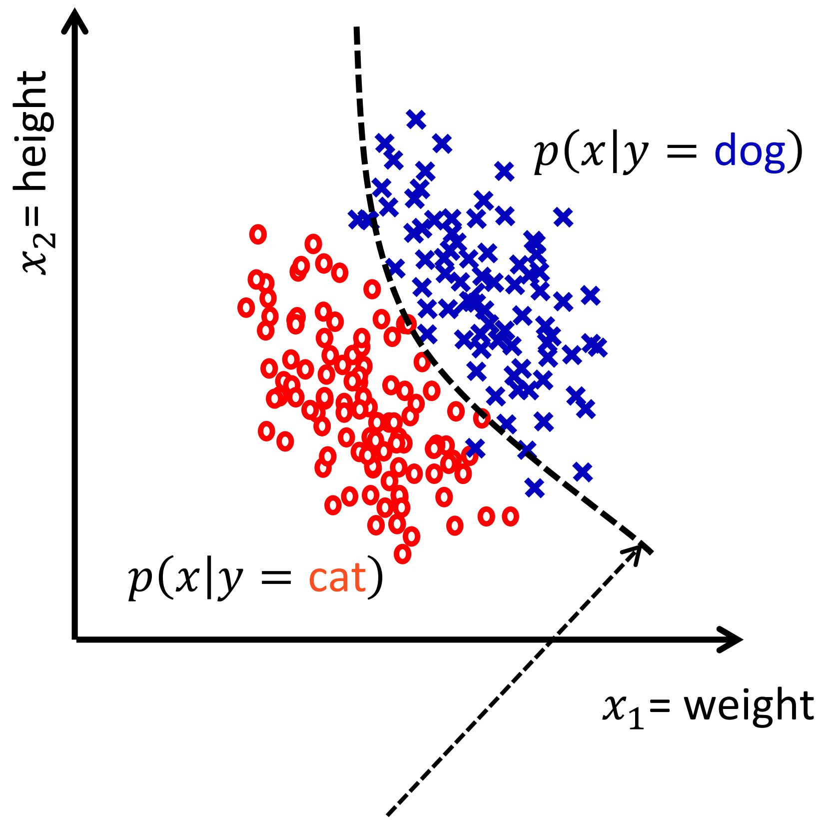
3. Construct the posterior distribution on class

$$P(y = \text{dog}|x) = \frac{p(x|y = \text{dog})p(y = \text{dog})}{\sum_{y\in\{doc,cat\}} p(x|y)p(y)}$$

$$P(y = \text{cat}|x) = \frac{p(x|y = \text{cat})p(y = \text{cat})}{\sum_{y\in\{doc,cat\}} p(x|y)p(y)}$$

4. Classify animal based on MAP estimation:

$$\hat{y} = \underset{y\in Y}{\operatorname{argmax}} P(y|x^{new})$$



Decision boundary

$$p(y = \text{dog}|x) = p(y = \text{cat}|x)$$

The shape of a decision boundary changes depending on the assumptions on the model (ex., linear, quadratic, …)

Multivariate Gaussian Distribution

Univariate Gaussian

$$N(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Mean $\mu = E[X]$

variance $\sigma^2 = \mathrm{var(X)} = \mathrm{E}[(X - E[X])^2]$

Multivariate Gaussian

$$N(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^n|\boldsymbol{\Sigma}|}}\exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)$$

Mean vector

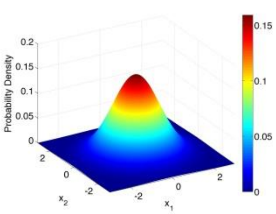
$$\boldsymbol{\mu} = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{bmatrix}$$

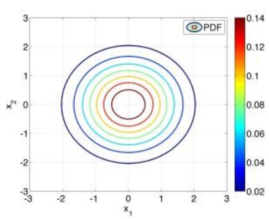
Covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} Cov[X_1,X_1] & \cdots & Cov[X_1,X_n] \\ \vdots & \ddots & \vdots \\ Cov[X_n,X_1] & \cdots & Cov[X_n,X_n] \end{bmatrix}$$

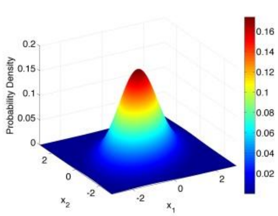
$$Cov[X,Z] = E[(X - E[X])(Z - E[Z])]$$

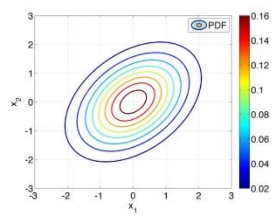
$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



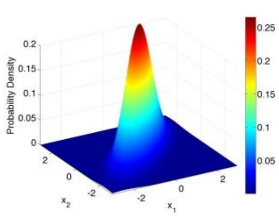


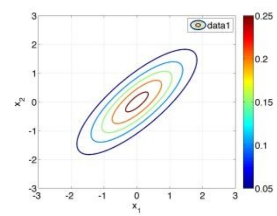
$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 1 \end{bmatrix}$$





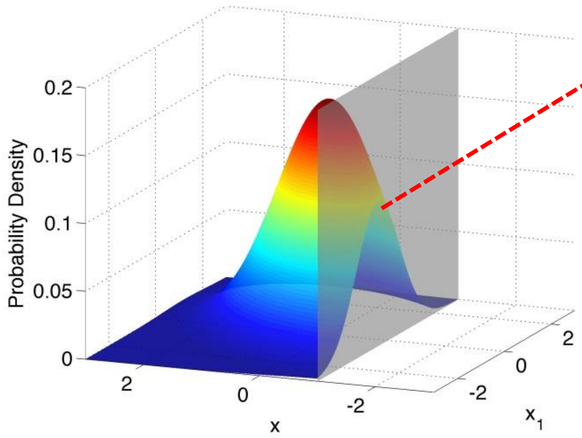
$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$





