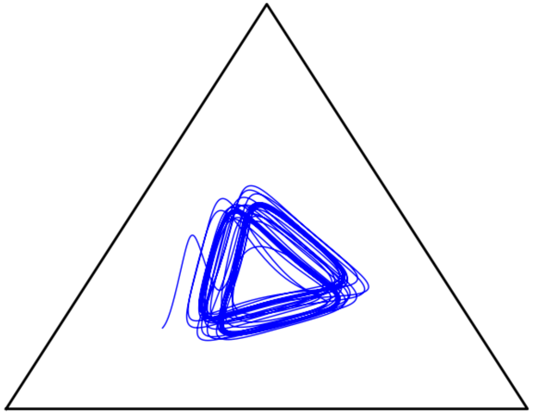
Lecture 16-The Replicator Dynamics

Dynamics

- The discussion of “dynamics” so far was largely heuristic.
- Are there actual dynamics of populations resulting from “game-theoretic” interactions that lead to evolutionarily stable strategies?
- Question at the intersection of game theory and population dynamics.
- The answer to this question is yes, and here we will discuss the simplest example, replicator dynamics.
- Throughout, we continue to focus on symmetric games.



Replicator Dynamics (Derivation)

- The replicator dynamics describes the evolution of the strategies under the assumption that the players are reactive to the environment
  - The players observe the population behavior and adopt their best-response strategies

- The replicator dynamics provides also an opportunity to look into asymptotic stability and to analyze the link with the evolutionarily stable strategies

Replicator Dynamics (Derivation)

- The evolutionary models encountered in the previous chapter do not consider any explicit dynamics
- Replicator dynamics model how the players change dynamically their strategies
- Let us enumerate the possible actions using an index $s = 1, 2, \dots, K$.
- Let us indicate with x_s the percentage of the population playing a strategy s
  - Obviously, it must hold that the sum of the percentages over the whole action space must sum up to one, i.e.,
$$\sum_{s=1}^{K} x_s = 1$$
  - Denote $x = (x_1, \dots, x_K)$ the population distribution
    - corresponds to a polymorphic strategy (evolutionary biology perspective)
    - corresponds to a mixed strategy (game theoretic perspective)

Replicator Dynamics (Derivation)

- Define the fitness at time t for playing a generic strategy s against a population playing x

$$u(s, x(t)) = \sum_{s'=1}^{K} u(s, s')x_{s'}(t)$$
$$= \mathbf{e}_s^T U x(t)$$

e.x., $\mathbf{e}_3 = [0,0,1]^T$

- Define the average fitness (payoff of population) at time t resulting from a population distribution given by $x(t)$, which can be computed as

$$\bar{u}(x(t)) = \sum_{s=1}^{K} x_s(t) u(s, x(t))$$

- Using the definition of fitness $u(s, x(t))$, we can express $\bar{u}(x(t))$ as

$$\bar{u}(x(t)) = \sum_{s=1}^{K} x_s(t) u(s, x(t)) = \sum_{s=1}^{K} x_s(t) \sum_{s'=1}^{K} u(s, s')x_{s'}(t) = x(t)^T U x(t)$$

with $U_{ss'} = u(s, s')$

Replicator Dynamics (Derivation)

- Let $n_s(t)$ the number of individual playing the strategy s
- Let $N(t)$ the total number of individual
- $x_s(t) = n_s(t)/N(t)$
- $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_s(t) \\ \vdots \\ x_K(t) \end{bmatrix} = \begin{bmatrix} n_1(t)/N(t) \\ \vdots \\ n_s(t)/N(t) \\ \vdots \\ n_K(t)/N(t) \end{bmatrix}$
- We assume that during the small time interval τ, only a τ fraction of the population takes part in pairwise competition, that is playing games as:

$$n_s(t+\tau) = (1-\tau)n_s(t) + \tau n_s(t)u(s, x(t)) \qquad (1)$$

$$N(t+\tau) = (1-\tau)N(t) + \tau N(t)\bar{u}(x(t)) \qquad (2)$$

Replicator Dynamics (Derivation)

$$n_s(t+\tau) = (1-\tau)n_s(t) + \tau n_s(t)u(s,x(t)) \quad (1)$$

$$N(t+\tau) = (1-\tau)N(t) + \tau N(t)\bar{u}(x(t)) \quad (2)$$

$$\text{LHS of } \frac{(1)}{(2)} = \frac{n_s(t+\tau)}{N(t+\tau)} = x_s(t+\tau)$$

$$\text{RHS of } \frac{(1)}{(2)} = \frac{(1-\tau)n_s(t) + \tau n_s(t)u(s,x(t))}{(1-\tau)N(t) + \tau N(t)\bar{u}(x(t))}$$

$$= \frac{(1-\tau)x_s(t)\cancel{N}(t) + \tau x_s(t)\cancel{N}(t)u(s,x(t))}{(1-\tau)\cancel{N}(t) + \tau \cancel{N}(t)\bar{u}(x(t))} \qquad \because x_s(t) = n_s(t)/N(t)$$

$$= \frac{x_s(t)\{(1-\tau) + \tau\bar{u}(x(t))\} - x_s(t)\tau\bar{u}\big(x(t)\big) + +\tau x_s(t)u(s,x(t))}{(1-\tau) + \tau\bar{u}(x(t))}$$

$$= x_s(t) - \frac{-x_s(t)\tau\bar{u}\big(x(t)\big) + \tau x_s(t)u(s,x(t))}{(1-\tau) + \tau\bar{u}(x(t))}$$

$$= x_s(t) + \tau x_s(t)\frac{\big[u(s,x(t)) - \bar{u}\big(x(t)\big)\big]}{(1-\tau) + \tau\bar{u}(x(t))}$$

$$\text{LHS} = \text{RHS} \Rightarrow x_s(t+\tau) - x_s(t) = \tau x_s(t)\frac{[u(s,x(t)) - \bar{u}(x(t))]}{(1-\tau) + \tau\bar{u}(x(t))}$$

Replicator Dynamics (Derivation)

$$\Rightarrow x_s\ (t+\tau) - x_s(t) = \tau x_s(t)\frac{[u(s,x(t)) - \bar{u}(x(t))]}{(1-\tau)+\tau\bar{u}(x(t))}$$

$$\Rightarrow \frac{x_s(t+\tau) - x_s(t)}{\tau} = x_s(t)\frac{[u(s,x(t)) - \bar{u}(x(t))]}{(1-\tau)+\tau\bar{u}(x(t))}$$

$$\Rightarrow \lim_{\tau\to 0}\frac{x_s(t+\tau) - x_s(t)}{\tau} = \lim_{\tau\to 0} x_s(t)\frac{[u(s,x(t)) - \bar{u}(x(t))]}{(1-\tau)+\tau\bar{u}(x(t))}$$

$$\Rightarrow \dot{x}_s(t) = x_s(t)[u(s,x(t)) - \bar{u}(x(t))]$$

Replicator Dynamics

Definition (Continuous-time replicator dynamics)

