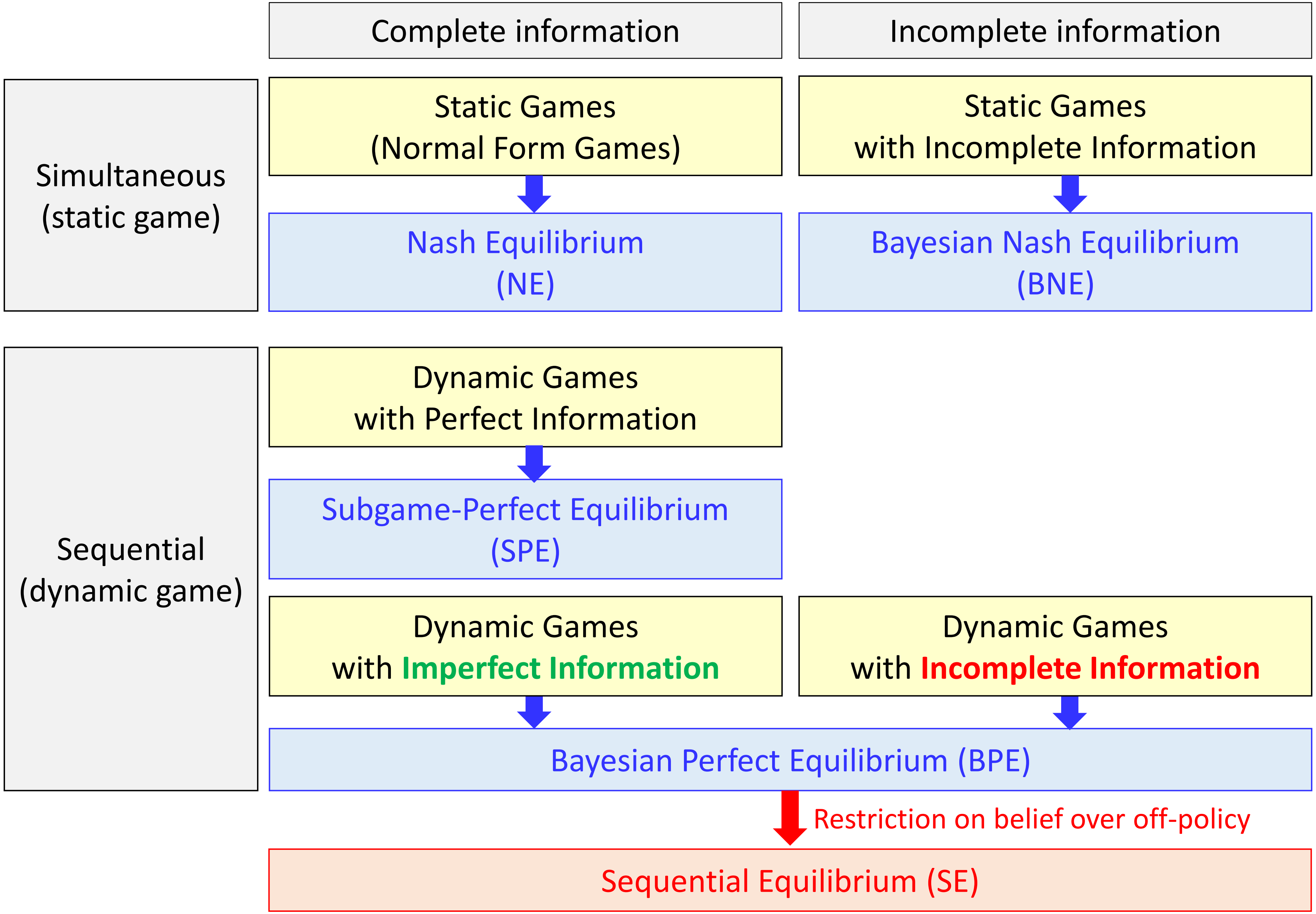
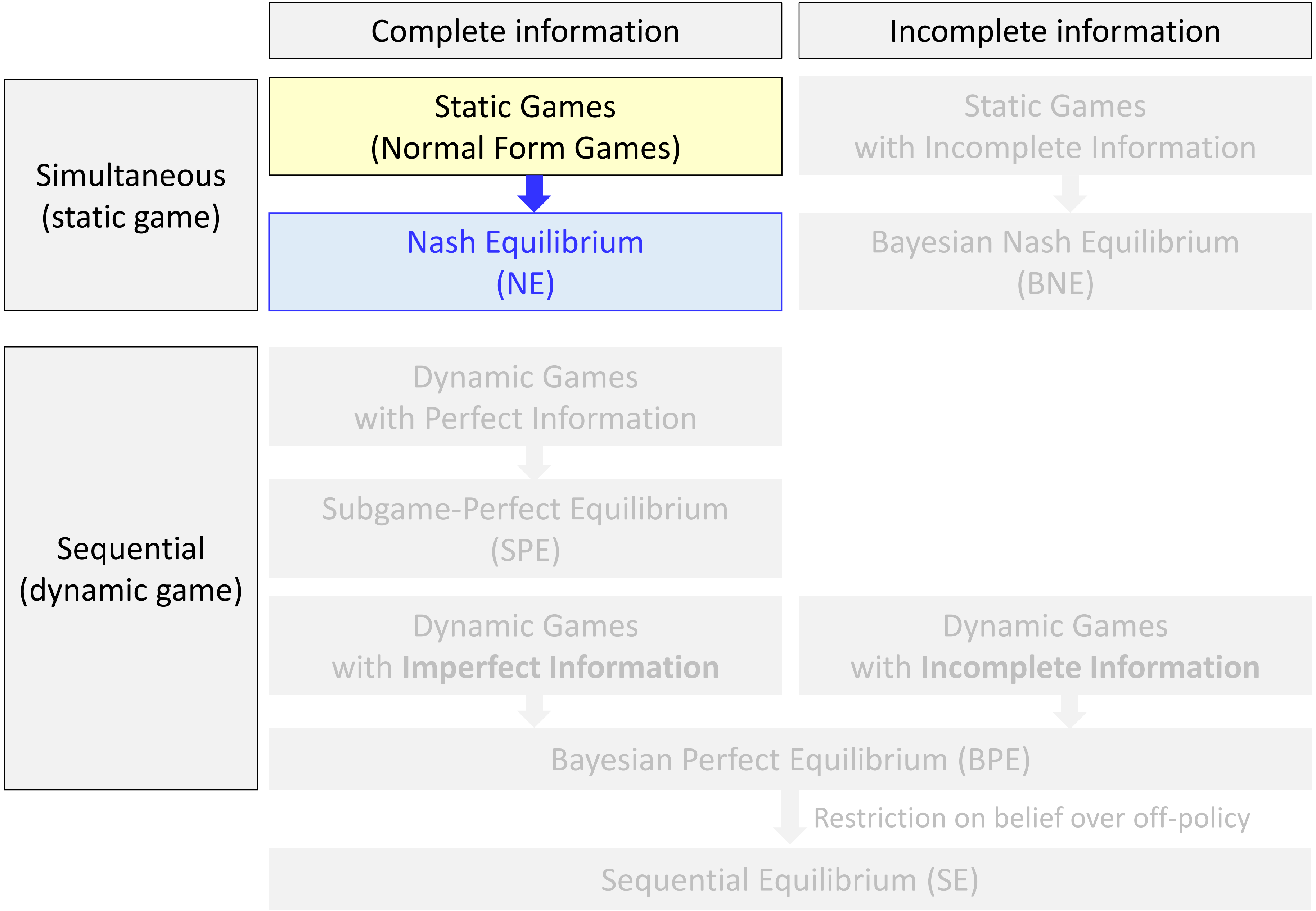
1. Static Games with Complete Information

Introduction



Introduction



Games in Normal Form

Self interested agents

What does it mean to say what an agent is self-interested?

- It does not necessarily mean that they want to cause harm to each other
- It does not necessarily mean that they care only about themselves
- It means that each agent has his own description of which sates of the world he likes, and acts based on this description

The dominant approach to modeling an agent's interest is utility theory

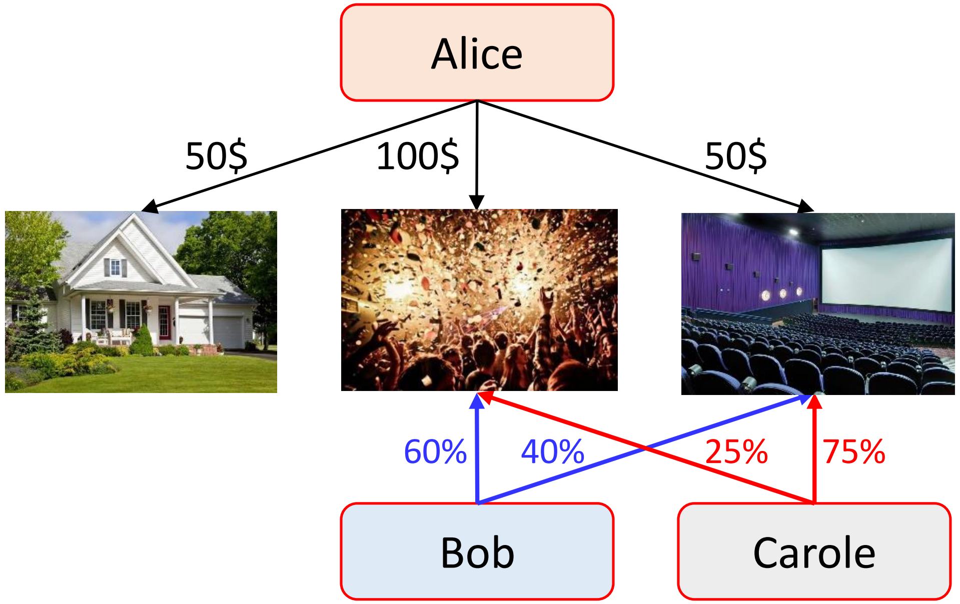
- Aims to quantify an agent's degree of preference across a set of available alternatives
- Aims to understand how these preferences change when an agent faces uncertainty about which alternatives he will receive
- We assume that agents actions are consistent with utility –theoretic assumptions.

A utility function is a mapping from states of the world to real numbers

- These numbers are interpreted as agent's level of happiness in the given states
- Confronting uncertainties, utility is defined as the expected value of his utility function with respect to the appropriate probability distribution over states

Utility function can be used as a basis for making decision

- Agents simply need to choose the course of action that maximizes expected utility
- When the world contains two or more utility-maximizing agents whose sections can affect each others' utility, things become complicated



Alice Hates Bob

Alice likes Carole

- When Alice see Bob in the club, utility will be decreased to 10
- When Alice see Bob in the movie, utility decreases to 10
- When Alice see Carole in any place, her utility will be increase by a factor 1.5

Utility function can be used as a basis for making decision

$A = h$

	$B = c$	$B = m$
$C = c$	50	50
$C = m$	50	50

$A = c$

	$B = c$	$B = m$
$C = c$	15	150
$C = m$	10	100

$A = m$

	$B = c$	$B = m$
$C = c$	50	10
$C = m$	75	15

- It will be easier to determine Alice's best course of action if we list Alice's utility for each possible state of the world.
- There are 12 outcomes that can occur: Bob and Carol can each be in either the club or the movie theater, and Alice can be in the club, the movie theater, or at home.
- Alice has a baseline level of utility for each of her three actions, and this baseline is adjusted if either Bob, Carol, or both are present.
- Following the description of our example, we see that Alice's utility is always 50 when at home

Utility function can be used as a basis for making decision

	0.6 $B=c$	0.4 $B=m$
0.25 $C=c$	50	50
0.75 $C=m$	50	50

$A=h$

	0.6 $B=c$	0.4 $B=m$
0.25 $C=c$	15	150
0.75 $C=m$	10	100

$A=c$

	0.6 $B=c$	0.4 $B=m$
0.25 $C=c$	50	10
0.75 $C=m$	75	15

$A=m$

- So how should Alice choose among her three activities?
- To answer this question we need to combine her utility function with her knowledge of Bob and Carol's randomized entertainment habits.
  - $\bar{u}(A=h)=50$
  - $\bar{u}(A=c)=0.25(0.6\cdot 15\ +\ 0.4\cdot 150)\ +\ 0.75(0.6\cdot 10\ +\ 0.4\cdot 100)\ =\ 51.75.$
  - $\bar{u}(A=m)=0.25(0.6\cdot 50\ +\ 0.4\cdot 10)\ +\ 0.75(0.6(75)\ +\ 0.4(15))\ =\ 46.75.$
- Thus, Alice prefers to go to the club (even though Bob is often there and Carol rarely is) and prefers staying home to going to the movies

Preferences and utility

- why should a single-dimensional function be enough to explain preferences over an arbitrarily complicated set of alternatives?

- why should an agent's response to uncertainty be captured purely by the expected value of his utility function, rather than also depending on other properties of the distribution such as its standard deviation or number of modes?

- Utility theorists respond to such questions by showing that the idea of utility can be grounded in a more basic concept of preferences.

- We need a way to talk about how preferences interact with uncertainty about which outcome will be selected.

- In utility theory this is achieved through the concept of lottery.
  - A lottery is the random selection of one of a set of outcomes according to specified probabilities
  - Formally, a lottery is a probability distribution over outcomes written

$$[p_1 : o_1, \ldots, p_k : o_k], \text{ where each } o_i \in O, \text{ each } p_i > 0 \text{ and } \sum_{i=1}^{k} p_i = 1$$

Axioms of utility theory

- **Axiom 1 (Completeness)** $\forall\, o_1, o_2,\ \ o_1 \succ o_2\ \ or\ \ o_2 \succ o_1\ \ or\ \ o_1 \sim o_2$

- **Axiom 2 (Transitivity)** $if\ \ o_1 \succcurlyeq o_2\ and\ \ o_2 \succcurlyeq o_3,\ \ then\ \ o_1 \succcurlyeq o_3$

- **Axiom 3 (Substitutability)**
$if\ \ o_1 \sim o_2\ then\ for\ all\ sequences\ of\ one\ or\ more\ outcomes\ o_3, \dots, o_k\ and\ sets\ of$
$probabilities\ p, p_3, \dots, p_k\ for\ which\ p + \sum_{i=3}^{k} p_i = 1,$
$$[p: o_1, p_3: o_3, \dots, p_k: o_k] \sim [p: o_2, p_3: o_3, \dots, p_k: o_k]$$

- **Axiom 4 (Decomposability)** $if\ \forall o_i \in O,\ \ P_{l_1}(o_i) = P_{l_2}(o_i)\ then\ l_1 \sim l_2.$

- **Axiom 5 (Monotonicity)** $if\ o_1 \succ o_2\ \ and\ \ p \succ q\ then\ [p: o_1, 1-p: o_2] \succ [q: o_1, 1-q: o_2]$

- **Lemma 1** $if\ a\ preference\ relation\ \ \succcurlyeq$
$satisfies\ the\ axioms\ completeness, transitivity, decomposabiity, and\ monotoniciy,$
$and\ if\ o_1 \succ o_2\ \ and\ o_2 \succ o_3,\ \ then\ there\ eixists\ some\ probability\ p\ such\ that\ for\ all$
$p' < p,\ \ o_2 \succ [p': o_1, 1-p': o_3], and\ for\ all\ p'' > p, [p'': o_1, 1-p'': o_3] \succ o_2.$

- **Axiom 6 (Continuity)** $if\ o_1 \succ o_2\ \ and\ o_2 \succ o_3,\ \ then\ \exists p \in [0,1]\ such\ that$
$o_2 \sim [p: o_1, 1-p: o_3].$

Axioms of utility theory

Theorem (Von Neumann and Morgenstern, 1944)

$if\ a\ preference\ relation \succcurlyeq satisfies\ the\ axioms\ completeness, transitivity,$
$substiutability, decomposabiity, monotonicity, and\ continuity, then\ there\ exists$
$a\ function\ u: O \longmapsto [0,1]\ with\ the\ properties\ that$

1. $u(o_1) \geq u(o_2)\ iff\ o_1 \succcurlyeq o_2, and$
2. $u([p_1: o_1, \dots, p_k: o_k]) = \sum_{i=1}^{k} p_i u(o_i)$

Key ingredients of a game

Players: who are the decision makers?

- People? Robots? Governments? Companies? Employees?

Actions: what can the players do?

- Enter a bid in an auction? Decide whether to start up a company? Decide when to buy car? Decide to sell a stock? Decide how to vote?

Payoffs: what motivates players?

- Do they care about some profit? Do they care about other players?

Defining Games – Two standard representations

- Normal Form (a.k.a. Matrix Form, Strategic Form) List what payoffs get as a function of their actions
  - It is as if players moved simultaneously
  - But strategies encode many things...

- Extensive Form Includes timing of moves (later in course)
  - Players move sequentially, represented as a tree
  - Chess: white player moves, then black player can see white's move and react...
  - Keeps track of what each player knows when he or she makes each decision
  - Poker: bet sequentially – what can a given player see when they bet?

- Above classification is not based on the type of game, but on the way how we represent it!

- Because most other representations of interest can be reduced to it, the normal – form representation is arguably the most fundamental in game theory

Definition (Normal-form game)

Finite, n-person normal form game: $\mathrm{G}(N, A, u)$:

- $N = \{1, \dots, n\}$ is a finite set of n, indexed by i
- $A = A_1 \times \dots \times A_n$, where A_i is a finite set of actions available to plyer i
  - Each vector $a = (a_1, \dots, a_n) \in A$ is an action profile
- $u = (u_1, \dots, u_n)$, where $u_i: A \mapsto \mathbb{R}$ is real-valued utility (or payoff) function for player i

Normal form game – the standard matrix representation

- Writing a 2-player game as a matrix:
  - "row" player is player 1,
  - "column" player is player 2
  - rows correspond to actions $a_1 \in A_1$
  - columns correspond to actions $a_2 \in A_2$
  - cells listing utility or payoff values for each player:
  (the row player first, then the column)

- Prisoner's Dilemma game can be represented as

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

- $N = \{1, 2\}$
- $A = A_1 \times A_2$, where $A_1, A_2 = \{C, D\}$
- $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$ are given as numbers

C: Cooperate D: Defect

Prisoner's Dilemma

	C	D
C	a, a	b, c
D	c, b	d, d

- Any $c > a > d > b$ defines an instance of Prisoner's Dilemma
- Why Dilemma?