Multivariate Gaussian Distribution

Conditionalization→ Gaussian



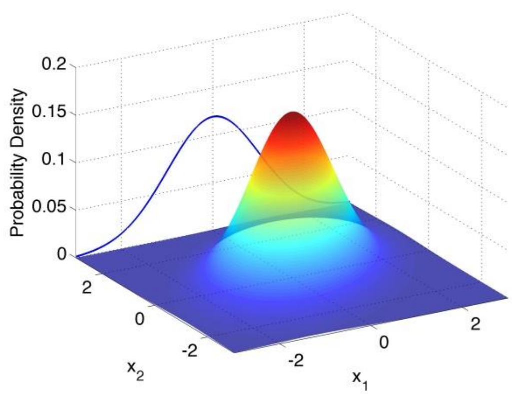
$$P(x_1|x_2) = \frac{P(x_1,x_2)}{P(x_2)}$$

(Graph does not show normalization)

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$$

$$X_1|\{X_2 = x_2\} \sim N\left(\Sigma_{21}\Sigma_{11}^{-1}(x_2 - \mu_1) + \mu_{2,} \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)$$

Marginalization→ Gaussian



$$P(X_1) = \int_{x_2=-\infty}^{x_2=\infty} P(X_1,X_2 = x_2)\,dx_2$$

(Graph does not show normalization)

Concept

1. The class **prior** is represented as multinomial distribution

$$p(y = j) = \phi_j, \ \left(\sum_{j=1}^{N} \phi_j = 1\right)$$

2. The distribution of input feature $\boldsymbol{x}$ conditional on the ouput class y is modeled as multivariate Gaussian distribution

$$p(\boldsymbol{x}|y = j) = N\left(\boldsymbol{x}; \boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j\right) = \frac{1}{\sqrt{(2\pi)^n |\boldsymbol{\Sigma}_j|}} \exp\left(-\frac{1}{2}\left(\boldsymbol{x} - \boldsymbol{\mu}_j\right)^T \boldsymbol{\Sigma}_j^{-1}\left(\boldsymbol{x} - \boldsymbol{\mu}_j\right)\right)$$

$\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j$: the mean vector and covariance matrix for the jth class

Parameter learning for GDA

- Using the training data $\boldsymbol{D} = \{(x_i, y_i); i = 1, \dots, m\}$, the parameter sets for GDA are:

$$\boldsymbol{\phi} = \{\phi_1, \dots, \phi_N\}: \text{set of priors}$$
$$\boldsymbol{\mu} = \{\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_N\}: \text{set of mean vectors}$$
$$\boldsymbol{\Sigma} = \{\boldsymbol{\Sigma}_1, \dots, \boldsymbol{\Sigma}_N\}: \text{set of covariance matrices}$$

- The parameters are found as ones maximizing the log-likelihood of data

$$\sum_{i=1}^{m} \log p(x_i, y_i | \boldsymbol{\phi}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \sum_{i=1}^{m} \log\, p(x_i | y_i, \boldsymbol{\mu}, \boldsymbol{\Sigma}) P(y_i | \boldsymbol{\phi})$$

The log-likelihood function is **concave function** in terms of the parameters
→ the optimum parameters are analytically derived as

$$\phi_j = \frac{1}{m} \sum\nolimits_{i=1}^{m} 1\{y_i = j\}$$

$$\boldsymbol{\mu}_j = \frac{\sum_{i=1}^{m} 1\{y_i = j\}\, x_i}{\sum_{i=1}^{m} 1\{y_i = j\}}$$

$$\boldsymbol{\Sigma}_j = \frac{1}{\sum_{i=1}^{m} 1\{y_i = j\}} \sum\nolimits_{i=1}^{m} 1\{y_i = j\} \left(x_i - \boldsymbol{\mu}_{y^{(i)}}\right) \left(x_i - \boldsymbol{\mu}_{y^{(i)}}\right)^T$$

Indication function

$$1\{y_i = j\} = \begin{cases} 1, \text{if} \quad y_i = j \\ 0, \text{otherwise} \end{cases}$$

Class Prediction

- The probability of class $y = j$ given the new input x^{new} can be computed

$$P(y = j\,|x^{new}) \sim P(x^{new}|y = j)P(y = j) \qquad p(y|x^{new}) = \frac{P(x^{new}|y)P(y)}{P(x^{new})}$$

$$= \frac{1}{\sqrt{(2\pi)^n|\boldsymbol{\Sigma}_j|}} \exp\left(-\frac{1}{2}\left(x^{new} - \boldsymbol{\mu}_j\right)^T \boldsymbol{\Sigma}_j^{-1}\left(x^{new} - \boldsymbol{\mu}_j\right)\right)\phi_j$$

- The class can be selected using MAP estimation:

$$\hat{y} = \underset{y \in Y}{\operatorname{argmax}}\, p(y|x^{new})$$

- The boundary surface between two neighboring classes i and j $\left(\text{i.e.}, P(y = i|\boldsymbol{x}) = P(y = j|\boldsymbol{x})\right)$

