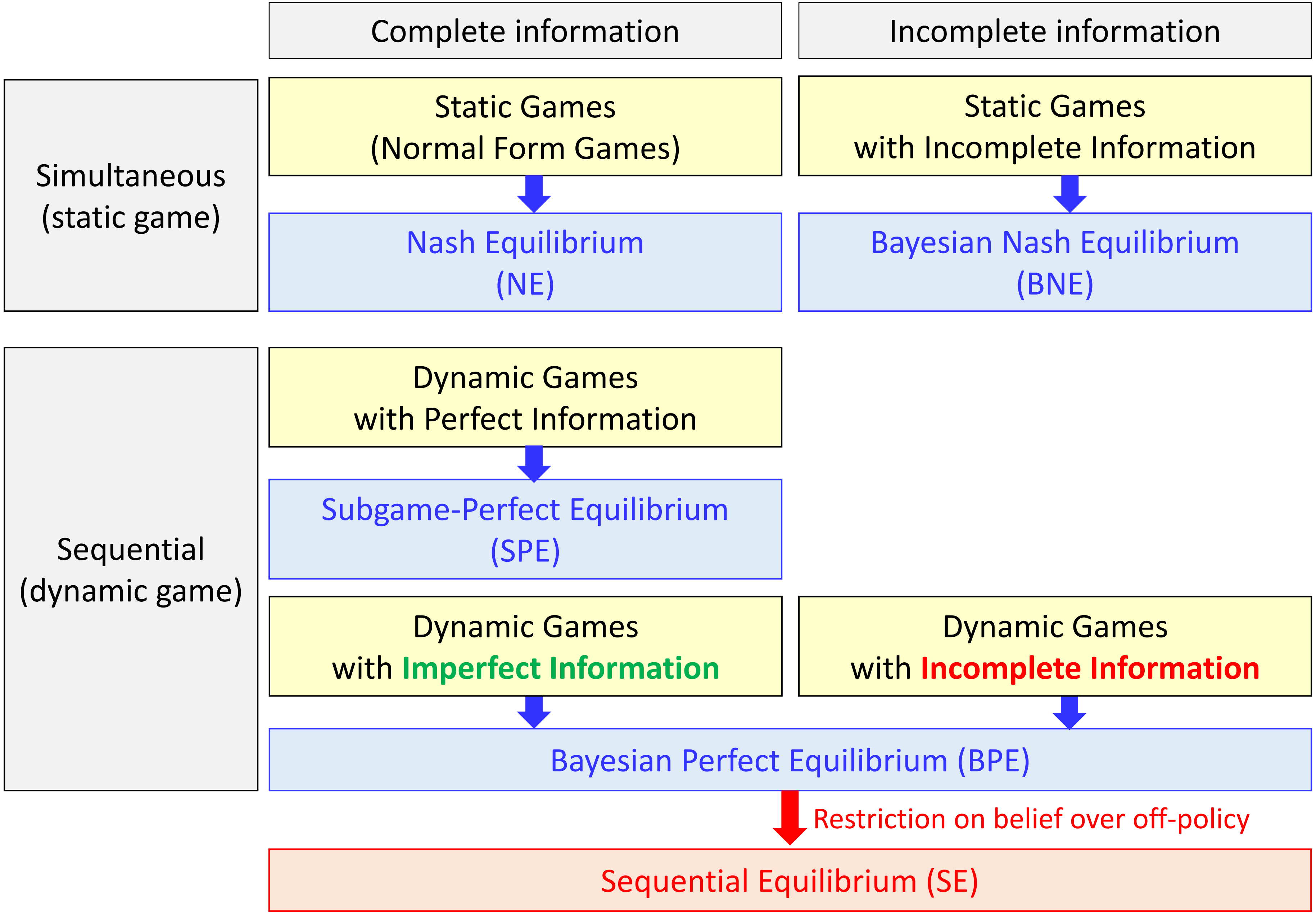
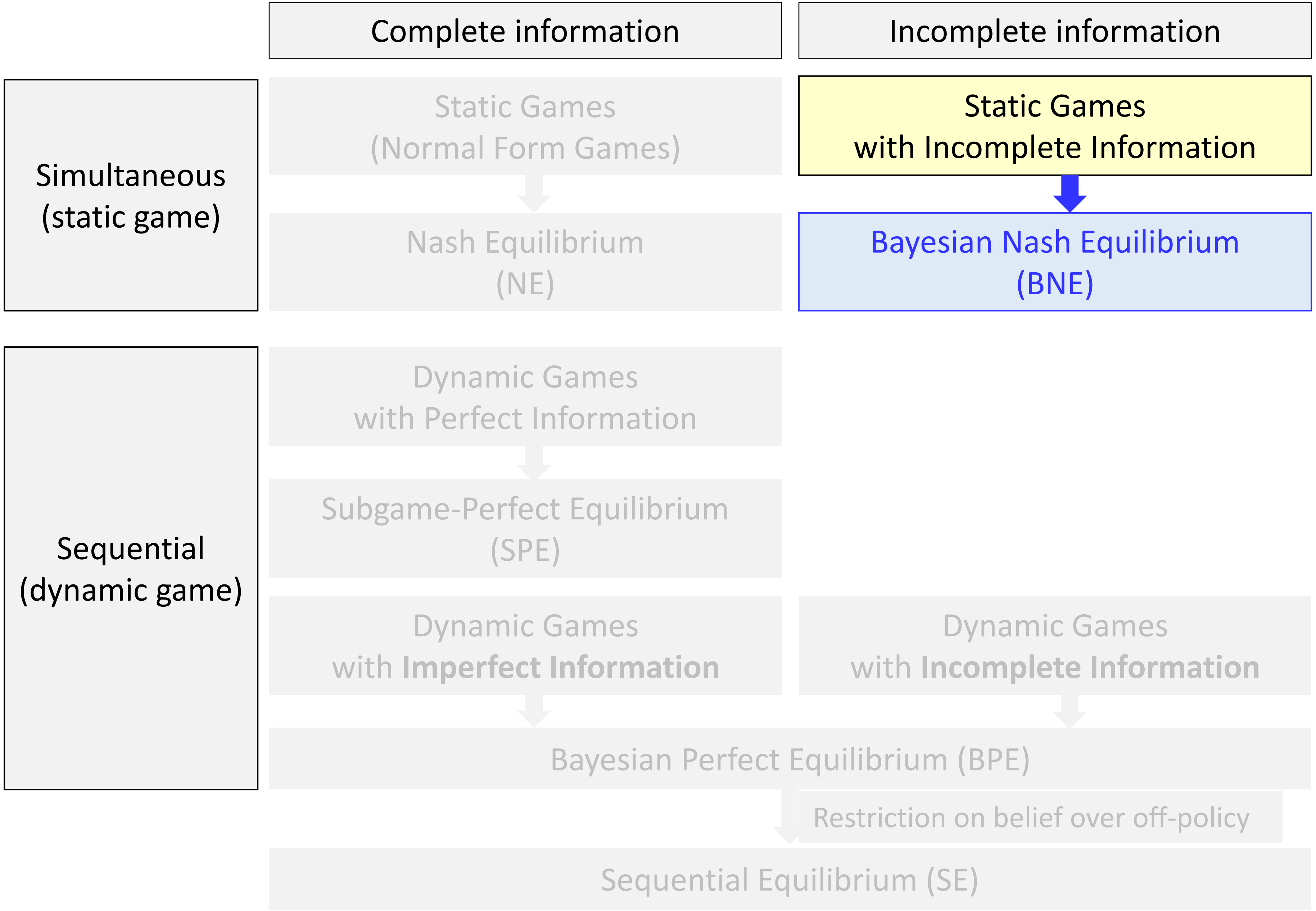
3. Static Games with Incomplete Information