The continuous-time version of the dynamics takes the form

$$\dot{x}_s(t) = x_s(t)[u(s, x(t)) - \bar{u}(x(t))]$$

which is called continuous replicator

Definition (Discrete-time Replicator dynamics)

For each $s = 1,2, \ldots, K$ and for all t and τ, the replicator dynamic is given by

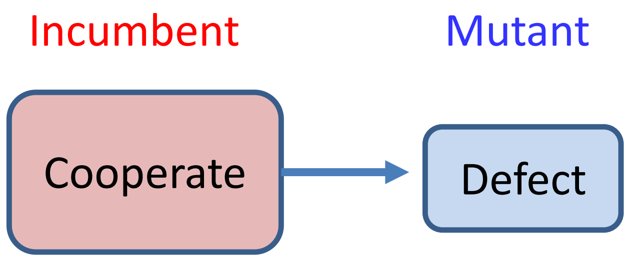
$$x_s(t + \tau) - x_s(t) = \tau x_s(t)[u(s, x(t)) - \bar{u}(x(t))]$$

- The greater the fitness of a strategy relative to the average fitness, the greater its relative increase in the population.

Replicator dynamics example: Prisoner’s Dilemma

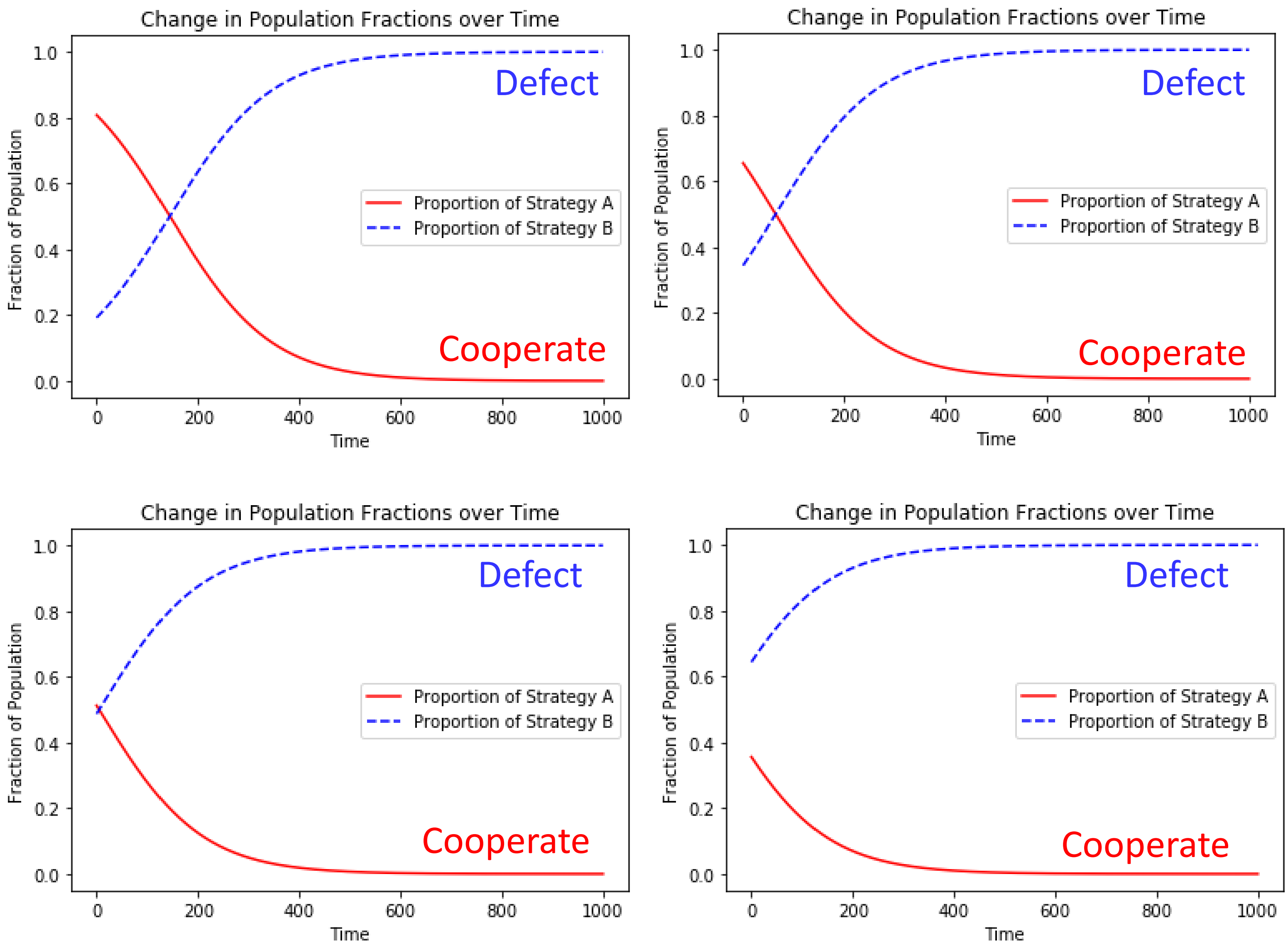
	Populations: $1-\epsilon$	ϵ
	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

<Prisoner’s dilemma as an evolutionarily game>



- Case 1: Incumbent (playing C) vc. the population:
  - C vc. $[(1-\epsilon)C + \epsilon D]$
  - The expected payoff : $(1-\epsilon)3 + \epsilon 0 = 3(1-\epsilon)$
- Case 2: Mutant (playing D) vc. the population:
  - D vc. $[(1-\epsilon)C + \epsilon D]$
  - The expected payoff : $(1-\epsilon)4 + \epsilon 1 = 4(1-\epsilon) + \epsilon$
- We need to compare the expected payoffs of incumbents and mutants when involved in random matchings with other individuals
  - Since $4(1-\epsilon) + \epsilon > 3(1-\epsilon)$, mutant performs better than the incumbent
    - ➢ Cooperation (Incumbent) is not a an evolutionarily stable strategy

Replicator dynamics example: Prisoner's Dilemma



- Regardless of the initial percentages, it always converges to Nash Equilibrium strategy

Replicator dynamics example: Rock Paper Scissor game

- Consider the modified rock-paper-scissors game:

	Rook	Paper	Scissors
Rook	γ, γ	-1, 1	1, -1
Paper	1, -1	γ, γ	-1, 1
Scissors	-1, 1	1, -1	γ, γ

