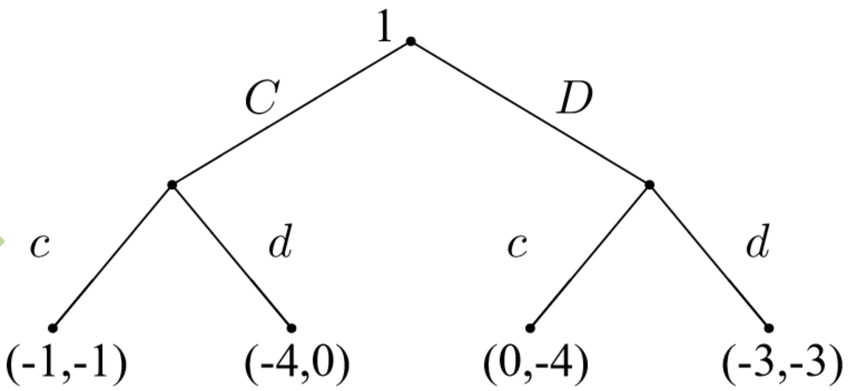
Lecture 7. Imperfect information extensive-form game

Motivations

Can we represent prisoner’s dilemma game into extensive form?

Player 2

		c	d
Player 1	C	-1, -1	-4, 0
	D	0, -4	-3, -3



Motivations

- Up to this point, in our discussion of extensive-form games we have allowed players to specify the action that they would take at every choice node of the game.

- This implies that players know the node they are in and all the prior choices, including those of other agents (perfect information game)

- We may want to model agents needing to act with partial or no knowledge of the actions taken by others, or even with limited memory of their own past actions.

- This is possible using imperfect information extensive-form games.
  - each player's choice nodes are partitioned into **information sets**
  - if two choice nodes are in the same information set then the agent cannot distinguish between them.

Formal definition

Definition (Imperfect-information game)

An imperfect-information game (in extensive form) is a tuple $(N, A, H, Z, \chi, \rho, u, I)$, where:

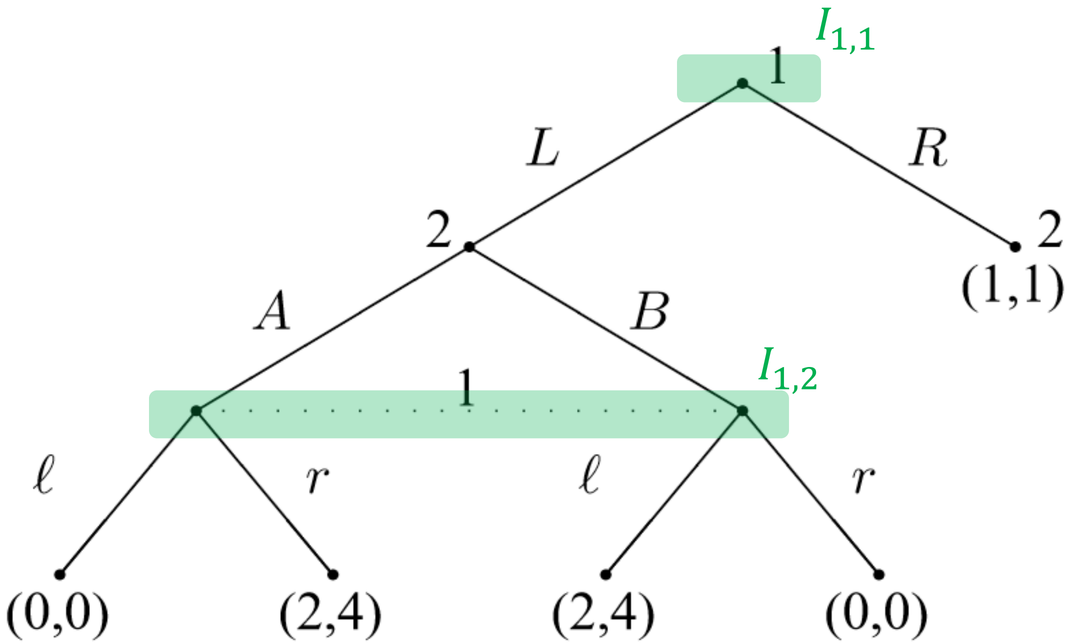
- $(N, A, H, Z, \chi, \rho, u)$ is a perfect-information extensive-form game; and
- $I = (I_1, \dots, I_n)$, where $I_i = \left(I_{i,1}, \dots, I_{i,k_i}\right)$ is a set of equivalence classes on (i.e., a partition of) $\{h \in H: \rho(h) = i\}$ with the property that $\chi(h) = \chi(h')$ and $\rho(h) = \rho(h')$ whenever there exists a j for which $h \in I_{i,j}$ and $h' \in I_{i,j}$.

- in order for the choice nodes to be truly indistinguishable, we require that the set of actions at each choice node in an information set be the same (otherwise, the player would be able to distinguish the nodes)

$$\chi(h) = \chi(h') \text{ and } \rho(h) = \rho(h')$$

- Thus, if $I_{i,j} \in I_i$ is an equivalence class, we can unambiguously use the notation $\chi\left(I_{i,j}\right)$ to denote the set of actions available to player i at any node in information set $I_{i,j}$

Example



An imperfect-information game

- player 1 has two information sets:

$$I_1 = (I_{1,1}, I_{1,2})$$

- The information set $I_{1,2}$ has the same set of possible actions

$$\chi(I_{1,2}) = \{l, r\}$$

- We can regard player 1 as not knowing whether player 2 chose A or B when he makes his choice between l and r

Pure strategies

Definition (Pure strategies for imperfect-information game)

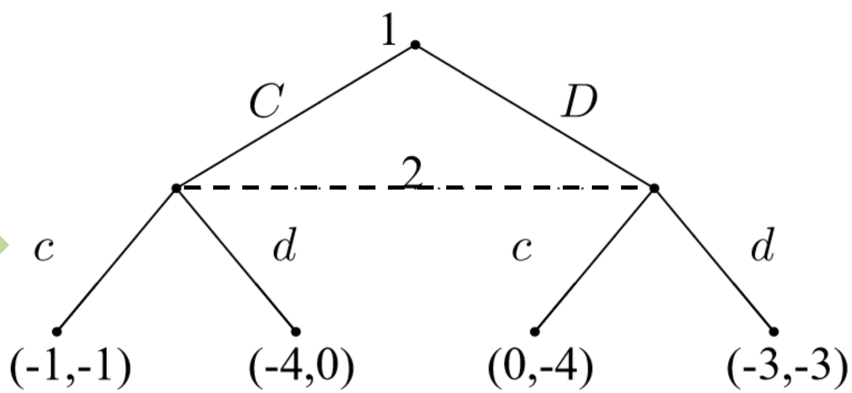
Let $G = (N, A, H, Z, \chi, \rho, u, I)$ be an imperfect-information game. Then the pure strategies of player i consist of the Cartesian product $\prod_{I_{i,j}} \chi(I_{i,j})$

- In other words, a pure strategy for an agent in an imperfect-information game selects one of the available actions in each information set of that agent
- Thus perfect-information games can be thought of as a special case of imperfect-information games, in which every equivalence class of each partition is a singleton.