Complete information
Incomplete information
Simultaneous
(static game)
Static Games
(Normal Form Games)
Nash Equilibrium
(NE)
Static Games
with Incomplete Information
Bayesian Nash Equilibrium
(BNE)
Sequential
(dynamic game)
Dynamic Games
with Perfect Information
Subgame-Perfect Equilibrium
(SPE)
Dynamic Games
with Imperfect Information
Dynamic Games
with Incomplete Information
Bayesian Perfect Equilibrium (BPE)
Restriction on belief over off-policy
Sequential Equilibrium (SE)

Introduction



Bayesian games

Introduction

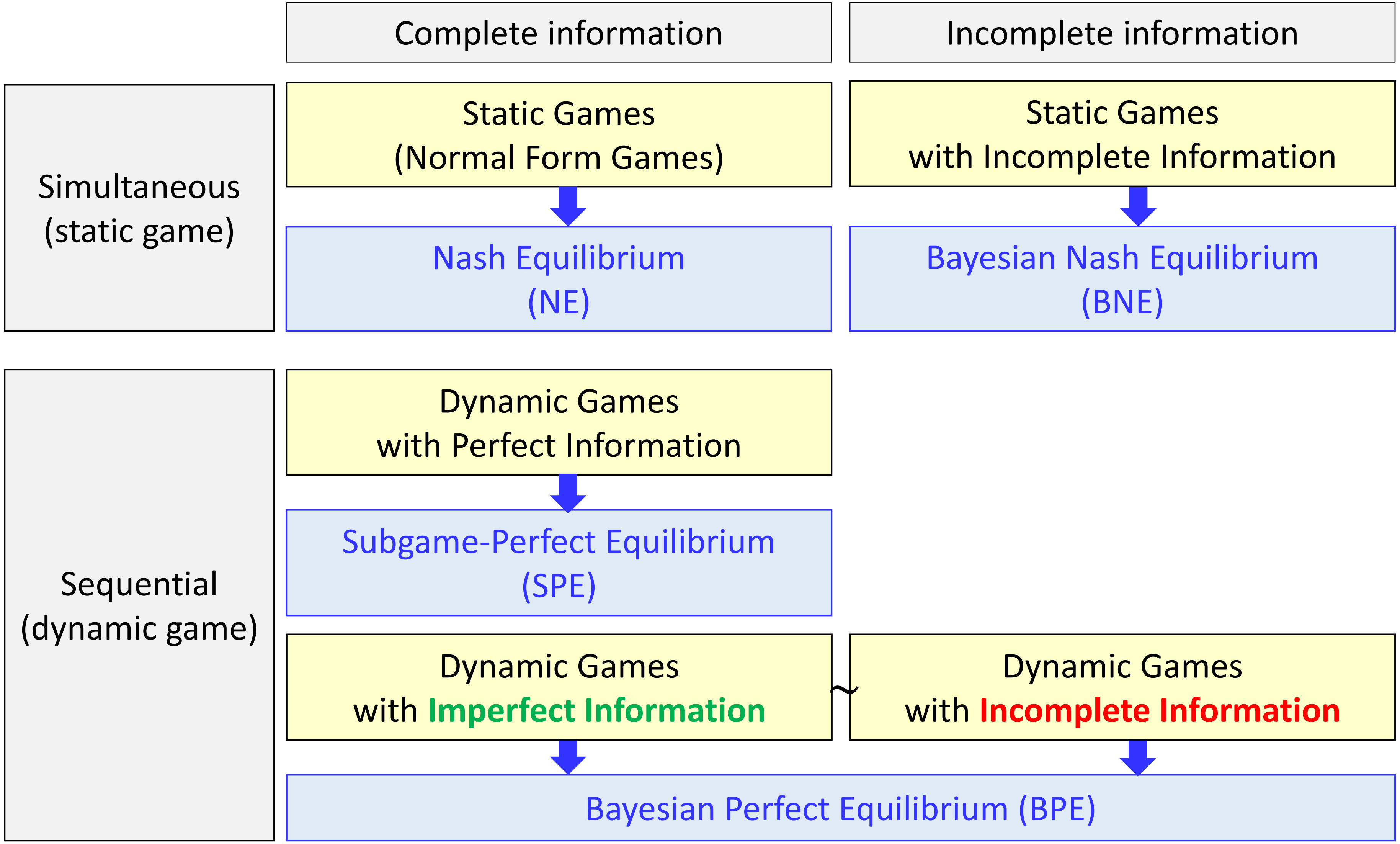
- Static games with complete information
  - Normal form games
- Dynamic games (sequential games) with complete information
  - Perfect information games
  - Imperfect information games
- Static games with incomplete information
  - Bayesian games
- Dynamic games with incomplete information

- Imperfect Information: Players do not perfectly observe the actions of other players or forget their own actions.
- Incomplete Information: Players have private information about something relevant to his decision making.
  - Incomplete information introduces uncertainty about the game being played
  - Various way to represent the uncertainty about the game (will be discussed)

