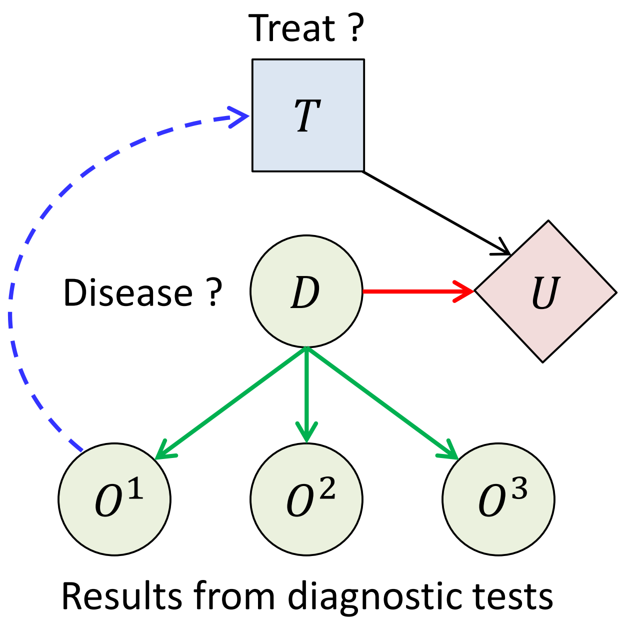
L11. Influential Diagram

Bayesian Network + **Decision node** + **Utility node** = **Decision network (Influential Diagram)**

- make rational decisions based on a probabilistic model and utility function



A chance node corresponds to a random variable

A decision node corresponds to each decision to be made

A utility node corresponds to an additive utility component

Utility theory

How to compare the plausibility of different statements?

A : "We can be a millionaire if we go to graduate school"

vs

B : "We can be a millionaire if we go to Samsung"

- If you believe A more than B, you can write $A \succ B$
- If you believe B more than A, you can write $A \prec B$
- If you have the same belief, you can write $A \sim B$

Constraints on Rational Preference

Assumptions about relationships of $\succ$ and $\sim$

- Completeness (comparability) : either $A \succ B$, $A \prec B$ or $A \sim B$
- Transitivity : if $\mathrm{A} \succ B$ and $B \succ C$, then $A \succ C$
- Continuity : if $A \succ B \succ C$, there exists a probability p such that $[A\!: p; C\!: 1-p] \sim B$
- Monotonicity: if $\mathrm{A} \succ B$, then for any C and probability p, $[A\!: p; C\!: 1-p] \succ [B\!: p; C\!: 1-p]$

The degree of belief can be represented by a real-valued function:

- $P(A) > P(B)$ if and only if $A \succ B$
- $P(A) = P(B)$ if and only if $A \sim B$

Utility functions

- Just as beliefs can be subjective, so can preferences.

- Preference operators can be used to compare preferences over uncertain outcomes. That is, a lottery is a set of probabilities $p_{1:n}$ associated with a set of outcomes $S_{1:n}$.

$$[S_1 : p_1 ; S_2 : p_2 ; \ldots ; S_n : p_n]$$

- The utility of a lottery is given by

$$U([S_1 : p_1 ; S_2 : p_2 ; \ldots ; S_n : p_n]) = \sum_{i=1}^{n} p_i U(S_i)$$

Maximum Expected Utility Principle

- Reach rational decisions with imperfect knowledge of the state of the world.

- Expected utility of taking action a given that we observe o and take action a:

Probabilistic model $P(s'|o,a)$

Utility function $U(s')$

$$\mathrm{EU}(a|o) = \sum_{s'} P(s'|o,a)\,U(s')$$

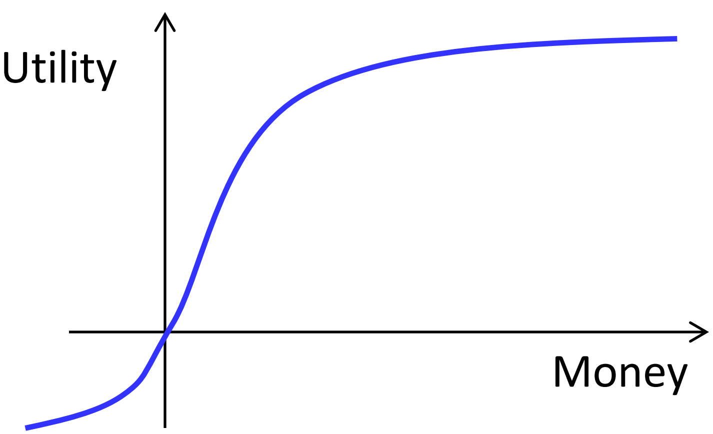
- The principle of maximum expected utility : a rational agent should choose the action that maximizes expected utility:

