

Lecture 3: Analyzing games

Solution concept

- A solution concept is a method of analyzing games with the objective of restricting the set of all possible outcomes to those that are more reasonable than others.
- A solution concept results in one or more strategy profiles, which we call equilibrium
- An equilibrium is prediction emerged by applying a solution concept to a target game

$$\text{equilibrium}_1 = f_1(G)$$

$$\text{equilibrium}_2 = f_2(G)$$

$$\text{equilibrium}_n = f_n(G)$$

f_1, \dots, f_n are solution concepts

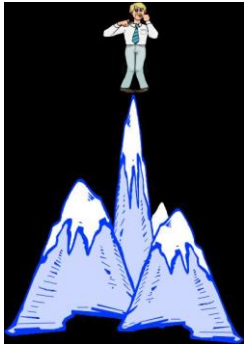
Assumptions and setup for analyzing game

- To set up the background for equilibrium analysis, it is useful to summarize the assumptions that we will be using in the lecture
 - **Players are “rational”**: A rational player is one who chooses his strategy $s_i \in S_i$, to maximize his payoff consistent with his beliefs about what is going on in the game
 - **Players are “intelligent”**: An intelligent player knows everything about the game: the actions, the outcomes, and the preferences of all the players.
 - **Common knowledge** : The fact that players are rational and intelligent is common knowledge among the players of the game
 - **Self-enforcing**: Any prediction (or equilibrium) of a solution concept must be self-enforcing
 - Core of our analysis and at the heart of non-cooperative game theory
 - Each player is in control of his own actions, and he will stick to an action only if he finds the action is in his best interest

Solution concepts

- Pareto optimality
- Nash equilibrium
- Maximin and minmax strategies
- Minimax regret
- Removal of dominated strategies
- Rationalizability
- Correlated equilibrium
- Trembling-hand perfect equilibrium
- Etc.

Analyzing games: from optimality to equilibrium



- **Single agent decision making:**

- Optimal strategy is one that maximizes the agent's expected utility for a given environment
- Uncertainties arose from stochastic environment, partially observable states, uncertain rewards, etc., which can be dealt with probability concepts.

$$a^* = \operatorname{argmax}_a E_s[u(a, s)]$$

- **Multiagents decision making:**



- The environment includes other agents, each of which tries to maximize its own utility
- Thus the notion of an optimal strategy for a given agent is not meaningful because *the best strategy depends on the choices of others*
- We need to identify certain subsets of outcomes, called solution concepts
- Two of the most fundamental solution concepts are
 - *Pareto optimality*
 - *Nash equilibrium*

$$\begin{aligned} u_1(a_1^*, a_2^*) &\geq u_1(a_1, a_2^*) \quad \forall a_1 \\ u_2(a_1^*, a_2^*) &\geq u_2(a_1^*, a_2) \quad \forall a_2 \end{aligned}$$

Pareto optimality

- We've defined some canonical games and thought about how to play them.
- Now let's examine the games from the outside:
 - From the point of view of an outside observer, can some outcomes of a game be said to be better than others?
 - Can we say that one agent's interests are more important than another's
 - Imagine trying to find the revenue-maximizing outcome when you don't know what currency is used to express each agent's payoff
 - Are there ways to still prefer one outcome to another?

Pareto optimality

Outcome of strategy s

Agent 1's utility : 10 unit of currency x

Agent 2's utility : 500 unit of currency y



Outcome of strategy s'

Agent 1's utility : 20 unit of currency x

Agent 2's utility : 10 unit of currency y



- Can we insist that the outcome of strategy s is better than that of strategy s' ?

Pareto optimality

Outcome of strategy s

Agent 1's utility : 10 unit of currency x
Agent 2's utility : 500 unit of currency y



Outcome of strategy s'

Agent 1's utility : 20 unit of currency x
Agent 2's utility : 10 unit of currency y



- Can we insist that the outcome of strategy s is better than that of strategy s' ?
 - No, because we cannot say that one agent's utility is more important than the other's

- Is there any situation that we can be sure that one outcome is better than another ?

Outcome of strategy s

Agent 1's utility : 10 unit of currency x
Agent 2's utility : 500 unit of currency y



Outcome of strategy s'

Agent 1's utility : 20 unit of currency x
Agent 2's utility : 1000 unit of currency y

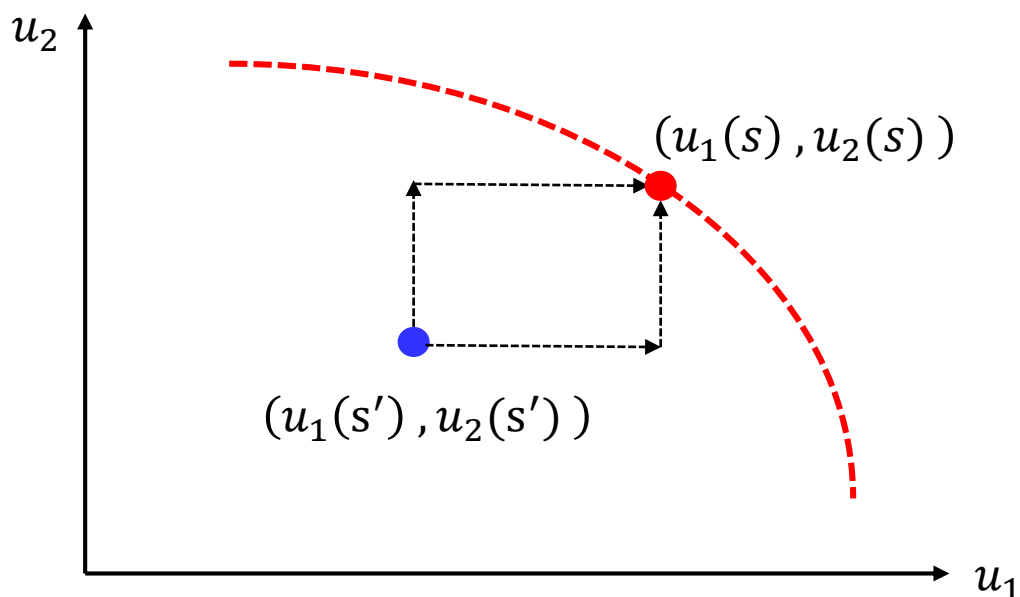


- The outcome of s' is always better than the outcome of s

Pareto optimality

Definition (Pareto domination)

Strategy profile s **Pareto dominates** strategy profile s' if for all $i \in N$, $u_i(s) \geq u_i(s')$, and there exists some $j \in N$ for which $u_j(s) > u_j(s')$.