Normal-form game

- Consider the Prisoner's Dilemma game, represented as the following imperfect information game:

		Player 2	
		c	d
Player 1	C	-1, -1	-4, 0
	D	0, -4	-3, -3



- Recall that perfect-information games were not expressive enough to capture the Prisoner's Dilemma game and many other ones

- In contrast, as is obvious from this example, any normal-form game can be trivially transformed into an equivalent imperfect-information game.

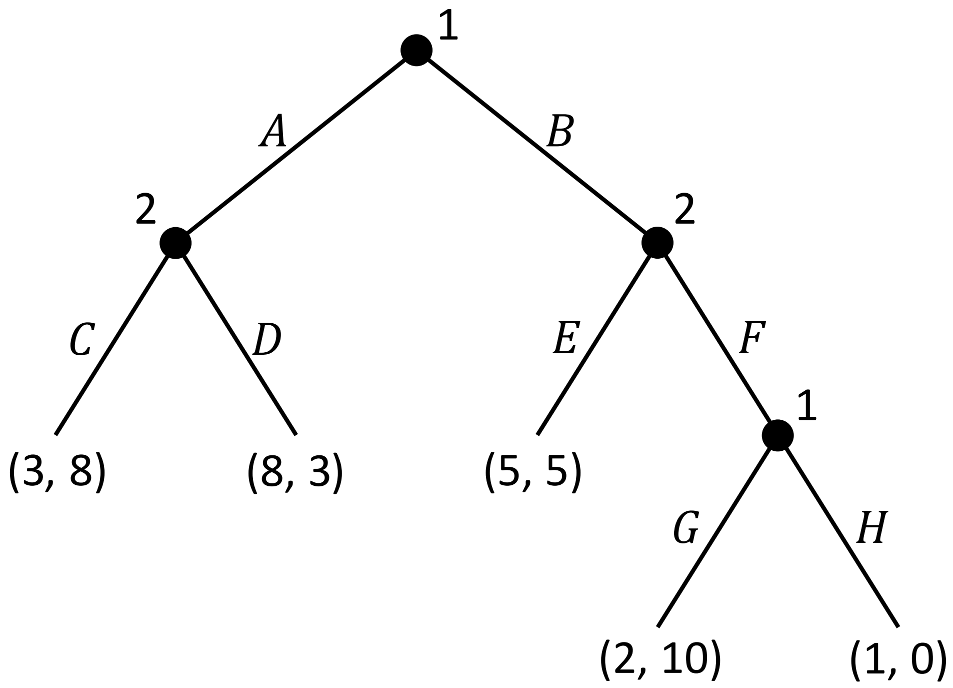
Induced normal-form game

- Same as before: enumerate pure strategies for all agents
- Mixed strategies are just mixtures over the pure strategies as before.
- Nash equilibria are also preserved.
- Note that we've now defined both mapping from normal form games (NF) to Imperfect information extensive form games (IIEF) and a mapping from IIEF to NF.
    - what happens if we apply each mapping in turn?
    - we might not end up with the same game, but we do get one with the same strategy spaces and equilibria.

Randomized strategies

- It turns out there are two meaningfully different kinds of randomized strategies in imperfect information extensive form games
  - Mixed strategy: randomize over pure strategies (i.e., distribution over vectors)
  - Behavioral strategy: **independent coin toss** every time an information set is encountered (vector of distribution)

Randomized strategies example



- Set of all pure strategy
  - $S_1 = \{(A, G), (A, H), (B, G), (B, G)\}$
  - $S_2 = \{(C, E), (C, F), (D, E), (D, F)\}$

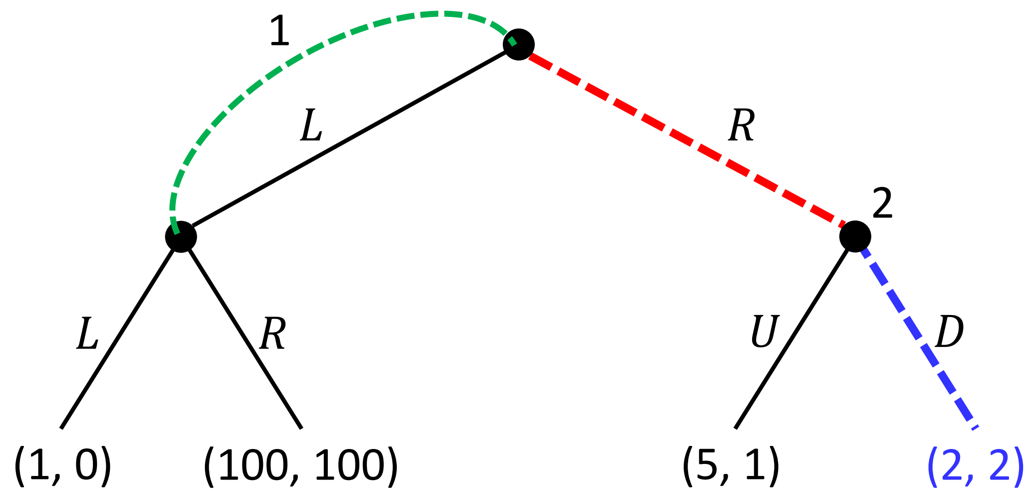
- Give an example of a mixed strategy
  - $\{0.15(A, G), 0.35(A, H), 0.15(B, G), 0.35(B, H)\}$

- Give an example of a behavioral strategy:
  - A with probability 0.5 and G with probability 0.3

- In this game every behavioral strategy corresponds to a mixed strategy...