Prisoner's Dilemma

		C	D
	C	$a,$	$b,$
Player 1	$\boldsymbol{D}$	$c,$	$d,$

- Any $c > a > d > b$ defines an instance of Prisoner's Dilemma
- Why Dilemma?
  - For player 1, playing D is always better!

Prisoner's Dilemma

Player 2

	C		$\boldsymbol{D}$
C	$, a$	$<$	$, c$
D	$, b$	$<$	$, d$

- Any $c > a > d > b$ defines an instance of Prisoner's Dilemma

- Why Dilemma?
  - For player 1, playing D is always better!
  - For player 2, playing D is also always better!

Prisoner's Dilemma

	C	D
C	$\boldsymbol{a}, \boldsymbol{a}$	b, c
D	c, b	$\boldsymbol{d}, \boldsymbol{d}$

- Any $c > a > d > b$ defines an instance of Prisoner's Dilemma

- Why Dilemma?
  - For player 1, playing D is always better!
  - For player 2, playing D is also always better!
  - However, the outcomes (d, d) of playing (D, D) is dominated by the outcomes (a, a) of playing (C, C)

Common-payoff game

Definition (Common-payoff game)

A common-payoff game is a game in which for all action profiles $a \in A_1 \times \cdots \times A_n$ and any pair of agents i, j, it is the case that $u_i(a) = u_j(a)$

- Represents **pure coordination**
- Sometimes called pure coordination games or team games
- The agents have no conflicting interests;
  - their sol challenge is to coordinate on an action that is maximally beneficial to all

	Left	Right
Left	$1, 1$	$0, 0$
Right	$0, 0$	$1, 1$

<Example: Traffic game>

Zero-sum games

Definition (Constant-sum game)

A two-player normal-form game is constant-sum if there exists a constant c such that for each strategy profile $a \in A_1 \times A_2$ it is the case that $u_1(a) + u_2(a) = c$.

- Represents **pure competition**
- In general, $c = 0$ and the game is called zero-sum game
  - Positive affine transformations can always make a general sum G into zero-sum G.

	Head	Tails
Head	1, -1	-1, 1
Tails	-1, 1	1, -1

<Matching Pennies game>

	Rook	Paper	Scissors
Rook	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

<Rock, Paper, Scissors game>

Battle of the sexes

- In general games includes elements of both **coordination** and **competition**

		Player 2 (Husband)	
		TF	LA
Player 1 (wife)	TF	$\boxed{2, 1}$	0, 0
	LA	0, 0	$\boxed{1, 2}$

- A husband and wife wish to go to the movies, and they can select among two movies
  - They prefer to go together rather than to separate movies
  - Wife prefer “Transformer” (TF), the husband prefers “LALALAND” (LA)

- **Pure strategy**:
  - Select a single action and play
  - Call a set of pure strategy for each agent a pure-strategy profile
- **Mixed strategy:**
  - Randomizing over the set of available actions according to some probability distribution
  - In a multiagent setting, the role of mixed strategy is critical
- **Why do we need a mixed strategy?**
  - It would be a pretty bad idea to play any deterministic strategy in matching pennies game or Rock-Paper-Scissor game

	Head	Tails
Head	1, -1	-1, 1
Tails	-1, 1	1, -1

Mixed strategy

Definition (Mixed strategy)

Let (N, A, u) be a normal-form game, and for any set X let $\Pi(X)$ be the set of all probability distributions over X. Then, the set of mixed strategies for player i is $S_i = \Pi(A_i)$

Definition (Mixed strategy profile)

The set of mixed-strategy profile is simply the Cartesian product of the individual mixed-strategy sets, $S = S_1 \times \ldots \times S_n$.

- $s_i(a_j)$ denote the probability that an action a_j will be played under mixed strategy s_i
  - For example, $A = \{\text{Rook, Paper,Scissors}\}$, $s_i(\mathrm{R}) = 0.2$, $s_i(\mathrm{P}) = 0.3$, $s_i(\mathrm{S}) = 0.5$

Definition (Support)

The support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$

- A pure strategy is a special case of randomized strategy, in which the support is a single action
- A strategy is fully mixed if it has full support (i.e., if it assigns every action a nonzero probability)

Mixed strategy

	Rook	Paper	Scissors
Rook	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

$$s_1 = \{\overset{\text{Rook}}{0.2}, \overset{\text{Paper}}{0.3}, \overset{\text{Scissors}}{0.5}\}$$

$$s_2 = \{\overset{\text{Rook}}{0.3}, \overset{\text{Paper}}{0.7}, \overset{\text{Scissors}}{0}\}$$

- Support for s_1 is {Rook, Paper, Scissors} and s_1 is fully mixed strategy
- Support for s_2 is {Rook, Paper}

Definition (Expected utility of a mixed strategy)

Given a normal-form game (N, A, u), the expected utility u_i for player i of the mixed strategy profile $s = (s_1, \dots, s_n)$ is defined as

$$u_i(s) = \sum_{a \in A} u_i(a) \Pr(a|s)$$

where $\Pr(a|s)$ is probability that action $a = (a_1, \dots, a_n)$ is selected given strategy s. That is,

$$\Pr(a|s) = \prod_{j=1}^{n} s_j(a_j) = s_1(a_1) \times \ldots \times s_n(a_n)$$

	C	D
C	-1, 1	-4, 0
D	0, -4	-3, -3

$$s = (s_1, s_2)$$

$$s_1 = \{\overset{C}{0.3}, \overset{D}{0.7}\} \qquad s_2 = \{\overset{C}{0.6}, \overset{D}{0.4}\}$$

$$\begin{aligned} u_1(s) = {} & u_1(C,C)s_1(C)s_2(C) \\ & + u_1(C,D)s_1(C)s_2(D) \\ & + u_1(D,C)s_1(D)s_2(C) \\ & + u_1(D,D)s_1(D)s_2(D) \end{aligned} \qquad \begin{aligned} u_1(s) = {} & -1 \times 0.3 \times 0.6 \\ & -4 \times 0.3 \times 0.4 \\ & +0 \times 0.7 \times 0.6 \\ & -3 \times 0.7 \times 0.4 \end{aligned}$$

- It can be represented compactly as $u_1(s) = s_1^T U_1 s_2$

$$\text{with } U_1 = \begin{bmatrix} u_1(C,C) & u_1(C,D) \\ u_1(D,C) & u_1(D,D) \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 0 & -3 \end{bmatrix}$$

Analyzing games

Solution concept

- A solution concept is a method of analyzing games with the objective of restricting the set of all possible outcomes to those that are more reasonable than others.

- A solution concept results in one or more strategy profiles, which we call equilibrium

- An equilibrium is prediction emerged by applying a solution concept to a target game

$$equilibrium_1 = f_1(G)$$
$$equilibrium_2 = f_2(G)$$

$$equilibrium_n = f_n(G)$$

$f_1, \ldots, f_n$ are solution concepts

Assumptions and setup for analyzing game

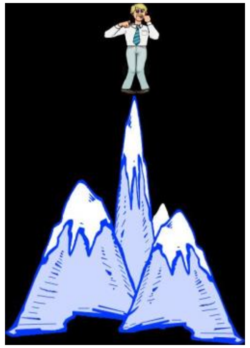
- To set up the background for equilibrium analysis, it is useful to summarize the assumptions that we will be using in the lecture

  - **Players are "rational"**: A rational player is one who chooses his strategy $s_i \in S_i$, to maximize his payoff consistent with his beliefs about what is going on in the game
  - **Players are "intelligent"**: An intelligent player knows everything about the game: the actions, the outcomes, and the preferences of all the players.
  - **Common knowledge** : The fact that players are rational and intelligent is common knowledge among the players of the game
  - **Self-enforcing:** Any prediction (or equilibrium) of a solution concept must be self-enforcing
    - Core of our analysis and at the heart of non-cooperative game theory
    - Each player is in control of his own actions, and he will stick to an action only if he finds the action is in his best interest

Solution concepts

- Pareto optimality
- Nash equilibrium
- Maximin and minmax strategies
- Minimax regret
- Removal of dominated strategies
- Rationalizability
- Correlated equilibrium
- Trembling-hand perfect equilibrium
- Etc.

Analyzing games: from optimality to equilibrium



- **Single agent decision making:**
  - Optimal strategy is one that maximizes the agent's expected utility for a given environment
  - Uncertainties arose from stochastic environment, partially observable states, uncertain rewards, etc., which can be dealt with probability concepts.

$$a^* = \underset{a}{\operatorname{argmax}}\, E_s[u(a,s)]$$

- **Multiagents decision making:**
  - The environment includes other agents, each of which tries to maximize its own utility
  - Thus the notion of an optimal strategy for a given agent is not meaningful because *the best strategy depends on the choices of others*
  - We need to identify certain subsets of outcomes, called solution concepts
  - Two of the most fundamental solution concepts are
    - *Pareto optimality*
    - *Nash equilibrium*

$$u_1(a_1^*, a_2^*) \geq u_1(a_1, a_2^*) \;\; \forall a_1$$
$$u_2(a_1^*, a_2^*) \geq u_2(a_1^*, a_2) \;\; \forall a_2$$

Pareto optimality

- We’ve defined some canonical games and thought about how to play them.
- Now let’s examine the games from the outside:
  - From the point of view of an outside observer, can some outcomes of a game be said to be better than others?
  - Can we say that one agent’s interests are more important than another’s
  - Imagine trying to find the revenue-maximizing outcome when you don’t know what currency is used to express each agent’s payoff
    - Are there ways to still prefer one outcome to another?

Pareto optimality

Outcome of strategy s	**Outcome of strategy s'**
Agent 1's utility : 10 unit of currency x	Agent 1's utility : 20 unit of currency x
Agent 2's utility : 500 unit of currency y	Agent 2's utility : 10 unit of currency y

- Can we insist that the outcome of strategy s is better than that of strategy s'?

Pareto optimality

Outcome of strategy s

Agent 1's utility : 10 unit of currency x

Agent 2's utility : 500 unit of currency y

Outcome of strategy s'

Agent 1's utility : 20 unit of currency x

Agent 2's utility : 10 unit of currency y

- Can we insist that the outcome of strategy s is better than that of strategy s'?
  - No, because we cannot say that one agent's utility is more important than the other's

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- Is there any situation that we can be sure that one outcome is better than another ?

Outcome of strategy s

Agent 1's utility : 10 unit of currency x

Agent 2's utility : 500 unit of currency y

Outcome of strategy s'

Agent 1's utility : 20 unit of currency x

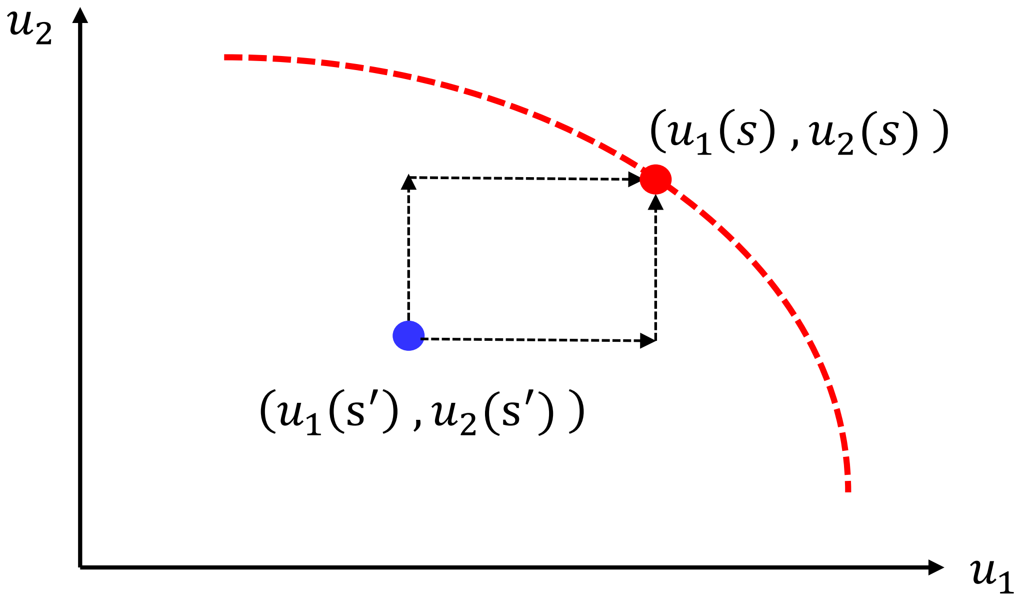
Agent 2's utility : 1000 unit of currency y

- The outcome of s' is always better than the outcome of s

Pareto optimality

Definition (Pareto domination)

Strategy profile s Pareto dominates strategy profile s' if for all $i \in N$, $u_i(s) \geq u_i(s')$, and there exists some $j \in N$ for which $u_j(s) > u_j(s')$.

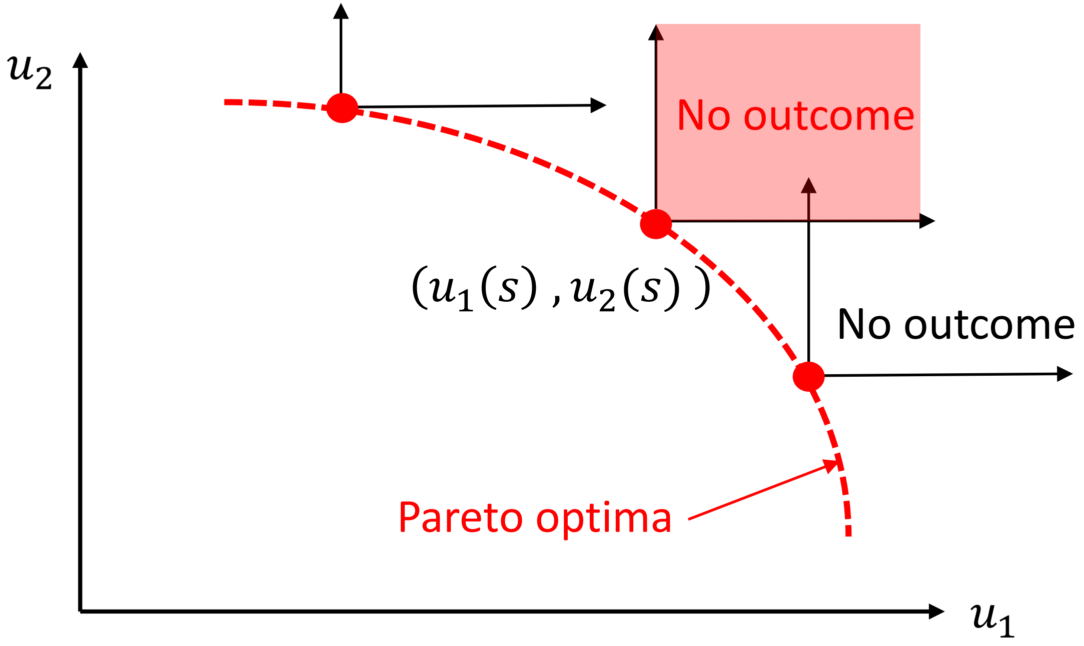


- In other words, in a Pareto-dominated strategy profile some players can be made better off without making any other player worse off
- We cannot generally identify a single "best" outcomes; instead we may have a set of non-comparable optima

Pareto optimality

Definition (Pareto optimality)

Strategy profile s is Pareto optimal, or strictly Pareto efficient, if there does not exist another strategy profile $s' \in S$ that Pareto dominates s

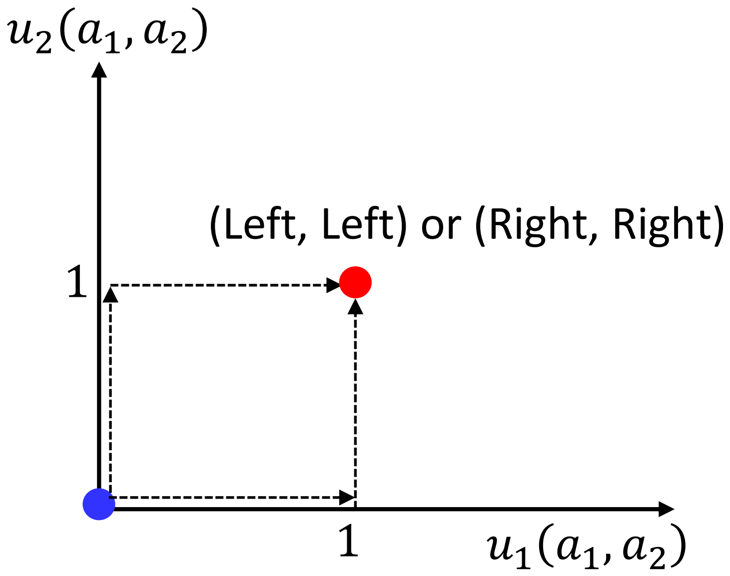


- Every game must have at least one Pareto optimal strategy profile, and there must always exist at least one such optimum in which all players adopt pure strategies.
- Some agent will have multiple optima
  (for example, in zero-sum games, all strategy profiles are strictly Pareto efficient. **Why?**)