- Here $0 \leq \gamma < 1$ (If $\gamma = 1$, this is the standard rock-paper-scissors game)
- For all such γ, there is a unique mixed strategy equilibrium $s^* = (1/3, 1/3, 1/3)$, with expected payoff $u(s^*, s^*) = \frac{\gamma}{3}$.
- But for $\gamma > 0$, this is not ESS. For example, $s = R$ would invade, since

$$u(s^*, s) = \gamma \times \frac{1}{3} - 1 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{\gamma}{3} < u(s, s) = \gamma$$

Under ESS: if $\underbrace{u(s^*, s^*) = u(s, s^*)}$, then $u(s^*, s) > u(s, s)$

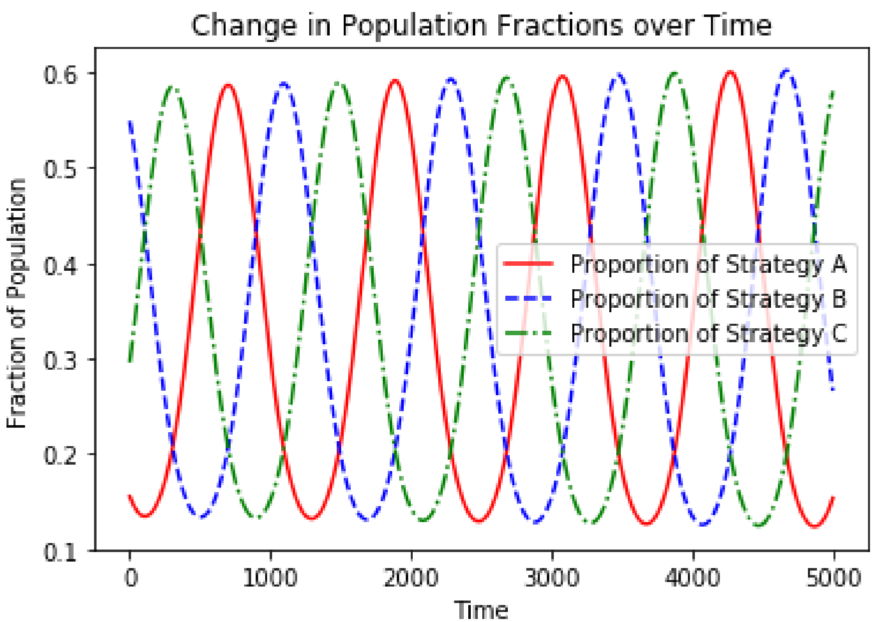
Hold when one player is plying a mixed Nash: $u(s^*, s^*) = u(R, s^*) = u(P, s^*) = \mathrm{u}(S, s^*)$

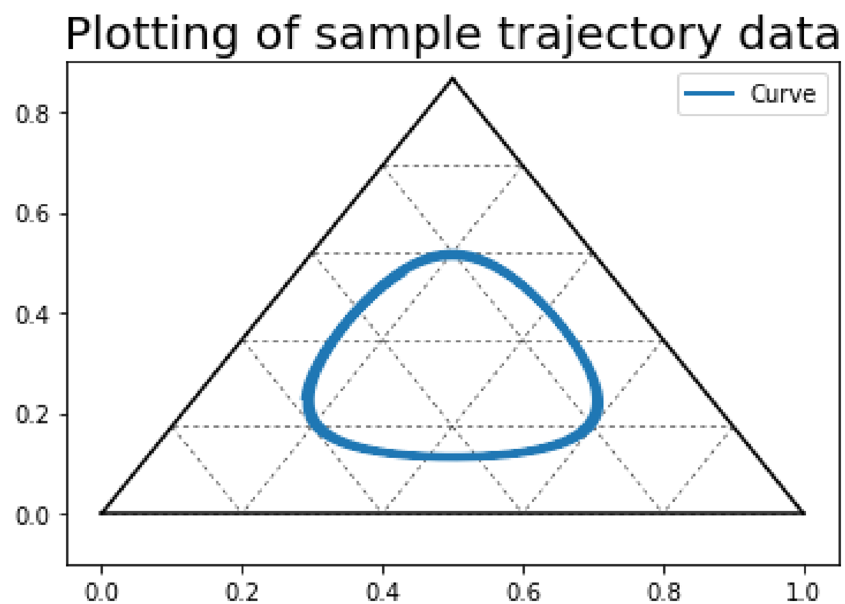
- This also shows that ESS doesn't necessarily exist

Replicator dynamics example: Rock Paper Scissor game

	Rook	Paper	Scissors
Rook	γ, γ	-1, 1	1, -1
Paper	1, -1	γ, γ	-1, 1
Scissors	-1, 1	1, -1	γ, γ