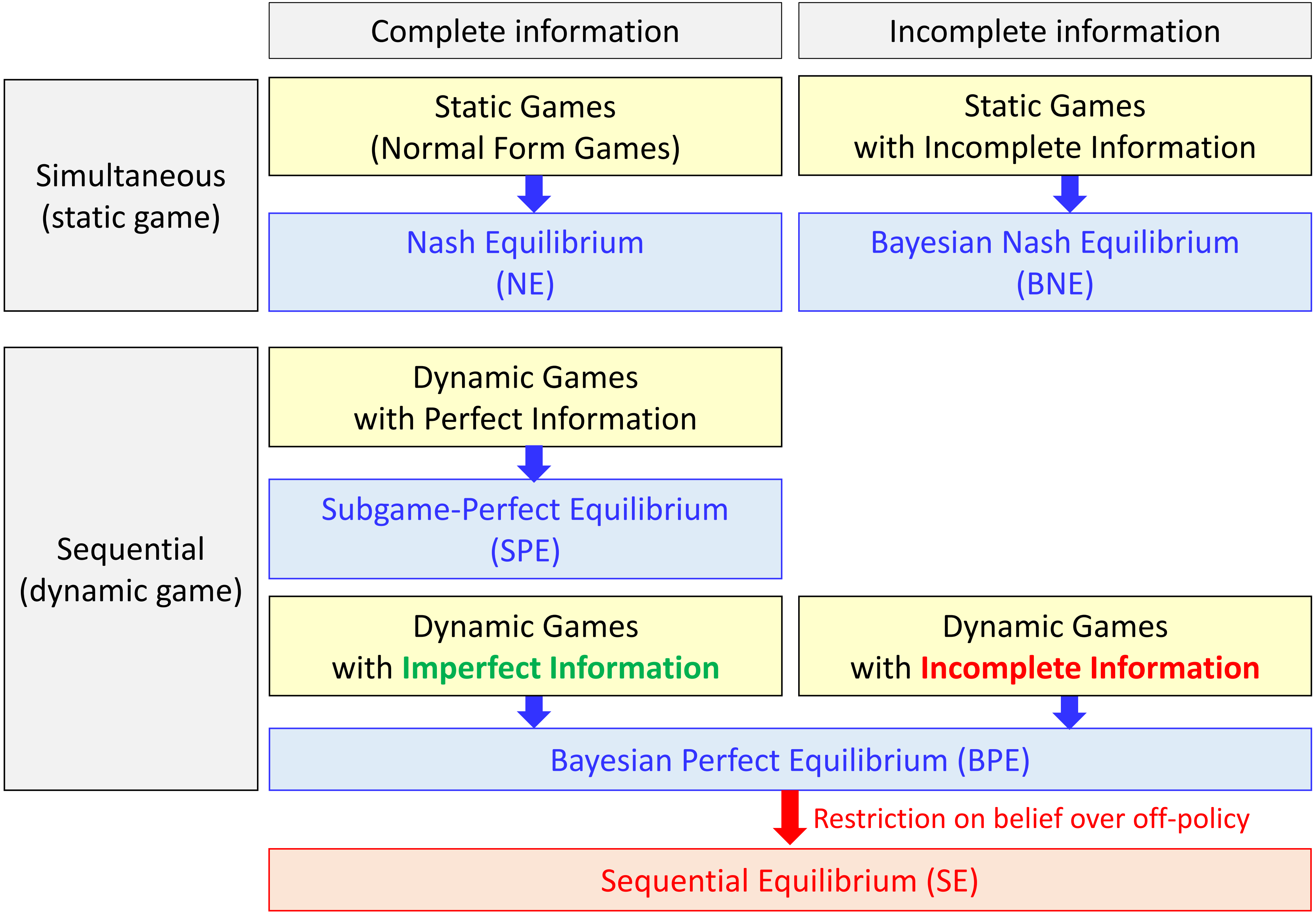
Introduction

	Complete information	Incomplete information
Simultaneous (static game)	Static Games (Normal Form Games) ↓ Nash Equilibrium (NE)	Static Games with Incomplete Information ↓ Bayesian Nash Equilibrium (BNE)
Sequential (dynamic game)	Dynamic Games with Perfect Information ↓ Subgame-Perfect Equilibrium (SPE)	Dynamic Games with Incomplete Information ↓ Bayesian Perfect Equilibrium (BPE)