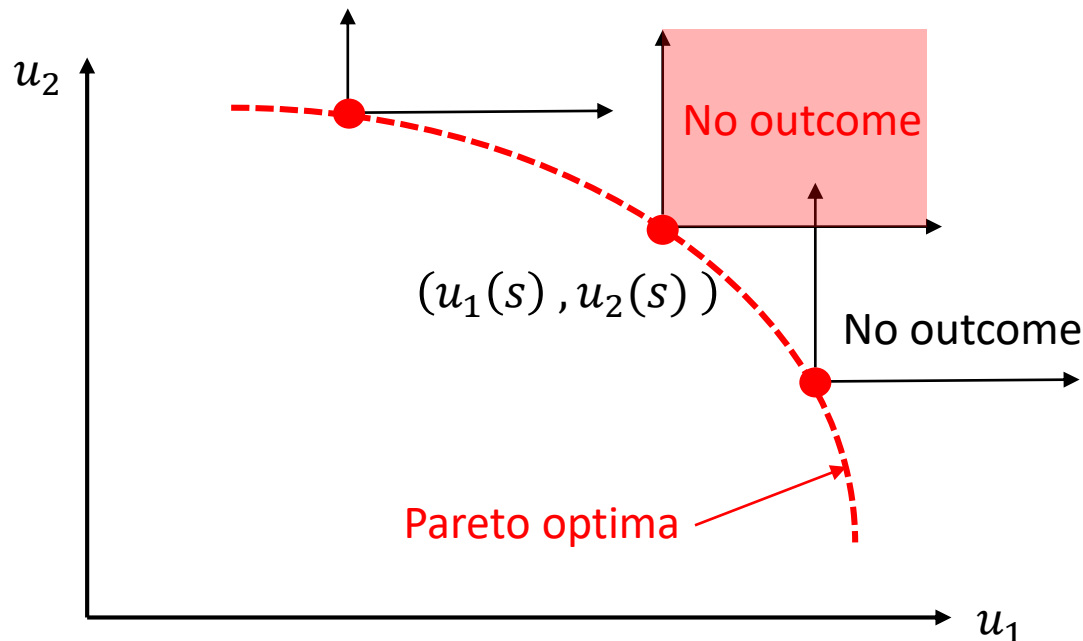


- In other words, in a **Pareto-dominated strategy profile** some players can be made better off without making any other player worse off
- We cannot generally identify a single “best” outcomes; instead we may have a set of non-comparable optima

Pareto optimality

Definition (Pareto optimality)

Strategy profile s is **Pareto optimal, or strictly Pareto efficient**, if there does not exist another strategy profile $s' \in S$ that Pareto dominates s

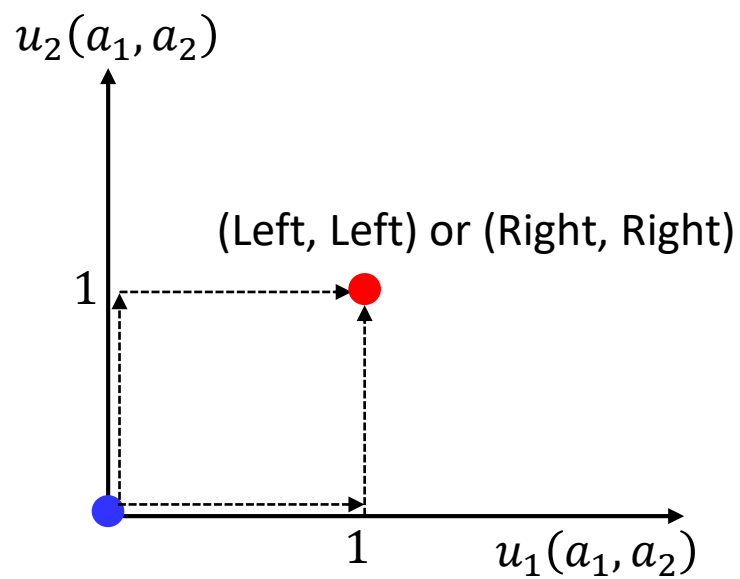


- Every game must have at least one Pareto optimal strategy profile, and there must always exist at least one such optimum in which all players adopt pure strategies.
- Some agent will have multiple optima
(for example, in zero-sum games, all strategy profiles are strictly Pareto efficient. **Why?**)