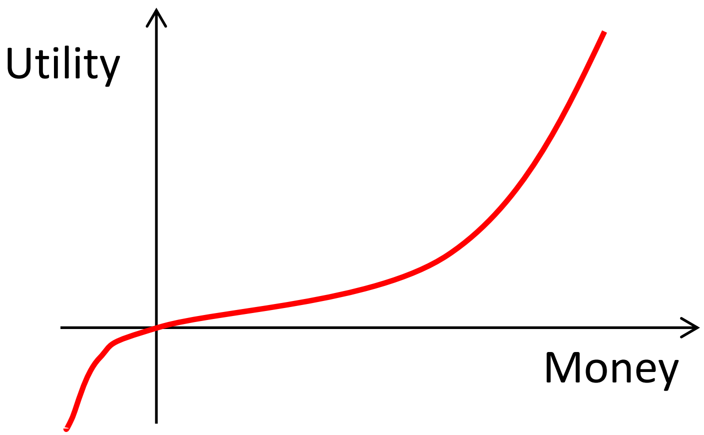
$$a^* = \underset{a}{\operatorname{argmax}}\, \mathrm{EU}(a|o)$$

Utility of Money

- Monetary values are often used to infer utility function.
  Ex) cost of property, human loss damage caused by natural disasters

- It is well known that the relationship between utility and money may not be linear as shown below;

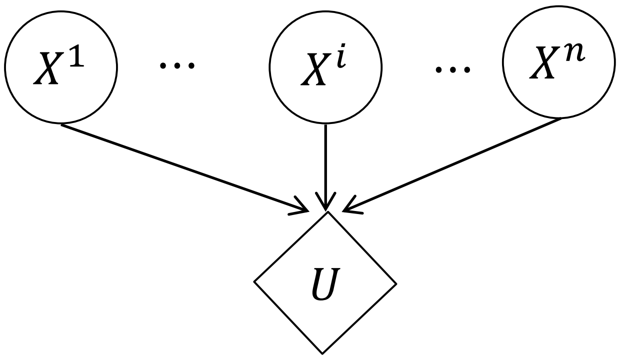




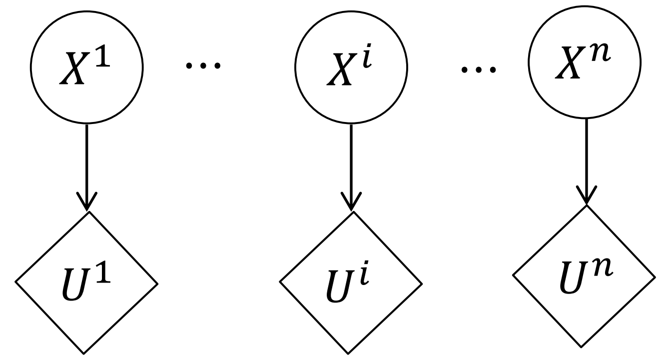
A: Winning 1$ with a probability 1 vs **B**: **Winning** 100$ with a probability 0.01

- Risk averse : Preference for A (Utility function is concave)
- Risk neutral : There is no difference between A and B (Utility function is a linear)
- Risk seeking : Preference for B (Utility function is convex)

Multiple Variable Utility Function



Assumption:
Sum of single-variable
utility function



$$U(X^{1:n})$$

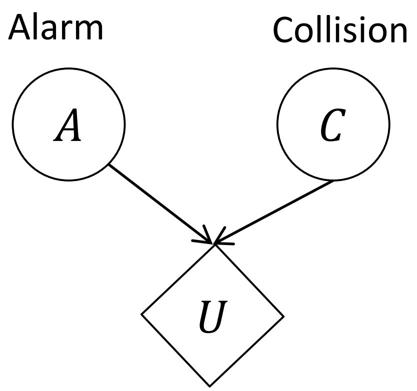
$$U(X^{1:n}) = \sum_{i=1}^{n} U(X^i)$$

- If X^i is a binary variable,
  2^n parameters are required to specify $U(X^{1:n})$
- If X^i is a binary variable,
  $2n$ parameters are required to specify $U(X^{1:n})$