Pareto optimal outcomes in various games

	Left	Right
Left	$1, 1$	$0, 0$
Right	$0, 0$	$1, 1$

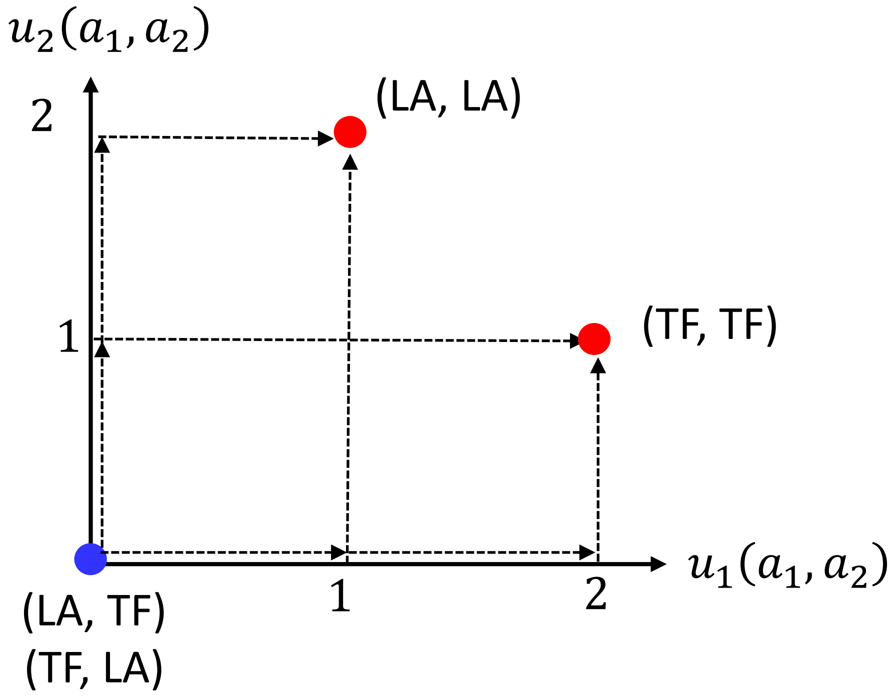
Coordination game



Pareto optimal outcomes in various games

	TF	LA
TF	2, 1	0, 0
LA	0, 0	1, 2

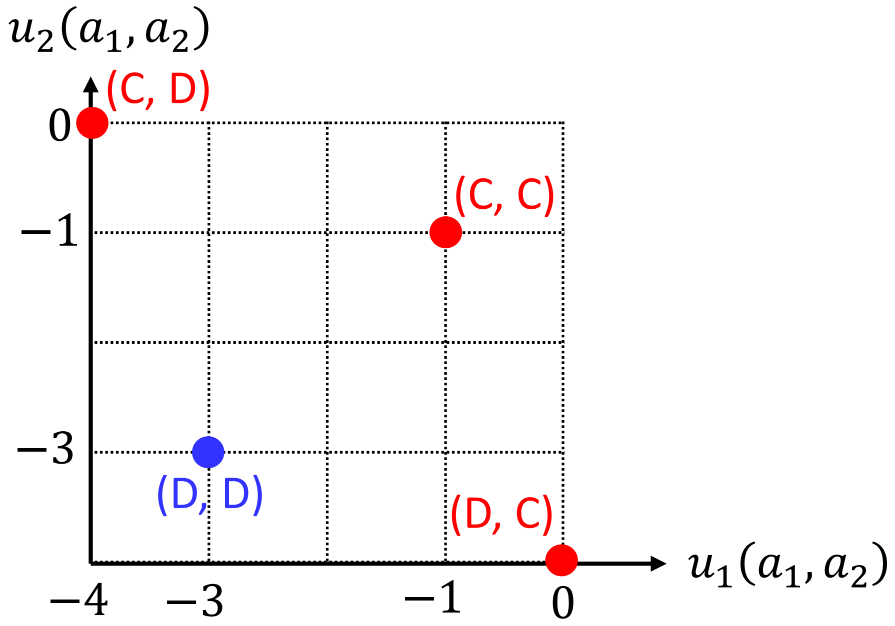
Battle of Sex game



Pareto optimal outcomes in various games

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

Prisoner's Dilemma game



Best response

- If you knew what everyone else was going to do, it would be easy to pick your own action
- Let $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ to be strategy profiles of other agents (all agents except i)
  - then, $s = (s_i, s_{-i})$

Definition (Best response)

Player i's best response to the strategy profile s_{-i} is a mixed strategy $s_i^* \in S_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$

$$s_i^* \in BR(s_{-i})$$

- The best response is not necessarily unique
- Indeed, except in the extreme case in which there is a unique best response that is a pure strategy, the number of best responses is always infinite.
- When the support of a best response s_i^* includes two or more actions, any mixture of these actions must also be a best response
- If there are two pure strategies that are individually best responses, any mixture of the two is necessarily also a best response

Nash equilibrium

- Really, no agent knows what the others will do
- What can we say about which actions will occur ?

Definition (Nash Equilibrium)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i and for all strategies s_i, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$.

- A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i, s_i^* is a best response to s_{-i}^*, i.e., $s_i^* \in BR(s_{-i}^*)$
- A Nash equilibrium is a stable strategy profile:
  - no agent would want to change his strategy if he knew what strategies the other agents were following

Nash equilibrium

Definition (Strict Nash)

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a strict Nash Equilibrium if, for all agents i and for all strategies $s_i \neq s_i^*$, $u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$.

Definition (Week Nash)

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a week Nash Equilibrium if, for all agents i and for all strategies $s_i \neq s_i^*$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$, and s^* is not a strict Nash equilibrium.

- Mixed-strategy Nash equilibria are necessarily week
- Pure-strategy Nash equilibria can be either strict or week, depending on the game.

Nash equilibrium examples

Pure-strategy Nash equilibria in the Battle of the Sexes game

		Player 2	
		TF	LA
Player 1	TF	$\boxed{2, 1}$	0, 0
	LA	0, 0	$\boxed{1, 2}$

- We immediately see that it has two pure-strategy Nash equilibria

Pure-strategy Nash equilibria in the Battle of the Sexes game

Player 2

Player 1	TF	LA
TF	2, 1	0, 0
LA	0, 0	1, 2

- We can check that these are Nash equilibria by confirming that whenever one of the players play the given (pure) strategy, the other player would only lose by deviating

$$a^* = (\mathrm{TF}, \mathrm{TF}) \qquad \begin{aligned} u_1(\mathrm{TF}, \mathrm{TF}) &> u_1(\mathrm{LA}, \mathrm{TF}) \\ u_2(\mathrm{TF}, \mathrm{TF}) &> u_2(\mathrm{TF}, \mathrm{LA}) \end{aligned}$$

Nash equilibrium examples

Pure-strategy Nash equilibria in the Battle of the Sexes game

		Player 2	
		TF	LA
Player 1	TF	2, 1	0, 0
	LA	0, 0	$\boxed{1, 2}$

- We can check that these are Nash equilibria by confirming that whenever one of the players play the given (pure) strategy, the other player would only lose by deviating

$$a^* = (\mathrm{LA}, \mathrm{LA}) \qquad \begin{aligned} u_1(\mathrm{LA}, \mathrm{LA}) &> u_1(\mathrm{TF}, \mathrm{LA}) \\ u_2(\mathrm{LA}, \mathrm{LA}) &> u_2(\mathrm{LA}, \mathrm{TF}) \end{aligned}$$

Nash equilibrium examples

How to easily find pure Nash equilibria?

		Player 2	
		TF	LA
Player 1	TF	2, 1	0, 0
	LA	0, 0	1, 2

Nash equilibrium examples

How to easily find pure Nash equilibria?

		Player 2: TF	Player 2: LA
Player 1	TF	2, 1	0, 0
	LA	0, 0	1, 2

- **Step 1**: For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column

How to easily find pure Nash equilibria?

		Player 2 TF	Player 2 LA
Player 1	TF	$\overline{\underline{2, 1}}$	0, 0
	LA	0, 0	$\overline{\underline{1, 2}}$

- **Step 1**: For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column
- **Step 2**: For every row, which is strategy for player 1, **find the highest payoff entry for player 2** and over line the pair of payoffs

How to easily find pure Nash equilibria?

		Player 2: TF	Player 2: LA
Player 1	TF	$\overline{\underline{2, 1}}$	0, 0
	LA	0, 0	$\overline{\underline{1, 2}}$

- **Step 1**: For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column
- **Step 2**: For every row, which is strategy for player 1, **find the highest payoff entry for player 2** and over line the pair of payoffs
- **Step 3**: If any matrix entry has both an under- and an over line, it is the outcome of a **Nash equilibrium in pure strategies**

Nash equilibrium examples

How to easily find pure Nash equilibria?

Player 2

		L	C	R
	U	7, 0	4, 2	1, 8
Player 1	M	2, 4	5, 5	2, 3
	D	8, 1	3, 2	0, 0

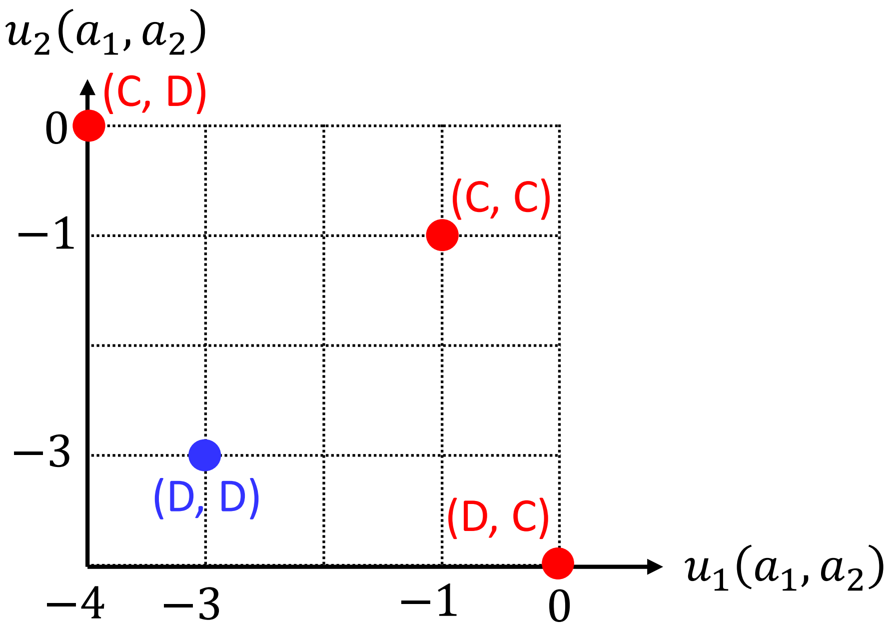
Find pure Nash equilibria by yourself

How many did you get?

Evaluating Nash equilibria solution

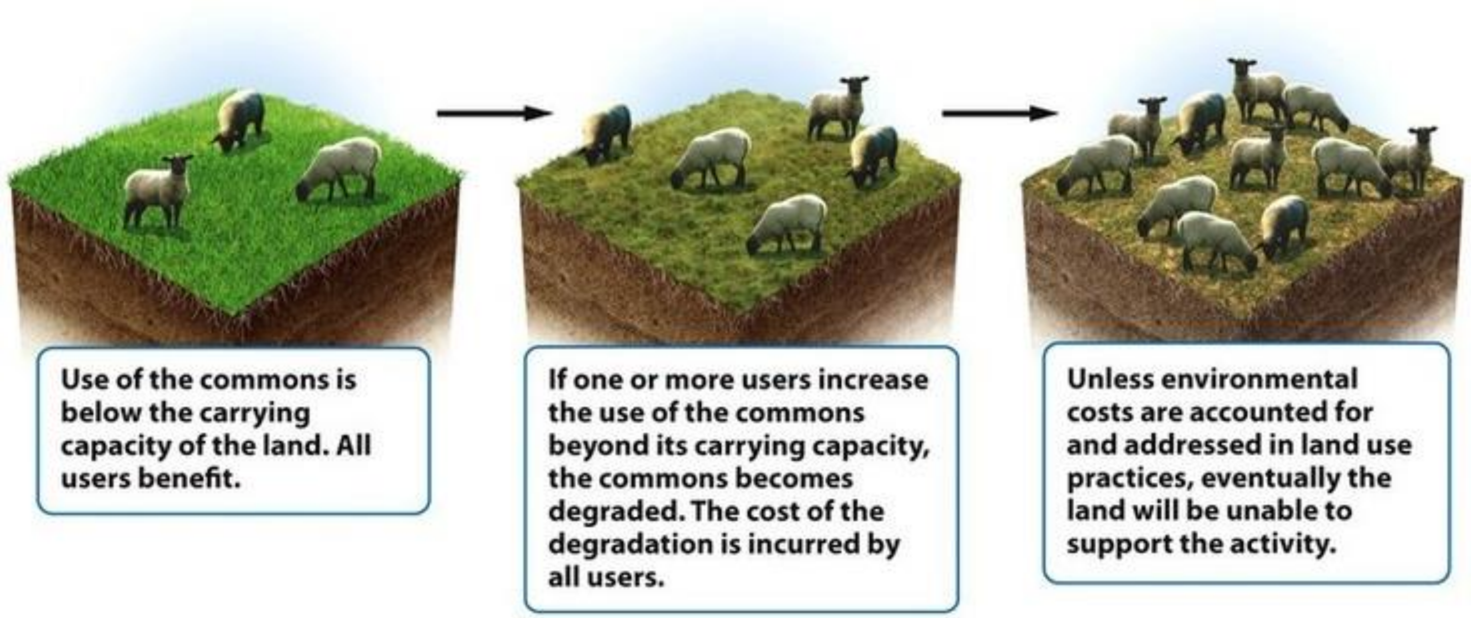
	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

Prisoner's Dilemma game



- As seen in Prisoner's Dilemma game, Nash equilibrium does not guarantee Pareto optimality
- People in many situations will do what is best for them, *at the expense of social efficiency*
- The solution concepts took the game as given, and they impose rationality and common knowledge of rationality to try to see what players would choose to do.
- **If each player seeks to maximize their individual well-being then the players may hinder their ability to achieve socially optimal outcomes**

Nash equilibrium examples : The Tragedy of the Commons



Nash equilibrium examples : The Tragedy of the Commons

- There are n players, say firms, in the world, each choosing how much to produce
- Their production activity in turn consumes some of the clean air that surrounds our planet
- There is a total amount of clean air equal to K, and any consumption of clean air comes out of this common resource
- Each player i chooses his own consumption of clean air for production, k_i
- The clean air left is $K - \sum_{i=1}^{n} k_i$
- The payoff for player i from the choice $k = (k_1, k_2, \dots, k_n)$ is equal to

$$u_i(k_i, k_{-i}) = \ln(k_i) + \ln\left(K - \sum_{j=1}^{n} k_j\right)$$

The benefit of consuming individual air consumption

The benefit of consuming the remainder of the clean air

Nash equilibrium examples : The Tragedy of the Commons

- To solve for a Nash equilibrium, we need to find some profile of choices $k^* = (k_1^*, k_2^*, \dots, k_n^*)$ for which $k_i^* = BR_i(k_{-i}^*)$ for all $i \in N$

- Then we have a system of n equations, on for each player's best-response function, with n unknowns, the choices of each player.

- For example, to get player $i's$ best-response function, the following first-order condition of his payoff function should be satisfied

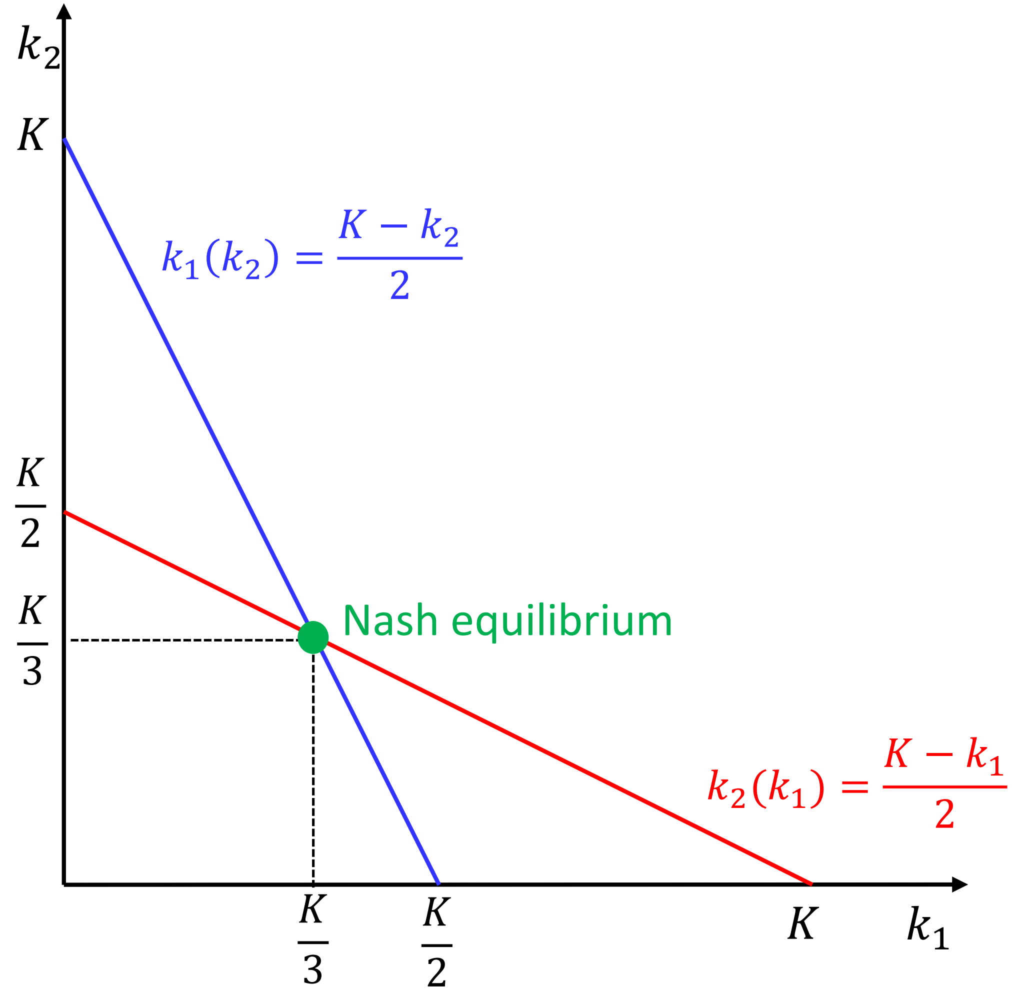
$$\frac{\partial u_i(k_i, k_{-i})}{\partial k_i} = \frac{1}{k_i} - \frac{1}{K - \sum_{j=1}^{n} k_j} = 0$$

which gives player i's best response function,

$$BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2}$$

- In case there are two firms, we have two best-response equations:

$$k_1(k_2) = \frac{K - k_2}{2} \quad \text{and} \quad k_2(k_1) = \frac{K - k_1}{2}$$



Nash equilibrium examples : The Tragedy of the Commons

- Now we can ask whether this two-player behave to make the society better
  - Is consuming $K/3$ for each player too much or too little?
  - Can we find another consumption profile that will make everyone better off?