Pareto optimal outcomes in various games

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

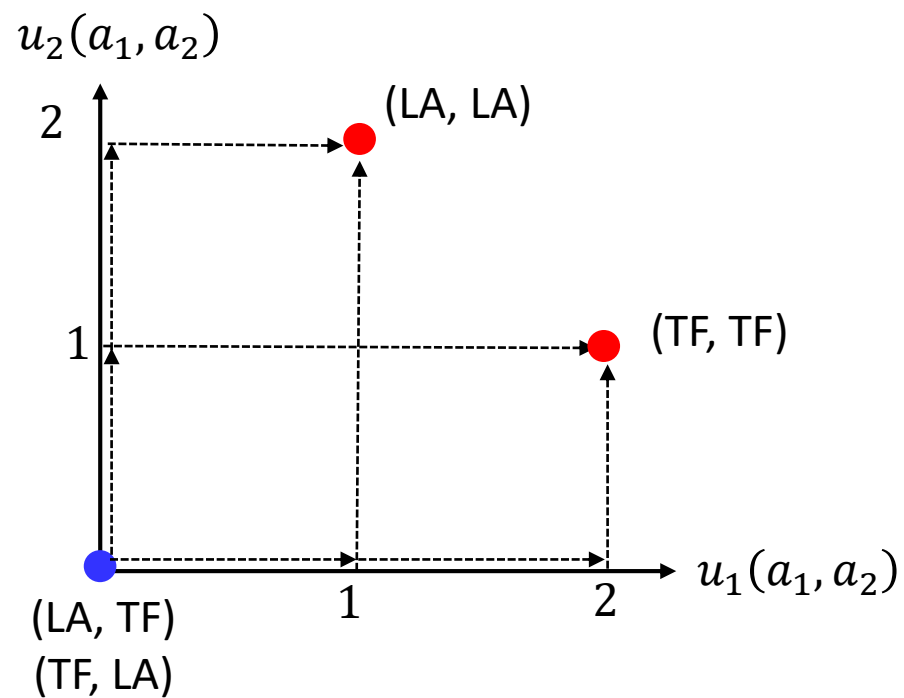
Coordination game



Pareto optimal outcomes in various games

	TF	LA
TF	2, 1	0, 0
LA	0, 0	1, 2

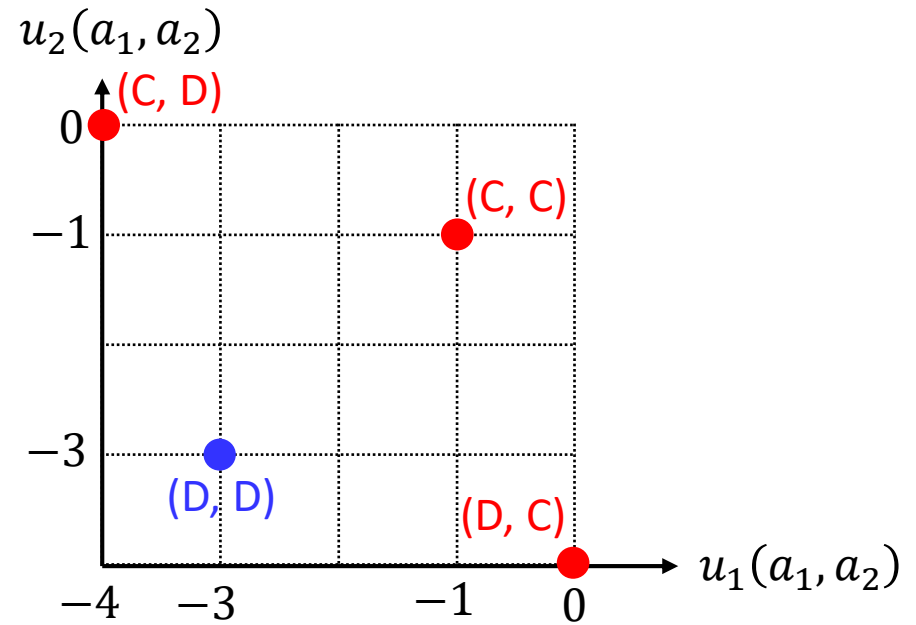
Battle of Sex game



Pareto optimal outcomes in various games

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

Prisoner's Dilemma game



Best response

- If you knew what everyone else was going to do, it would be easy to pick your own action
- Let $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ to be strategy profiles of other agents (all agents except i)
 - then, $s = (s_i, s_{-i})$

Definition (Best response)

Player i 's **best response** to the strategy profile s_{-i} is a mixed strategy $s_i^* \in S_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$

$$s_i^* \in BR(s_{-i})$$

- The best response is not necessarily unique
- Indeed, except in the extreme case in which there is a unique best response that is a pure strategy, the number of best responses is always infinite.
- When the support of a best response s_i^* includes two or more actions, any mixture of these actions must also be a best response
- If there are two pure strategies that are individually best responses, any mixture of the two is necessarily also a best response

Nash equilibrium

- Really, no agent knows what the others will do
- What can we say about which actions will occur ?

Definition (Nash Equilibrium)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i and for all strategies s_i , $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$.

- A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, **for all agents i** , s_i^* is a best response to s_{-i}^* , i.e., $s_i^* \in BR(s_{-i}^*)$
- A Nash equilibrium is a stable strategy profile:
 - no agent would want to change his strategy if he knew what strategies the other agents were following

Nash equilibrium

Definition (Strict Nash)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a strict Nash Equilibrium if, for all agents i and for all strategies $s_i \neq s_i^*$, $u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$.

Definition (Weak Nash)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a weak Nash Equilibrium if, for all agents i and for all strategies $s_i \neq s_i^*$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$, and s^* is not a strict Nash equilibrium.

- Mixed-strategy Nash equilibria are necessarily weak
- Pure-strategy Nash equilibria can be either strict or weak, depending on the game.

Nash equilibrium examples

Pure-strategy Nash equilibria in the Battle of the Sexes game

		Player 2	
		TF	LA
Player 1	TF	2, 1	0, 0
	LA	0, 0	1, 2

- We immediately see that it has two pure-strategy Nash equilibria

Nash equilibrium examples

Pure-strategy Nash equilibria in the Battle of the Sexes game

		Player 2	
		TF	LA
Player 1	TF	2, 1	0, 0
	LA	0, 0	1, 2

- We can check that these are Nash equilibria by confirming that whenever one of the players play the given (pure) strategy, the other player would only lose by deviating

$$a^* = (\text{TF}, \text{TF})$$
$$u_1(\text{TF}, \text{TF}) > u_1(\text{LA}, \text{TF})$$
$$u_2(\text{TF}, \text{TF}) > u_2(\text{TF}, \text{LA})$$

Nash equilibrium examples

Pure-strategy Nash equilibria in the Battle of the Sexes game

		Player 2	
		TF	LA
Player 1	TF	2, 1	0, 0
	LA	0, 0	1, 2

- We can check that these are Nash equilibria by confirming that whenever one of the players play the given (pure) strategy, the other player would only lose by deviating

$$a^* = (\text{LA}, \text{LA})$$
$$u_1(\text{LA}, \text{LA}) > u_1(\text{TF}, \text{LA})$$
$$u_2(\text{LA}, \text{LA}) > u_2(\text{LA}, \text{TF})$$

Nash equilibrium examples

How to easily find pure Nash equilibria?

		Player 2	
		TF	LA
Player 1	TF	2, 1	0, 0
	LA	0, 0	1, 2

Nash equilibrium examples

How to easily find pure Nash equilibria?

		Player 2	
		TF	LA
Player 1	TF	<u>2, 1</u>	0, 0
	LA	0, 0	<u>1, 2</u>

- **Step 1:** For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column

Nash equilibrium examples

How to easily find pure Nash equilibria?

		Player 2	
		TF	LA
Player 1	TF	<u>2, 1</u>	0, 0
	LA	0, 0	<u>1, 2</u>

- **Step 1:** For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column
- **Step 2:** For every row, which is strategy for player 1, **find the highest payoff entry for player 2** and over line the pair of payoffs

Nash equilibrium examples

How to easily find pure Nash equilibria?

		Player 2	
		TF	LA
Player 1	TF	<u>2, 1</u>	0, 0
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- **Step 1:** For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column
- **Step 2:** For every row, which is strategy for player 1, **find the highest payoff entry for player 2** and over line the pair of payoffs
- **Step 3:** If any matrix entry has both an under- and an over line, it is the outcome of a **Nash equilibrium in pure strategies**

Nash equilibrium examples

How to easily find pure Nash equilibria?