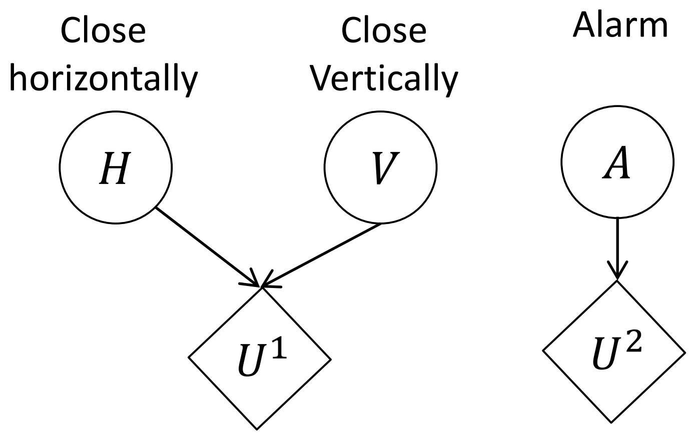
Different additive decomposition can be explicitly imposed on the network structure!

Multiple Variable Utility Function

Example : Collision avoidance system



A	C	U
a^0	c^0	$U(a^0, c^0)$
a^0	c^1	$U(a^0, c^1)$
a^1	c^0	$U(a^1, c^0)$
a^1	c^1	$U(a^1, c^1)$

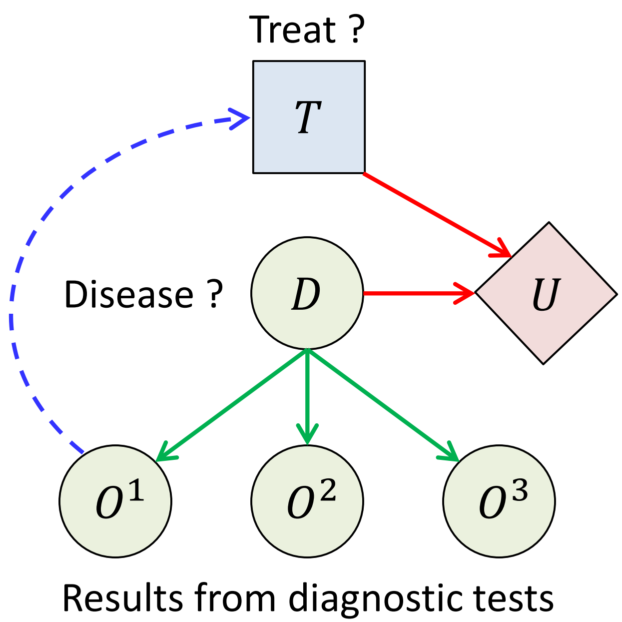


Additive decomposition of utility function

$$U(h, v, a) = U^1(h, v) + U^2(a)$$

Decision Network

Bayesian Network + **Decision node** + **Utility node** = **Decision network (Influential Diagram)**



T	D	$U(T,D)$
0	0	0
0	1	-10
1	0	-1
1	1	-1

→ Conditional edge
→ Functional edge
⇢ Information edge (often omitted)

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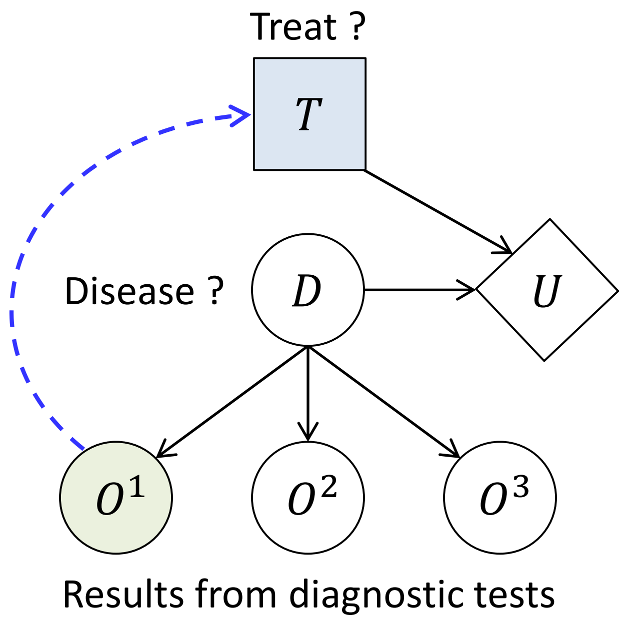
Decision Network

