Lecture 3: Analyzing games

Solution concept

- A solution concept is a method of analyzing games with the objective of restricting the set of all possible outcomes to those that are more reasonable than others.
- A solution concept results in one or more strategy profiles, which we call equilibrium
- An equilibrium is prediction emerged by applying a solution concept to a target game

$$equilibrium_1 = f_1(G)$$
$$equilibrium_2 = f_2(G)$$

$$equilibrium_n = f_n(G)$$

$f_1, \ldots, f_n$ are solution concepts

Assumptions and setup for analyzing game

- To set up the background for equilibrium analysis, it is useful to summarize the assumptions that we will be using in the lecture

  - **Players are "rational"**: A rational player is one who chooses his strategy $s_i \in S_i$, to maximize his payoff consistent with his beliefs about what is going on in the game

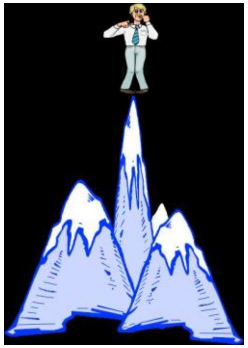
  - **Players are "intelligent"**: An intelligent player knows everything about the game: the actions, the outcomes, and the preferences of all the players.

  - **Common knowledge** : The fact that players are rational and intelligent is common knowledge among the players of the game

  - **Self-enforcing:** Any prediction (or equilibrium) of a solution concept must be self-enforcing
    - Core of our analysis and at the heart of non-cooperative game theory
    - Each player is in control of his own actions, and he will stick to an action only if he finds the action is in his best interest

Solution concepts

- Pareto optimality
- Nash equilibrium
- Maximin and minmax strategies
- Minimax regret
- Removal of dominated strategies
- Rationalizability
- Correlated equilibrium
- Trembling-hand perfect equilibrium
- Etc.



- **Single agent decision making:**
  - Optimal strategy is one that maximizes the agent's expected utility for a given environment
  - Uncertainties arose from stochastic environment, partially observable states, uncertain rewards, etc., which can be dealt with probability concepts.

$$a^* = \underset{a}{\operatorname{argmax}}\, E_s[u(a,s)]$$

- **Multiagents decision making:**
  - The environment includes other agents, each of which tries to maximize its own utility
  - Thus the notion of an optimal strategy for a given agent is not meaningful because *the best strategy depends on the choices of others*
  - We need to identify certain subsets of outcomes, called solution concepts
  - Two of the most fundamental solution concepts are
    - *Pareto optimality*
    - *Nash equilibrium*

$$u_1(a_1^*, a_2^*) \geq u_1(a_1, a_2^*) \ \ \forall a_1$$
$$u_2(a_1^*, a_2^*) \geq u_1(a_1^*, a_2) \ \ \forall a_2$$

Pareto optimality

- We’ve defined some canonical games and thought about how to play them.
- Now let’s examine the games from the outside:
  - From the point of view of an outside observer, can some outcomes of a game be said to be better than others?
  - Can we say that one agent’s interests are more important than another’s
  - Imagine trying to find the revenue-maximizing outcome when you don’t know what currency is used to express each agent’s payoff
    - Are there ways to still prefer one outcome to another?

Pareto optimality

Outcome of strategy s

Agent 1's utility : 10 unit of currency x

Agent 2's utility : 500 unit of currency y

Outcome of strategy s'

Agent 1's utility : 20 unit of currency x

Agent 2's utility : 10 unit of currency y

- Can we insist that the outcome of strategy s is better than that of strategy s'?

Pareto optimality

Outcome of strategy s

Agent 1's utility : 10 unit of currency x

Agent 2's utility : 500 unit of currency y

Outcome of strategy s'

Agent 1's utility : 20 unit of currency x

Agent 2's utility : 10 unit of currency y

- Can we insist that the outcome of strategy s is better than that of strategy s'?
  - No, because we cannot say that one agent's utility is more important than the other's

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- Is there any situation that we can be sure that one outcome is better than another ?

Outcome of strategy s

Agent 1's utility : 10 unit of currency x

Agent 2's utility : 500 unit of currency y

Outcome of strategy s'

Agent 1's utility : 20 unit of currency x

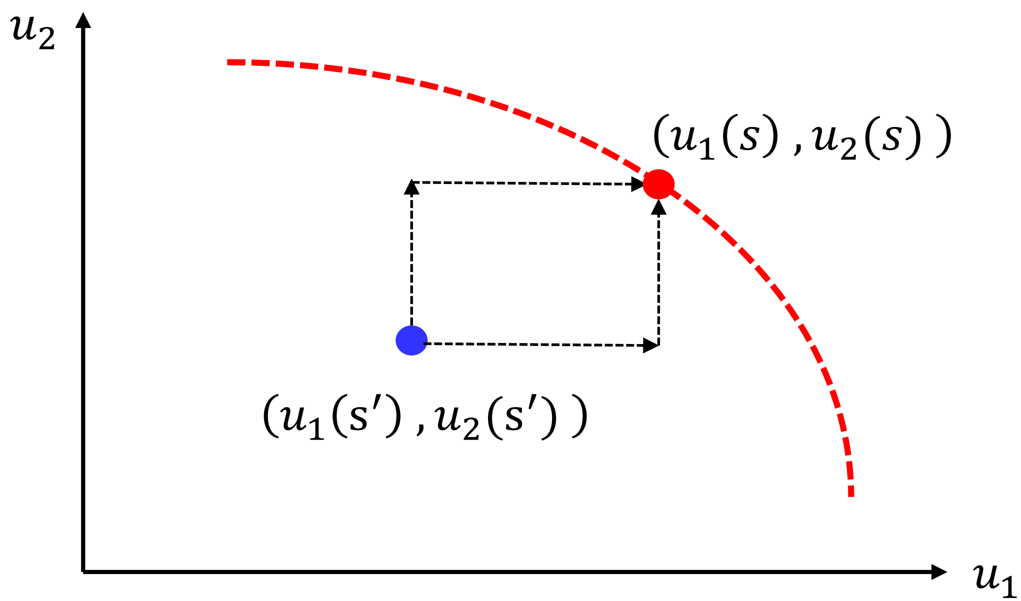
Agent 2's utility : 1000 unit of currency y

- The outcome of s' is always better than the outcome of s

Pareto optimality

Definition (Pareto domination)

Strategy profile s Pareto dominates strategy profile s' if for all $i \in N$, $u_i(s) \geq u_i(s')$, and there exists some $j \in N$ for which $u_j(s) > u_j(s')$.