- We will maximize the sum of all the payoff functions, which we can think of as the "world's payoff function $w(k_1, k_2)$

- We can maximize

$$\max_{k_1,k_2} w(k_1, k_2) = \sum_{i=1}^{2} u_i(k_1, k_2) = \sum_{i=1}^{2} \left\{ \ln(k_i) + \ln\left(K - \sum_{j=1}^{n} k_j\right) \right\}$$

- The first-order conditions of this problem are

$$\frac{\partial w(k_1, k_2)}{\partial k_1} = \frac{1}{k_1} - \frac{2}{K - k_1 - k_2} = 0$$

$$\frac{\partial w(k_1, k_2)}{\partial k_2} = \frac{1}{k_2} - \frac{2}{K - k_1 - k_2} = 0$$

- The solution for this is $k_1 = k_2 = \frac{K}{4}$, that gives $u_1 = u_2 = \ln\frac{K}{4} + \ln\frac{K}{2} = \ln(\frac{K^2}{8})$
  - ➢ which is larger that $u_1 = u_2 = \ln\frac{K}{3} + \ln(\frac{K}{3}) = \ln(\frac{K^2}{9})$ for Nash equilibrium

Nash equilibrium examples: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good
- Firm i produce quantity q_i
- Cost for production is $c_i(q_i) = c_i q_i$
- Price is given by $d = a - b(q_1 + q_2)$
- The profit of company i given its opponent chooses quantity q_j is

$$u_i(q_i, q_j) = (a - bq_i - bq_j)q_i - c_i q_i = -bq_i^2 + (a - c_i)q_i - bq_j q_i$$

- The best-response function for each firm is given by the first-order condition

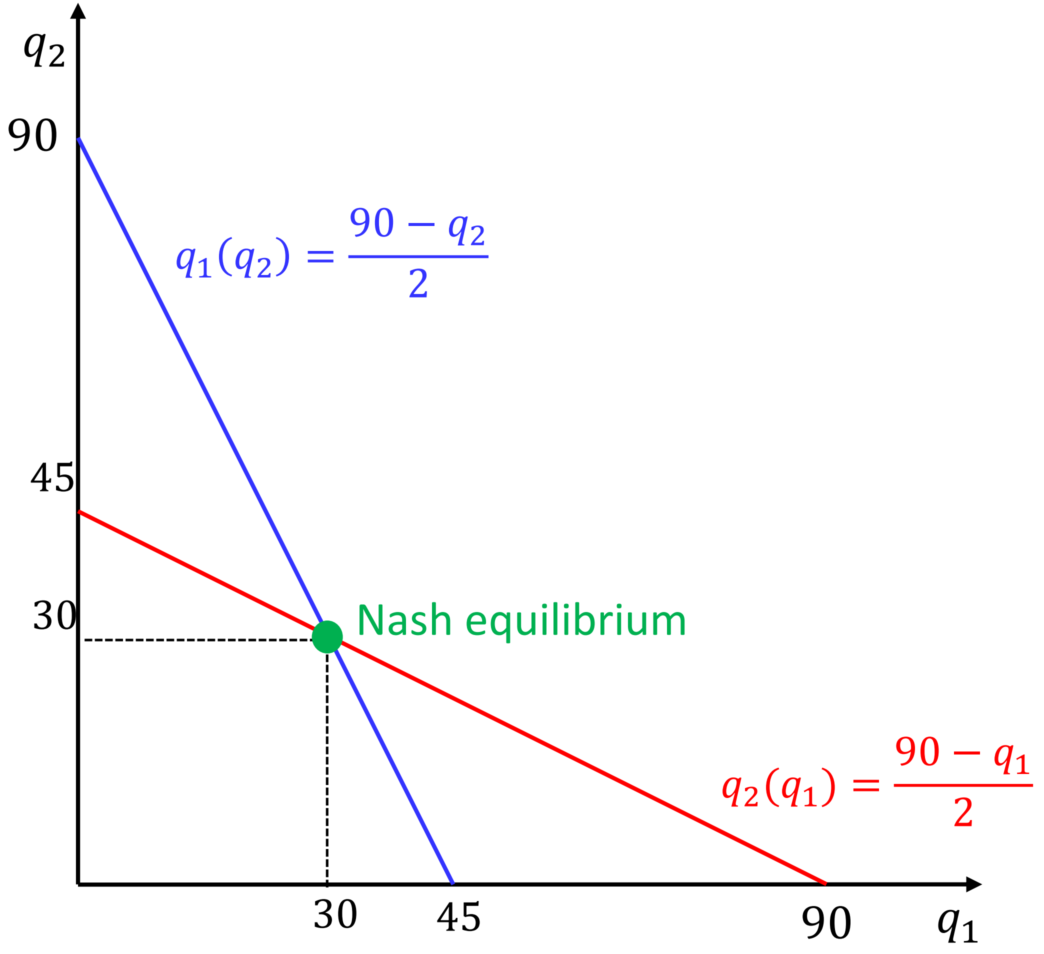
$$BR_i(q_j) = \frac{a - bq_j - c_i}{2b}$$

Nash equilibrium examples: Cournot Duopoly

- In case there are two firms, we have two best-response equations:

$$q_1 = \frac{a - bq_2 - c_1}{2b} \quad \text{and} \quad q_2 = \frac{a - bq_1 - c_2}{2b}$$

$a = 100, b = 1, c_1 = c_2 = 10$



Mixed strategy Nash equilibrium

- Why anyone would wish to randomize between actions?
- We will see mixed (stochastic) strategies turns out to be an important type of behavior to consider , with interesting implications and interpretations.
- No pure strategy Nash equilibria exists for the following Matching Pennies game

Player 2

		Heads	Tails
Player 1	Heads	$\underline{1}, -1$	$-1, \overline{1}$
	Tails	$-1, \overline{1}$	$\underline{1}, -1$

- Nash equilibrium will indeed exist if we allow players to choose random strategies

Revisit: mixed strategy

Definition (Mixed strategy)

Let (N, A, u) be a normal-form game, and for any set X let $\Pi(X)$ be the set of all probability distributions over X. Then, the set of mixed strategies for player i is $S_i = \Pi(A_i)$

Definition (Mixed strategy profile)

The set of mixed-strategy profile is simply the Cartesian product of the individual mixed-strategy sets, $S = S_1 \times \ldots \times S_n$.

- $s_i(a_j)$ denote the probability that an action a_j will be played under mixed strategy s_i
  - For example, $A = \{\text{Rook, Paper,Scissors}\}$, $s_i(\text{R}) = 0.2$, $s_i(\text{P}) = 0.3$, $s_i(\text{S}) = 0.5$

Definition (Support)

The support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$

$A_1 = \{L, R\}$
$S_1 = \Pi(A_1) = \{(s_1(L), s_1(R)): s_1(L), s_1(R) \geq 0, s_1(L) + s_1(R) = 1\}$
$s_1 \in S_1, \text{i.e.}, s_1 = (q, 1 - q)$

- Introducing probability distributions not only enriches the set of actions from which a player can choose but also allows us to enrich the beliefs that players can have

Definition (Belief)

A belief for player i is given by a probability distribution $\pi_i \in \Pi(A_{-i})$ over the actions of his opponents. We denote by $\pi_i(a_{-i})$ the probability player i assigns to his opponents playing $a_{-i} \in A_{-i}$

How to find mixed strategy Nash equilibrium?

Definition (Nash Equilibrium)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i and for all strategies $s_i \in \Pi(A_i)$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$.

- A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i, s_i^* is a best response to s_{-i}^*, i.e., $s_i^* \in BR(s_{-i}^*)$
- We can think of s_{-i}^* as the belief of player i about his opponents, π_i, which captures the idea that player i is uncertain of his opponent's behavior
  - The profile of mixed strategies s_{-i}^* thus captures this uncertain belief over all of the pure strategies that player i's opponent can play
  - Rationality requires that a player play a best response given his belief (Nash equilibrium requires that these beliefs are correct, i.e., a system of equations should be satisfied)

How to find mixed strategy Nash equilibrium?

- Recall that the support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$

- Imagine that the Nash equilibrium profile s_i^* contains more than one pure strategy -say a_i and a_i' as supports.

- What must we conclude about a rational player i if s_i^* is indeed part of a Nash equilibrium (s_i^*, s_{-i}^*)?

How to find mixed strategy Nash equilibrium?

- Recall that the support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$
- Imagine that the Nash equilibrium profile s_i^* contains more than one pure strategy -say a_i and a_i' as supports.
- What must we conclude about a rational player i if s_i^* is indeed part of a Nash equilibrium (s_i^*, s_{-i}^*)?

if s_i^* is a Nash equilibrium, and if a_i and a_i' are in the support of s_i^*, then

$$u_i(a_i, s_{-i}^*) = u_i(a_i', s_{-i}^*) = u_i(s_i^*, s_{-i}^*)$$

Proof:

- assume $u_i(a_i, s_{-i}^*) > u_i(a_i', s_{-i}^*)$ and a_i and a_i' are support of s_i^*
- Adjusting the mixed strategy $s_i = \{s_i(a_i), s_i(a_i')\} \rightarrow \{s_i(a_i) + s_i(a_i'), 0\}$ will increase u_i
- s_i^* could not have been a best response to s_{-i}^*
- Therefore, by contradiction, $u_i(a_i, s_{-i}^*) = u_i(a_i', s_{-i}^*)$

How to find mixed strategy Nash equilibrium?

if s_i^* is a Nash equilibrium, and both if a_i and if a_i' are in the support of s_i^*, then

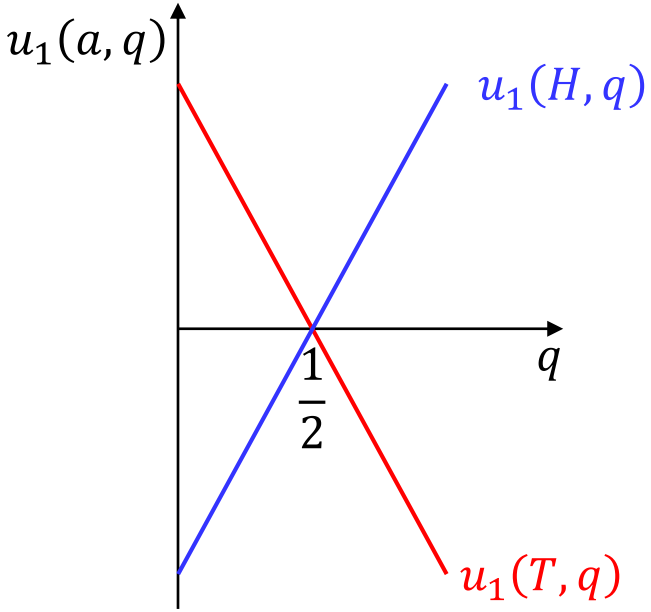
$$u_i(a_i, s_{-i}^*) = u_i(a_i', s_{-i}^*) = u_i(s_i^*, s_{-i}^*)$$

- This result will play an important role in computing mixed-strategy Nash equilibria
  - If a player is playing a mixed strategy then he must be indifferent between the actions he is choosing with positive probability (i.e., actions in the support)
- One player's indifference will impose restrictions on the behavior or other players
  - This restriction will help us find the mixed-strategy Nash equilibrium

Finding mixed Nash equilibria : Matching Pennies

		q H	$1-q$ T
p	H	$1,-1$	$-1,1$
$(1-p)$	T	$-1,1$	$1,-1$

$$u_1(H,q) = q \times 1 + (1-q) \times (-1) = 2q - 1$$
$$u_1(T,q) = q \times (-1) + (1-q) \times 1 = 1 - 2q$$

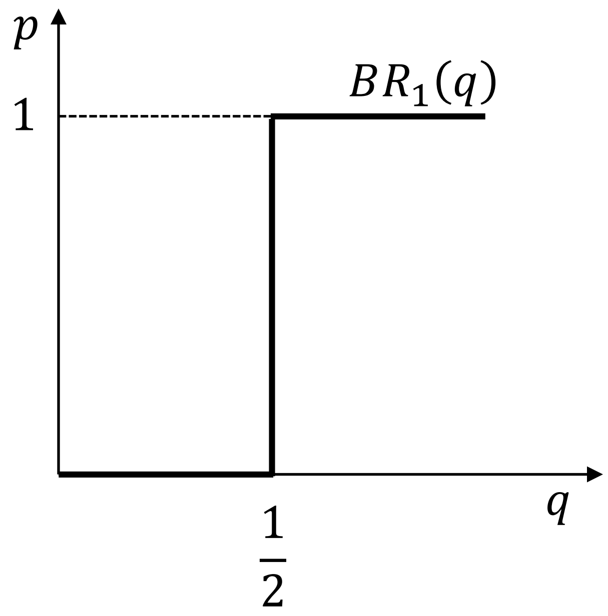


$$BR_1(q) = \begin{cases} p = 0 & if \;\; q < 1/2 & \text{(Playing } T\text{)} \\ p \in [0,1] & if \;\; q = 1/2 & \text{(Indifferent)} \\ p = 1 & if \;\; q > 1/2 & \text{(Playing } H\text{)} \end{cases}$$

Finding mixed Nash equilibria : Matching Pennies

		q **H**	$1-q$ **T**
p	**H**	$1,-1$	$-1,1$
$(1-p)$	**T**	$-1,1$	$1,-1$

$$u_1(H,q) = q \times 1 + (1-q) \times (-1) = 2q - 1$$
$$u_1(T,q) = q \times (-1) + (1-q) \times 1 = 1 - 2q$$

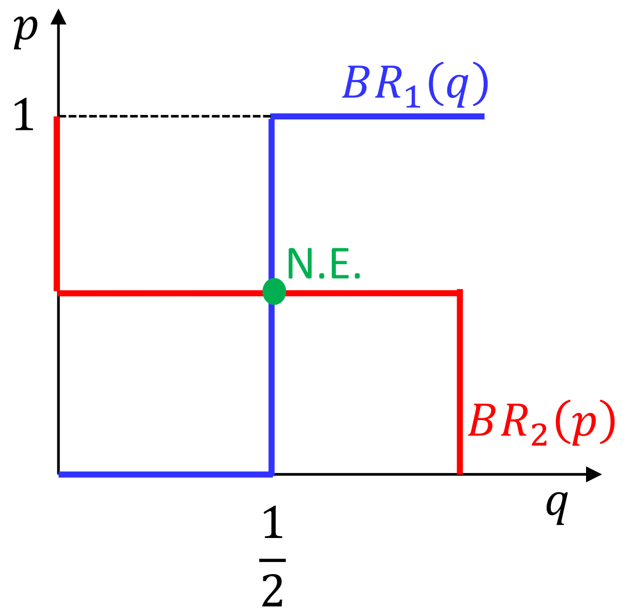


$$BR_1(q) = \begin{cases} p = 0 & if \;\; q < 1/2 & \text{(Playing } T\text{)} \\ p \in [0,1] & if \;\; q = 1/2 & \text{(Indifferent)} \\ p = 1 & if \;\; q > 1/2 & \text{(Playing } H\text{)} \end{cases}$$

Finding mixed Nash equilibria : Matching Pennies

		q **H**	$1-q$ **T**
p	**H**	$1,-1$	$-1,1$
$(1-p)$	**T**	$-1,1$	$1,-1$

$$u_1(H,q) = q \times 1 + (1-q) \times (-1) = 2q - 1$$
$$u_1(T,q) = q \times (-1) + (1-q) \times 1 = 1 - 2q$$
$$u_2(H,p) = p \times (-1) + (1-p) \times 1 = 1 - 2p$$
$$u_2(T,p) = p \times 1 + (1-p) \times (-1) = 2p - 1$$



$$BR_1(q) = \begin{cases} p = 0 & if\ \ q < 1/2 \quad \text{(Playing } T\text{)} \\ p \in [0,1] & if\ \ q = 1/2 \quad \text{(Indifferent)} \\ p = 1 & if\ \ q > 1/2 \quad \text{(Playing } H\text{)} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & if\ \ p < 1/2 \quad \text{(Playing } H\text{)} \\ q \in [0,1] & if\ \ p = 1/2 \quad \text{(Indifferent)} \\ q = 0 & if\ \ p > 1/2 \quad \text{(Playing } T\text{)} \end{cases}$$

The intersections of two best response curve→ Nash equilibria

To find Nash equilibrium, make other player indifferent between some of his pure actions

Mixed-strategy Nash equilibria in the **Matching Pennies game**

Player 2

Player 1		p Heads	$1-p$ Tails
	q Heads	$1, -1$	$-1, 1$
	$1-q$ Tails	$-1, 1$	$1, -1$

No pure strategy Nash equilibria exists

- Player 1 should randomize his action to make Player 2 to be indifferent between her actions
  → otherwise, player 2 would play the action that is superior than the other
- Let us assume Player 1' strategy is to play Heads with probability q and Tails with $1-q$

$$u_2(\text{Heads}) = u_2(\text{Tails})$$
$$-1 \times q + 1 \times (1-q) = 1 \times q - 1 \times (1-q)$$
$$q = \frac{1}{2}$$