$\gamma = 0$

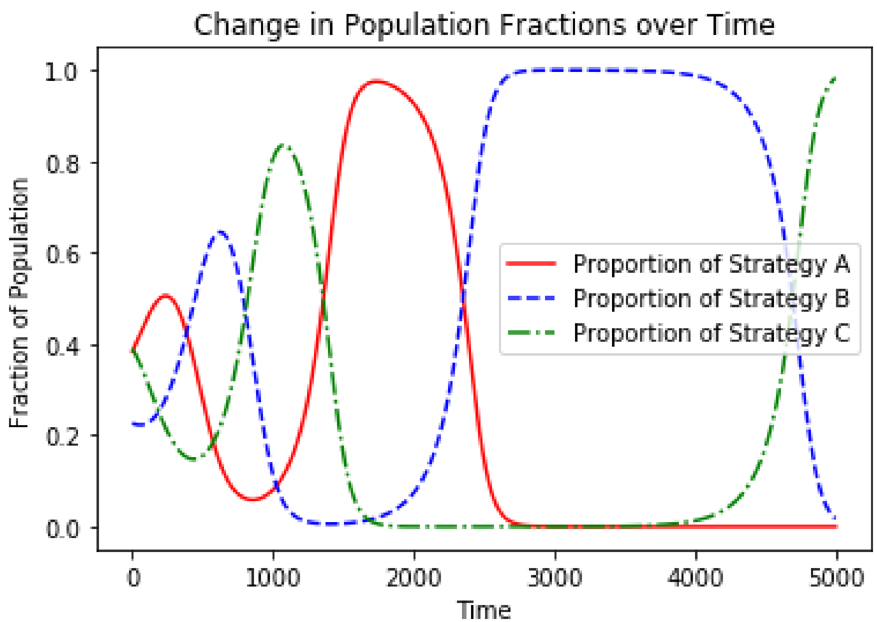


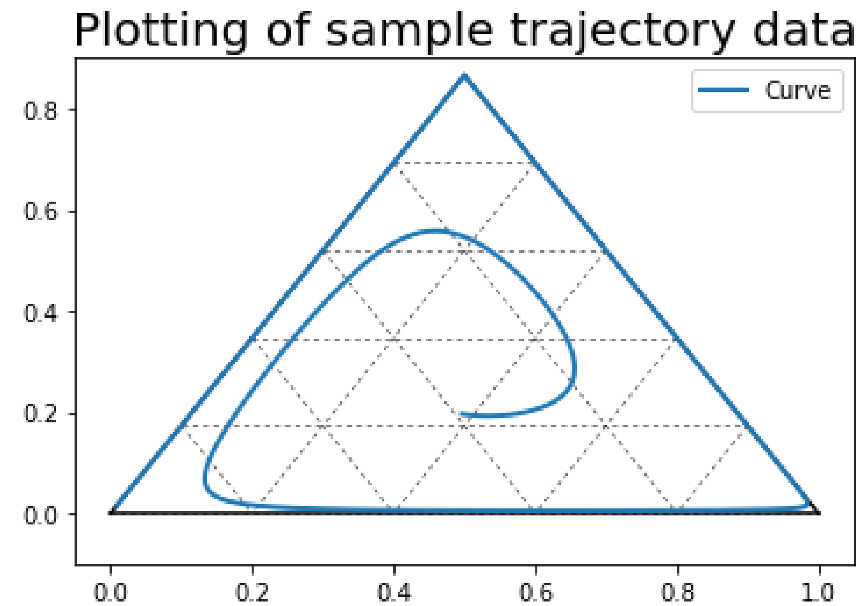


Replicator dynamics example: Rock Paper Scissor game

	Rook	Paper	Scissors
Rook	γ, γ	-1, 1	1, -1
Paper	1, -1	γ, γ	-1, 1
Scissors	-1, 1	1, -1	γ, γ

$\gamma = 0.1$

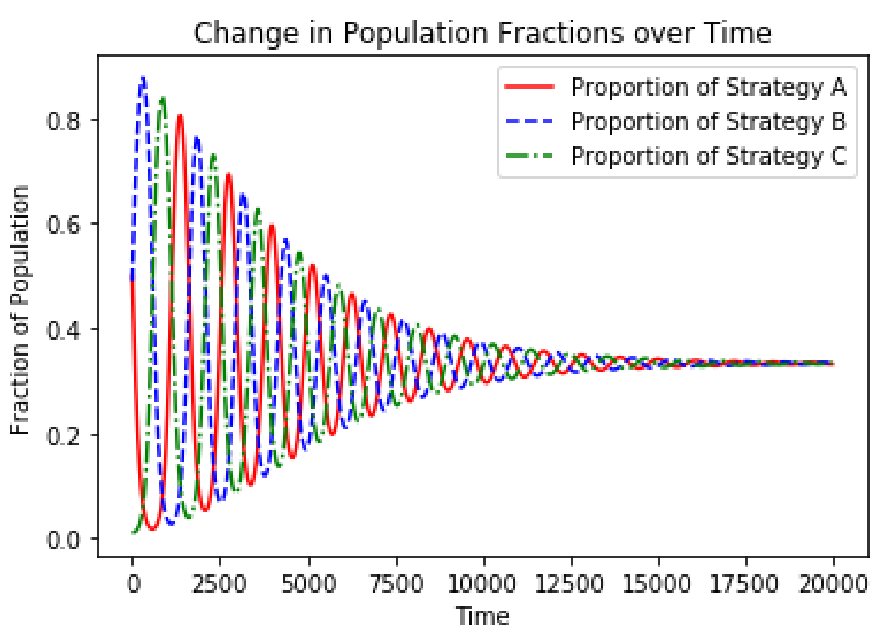


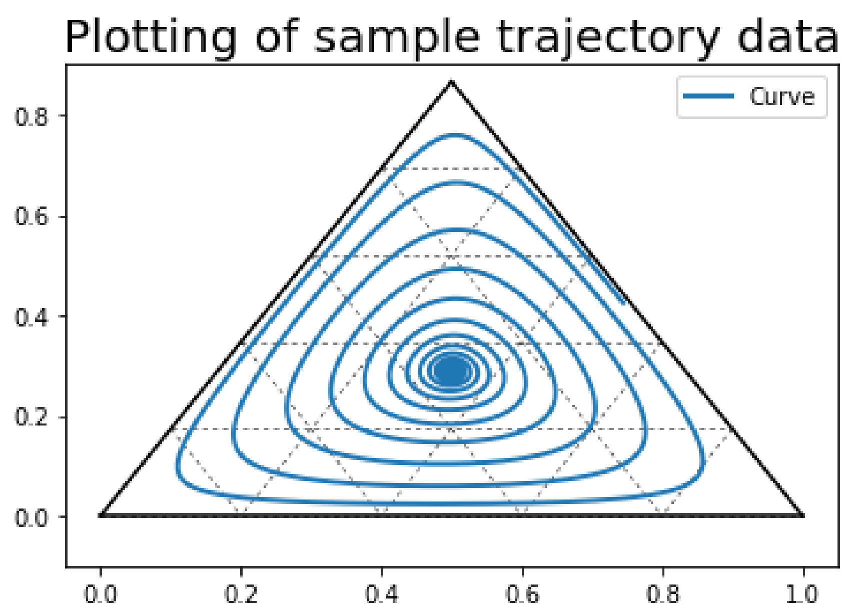


Replicator dynamics example: Rock Paper Scissor game

	Rook	Paper	Scissors
Rook	γ, γ	-1, 1	1, -1
Paper	1, -1	γ, γ	-1, 1
Scissors	-1, 1	1, -1	γ, γ

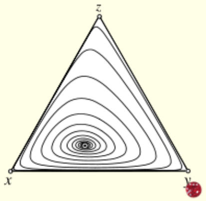
$\gamma = -0.1$





Further population dynamics

Deterministic Dynamics



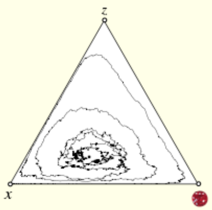
Replicator dynamics - Attractor

In the limit $N \to \infty$ demographic stochasticity arising in finite populations disappears and the dynamics becomes deterministic. For $s > 1$ the interior fixed point $\hat{x}$ is a stable focus of the replicator dynamics. All trajectories spiral toward $\hat{x}$.