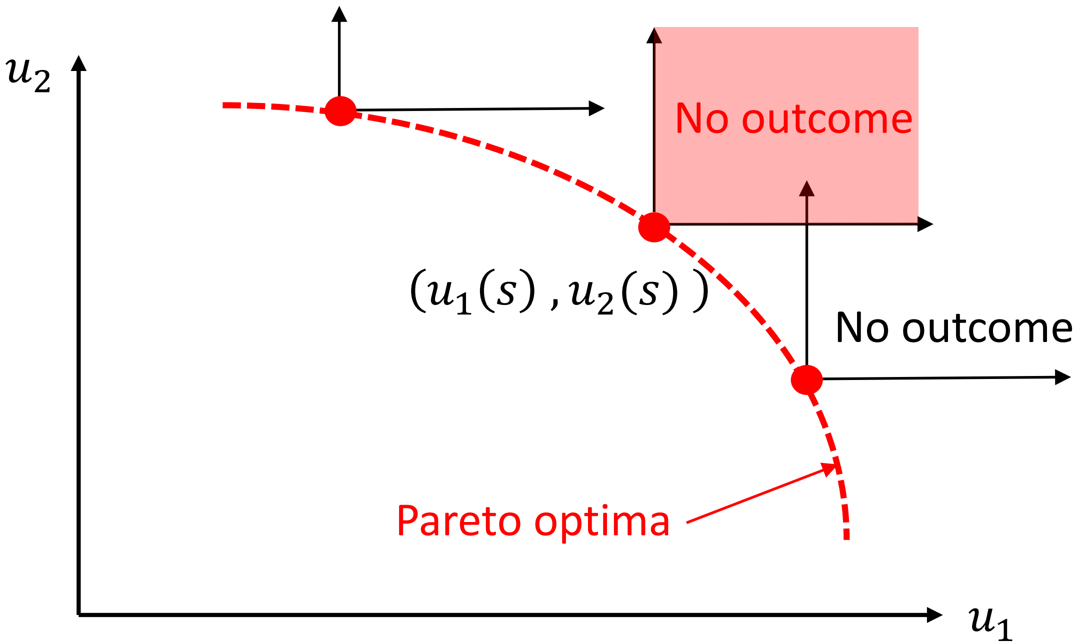


- In other words, in a Pareto-dominated strategy profile some players can be made better off without making any other player worse off
- We cannot generally identify a single "best" outcomes; instead we may have a set of non-comparable optima

Pareto optimality

Definition (Pareto optimality)

Strategy profile s is Pareto optimal, or strictly Pareto efficient, if there does not exist another strategy profile $s' \in S$ that Pareto dominates s

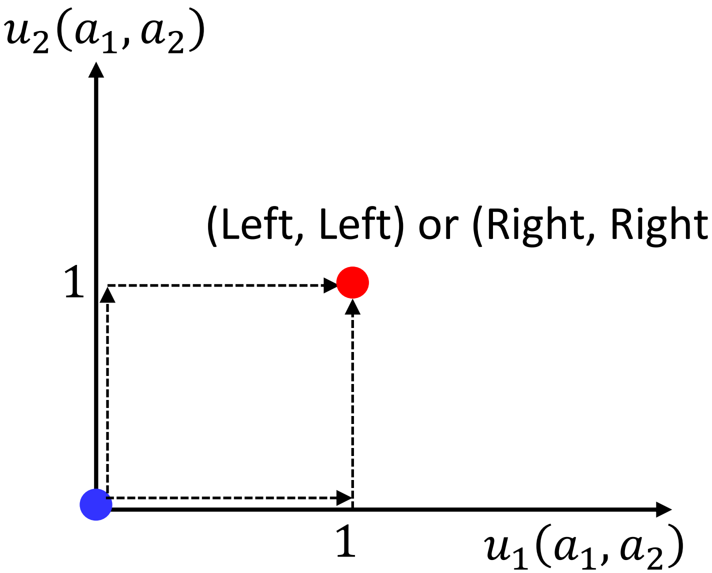


- Every game must have at least one Pareto optimal strategy profile, and there must always exist at least one such optimum in which all players adopt pure strategies.
- Some agent will have multiple optima
  (for example, in zero-sum games, all strategy profiles are strictly Pareto efficient. **Why?**)