		Player 2		
		<i>L</i>	<i>C</i>	<i>R</i>
Player 1	<i>U</i>	7, 0	4, 2	1, 8
	<i>M</i>	2, 4	5, 5	2, 3
	<i>D</i>	8, 1	3, 2	0, 0

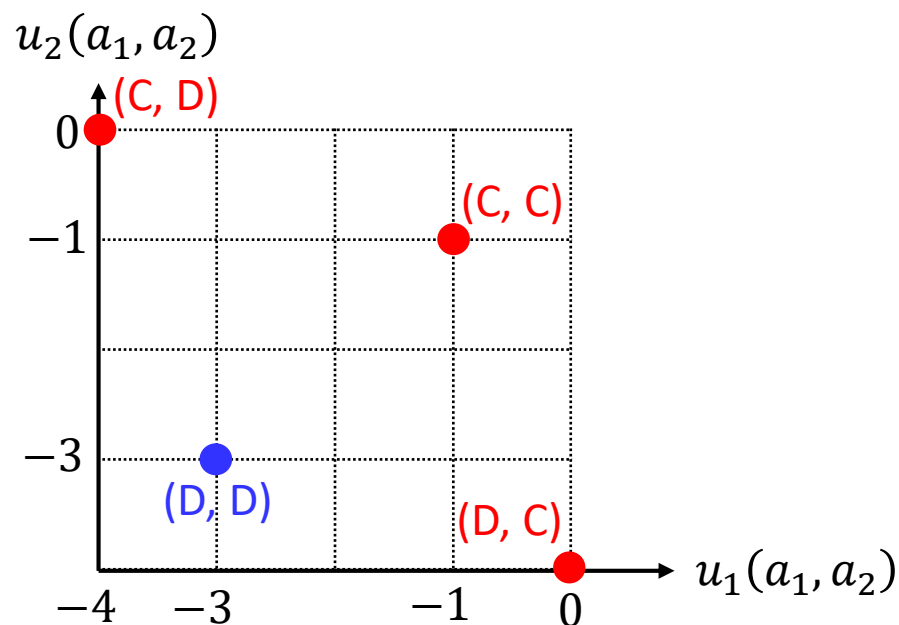
Find pure Nash equilibria by yourself

How many did you get?

Evaluating Nash equilibria solution

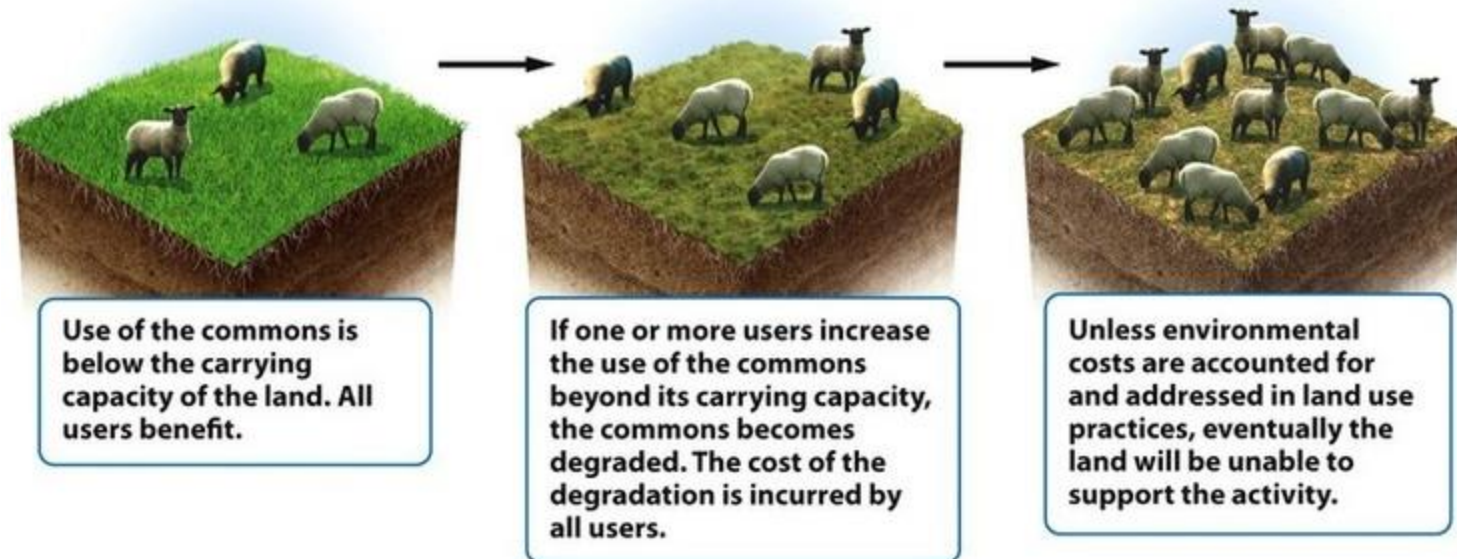
	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

Prisoner's Dilemma game



- As seen in Prisoner's Dilemma game, Nash equilibrium does not guarantee Pareto optimality
- People in many situations will do what is best for them, *at the expense of social efficiency*
- The solution concepts took the game as given, and they impose rationality and common knowledge of rationality to try to see what players would choose to do.
- If each player seeks to maximize their individual well-being then the players may hinder their ability to achieve socially optimal outcomes

Nash equilibrium examples : The Tragedy of the Commons



Nash equilibrium examples : The Tragedy of the Commons

- There are n players, say firms, in the world, each choosing how much to produce
- Their production activity in turn consumes some of the clean air that surrounds our planet
- There is a total amount of clean air equal to K , and any consumption of clean air comes out of this common resource
- Each player i chooses his own consumption of clean air for production, k_i
- The clean air left is $K - \sum_{i=1}^n k_i$
- The payoff for player i from the choice $k = (k_1, k_2, \dots, k_n)$ is equal to

$$u_i(k_i, k_{-i}) = \ln(k_i) + \ln\left(K - \sum_{j=1}^n k_j\right)$$

The benefit of consuming individual air consumption

The benefit of consuming the remainder of the clean air

Nash equilibrium examples : The Tragedy of the Commons

- To solve for a **Nash equilibrium**, we need to find some profile of choices $k^* = (k_1^*, k_2^*, \dots, k_n^*)$ for which $k_i^* = BR_i(k_{-i}^*)$ for all $i \in N$
- Then we have a system of n equations, one for each player's best-response function, with n unknowns, the choices of each player.
- For example, to get player i 's best-response function, the following first-order condition of his payoff function should be satisfied

$$\frac{\partial u_i(k_i, k_{-i})}{\partial k_i} = \frac{1}{k_i} - \frac{1}{K - \sum_{j=1}^n k_j} = 0$$