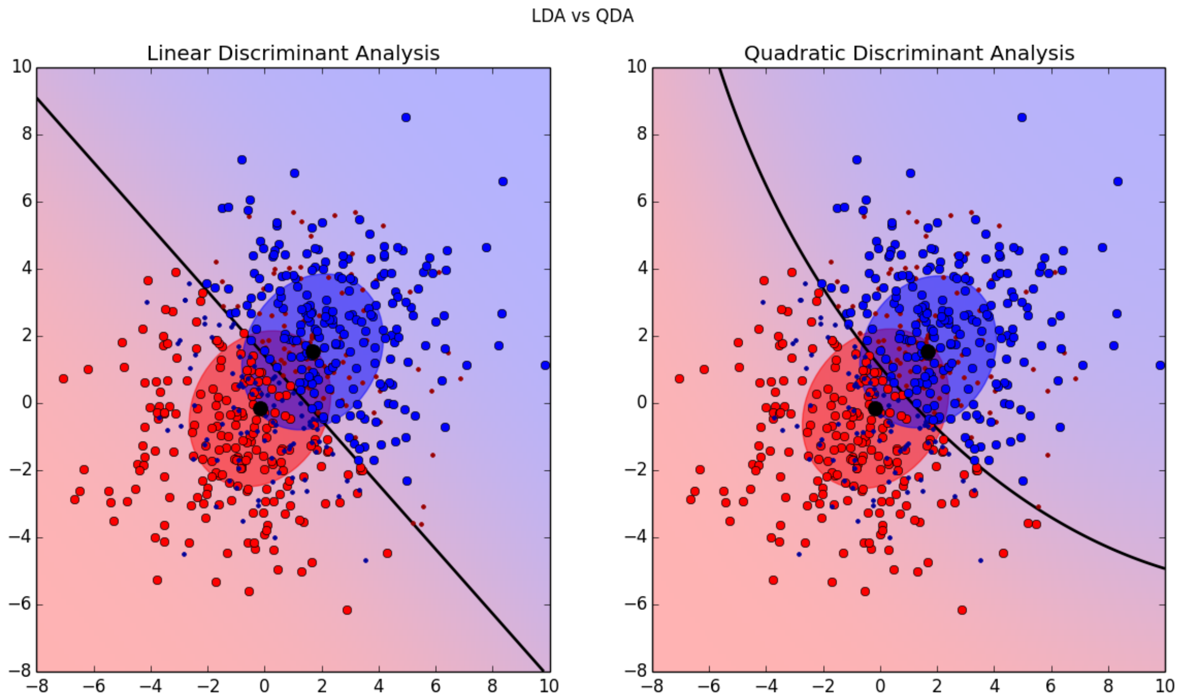
$$\frac{1}{\sqrt{(2\pi)^n|\boldsymbol{\Sigma}_i|}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_i)^T \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_i)\right)\phi_i = \frac{1}{\sqrt{(2\pi)^n|\boldsymbol{\Sigma}_j|}} \exp\left(-\frac{1}{2}\left(\boldsymbol{x} - \boldsymbol{\mu}_j\right)^T \boldsymbol{\Sigma}_j^{-1}\left(\boldsymbol{x} - \boldsymbol{\mu}_j\right)\right)\phi_j$$

$$\rightarrow \boldsymbol{x}^T\left(\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_j^{-1}\right)\boldsymbol{x} - 2\left(\boldsymbol{\mu}_i^T\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\mu}_j^T\boldsymbol{\Sigma}_j^{-1}\right)\boldsymbol{x} + \boldsymbol{\mu}_i^T\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i - \boldsymbol{\mu}_j^T\boldsymbol{\Sigma}_j^{-1}\boldsymbol{\mu}_j + \log\frac{\phi_j|\boldsymbol{\Sigma}_i|}{\phi_i|\boldsymbol{\Sigma}_j|} = 0$$

Example (binary-classes)

The boundary surface between two neighboring classes i and j $\left(\text{i.e.}, P(y=i|\boldsymbol{x}) = P(y=j|\boldsymbol{x})\right)$

$$\boldsymbol{x}^T\left(\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_j^{-1}\right)\boldsymbol{x} - 2\left(\boldsymbol{\mu}_i^T\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\mu}_j^T\boldsymbol{\Sigma}_j^{-1}\right)\boldsymbol{x} + \boldsymbol{\mu}_i^T\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i - \boldsymbol{\mu}_j^T\boldsymbol{\Sigma}_j^{-1}\boldsymbol{\mu}_j + \log\frac{\phi_j|\boldsymbol{\Sigma}_i|}{\phi_i|\boldsymbol{\Sigma}_j|} = 0$$



Linear discriminant analysis

$\Sigma_i = \Sigma$ for all i

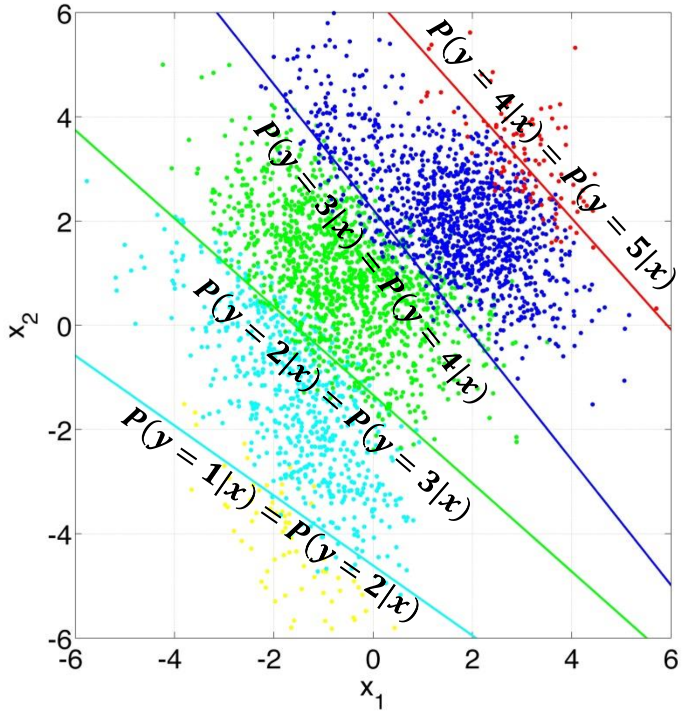
Quadratic discriminant analysis

Σ_i for each i

Example (multi-classes)