Pareto optimal outcomes in various games

	Left	Right
Left	$1, 1$	$0, 0$
Right	$0, 0$	$1, 1$

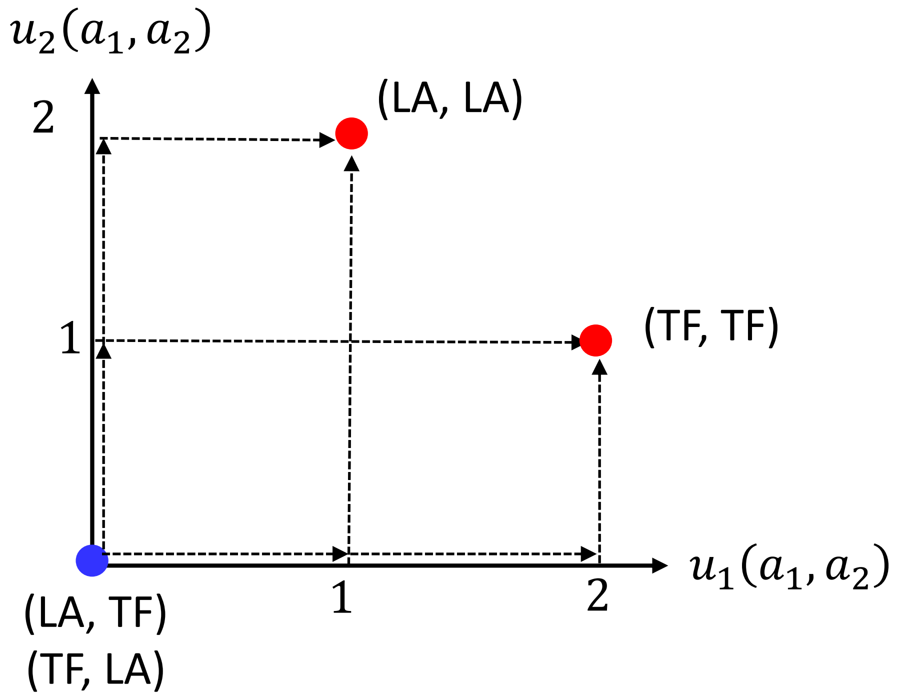
Coordination game



Pareto optimal outcomes in various games

	TF	LA
TF	2, 1	0, 0
LA	0, 0	1, 2

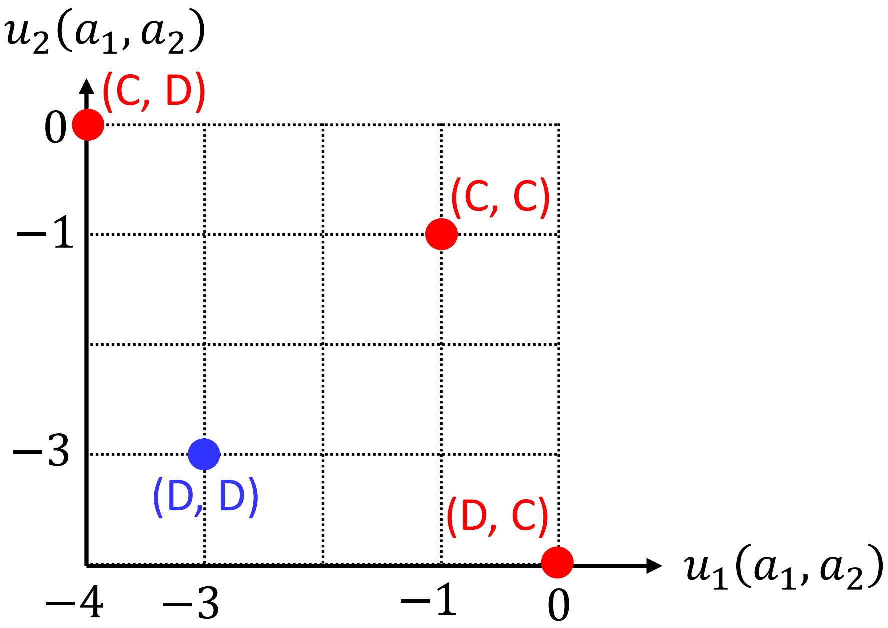
Battle of Sex game



Pareto optimal outcomes in various games

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

Prisoner's Dilemma game



Best response

- If you knew what everyone else was going to do, it would be easy to pick your own action
- Let $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ to be strategy profiles of other agents (all agents except i)
  - then, $s = (s_i, s_{-i})$

Definition (Best response)

Player i's best response to the strategy profile s_{-i} is a mixed strategy $s_i^* \in S_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$

$$s_i^* \in BR(s_{-i})$$

- The best response is not necessarily unique
- Indeed, except in the extreme case in which there is a unique best response that is a pure strategy, the number of best responses is always infinite.
- When the support of a best response s_i^* includes two or more actions, any mixture of these actions must also be a best response
- If there are two pure strategies that are individually best responses, any mixture of the two is necessarily also a best response

Nash equilibrium

- Really, no agent knows what the others will do
- What can we say about which actions will occur ?

Definition (Nash Equilibrium)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i and for all strategies s_i, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$.

- A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i, s_i^* is a best response to s_{-i}^*, i.e., $s_i^* \in BR(s_{-i}^*)$
- A Nash equilibrium is a stable strategy profile:
  - no agent would want to change his strategy if he knew what strategies the other agents were following

Nash equilibrium

Definition (Strict Nash)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a strict Nash Equilibrium if, for all agents i and for all strategies $s_i \neq s_i^*$, $u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$.

Definition (Week Nash)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a week Nash Equilibrium if, for all agents i and for all strategies $s_i \neq s_i^*$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$, and s^* is not a strict Nash equilibrium.

- Mixed-strategy Nash equilibria are necessarily week
- Pure-strategy Nash equilibria can be either strict or week, depending on the game.

Nash equilibrium examples

Pure-strategy Nash equilibria in the Battle of the Sexes game

Player 2

		TF	LA
Player 1	TF	$\boxed{2, 1}$	0, 0
	LA	0, 0	$\boxed{1, 2}$

- We immediately see that it has two pure-strategy Nash equilibria

Pure-strategy Nash equilibria in the Battle of the Sexes game

		Player 2	
		TF	LA
Player 1	TF	$\boxed{2, 1}$	0, 0
	LA	0, 0	1, 2

- We can check that these are Nash equilibria by confirming that whenever one of the players play the given (pure) strategy, the other player would only lose by deviating

$$a^* = (\mathrm{TF}, \mathrm{TF})$$

$$u_1(\mathrm{TF}, \mathrm{TF}) > u_1(\mathrm{LA}, \mathrm{TF})$$
$$u_2(\mathrm{TF}, \mathrm{TF}) > u_2(\mathrm{TF}, \mathrm{LA})$$

Nash equilibrium examples

Pure-strategy Nash equilibria in the Battle of the Sexes game

		Player 2	
		TF	LA
Player 1	TF	2, 1	0, 0
	LA	0, 0	$\boxed{1, 2}$

- We can check that these are Nash equilibria by confirming that whenever one of the players play the given (pure) strategy, the other player would only lose by deviating

$$a^* = (\mathrm{LA}, \mathrm{LA}) \qquad \begin{array}{l} u_1(\mathrm{LA}, \mathrm{LA}) > u_1(\mathrm{TF}, \mathrm{LA}) \\ u_2(\mathrm{LA}, \mathrm{LA}) > u_2(\mathrm{LA}, \mathrm{TF}) \end{array}$$

Nash equilibrium examples

How to easily find pure Nash equilibria?

Player 2

Player 1		TF	LA
	TF	2, 1	0, 0
	LA	0, 0	1, 2

Nash equilibrium examples

How to easily find pure Nash equilibria?

		Player 2	
		TF	LA
Player 1	TF	<u>2, 1</u>	0, 0
	LA	0, 0	<u>1, 2</u>

- **Step 1**: For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column

Nash equilibrium examples

How to easily find pure Nash equilibria?

		Player 2	
		TF	LA
Player 1	TF	$\overline{\underline{2, 1}}$	0, 0
	LA	0, 0	$\overline{\underline{1, 2}}$

- **Step 1**: For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column

- **Step 2**: For every row, which is strategy for player 1, **find the highest payoff entry for player 2** and over line the pair of payoffs

Nash equilibrium examples

How to easily find pure Nash equilibria?

		Player 2	
		TF	LA
Player 1	TF	$\overline{\underline{2, 1}}$	0, 0
	LA	0, 0	$\overline{\underline{1, 2}}$

- **Step 1**: For every column, which is strategy for player 2, **find the highest payoff entry for player 1** and underline the pair of payoffs in this row under this column

- **Step 2**: For every row, which is strategy for player 1, **find the highest payoff entry for player 2** and over line the pair of payoffs

- **Step 3**: If any matrix entry has both an under- and an over line, it is the outcome of a **Nash equilibrium in pure strategies**

Nash equilibrium examples

How to easily find pure Nash equilibria?