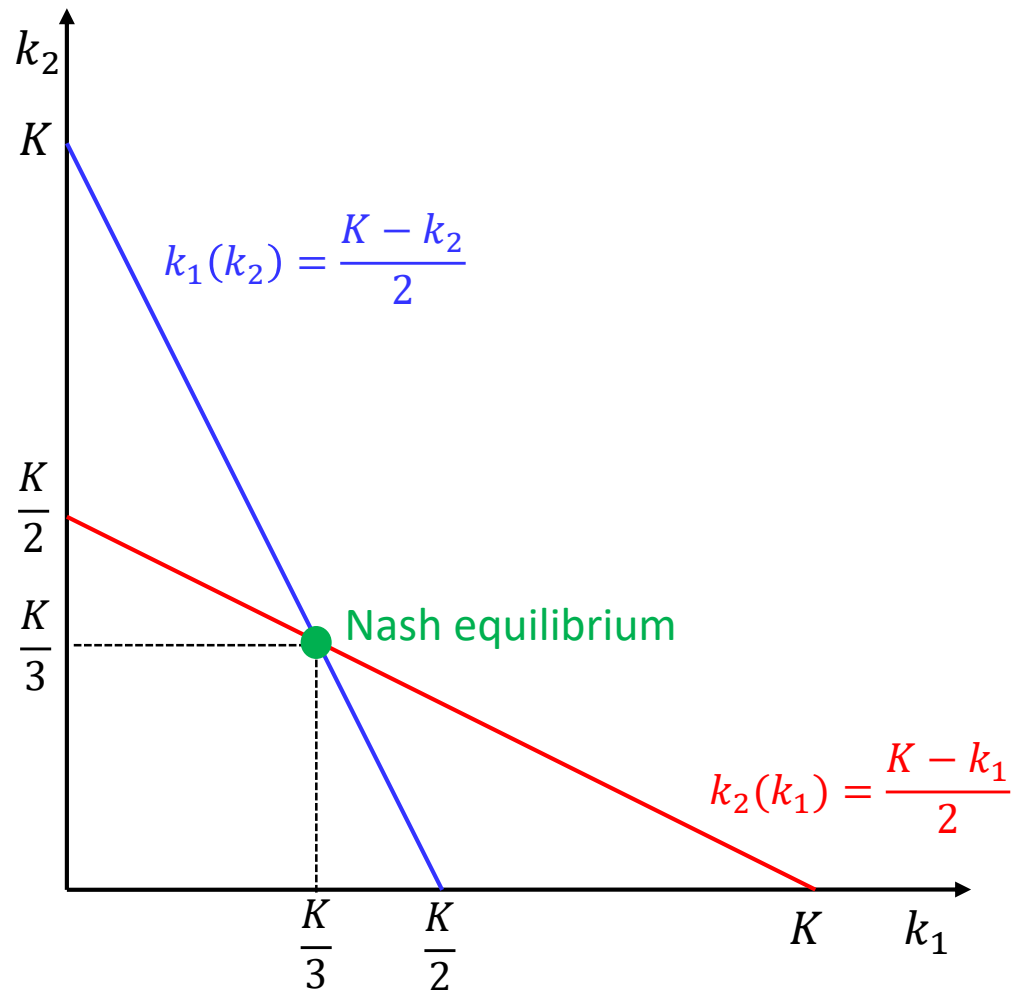
which gives player i 's best response function,

$$BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2}$$

Nash equilibrium examples : The Tragedy of the Commons

- In case there are two firms, we have two best-response equations:

$$k_1(k_2) = \frac{K - k_2}{2} \quad \text{and} \quad k_2(k_1) = \frac{K - k_1}{2}$$



Nash equilibrium examples : The Tragedy of the Commons

- Now we can ask whether this two-player behave to make the **society better**
 - Is consuming $K/3$ for each player too much or too little?
 - Can we find another consumption profile that will make **everyone better off**?
- We will maximize **the sum of all the payoff functions**, which we can think of as the “world’s payoff function $w(k_1, k_2)$ ”
- We can maximize

$$\max_{k_1, k_2} w(k_1, k_2) = \sum_{i=1}^2 u_i(k_1, k_2) = \sum_{i=1}^2 \left\{ \ln(k_i) + \ln \left(K - \sum_{j=1}^n k_j \right) \right\}$$

-
- The first-order conditions of this problem are

$$\frac{\partial w(k_1, k_2)}{\partial k_1} = \frac{1}{k_1} - \frac{2}{K - k_1 - k_2} = 0$$

$$\frac{\partial w(k_1, k_2)}{\partial k_2} = \frac{1}{k_2} - \frac{2}{K - k_1 - k_2} = 0$$

- The solution for this is $k_1 = k_2 = \frac{K}{4}$, that gives $u_1 = u_2 = \ln \frac{K}{4} + \ln \frac{K}{2} = \ln \left(\frac{K^2}{8} \right)$
 - which is larger than $u_1 = u_2 = \ln \frac{K}{3} + \ln \left(\frac{K}{3} \right) = \ln \left(\frac{K^2}{9} \right)$ for Nash equilibrium

Nash equilibrium examples: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good
- Firm i produce quantity q_i
- Cost for production is $c_i(q_i) = c_i q_i$
- Price is given by $d = a - b(q_1 + q_2)$
- The profit of company i given its opponent chooses quantity q_j is

$$u_i(q_i, q_j) = (a - bq_i - bq_j)q_i - c_i q_i = -bq_i^2 + (a - c_i)q_i - bq_j q_i$$

- The best-response function for each firm is given by the first-order condition

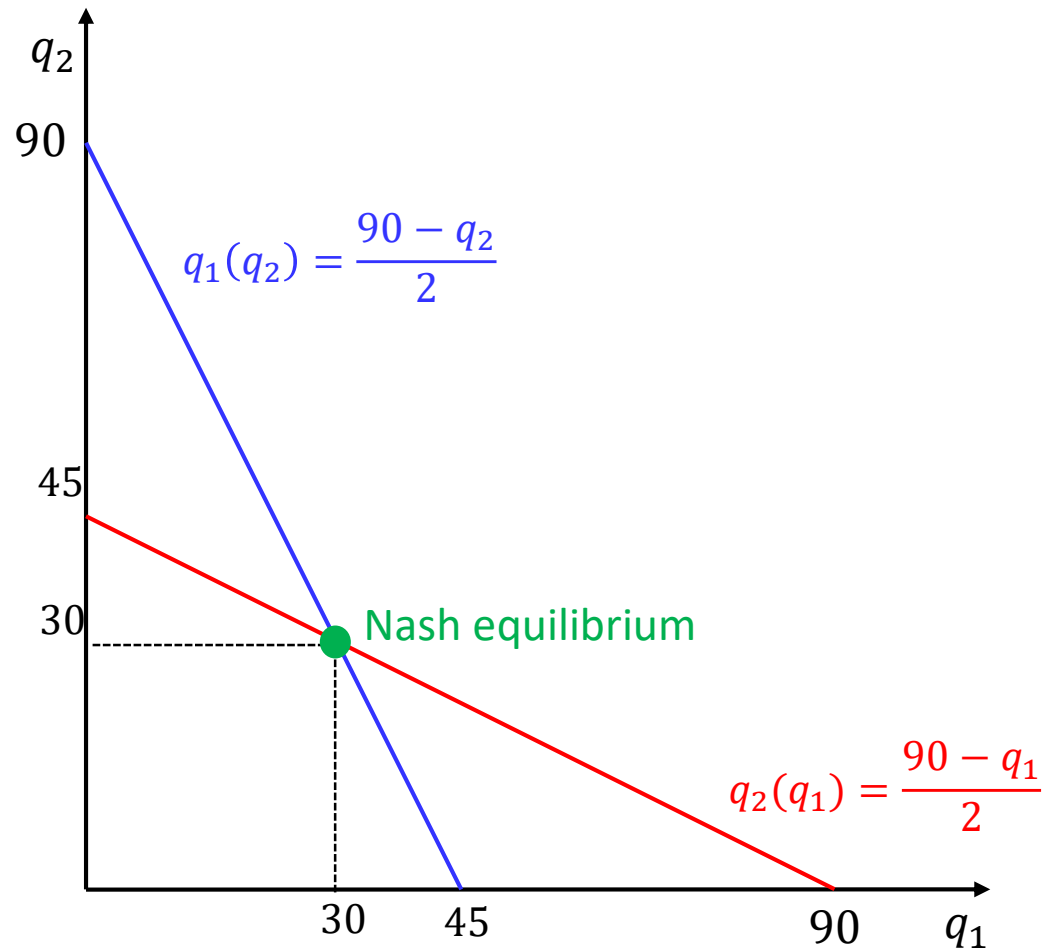
$$BR_i(q_j) = \frac{a - bq_j - c_i}{2b}$$