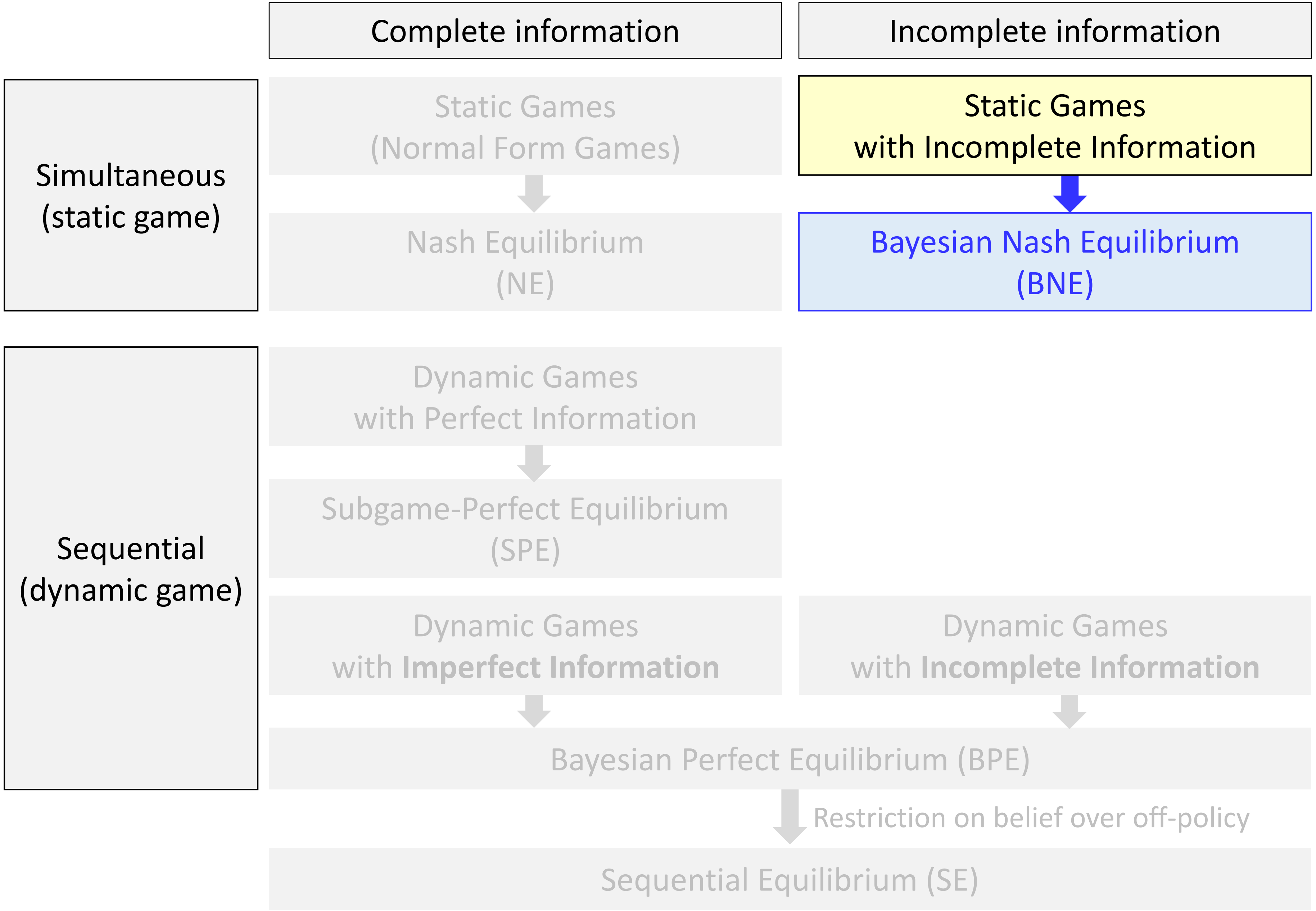
Introduction



Introduction



Introduction



Summary so far….

- So far, we have made an important assumption: The game played is common knowledge. That is, the players in the games aware of
  - ✓ Who is playing
  - ✓ What the possible actions of each player are
  - ✓ How outcomes translate into payoffs

- This is true even for imperfect-information games
  - the actual moves of agents are not common knowledge, but the game itself is

- Furthermore, we have assumed that this knowledge of the game is itself common knowledge

- These assumptions enabled us to lay the methodological foundation for such solution concepts as
  - ✓ Iterated elimination of dominated strategies,
  - ✓ Rationalizability
  - ✓ Nash equilibrium
  - ✓ Subgame-perfect equilibrium
  - ✓ …

Motivations

- These idealized situations are rarely encountered in reality
- For example, Cournot models of duopolistic completion, each firm may not know
  - ✓ Payoffs of the firms
  - ✓ Actions spaces
  - ✓ Production technologies
  - ✓ Productivity of workers
  - ✓ Cost function of each firm
  - ✓ …
- Firms have a reasonably good idea about their opponents' cost but do not know exactly what they are
- In general, players have some idea about their opponents' characteristics but don't know for sure
- The situation is somewhat similar to imperfect information game, in which we analyzed game as follows:
  - ➢ Conjecture about the behavior (strategy) of opponents, captured by belief
  - ➢ **Beliefs** and appropriate **best Reponses** should be **mutually consistent** and correct

Bayesian games

- Bayesian games (games of incomplete information) allow us to represent players' uncertainties about the game. We assumes:
    1. All possible games have the same number of agents and the same strategy space for each agent; they differ only in their payoffs.
    2. The beliefs of the different agents are posteriors, obtained by conditioning **a common prior** on individual private signals

- One can imagine many other potential types of uncertainty that players might have about the game
    - how many players are involved
    - what actions are available to each player
    - other aspects of the situation
    - …

- It turns out that these other types of uncertainty can be reduced to **uncertainty only about payoffs** via problem reformulation

Uncertainty regarding a game

- Imagine that we want to model a situation in which one player is uncertain about the number of actions available to the other players
- We can reduce this uncertainty to uncertainty about payoffs by padding the game with irrelevant actions

	L	R
U	$1, 1$	$1, 3$
D	$0, 5$	$1, 13$

	L	C	R
U	$1, 1$	$0, 2$	$1, 3$
D	$0, 5$	$2, 8$	$1, 13$

- Consider the following two-player game, in which the row player does not know whether his opponent has only the two strategies L and R or also the third one C:
  - the newly added column is dominated by the others and will not participate in any Nash equilibrium
  - Indeed, Nash equilibria of the original game are equal to that of the modified game (padded one)

Definition

- There are several ways of presenting Bayesian games:
  - Information sets
  - Extensive form with chance moves (Nature)
  - Epistemic (i.e., relating to knowledge or to the degree of its validation)

Bayesian games represented by information set

Definition (Bayesian game: information sets)

A Bayesian game is a tuple (N, G, P, I) where:

- N is a set of agents
- G is a set of games with n agents each such that if $g, g' \in G$ then for each agent $i \in N$ the strategy space in g is identical to the strategy space in g'
- $P \in \Pi(G)$ is a common prior over games, where $\Pi(G)$ is the set of all probability distributions over G; and
- $I = (I_1, \ldots, I_N)$ is a tuple of partitions of G, one for each agent

- Equilibrium analysis is possible because,
  - Players have a common prior over games
  - Everything about a game must be common knowledge
- Players know their own preferences (information set, types, …), which in turn will allow us to analyze a players' best response given his assumptions about the behavior of his opponents

Bayesian games represented by information set

Example

$I_{2,1}$ (MP, Coord) — $I_{2,2}$ (PD, BoS)

$I_{1,1}$ (MP, PD)

MP

	L	R
U	2, 0	0, 2
D	0, 2	2, 0

$p = 0.3$

PD

	L	R
U	2, 2	0, 3
D	3, 0	1, 1

$p = 0.1$

$I_{1,2}$ (Coord, BoS)

Coord

	L	R
U	2, 2	0, 0
D	0, 0	1, 1

$p = 0.2$

BoS

	L	R
U	2, 1	0, 0
D	0, 0	1, 2

$p = 0.4$

- It consists of four 2× 2 games (Matching Pennies, Prisoner's Dilemma, Coordination and Battle of the Sexes), and each agent's partition consists of two equivalence classes

Bayesian games represented in a extensive form game with imperfect information

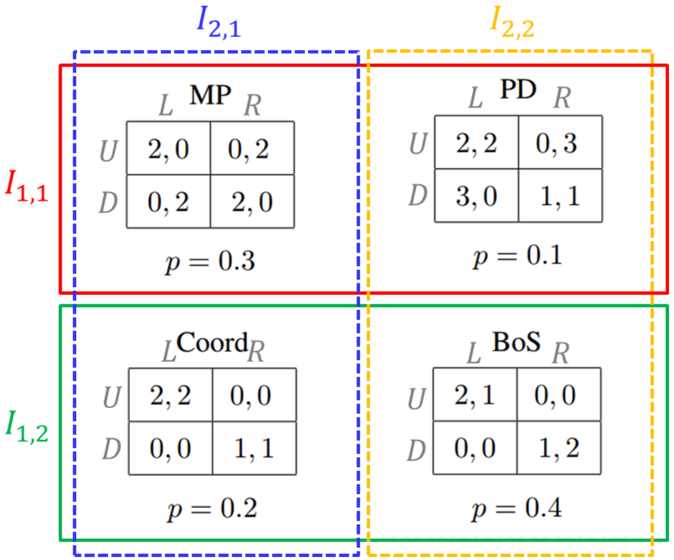
- A second way of capturing the common prior is to hypothesize a special agent called Nature who makes probabilistic choices

Following **Harsanyi (1967),** we can **change a game of incomplete information** into a **dynamic game of imperfect information**, by making nature as a mover in the game. In such a game, nature chooses player i's type, but another player j is not perfectly informed about this choice.