Mixed-strategy Nash equilibria in the **Matching Pennies game**

		Player 2	
		p Heads	$1-p$ Tails
Player 1	q Heads	$1, -1$	$-1, 1$
	$1-q$ Tails	$-1, 1$	$1, -1$

No pure strategy Nash equilibria exists

- Player 2 should randomize his action to make Player 1 to be indifferent between her actions
  → otherwise, player 1 would play the action that is superior than the other
- Let us assume Player 2' strategy is to play Heads with probability p and Tails with $1-p$

$$u_1(\text{Heads}) = u_1(\text{Tails})$$
$$1 \times p - 1 \times (1-p) = -1 \times p + 1 \times (1-p)$$
$$p = \frac{1}{2}$$

Mixed-strategy Nash equilibria in the **Battle of the Sexes game**

Player 2

		p TF	$1-p$ LA
Player 1	q TF	2, 1	0, 0
	$1-q$ LA	0, 0	1, 2

- Player 1 should randomize his action to make Player 2 to be indifferent between her actions
  → otherwise, player 2 would play the action that is superior than the other
- Let us assume Player 1' strategy is to play TF with probability q and LA with $1-q$

$$u_2(\mathrm{TF}) = u_2(\mathrm{LA})$$
$$1 \times q + 0 \times (1-q) = 0 \times q + 2 \times (1-q)$$
$$q = \frac{2}{3}$$

Mixed-strategy Nash equilibria in the **Battle of the Sexes game**

		Player 2: TF (p)	Player 2: LA ($1-p$)
Player 1	TF (q)	2, 1	0, 0
	LA ($1-q$)	0, 0	1, 2

- Player 2 should randomize his action to make Player 1 to be indifferent between her actions
  → otherwise, player 1 would play the action that is superior than the other
- Let us assume Player 2' strategy is to play TF with probability p and LA with $1-p$

$$u_1(\mathrm{LA}) = u_1(\mathrm{TF})$$
$$2 \times p + 0 \times (1-p) = 0 \times p + 1 \times (1-p)$$
$$p = \frac{1}{3}$$

Mixed-strategy Nash equilibria in the **Battle of the Sexes game**

			Player 2	
			$1/3$	$2/3$
			TF	LA
Player 1	$2/3$	TF	2, 1	0, 0
	$1/3$	LA	0, 0	1, 2

- Now, we can confirm that we have indeed found an equilibrium:
  - Both players play in a way that makes the other indifferent, they are both best responding to each other
- Expected payoff for both agents is 2/3 in this equilibrium
  - Each of the pure-strategy equilibria Pareto-dominates the mixed strategy equilibrium
- **This mixed strategy, as all other mixed strategies, is a week Nash equilibrium**

$$u_1(s_i^*, s_{-i}^*) \geq u_1(s_i, s_{-i}^*) \qquad u_1\left(\left(\frac{2}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{2}{3}\right)\right) \geq u_1\left((x, 1-x),\left(\frac{1}{3},\frac{2}{3}\right)\right)$$ For any $0 \leq x \leq 1$

Finding Nash equilibria

Mixed-strategy Nash equilibria in the Rock-Paper-Scissor game

	p_R Rook	p_P Paper	p_S Scissors
Rook	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

$$u_1(\mathrm{R}) = u_1(\mathrm{P}) = u_1(\mathrm{S})$$
$$\Rightarrow 0p_R + (-1)p_P + 1p_s = 1p_R + 0p_P + (-1)p_S = -1p_R + 1p_P + 0p_s$$
$$\Rightarrow 0p_R + (-1)p_P + 1p_s = 1p_R + 0p_P + (-1)p_S \Rightarrow 2p_S = p_R + p_P$$
$$\Rightarrow 1p_R + 0p_P + (-1)p_S = -1p_R + 1p_P + 0p_s \quad \Rightarrow 2p_R = p_S + p_P$$
$$\Rightarrow p_R = p_P = p_S \qquad (1)$$

$$p_R + p_P + p_S = 1 \qquad (2)$$

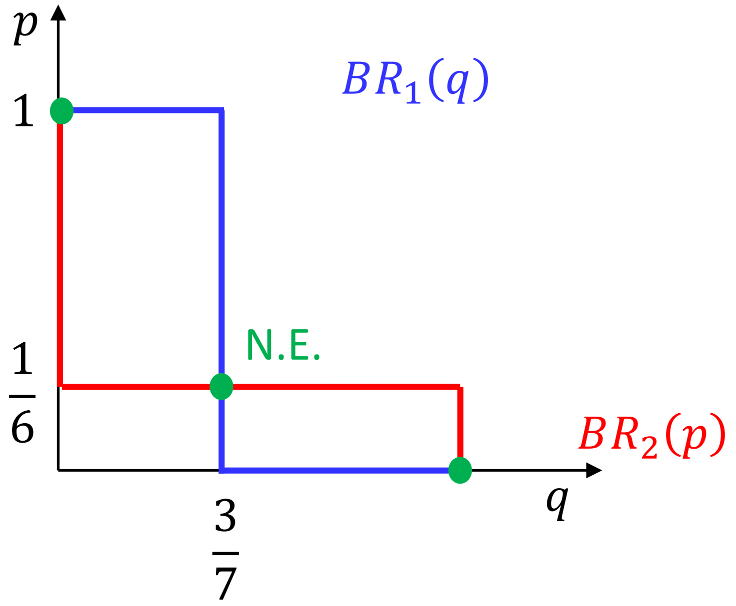
- Due to (1) and (2), $p_R = p_P = p_S = 1/3$ (Mixed strategy Nash Equilibrium)

Multiple mixed strategies

		q **H**	$1-q$ **T**
p	**H**	$0,0$	$3,5$
$(1-p)$	**T**	$4,4$	$0,3$

$$u_1(H,q) = q \times 0 + (1-q) \times (3) = 3q - 3$$
$$u_1(T,q) = q \times (4) + (1-q) \times 0 = 4q$$

$$u_2(H,p) = p \times (0) + (1-p) \times 4 = 4 - 4p$$
$$u_2(T,p) = p \times 5 + (1-p) \times (3) = 2p + 3$$



Nash equilibriums are $\left\{(1,0), \left(\frac{1}{6}, \frac{3}{7}\right), (0,1)\right\}$

The meaning of playing mixed-strategy

- Randomize to confuse your opponent
  - consider the matching pennies example

- Randomize when uncertain about the other’s action
  - consider battle of the sexes

- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit

- Mixed strategies describe population dynamics:
  - agents chosen from a population have deterministic strategies.
  - Mixed strategies gives the probability of getting each pure strategies.

The existence of Nash equilibria

Theorem (Nash, 1951)

Every game with a finite number of players and action profiles has at least one Nash equilibrium

Further solution concepts

Motivations

- We reason about multiplayer games using solution concepts, principles according to which we identify interesting subsets of the outcomes of a game

- Nash equilibrium is the most important solution concept

- There are also a large number of others:
  - Maximin and minmax strategies
  - Minimax regret
  - Removal of dominated strategies
  - Rationalizability
  - Correlated equilibrium
  - Trembling-hand perfect equilibrium

Maxmin and minmax strategies

Definition (Maxmin)

The maxmin strategy for player i is $s_i^* = \arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the maxmin value for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$

- The ***maxmin strategy*** of player i in an n-players game is a strategy that maximizes $i's$ worst – case payoff, in the situation where all the others players happen to play the strategies which cause the greatest harm to i
- The ***maxmin strategy*** is a sensible choice for a conservative agent who wants to maximize his expected utility without having to make any assumptions about the other agents
- The ***maxmin value*** (or security level) of the game for player i is that minimum amount of payoff guaranteed by a maxmin strategy
- It is strategy that **defends against** other agents (defensive strategy)
- Player i set the mixed strategy ⇒ player $-i$ observe this strategy (not an action) and choose their own strategies to minimize i's expected payoff

(temporal interpretation)

Maxmin and minmax strategies

Definition (Minmax, two-player**)**

In an two-player game, the *minmax strategy* for player i against player $-i$ is $s_i^* = \arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$ and the minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$

- The *minnmax strategy* of player i in an two-players game is a strategy that keeps the maximum payoff of $-i$ at a minimum
- The *minmax value* of player $-i$ is that minimum
- It is strategy that attack against other agents (offensive strategy)

Maxmin and minmax strategies

In agent $i's$ perspective

$$\boxed{\max_{s_i}} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- Agent always maximizes its payoff
- Defensive strategy (if max is first)

$$\boxed{\min_{s_i}} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

- Agent always maximizes its payoff
- offensive strategy (if min is first)

Definition (Minmax, n-player)

In an n-player game, the minmax strategy for player i against player $j \neq i$ is i-th component of the mixed-strategy profile s_{-j} in the expression $\arg\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$. As before, the minmax value for player j is $\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$

- Here, we assume that all the players other than j choose to "gang up" on j
  - They are able to coordinate appropriately when there is more than one strategy profile that would yield the same minimal payoff for j

Maxmin and minmax strategies

- An agent's maximum strategy nor his minmax strategy depends on the strategies that the other agents actually choose
- the maximin and minmax strategies give rise to solution concept
- Call $s = (s_1, \ldots, s_n)$ a maxmin strategy profile of a given game if s_1 is a maxmin strategy for player 1, s_2 is a maxmin strategy for player 2 and so on.
  - Similar to minmax strategy profile

- In two-player games, a player's minmax value is always equal to his maxmin value

$$\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i}) = \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- For games with more than two players, a weaker condition holds:

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) \leq \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

- See that player $-i$ chooses first, allowing player i to best respond to it.

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$$\min_{s_{-i}} u_i(s_i^{\max}, s_{-i}) \leq u_i\left(s_i^{\max}, s_{-i}^{\min}\right) \leq \max_{s_i} u_i(s_i, s_{-i}^{\min})$$

- See that player $-i$ chooses first, allowing player i to best respond to it.

Minimax theorem (von Neumann, 1928)

Theorem (Minmax theorem by von Neumann, 1928)

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

- Minmax theorem states that in a two-player zero-sum game:

maximin value = minmax value = Nash equilibrium value

- Any *maximin* strategy profile is a Nash equilibrium. Furthermore, these are all the Nash equilibria
  - ➢ Consequently, all Nash equilibria have the same payoff vector

Minimax theorem (von Neumann, 1928)

Proof:

- Let's assume (s_i', s_{-i}') be an arbitrary Nash equilibrium and denote v_i to be the i's equilibrium payoff
- Denote i's maxmin value as $\overline{v}_i = \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$
- Denote i's minmax value as $\underline{v} = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$
- First, we show that $\overline{v}_i = v_i$

✓ $\overline{v}_i \leq v_i$

$$\overline{v}_i = \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) \leq \max_{s_i} u_i(s_i, s_{-i}') = v_i$$

✓ $\overline{v}_i \geq v_i$

$$v_{-i} = \max_{s_{-i}} u_{-i}(s_i', s_{-i})$$

$$-v_{-i} = \min_{s_{-i}} -u_{-i}(s_i', s_{-i}) \qquad \because \max f(x) = -\min\{-f(x)\}$$

since the game is zero sum, $-v_{-i} = v_i$ and $u_i = -u_{-i}$, thus

$$v_i = \min_{s_{-i}} u_i(s_i', s_{-i})$$

$$\overline{v}_i = \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) \geq \min_{s_{-i}} u_i(s_i', s_{-i}) = v_i$$

➢ As a result, $v_i = \overline{v}_i$

Minimax theorem example

		Player 2	
		q Heads	$1-q$ Tails
Player 1	p Heads	$1, -1$	$-1, 1$
	$1-p$ Tails	$-1, 1$	$1, -1$

- Player 1's maxmin value :

$$\bar{u}_1 = \max_{s_1} \min_{s_2} u_1(s_1, s_2)$$
$$= \max_{p} \min_{q} \{pq - p(1-p) - (1-p)q + (1-p)(1-q)\}$$

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Minimax theorem example

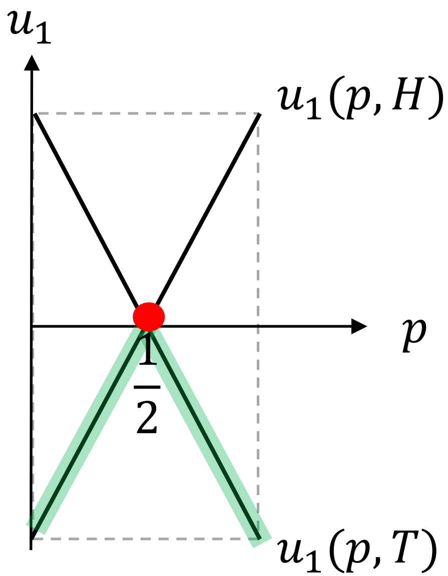
- Player 1's maxmin value : $\bar{u}_1 = \max_{s_1} \min_{s_2} u_1(s_1, s_2)$
$= \max_{p} \min_{q} \{pq - p(1-p) - (1-p)q + (1-p)(1-q)\}$

- For any p set by player 1, player 2 tries to chooses q **deterministically** to minimize u_1
- $\min_{q} \{pq - p(1-p) - (1-p)q + (1-p)(1-q)\} \Rightarrow$

$$\min_{q \in \{0,1\}} \{pq - p(1-p) - (1-p)q + (1-p)(1-q)\} = \min\{2p-1, 1-2p\}$$

  - When player 2 plays Heads $(q = 1)$: $u_1(p, H) = 2p - 1$
  - When player 2 plays Tails $(q = 0)$: $u_1(p, T) = 1 - 2p$

- Thus, $\bar{u}_1 = \max_{p} \min\{2p-1, 1-2p\} = 0$



- Player 1's maxmin strategy:

$$\bar{s}_1 = \operatorname{argmax}_{s_1} \min_{s_2} u_1(s_1, s_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

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$$\bar{u}_1 = \max_{s_1} \min_{s_2} u_1(s_1, s_2) = 0$$

Minimax theorem example

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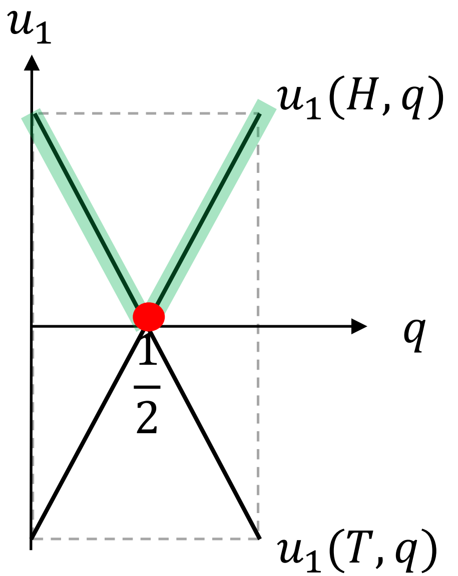
- Thus, $\underline{u}_1 = \min_q \max\{2q-1, 1-2q\} = 0$



- Player 2's minmax strategy:

$$\underline{s}_2 = \operatorname{argmin}_{s_2} \max_{s_1} u_1(s_1, s_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

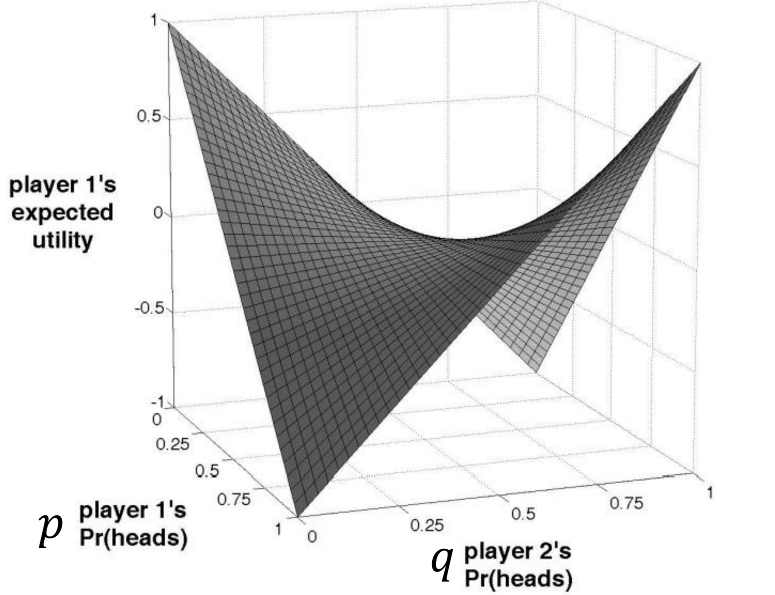
- Player 1's minmax value:

$$\underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2) = 0$$

Minimax theorem graphical representation

		Player 2	
		q Heads	$1-q$ Tails
Player 1	p Heads	$1, -1$	$-1, 1$
	$1-p$ Tails	$-1, 1$	$1, -1$

$$u_1(p,q) = pq - p(1-p) - (1-p)q + (1-p)(1-q)$$



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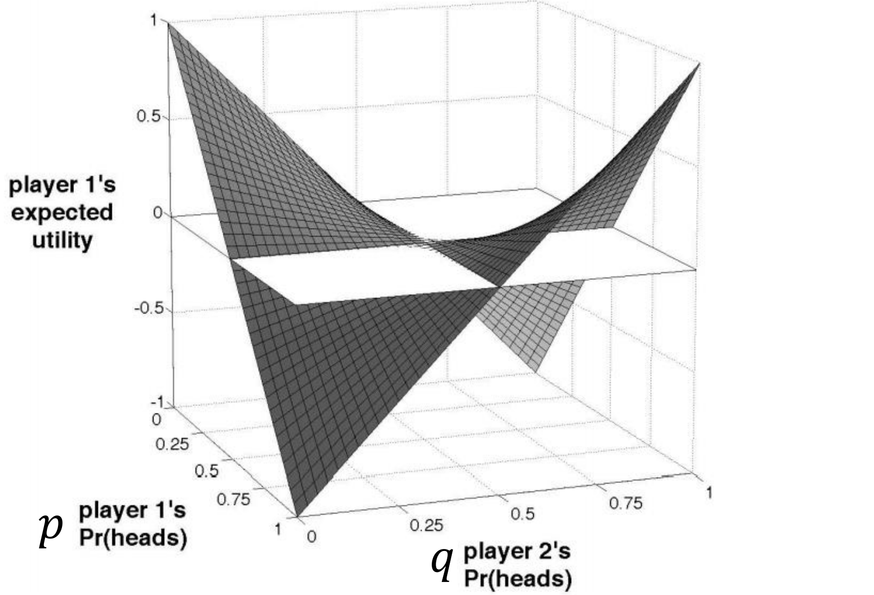

- Nash equilibria in zero-sum games can be viewed graphically as a “saddle” in a high-dimensional space
- At a saddle point, any deviation of the agent lowers his utility and increases the utility of the other agent.