Games of imperfect recall

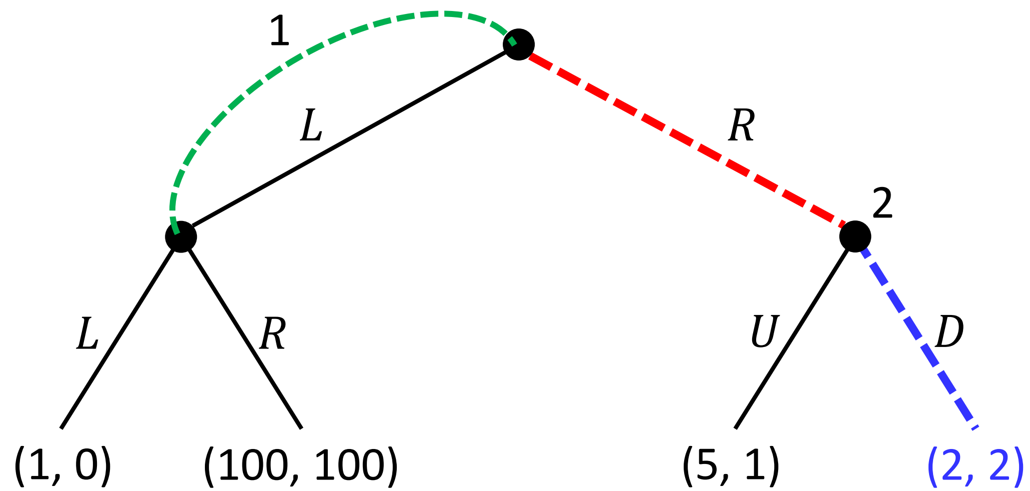
Consider the following game with imperfect recall



- the space of pure strategies
  - Agent 1: $\{L, R\}$,
  - Agent 2: $\{U, D\}$

Games of imperfect recall

Consider the following game with imperfect recall

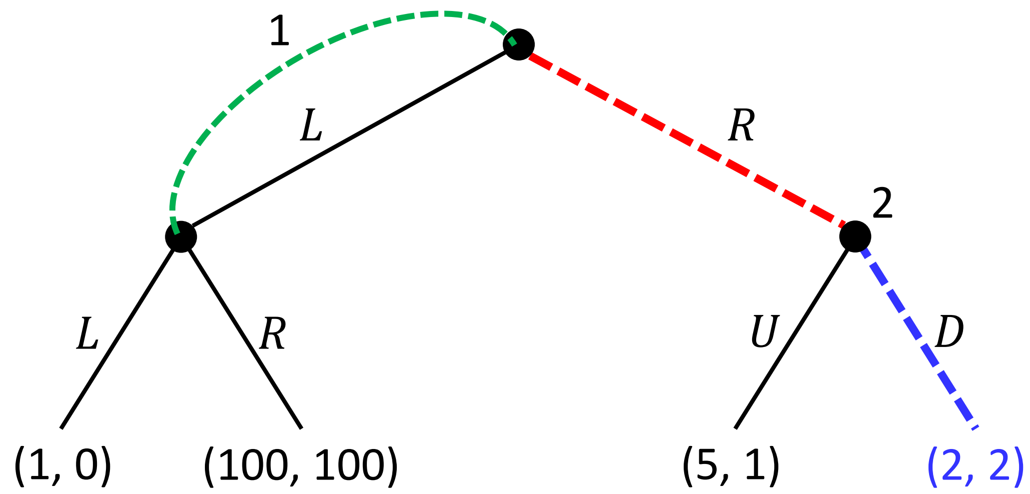


- the space of pure strategies
  - Agent 1: $\{L, R\}$,
  - Agent 2: $\{U, D\}$

- What is the **mixed strategy equilibrium** ?
  - Agent 1 decides probabilistically whether to play L or R in his information set
  - Once he decides, he plays that pure strategy consistently.
    - ✓ R is a strictly dominant strategy for agent 1,
    - ✓ D is agent 2's strict best response
  - thus (R,D) is the unique Nash equilibrium.
  - thus the payoff of 100 is irrelevant in the game

Games of imperfect recall

Consider the following game with imperfect recall



- the space of pure strategies
  - Agent 1: $\{L, R\}$,
  - Agent 2: $\{U, D\}$

- What is an equilibrium in **behavioral strategies**?
- With behavioral strategies agent 1 gets to randomize afresh each time he finds himself in the information set
  - Again, D strongly dominant for player 2
  - If 1 uses the behavioral strategy $(p, 1-p)$, his expected utility is $1 \times p^2 + 100 \times p(1-p) + 2 \times (1-p) = -99p^2 + 98p + 2$
  - Maximum at $p = 98/198$
  - Thus equilibrium is $\big((98/198, 100/198), (0,1)\big)$

Perfect recall

- In a sequential game, perfect recall refers to the assumption that, at every opportunity to act,
    - each Player remembers what he did in prior moves,
    - each player remembers everything that he knew before.
- Effectively, the assumption is one that players never forget information once it is acquired.

Theorem

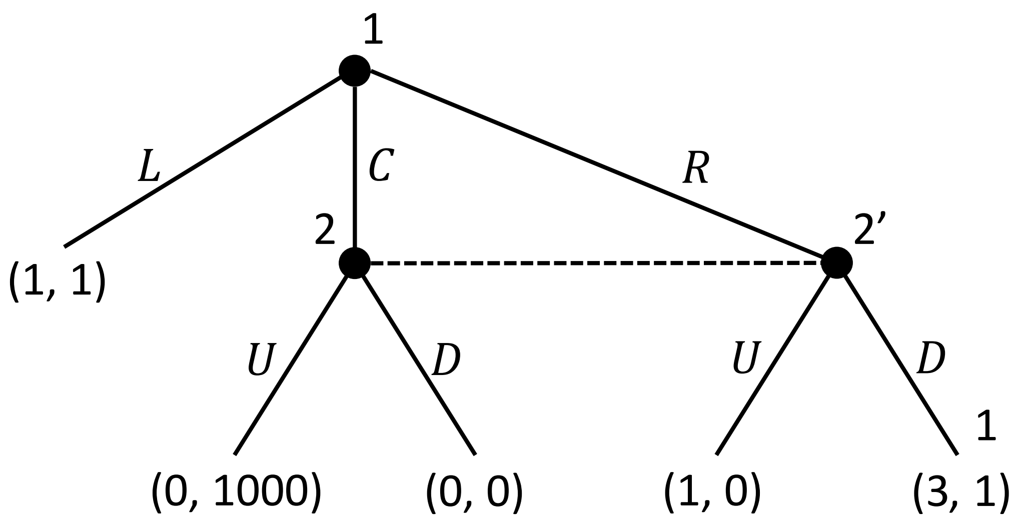
In a game of perfect recall, any mixed strategy of given agent can be replaced by an equivalent behavioral strategy, and any behavioral strategy can be replaced by an equivalent mixed strategy. Here two strategies are equivalent in the sense that they induce the same probabilities on outcomes, for any fixed strategy profile (mixed or behavioral) of the remaining agents

- Perfect information game is a perfect recall game
    - We can find behavioral strategies to find the Nash equilibrium

- In general, imperfect-information games, mixed and behavioral strategies yield noncom parable sets of equilibria

Sequential equilibrium

Consider the following imperfect information game in a extensive form game



- In a subgame-perfect equilibrium, we require that the strategy of each agent be a best response in every subgame
    - ✓ It cannot be applied to imperfect information game because a subgame cannot be well defined

