

## **Non-Cooperative Game:**

## Overview

|         | Single Agent                | Multi Agent  |
|---------|-----------------------------|--------------|
| Static  | Static optimization         | Static Game  |
| Dynamic | <b>Dynamic Optimization</b> | Dynamic Game |

## Action space

| Time space | Model based | Finite                                       | Infinite  |
|------------|-------------|--|---|
|            | Discrete    | Discrete time MDP<br>$P(s_{t+1} s_t, a_t)$   | Discrete-time dynamic system<br>$x_{t+1} = f(x_t, u_t)$     |
|            | Continuous  | Continuous time MDP<br>$P(s_{t+h} s_t, a_t)$ | Continuous-time dynamic system<br>$\dot{x}_t = f(x_t, u_t)$ |

## Overview

|         | Single Agent                | Multi Agent  |
|---------|-----------------------------|--------------|
| Static  | Static optimization         | Static Game  |
| Dynamic | <b>Dynamic Optimization</b> | Dynamic Game |

## Action space

| Time space | Model free | Finite                             | Infinite                            |
|------------|------------|------------------------------------|-------------------------------------|
|            | Discrete   | Value-based Reinforcement Learning | Policy-based Reinforcement Learning |
|            | Continuous |                                    |                                     |

## Overview

|         | Single Agent         | Multi Agent         |
|---------|----------------------|---------------------|
| Static  | Static optimization  | Static Game         |
| Dynamic | Dynamic Optimization | <b>Dynamic Game</b> |

|            |             | Action space                  |   |
|------------|-------------|-------------------------------|---|
|            |             | Finite                        | Infinite  |
| Time space | Model based |                               |   |
|            | Discrete    | Markov Game (Stochastic Game) | DT Infinite dynamic game (Stochastic Game)        |
|            | Continuous  | Continuous time Markov Game   | CT-time Infinite dynamic game (differential game) |

Overview

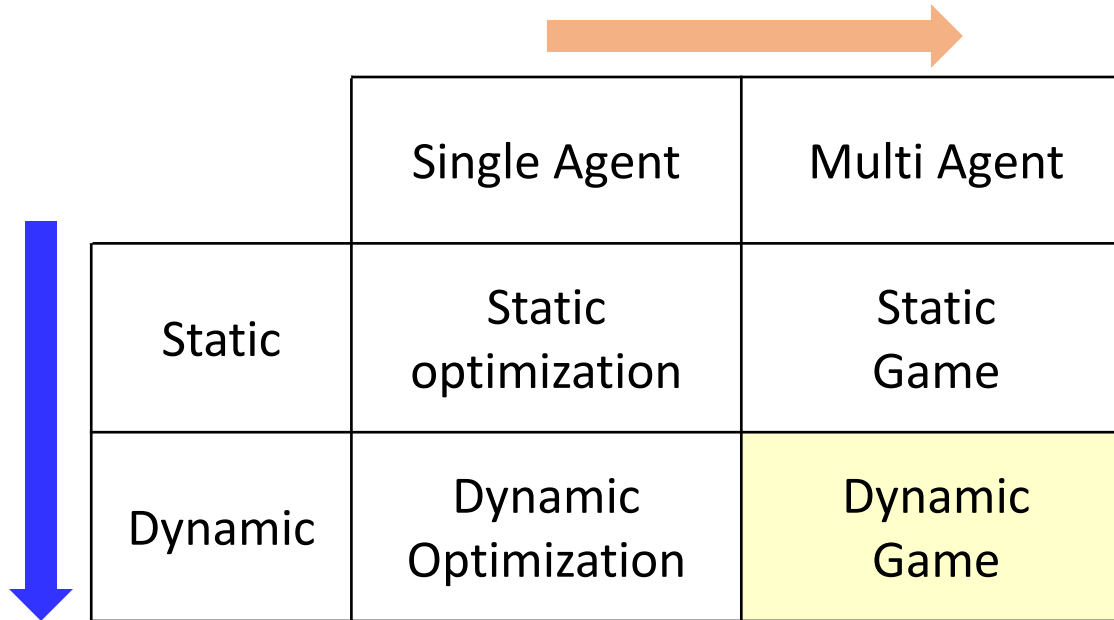
|         |                      |                     |
|---------|----------------------|---------------------|
|         | Single Agent         | Multi Agent         |
| Static  | Static optimization  | Static Game         |
| Dynamic | Dynamic Optimization | <b>Dynamic Game</b> |

|                     |                   |                            |                             |
|---------------------|-------------------|----------------------------|-----------------------------|
| <b>Action space</b> |                   |                            |                             |
| <b>Time space</b>   | <b>Model free</b> | Finite                     | Infinite                    |
|                     | Discrete          | Multi-Agent Value-based RL | Multi-Agent Policy-based RL |
|                     | Continuous        |                            |                             |

## Basic Principle to Analyze Dynamic Games

### Equilibrium concept:

-Nash; Zero-sum; Stackelberg; Correlated



|         | Single Agent         | Multi Agent  |
|---------|----------------------|--------------|
| Static  | Static optimization  | Static Game  |
| Dynamic | Dynamic Optimization | Dynamic Game |

### Dynamic optimization as a static optimization concept:

- Minimum principle (necessary condition)
- Dynamic programming principle (sufficient condition)
- Need to specify information structure

$$\begin{aligned} L^{1*} &\triangleq L^1(u^{1*}; u^{2*}; \dots; u^{N*}) \leq L^1(u^1; u^{2*}; \dots; u^{N*}), \\ L^{2*} &\triangleq L^2(u^{1*}; u^{2*}; \dots; u^{N*}) \leq L^2(u^{1*}; u^2; \dots; u^{N*}), \\ &\dots \\ L^{N*} &\triangleq L^N(u^{1*}; u^{2*}; \dots; u^{N*}) \leq L^N(u^{1*}; u^{2*}; \dots; u^{N*}) \end{aligned}$$

(Think in normal form game setting)

## Basic Principle to Analyze Dynamic Games

### Equilibrium concept:

|                              | Cooperative  | Nash                       | Zero-sum                       | Stackelberg |
|------------------------------|--|----------------------------|--------------------------------|-------------|
| Open-loop<br>(perfect state) | Open-loop<br>Cooperative-<br>Strategy<br>(decentralized control) | Open-loop<br>Nash-Strategy | Open-loop<br>Zero-sum Strategy |             |
| Feedback<br>(perfect state)  | Feedback<br>Cooperative-<br>Strategy<br>(decentralized control)  | Feedback<br>Nash-Strategy  | Feedback<br>Zero-sum Strategy  |             |
| ⋮                            |  |                            |                                |             |

(Dynamic)  
Information  
structure

- We need to specify *information structure*
  - ✓ Open-loop vs. close-loop (feedback)
  - ✓ Perfect vs. imperfect
- We need to *equilibrium concept*
  - ✓ Cooperative, Nash, Zero-sum, Stackelberg, Correlated,...