Minimax theorem graphical representation

		Player 2 q Heads	Player 2 $1-q$ Tails
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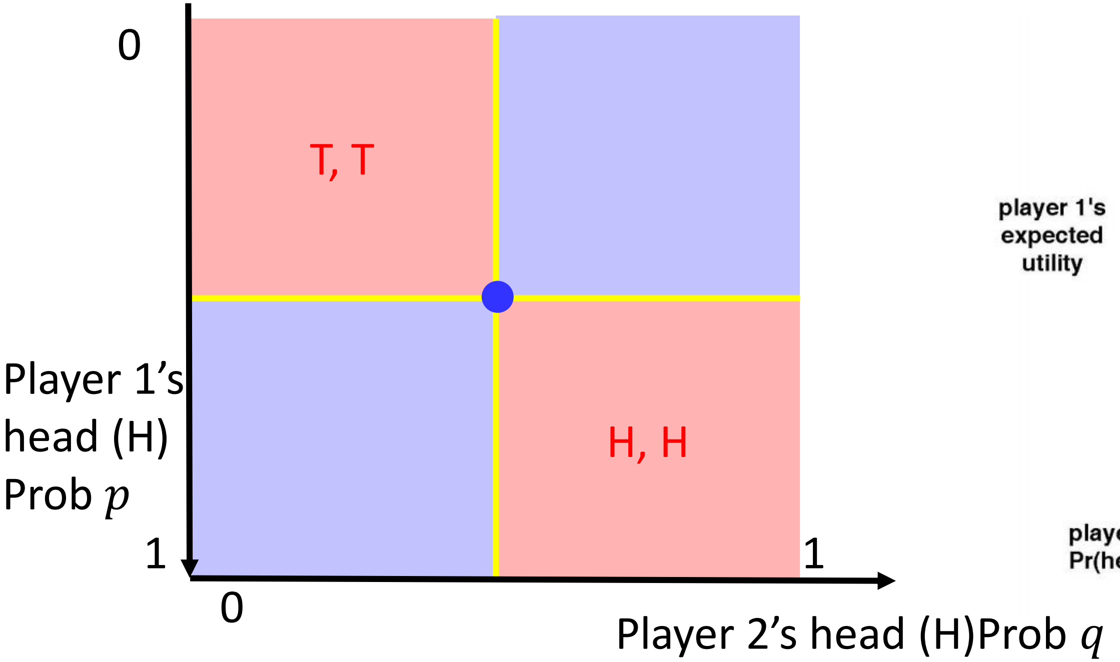

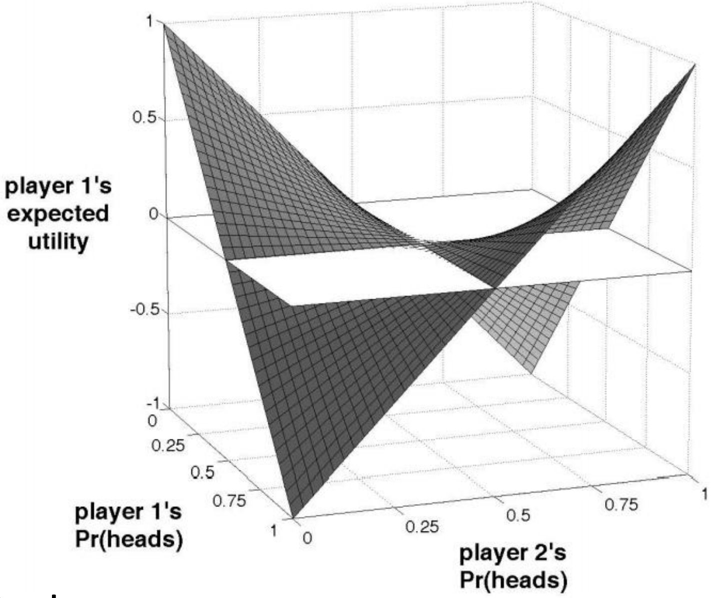

- Nash equilibria in zero-sum games can be viewed graphically as a "saddle" in a high-dimensional space
- At a saddle point, any deviation of the agent lowers his utility and increases the utility of the other agent.

Minimax regret

		Player 2	
		L	R
Player 1	T	$100, a$	$1-\epsilon, b$
	B	$2, c$	$1, d$

- We argued agents might play maxmin strategies to achieve good payoffs in the worst case

- Player 1's maximin strategy is to play B (to receive 1 rather than $1-\epsilon$):
  - If player 1 play T, then player 2 will chose R to minimize player 1's payoff: $u_1 = 1-\epsilon$
  - If player 1 play B, then player 2 will chose R to minimize player 1's payoff: $u_1 = 1$
  - Thus, maximin strategy for player 1 is to play B, giving him a payoff of 1

Minimax regret

		Player 2	
		L	R
Player 1	T	$100, a$	$1-\epsilon, b$
	B	$2, c$	$1, d$

- However, the other agent is not believed to be malicious, but is instead unpredictable
- In this case, agents might care about minimizing their worst-case losses, rather than maximizing their worst case payoffs

- Player 1's Minmax regret strategy is to play T:
  - If player 2 were to play R, then it would not matter very much how player 1 plays
    - ✓ The most he could lose by playing the wrong way would be ϵ
  - If player 2 were to play L, then player 1's action would be very significant
    - ✓ If player makes wrong choice, his utility would be decreased by 98
  - Thus, given that player can maximize your regret, player 1 might choose to play T in order to minimize his worst-case loss

Minimax regret

Definition (Regret)

An agent i's regret for playing an action a_i if the other agents adopt action profile a_{-i} is defined as

$$\left[\max_{a_i' \in A_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i})$$

- In words, this is the amount that i loses by playing a_i, rather than playing his best response to a_{-i}. Of course, i does not know that actions the other players will take.
- But, we can consider those actions that would give him the highest regret for playing a_i

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- This is the amount that i loses by playing a_i rather than playing his best response to a_{-i}, if the other agents chose the a_{-i} that makes this loss as large as possible

Minimax regret

Definition (Regret)

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Definition (Minmax Regret)

Minmax regret actions for agent i are defined as

$$\underset{a_i \in A_i}{\operatorname{argmin}} \left[\max_{a_{-i} \in A_{-i}} \left(\left[\max_{a_i' \in A_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i}) \right) \right]$$

- Minmax regret actions are one that yields the smallest maximum regret

Removal of dominated strategies

Definition (Domination)

Let s_i and s_i' be two strategies of player i, and S_{-i} the set of all strategy profiles of the remaining players. Then,

1. s_i strictly dominates s_i' if for all $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
2. s_i weekly dominates s_i' if for all $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$, and for at least one $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
3. s_i very weekly dominates s_i' if for all $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$

- Domination is comparison between two strategies s_i and s_i' given others $s_{-i} \in S_{-i}$

Definition (Pareto domination)

Strategy profile s Pareto dominates strategy profile s' if for all $i \in N$, $u_i(s) \geq u_i(s')$, and there exists some $j \in N$ for which $u_j(s) > u_j(s')$.

Removal of dominated strategies

Definition (Dominant strategy)

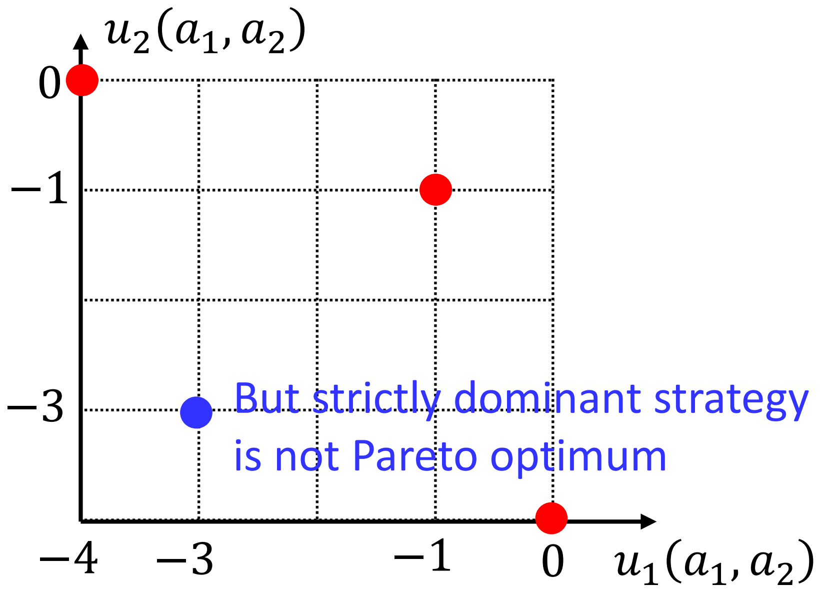
A strategy is strictly (resp., weekly; very weakly) dominant for an agent if it strictly (weakly; very weakly) dominates any other strategy for that agent.

- A strategy profile $(s_1, \ldots, s_n)$ in which every s_i is dominant for player i (whether strictly, weakly, or very weakly) is a Nash equilibrium.
- A strategy profile consisting of dominant strategies for every player must be a Nash equilibrium
  - ➢ An equilibrium in strictly dominant strategies must be unique.

	C	D
C	-1, -1	-4, 0
D	0, -4	$\boxed{-3, -3}$

Strictly dominant strategy profile

Prisoner's Dilemma game



Removal of dominated strategies

Definition (Dominated strategy)

A strategy s_i is strictly (weakly; very weakly) dominated for an agent i if some other strategy s_i' strictly (weakly; very weakly) dominates s_i

- Note that it is easy to check if a strategy is dominated because we need to find any strategy that dominate it

Removal of dominated strategies

	L	C	R
U	$3, 1$	$0, 1$	$0, 0$
M	$1, 1$	$1, 1$	$5, 0$
D	$0, 1$	$4, 1$	$0, 0$

- R is dominated by L

Removal of dominated strategies

	L	C	R
U	$3, 1$	$0, 1$	$0, 0$
M	$1, 1$	$1, 1$	$5, 0$
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Removal of dominated strategies

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	L	C
U	$3, 1$	$0, 1$
M	$1, 1$	$1, 1$
D	$0, 1$	$4, 1$

- M is dominated by the mixed strategy that selects U and D with equal probability

	L	C
U	$3, 1$	$0, 1$
D	$0, 1$	$4, 1$

- No other strategies are dominated.

Removal of dominated strategies

	L	C	R
U	$4, 3$	$5, 1$	$6, 2$
M	$2, 1$	$8, 4$	$3, 6$
D	$3, 0$	$9, 6$	$2, 8$

- Find an equilibrium by yourself

Removal of dominated strategies

- This process preserves Nash equilibria.
  - strict dominance: all equilibria preserved.
  - weak or very weak dominance: at least one equilibrium preserved.

- Thus, it can be used as a preprocessing step before computing an equilibrium
  - Some games are solvable using this technique.
  - Example: Prisoner's Dilemma!

- What about the order of removal when there are multiple dominated strategies?
  - strict dominance: doesn't matter.
  - weak or very weak dominance: can affect which equilibria are preserved.

	L	C
U	$1, 1$	$2, 1$
D	$1, 2$	$3, 1$

- Remove the action of the column player first
- Remove the action of the row player first

What is the result?

Cournot duopoly

- Two identical firms, players 1 and 2, produce some good
- Firm i produce quantity q_i
- Cost for production is $c_i(q_i) = 10q_i$
- Price is given by $d = 100 - (q_1 + q_2)$
- The profit of company 1 is $u_1(q_1, q_2) = (100 - q_1 - q_2)q_1 - 10q_1 = 90q_1 - q_1^2 - q_1q_2$

What should firm 1 do in order to maximize their profit?

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- Firm i produce quantity q_i
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What should firm 1 do in order to maximize their profit?

- As the payoff is concave in q_1, the maximum is obtained by imposing the derivative of the payoff with respect q_1 for any given value of q_2

$$q_1 = \frac{90 - q_2}{2}$$

  - That is, for any given q_2 chosen by company 2, company maximize its payoff

- The same applied to company 2

$$q_2 = \frac{90 - q_1}{2}$$

Cournot duopoly

- The profit of company 1 is $u_1(q_1, q_2) = (100 - q_1 - q_2)q_1 - 10q_1 = 90q_1 - q_1^2 - q_1q_2$
- As the payoff is concave in q_1, the maximum is obtained by imposing the derivative of the payoff with respect q_1 for any given value of q_2

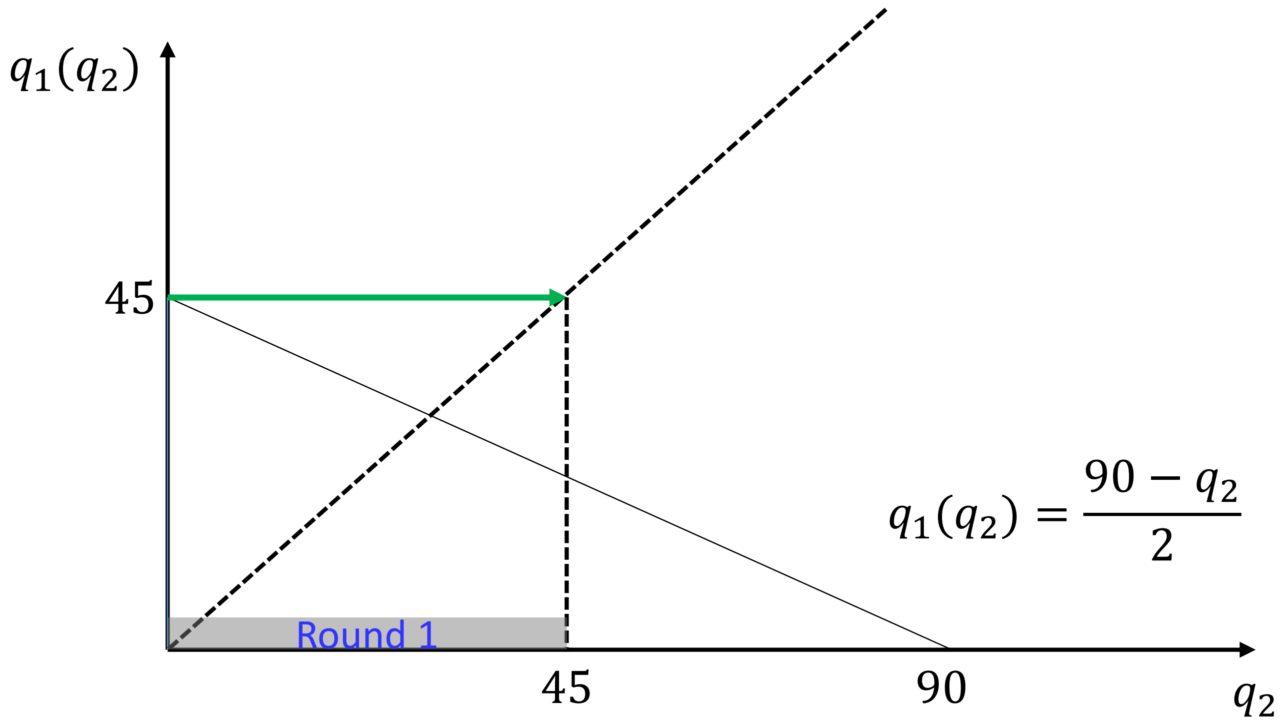
$$q_1 = \frac{90 - q_2}{2}$$

- Company 1 will never choose to produce more than $q_1 > 45$ because any quantity $q_1 > 45$ is strictly dominated by $q_1 = 45$ as follows:
  - $u_1(q_1 = 45, q_2) = (100 - 45 - q_2)45 - 450 = 2025 - 45q_2$
  - $u_1(q_1, q_2) = (100 - q_1 - q_2)q_1 - 10q_1 = 90q_1 - q_1^2 - q_1q_2$
  - $u_1(45, q_2) - u_1(q_1, q_2) = 2025 - q_1(90 - q_1) - q_2(45 - q_1) > 0$ for any $q_1 > 45$ regardless of q_2
- Due to symmetry, any $q_2 > 45$ is strictly dominated by $q_2 = 45$

- **The first round of iterated elimination**:
  - ➢ A rational produces no more than 45 units, implying that the effective strategy space that survives one round of elimination is $q_i \in [0,45]$ for $i \in \{1,2\}$