Player 2

		L	C	R
	U	7, 0	4, 2	1, 8
Player 1	M	2, 4	5, 5	2, 3
	D	8, 1	3, 2	0, 0

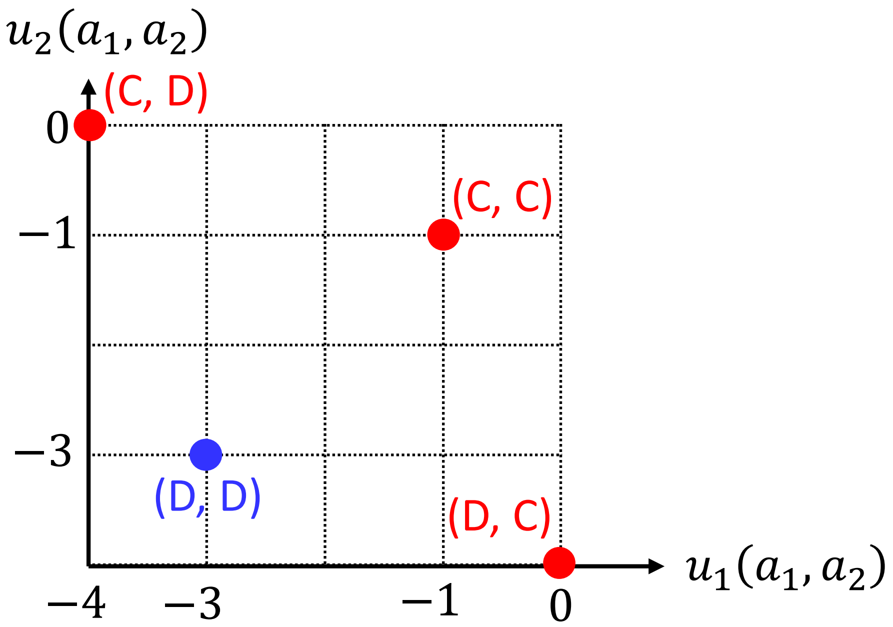
Find pure Nash equilibria by yourself

How many did you get?

Evaluating Nash equilibria solution

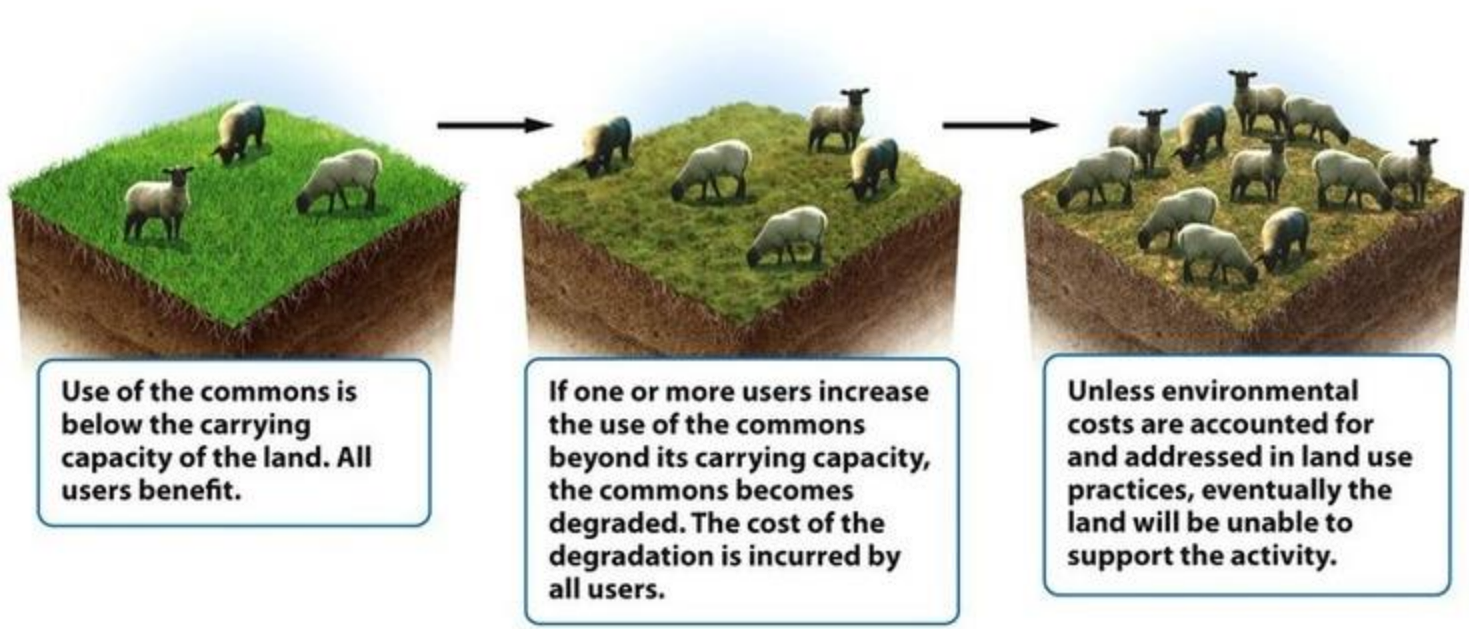
	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

Prisoner's Dilemma game



- As seen in Prisoner's Dilemma game, Nash equilibrium does not guarantee Pareto optimality
- People in many situations will do what is best for them, *at the expense of social efficiency*
- The solution concepts took the game as given, and they impose rationality and common knowledge of rationality to try to see what players would choose to do.
- **If each player seeks to maximize their individual well-being then the players may hinder their ability to achieve socially optimal outcomes**

Nash equilibrium examples : The Tragedy of the Commons



Nash equilibrium examples : The Tragedy of the Commons

- There are n players, say firms, in the world, each choosing how much to produce
- Their production activity in turn consumes some of the clean air that surrounds our planet
- There is a total amount of clean air equal to K, and any consumption of clean air comes out of this common resource
- Each player i chooses his own consumption of clean air for production, k_i
- The clean air left is $K - \sum_{i=1}^{n} k_i$
- The payoff for player i from the choice $k = (k_1, k_2, \dots, k_n)$ is equal to

$$u_i(k_i, k_{-i}) = \ln(k_i) + \ln\left(K - \sum_{j=1}^{n} k_j\right)$$

The benefit of consuming individual air consumption

The benefit of consuming the remainder of the clean air

Nash equilibrium examples : The Tragedy of the Commons

- To solve for a Nash equilibrium, we need to find some profile of choices $k^* = (k_1^*, k_2^*, \dots, k_n^*)$ for which $k_i^* = BR_i(k_{-i}^*)$ for all $i \in N$

- Then we have a system of n equations, on for each player's best-response function, with n unknowns, the choices of each player.