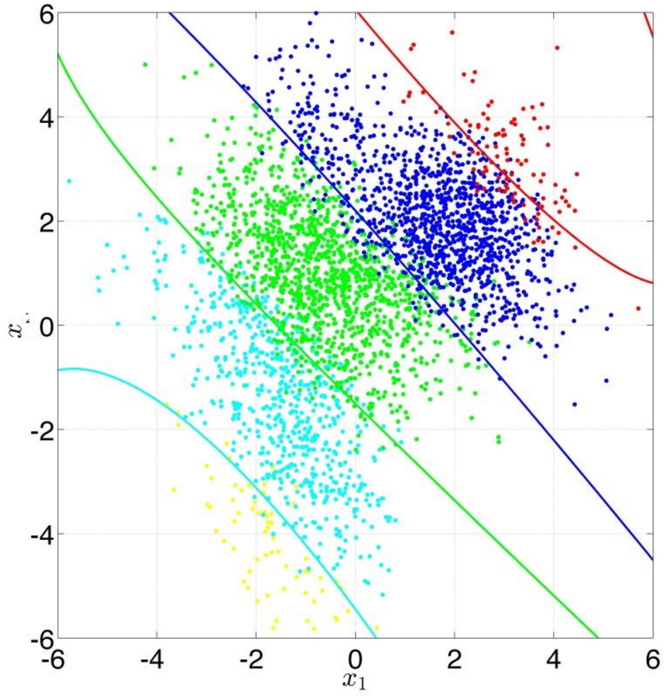
The boundary between the two neighboring classes i and j by setting $P(y=i|\boldsymbol{x}) = P(y=j|\boldsymbol{x})$, which yields

$$\boldsymbol{x}^T\left(\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_j^{-1}\right)\boldsymbol{x} - 2\left(\boldsymbol{\mu}_i^T\boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\mu}_j^T\boldsymbol{\Sigma}_j^{-1}\right)\boldsymbol{x} + \boldsymbol{\mu}_i^T\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i - \boldsymbol{\mu}_j^T\boldsymbol{\Sigma}_j^{-1}\boldsymbol{\mu}_j + \log\frac{|\boldsymbol{\Sigma}_i|}{|\boldsymbol{\Sigma}_j|} = 0$$