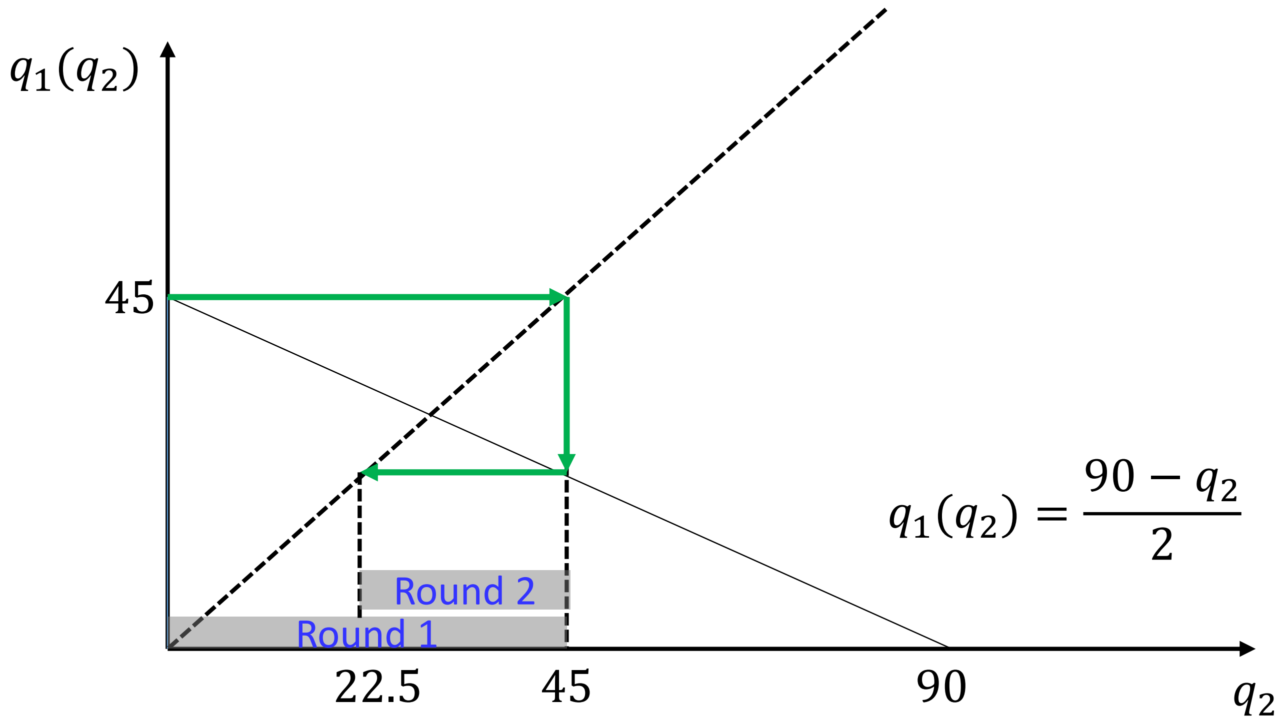
Cournot duopoly

- **The first round of iterated elimination**:
  - $q_2 > 45$ is strictly dominated by $q_2 \leq 45$



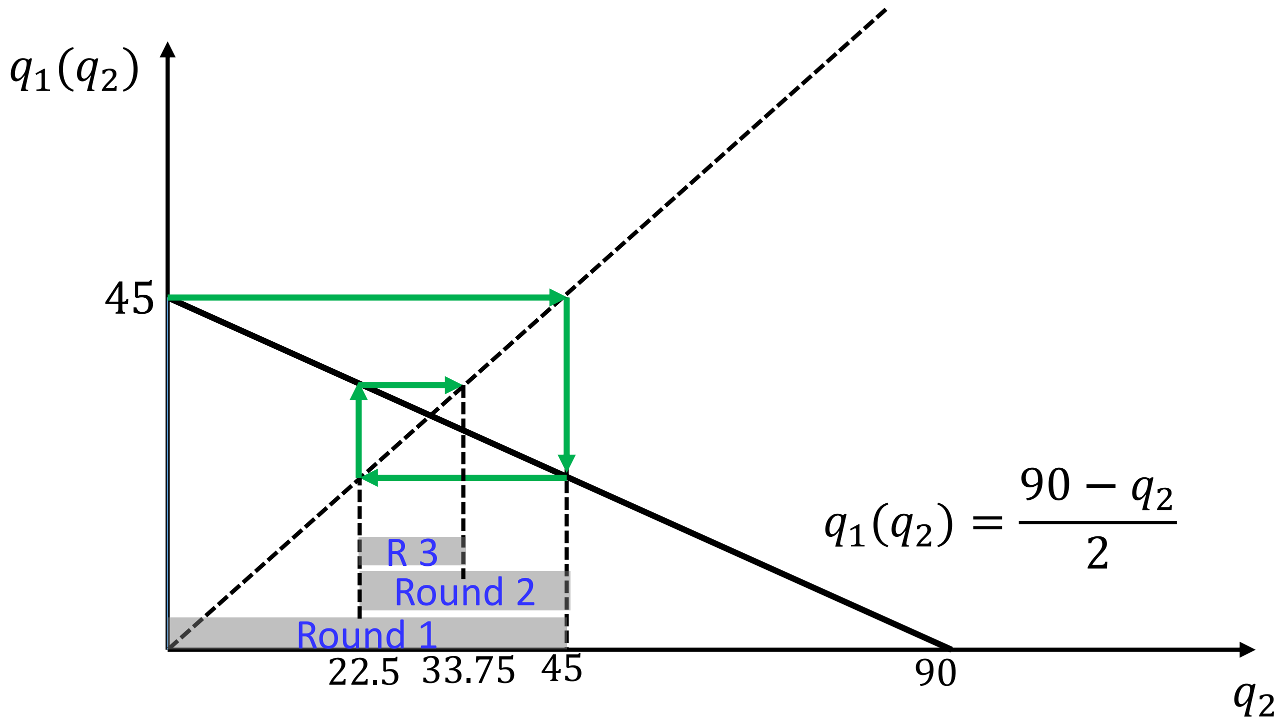
Cournot duopoly

- **The second round of iterated elimination**:
  - Because $q_2 \leq 45$, the equation $q_1 = \frac{90-q_2}{2}$ implies that company 1 will chose $q_1 \geq 22.5$
  - Symmetric argument applies to $q_2 \geq 22.5$
  - Therefore the second round of elimination implies that the surviving strategy sets are $q_i \in [22.5, 45]$ for $i \in \{1,2\}$



Cournot duopoly

- **The third round of iterated elimination**:
  - Because $q_2 \geq 22.5$, the equation $q_1 = \frac{90-q_2}{2}$ implies that company 1 will chose $q_1 \leq 33.75$
  - Symmetric argument applies to $q_2 \leq 33.75$
  - Therefore the second round of elimination implies that the surviving strategy sets are $q_i \in [22.5, 33.75]$ for $i \in \{1,2\}$



Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is heads rational in matching pennies?
  - is cooperate rational in prisoner’s dilemma?
- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.
- Furthermore, in two-player games, rationalizable $\Leftrightarrow$ survives iterated removal of strictly dominated strategies.

If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium.

– Roger Myerson

Correlated equilibrium

- Consider again Battle of the Sexes.
  - The values of each player under mixed Nash equilibrium is 2/3

		Player 2	
		p TF	$1-p$ LA
Player 1	q TF	2, 1	0, 0
	$1-q$ LA	0, 0	1, 2

$$u_1(\mathrm{TF}) = u_1(\mathrm{LA})$$
$$2 \times p + 0 \times (1-p) = 0 \times p + 1 \times (1-p)$$
$$p = \frac{1}{3}$$

$$u_2(\mathrm{TF}) = u_2(\mathrm{LAL})$$
$$1 \times q + 0 \times (1-q) = 0 \times q + 2 \times (1-q)$$
$$q = \frac{2}{3}$$

- The mixed Nash equilibrium is $s^* = (s_1^*, s_2^*) = \left\{\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right\}$
- The expected payoff under s^* are $u_1^* = \frac{2}{3} = u_2^*$

Can we do better?

Correlated equilibrium

- Consider again Battle of the Sexes.
  - The values of each player under mixed Nash equilibrium is 2/3

			Player 2	
			p TF	$1-p$ LA
Player 1	q	TF	2, 1	0, 0
	$1-q$	LA	0, 0	1, 2

- We could use the same idea to achieve the fair outcome in battle of the sexes.
  - Intuitively, the best outcome seems a 50-50 split between (TF, TF) and (LA, LA).

$$u_1^{CE} = \frac{1}{2}(2+1) = \frac{3}{2} > u_1^{NE} = \frac{2}{3}$$

$$u_2^{CE} = \frac{1}{2}(2+1) = \frac{3}{2} > u_2^{NE} = \frac{2}{3}$$

We show that no player has an incentive to deviate from the "recommendation" of the coin.

Correlated equilibrium

- Another classic example: traffic game

	Go	Wait
Go	-100, -100	10, 0
Wait	0, 10	-10, -10

Traffic game

- What is the natural solution here?
    - A traffic light: a fair randomizing device that tells one of the agents to go and the other to wait.
- Benefits:
    - the negative payoff outcomes are completely avoided
    - fairness is achieved
    - the sum of social welfare exceeds that of mixed Nash equilibrium

- More complex example

	L	R
U	5, 1	0, 0
D	4, 4	1, 5

- There are two pure strategy Nash equilibria: (U, L) and (D, R).
- This implies that there is a unique mixed strategy equilibrium with expected payoff (5/2,5/2).
- Suppose the players find a mediator who chooses $x \in \{1, 2, 3\}$ with equal probability 1/3. She then sends the following messages:
  - If $x = 1$, player 1 plays U, player 2 plays L.
  - If $x = 2$, player 1 plays D, player 2 plays L.
  - If $x = 3$, player 1 plays D, player 2 plays R.

  Actions are correlated

- Our first example presumed that everyone perfectly observes the random event; not required.
- More generally, some random variable with a commonly known distribution, and a private signal to each player about the outcome.
  - signal doesn't determine the outcome or others' signals; however, correlated:
    - ✓ Actions for agents are jointly determined by a drawn random variable

Correlated equilibrium

	L	R
U	5, 1	0, 0
D	4, 4	1, 5

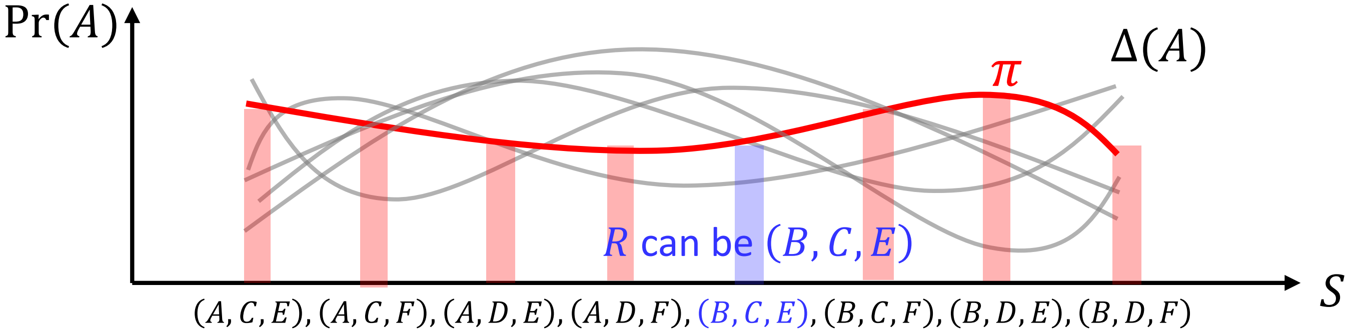
- If $x = 1$, player 1 plays U, player 2 plays L.
- If $x = 2$, player 1 plays D, player 2 plays L.
- If $x = 3$, player 1 plays D, player 2 plays R.

- We show that no player has an incentive to deviate from the "recommendation" of the mediator:
  - If player 1 gets the recommendation U, he believes player 2 will play L, so his best response is to play U.
  - If player 1 gets the recommendation D, he believes player 2 will play L, R with equal probability, so playing D is a best response.
  - If player 2 gets the recommendation L, he believes player 1 will play U, D with equal probability, so playing L is a best response.
  - If player 2 gets the recommendation R, he believes player 1 will play D, so his best response is to play R.
- Thus the players will follow the mediator's recommendations.
- With the mediator, the expected payoffs are (10/3, 10/3), strictly higher than what the players could get by randomizing between Nash equilibria.

Correlated equilibrium

- The preceding examples lead us to the notions of correlated strategies and "correlated equilibrium".
- Let $\Delta(A)$ denote the set of probability measures over the set A. Let R be a random variable taking values in $A = \Pi_{i=1}^{n} A_i$ distributed according to $\pi \in \Delta(A)$.
  - An instantiation of R is a pure strategy profile and the i th component of the instantiation will be called the recommendation to player i.
  - Given such a recommendation, player i can use conditional probability to form a posteriori beliefs about the recommendations given to the other players.

- $A_1 = \{A, B\}$, $A_2 = \{C, D\}$, $A_3 = \{E, F\}$
- $A = \{(A,C,E),(A,C,F),(A,D,E),(A,D,F),(B,C,E),(B,C,F),(B,D,E),(B,D,F)\}$
- $\Delta(A)$ is a set of probability mass function (PMF) over A
- $\pi \in \Delta(A)$ is a PMF over A
- $R \sim \pi(A)$ is a random variable distributed according to π and represents the joint action



Definition (Correlated equilibrium)

A correlated equilibrium of finite game is a joint probability distribution $\pi \in \Delta(A)$ such that if R is random variable distributed according to π then

$$\sum_{a_{-i} \in A_{-i}} \mathrm{Prob}(R = a | R_i = a_i) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \mathrm{Prob}(R = a | R_i = a_i) u_i(a_i', a_{-i})$$

For all players i, all $a_i \in A_i$ such that $\mathrm{Prob}(R_i = a_i) > 0$, and all $a_i' \in A_i$

- A distribution π is defined to be a correlated equilibrium if no player can ever expect to unilaterally gain by deviating from his recommendation, assuming the other players play according to their recommendations.
  - a_i is a recommendation by R drawn from $\pi \in \Delta(A)$
  - a_i' is a deviation from this recommendation

Correlated equilibrium

Proposition

A joint probability distribution $\pi \in \Delta(S)$ is a correlated equilibrium of a finite game if and only if

$$\sum_{a_{-i} \in A_{-i}} \pi(a) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \pi(a) u_i(a_i', a_{-i})$$

For all players i, all $a_i \in A_i$, $a_i' \in A_i$ such that $a_i \neq a_i'$

Proof:

$$\mathrm{Prob}(R = a | R_i = a_i) = \frac{\pi(a_i, a_{-i})}{\pi(a_i)} = \frac{\pi(a)}{\sum_{t_{-i} \in S_{-i}} \pi(a_i, t_{-i})}$$

$$\sum_{s_{-i} \in S_{-i}} \mathrm{Prob}(R = a | R_i = a_i) u_i(a_i, a_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \mathrm{Prob}(R = a | R_i = a_i) u_i(a_i', a_{-i})$$

$$\sum_{s_{-i} \in S_{-i}} \frac{\pi(a)}{\sum_{t_{-i} \in S_{-i}} \pi(a_i, t_{-i})} u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \frac{\pi(a)}{\sum_{t_{-i} \in S_{-i}} \pi(a_i, t_{-i})} u_i(a_i', a_{-i})$$

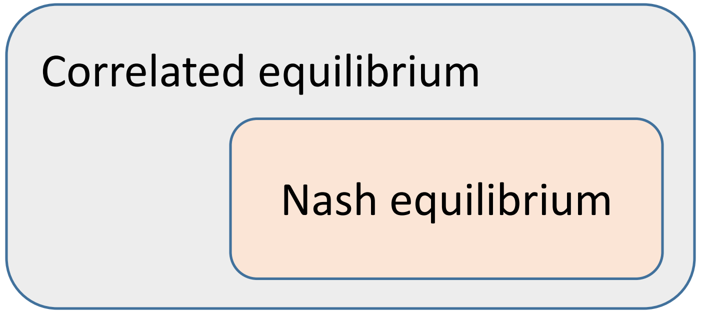
- The denominator does not depend on the variable of summation so it can be factored out of the sum and cancelled

Correlated equilibrium

Theorem (Correlated equilibrium)

For every Nash equilibrium s^* there exists a corresponding correlated equilibrium σ

- Not every correlated equilibrium is equivalent to a Nash equilibrium, (e.g., Battle of Sex game)
  - Correlated equilibrium is a strictly weaker notion than Nash



- Any convex combination of the payoffs achievable under correlated equilibria is itself realizable under a correlated equilibrium

ϵ – Nash equilibrium

- Players might not care about changing their strategies to a best response when the amount of utility that they could gain by doing so is very small.

Definition (ϵ – Nash equilibrium)

Fix $\epsilon > 0$. A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is an ϵ-Nash equilibrium if, for all agents i and for all strategies $s_i \neq s_i^*$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) - \epsilon$

	L	R
U	$1, 1$	$0, 0$
D	$1 + \frac{\epsilon}{2}, 1$	$500, 500$

A game with interesting ϵ – Nash equilibrium

Computing Solution Concepts for Normal Form Games

Motivations

- So far, we have ignored the issues of computation for finding equilibriums
- How hard is it to compute the Nash equilibria of a game?