- For example, to get player $i's$ best-response function, the following first-order condition of his payoff function should be satisfied

$$\frac{\partial u_i(k_i, k_{-i})}{\partial k_i} = \frac{1}{k_i} - \frac{1}{K - \sum_{j=1}^{n} k_j} = 0$$

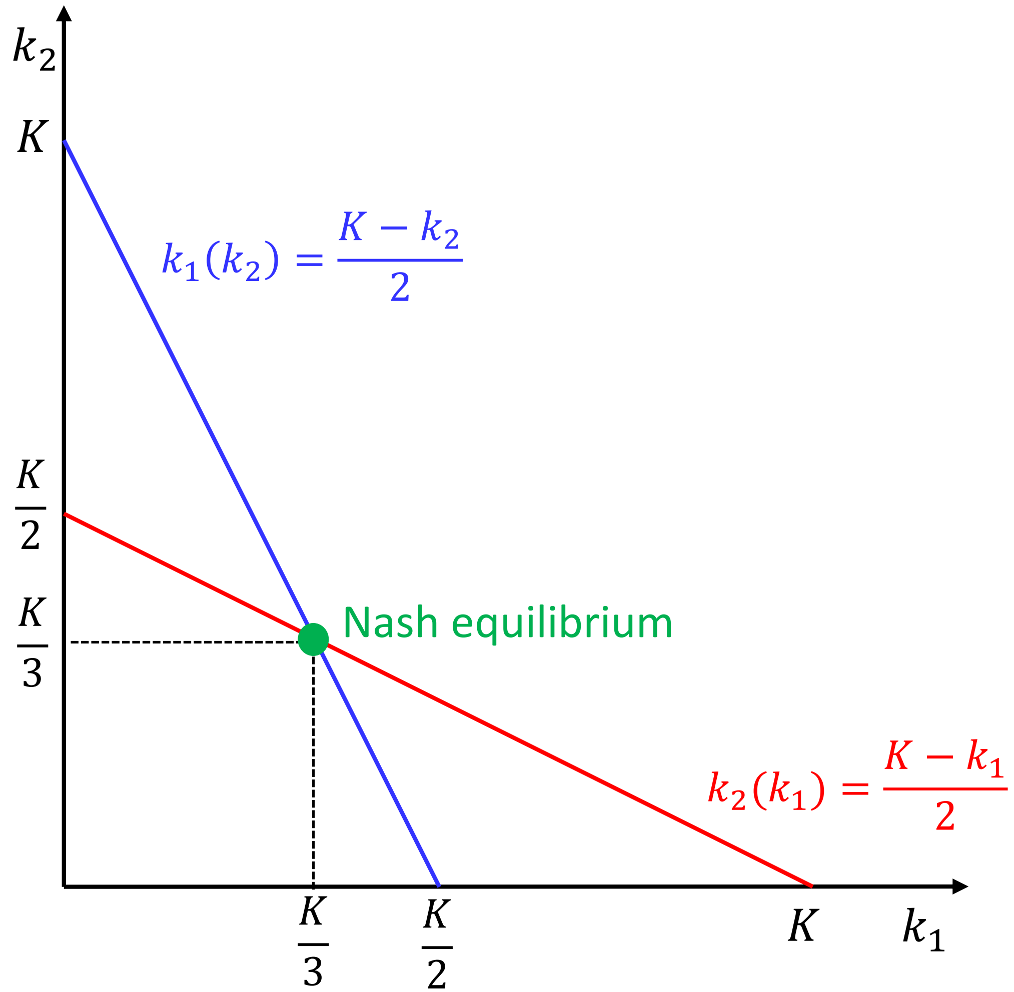
which gives player i's best response function,

$$BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2}$$

Nash equilibrium examples : The Tragedy of the Commons

- In case there are two firms, we have two best-response equations:

$$k_1(k_2) = \frac{K - k_2}{2} \qquad \text{and} \qquad k_2(k_1) = \frac{K - k_1}{2}$$



Nash equilibrium examples : The Tragedy of the Commons

- Now we can ask whether this two-player behave to make the society better
  - Is consuming $K/3$ for each player too much or too little?
  - Can we find another consumption profile that will make everyone better off?
- We will maximize the sum of all the payoff functions, which we can think of as the "world's payoff function $w(k_1, k_2)$
- We can maximize

$$\max_{k_1,k_2} w(k_1,k_2) = \sum_{i=1}^{2} u_i(k_1,k_2) = \sum_{i=1}^{2} \left\{ \ln(k_i) + \ln\left(K - \sum_{j=1}^{n} k_j\right) \right\}$$

- The first-order conditions of this problem are

$$\frac{\partial w(k_1,k_2)}{\partial k_1} = \frac{1}{k_1} - \frac{2}{K - k_1 - k_2} = 0$$

$$\frac{\partial w(k_1,k_2)}{\partial k_2} = \frac{1}{k_2} - \frac{2}{K - k_1 - k_2} = 0$$

- The solution for this is $k_1 = k_2 = \frac{K}{4}$, that gives $u_1 = u_2 = \ln\frac{K}{4} + \ln\frac{K}{2} = \ln(\frac{K^2}{8})$
  - which is larger that $u_1 = u_2 = \ln\frac{K}{3} + \ln(\frac{K}{3}) = \ln(\frac{K^2}{9})$ for Nash equilibrium

Nash equilibrium examples: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good
- Firm i produce quantity q_i
- Cost for production is $c_i(q_i) = c_i q_i$
- Price is given by $d = a - b(q_1 + q_2)$
- The profit of company i given its opponent chooses quantity q_j is

$$u_i(q_i, q_j) = (a - bq_i - bq_j)q_i - c_i q_i = -bq_i^2 + (a - c_i)q_i - bq_j q_i$$

- The best-response function for each firm is given by the first-order condition

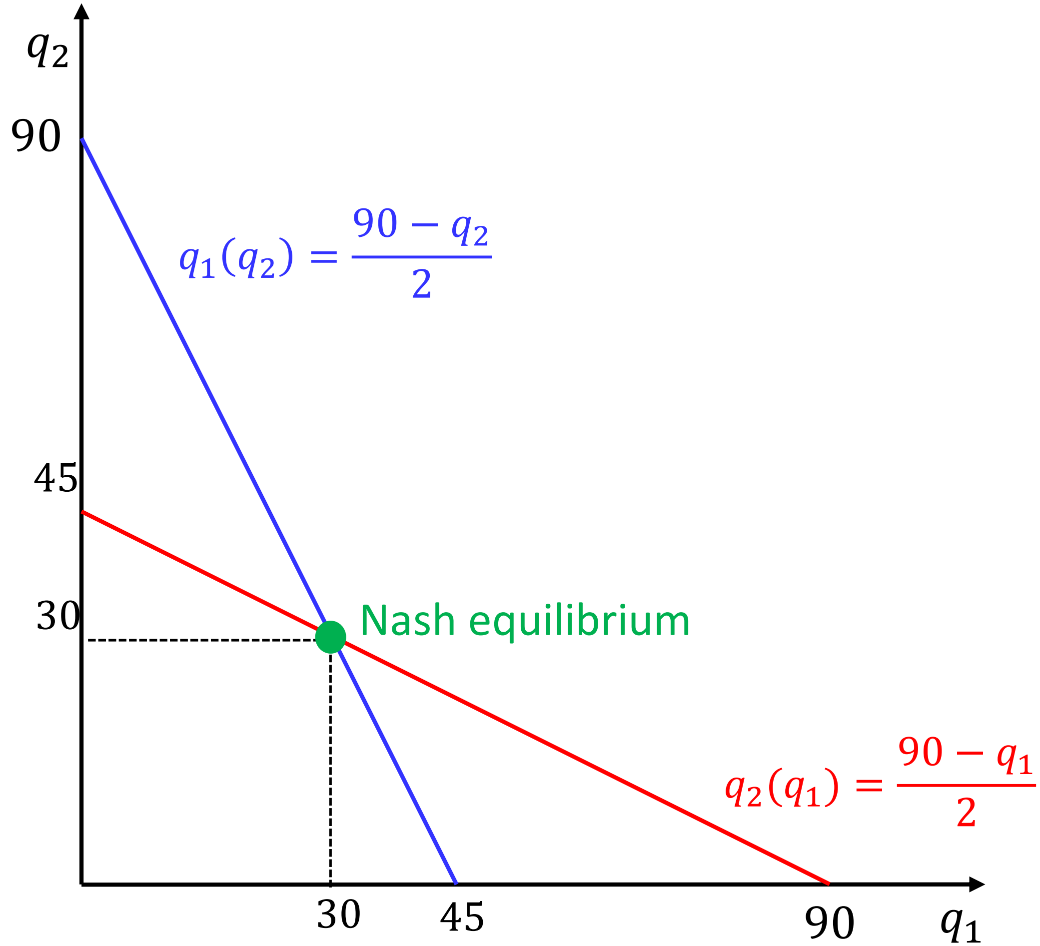
$$BR_i(q_j) = \frac{a - bq_j - c_i}{2b}$$