Sequential equilibrium

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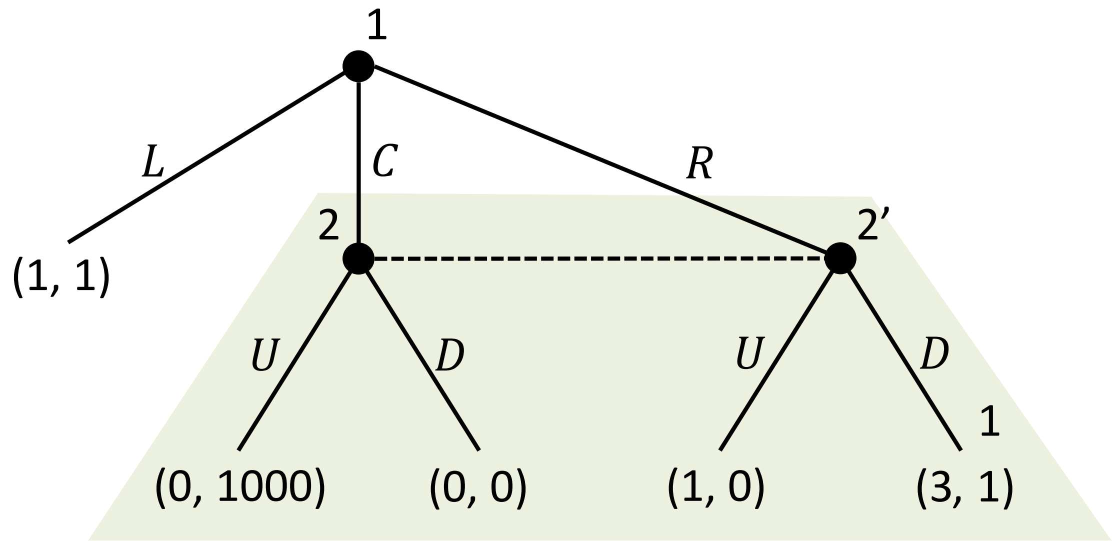
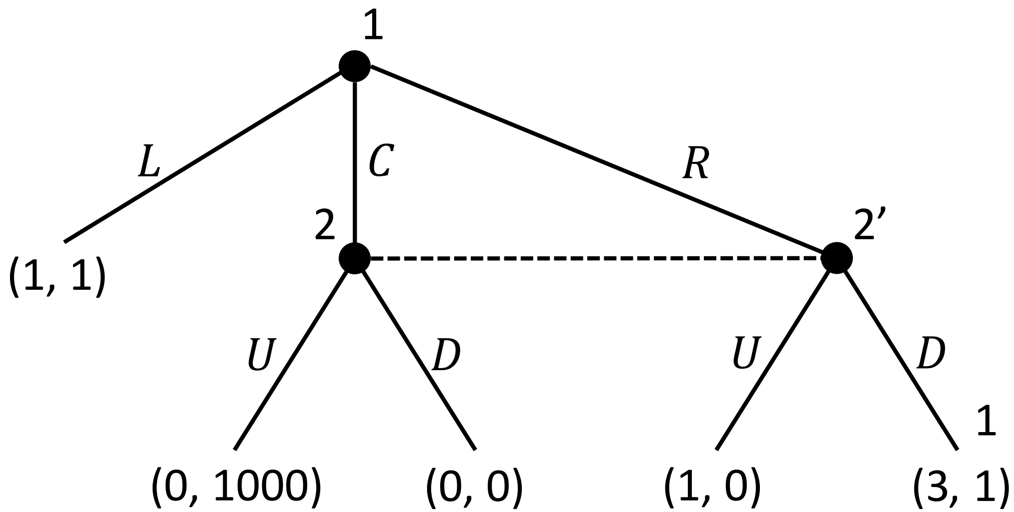


- In a subgame-perfect equilibrium, we require that the strategy of each agent be a best response in every subgame
  - ✓ It cannot be applied to imperfect information game because a subgame cannot be well defined
  - ✓ What we have at each information set is a "subforest" or a collection of subgames
- Can we require that each player's strategy be a best response in each subgame in each forest?
  - ✓ Sometimes it is yes, but not in general
  - ✓ U dominates D in the left subgame, but D dominate U in the right subgame in the forest

Sequential equilibrium

Consider the following imperfect information game in a extensive form game

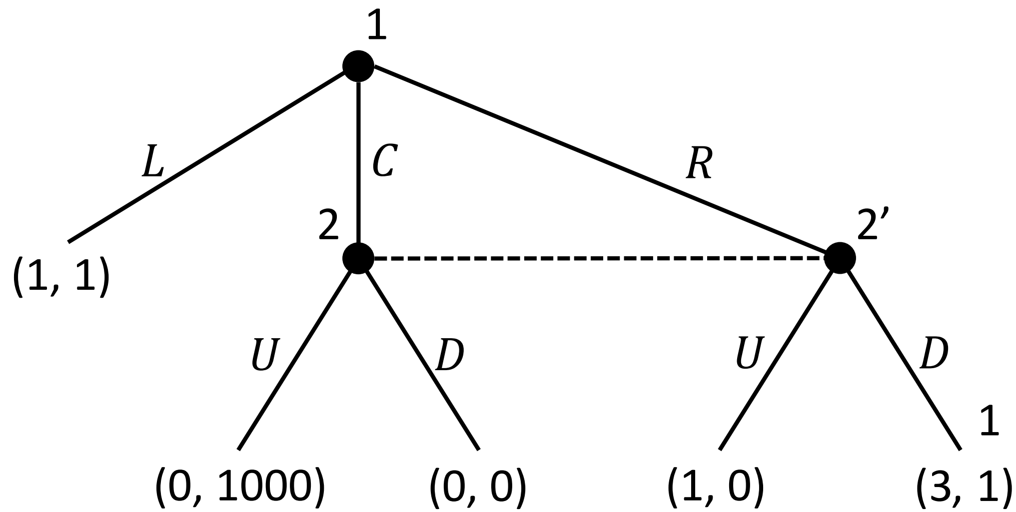


- An imperfect-information extensive-form game
  ⇒ A normal-form game
- The Nash Equilibrium (both pure and mixed) concept remains the same for imperfect-information extensive-form games

	U	D
L	1, 1	1, 1
C	0, 1000	0, 0
R	1, 0	3, 1

Sequential equilibrium

Consider the following imperfect information game in a extensive form game



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	U	D
L	1, 1	1, 1
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- The pure strategies of player 1 are $\{L, \mathrm{C}, \mathrm{R}\}$
- The pure strategies of paler 2 are $\{U, D\}$
- The two pure strategy Nash equilibria are”
  - (L, U) and (R, D)