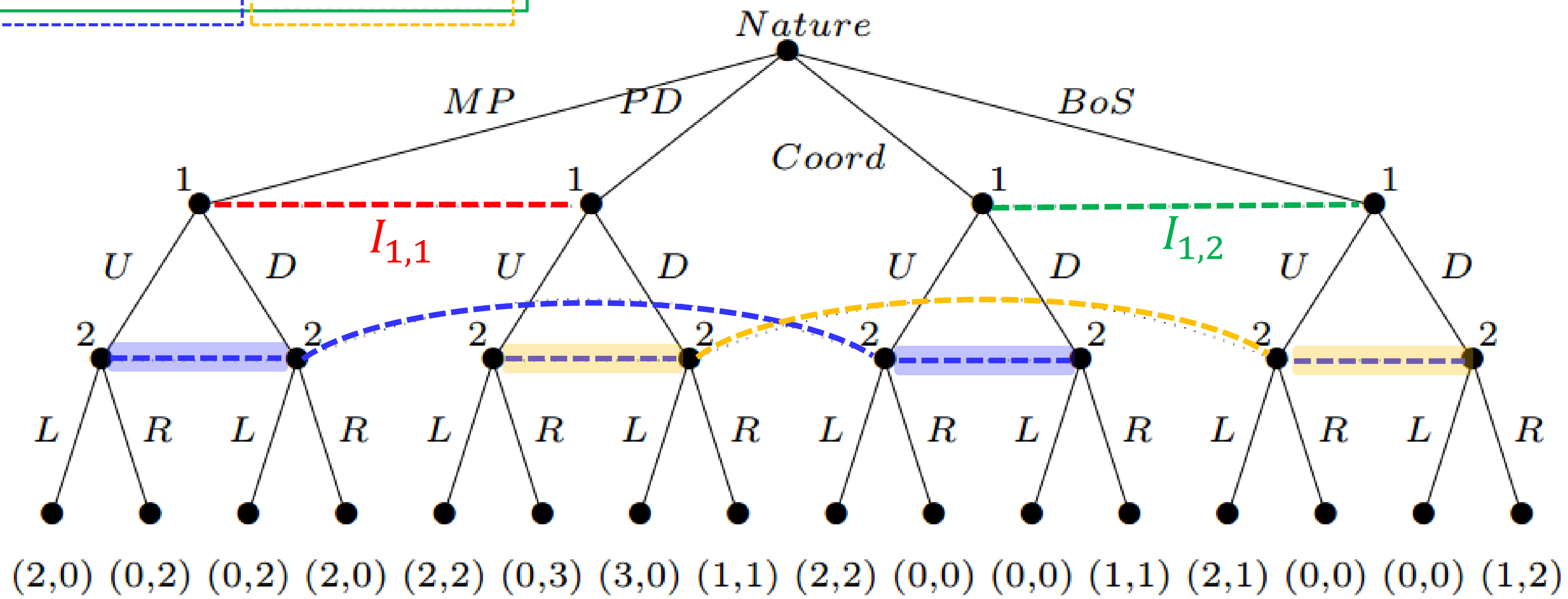
- Nature does not have a utility function
- Nature has the unique strategy of randomizing in a **commonly known way**
- The agents receive individual signals about Nature's choice, and these are captured by their information sets in a standard way
- The information sets capture the fact that agents make their choices without knowing the choices of others

- Thus, we have reduced games of incomplete information to games of imperfect information, albeit ones with chance moves.

Bayesian games represented in a extensive form game with imperfect information

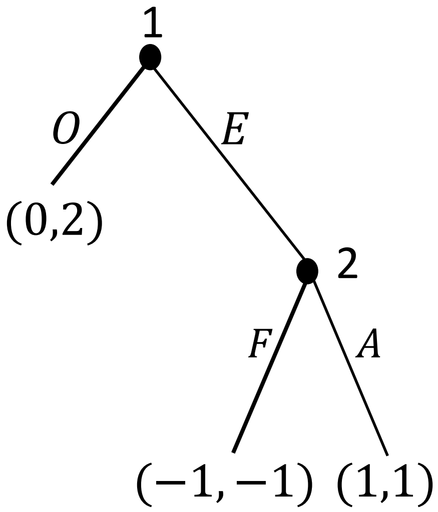


- The Bayesian game defined using information set can be represented in extensive form as follows(Harsanyi, Novel prize 1994)
- Each player knows the distribution of his opponents' type (the common prior assumptions), thus, we can perform analysis



Bayesian games represented in a extensive form game with imperfect information

Entry game example

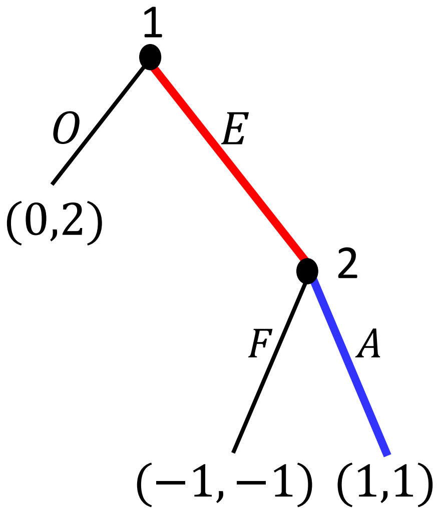


- **Player 1**: A potential entrant to an industry that has a monopolistic incumbent, player 2
  - Can decide to enter the market (Enter)
  - Can decide not to enter (Stay out)
- **Player 2**: If player 1 enters the market, player 2
  - Can Fight with player 1
  - Can Accommodate with player 1

Extensive form game
Subgame-Perfect equilibrium

Bayesian games represented in a extensive form game with imperfect information

Entry game example



Extensive form game
Subgame-Perfect equilibrium

- **Player 1**: A potential entrant to an industry that has a monopolistic incumbent, player 2
  - Can decide to enter the market (Stay out)
  - Can decide not to enter (Enter)
- **Player 2**: If player 1 enters the market, player 2
  - Can Fight with player 1
  - Can Accommodate with player 1