Nash equilibrium examples: Cournot Duopoly

- In case there are two firms, we have two best-response equations:

$$q_1 = \frac{a - bq_2 - c_1}{2b} \quad \text{and} \quad q_2 = \frac{a - bq_1 - c_2}{2b}$$

$$a = 100, b = 1, c_1 = c_2 = 10$$



Mixed strategy Nash equilibrium

- Why anyone would wish to randomize between actions?
- We will see mixed (stochastic) strategies turns out to be an important type of behavior to consider , with interesting implications and interpretations.
- No pure strategy Nash equilibria exists for the following Matching Pennies game

		Player 2	
		Heads	Tails
Player 1	Heads	<u>1, -1</u>	<u>-1, 1</u>
	Tails	<u>-1, 1</u>	<u>1, -1</u>

- Nash equilibrium will indeed exist if we allow players to choose random strategies

Revisit: mixed strategy

Definition (Mixed strategy)

Let (N, A, u) be a normal-form game, and for any set X let $\Pi(X)$ be the set of all probability distributions over X . Then, the set of mixed strategies for player i is $S_i = \Pi(A_i)$

Definition (Mixed strategy profile)

The set of mixed-strategy profile is simply the Cartesian product of the individual mixed-strategy sets, $S = S_1 \times \dots \times S_n$.

- $s_i(a_j)$ denote the probability that an action a_j will be played under mixed strategy s_i
 - For example, $A = \{\text{Rock, Paper, Scissors}\}$, $s_i(R) = 0.2$, $s_i(P) = 0.3$, $s_i(S) = 0.5$

Definition (Support)

The support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$

$$A_1 = \{L, R\}$$

$$S_1 = \Pi(A_1) = \{(s_1(L), s_1(R)) : s_1(L), s_1(R) \geq 0, s_1(L) + s_1(R) = 1\}$$

$$s_1 \in S_1, \text{ i. e., } s_1 = (q, 1 - q)$$

Beliefs and mixed strategies

- Introducing probability distributions not only enriches the set of actions from which a player can choose but also allows us to **enrich the beliefs that players can have**

Definition (Belief)

A belief for player i is given by a probability distribution $\pi_i \in \Pi(A_{-i})$ over the actions of his opponents. We denote by $\pi_i(a_{-i})$ the probability player i assigns to his opponents playing $a_{-i} \in A_{-i}$

How to find mixed strategy Nash equilibrium?

Definition (Nash Equilibrium)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i and for all strategies $s_i \in \Pi(A_i)$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$.

- A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, **for all agents i , s_i^* is a best response to s_{-i}^* , i.e., $s_i^* \in BR(s_{-i}^*)$**
- We can think of s_{-i}^* as the belief of player i about his opponents, π_i , **which captures the idea that player i is uncertain of his opponent's behavior**
 - **The profile of mixed strategies s_{-i}^*** thus captures this uncertain belief over all of the pure strategies that player i 's opponent can play
 - Rationality requires that a player play a best response given his belief (Nash equilibrium requires that these beliefs are correct, i.e., a system of equations should be satisfied)

How to find mixed strategy Nash equilibrium?

- Recall that the support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$
- Imagine that the Nash equilibrium profile s_i^* contains more than one pure strategy -say a_i and a'_i as supports.
- What must we conclude about a rational player i if s_i^* is indeed part of a Nash equilibrium (s_i^*, s_{-i}^*) ?

How to find mixed strategy Nash equilibrium?

- Recall that the support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$
- Imagine that the Nash equilibrium profile s_i^* contains more than one pure strategy -say a_i and a'_i as supports.
- What must we conclude about a rational player i if s_i^* is indeed part of a Nash equilibrium (s_i^*, s_{-i}^*) ?

if s_i^* is a Nash equilibrium, and if a_i and a'_i are in the support of s_i^* , then

$$u_i(a_i, s_{-i}^*) = u_i(a'_i, s_{-i}^*) = u_i(s_i^*, s_{-i}^*)$$

Proof:

- assume $u_i(a_i, s_{-i}^*) > u_i(a'_i, s_{-i}^*)$ and a_i and a'_i are support of s_i^*
- Adjusting the mixed strategy $s_i = \{s_i(a_i), s_i(a'_i)\} \rightarrow \{s_i(a_i) + s_i(a'_i), 0\}$ will increase u_i
- s_i^* could not have been a best response to s_{-i}^*
- Therefore, by contradiction, $u_i(a_i, s_{-i}^*) = u_i(a'_i, s_{-i}^*)$

How to find mixed strategy Nash equilibrium?