Sequential equilibrium

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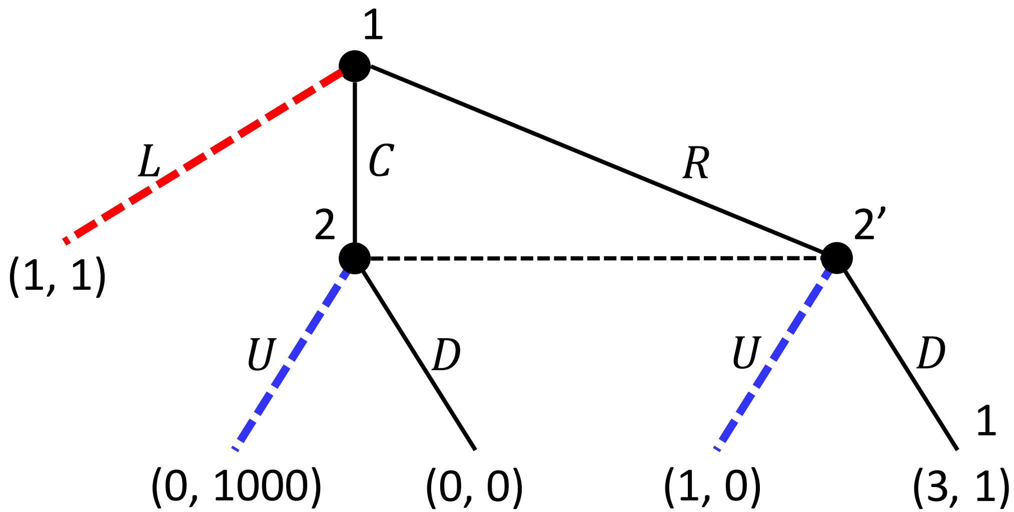

- The two pure strategy Nash equilibria are: (L, U) and (R, D)
- (L, U) is not a subgame perfect equilibrium
  - ➢ Agent 2 should select D instead of U at the right node in his information set

Sequential equilibrium

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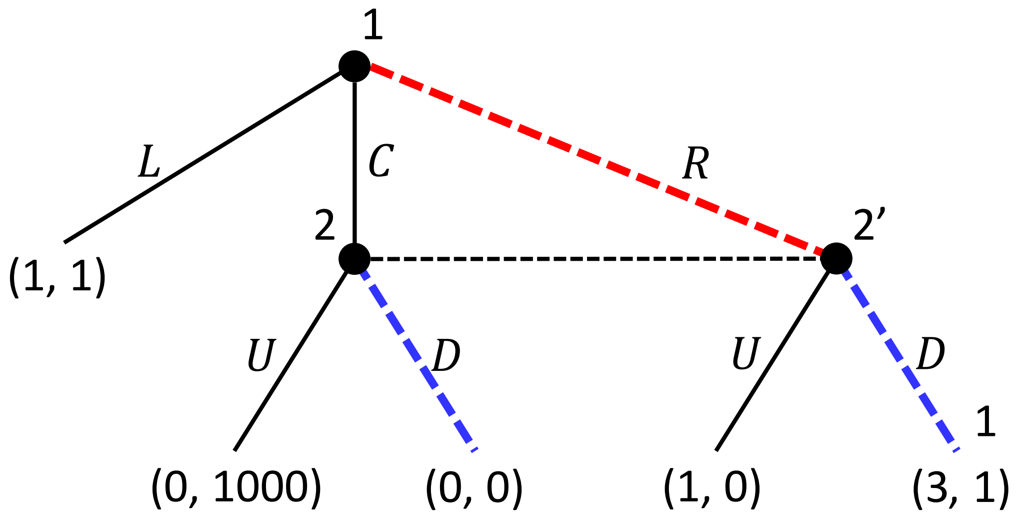
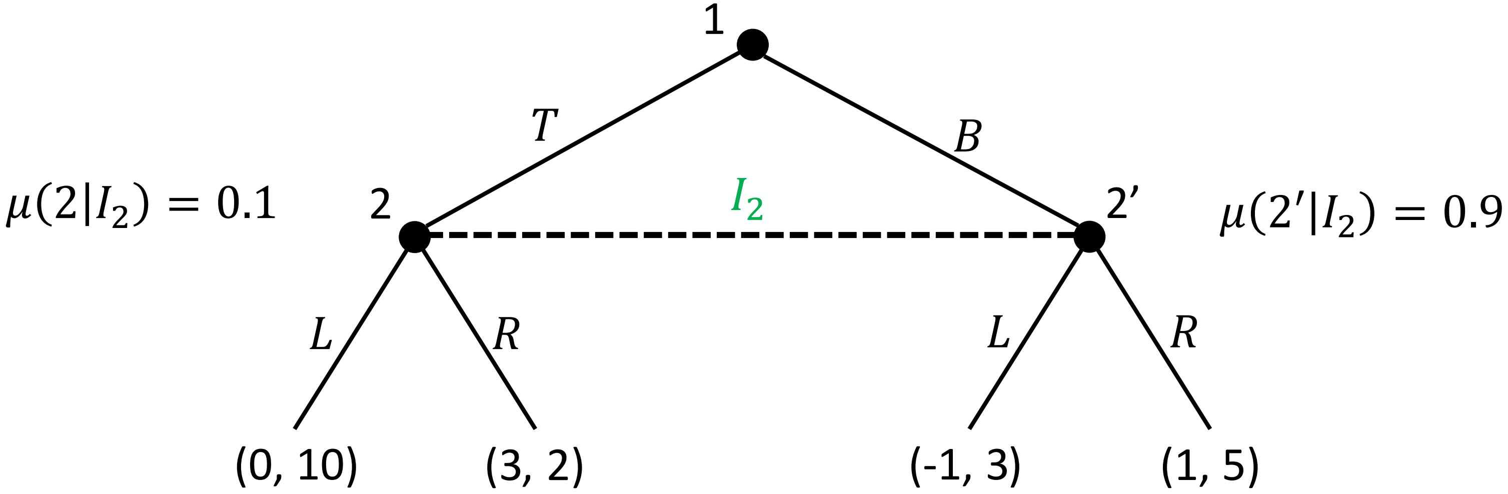


- The two pure strategy Nash equilibria are: (L, U) and (R, D)
- (L, U) is not a subgame perfect equilibrium
  - Agent 2 should select D instead of U at the right node in his information set
- (R, D) is the only **sequential equilibrium**
  - R dominates C for player 1, and player 2 knows this
  - player 2 can deduce that he is in the rightmost one based on player 1's incentives, and hence will go D : (player 2's belief)->(player 2's best action): **sequential rationality**
  - player 1 knows that player 2 can deduce this, and therefore player 1 should go R (rather than L) : player 2's belief is consistent with player 1's strategy: **consistency**

Beliefs

- A belief μ is a function that assigns to every information set a probability measure on the set of histories in the information set.