Nash equilibrium examples: Cournot Duopoly

- In case there are two firms, we have two best-response equations:

$$q_1 = \frac{a - bq_2 - c_1}{2b} \quad \text{and} \quad q_2 = \frac{a - bq_1 - c_2}{2b}$$

$a = 100, b = 1, c_1 = c_2 = 10$



Mixed strategy Nash equilibrium

- Why anyone would wish to randomize between actions?
- We will see mixed (stochastic) strategies turns out to be an important type of behavior to consider , with interesting implications and interpretations.
- No pure strategy Nash equilibria exists for the following Matching Pennies game

		Player 2	
		Heads	Tails
Player 1	Heads	$\underline{1}, -1$	$-1, \overline{1}$
	Tails	$-1, \overline{1}$	$\underline{1}, -1$

- Nash equilibrium will indeed exist if we allow players to choose random strategies

Revisit: mixed strategy

Definition (Mixed strategy)

Let (N, A, u) be a normal-form game, and for any set X let $\Pi(X)$ be the set of all probability distributions over X. Then, the set of mixed strategies for player i is $S_i = \Pi(A_i)$

Definition (Mixed strategy profile)

The set of mixed-strategy profile is simply the Cartesian product of the individual mixed-strategy sets, $S = S_1 \times \ldots \times S_n$.

- $s_i(a_j)$ denote the probability that an action a_j will be played under mixed strategy s_i
  - For example, $A = \{\text{Rook}, \text{Paper,Scissors}\}$, $s_i(\mathrm{R}) = 0.2$, $s_i(\mathrm{P}) = 0.3$, $s_i(\mathrm{S}) = 0.5$

Definition (Support)

The support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$

$$A_1 = \{L, R\}$$
$$S_1 = \Pi(A_1) = \{(s_1(L), s_1(R)) : s_1(L), s_1(R) \geq 0, s_1(L) + s_1(R) = 1\}$$
$$s_1 \in S_1, \text{i.e.}, s_1 = (q, 1 - q)$$

Beliefs and mixed strategies

- Introducing probability distributions not only enriches the set of actions from which a player can choose but also allows us to enrich the beliefs that players can have

Definition (Belief)

A belief for player i is given by a probability distribution $\pi_i \in \Pi(A_{-i})$ over the actions of his opponents. We denote by $\pi_i(a_{-i})$ the probability player i assigns to his opponents playing $a_{-i} \in A_i$

How to find mixed strategy Nash equilibrium?

Definition (Nash Equilibrium)

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i and for all strategies $s_i \in \Pi(A_i)$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$.

- A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a Nash Equilibrium if, for all agents i, s_i^* is a best response to s_{-i}^*, i.e., $s_i^* \in BR(s_{-i}^*)$
- We can think of s_{-i}^* as the belief of player i about his opponents, π_i, which captures the idea that player i is uncertain of his opponent's behavior
  - The profile of mixed strategies s_{-i}^* thus captures this uncertain belief over all of the pure strategies that player i's opponent can play
  - Rationality requires that a player play a best response given his belief (Nash equilibrium requires that these beliefs are correct, i.e., a system of equations should be satisfied)

How to find mixed strategy Nash equilibrium?

- Recall that the support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$

- Imagine that the Nash equilibrium profile s_i^* contains more that one pure strategy -say a_i and a_i' as supports.

- What must we conclude about a rational player i if s_i^* is indeed part of a Nash equilibrium (s_i^*, s_{-i}^*)?

How to find mixed strategy Nash equilibrium?

- Recall that the support of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i | s_i(a_i) > 0\}$
- Imagine that the Nash equilibrium profile s_i^* contains more that one pure strategy -say a_i and a_i' as supports.
- What must we conclude about a rational player i if s_i^* is indeed part of a Nash equilibrium (s_i^*, s_{-i}^*)?

if s_i^* is a Nash equilibrium, and both if a_i and if a_i' are in the support of s_i^*, then

$$u_i(a_i, s_{-i}^*) = u_i(a_i', s_{-i}^*) = u_i(s_i^*, s_{-i}^*)$$

Proof:

- assume $u_i(a_i, s_{-i}^*) > u_i(a_i', s_{-i}^*)$ and a_i and a_i' are support of s_i^*
- Adjusting the mixed strategy $s_i = \{s_i(a_i), s_i(a_i')\} \rightarrow \{s_i(a_i) + s_i(a_i'), 0\}$ will increase u_i
- s_i^* could not have been a best response to s_{-i}^*
- Therefore, by contradiction, $u_i(a_i, s_{-i}^*) = u_i(a_i', s_{-i}^*)$

How to find mixed strategy Nash equilibrium?

if s_i^* is a Nash equilibrium, and both if a_i and if a_i' are in the support of s_i^*, then