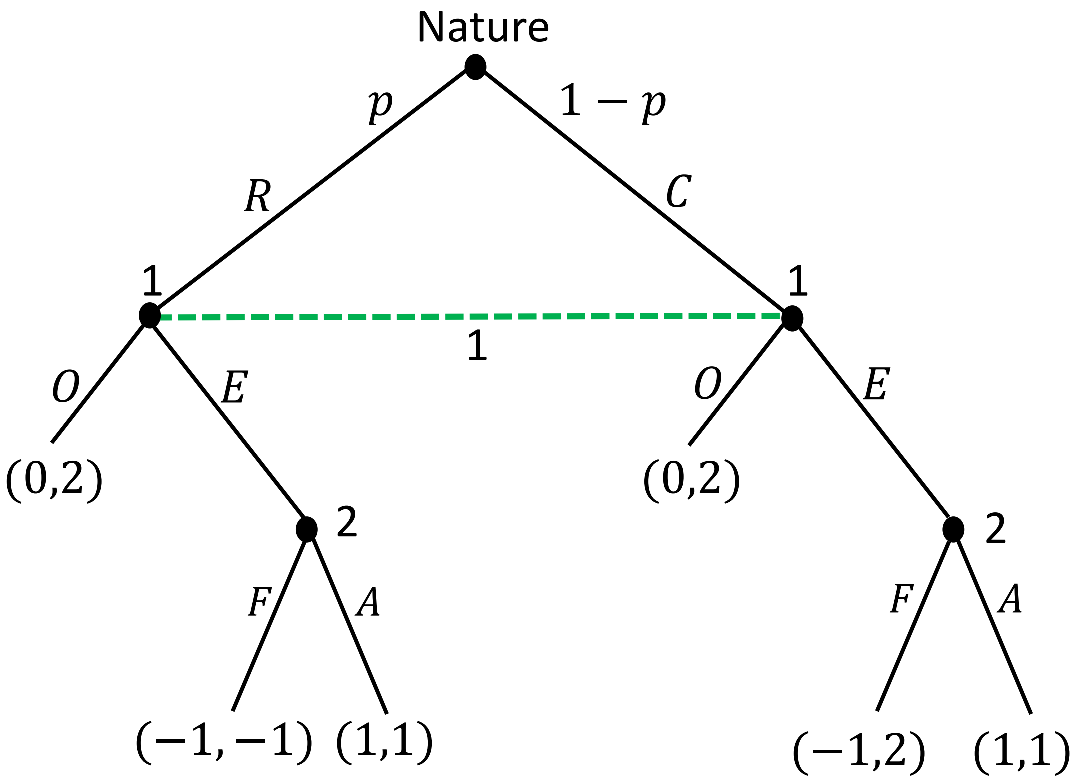
	F	A
O	$\boxed{0, 2}$	$0, 2$
E	$-1, -1$	$\boxed{1, 1}$

Nash equilibria = $\{(O, F)\ , (E, A)\ \}$

Subgame perfect equilibrium = (E, A)

Bayesian games represented in a extensive form game with imperfect information

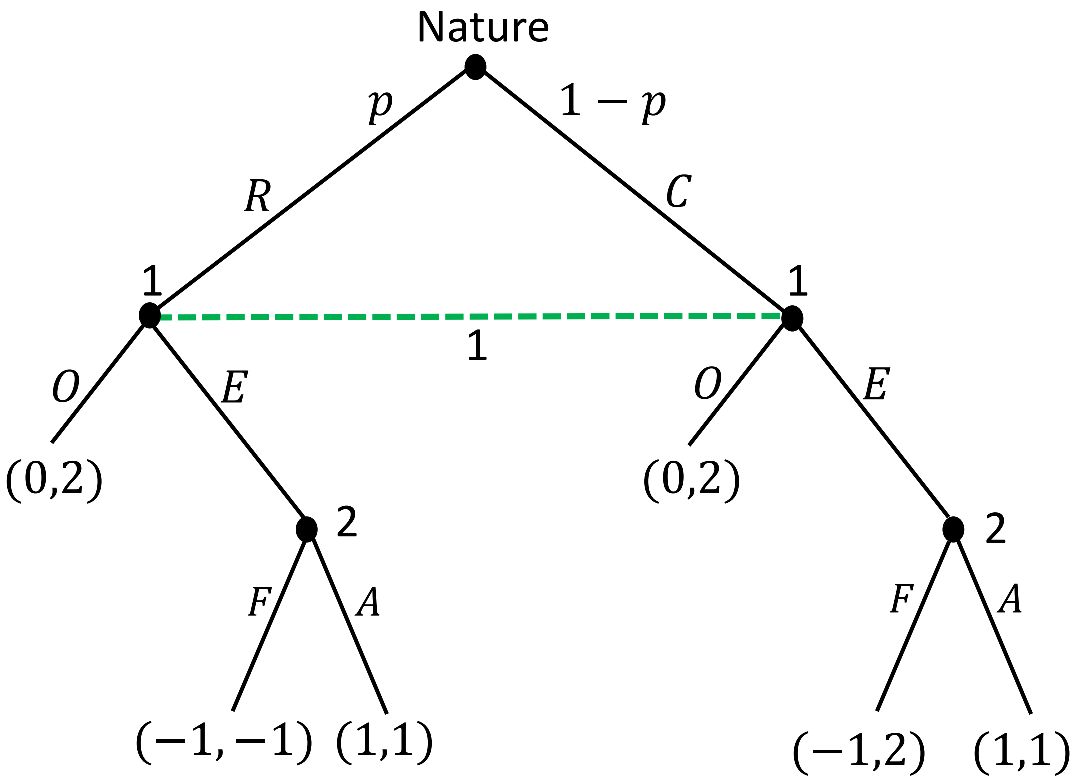
Entry game example



- **There are two types of player 2** : $\theta_2 \in \{R, C\}$, and let $P\{\theta_2 = R\} = p$
  - Rational player: Hate fighting
  - Crazy player: Enjoy fighting
- **Nature chooses which type of player 2** is playing the game with player 1