**Equilibrium concept** + **information structure** → **solution method**

- **Discrete-time Infinite Dynamic Game**
  - ✓ Definition
  - ✓ Information Structure
  - ✓ **Open-loop Nash Equilibrium** Strategy
    - ✓ **Minimum principle** to derive the equilibrium strategy
  - ✓ **Feedback Nash Equilibrium** Strategy
    - ✓ **Dynamic Programming principle (HJB)** to derive the equilibrium strategy
- **Continuous-time Infinite Dynamic Game**
  - ✓ Definition
  - ✓ Information Structure
  - ✓ **Open-loop Nash Equilibrium** Strategy
    - ✓ **Minimum principle** to derive the equilibrium strategy
    - ✓ Linear Quadratic game
  - ✓ **Feedback Nash Equilibrium** Strategy
    - ✓ **Dynamic Programming principle (HJB)** to derive the equilibrium strategy
  - ✓ Structural Dynamic Game and its various solutions



## **Discrete-time Infinite Dynamic Game**

**Definition (N-person discrete-time deterministic infinite dynamic game)**

N-person discrete-time deterministic infinite dynamic game involves:

- players' index set  $\mathbf{N} = \{1, \dots, N\}$
- The stage of game index set  $\mathbf{K} = \{1, \dots, K\}$
- state of game at stage  $k \in \mathbf{K}$ ,  $x_k$
- Action of  $\mathbf{P}_i$  at stage  $k \in \mathbf{K}$ ,  $u_k^i \in U_i^k$  where  $U_i^k$  denotes permissible action set
- A state equation of dynamic game,  $f_k: X \times U_1^1 \times \dots \times U_1^N \rightarrow X$  defined for  $\forall k \in \mathbf{K}$ ,

$$x_{t+1} = f_k(x_k, u_k^1, \dots, u_k^N),$$

defined for each  $k \in \mathbf{K}$  describes the evolution of the game

- Observation of  $\mathbf{P}_i$  at stage  $k$ ,  $y_k^i \in Y_k^i$
- state measurement  $h_k^i: X \rightarrow Y_k^i$

$$y_k^i = h_k^i(x_k), \quad k \in \mathbf{K}, i \in \mathbf{N}$$

## DT: Definition

### Definition (N-person discrete-time deterministic infinite dynamic game)

- Information structure(pattern)  $\eta_k^i \in N_k^i$

$$N_k^i \subset \{Y_1^1, \dots, Y_k^1; \dots; Y_1^N, \dots, Y_k^N; U_1^1, \dots, U_{k-1}^1; \dots; U_1^N, \dots, U_{k-1}^N\}$$

determines the information gained and recalled by  $P_i$  at stage  $k$

- A strategy of  $\mathbf{P}_i$  at stage  $k$ ,  $\gamma_k^i: N_k^i \rightarrow U_k^i$  maps information to action set, and its aggregation  $\gamma^i = \{\gamma_1^i, \dots, \gamma_K^i\}$  defines strategy of  $\mathbf{P}_i$  in game
- A cost function of  $P_i$ ,

$$L^i: (X \times U_1^1 \times \dots \times U_1^N) \times \dots \times (X \times U_K^1 \times \dots \times U_K^N) \rightarrow R$$

defined for each  $i \in \mathbf{N}$

- ✓  $L^i$  is the accumulated cost for player  $i$
- ✓ Goal :  $\mathbf{P}_i$  wants to find strategy  $\gamma^i = \{\gamma_1^i, \dots, \gamma_K^i\}$  which minimize  $L_i$  given available information  $\eta_k^i$

## DT: Definition

### Normal form description of a dynamic game

- For each fixed initial state  $x_1$  and for each fixed  $N$  –tuple permissible strategies  $\{\gamma^i \in \Gamma^i; i \in \mathbf{N}\}$  the extensive form description leads to a unique set of vectors  $\{u_k^i \triangleq \gamma_k^i(\eta_k^i), x_{k+1}; i \in \mathbf{N}, k \in \mathbf{K}\}$ 
  - ✓ because of the causal nature of the information structure
  - ✓ the state evolves according to a difference equation.
- Substitution of these quantities into  $L^i(i \in \mathbf{N})$  clearly leads to a unique  $N$  –tuple of numbers reflecting the corresponding costs to the players.
- This further implies existence of a composite mapping

$$J^i: \Gamma^1 \times \dots \times \Gamma^N \rightarrow \mathbf{R}, \text{ for each } i \in \mathbf{N}$$

which is known as the cost functional of  $\mathbf{P}i$  ( $i \in \mathbf{N}$ )

- Hence, the permissible strategy spaces of the players  $(\Gamma^1, \dots, \Gamma^N)$  together with these cost functions  $(J^1, \dots, J^N)$  constitute the normal form description of the dynamic game for each fixed initial state vector  $x_1$

There is no difference between **infinite discrete-time dynamic games** and **finite games**

➤ **allows us to use static game equilibrium concept to analyze the dynamic game**

## DT: Definition

### Definition (stage-additive cost function)

In a  $N$ -person discrete-time deterministic dynamic game of pre-specified fixed duration (i.e.,  $K$  stages),  $P^i$ 's cost functional is said to be stage-additive if there exist  $g_k^i: X \times X \times U_k^1 \times \dots \times U_k^N \rightarrow \mathbf{R}$ , ( $k \in K$ ), so that

$$L^i(u^1, \dots, u^N) = \sum_{k=1}^K g_k^i(x_{k+1}, u_k^1, \dots, u_k^N, x_k)$$

where

$$u^j = (u_1^{j'}, \dots, u_K^{j'})'$$

Furthermore, if  $L^i(u^1, \dots, u^N)$  depends on only on  $x_{K+1}$ , (the termination state), then we call it a terminal cost functional.