Try to identify some pure strategy that is strictly better than s_i for any pure strategy profile of the others.

```
for all pure strategies a_i ∈ A_i for player i where a_i ≠ s_i do
    dom ← true
    for all pure strategy profiles a_{-i} ∈ A_{-i} for the players other than i
    do
        if u_i(s_i, a_{-i}) ≥ u_i(a_i, a_{-i}) then
            dom ← false
            break
        end if
    end for
    if dom = true then return true
end for
return false
```

- **We will discuss the computation methods for:**
  - Nash equilibria of two-player, zero-sum game
  - Nash equilibria of two-player, general-sum game
  - Nash equilibria of n-player, general-sum game
  - maximin and minmax strategies for two-player, general-sum games
  - Computing correlated equilibria

Linear Programming (LP)

- Mathematical optimization problem can be expressed as

$$\begin{array}{ll} \text{minmize} & f_o(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \dots, m \end{array}$$

- $x = (x_1, \dots, x_n)$: optimization variables
- $f_0 \colon \mathbf{R}^n \to \mathbf{R}$: objective function
- $f_i \colon \mathbf{R}^n \to \mathbf{R}$, $i = 1, \dots, m$: constraint functions

- A linear program is defined by:
  - a set of real-valued variables
  - a linear objective function
    - a weighted sum of the variables
  - a set of linear constraints
    - the requirement that a weighted sum of the variables must be greater than or equal to some constant

$$\begin{array}{ll} \text{minmize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, m \end{array}$$

Computing Nash equilibria of two-player, zero-sum game

Theorem (Minmax theorem by von Neumann, 1928)

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

- Consider a two-player, zero-sum game $G = \left(\{1,2\}, A_1 \times A_2, (u_1, u_2)\right)$.
- Let $U_1^* = -U_2^*$
- By the minmax theorem, U_1^* holds constant in all equilibria and that it is the same as the value that player 1 achieves under a minmax strategy by player 2

$$\bar{u}_1 = \max_{s_1} \min_{s_2} u_1(s_1, s_2) \quad = \quad \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

minmax strategy by player 2

Computing Nash equilibria of two-player, zero-sum game

- Standard form convex optimization problem can be converted into epigraph form:

Using slack variables

$$\begin{array}{ll} \text{minimize (over } x,\ s) & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\ & s_i \geq 0, \quad i = 1, \ldots m \end{array}$$

Standard convex optimization from

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

Epigraph form

$$\begin{array}{ll} \text{minimize (over } x,\ t) & t \\ \text{subject to} & f_0(x) - t \leq 0 \\ & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

$$\begin{array}{lll} \text{minimize} & U_1^* & \\ \text{subject to} & \sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* & \forall a_1 \in A_1 \\ & \sum_{a_2 \in A_2} s_2^{a_2} = 1 & \\ & s_2^{a_2} \geq 0 & \forall a_2 \in A_2 \end{array}$$

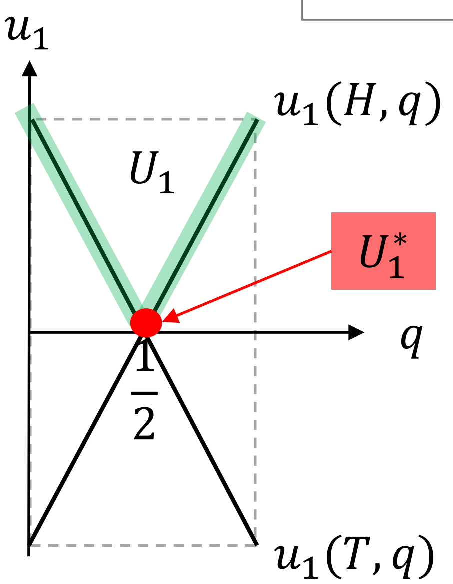
- First, identify the variables:
  - U_1^* is the expected utility for player 1
  - $s_2^{a_2}$ is player 2's probability of playing action a_2 under his mixed strategy
- each $u_1(a_1, a_2)$ is a constant
- Decision variables are U_1^* and $s_2^{a_2}$ for $\forall a_2 \in A_2$
- The LP will choose player 2's mixed strategy in order to minimize U_1^*

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

$$\begin{aligned} \text{minimize} \quad & U_1^* \\ \text{subject to} \quad & \sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \le U_1^* \qquad \forall a_1 \in A_1 \\ & \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\ & s_2^{a_2} \ge 0 \qquad \forall a_2 \in A_2 \end{aligned}$$



- Player 2's minmax strategy:

$$\underline{s}_2 = \operatorname*{argmin}_{s_2} \max_{s_1} u_1(s_1, s_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

minimize $\quad U_1^*$

subject to $\quad \sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \qquad \forall a_1 \in A_1$

$$\sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_2^{a_2} \geq 0 \qquad \forall a_2 \in A_2$$

- For every pure strategy j of player 1, his expected utility for playing any action $j \in A_1$ given player 2's mixed strategy s_2 is at most U_1^*
  - Those pure strategies for which the expected utility is exactly U_1^* will be in player 1's best response set

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

minimize $\quad U_1^*$

subject to $\quad \sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \le U_1^* \qquad \forall a_1 \in A_1$

$$\sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_2^{a_2} \ge 0 \qquad \forall a_2 \in A_2$$

- Player 2 plays the mixed strategy that minimizes the utility player 1 can gain by playing his best response

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

$$\begin{aligned}
&\text{minimize} && U_1^* && \\
&\text{subject to} && \sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* && \forall a_1 \in A_1 \\
& && \sum_{a_2 \in A_2} s_2^{a_2} = 1 && \\
& && s_2^{a_2} \geq 0 && \forall a_2 \in A_2
\end{aligned}$$

- s_2 is a valid probability distribution

Computing Nash equilibria of two-player, zero-sum game

For player 2's strategy

$$U_1^* = \underline{u}_1 = \min_{s_2} \max_{s_1} u_1(s_1, s_2)$$

minimize U_1^*

subject to
$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \quad \forall a_1 \in A_1$$
$$\sum_{a_2 \in A_2} s_2^{a_2} = 1$$
$$s_2^{a_2} \geq 0 \quad \forall a_2 \in A_2$$

Introduce slack variables $r_1^{a_1}$ for every $a_1 \in A_1$

minimize U_1^*
$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} + r_1^{a_1} = U_1^* \quad \forall a_1 \in A_1$$
$$\sum_{a_2 \in A_2} s_2^{a_2} = 1$$
$$s_2^{a_2} \geq 0 \quad \forall a_2 \in A_2$$
$$r_1^{a_1} \geq 0 \quad \forall a_1 \in A_1$$

Computing Nash equilibria of two-player, zero-sum game

For player 1's strategy

$$U_1^* = \bar{u}_1 = \max_{s_1} \min_{s_2} u_1(s_1, s_2)$$

maximize U_1^*

subject to $\sum_{a_1 \in A_1} u_1(a_1, a_2) \times s_1^{a_1} \geq U_1^* \quad \forall a_2 \in A_2$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1$$

$$s_1^{a_1} \geq 0 \qquad \forall a_1 \in A_1$$

- First, identify the variables:
  - U_1^* is the expected utility for player 1
  - $s_1^{a_1}$ is player 1's probability of playing action a_1 under his mixed strategy
- each $u_1(a_1, a_2)$ is a constant
- Decision variables are U_1^* and $s_1^{a_1}$ for $\forall a_1 \in A_1$
- The LP will choose player 1's mixed strategy in order to maximize U_1^*

Computing Nash equilibria of two-player, general-sum game

- The problem of finding a Nash equilibrium of a two-player, general-sum game cannot be formulated as a linear programming
  - The two players' interests are no longer directly opposed
  - We cannot state our problem as an optimization problem: one player is not trying to minimize the other's utility

Computing Nash equilibria of two-player, general-sum game

Let's define (s_1, s_2) is NE with $u_1(s_1, s_2) = U_1^*$

If $a_1 \in$ support for s_1

$$u_1(a_1, s_2) = U_1^*$$

Otherwise

$$u_1(a_1, s_2) \leq U_1^*$$

Let's define (s_1, s_2) is NE with $u_2(s_1, s_2) = U_2^*$

If $a_2 \in$ support for s_2

$$u_2(s_1, a_2) = U_2^*$$

Otherwise

$$u_1(s_1, a_2) \leq U_2^*$$

Computing Nash equilibria of two-player, general-sum game

Let's define (s_1, s_2) is NE with $u_1(s_1, s_2) = U_1^*$

If $a_1 \in$ support for s_1

$$u_1(a_1, s_2) = U_1^*$$

Otherwise

$$u_1(a_1, s_2) \leq U_1^*$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \quad \forall a_1 \in A_1$$

Let's define (s_1, s_2) is NE with $u_2(s_1, s_2) = U_2^*$

If $a_2 \in$ support for s_2

$$u_2(s_1, a_2) = U_2^*$$

Otherwise

$$u_1(s_1, a_2) \leq U_2^*$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} \leq U_2^* \quad \forall a_2 \in A_2$$

Computing Nash equilibria of two-player, general-sum game

Let's define (s_1, s_2) is NE with $u_1(s_1, s_2) = U_1^*$

If $a_1 \in$ support for s_1

$$u_1(a_1, s_2) = U_1^*$$

Otherwise

$$u_1(a_1, s_2) \le U_1^*$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \le U_1^* \quad \forall a_1 \in A_1$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} + r_1^{a_1} = U_1^*, \forall a_1 \in A_1, r_1^{a_1} \ge 0$$

Let's define (s_1, s_2) is NE with $u_2(s_1, s_2) = U_2^*$

If $a_2 \in$ support for s_2

$$u_2(s_1, a_2) = U_2^*$$

Otherwise

$$u_1(s_1, a_2) \le U_2^*$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} \le U_2^* \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} u_1(a_1, a_2) \times s_1^{a_1} + r_2^{a_2} = U_1^*, \forall a_2 \in A_2, r_2^{a_2} \ge 0$$

Computing Nash equilibria of two-player, general-sum game

Let's define (s_1, s_2) is NE with $u_1(s_1, s_2) = U_1^*$

If $a_1 \in$ support for s_1

$$u_1(a_1, s_2) = U_1^*$$

Otherwise

$$u_1(a_1, s_2) \leq U_1^*$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \quad \forall a_1 \in A_1$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} + r_1^{a_1} = U_1^*, \forall a_2 \in A_2, r_1^{a_1} \geq 0$$

$$s_1^{a_1} > 0 \rightarrow r_1^{a_1} = 0; s_1^{a_1} \times r_1^{a_1} = 0$$

Let's define (s_1, s_2) is NE with $u_2(s_1, s_2) = U_2^*$

If $a_2 \in$ support for s_2

$$u_2(s_1, a_2) = U_2^*$$

Otherwise

$$u_1(s_1, a_2) \leq U_2^*$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} \leq U_2^* \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} u_1(a_1, a_2) \times s_1^{a_1} + r_2^{a_2} = U_1^*, \forall a_2 \in A_2, r_2^{a_2} \geq 0$$

$$s_2^{a_2} > 0 \rightarrow r_2^{a_2} = 0; s_2^{a_2} \times r_2^{a_2} = 0$$

Linear complementarity problem (LCP) formulation for two-player, general-sum game

$$u_1(a_1, s_2) \leq u_1(a_1^*, s_2) \quad \forall a_1 \in A_1$$

$$u_2(s_1, a_2) \leq u_2(s_1, a_2^*) \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \geq 0, \ s_2^{a_2} \geq 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} \leq U_1^* \qquad \forall a_1 \in A_1$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} \leq U_2^* \qquad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \geq 0, \ s_2^{a_2} \geq 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

Linear complementarity problem (LCP) formulation for two-player, general-sum game

$$u_1(a_1, s_2) \le u_1(a_1^*, s_2) \quad \forall a_1 \in A_1$$

$$u_2(s_1, a_2) \le u_2(s_1, a_2^*) \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \ge 0, \; s_2^{a_2} \ge 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} + r_1^{a_1} = U_1^* \qquad \forall a_1 \in A_1$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} + r_2^{a_2} = U_2^* \qquad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \ge 0, \; s_2^{a_2} \ge 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$r_1^{a_1} \ge 0, \; r_2^{a_2} \ge 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

- The slack variables are introduced to convert inequality constraints to equality constrains

Issues

- The variables U_1^* and U_2^* would be insufficiently constrained
  - ➢ We want these values to express the expected utility that each player would achieve by playing his best responses to the other player's chosen mixed strategy

Computing Nash equilibria of two-player, general-sum game

Linear complementarity problem (LCP) formulation for two-player, general-sum game

$$u_1(a_1, s_2) \leq u_1(a_1^*, s_2) \quad \forall a_1 \in A_1$$

$$u_2(s_1, a_2) \leq u_2(s_1, a_2^*) \quad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \geq 0, \; s_2^{a_2} \geq 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2) \times s_2^{a_2} + r_1^{a_1} = U_1^* \qquad \forall a_1 \in A_1$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2) \times s_1^{a_1} + r_2^{a_2} = U_2^* \qquad \forall a_2 \in A_2$$

$$\sum_{a_1 \in A_1} s_1^{a_1} = 1, \sum_{a_2 \in A_2} s_2^{a_2} = 1$$

$$s_1^{a_1} \geq 0, \; s_2^{a_2} \geq 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$r_1^{a_1} \geq 0, \; r_2^{a_2} \geq 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

$$r_1^{a_1} \cdot s_1^{a_1} = 0, \; r_2^{a_2} \cdot s_2^{a_2} = 0 \qquad \forall a_1 \in A_1, \forall a_2 \in A_2$$

- Add the nonlinear constraints, called the complementarity condition (non-linear programing)
- This constraint requires that whenever an action is played by a given player with positive probability (supports for a strategy) then the corresponding slack variable must be zero
  - It capture the fact that, in equilibrium, all strategies that are played with positive probability must yield the same expected payoff
  - all strategies that lead to lower expected payoffs are not played

Computing Nash equilibria of two-player, general-sum game

- LCP problem can be formulated in a Quadratic programming that can be solved using an optimization solver (for this class, we can use a library for LCP solver)

- Classical algorithm to solve LCP is Lemke-Howson algorithm, which is similar to simplex method for Linear Programming (LP)

Computing Nash equilibria of n-player, general-sum game

- For n-player games where $n \geq 3$, the problem of finding a Nash equilibrium can no longer be represented even as an LCP
    - Hopelessly impractical to solve exactly
- Textbook discusses how to formulate the problem to find NEs using heuristic methods

Computing maximin and minmax strategies for two-player, general-sum games

- Let's say we want to compute a maxmin strategy for player 1 in an arbitrary 2-player game G
  - Create a new game G' where player 2's payoffs are just the negatives of player 1's payoffs.
  - The maxmin strategy for player 1 in G does not depend on player 2's payoffs
    - Thus, the maxmin strategy for player 1 in G is the same as the maxmin strategy for player 1 in G'
  - By the minmax theorem, equilibrium strategies for player 1 in G' are equivalent to a maxmin strategies
  - Thus, to find a maxmin strategy for G, find an Nash equilibrium strategy for G'

$$G = \left(\{1,2\}, A_1 \times A_2, (u_1, u_2)\right) \longrightarrow G' = \left(\{1,2\}, A_1 \times A_2, (u_1, -\boldsymbol{u_1})\right)$$

Computing correlated equilibria

- A sample correlated equilibrium can be found in polynomial time using a linear programming formulation
- Every game has at least one correlated equilibrium in which the value of the random variable can be interpreted as a recommendation to each agent of what action to play, and in equilibrium the agents all follow these recommendations.
- Thus, we can find a sample correlated equilibrium if we can find a probability distribution over pure action profiles satisfying

$$\sum_{a_{-i}\in A_{-i}} \pi(a)u_i(a_i,a_{-i}) \geq \sum_{a_{-i}\in A_{-i}} \pi(a)u_i(a_i',a_{-i}) \qquad \forall i \in N, \forall a_i, a_i' \in A_i \qquad (1)$$

$$\pi(a) > 0 \qquad \forall a \in A \qquad (2)$$

$$\sum_{a\in A} \pi(a) = 1 \qquad (3)$$

- Variables: $\pi(a)$, constants: $u_i(a)$
- Constraint (1) requires player i must be better off playing action a_i when he is told to do so than playing any other action a_i', given that other players play their prescribed action
- Constraint (2) and (3) requires p is a valid probability distribution

Computing correlated equilibria

- One can select a desired correlated equilibrium by adding an objective function to the linear program.
  - For example, the problem maximizes the sum of the agents' expected utilities by adding the objective function (social-welfare maximizing CE)

$$
\begin{aligned}
&\text{maximize} && \sum_{a \in A} \pi(a) \sum_{i \in N} u_i(a) && \\
&\text{subject to} && \sum_{a_{-i} \in A_{-i}} \pi(a) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \pi(a) u_i(a_i', a_{-i}) \quad \forall i \in N, \forall a_i, a_i' \in A_i && (1) \\
& && \pi(a) > 0 \quad \forall a \in A && (2) \\
& && \sum_{a \in A} \pi(a) = 1 && (3)
\end{aligned}
$$

- Utilitarian equilibrium: an equilibrium which maximizes the sum of the expected payoffs of the players
- Libertarian i equilibrium: an equilibrium which maximizes the expected payoff of Player i
- Egalitarian equilibrium: an equilibrium which maximizes the minimum expected payoff of a player is called an.

Computing correlated equilibria : Example

	C	F
C	$2, 5$	$0, 0$
F	$0, 0$	$5, 2$

- Formulate LP to find the Libertarian 1 equilibrium (do it by your self):

Difference between Nash and Correlated equilibrium?

Why are CE easier to compute than NE?

- Intuitively, correlated equilibrium has only a single randomization over outcomes, whereas in NE this is constructed as a product of independent probabilities.
- To change this program so that it finds NE, the first constraint would be

$$\sum_{a_{-i}\in A_{-i}} \pi(a)u_i(a_i,a_{-i}) \geq \sum_{a_{-i}\in A_{-i}} \pi(a)u_i(a_i',a_{-i}) \qquad \forall i\in N, \forall a_i, a_i' \in A_i$$

$$\sum_{a\in A}\left(\prod_{j\in N} s_j(a_j)\right)u_i(a_i,a_{-i}) \geq \sum_{a\in A}\left(\prod_{j\in N} s_j(a_j)\right)u_i(a_i',a_{-i}) \qquad \forall i\in N, a_i' \in A_i$$

The constrain is non-linear!

$\pi(a_1^1,a_2^1)$	$\pi(a_1^1,a_2^2)$
$\pi(a_1^2,a_2^1)$	$\pi(a_1^2,a_2^2)$

Joint distribution

$s_1(a_1^1)s_2(a_2^1)$	$s_1(a_1^1)s_2(a_2^2)$
$s_1(a_1^2)s_2(a_2^1)$	$s_1(a_1^2)s_2(a_2^2)$

independent distribution