Linear discriminant analysis

$\Sigma_i = \Sigma$ for all i

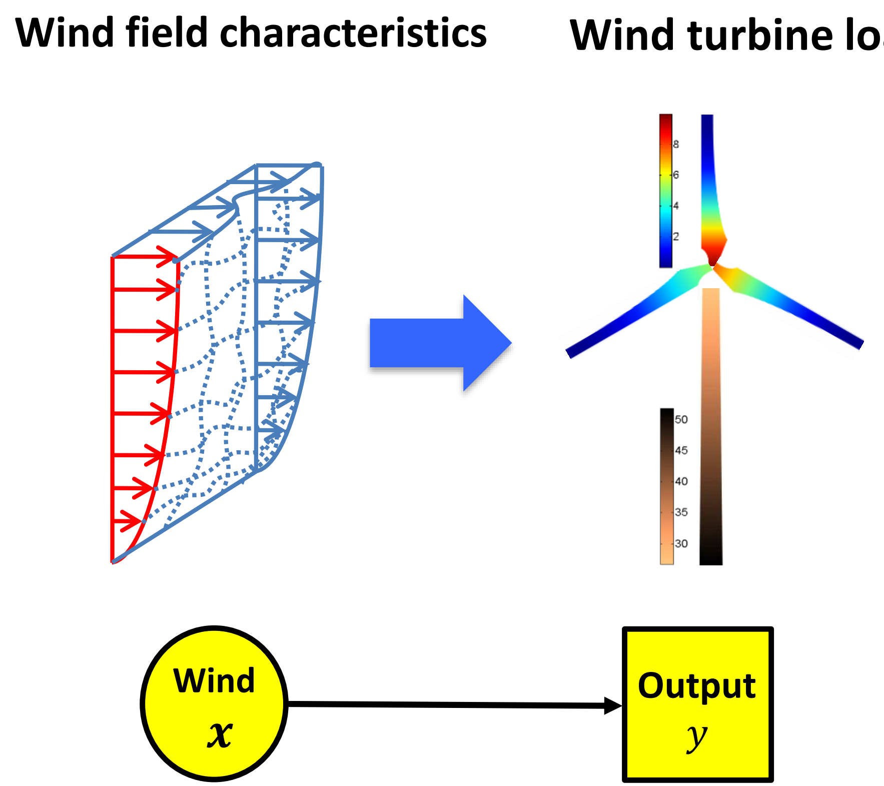


Quadratic discriminant analysis

Σ_i for each i

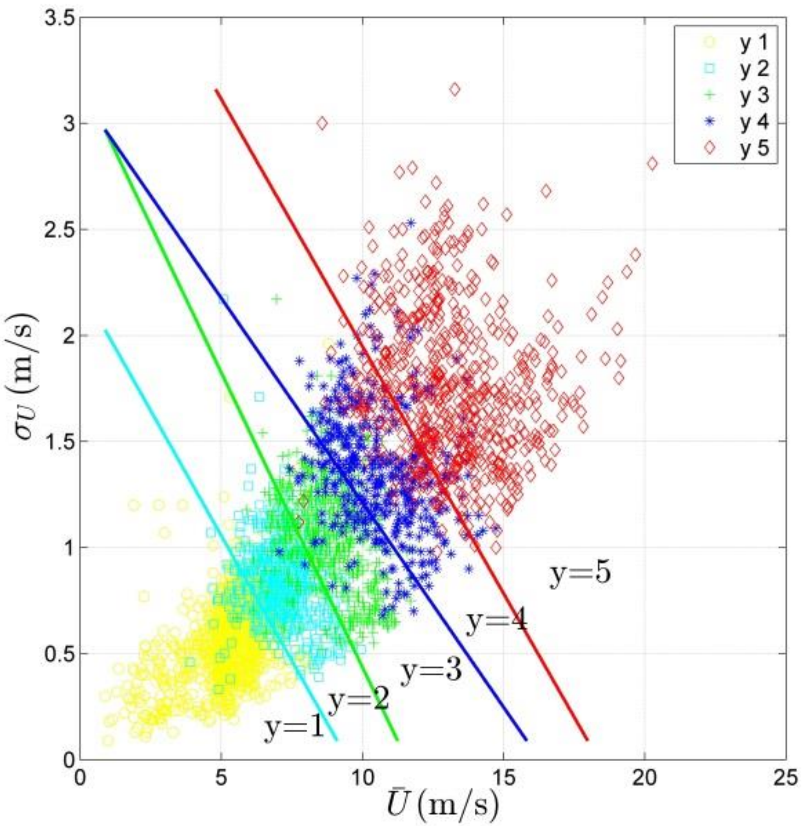
Application to wind turbine monitoring data

Study how wind field characteristics affect wind turbine response class.



→ Construct the posterior probability mass function for the response class $p(y|x)$

Application to wind turbine monitoring data

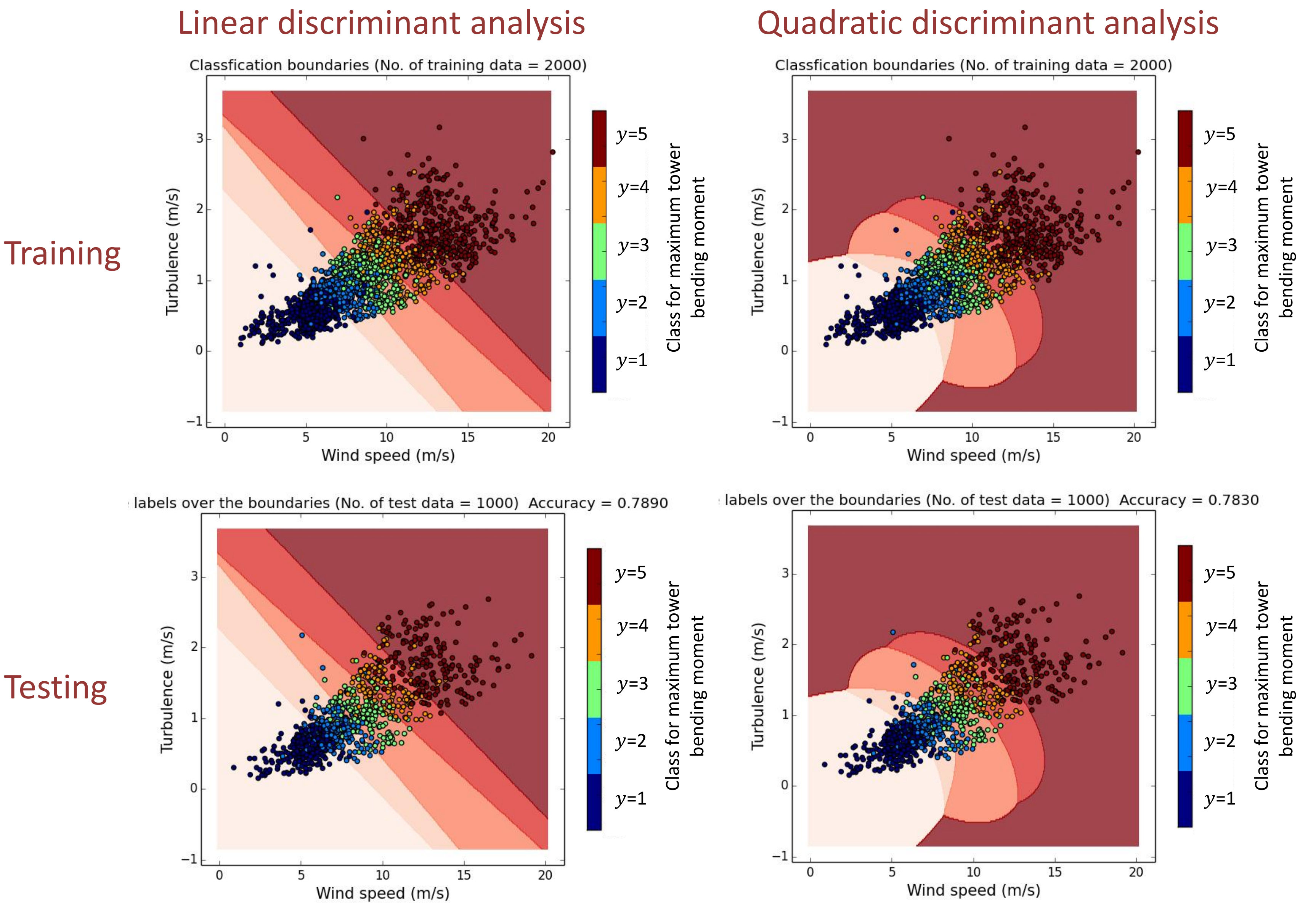


Classification boundaries between the ith and the jth class is determined as:

$$p(y = i|\boldsymbol{x}) = p(y = j|\boldsymbol{x})$$

- Higher wind speed and higher turbulence tend to cause higher blade bending moment.
- Accuracy for the classification is approximately 80 %.
- Including more input features can increases the accuracy ration.