Replicator-Mutator dynamics - Stable limit cycle

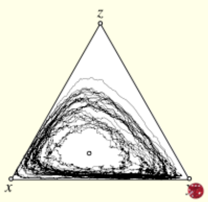
For $s < 1$ the interior fixed point $\hat{x}$ is an unstable focus. The trajectories spiral away from $\hat{x}$ and, in the absence of mutations, approach the heteroclinic cycle along the boundary of the simplex S_3. With mutation rates $\mu > 0$, however, the boundary of S_3 becomes repelling, which can give rise to stable limit cycles. If the mutation rate is sufficiently high, the interior fixed point is stable again. The image shows a sample trajectory for $s = 0.2$, $\mu = 0.001$.

Stochastic Dynamics



Stochastic differential equations

The interior fixed point $\hat{x}$ is a stable focus of the replicator dynamics, $s < 1$. Demographic stochasticity arises from the finite population size of $N = 100$. In the absence of mutations, the boundaries remain absorbing and even though the interior fixed point is attracting, stochastic fluctuations nevertheless eventually drive the population to the absorbing boundaries.



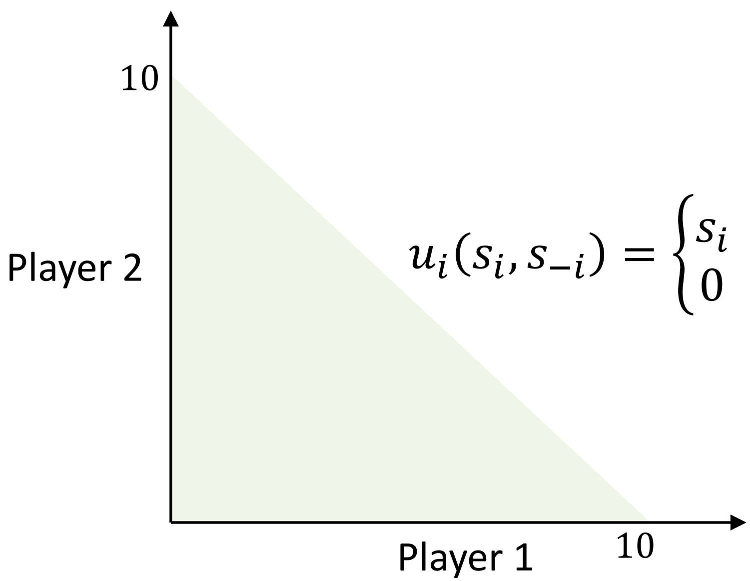
Stochastic differential equations, with mutations

The interior fixed point $\hat{x}$ of the replicator dynamics is an unstable focus. Even without stochasticity all trajectories spiral away from $\hat{x}$ toward the boundary of the simplex S_3. However, due to mutations, the boundary is repelling, which results in a stochastic analog of a stable limit cycle. For larger mutation rates the interior fixed point becomes stable again even for $s < 1$.

http://wiki.evoludo.org/index.php?title=Stochastic_dynamics_in_finite_populations

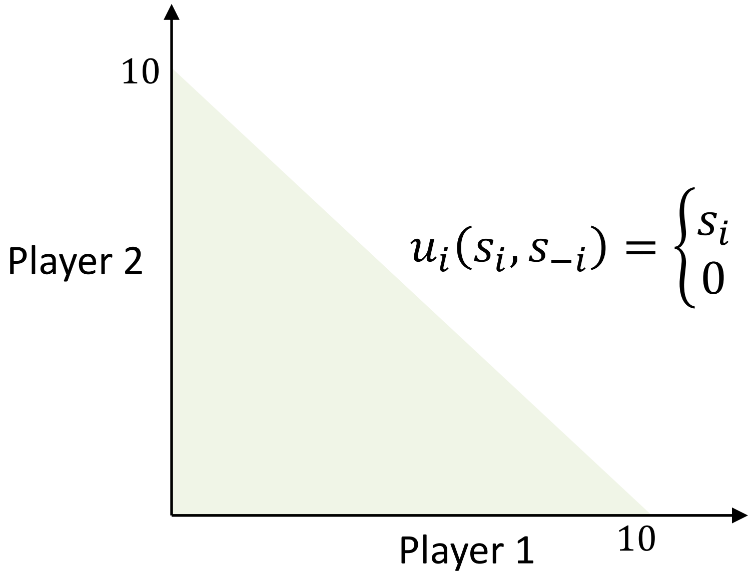
Replicator dynamics example: A sense of fairness

- One natural game to use for investigating the evolution of fairness is *divide-the-cake*
  - Suppose that two individuals are presented with a resource of size C by a third party.
  - A strategy for a player, in this game, consists of an amount of cake that he would like, thus $a_1 \in [0, C]$
  - If the sum of strategies for each player is less than or equal to C, each player receives the amount he asked for. However, if the sum of strategies exceeds C, players receive nothing
  - The feasible set of the game is as follow



$$u_i(s_i, s_{-i}) = \begin{cases} s_i & \text{if } s_i + s_{-i} \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

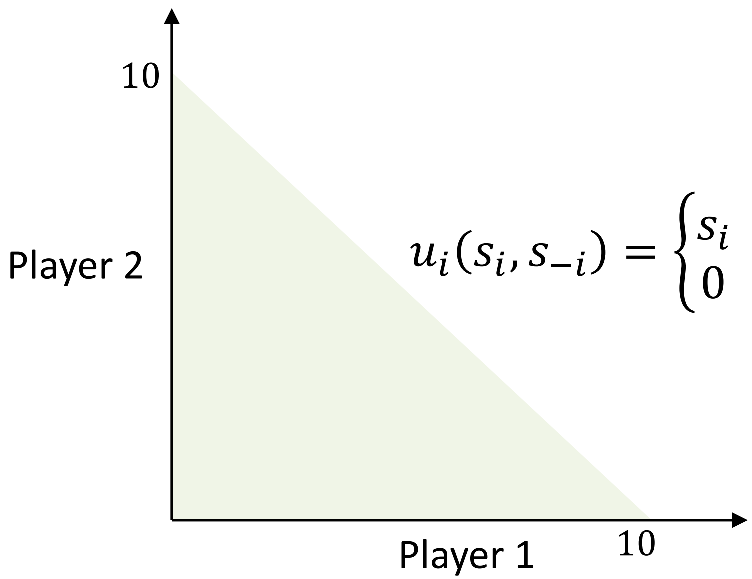
Replicator dynamics example: A sense of fairness



$$u_i(s_i, s_{-i}) = \begin{cases} s_i & \text{if } s_i + s_{-i} \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

- What is the Nash equilibrium?

Replicator dynamics example: A sense of fairness



$$u_i(s_i, s_{-i}) = \begin{cases} s_i & \text{if } s_i + s_{-i} \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

- What is the Nash equilibrium?