- For any information set I, the player who moves at I believes that he is at node $h \in I$ with probability $\mu(h|I)$



Sequential equilibrium

Definition (Sequential equilibrium)

A strategy profile s is a sequential equilibrium of an extensive-form game G if there exist probability distributions $\mu(I)$ for each information set I in G, such that the following two conditions hold:

1. Sequentially rational: The assessment (s, μ) is sequentially rational if for every player i and every information set $I_{i,j} \in I_i$ we have

$$E\left[u_i(s_i, s_{-i})|I_{i,j}\right] \geq E\left[u_i(s_i', s_{-i})|I_{i,j}\right] \quad \text{for any } s_i' \neq s_i$$

2. Consistency: $(s, \mu) = \lim_{m\to\infty} (s^m, \mu^m)$ for some sequence $(s^m, \mu^m), (s^m, \mu^m), \dots$, where s^m is fully mixed, and μ^m is consistent with s^m (derived from s^n using Bayes rule); and

- Sequential equilibrium is a pair, not just a strategy profile
- Hence, in order to identify a sequential equilibrium, one must identify
  - a strategy profile s, which describes what a player does at every information set
  - A belief assessment μ, which describes what a player believes at every information set

Sequential Rationality

1. Sequentially rational: The assessment (s, μ) is sequentially rational if for every player i and every information set $I_{i,j} \in I_i$ we have

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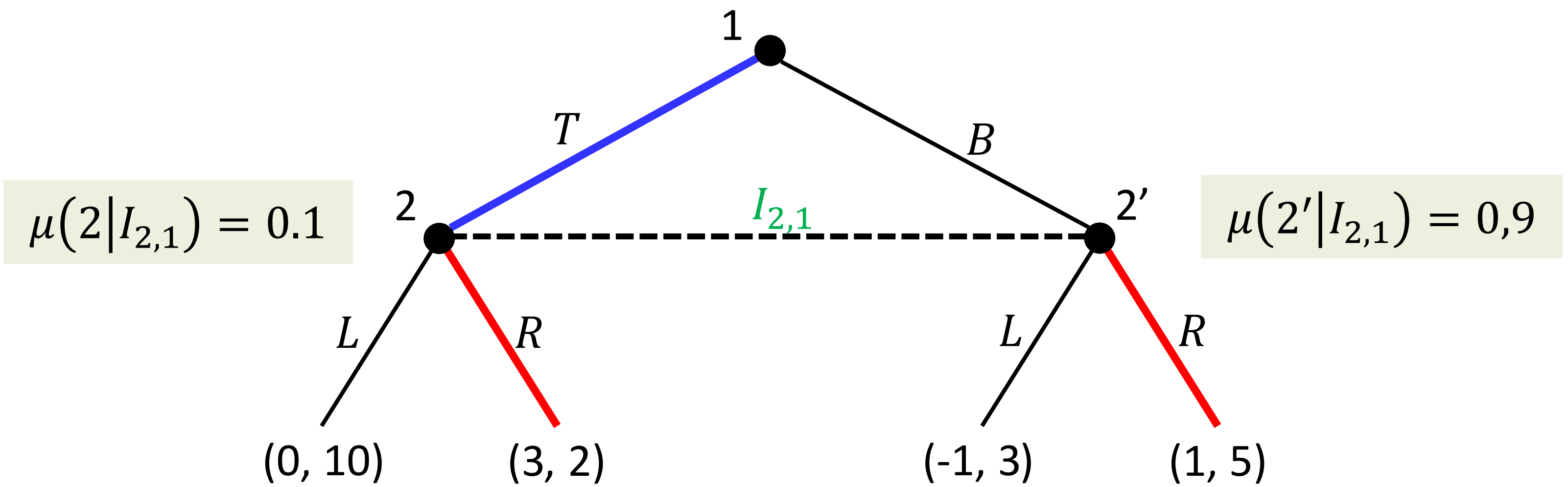

Sequential Rationality

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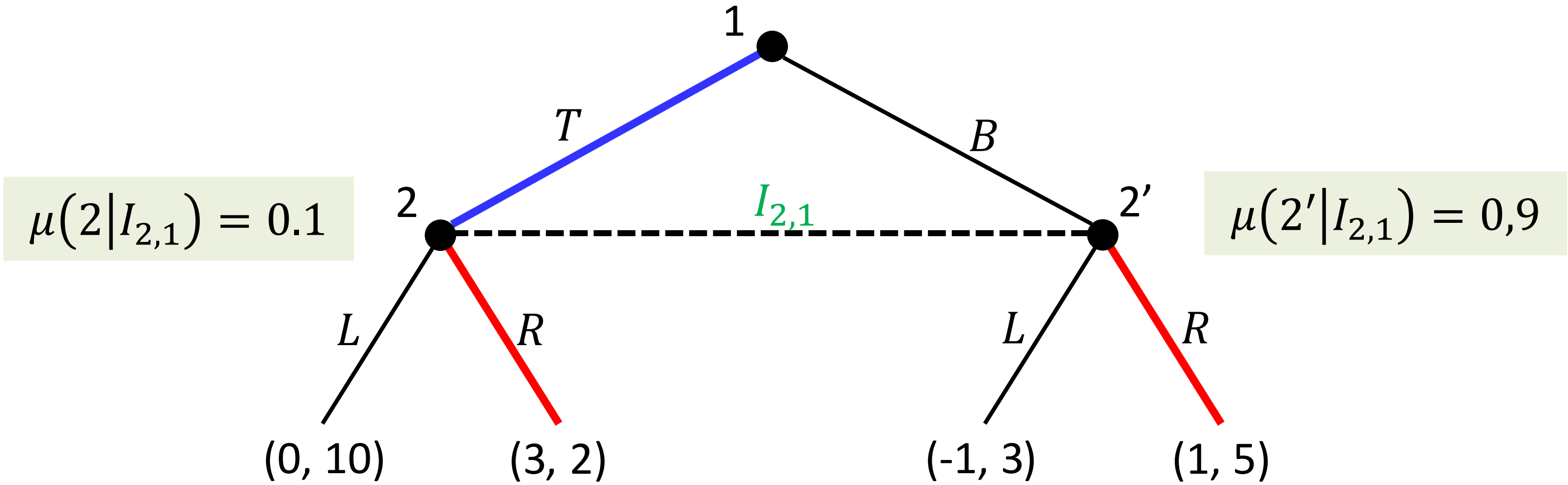

- Player 2:

$$E\left[u_2(s_1, L) | I_{2,1}\right] = 0.1 \times 10 + 0.9 \times 3 = 3.7$$
$$E\left[u_2(s_1, R) | I_{2,1}\right] = 0.1 \times 2 + 0.9 \times 5 = 4.7$$

  - ➢ Playing R is sequential rational

- Player 1: Deterministically at node 1 (deterministic belief)

$$u_1(s_1 = T, R) > u_1(s_1 = B, R) \quad \text{for any } s_{-i}$$

- Is playing (T, R) Sequential Equilibrium?