Application to wind turbine monitoring data



Naïve Bayes Classification

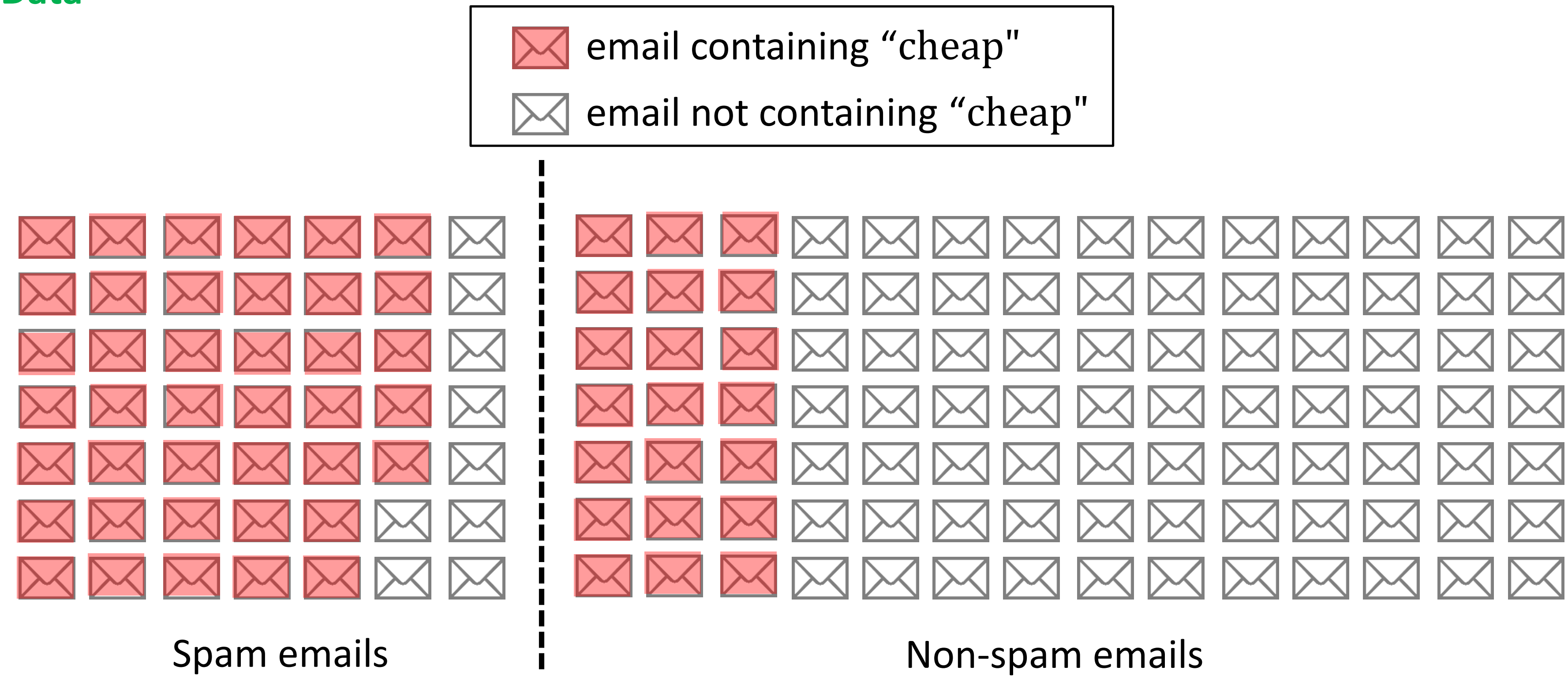
Detecting Spam e-mails

- Input: $x =$ email message
- Output $y \in$ {Spam, non-spam}
- Objective: Obtain a classifier f