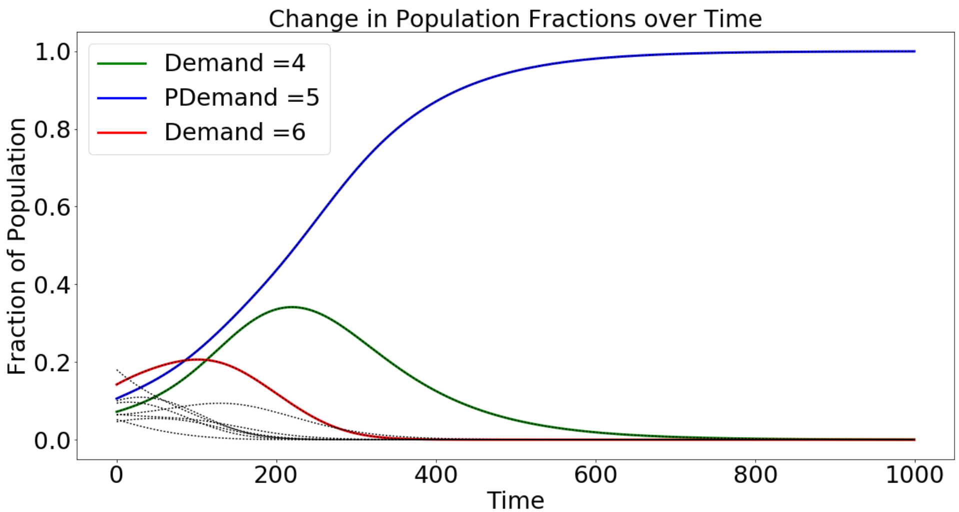
  - There are an infinite number of Nash equilibria for this game
    - Player 1 asks for $p \in [0, C]$ of the cake
    - Player 2 asks for $C - p$

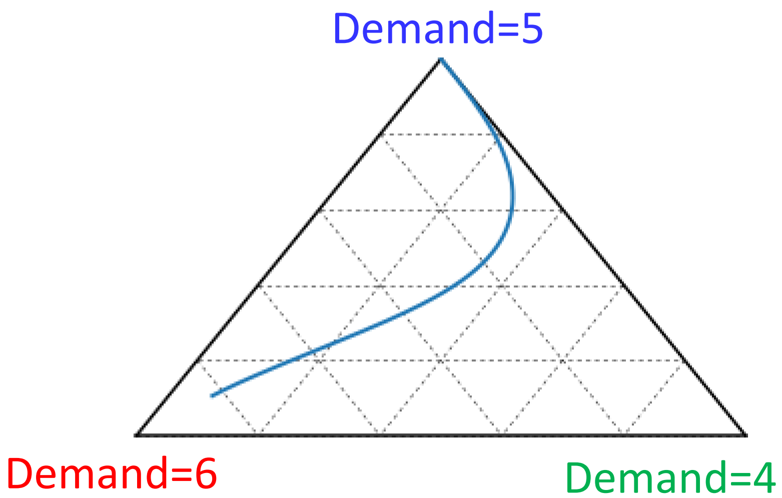
  - Therefore, the equal split is only one of infinitely many Nash equilibria

Replicator dynamics example: A sense of fairness

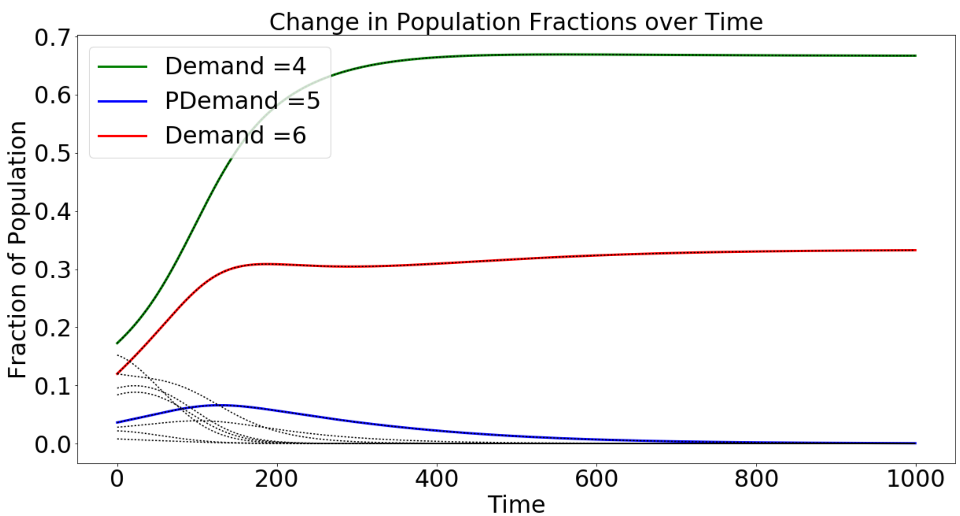
- Let's model this game using Replicator dynamics
  - Let's assume that the cake is divided into 10 equally sized slices
  - Each player's strategy conforms to one of the following 11 possible types:
    - Demand 0, Demand 1, …, Demand 10
  - For the replicator dynamics, the state of the population is represented by a vector $[p_0, p_1, \ldots, p_{10}]$, where each p_i denotes the frequency of the strategy "Demand i slices" in the population
- The replicator dynamics allows us to model how the distribution of strategies in the population changes over time, beginning from a particular initial condition

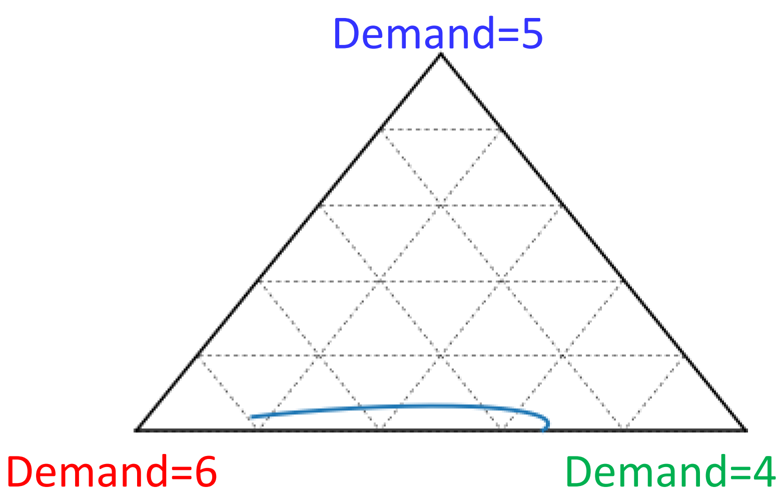
Replicator dynamics example: A sense of fairness