- State-additive cost function is widely used for optimal control or dynamic game

We call that  $P_i'$ 's information structure  $\eta_k^i$  is

- |  |  |
|--|--|
| i. (OL) <b>open-loop information pattern</b> if                | $\eta_k^i = \{x_1\}$                                   |
| ii. (CLPS) closed-loop perfect state information pattern if    | $\eta_k^i = \{x_1, \dots, x_k\}, k \in \mathbf{K}$     |
| iii. (CLIS) closed-loop imperfect state information pattern if | $\eta_k^i = \{y_1^i, \dots, y_k^i\}, k \in \mathbf{K}$ |
| iv. (MPS) memoryless perfect state information pattern if      | $\eta_k^i = \{x_1, x_k\}, k \in \mathbf{K}$            |
| v. (FB) <b>feedback perfect state information</b> pattern if   | $\eta_k^i = \{x_k\}, k \in \mathbf{K}$                 |
| vi. (FIS) feedback imperfect state information pattern if      | $\eta_k^i = \{y_k^i\}, k \in \mathbf{K}$               |

- With each information structure  $\eta_k^i$ , action  $u_k^i \triangleq \gamma_k^i(\eta_k^i)$  can be realized
- Under the information structure, the Nash solution is referred “*open-loop Nash equilibrium solution*” or “*feedback Nash equilibrium solution*”

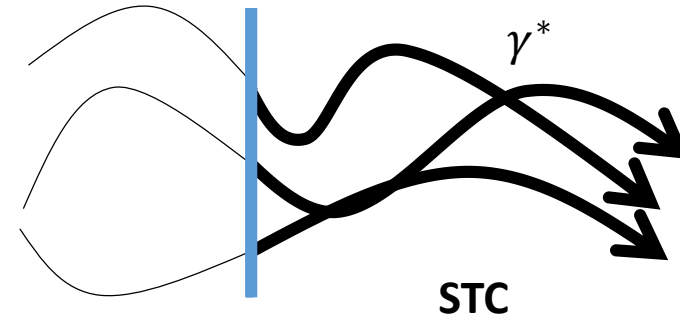
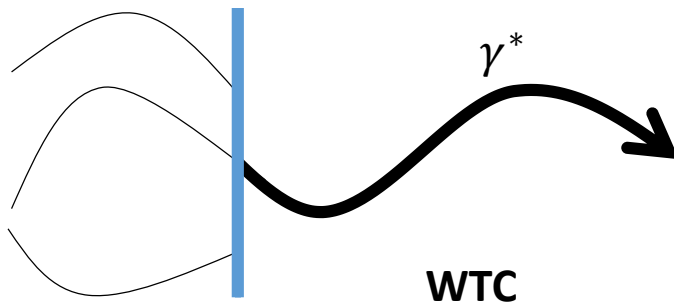
## Time Consistency

### Definition (Weakly time consistent)

An N-tuple of policies  $\gamma^*$  is **weakly time consistent** if its truncation to the interval  $[s, T]$ ,  $\gamma_{[s,T]}^*$  solves the truncated game  $D_{[s,T]}^{\gamma^*}$ , this being so for all  $s \in (0, T]$

### Definition (Strongly time consistent on subgame perfect)

An N-tuple of policies  $\gamma^*$  is **strongly time consistent** if its truncation to the interval  $[s, T]$ ,  $\gamma_{[s,T]}^*$  solves the truncated game  $D_{[s,T]}^{\gamma^*}$ , **for every  $\gamma_{[0,s]}$** , this being so for all  $s \in (0, T]$



- In both case, players have no reason to deviate from strategy
- Difference lies in the consistency of past actions with **the adopted strategies**

## DT: Nash Equilibrium Strategy (Formulation)

### Definition (Nash equilibrium in **discrete time** dynamic game : action space)

$N$  – tuple of strategies  $\{\gamma^{i*}(\cdot) \in \Gamma^i; i \in N\}$  constitutes a Nash equilibrium (for any information set) if it satisfies following inequalities for all  $u^{i*} = \gamma^{i*}(\cdot), i \in N$

$$\begin{aligned} L^{1*} &\triangleq L^1(u^{1*}; u^{2*}; \dots; u^{N*}) \leq L^1(u^1; u^{2*}; \dots; u^{N*}), \\ L^{2*} &\triangleq L^2(u^{1*}; u^{2*}; \dots; u^{N*}) \leq L^2(u^{1*}; u^2; \dots; u^{N*}), \\ &\vdots \\ L^{N*} &\triangleq L^N(u^{1*}; u^{2*}; \dots; u^{N*}) \leq L^N(u^{1*}; u^{2*}; \dots; u^N), \end{aligned}$$

Here,  $u^i \triangleq \{u_i^1, \dots, u_i^K\}$  is the aggregate action of  $\mathbf{P}_i$

### Definition (Nash equilibrium in **discrete time** dynamic game : strategy space)

$N$  – tuple permissible strategies  $\{\gamma^{i*} \in \Gamma^i; i \in \mathbf{N}\}$  constitutes a Nash equilibrium solution if, and only if, the following inequalities are satisfied for all  $\{\gamma^i \in \Gamma^i; i \in \mathbf{N}\}$

$$\begin{aligned} J^{1*} &\triangleq J^1(\gamma^{1*}; \gamma^{2*}; \dots; \gamma^{N*}) \leq J^1(\gamma^1; \gamma^{2*}; \dots; \gamma^{N*}), \\ J^{2*} &\triangleq J^2(\gamma^{1*}; \gamma^{2*}; \dots; \gamma^{N*}) \leq J^2(\gamma^{1*}; \gamma^2; \dots; \gamma^{N*}), \\ &\vdots \\ J^{N*} &\triangleq J^N(\gamma^{1*}; \gamma^{2*}; \dots; \gamma^{N*}) \leq J^N(\gamma^{1*}; \gamma^{2*}; \dots; \gamma^N), \end{aligned}$$

Here,  $\gamma^i \triangleq \{\gamma_i^1, \dots, \gamma_i^K\}$  is the aggregate strategy of  $\mathbf{P}_i$



## DT: Open-loop Nash Equilibrium Strategy (Solution Method)

**Open-Loop Nash equilibria** : Information set  $\eta_k^i = \{x_1\}$

- Identical to optimal control problem for each  $\mathbf{P}_i$ , since open-loop control does not depend on other's control
- The minimum principle provides optimal control  $u^{i*} = (u_1^{i*}, \dots, u_K^{i*}) \forall i \in N$  and corresponding state trajectory,  $(x_1^{i*}, \dots, x_K^{i*})$
- Optimal open-loop NE strategy  $\gamma^{i*}(x_1)$  is weekly time consistent, as it cannot provide optimal strategy out of optimal trajectory

## The Minimum Principle

Consider the following optimal control problem defined by

$$L(u) = \int_0^T g(t, x(t), u(t)) dt + q(T, x(T))$$

where the state variable  $x(t)$  satisfies the differential equation:

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0$$

### Theorem (The minimum principle)

In the continuous time dynamic system defined by equation (1) and (2), optimal control  $u^*(t)$  and corresponding trajectory  $x^*(t)$  satisfy following equations:

$$H(t, \lambda, x, u) := g(t, x, u) + \lambda(t)f(t, x, u)$$

$$\dot{x}^*(t) = f(t, x^*, u^*) \left( = \frac{\partial H(t, x^*, u^*, \lambda)}{\partial \lambda} \right), x^*(0) = x_0;$$

$$\dot{\lambda}^*(t) = -\frac{\partial H(t, \lambda, x^*, u^*)}{\partial x}; \lambda^*(T) = \frac{\partial h(x^*(T))}{\partial x}$$

$$u^*(t) = \operatorname{argmin}_{u \in U} H(t, \lambda^*, x^*, u)$$

## DT: Open-loop Nash Equilibrium Strategy (Solution Method)

### Definition (open-loop Nash equilibria in discrete time dynamic game)

If  $\gamma^{i*}(x_1) = u^{i*}$  provides an open-loop Nash equilibrium, and  $\{x_k^*, k \in \mathbf{K}\}$  is corresponding state trajectory, there exists costate vectors  $\{p_1^i, \dots, p_K^i\}$  for each  $i \in \mathbf{N}$  such that:

$$H_k^i(x_k, u_k^1, \dots, u_k^N, p_{k+1}^i) := g_k^i(x_k, u_k^1, \dots, u_k^N) + p_{k+1}^i f_k(x_k, u_k^1, \dots, u_k^N)$$

$$x_{k+1}^* = f_k(x_k^*, u_k^{1*}, \dots, u_k^{N*}), \quad x_1^* = x_1$$

$$\gamma_k^{i*}(x_1) = u_k^{i*} = \arg \min_{u_k^i \in U_k^i} H_k^i(x_k^*, u_k^{1*}, \dots, u_k^{i-1*}, u_k^i, u_k^{i+1*}, \dots, u_k^{N*}, p_{k+1}^i)$$

$$p_k^i = \frac{\partial}{\partial x_k} f_k(x_k^*, u_k^{1*}, \dots, u_k^{N*}) p_{k+1}^i + \frac{\partial}{\partial x_k} g_k^i(x_k^*, u_k^{1*}, \dots, u_k^{N*}), \quad p_K^i = 0$$

$$\forall k \in \mathbf{K}, i \in \mathbf{N}$$

상대방의 action이 optimal하게 고정되었다는 가정하에 minimum principle이 모든 agent와 모든 시간 instance에 대해 정의된다.

- **Feedback Nash equilibria**
  - initial state information is known a priori
  - *depend only on the time variable and current value of the state*
  - $x_k \in \eta_k^i$
  - Feedback NE solution provides NE for any subgame defined in  $\{s, s + 1, \dots, K\}$  for all  $s \in \mathbf{K}$
- $N$  person feedback game in extensive form
  - Recursive procedure to obtain NE of finite game
- Feedback strategy  $\gamma^{i*}(\cdot)$  is strongly time consistent

## DT: Feedback Nash Equilibrium Strategy (Solution Method)

### Definition (feedback Nash equilibria in discrete time dynamic game)

#### Level $K$

$$\begin{cases} L^1(\gamma_1^1, \dots, \gamma_{K-1}^1, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^2, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^N, \gamma_K^{N*}) \leq L^1(\gamma_1^1, \dots, \gamma_{K-1}^1, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^2, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^N, \gamma_K^{N*}) \\ L^2(\gamma_1^1, \dots, \gamma_{K-1}^1, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^2, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^N, \gamma_K^{N*}) \leq L^2(\gamma_1^1, \dots, \gamma_{K-1}^1, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^2, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^N, \gamma_K^{N*}) \\ \vdots \\ L^K(\gamma_1^1, \dots, \gamma_{K-1}^1, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^2, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^N, \gamma_K^{N*}) \leq L^K(\gamma_1^1, \dots, \gamma_{K-1}^1, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^2, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^N, \gamma_K^{N*}) \end{cases}$$

#### Level $K - 1$

$$\begin{cases} L^1(\gamma_1^1, \dots, \gamma_{K-1}^{1*}, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^2, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^N, \gamma_K^{N*}) \leq L^1(\gamma_1^1, \dots, \gamma_{K-1}^{1*}, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^2, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^N, \gamma_K^{N*}) \\ L^2(\gamma_1^1, \dots, \gamma_{K-1}^{1*}, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^{2*}, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^N, \gamma_K^{N*}) \leq L^2(\gamma_1^1, \dots, \gamma_{K-1}^{1*}, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^{2*}, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^N, \gamma_K^{N*}) \\ \vdots \\ L^K(\gamma_1^1, \dots, \gamma_{K-1}^{1*}, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^{2*}, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^{N*}, \gamma_K^{N*}) \leq L^K(\gamma_1^1, \dots, \gamma_{K-1}^{1*}, \gamma_K^{1*}; \gamma_1^2, \dots, \gamma_{K-1}^{2*}, \gamma_K^{2*}; \dots; \gamma_1^N, \dots, \gamma_{K-1}^{N*}, \gamma_K^{N*}) \end{cases}$$

#### Level 1

$$\begin{cases} L^1(\gamma_1^{1*}, \gamma_2^{1*}, \dots, \gamma_K^{1*}; \gamma_1^{2*}, \gamma_2^{2*}, \dots, \gamma_K^{2*}; \dots; \gamma_1^{N*}, \gamma_2^{N*}, \dots, \gamma_K^{N*}) \leq L^1(\gamma_1^{1*}, \gamma_2^{1*}, \dots, \gamma_K^{1*}; \gamma_1^{2*}, \gamma_2^{2*}, \dots, \gamma_K^{2*}; \dots; \gamma_1^{N*}, \gamma_2^{N*}, \dots, \gamma_K^{N*}) \\ L^2(\gamma_1^{1*}, \gamma_2^{1*}, \dots, \gamma_K^{1*}; \gamma_1^{2*}, \gamma_2^{2*}, \dots, \gamma_K^{2*}; \dots; \gamma_1^{N*}, \gamma_2^{N*}, \dots, \gamma_K^{N*}) \leq L^2(\gamma_1^{1*}, \gamma_2^{1*}, \dots, \gamma_K^{1*}; \gamma_1^{2*}, \gamma_2^{2*}, \dots, \gamma_K^{2*}; \dots; \gamma_1^{N*}, \gamma_2^{N*}, \dots, \gamma_K^{N*}) \\ \vdots \\ L^N(\gamma_1^{1*}, \gamma_2^{1*}, \dots, \gamma_K^{1*}; \gamma_1^{2*}, \gamma_2^{2*}, \dots, \gamma_K^{2*}; \dots; \gamma_1^{N*}, \gamma_2^{N*}, \dots, \gamma_K^{N*}) \leq L^N(\gamma_1^{1*}, \gamma_2^{1*}, \dots, \gamma_K^{1*}; \gamma_1^{2*}, \gamma_2^{2*}, \dots, \gamma_K^{2*}; \dots; \gamma_1^{N*}, \gamma_2^{N*}, \dots, \gamma_K^{N*}) \end{cases}$$