$$x \longrightarrow \boxed{f} \longrightarrow y$$

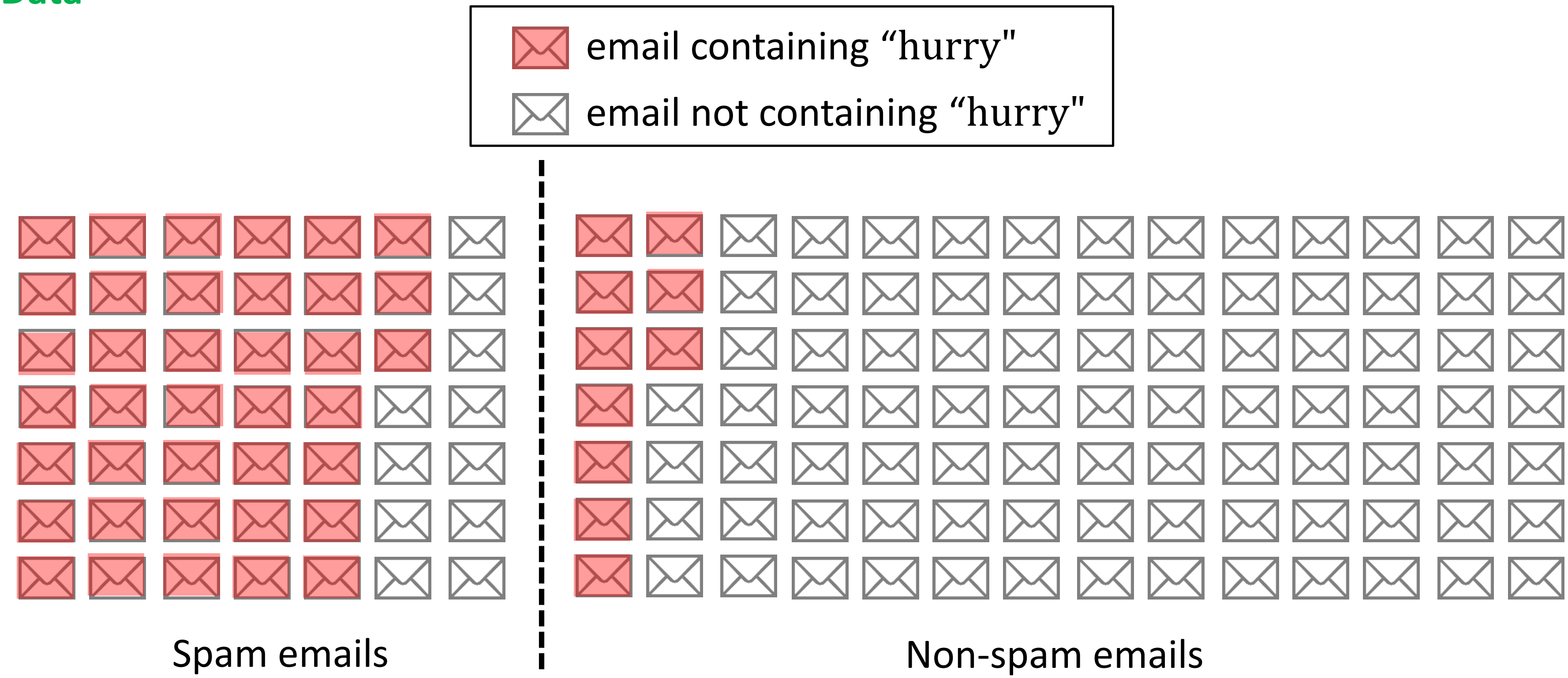
Naïve Bayes Classification

Data



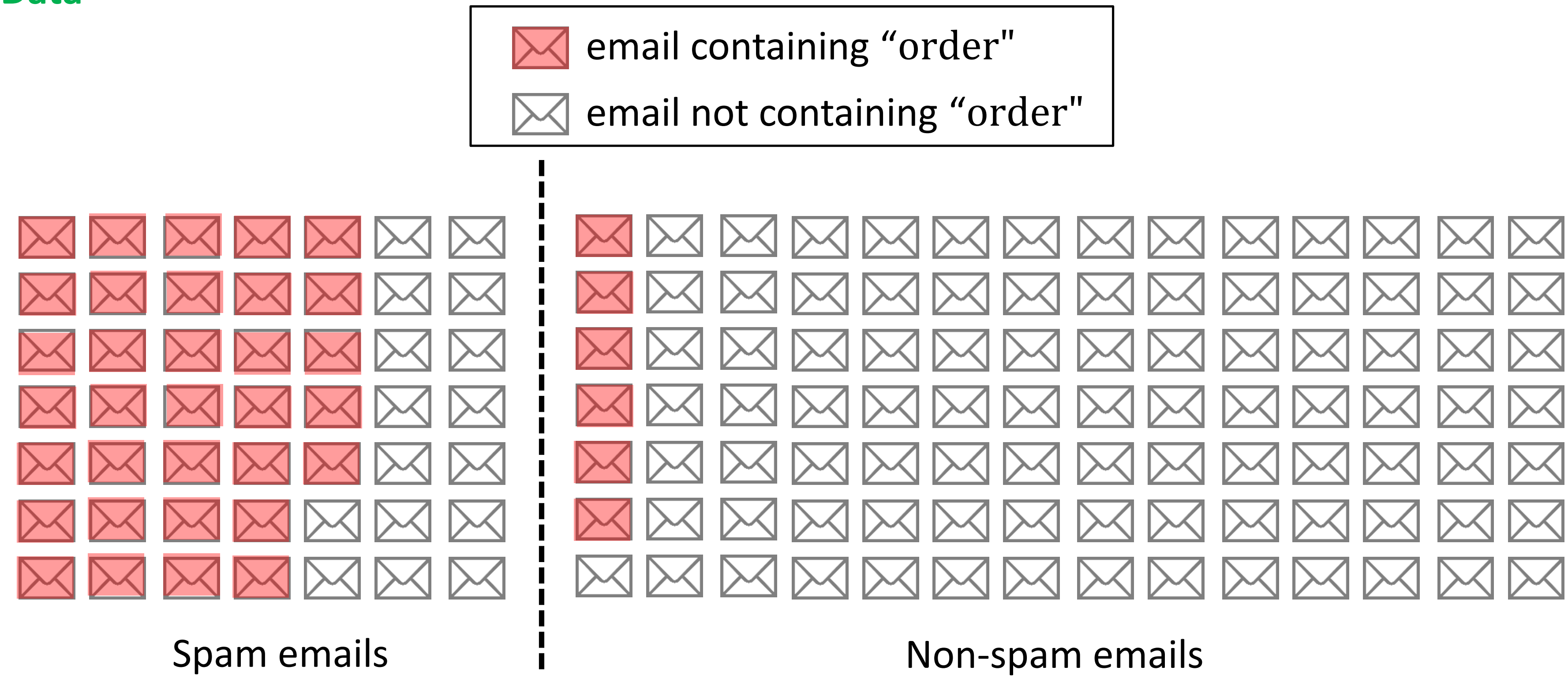
$$P(\text{cheap}|y = \text{spam}) = \frac{40}{49} \qquad P(\text{cheap}|y = \text{nonspam}) = \frac{21}{98}$$