Evolution to equal division





Evolution to unequal division

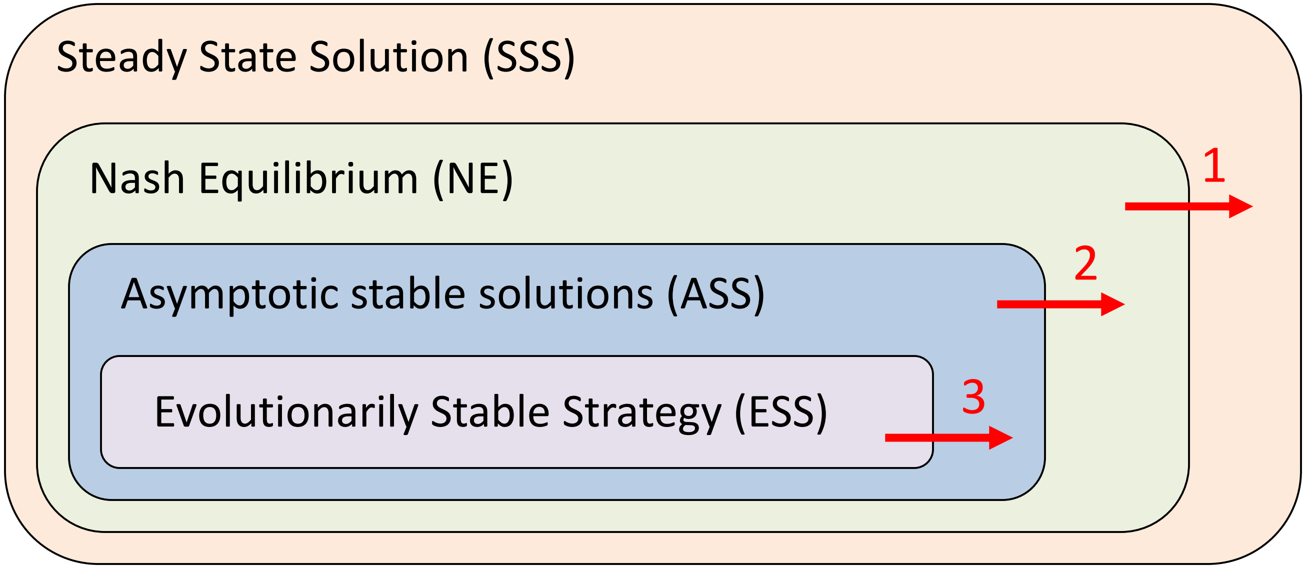
Replicator dynamics example: A sense of fairness

- In a population of bounded rational agents who modify their behaviors in a manner described by the replicator dynamics, fair division one, although not the only, evolutionary outcome.
- Evolves differently depending on the initial distribution
- The tendency of fair division to emerge can be measured by determining the size of the basin of attraction of the state where everyone in the population uses the strategy Demand 5 slices
- Skyrms (1996) measured this size using MC method, 62%

Stationarity, equilibria, and asymptotic stability

- Given a vector of distribution x^*, is such a vector a stationary state?
  - ➢ In systems theory, a stationary state is a state for which the first-order derivative is null, namely $\dot{x}^*(t) = 0$.
- Is a given vector of distribution x^* asymptotically stable?
  - ➢ There exists a neighborhood of x^* such that any trajectory starting from any x_0 in this neighborhood is such that the continuous replicator dynamics provides a trajectory that converges to x^*

Stationarity, equilibria, and asymptotic stability



Theorem

If x^* is a Nash equilibrium, then it is a stationary state.

Theorem

If x^* is asymptotically stable, then it is a Nash equilibrium.

Theorem

If x^* is Evolutionarily stable, then it is a asymptotically stable.

Stationarity, equilibria, and asymptotic stability

Theorem

If x^* is a Nash equilibrium, then it is a stationary state.

Proof:

- Assume x^* is a Nash equilibrium. Consequently, x^* must also be a best-response to itself:

$$(1)\quad u(s,x^*) - \bar{u}(x^*) \leq 0 \text{ for all } s$$
$$(2)\quad u(s,x^*) - \bar{u}(x^*) = 0 \text{ for all } s \text{ in the support of } x^*$$

(1) is true : $u(s,x^*) \leq \sum_{s=1}^{K} x_s^* u(s,x^*) = \bar{u}(x^*)$ because $x^* = BR(x^*)$

(2) is true due to indifference principle under Nash mixed strategy

- when $x_s^*(t) \neq 0, u(s,x^*) - \bar{u}(x^*) = 0$

- The two conditions (1) and (2) guarantees for any s, either $u(s,x^*) - \bar{u}(x^*) = 0$ or $x_s^*(t) = 0$, which makes

$$\dot{x}_s(t) = x_s(t)\frac{[u(s,x(t)) - \bar{u}(x(t))]}{\bar{u}(x(t))} = 0$$

Stationarity, equilibria, and asymptotic stability

Theorem

If x^* is asymptotically stable, then it is a Nash equilibrium.

- The proof is immediate if x^* corresponds to a pure strategy(monomorphic population).
- In the case where x^* corresponds to a mixed strategy Nash equilibrium, the proof is also straightforward but long. The basic idea is that equation (Continuous replicator) implies that we are moving in the direction of "better replies"—relative to the average. If this process converges, then there must not exist any more (any other) strict better replies, and thus we must be at a Nash equilibrium.
- The converse is again not true.

Stationarity, equilibria, and asymptotic stability

Theorem

If x^* is asymptotically stable, then it is a Nash equilibrium.

- Example showing the converse is not true.

	A	B
A	$1, 1$	$0, 0$
B	$0, 0$	$\boxed{0, 0}$

- Here $(B, B)$is a Nash equilibrium, but clearly it is not asymptotically stable, since B is weakly dominated, and thus any perturbation away from (B, B)will start a process in which the fraction of agents playing A steadily increases.
- Take $x(t) = (\epsilon, 1 - \epsilon)$; then
  - $x_A = \epsilon, u\big(A, x(t)\big) - \bar{u}\big(x(t)\big) = \epsilon - \epsilon^2 > 0$
  - Playing A returns a payoff higher than the average payoff computed over the population, and therefore the percentage of people playing A increases

Stationarity, equilibria, and asymptotic stability

Theorem

If x^* is Evolutionarily stable, then it is a asymptotically stable.

- The proof is again somewhat delicate, but intuitively straightforward. The first definition of ESS states that for small enough perturbations, the evolutionarily stable strategy is a strict best response. This essentially implies that in the neighborhood of the ESS x^*, x^* will do better than any other strategy x, and thus according to (Continuous replicator), the fraction of those playing x^* should increase, thus implying asymptotic stability.