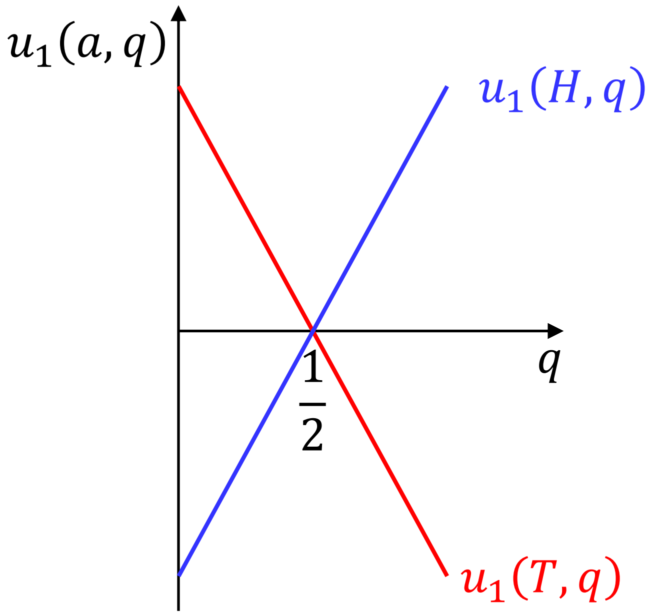
$$u_i(a_i, s_{-i}^*) = u_i(a_i', s_{-i}^*) = u_i(s_i^*, s_{-i}^*)$$

- This result will play an important role in computing mixed-strategy Nash equilibria
  - If a player is playing a mixed strategy then he must be indifferent between the actions he is choosing with positive probability (i.e., actions in the support)
- One player's indifference will impose restrictions on the behavior or other players
  - This restriction will help us find the mixed-strategy Nash equilibrium

Finding mixed Nash equilibria : Matching Pennies

		q **H**	$1-q$ **T**
p	**H**	$1,-1$	$-1,1$
$(1-p)$	**T**	$-1,1$	$1,-1$

$$u_1(H,q) = q\times 1 + (1-q)\times(-1) = 2q-1$$
$$u_1(T,q) = q\times(-1) + (1-q)\times 1 = 1-2q$$



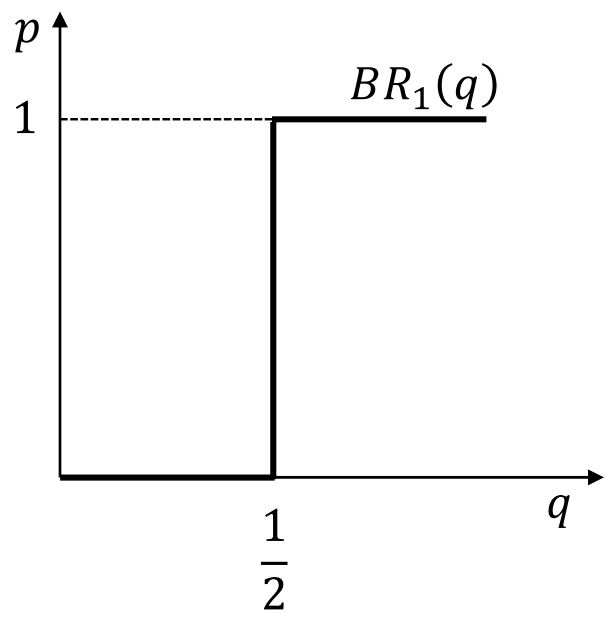
$$BR_1(q) = \begin{cases} p=0 & if\ \ q<1/2 \quad \text{(Playing } T\text{)} \\ p\in[0,1] & if\ \ q=1/2 \quad \text{(Indifferent)} \\ p=1 & if\ \ q>1/2 \quad \text{(Playing } H\text{)} \end{cases}$$

Finding mixed Nash equilibria : Matching Pennies

		q H	$1-q$ T
p	**H**	$1,-1$	$-1,1$
$(1-p)$	**T**	$-1,1$	$1,-1$

$u_1(H,q) = q \times 1 + (1-q) \times (-1) = 2q - 1$

$u_1(T,q) = q \times (-1) + (1-q) \times 1 = 1 - 2q$



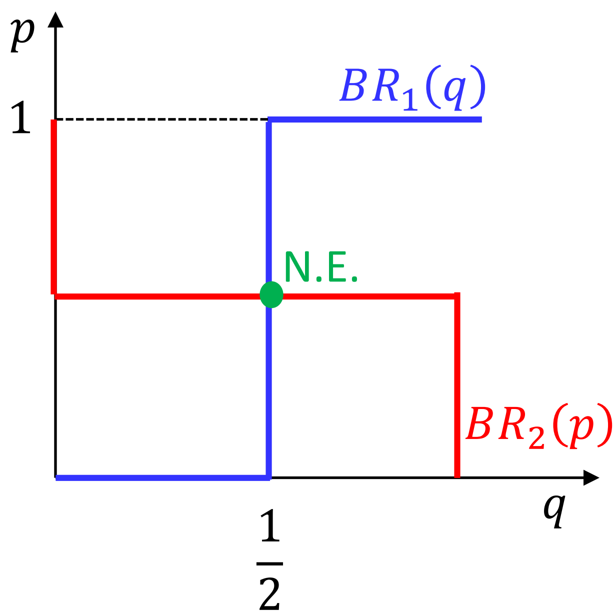
$$BR_1(q) = \begin{cases} p = 0 & if \quad q < 1/2 & \text{(Playing } T\text{)} \\ p \in [0,1] & if \quad q = 1/2 & \text{(Indifferent)} \\ p = 1 & if \quad q > 1/2 & \text{(Playing } H\text{)} \end{cases}$$

Finding mixed Nash equilibria : Matching Pennies

		q **H**	$1-q$ **T**
p	**H**	$1,-1$	$-1,1$
$(1-p)$	**T**	$-1,1$	$1,-1$

$$u_1(H,q) = q \times 1 + (1-q) \times (-1) = 2q - 1$$
$$u_1(T,q) = q \times (-1) + (1-q) \times 1 = 1 - 2q$$

$$u_2(H,p) = p \times (-1) + (1-p) \times 1 = 1 - 2p$$
$$u_2(T,p) = p \times 1 + (1-p) \times (-1) = 2p - 1$$



$$BR_1(q) = \begin{cases} p = 0 & if \;\; q < 1/2 & \text{(Playing } T\text{)} \\ p \in [0,1] & if \;\; q = 1/2 & \text{(Indifferent)} \\ p = 1 & if \;\; q > 1/2 & \text{(Playing } H\text{)} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & if \;\; p < 1/2 & \text{(Playing } H\text{)} \\ q \in [0,1] & if \;\; p = 1/2 & \text{(Indifferent)} \\ q = 0 & if \;\; p > 1/2 & \text{(Playing } T\text{)} \end{cases}$$

The intersections of two best response curve→ Nash equilibria

To find Nash equilibrium, make other player indifferent between some of his pure actions

Finding mixed Nash equilibria : Matching Pennies

Mixed-strategy Nash equilibria in the **Matching Pennies game**

		Player 2: Heads (p)	Player 2: Tails ($1-p$)
Player 1	Heads (q)	$1,-1$	$-1,1$
	Tails ($1-q$)	$-1,1$	$1,-1$