Data



$$P(\text{"hurry"}|y = \text{spam}) = \frac{38}{49} \qquad P(\text{"hurry"}|y = \text{nonspam}) = \frac{10}{98}$$

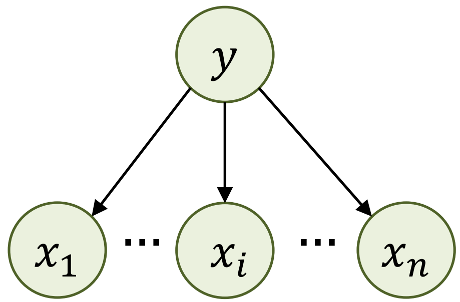
Data



$$P(\text{"order"}|y = \text{spam}) = \frac{33}{49} \qquad P(\text{"order"}|y = \text{nonspam}) = \frac{6}{98}$$

Naïve Bayes Classification

- Now it's a time to build a model for spam classifier

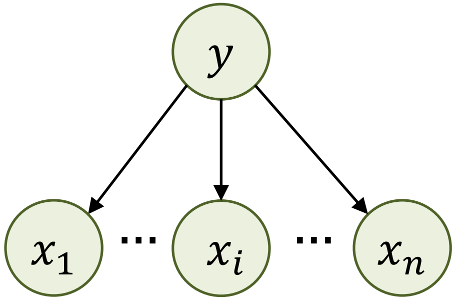


- Input $x = \{x_1, x_2, \dots, x_i, \dots, x_n\}$ $\quad x_i = \begin{cases} 1 & \text{if } i\text{th word is in the email} \\ 0 & \text{otherwise} \end{cases}$
- Output $y = \begin{cases} 1 & \text{if Spam} \\ 0 & \text{otherwise} \end{cases}$
- $p(y)$ is a prior on class
- Naïve Bayes Model assumes x_i (attributes) are conditionally independent given y model. Thus, the likelihood is

$$P(x|y) = \prod_{i=1}^{m} P(x_i|y)$$

Naïve Bayes Classification

- Now it's a time to build a model for spam classifier



- Posterior : $P(y|x) = \frac{P(x|y)P(y)}{P(x)} = \frac{P(x|y)P(y)}{\sum_{y \in Y} P(x|y)P(y)}$

- Class prediction : $\hat{y} = \underset{y}{\operatorname{argmax}} P(y|x) = \underset{y}{\operatorname{argmax}} \frac{P(x|y)P(y)}{\sum_y P(x|y)P(y)}$

$$= \underset{y}{\operatorname{argmax}} \prod_{i=1}^{m} P(x_i|y)\, P(y)$$

$$= \underset{y}{\operatorname{argmax}} \prod_{i=1}^{m} P(x_i|y)\, P(y)$$

Naïve Bayes Classification

- Training the model

$$D = (x_1, y_1), \dots, (x_i, y_i), \dots, (x_m, y_m) \qquad\qquad x_i = (x_{i1}, \dots, x_{in})$$

- Parameterization

$$\phi_{j|y=1} = p(x_j = 1|y = 1) \qquad 1 - \phi_{j|y=1} = p(x_j = 0|y = 1)$$

$$\phi_{j|y=0} = p(x_j = 1|y = 0) \qquad 1 - \phi_{j|y=0} = p(x_j = 0|y = 0)$$

$$\phi_y = p(y = 1) \qquad 1 - \phi_y = p(y = 0)$$

- Cost function = likelihood

$$L\left(\phi_{j|y=1}, \phi_{j|y=0}, \phi_y\right) = \prod_{i=1}^{m} P(x_i, y_i) = \prod_{i=1}^{m} \prod_{j=1}^{n} P\left(x_{ij} \middle| y_i\right) p(y_i)$$

- Maximizing log likelihood with respect to the parameters leads

$$\phi_{j|y=1} = \frac{\sum_{i=1}^{m} 1\{x_{ij} = 1 \ \cap y_i = 1\}}{\sum_{i=1}^{m} 1\{y_i = 1\}} \qquad \phi_{j|y=0} = \frac{\sum_{i=1}^{m} 1\{x_{ij} = 1 \ \cap y_i = 0\}}{\sum_{i=1}^{m} 1\{y_i = 0\}} \qquad \phi_{y=0} = \frac{\sum_{i=1}^{m} 1\{y_i = 1\}}{m}$$

- Example

$$P(\text{cheap}|y=\text{spam})=\frac{40}{49} \qquad P(\text{cheap}|y=\text{nonspam})=\frac{21}{98}$$

$$P(\text{"hurry"}|y=\text{spam})=\frac{38}{49} \qquad P(\text{"hurry"}|y=\text{nonspam})=\frac{10}{98}$$

$$P(\text{"order"}|y=\text{spam})=\frac{33}{49} \qquad P(\text{"order"}|y=\text{nonspam})=\frac{6}{98}$$

x =(if cheap, if hurry, if order)

$p(y=\text{spam})=p(y=\text{non}-\text{spam})=0.5$

- The new email has been arrived with $x=(1,1,0)$

$$p\{y=1|x=(1,1,0)\}\propto P\{x=(1,1,0)\ |y=1\}P(y=1)=\frac{40}{49}\frac{38}{49}\left(1-\frac{33}{49}\right)=0.206$$

$$p\{y=0|x=(1,1,0)\}\propto P\{x=(1,1,0)\ |y=0\}P(y=0)=\frac{21}{49}\frac{10}{49}\left(1-\frac{6}{49}\right)=0.077$$

$$p\{y=0|x=(1,1,0)\}=\frac{0.206}{0.206+0.077}=0.730, \qquad p\{y=0|x=(1,1,0)\}=0.270$$

Naïve Bayes Classification

Jupyter Demo Simulation