if s_i^* is a Nash equilibrium, and both if a_i and if a'_i are in the support of s_i^* , then

$$u_i(a_i, s_{-i}^*) = u_i(a'_i, s_{-i}^*) = u_i(s_i^*, s_{-i}^*)$$

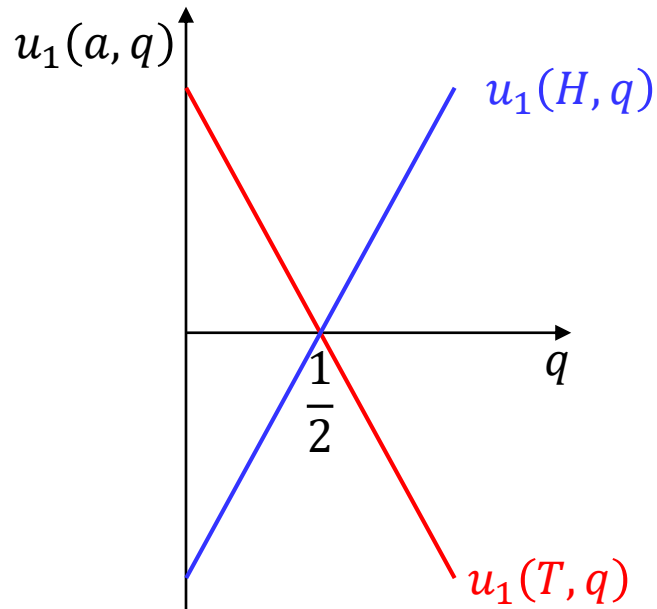
- This result will play an important role in computing mixed-strategy Nash equilibria
 - If a player is playing a mixed strategy then he must be indifferent between the actions he is choosing with positive probability (i.e., actions in the support)
- One player's indifference will impose restrictions on the behavior of other players
 - This restriction will help us find the mixed-strategy Nash equilibrium

Finding mixed Nash equilibria : Matching Pennies

		q H	$1 - q$ T
p H	H	1, -1	-1, 1
	T	-1, 1	1, -1
$(1 - p)$		H	T

$$u_1(H, q) = q \times 1 + (1 - q) \times (-1) = 2q - 1$$

$$u_1(T, q) = q \times (-1) + (1 - q) \times 1 = 1 - 2q$$



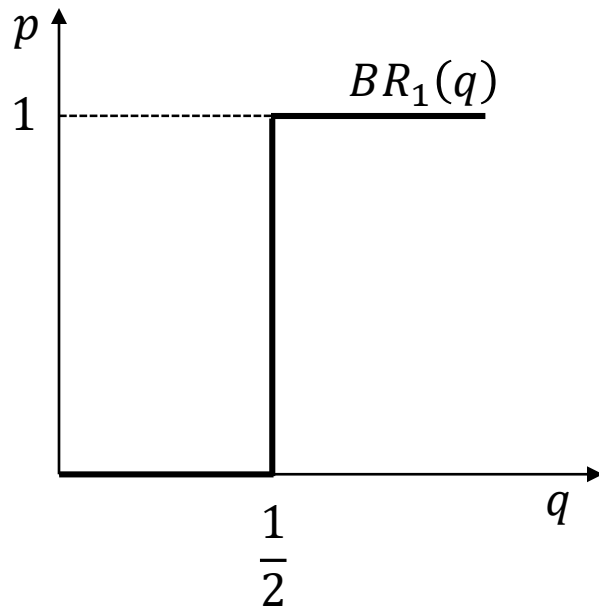
$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < 1/2 & \text{(Playing T)} \\ p \in [0, 1] & \text{if } q = 1/2 & \text{(Indifferent)} \\ p = 1 & \text{if } q > 1/2 & \text{(Playing H)} \end{cases}$$

Finding mixed Nash equilibria : Matching Pennies

		q H	$1 - q$ T
p H	H	1, -1	-1, 1
	T	-1, 1	1, -1
		$(1 - p)$ T	

$$u_1(H, q) = q \times 1 + (1 - q) \times (-1) = 2q - 1$$

$$u_1(T, q) = q \times (-1) + (1 - q) \times 1 = 1 - 2q$$



$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < 1/2 & \text{(Playing T)} \\ p \in [0, 1] & \text{if } q = 1/2 & \text{(Indifferent)} \\ p = 1 & \text{if } q > 1/2 & \text{(Playing H)} \end{cases}$$

Finding mixed Nash equilibria : Matching Pennies

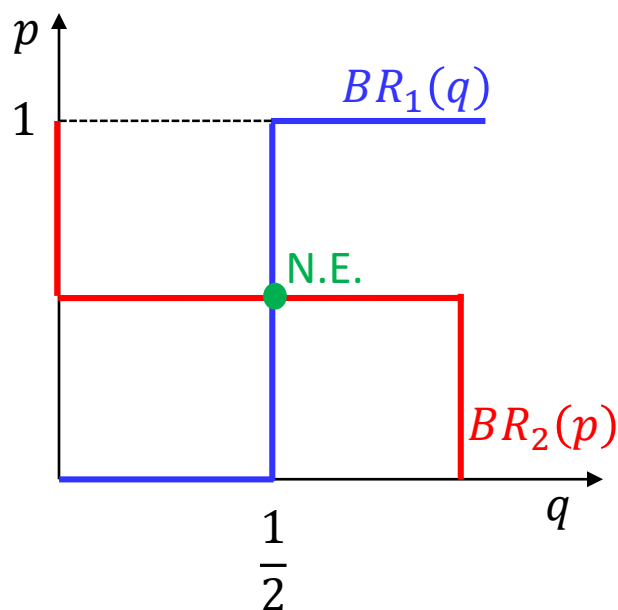
		q H	$1 - q$ T
p	H	1, -1	-1, 1
	T	-1, 1	1, -1

$$u_1(H, q) = q \times 1 + (1 - q) \times (-1) = 2q - 1$$

$$u_1(T, q) = q \times (-1) + (1 - q) \times 1 = 1 - 2q$$

$$u_2(H, p) = p \times (-1) + (1 - p) \times 1 = 1 - 2p$$

$$u_2(T, p) = p \times 1 + (1 - p) \times (-1) = 2p - 1$$



$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < 1/2 & \text{(Playing T)} \\ p \in [0,1] & \text{if } q = 1/2 & \text{(Indifferent)} \\ p = 1 & \text{if } q > 1/2 & \text{(Playing H)} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < 1/2 & \text{(Playing H)} \\ q \in [0,1] & \text{if } p = 1/2 & \text{(Indifferent)} \\ q = 0 & \text{if } p > 1/2 & \text{(Playing T)} \end{cases}$$

The intersections of two best response curve \rightarrow Nash equilibria

To find Nash equilibrium, make other player indifferent between some of his pure actions

Finding mixed Nash equilibria : Matching Pennies

Mixed-strategy Nash equilibria in the Matching Pennies game

		Player 2	
		p Heads	$1 - p$ Tails
Player 1	q Heads	1, -1	-1, 1
	$1 - q$ Tails	-1, 1	1, -1

No pure strategy Nash equilibria exists

- Player 1 should randomize his action to make Player 2 to be indifferent between her actions
→ otherwise, player 2 would play the action that is superior than the other
- Let us assume Player 1' strategy is to play Heads with probability q and Tails with $1 - q$

$$\begin{aligned}u_2(\text{Heads}) &= u_2(\text{Tails}) \\ -1 \times q + 1 \times (1 - q) &= 1 \times q - 1 \times (1 - q) \\ q &= \frac{1}{2}\end{aligned}$$