- This is the key result that justifies the focus on Evolutionary stable strategies

Examples

Stag-hunt game

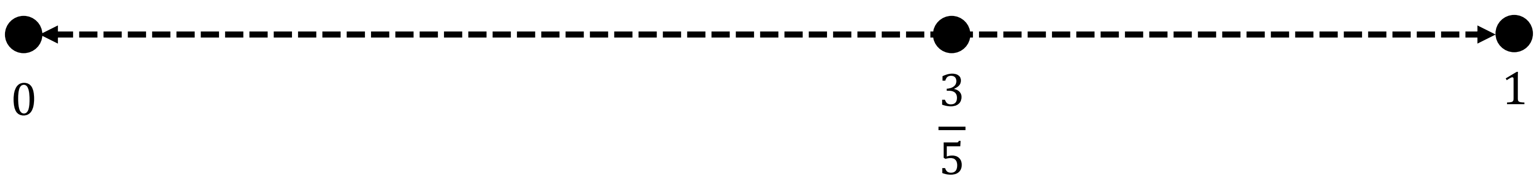
		S	H
$x_S(t)$	S	$5, 5$	$0, 3$
$1 - x_S(t)$	H	$3, 0$	$3, 3$

$$x(t) = \begin{bmatrix} x_s(t) \\ 1 - x_s(t) \end{bmatrix}$$

$$\dot{x}_s(t) = x_s(t)[u(x_s(t), x(t)) - \bar{u}(x(t))]$$

$$= x_s(t)\left([1, 0]\begin{bmatrix} 5 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_s(t) \\ 1 - x_s(t) \end{bmatrix} - [x_s(t), 1 - x_s(t)]\begin{bmatrix} 5 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_s(t) \\ 1 - x_s(t) \end{bmatrix}\right)$$

$$= x_s(t)\big(1 - x_s(t)\big)(5x_s(t) - 3)$$



Examples

Hawk-Dove game

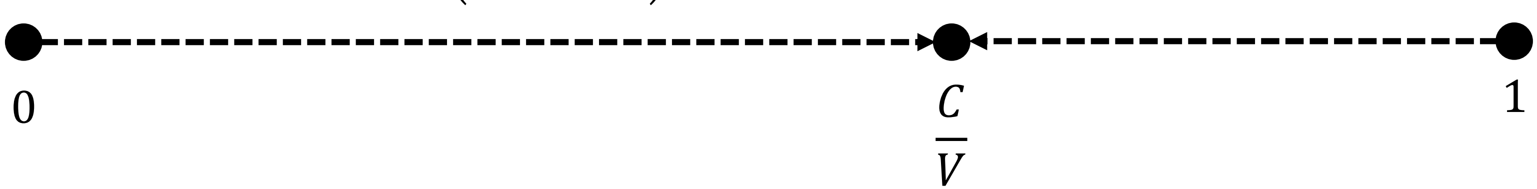
		H	D
$x_H(t)$	H	$\frac{V-C}{2}$	V
$1-x_H(t)$	D	0	$\frac{V}{2}$

$$x(t) = \begin{bmatrix} x_H(t) \\ 1-x_H(t) \end{bmatrix}$$

$$\dot{x}_H(t) = x_H(t)[u(x_H(t), x(t)) - \bar{u}(x(t))]$$

$$= x_H(t)\left([1,0]\begin{bmatrix} \frac{(V-C)}{2} & V \\ 0 & \frac{V}{2} \end{bmatrix}\begin{bmatrix} x_H(t) \\ 1-x_H(t) \end{bmatrix} - [x_H(t), 1-x_H(t)]\begin{bmatrix} \frac{(V-C)}{2} & V \\ 0 & \frac{V}{2} \end{bmatrix}\begin{bmatrix} x_H(t) \\ 1-x_H(t) \end{bmatrix} \right)$$

$$= -x_H(t)\big(1-x_H(t)\big)\left(x_H(t) - \frac{C}{V}\right)$$



Examples

Prisoner's Dilemma game

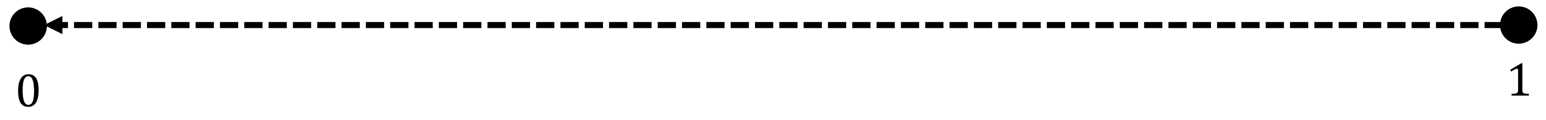
		C	D
$x_C(t)$	C	3	0
$1 - x_C(t)$	D	5	1

$$x(t) = \begin{bmatrix} x_C(t) \\ 1 - x_C(t) \end{bmatrix}$$

$$\dot{x}_H(t) = x_C(t)[u(x_C(t), x(t)) - \bar{u}(x(t))]$$

$$= x_C(t)\left([1,0]\begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x_C(t) \\ 1 - x_C(t) \end{bmatrix} - [x_C(t), 1 - x_C(t)]\begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x_C(t) \\ 1 - x_C(t) \end{bmatrix}\right)$$

$$= -x_H(t)\big(1 - x_H(t)\big)(x_H(t) + 1)$$



Why Evolutionary Game Theory?

- A numerous researchers hope evolutionary game theory will provide tools for addressing a number of deficiencies in the traditional theory of games

  (1) The equilibrium selection problem
  - If one restrict to select only pure strategies, some games lose solutions
  - When there are multiple NEs, various refinement concepts exist

  (2) The problem of hype rational agents
  - Numerous experiment results show rationality assumptions does not describe the behavior of real human subjects.
  - Evolutionary game theory can better describe and predict the choices of human subjects

  (3) The lack of a dynamical theory in the traditional theory of games
  - Most game theoretic equilibrium concepts are static
    - Even extensive form game, traditional game theory represents an individual’s strategy as a specification of what choice that individual would make at each information set in the game (selection can be made prior the game)

Evolution vs. Learning

- Evolution is a good model for fully myopic behavior. But even when individuals follow rules of thumb, they are not fully myopic.

- Moreover, in evolution, the time scales are long. We need "mutations," which are random and, almost by definition, rare.

- It would be better that one player observes his opponents' behavior, learns from these observations, and makes the best move in response to what he has learned

- In most (human) game-theoretic situations, even if individuals are not fully rational, they can imitate more successful strategies quickly, and learn the behavior of their opponents and best respond to those.

- This suggests a related but distinct approach to dynamic game-theoretic behavior, which is taken in the literature on *learning in games*.
  - Note that this is different from Bayesian game-theoretic learning, which focuses on learning the game structure but not strategies