- **Backward Induction:**

## DT: Feedback Nash Equilibrium Strategy (Solution Method)

### Definition (feedback Nash equilibria in discrete time dynamic game)

For N-person discrete time infinite dynamic game, the set of strategies  $\{\gamma_k^{i*}(x_k); k \in K, i \in N\}$  provides **feedback Nash equilibrium** solution if and only if there exists functions  $V^i(k, \cdot): R^n \rightarrow R$  such that following recursive relations are satisfied:

$$\begin{aligned} V^i(k, x) &= \min_{u_k^i \in U_k^i} \left[ g_k^i \left( x, \gamma_k^{1*}(x), \dots, u_k^i, \dots, \gamma_k^{N*}(x) \right) + V^i \left( k + 1, \tilde{f}_k^{i*} \left( x, u_k^i \right) \right) \right] \\ &= g_k^i \left( x, \gamma_k^{1*}(x), \dots, \gamma_k^{i*}(x), \dots, \gamma_k^{N*}(x) \right) + V^i \left( k + 1, \tilde{f}_k^{i*} \left( x, \gamma_k^{i*}(x) \right) \right); \\ V^i(K + 1, x) &= 0, \quad \forall i \in N \end{aligned}$$

where

$$\tilde{f}_k^{i*}(x, u_k^i) \triangleq f_k \left( x, \gamma_k^{1*}(x), \dots, \gamma_k^{i-1*}(x), u_k^i, \gamma_k^{i+1*}(x), \dots, \gamma_k^{N*}(x) \right)$$

Every such equilibrium solution is strongly time consistent, and corresponding NE cost for  $\mathbf{P}_i$  is  $V^i(1, x_1)$

- **Employ HJB equation (Dynamic Programming Principle)**
- Apply Best response principle for every time step (**rationality**), and the best responses of all the players are consistent (**consistency**)

## DT: Feedback Nash Equilibrium Strategy (Solution Method)

- **Proof Sketch:**

In definition, the first set of  $N$  inequalities have to hold for all  $\gamma_k^i \in \Gamma_k^i$  implies that they have to hold for all state  $x_k$  which are reachable by combination of strategies.

At time  $k$ , set of inequalities becomes equivalent to the problem of seeking Nash equilibria of  $N$ -person static game with cost functional

$$g_{k-1}^i(x_{k-1}, u_{k-1}^1, \dots, u_{k-1}^N) + V^i(k, x_k), \quad i \in N,$$

where

$$x_k = f_{k-1}(x_{k-1}, u_{k-1}^1, \dots, u_{k-1}^N)$$

Here, we observe that the Nash equilibrium controls can only be functions of  $x_{k-1}$ , and previous theorem provides a set of necessary and sufficient conditions for  $\{\gamma_{k-1}^{i*}(x_{k-1}); i \in N\}$  to solve this static Nash game.

## **Continuous-time Infinite Dynamic Game**



## CT: Definition

### Definition (N-person continuous-time deterministic infinite dynamic game, **differential game**)

N-person differential game involves:

- players' index set  $\mathbf{N} = \{1, \dots, N\}$
- A time interval  $[0, T]$  which is specified *a priori*, duration of the evolution of game
- Permissible state trajectories of the game,  $\{x(t), 0 \leq t \leq T\}$
- control function(or simply control) of  $P_i$ ,  $\{u_i(t), 0 \leq t \leq T\}$
- A differential equation

$$\frac{dx(t)}{dt} = f(t, x(t), u^1(t), \dots, u^N(t)), \quad x(0) = x_0$$

whose solution describes the state trajectory of the game

- state information gained and recalled by  $\mathbf{P}_i$  at time  $t$ ,  $\eta^i(t)$
- strategy of  $\mathbf{P}_i$   $\gamma^i$ , with property  $u^i(t) = \gamma^i(t, \eta^i(t))$  cost function of  $\mathbf{P}_i$  in the differential game  $L^i$ ,

$$L^i(u^1, \dots, u^N) = \int_0^T g^i(t, x(t), u^1(t), \dots, u^N(t))dt + q^i(x(T))$$

**Goal** :player  $\mathbf{P}_i$  wants to find strategy  $\gamma^i$  which minimize  $L_i$  given available information  $\eta^i(t)$

## CT: Information Structure

- We call that  $\mathbf{P}_i'$ 's information structure is

|   |   |
|---|---|
| i. (OL) open-loop information pattern if                    | $\eta^i(t) = \{x_0\}, t \in [0, T]$                   |
| ii. (CLPS) closed-loop perfect state information pattern if | $\eta^i(t) = \{x(s), 0 \leq s \leq t\}, t \in [0, T]$ |
| iii. (MPS) memoryless perfect state information pattern if  | $\eta^i(t) = \{x_0, x(t)\}, t \in [0, T]$             |
| iv. (FB) feedback pattern if                                | $\eta^i(t) = \{x(t)\}, t \in [0, T]$                  |

- With each information structure  $\eta^i(t)$ ,  $u^i(t) \triangleq \gamma^i(t, \eta^i(t))$  can be realized
- Under the information structure, the Nash solution is referred “*open-loop Nash equilibrium solution*” or “*feedback Nash equilibrium solution*”

## CT: Open-loop Nash Equilibrium Strategy (Formulation)

- We consider  $N$  – person dynamic game defined in continuous time

$$\frac{dx(t)}{dt} = f(t, x(t), u^1(t), \dots, u^N(t)), \quad x(0) = x_0$$

and cost functional

$$L^i(u^1, \dots, u^N) = \int_0^T g^i(t, x(t), u^1(t), \dots, u^N(t))dt + q^i(x(T))$$

## CT: Open-loop Nash Equilibrium Strategy (Formulation)

### Definition (Nash equilibrium in discrete time dynamic game : action space)

$N$  – tuple of strategies  $\{\gamma^{i*}(\cdot) \in \Gamma^i; i \in N\}$  constitutes a Nash equilibrium (for any information set) if it satisfies following inequalities for all  $u^{i*} = \gamma^{i*}(\cdot), i \in N$

$$\begin{aligned} L^{1*} &\triangleq L^1(\mathbf{u}^{1*}; u^{2*}; \dots; u^{N*}) \leq L^1(\mathbf{u}^1; u^{2*}; \dots; u^{N*}), \\ L^{2*} &\triangleq L^2(u^{1*}; \mathbf{u}^{2*}; \dots; u^{N*}) \leq L^2(u^{1*}; \mathbf{u}^2; \dots; u^{N*}), \\ &\vdots \\ L^{N*} &\triangleq L^N(u^{1*}; u^{2*}; \dots; \mathbf{u}^{N*}) \leq L^N(u^{1*}; u^{2*}; \dots; \mathbf{u}^N), \end{aligned}$$