Assume we only have a single observation $O^1 = 1 (= o_1^1)$ from test 1



$$EU(t^1 | o_1^1) = \sum_d P(d | t^1, o_1^1) U(t^1, d)$$

$$= \sum_d \sum_{o_2} \sum_{o_3} P(d, o_2, o_3 | t^1, o_1^1) U(t^1, d)$$

Compare $EU(t^0 | o_1^1)$ and $EU(t^1 | o_1^1)$ and choose the treatment that leads to maximum EU

Value of Information

- It may be beneficial to administer additional diagnostic tests to reduce the uncertainty about the decease. Then, how to choose a test type to be conducted?

- Expected utility of optimal action given observation o :

$$EU^*(o) = \underset{a}{\operatorname{argmax}} EU(a|o)$$

- The value of information (VOI) about new variable O^{new} (unobserved) given the current observation o (observed):

$$\mathrm{VOI}(o^{new}|o) = \left(\sum_{o^{new}} P(o^{new}|o) EU^*(o^{new}, o) \right) - EU^*(o)$$

$$\mathrm{VOI}(o^{new}|o) \geq 0 \ \forall o^{new}, o$$

- The value of information about a variable is the increase in expected utility with the observation of that variable
- VPI can only captures the increase in expected utility→ need to consider the cost associated with observing the new information

Sequential Decision Making : Partial Ordering

- The sequential decision making problems can be solved by exploiting structure in the problem based on Bayesian Network and the corresponding inference routines

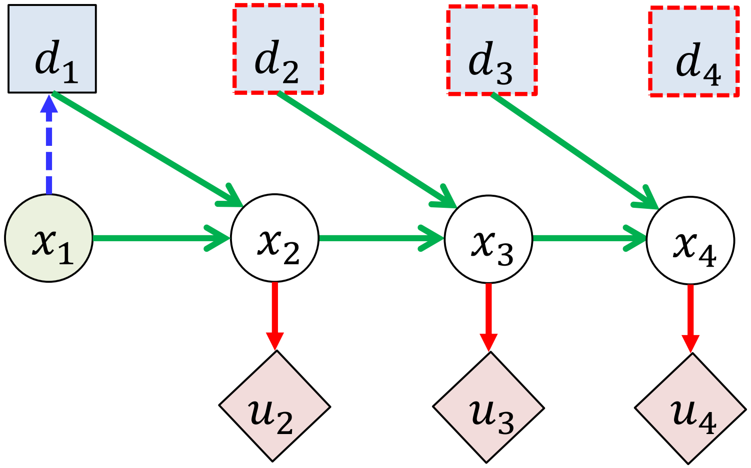
- The sequential decision making problem will be extended later to problems in **control theory** and **reinforcement learning**

- Influential Diagram defines a partial ordering of the nodes:

$$\mathcal{X}_0 \prec D_1 \prec \mathcal{X}_1 \prec D_2, \dots, \prec \mathcal{X}_{n-1} \prec D_n \prec \mathcal{X}_n$$

with $\mathcal{X}_k$ being the variables revealed between decision D_k and D_{k+1}

Sequential Decision Making : Partial Ordering



- Transition probability :

$$p(x_{t+1}|x_t, d_t)$$

- Utility is

$$U(x_{1:4}) = \sum_{t=2}^{4} u(x_t)$$

- The probability of the sequence

$$p(x_{2:4}|x_1, d_{1:3}) = \prod_{t=1}^{3} p(x_{t+1}|x_t, d_t) = p(x_2|x_1, d_1)p(x_3|x_2, d_2)p(x_4|x_3, d_3)$$

- At time $t = 1$, we want to made the decision d_1 that will lead to maximized expected total utility

$$U(d_1|x_1) = \sum_{x_2} \max_{d_2} \sum_{x_3} \max_{d_3} p(x_{2:4}|x_1, d_{1:3}) U(x_{2:4})$$