Bayesian games represented in a extensive form game with imperfect information

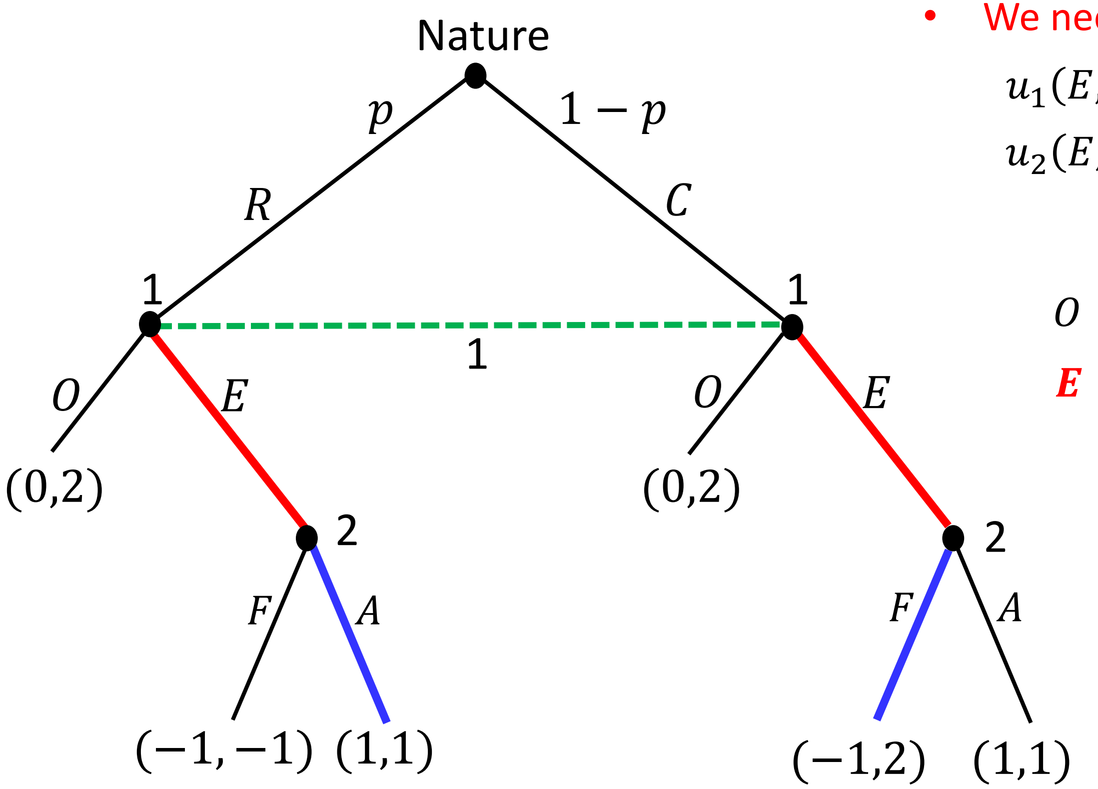
Entry game example



- We assume that players know their own preferences
  - Player 1 is uncertain about player 2's preference (he does not know the type for player2)
  - But player 2 knows what his preferences are when he needs to make a decision
- If players know their own preferences, but they do not know the preferences of others, What they have to do?
  - ➢ **Form correct beliefs about the preferences and types of their opponents**

Bayesian games represented in a extensive form game with imperfect information

Entry game example



- We need to compute the expected payoff

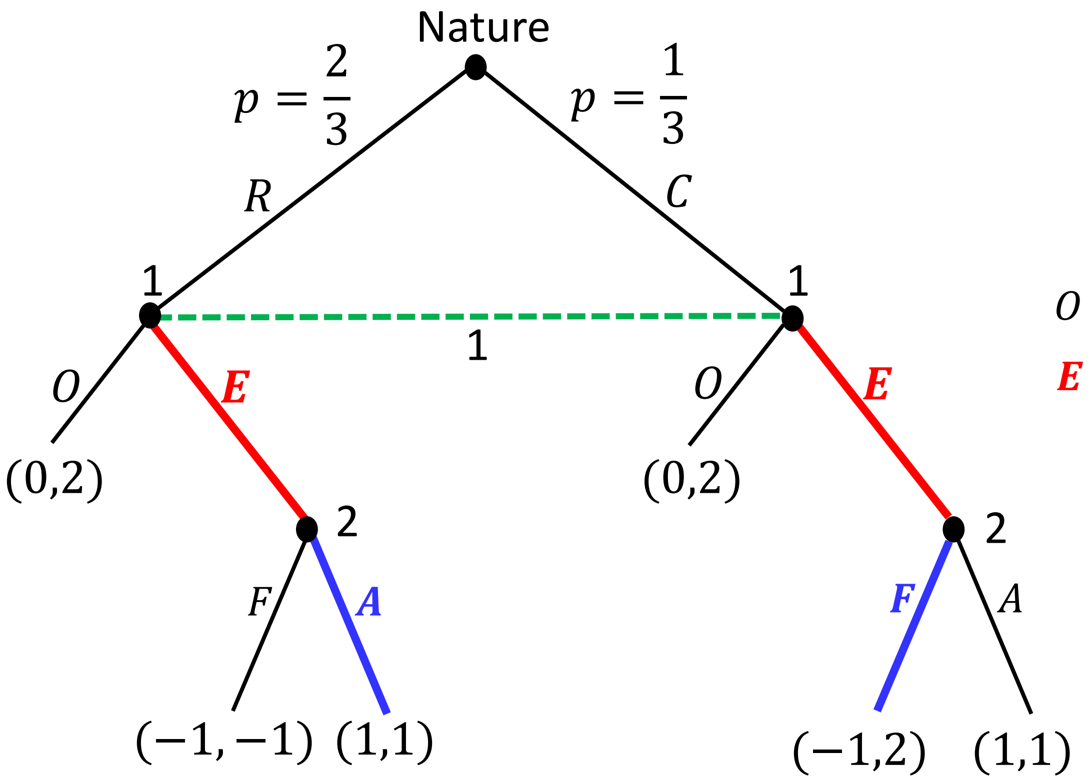
$$u_1(E, AF) = p \times 1 + (1-p) \times (-1) = 2p - 1$$
$$u_2(E, AF) = p \times 1 + (1-p) \times 2 = 2 - p$$

	AA	AF	FA	FF
O	0,2	0,2	0,2	0,2
E	1, 1	$\mathbf{2p-1,}$ $\mathbf{2-p}$	$1-2p,$ $1-2p$	$-1,$ $2-3p$

- Each player knows the probability distribution over types, and this itself is common knowledge among the players of the game (the common prior assumption)
  - ➢ All players know Nature will chose type 1 (rational) with a probability p
- Possible strategies for each player
  - ✓ $s_1 \in \{O, E\}$
  - ✓ $s_2 \in \{AA, AF, FA, FF\}$

Bayesian games represented in a extensive form game with imperfect information

Entry game example



	AA	AF	FA	FF
O	0,2	0,2	0,2	0,2
E	1,1	$\frac{1}{3},\frac{4}{3}$	$-\frac{1}{3},-\frac{1}{3}$	$-1,0$

- There are three pure strategy Nash equilibria
  - $\{(O,FA),(O,FF),(E,AF)\}$
- Only (E,AF) is a subgame-perfect equilibria