Here,  $u^i(t) \in S^i$  is the action of  $\mathbf{P}_i$  chosen at time  $t \in [0, T]$

### Definition (Nash equilibrium in discrete time dynamic game : strategy space)

$N$  – tuple permissible strategies  $\{\gamma^{i*} \in \Gamma^i; i \in \mathbf{N}\}$  constitutes a Nash equilibrium solution if, and only if, the following inequalities are satisfied for all  $\{\gamma^i \in \Gamma^i; i \in \mathbf{N}\}$

$$\begin{aligned} J^{1*} &\triangleq J^1(\boldsymbol{\gamma}^{1*}; \gamma^{2*}; \dots; \gamma^{N*}) \leq J^1(\boldsymbol{\gamma}^1; \gamma^{2*}; \dots; \gamma^{N*}), \\ J^{2*} &\triangleq J^2(\gamma^{1*}; \boldsymbol{\gamma}^{2*}; \dots; \gamma^{N*}) \leq J^2(\gamma^{1*}; \boldsymbol{\gamma}^2; \dots; \gamma^{N*}), \\ &\vdots \\ J^{N*} &\triangleq J^N(\gamma^{1*}; \gamma^{2*}; \dots; \boldsymbol{\gamma}^{N*}) \leq J^N(\gamma^{1*}; \gamma^{2*}; \dots; \boldsymbol{\gamma}^N), \end{aligned}$$

Here,  $\gamma^i(t, \eta^i(t))$  is the strategy of  $\mathbf{P}_i$  at time  $t \in [0, T]$

## CT: Open-loop Nash Equilibrium Strategy (Solution Method)

- Open-Loop Nash equilibria
  - Information set  $\eta_k^i = \{x_0\}$
- Identical to optimal control problem for each  $\mathbf{P}_i$ , since open-loop control does not depend on other's control
$$\begin{aligned} &\text{minimize } J^i \left( u^{1*}(t), \dots, u^{i-1*}(t), \mathbf{u}^i(t), u^{i+1*}(t), \dots, u^{N*}(t) \right) \\ &\text{s.t } \dot{x}^* = f(t, x^*(t), u^{1*}(t), \dots, u^{i-1*}(t), \mathbf{u}^i(t), u^{i+1*}(t), \dots, u^{N*}(t)) \end{aligned}$$
- The minimum principle provides optimal control  $u^{i*}(t) \forall i \in N$  and state trajectory  $x^*(t)$
- Optimal open-loop NE strategy  $\gamma^{i*}(t, x_0) = u^{i*}(t)$  is weakly time consistent, as it cannot provide optimal strategy out of optimal trajectory

## The Minimum Principle

Consider the following optimal control problem defined by

$$L(u) = \int_0^T g(t, x(t), u(t)) dt + q(T, x(T))$$

where the state variable  $x(t)$  satisfies the differential equation:

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0$$

### Theorem (The minimum principle)

In the continuous time dynamic system defined by equation (1) and (2), optimal control  $u^*(t)$  and corresponding trajectory  $x^*(t)$  satisfy following equations:

$$H(t, \lambda, x, u) := g(t, x, u) + \lambda(t)f(t, x, u)$$

$$\dot{x}^*(t) = f(t, x^*, u^*) \left( = \frac{\partial H(t, x^*, u^*, \lambda)}{\partial \lambda} \right), x^*(0) = x_0;$$

$$\dot{\lambda}^*(t) = -\frac{\partial H(t, \lambda, x^*, u^*)}{\partial x}; \lambda^*(T) = \frac{\partial h(x^*(T))}{\partial x}$$

$$u^*(t) = \operatorname{argmin}_{u \in U} H(t, \lambda^*, x^*, u)$$

## CT: Open-loop Nash Equilibrium Strategy (Solution Method)

### Definition (Open-loop Nash equilibria in continuous time dynamic game)

If  $\gamma^{i*}(t, x_0) = u^{i*}(t)$  provides an open-loop Nash equilibrium, there exists  $N$  co-state functions  $p^i(\cdot): [0, T] \rightarrow R^n$  for each  $i \in N$  such that:

$$\dot{x}^i = f(t, x^*(t), u^{1*}(t), \dots, u^{N*}(t)), x^*(0) = x_0$$

$$\gamma^{i*}(t, x_0) = u^{i*}(t) = \underset{u^i(t)}{\operatorname{argmin}} H^i(t, p^i(t), x^*(t), u^{1*}(t), \dots, u^i(t), \dots, u^{N*}(t))$$

$$\dot{p}^i = -\frac{\partial}{\partial x} H^i(t, p^i(t), x^*, u^{1*}(t), \dots, u^{N*}(t))$$

$$\dot{p}^i(T) = -\frac{\partial}{\partial x} q^i(x^*(T))$$

where

$$H^i(t, p^i, x, u^1, \dots, u^N) := g^i(t, x, u^1, \dots, u^N) + p^i f(t, x, u^1, \dots, u^N)$$

- **Feedback Nash equilibria**
  - ✓ initial state information is known a priori
  - ✓ *depend only on the time variable and current value of the state*
  - ✓  $x(t) \in \eta_t$
  - ✓ Feedback NE solution provides NE for any subgame defined in  $[t, T]$  for all  $t \in [0, T)$
- Definition of the feedback NE leads to a recursive derivation
  - Value function  $V^i(t, x)$ , minimum cost-to-go for player  $i$  at time  $t$  on state  $x$
- Optimal feedback NE strategy  $\gamma^{i*}(t, \eta_t)$  is strongly time consistent



## CT: Feedback Nash Equilibrium Formulation

### Definition (Feedback NE Solution)

An  $N$ -tuple of strategies  $\{\gamma^{i*} \in \Gamma^i; i \in N\}$  constitutes a feedback Nash equilibrium solution if there exists  $V^i(\cdot, \cdot)$  on  $[0, T] \times R^n$  s.t.

$$\begin{aligned} V^i(t, x) &= \int_t^T g^i(s, x^*(s), \gamma^{1*}(s, \eta_s), \dots, \gamma^{i*}(s, \eta_s), \dots, \gamma^{N*}(s, \eta_s)) ds + q^i(x^*(T)) \\ &\leq \int_t^T g^i(s, x^*(s), \gamma^{1*}(s, \eta_s), \dots, \gamma^i(s, \eta_s), \dots, \gamma^{N*}(s, \eta_s)) ds + q^i(x^i(T)), \forall \gamma^i \\ &\in \Gamma^i, x \in R^n \end{aligned}$$