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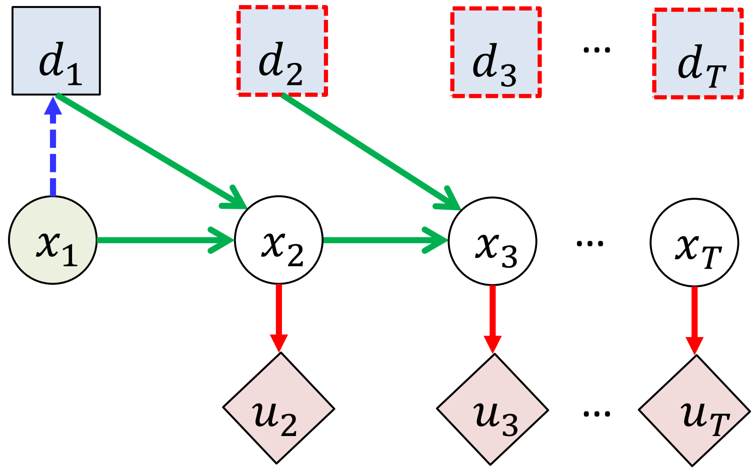
Sequential Decision Making : Partial Ordering



- Transition probability :

$$p(x_{t+1}|x_t, d_t)$$

- Utility is

$$U(x_{1:T}) = \sum_{t=2}^{T} u(x_t)$$

- The probability of the sequence

$$p(x_{2:T}|x_1, d_{1:T-1}) = \prod_{t=1}^{T-1} p(x_{t+1}|x_t, d_t)$$

- At time $t = 1$, we want to made the decision d_1 that will lead to maximized expected total utility

$$U(d_1|x_1) = \sum_{x_2} \max_{d_2} \sum_{x_3} \max_{d_3} \sum_{x_4} \dots \max_{d_{T-1}} \sum_{x_T} p(x_{2:T}|x_1, d_{1:T-1}) U(x_{2:T})$$

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with $\mathcal{X}_k$ being the variables revealed between decision D_k and D_{k+1}

- The optimal first decision $D_1 = d_1$ is determined by computing

$$U(d_1|x_0) \equiv \sum_{\mathcal{X}_1} \max_{D_1} \dots \sum_{\mathcal{X}_{n-1}} \max_{D_{n-1}} \sum_{\mathcal{X}_n} \prod_{i \in \mathcal{L}} p(x_i|\mathrm{pa}(x_i)) \sum_{j \in \mathcal{T}} U_j\left(\mathrm{pa}(u_j)\right)$$

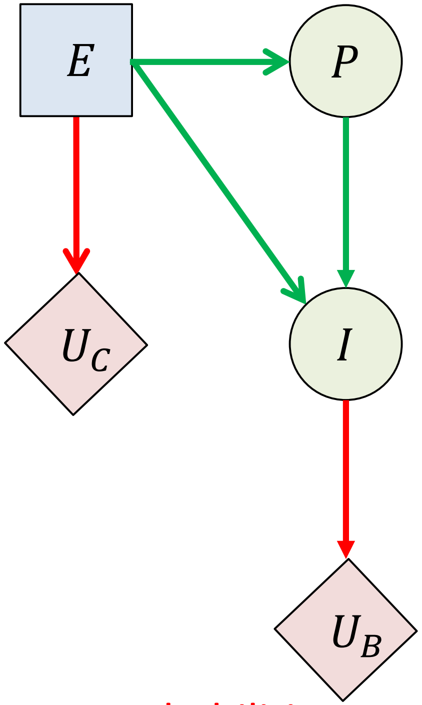
$\mathrm{pa}(x_i)$:=Parent nodes of x_i

$\mathcal{L}$ is a set of random variables and $\mathcal{T}$ is a set of utility variables

- The optimal first decision D_1^* is determined as

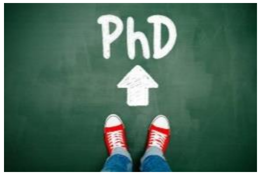
$$d_1^* = \underset{d_1}{\mathrm{argmax}}\, U(d_1|x_0)$$

Example: Should I do a PhD?



Do PhD to wind a Novel Prize?