Consistency (on the equilibrium path)

2. Consistency: $(s, \mu) = \lim_{m \to \infty} (s^m, \mu^m)$ for some sequence $(s^m, \mu^m), (s^m, \mu^m), \ldots$, where s^m is fully mixed, and μ^m is consistent with s^m (derived from s^n using Bayes rule); and

- Let discuss first a simple case (**on the equilibrium path**)
- Given any strategy profile s, belief μ, and any information set I that is reached with positive probability according to s, the beliefs $\mu(\cdot\,|I)$ at I is said to be consistent with s iff $\mu(\cdot\,|I)$ is derived using the **Bayes rule** and s. That is

$$\mu(h|I) = \frac{\Pr(h|I)}{\Pr(h|I) + \Pr(h'|I)}$$

  - $\Pr(h|I)$ is the probability that we reach node h according to s

Consistency (on the equilibrium path)

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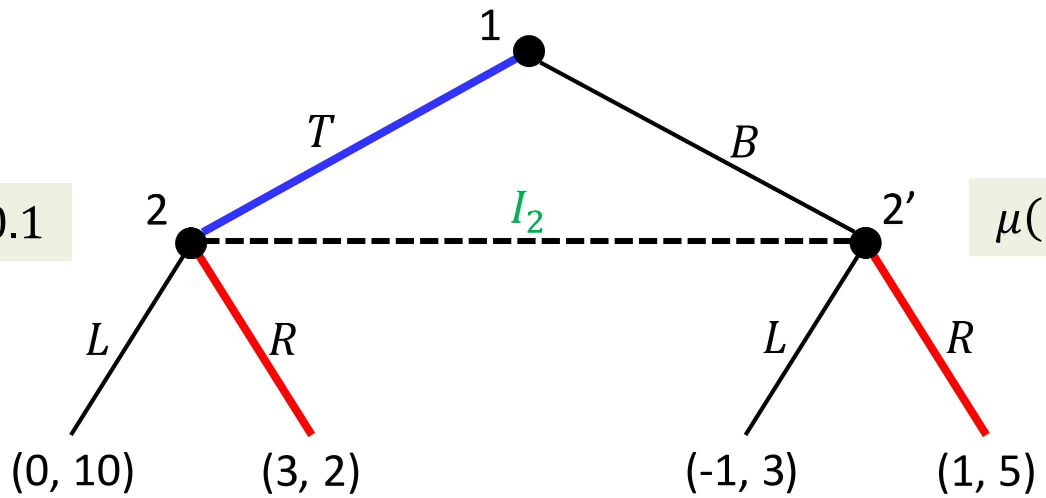


$\mu(2|I_2) = 0.1$

$\mu(2'|I_2) = 0,9$

- Let's go back to the example:

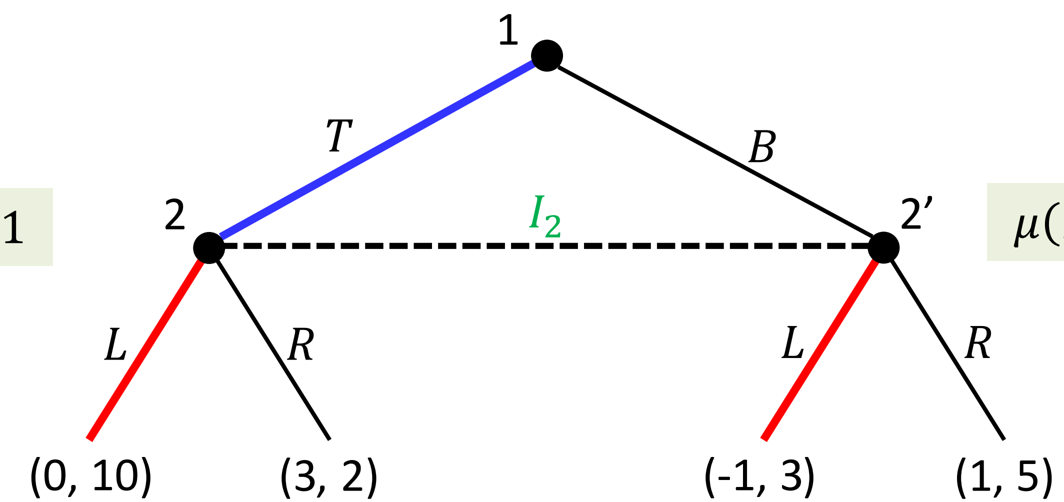
$$\mu(2|I_2) = \frac{\Pr(2|s = (T, R))}{\Pr(2|s = (T, R)) + \Pr(2'|s = (T, R))} = \frac{1}{1 + 0} = 1$$

- This contradicts to our previous belief $\mu(2|I_2) = 0.1$
- Thus, $s = (T, R)$ is not SE

Sequential Equilibrium

- A pair (s, μ) of a strategy profile s and a belief μ is said to be a sequential equilibrium if (s, μ) is sequentially rational and μ is consistent with s.

$\mu(2|I_2) = 1$



$\mu(2'|I_2) = 0$

- Based on the corrected belief

$$E[u_2(s_1, L)|I_2] = 1 \times 10 + 0 \times 3 = 10$$
$$E[u_2(s_1, R)|I_2] = 1 \times 2 + 0 \times 5 = 2$$

→Thus, player 2 will choose L
Player 1 will choose T

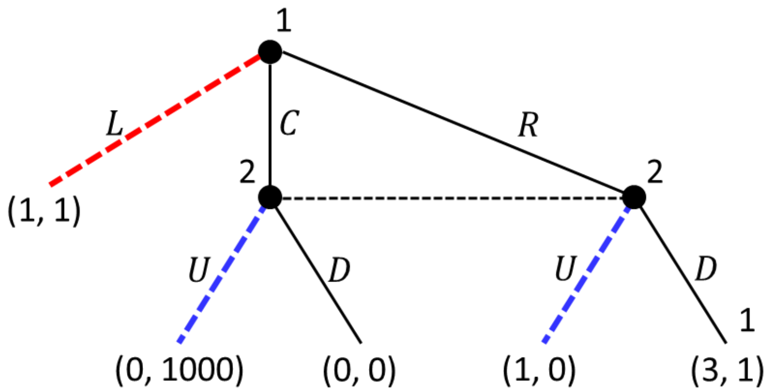
- Check the consistency of belief

$$\mu(2|I_2) = \frac{\Pr(2|s = (T, L))}{\Pr(2|s = (T, L)) + \Pr(2'|s = (T, L))} = \frac{1}{1+0} = 1$$

- Thus, $\left(s = (T, L), \mu(\cdot\, |I)\right)$ is sequential equilibrium

Consistency (on the equilibrium path)

- But, what about beliefs for information sets that are **off the equilibrium path**?
- We want beliefs for information sets that are off the equilibrium path to be reasonable. But what is reasonable?
- Consider the Nash Equilibrium (L, U) again:
  - Player 2's information set will not be reached at the equilibrium, because player 1 will play L with probability 1.
  - But assume that player 1 plays a completely mixed strategy, playing L, C, and R with probabilities $1-\epsilon$, $\frac{3\epsilon}{4}$, and $\frac{\epsilon}{4}$.
  - Then, the belief on player 2's information set is well defined. Now, if $\epsilon \to 0$, it's still well defined.