Finding mixed Nash equilibria : Matching Pennies

Mixed-strategy Nash equilibria in the Matching Pennies game

		Player 2	
		p Heads	$1 - p$ Tails
Player 1	q Heads	1, -1	-1, 1
	$1 - q$ Tails	-1, 1	1, -1

No pure strategy Nash equilibria exists

- Player 2 should randomize his action to make Player 1 to be indifferent between her actions
→ otherwise, player 1 would play the action that is superior than the other
- Let us assume Player 2' strategy is to play Heads with probability p and Tails with $1 - p$

$$\begin{aligned}u_1(\text{Heads}) &= u_1(\text{Tails}) \\1 \times p - 1 \times (1 - p) &= -1 \times p + 1 \times (1 - p) \\p &= \frac{1}{2}\end{aligned}$$

Finding Nash equilibria

Mixed-strategy Nash equilibria in the **Battle of the Sexes** game

		Player 2	
		p TF	$1 - p$ LA
Player 1	q TF	2, 1	0, 0
	$1 - q$ LA	0, 0	1, 2

- Player 1 should randomize his action to make Player 2 to be indifferent between her actions
→ otherwise, player 2 would play the action that is superior than the other
- Let us assume Player 1' strategy is to play TF with probability q and LA with $1 - q$

$$\begin{aligned}u_2(\text{TF}) &= u_2(\text{LA}) \\1 \times q + 0 \times (1 - q) &= 0 \times q + 2 \times (1 - q) \\q &= \frac{2}{3}\end{aligned}$$

Finding Nash equilibria

Mixed-strategy Nash equilibria in the **Battle of the Sexes** game

		Player 2	
		p TF	$1 - p$ LA
Player 1	q TF	2, 1	0, 0
	$1 - q$ LA	0, 0	1, 2

- Player 2 should randomize his action to make Player 1 to be indifferent between her actions
→ otherwise, player 1 would play the action that is superior than the other
- Let us assume Player 2' strategy is to play TF with probability p and LA with $1 - p$

$$\begin{aligned}u_1(\text{LA}) &= u_1(\text{TF}) \\2 \times p + 0 \times (1 - p) &= 0 \times p + 1 \times (1 - p) \\p &= \frac{1}{3}\end{aligned}$$

Finding Nash equilibria

Mixed-strategy Nash equilibria in the **Battle of the Sexes** game

		Player 2	
		1/3 TF	2/3 LA
Player 1	2/3 TF	2, 1	0, 0
	1/3 LA	0, 0	1, 2

- Now, we can confirm that we have indeed found an equilibrium:
 - Both players play in a way that makes the other indifferent, they are both best responding to each other
- Expected payoff for both agents is 2/3 in this equilibrium
 - Each of the pure-strategy equilibria Pareto-dominates the mixed strategy equilibrium
- This mixed strategy, as all other mixed strategies, is a weak Nash equilibrium**

$$u_1(s_i^*, s_{-i}^*) \geq u_1(s_i, s_{-i}^*) \quad u_1\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right) \geq u_1\left((x, 1-x), \left(\frac{1}{3}, \frac{2}{3}\right)\right) \text{ For any } 0 \leq x \leq 1$$

Finding Nash equilibria

Mixed-strategy Nash equilibria in the Rock-Paper-Scissor game

	p_R Rock	p_P Paper	p_S Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

$$u_1(R) = u_1(P) = u_1(S)$$

$$\Rightarrow 0p_R + (-1)p_P + 1p_S = 1p_R + 0p_P + (-1)p_S = -1p_R + 1p_P + 0p_S$$

$$\Rightarrow 0p_R + (-1)p_P + 1p_S = 1p_R + 0p_P + (-1)p_S \Rightarrow 2p_S = p_R + p_P$$

$$\Rightarrow 1p_R + 0p_P + (-1)p_S = -1p_R + 1p_P + 0p_S \Rightarrow 2p_R = p_S + p_P$$

$$\Rightarrow p_R = p_P = p_S \quad (1)$$

$$p_R + p_P + p_S = 1 \quad (2)$$

- Due to (1) and (2), $p_R = p_P = p_S = 1/3$ (Mixed strategy Nash Equilibrium)

Multiple mixed strategies

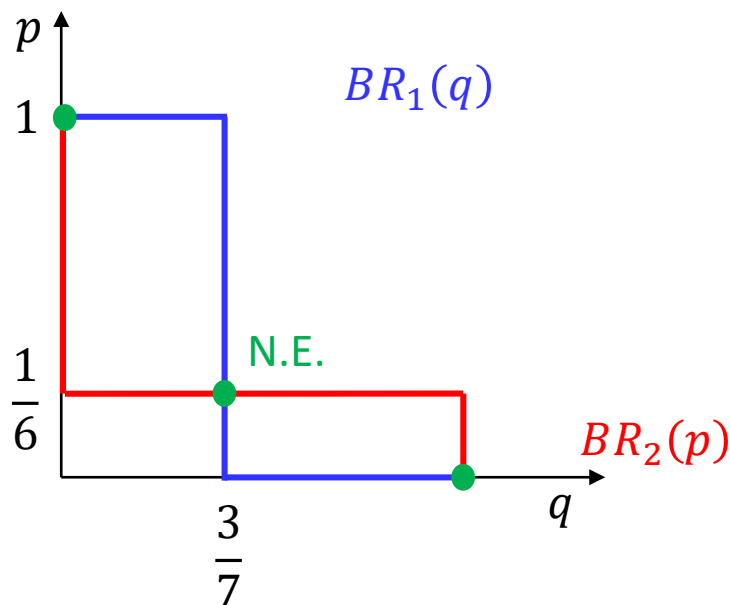
		q H	$1 - q$ T
p	H	0, 0	3, 5
	T	4, 4	0, 3

$$u_1(H, q) = q \times 0 + (1 - q) \times (3) = 3q - 3$$

$$u_1(T, q) = q \times (4) + (1 - q) \times 0 = 4q$$

$$u_2(H, p) = p \times (0) + (1 - p) \times 4 = 4 - 4p$$

$$u_2(T, p) = p \times 5 + (1 - p) \times (3) = 2p + 3$$



Nash equilibriums are $\left\{ (1, 0), \left(\frac{1}{6}, \frac{3}{7} \right), (0, 1) \right\}$

The meaning of playing mixed-strategy

- Randomize to confuse your opponent
 - consider the matching pennies example
- Randomize when uncertain about the other's action
 - consider battle of the sexes
- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit
- Mixed strategies describe population dynamics:
 - agents chosen from a population have deterministic strategies.
 - Mixed strategies gives the probability of getting each pure strategies.

The existence of Nash equilibria

Theorem (Nash, 1951)

Every game with a finite number of players and action profiles has at least one Nash equilibrium