Where, on  $[t, T]$ ,

$$\begin{aligned} \dot{x}^i(s) &= f(s, x^i(s), \gamma^{1*}(s, \eta_s), \dots, \gamma^i(s, \eta_s), \dots, \gamma^{N*}(s, \eta_s)); \quad x^i(t) = x, \\ \dot{x}^*(s) &= f(s, x^*(s), \gamma^{1*}(s, \eta_s), \dots, \gamma^{i*}(s, \eta_s), \dots, \gamma^{N*}(s, \eta_s)); \quad x^*(t) = x \end{aligned}$$

$\eta_s$  stands for data set  $\{x(s), x_0\}$  or  $\{x(\sigma), \sigma \leq s\}$ , depending on information pattern is MPS or CLPS

## CT: Feedback Nash Equilibrium Strategy (Solution Method)

- Time interval restriction,  $[t, T]$  provides same differential game with initial state  $x(t), \forall t$
- Under either MPS or CLPS information pattern, *feedback NE will depend only on the time variable and current value of the state*, but not on memory.
- If value functions  $V^i$  are continuously differentiable in  $x$  and  $t$ , then  $N$  partial differential equations replace previous equation (HJB equation)

## Obtaining the optimal control strategy from HJB equation

### Theorem

If a continuously differentiable function  $V(t, x)$  can be found that satisfies the HJB equation (9), then it generate the optimal strategy through the static (pointwise) minimization problem defined by the RHS of (9)

$$-\frac{\partial V(t, x)}{\partial t} = \min_{u \in U} \left[ \frac{\partial V(t, x)}{\partial x} f(t, x, u) + g(t, x, u) \right] \quad (9)$$



$$u^*(t) = \operatorname{argmin}_{u \in U} \left[ \frac{\partial V(t, x)}{\partial x} f(t, x, u) + g(t, x, u) \right] \quad (10)$$

## CT: Feedback Nash Equilibrium Strategy (Solution Method)

### Definition (Feedback NE solution with value function)

For an  $N$  person differential game for  $[0, T]$  and under either MPS or CLPS,  $N$  tuple of strategies  $\{\gamma^{i*} \in \Gamma^i, i \in N\}$  provides a feedback Nash equilibrium solution if there exists functions  $V^i: \{0, T\} \times R^n \rightarrow R, i \in N$  satisfying the partial differential equations

$$\begin{aligned} -\frac{\partial V^i(t, x)}{\partial t} &= \min_{u^i \in S^i} \left[ \frac{\partial V^i(t, x)}{\partial t} \tilde{f}^{i*}(t, x, u^i) + \tilde{g}^{i*}(t, x, u^i) \right] \\ &= \frac{\partial V^i(t, x)}{\partial t} \tilde{f}^{i*}(t, x, \gamma^{i*}(t, x)) + \tilde{g}^{i*}(t, x, \gamma^{i*}(t, x)) \end{aligned}$$

$$V^i(T, x) = q^i(x), \quad \forall i \in N$$

where

$$\tilde{f}^{i*}(t, x, u^i) \triangleq f(t, x, \{\gamma_{-i}^*(t, x), u^i\}),$$

$$\tilde{g}^{i*}(t, x, u^i) \triangleq g^i(t, x, \{\gamma_{-i}^*(t, x), u^i\}),$$

$$\{\gamma_{-i}^*(t, x), u^i\} \triangleq \gamma^{1*}(t, x), \dots, u^i, \dots, \gamma^{N*}(t, x)$$

Every such equilibrium solution is STC, and Nash equilibrium cost of  $P_i$  is  $V^i(0, x_0)$

**Example**

## Structural Dynamic Control using Game Theory

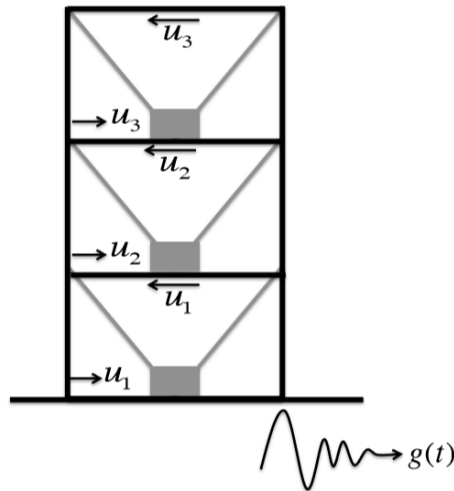
- Linear Quadratic Differential game is defined as follows
  - Cost function for each agent:

$$J_i = \frac{1}{2} \int_0^T \left\{ x^T(t) Q_i x(t) + \sum_{j=1}^N u_j(t)^T R_{ij} u_j(t) \right\} dt + \frac{1}{2} x^T(T) Q_T x(T)$$

- Dynamics of joint state

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^N B_j u_j(t) + Eg(t)$$

- Dynamic control of the three story building is expressed as:



$$J_1 = \int_0^\infty \{ x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2 + u_3^T R_{13} u_3 \} dt$$

$$J_2 = \int_0^\infty \{ x^T Q_2 x + u_1^T R_{21} u_1 + u_2^T R_{22} u_2 + u_3^T R_{23} u_3 \} dt$$

$$J_3 = \int_0^\infty \{ x^T Q_3 x + u_1^T R_{31} u_1 + u_2^T R_{32} u_2 + u_3^T R_{33} u_3 \} dt$$

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 + B_3 u_3 + Eg$$

## Structural Dynamic Control using Game Theory

### Cooperative Control

- The cooperative control policy is derived assuming each agent tries to minimize the commonly shared objective

$$J_1 + J_2 + J_3 = \int_0^\infty \left\{ x^T (Q_1 + Q_2 + Q_3) x + [u_1 u_2 u_3]^T \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right\} dt$$

$$J_1 + J_2 + J_3 = \int_0^\infty \{ z^T Q z + u^T R u \} dt$$

- ✓ where  $u = [u_1, u_2, u_3]$  is aggregated control action vector
- ✓  $Q$  and  $R$  are aggregated accordingly
- Due to the cooperation, the cooperative control problem can be formulated as an optimal control problem and can be solved using optimal control theory

$$PA + A^T P - PSP + Q = 0$$

$$S = BR^{-1}B^T$$

$$u(x) = -R^{-1}B^T Px(t) = F^* x(t)$$

## Structural Dynamic Control using Game Theory

### Nash Feedback Control

- Assuming the control actions are expressed as linear function of constant gain matrix such that  $u_i = F_i x$ , we can re-write the cost functions

$$J_1(F_1, F_2, F_3, x_0) = \int_0^\infty \{x^T (Q_1 + F_1^T R_{11} F_1 + F_2^T R_{12} F_2 + F_3^T R_{13} F_3) x\} dt$$

$$J_2(F_1, F_2, F_3, x_0) = \int_0^\infty \{x^T (Q_2 + F_1^T R_{21} F_1 + F_2^T R_{22} F_2 + F_3^T R_{23} F_3) x\} dt$$

$$J_3(F_1, F_2, F_3, x_0) = \int_0^\infty \{x^T (Q_3 + F_1^T R_{31} F_1 + F_2^T R_{32} F_2 + F_3^T R_{33} F_3) x\} dt$$