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L	1, 1	1, 1
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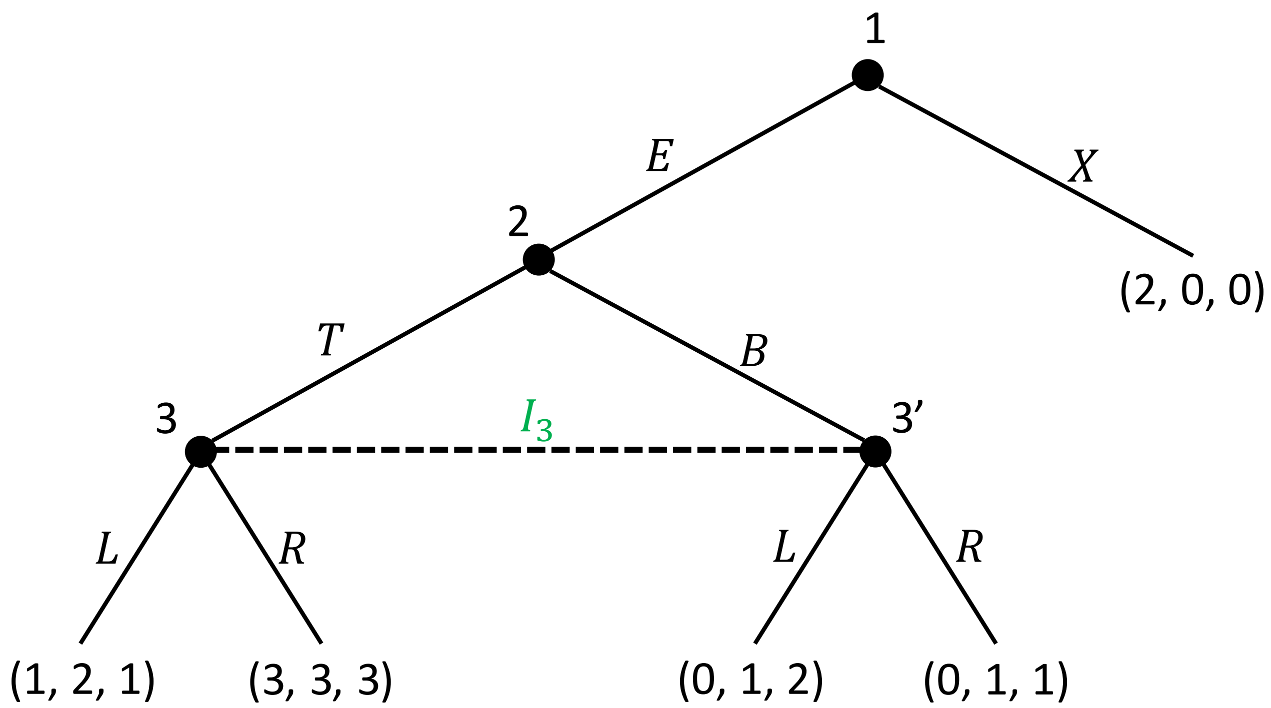
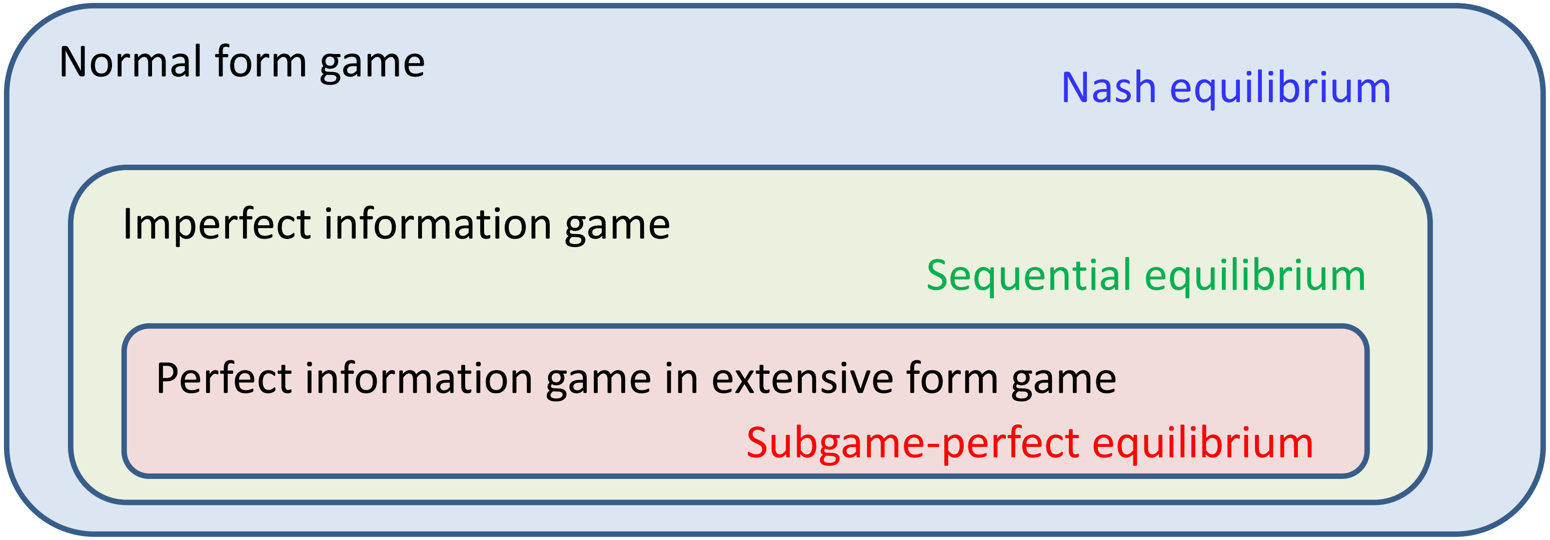
Exercise



- Find sequential equilibria

Summary



- Every finite extensive-form game with perfect recall has a sequential equilibrium.
- A sequential equilibrium is a Nash equilibrium.
- With perfect information, a subgame perfect equilibrium is a sequential equilibrium.
- Analogous to subgame-perfect equilibria in games of perfect information, sequential equilibria are guaranteed to always exist
- In extensive-form games of perfect information, the sets of subgame perfect equilibria and sequential equilibria are always equivalent