The ordering : $E^* \prec \{I, P\}$



Domains

- $\text{dom}(E) = \{\text{do PhD}, \text{no PhD}\}$
- $\text{dom}(P) = \{\text{prize}, \text{no prize}\}$
- $\text{dom}(I) = \{\text{low}, \text{average}, \text{high}\}$

Utilities

- $U_C(\text{do PhD}) = -50000,\ \ U_C(\text{no PhD}) = 0$
- $U_B(\text{low}) = 100000,\ U_B(\text{average}) = 200000,\ \ U_B(\text{high}) = 500000$

Probabilities

- $p(\text{win Novel Prize}|\text{no PhD}) = 0.0000001, p(\text{win Novel Prize}|\text{do PhD}) = 0.001$

- $p(\text{low}|\text{do PhD}, \text{no prize}) = 0.1,\ \ p(\text{average}|\text{do PhD}, \text{no prize}) = 0.5,\ \ p(\text{hight}|\text{do PhD}, \text{no prize}) = 0.4$
- $p(\text{low}|\text{no PhD}, \text{no prize}) = 0.2,\ \ p(\text{average}|\text{no PhD}, \text{no prize}) = 0.6,\ \ p(\text{high}|\text{no PhD}, \text{no prize}) = 0.2$
- $p(\text{low}|\text{do PhD}, \text{prize}) = 0.01,\ \ p(\text{average}|\text{do PhD}, \text{prize}) = 0.04,\ \ p(\text{hight}|\text{do PhD}, \text{prize}) = 0.95$
- $p(\text{low}|\text{no PhD}, \text{prize}) = 0.01,\ \ p(\text{average}|\text{no PhD}, \text{prize}) = 0.04,\ \ p(\text{high}|\text{no PhD}, \text{prize}) = 0.95$

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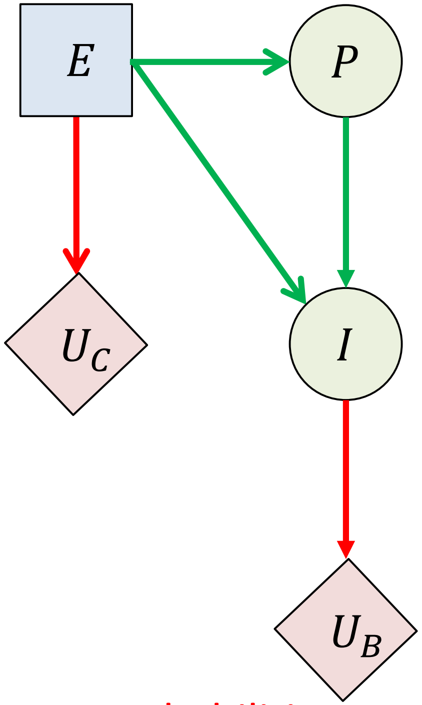

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Utilities

- $U_C(\text{do PhD}) = -50000, \; U_C(\text{no PhD}) = 0$
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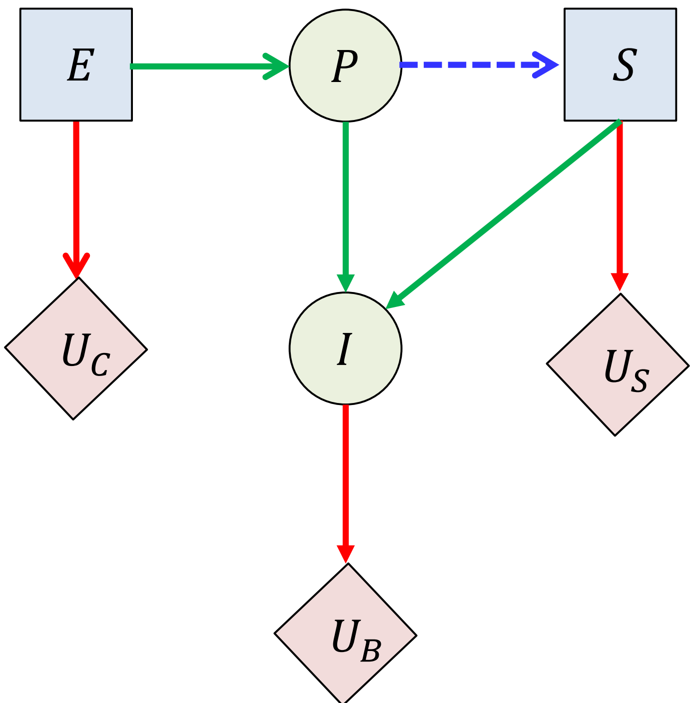
The expected utility of Education is

$$U(E) = \sum_{I,P} p(I|E, P) p(P|E) [U_C(E) + U_B(I)]$$

$U(\text{do PhD}) = 260174$

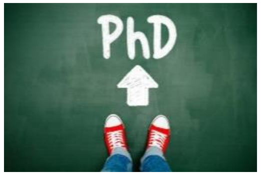
$U(\text{no PhD}) = 240000$

Example: PhD and start-up companies



Do PhD to wind a Novel Prize and start-up?