- Assuming the control actions are expressed as linear function of constant gain matrix such that  $u_i = F_i x$ , we can re-write the transition functions

$$\dot{x} = (A + B_1 F_1 + B_2 F_2 + B_3 F_3) x + E g$$



## Structural Dynamic Control using Game Theory

### Nash Feedback Control

- The Nash equilibrium  $(F_1^*, F_2^*, F_3^*)$  satisfies the following conditions:

$$J_1(F_1^*, F_2^*, F_3^*, x_0) \leq J_1(F_1, F_2^*, F_3^*, x_0)$$

$$J_2(F_1^*, F_2^*, F_3^*, x_0) \leq J_2(F_1^*, F_2, F_3^*, x_0)$$

$$J_3(F_1^*, F_2^*, F_3^*, x_0) \leq J_3(F_1^*, F_2^*, F_3, x_0)$$

- Nash equilibrium  $(F_1^*, F_2^*, F_3^*)$  strategy can be computed by solving the coupled inequality equations
- When agent 2 and 3 are assumed to follow the optimum strategy, the first agent should best respond to the fixed strategies as:

$$F_1^* = \arg \min_{F_1} J_1(F_1, F_2^*, F_3^*, x_0)$$

- When  $F_2^* = -R_{22}^{-1} B_2^T P_2$  and  $F_3^* = -R_{33}^{-1} B_3^T P_3$  ( $P_2$  and  $P_3$  are unknown matrices), the objection function of player 1 becomes:

$$J_1(F_1, F_2^*, F_3^*, x_0) = \int_0^\infty [x^T \{Q_1 + F_1^T R_{11} F_1 + (-R_{22}^{-1} B_2^T P_2)^T R_{12} (-R_{22}^{-1} B_2^T P_2) \\ (-R_{33}^{-1} B_3^T P_3)^T R_{13} (-R_{33}^{-1} B_3^T P_3)\} x] dt$$

## Structural Dynamic Control using Game Theory

### Nash Feedback Control

$$J_1(F_1, F_2^*, F_3^*, x_0) = \int_0^\infty [x^T \{Q_1 + F_1^T R_{11} F_1 + (-R_{22}^{-1} B_2^T P_2)^T R_{12} (-R_{22}^{-1} B_2^T P_2) \\ (-R_{33}^{-1} B_3^T P_3)^T R_{13} (-R_{33}^{-1} B_3^T P_3)\} x] dt$$

- Setting  $S_{ij} = B_i R_{ii}^{-1} R_{ij} R_{ii}^{-1} B_i^T$ , Agent 1 needs to maximize

$$\bar{J}_1(F_1, x_0) = \int_0^\infty \{x^T (Q_1 + P_2 S_{12} P_2 + P_3 S_{13} P_3) x + x^T F_1^T R_{11} F_1 x\} dt$$

- Assuming the following state dynamics

$$\begin{aligned} \dot{x} &= (A + B_1 F_1 + B_2 F_2 + B_3 F_3) x + E g \\ &= \{A + B_1 F_1 + B_2 (-R_{22}^{-1} B_2^T P_2) + B_3 (-R_{33}^{-1} B_3^T P_3)\} x + E g \\ &= (A - S_{22} P_2 - S_{33} P_3) x + B_1 F_1 x + E g \end{aligned}$$

- Having formulated the cost function and the system dynamics, while assuming the gain matrices for other controller, we can derive the Riccati equation for player 1:

$$\begin{aligned} (A')^T P_1 + P_1 A' - P_1 S_{11} P_1 + Q' &= 0 \\ (A - S_{22} P_2 - S_{33} P_3)^T P_1 + P_1 (A - S_{22} P_2 - S_{33} P_3) - P_1 S_{11} P_1 + (Q_1 + P_2 S_{12} P_2 + P_3 S_{13} P_3) &= 0 \end{aligned}$$

## Structural Dynamic Control using Game Theory

### Two Person Min-Max Game

- To explicitly account for an external load (earthquake), a two player zero-sum game framework can be used. In it, the external load is treated as a fictitious agent competing with controllers. The cost function for the controller is redefined as

$$J_c(u, g, x_0) = \int_0^\infty \{x^T Q x + u^T R u - g^T V g\} dt$$

or the external load is defined as the negative of that for the controller:

$$J_g(u, g, x_0) = -J_c(u, g, x_0) = \int_0^\infty \{-x^T Q x - u^T R u + g^T V g\} dt$$

- the Min-Max control problem is to find the gain matrix  $F^*(u^*(t) = F^*(x(t)))$  that minimizes the worst-case cost function incurred by the external load as follows [5]:

$$F^* = \min_{F \in F} \sup_{g \in L_2^q(0, \infty)} J_c(F, g, x_0)$$

$$J_c(F, g, x_0) = \int_0^\infty \{x^T (Q + F^T R F) x - g^T V g\} dt$$

## Structural Dynamic Control using Game Theory

### Two Person Min-Max Game

- Policies for the controllers and the earthquake are

$$u^*(t) = -R^{-1}B^T P_1 x(t)$$

$$g^*(t) = -V^{-1}E^T P_2 x(t)$$

- Inserting the policy into the cost function:

$$\bar{J}_c(F, x_0) = \int_0^\infty [x^T \{Q + F^T R F - (-V^{-1}E^T P_2)^T V (-V^{-1}E^T P_2)\} x] dt$$

- Setting  $M = EV^{-1}ET$

$$\bar{J}_c(F, x_0) = \int_0^\infty [x^T \{(Q - P_2 M P_2) + T F^T R F\} x] dt$$

- State dynamics is expressed as:

$$\dot{x} = Ax + BFx + Eg$$

$$= \{A - E(V^{-1}E^T P_2)\} x + BFx = (A - MP_2)x + BFx$$

## Structural Dynamic Control using Game Theory

### Two Person Min-Max Game

- Setting  $S = BR^{-1}B^T$ ,  $M = EV^{-1}E^T$ ,

$$(A - MP_2)^T P_1 + P_1(A - MP_2) - P_1SP_1 + (Q - P_2MP_2) = 0$$

$$(A - SP_1)^T P_2 + P_2(A - SP_1) - P_2MP_2 + (-Q - P_1SP_1) = 0$$

- Adding Two equations:

$$(P_1 + P_2)(A - SP_1 - MP_2) + (A - SP_1 - MP_2)^T (P_1 + P_2) = 0$$

- Substituting  $P_1 = -P_2 = P$ , the above equation becomes

$$A^T P + PA - P(S - M)P + Q = 0$$