No pure strategy Nash equilibria exists

- Player 1 should randomize his action to make Player 2 to be indifferent between her actions
  → otherwise, player 2 would play the action that is superior than the other
- Let us assume Player 1' strategy is to play Heads with probability q and Tails with $1-q$

$$u_2(\text{Heads}) = u_2(\text{Tails})$$
$$-1 \times q + 1 \times (1-q) = 1 \times q - 1 \times (1-q)$$
$$q = \frac{1}{2}$$

Finding mixed Nash equilibria : Matching Pennies

Mixed-strategy Nash equilibria in the **Matching Pennies game**

		Player 2	
		p Heads	$1-p$ Tails
Player 1	q Heads	$1, -1$	$-1, 1$
	$1-q$ Tails	$-1, 1$	$1, -1$

No pure strategy Nash equilibria exists

- Player 2 should randomize his action to make Player 1 to be indifferent between her actions
  → otherwise, player 1 would play the action that is superior than the other
- Let us assume Player 2' strategy is to play Heads with probability p and Tails with $1-p$

$$u_1(\text{Heads}) = u_1(\text{Tails})$$
$$1 \times p - 1 \times (1-p) = -1 \times p + 1 \times (1-p)$$
$$p = \frac{1}{2}$$

Mixed-strategy Nash equilibria in the **Battle of the Sexes game**

		Player 2	
		p TF	$1-p$ LA
Player 1	q TF	2, 1	0, 0
	$1-q$ LA	0, 0	1, 2

- Player 1 should randomize his action to make Player 2 to be indifferent between her actions
  → otherwise, player 2 would play the action that is superior than the other
- Let us assume Player 1' strategy is to play TF with probability q and LA with $1-q$

$$u_2(\mathrm{TF}) = u_2(\mathrm{LA})$$
$$1 \times q + 0 \times (1-q) = 0 \times q + 2 \times (1-q)$$
$$q = \frac{2}{3}$$

Mixed-strategy Nash equilibria in the **Battle of the Sexes game**

		Player 2	
		p TF	$1-p$ LA
Player 1	q TF	2, 1	0, 0
	$1-q$ LA	0, 0	1, 2

- Player 2 should randomize his action to make Player 1 to be indifferent between her actions
  → otherwise, player 1 would play the action that is superior than the other
- Let us assume Player 2' strategy is to play TF with probability p and LA with $1-p$

$$u_1(\text{LA}) = u_1(\text{TF})$$
$$2 \times p + 0 \times (1-p) = 0 \times p + 1 \times (1-p)$$
$$p = \frac{1}{3}$$

Finding Nash equilibria

Mixed-strategy Nash equilibria in the **Battle of the Sexes game**

		Player 2	
		$1/3$ TF	$2/3$ LA
Player 1	$2/3$ TF	2, 1	0, 0
	$1/3$ LA	0, 0	1, 2

- Now, we can confirm that we have indeed found an equilibrium:
  - Both players play in a way that makes the other indifferent, they are both best responding to each other
- Expected payoff for both agents is 2/3 in this equilibrium
  - Each of the pure-strategy equilibria Pareto-dominates the mixed strategy equilibrium
- **This mixed strategy, as all other mixed strategies, is a week Nash equilibrium**

$$u_1(s_i^*, s_{-i}^*) \geq u_1(s_i, s_{-i}^*) \qquad u_1\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right) \geq u_1\left((x, 1-x), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$$ For any $0 \leq x \leq 1$

Mixed-strategy Nash equilibria in the Rock-Paper-Scissor game

	p_R Rook	p_P Paper	p_S Scissors
Rook	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

$u_1(\mathrm{R}) = u_1(\mathrm{P}) = u_1(\mathrm{S})$

$\Rightarrow 0p_R + (-1)p_P + 1p_s = 1p_R + 0p_P + (-1)p_S = -1p_R + 1p_P + 0p_s$

$\Rightarrow 0p_R + (-1)p_P + 1p_s = 1p_R + 0p_P + (-1)p_S \Rightarrow 2p_S = p_R + p_P$

$\Rightarrow 1p_R + 0p_P + (-1)p_S = -1p_R + 1p_P + 0p_s \quad \Rightarrow 2p_R = p_S + p_P$

$\Rightarrow p_R = p_P = p_S \qquad (1)$

$p_R + p_P + p_S = 1 \qquad (2)$

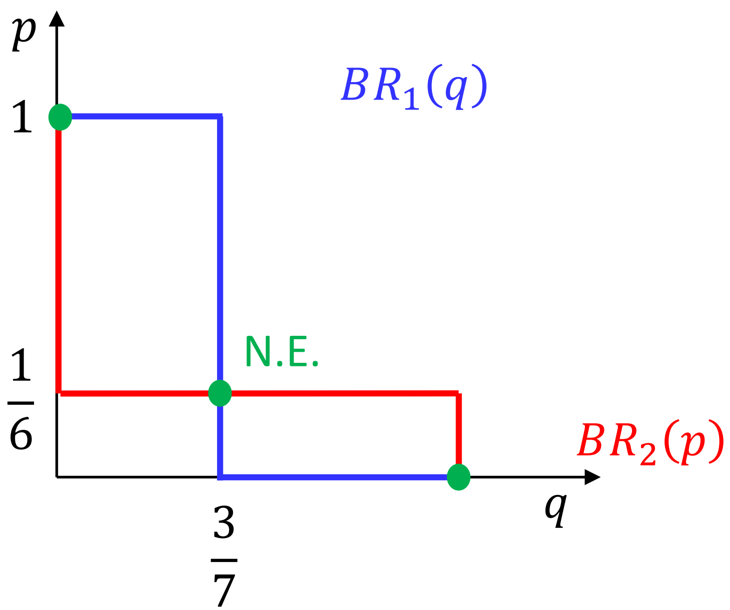
- Due to (1) and (2), $p_R = p_P = p_S = 1/3$ (Mixed strategy Nash Equilibrium)

Multiple mixed strategies

		q H	$1-q$ T
p	**H**	$0,0$	$3,5$
$(1-p)$	**T**	$4,4$	$0,3$

$$u_1(H,q) = q \times 0 + (1-q) \times (3) = 3q - 3$$
$$u_1(T,q) = q \times (4) + (1-q) \times 0 = 4q$$

$$u_2(H,p) = p \times (0) + (1-p) \times 4 = 4 - 4p$$
$$u_2(T,p) = p \times 5 + (1-p) \times (3) = 2p + 3$$



Nash equilibriums are $\left\{(1,0), \left(\frac{1}{6}, \frac{3}{7}\right), (0,1)\right\}$

The meaning of playing mixed-strategy

- Randomize to confuse your opponent
  - consider the matching pennies example

- Randomize when uncertain about the other’s action
  - consider battle of the sexes

- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit

- Mixed strategies describe population dynamics:
  - agents chosen from a population have deterministic strategies.
  - Mixed strategies gives the probability of getting each pure strategies.

The existence of Nash equilibria

Theorem (Nash, 1951)

Every game with a finite number of players and action profiles has at least one Nash equilibrium