The ordering : $E^* \prec P \prec S^* \prec I$



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- $\text{dom}(S) = \{\text{yes}, \text{no}\ \}$

Utilities

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- $U_B(\text{low}) = 100000,\ U_B(\text{average}) = 200000, U_B(\text{high}) = 500000$
- $U_S(\text{start up}) = -200000,\ \ U_S(\text{no start up}) = 0$

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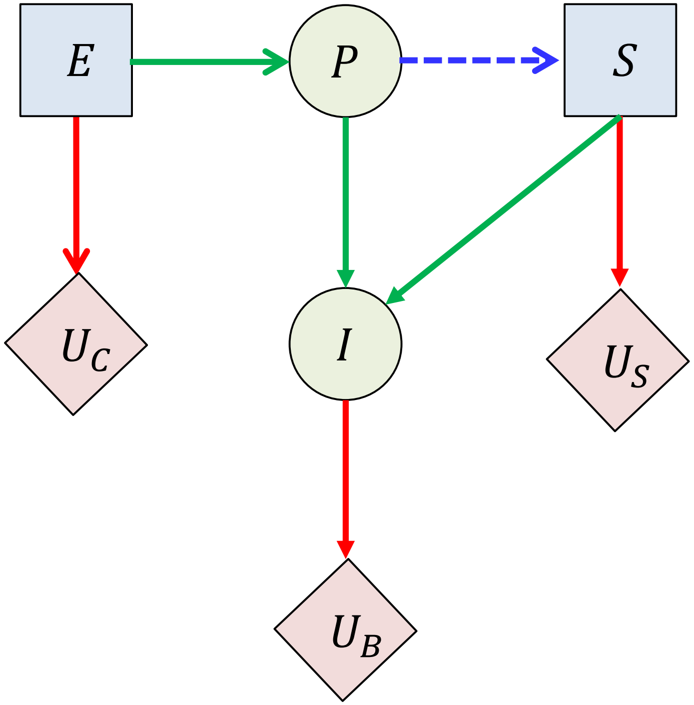
- $p(\text{low}|\text{start up}, \text{no prize}) = 0.1,\ \ p(\text{average}|\text{start up}, \text{no prize}) = 0.5,\ \ p(\text{hight}|\text{start up}, \text{no prize}) = 0.4$
- $p(\text{low}|\text{no start up}, \text{no prize}) = 0.2,\ \ p(\text{average}|\text{no start up}, \text{no prize}) = 0.6,\ \ p(\text{high}|\text{no start up}, \text{no prizeze}) = 0.2$
- $p(\text{low}|\text{start up}, \text{prize}) = 0.005,\ \ p(\text{average}|\text{start up}, \text{prize}) = 0.005,\ \ p(\text{hight}|\text{start up}, \text{prize}) = 0.99$
- $p(\text{low}|\text{no start up}, \text{prize}) = 0.05,\ \ p(\text{average}|\text{no start up}, \text{prize}) = 0.15,\ \ p(\text{high}|\text{no start up}, \text{prize}) = 0.8$

Example: PhD and start-up companies



Do PhD to wind a Novel Prize and start-up?

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- $U_B(\text{low}) = 100000,\ U_B(\text{average}) = 200000,\ U_B(\text{high}) = 500000$
- $U_S(\text{start up}) = -200000,\ U_S(\text{no start up}) = 0$

- Our interest is to advise whether or not it is desirable to take a PhD, bearing in mind that later one may or may not win the Novel Prize, and may or may not form a start-up company

- The expected optimal utility for any state E is

$$U(E) = \sum_P \max_S \sum_I p(I|S,P)p(P|E)[U_C(E) + U_B(I) + U_C(S)]$$

(where we assume that the optimal decisions are taken in the future)

$U(\text{do PhD}) = 190195$
$U(\text{no PhD}) = 240002$