Bayesian games represented in a extensive form game with imperfect information

Entry game example

- We modeled this situation as one in which players have uncertainty about the preferences of other players (i.e., uncertainty about the payoffs in the game)
- We assumed that players share the same beliefs about the uncertainty, which allowed us to create a new game for two players with expected payoffs
  - We cannot perform equilibrium analysis without having the common prior
- We have changed the complex and challenging concept of incomplete information into a well-known of imperfect information, in which
  - Nature choose the player’s types
  - We can then use our standard tools of analysis
- Nash equilibrium concept required players to form conjectures, or beliefs, that in equilibrium have to match the choices of their opponents.
- Here we must ask for more
  - All players agree on the way in which players' types different from each other
  - On the way in which Nature chooses among these profiles of types

Bayesian games represented by epistemic types

- Our third definition uses the notion of an epistemic type, or simply a type, as a way of defining uncertainty directly over a game's utility function

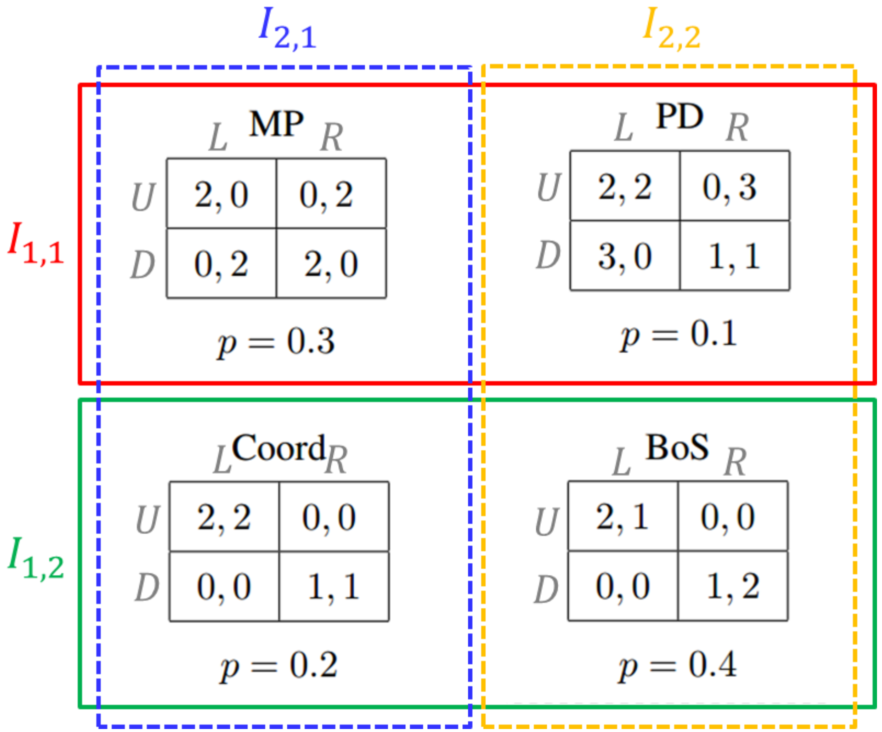
Definition (Bayesian game: epistemic types)

A Bayesian game is a tuple (N, A, Θ, p, u) where:

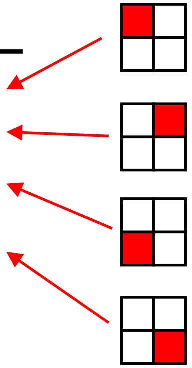
- N is a set of agents
- $A = A_1 \times \cdots \times A_n$, where A_i is the set of actions available to player i;
- $\Theta = \Theta_1 \times \cdots \times \Theta_n$, where Θ_i is the type space of player i;
- $p : \Theta \longmapsto [0,1]$ is a common prior over types, and
- $u = (u_1, \ldots, u_n)$, where $u_i : A \times \Theta_i \longmapsto \mathbb{R}$ (or $u_i : A \times \Theta \longmapsto \mathbb{R}$) is the utility function of player i, which is type dependent,
  - ✓ $u_i(\cdot, \theta)$ is the utility function with a type $\theta \in \Theta$ (common value setting)
  - ✓ $u_i(\cdot, \theta_i)$ is the utility function with a type $\theta_i \in \Theta_i$ (individual value setting)

- The assumption is that all of the above is common knowledge among the players, and that each agent knows his own type
- the type of agent encapsulates all the information possessed by the agent that is not common knowledge
  - $p(\theta_{-i} | \theta_i)$ is (posterior) conditional distribution on all other types but i given θ_i

Example



a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2



- The joint distribution on these types is as follows:
  - $p(\theta_{1,1}, \theta_{2,1}) = 0.3;\ p(\theta_{1,1}, \theta_{2,2}) = 0.1;\ p(\theta_{1,2}, \theta_{2,1}) = 0.2;\ p(\theta_{1,2}, \theta_{2,2}) = 0.4$
- The conditional distribution on player 2 given $\theta_{1,1}$:
  - $p(\theta_{2,1}|\theta_{1,1}) = \frac{3}{3+1}\ ; p(\theta_{2,2}|\theta_{1,1}) = \frac{1}{3+1};$

Conjecture about other player's game type given his own (private) type

Bayesian games represented by epistemic types

(static) Bayesian game as a procedure

1. Nature choose a profile of types $\theta = (\theta_1, \theta_2, \dots, \theta_n)$
2. Each player i learns his own type θ_i, which is his private information, and then form posterior beliefs over the other types of players
3. Players simultaneously (hence this is a static game) choose actions $a_i \in A_i, i \in N$.
4. Given the players' choices $a = (a_1, a_2, \dots, a_n)$, the payoffs $u_i(a; \theta_i)$ are realized for each player $i \in N$
   - private values case : $u_i(a_1, a_2, \dots, a_n; \theta_i)$: private values case
     - ✓ Player's payoff depends only on his type
   - common values case : $u_i(a_1, a_2, \dots, a_n; \theta_1, \theta_2, \dots, \theta_n)$
     - ✓ Player's payoff depends on the types of all players

Strategies

- In **imperfect-information extensive-form game**:
  - a pure strategy is a mapping from every information set to actions.

- In **a Bayesian game**:
  - A pure strategy $s_i \colon \Theta_i \longmapsto A_i$ is a mapping from every type agent i could have to the action he would play if he had that type, i.e., $a_i = s_i(\theta_i)$
  - A mixed strategies in the natural way as probability distributions over pure strategies
    - $s_i(a_i \mid \theta_i)$ denotes the probability under mixed strategy s_i that agent i plays action a_i, given that i's type is θ_i.

Expected utility

- Due to uncertainties in Bayesian game, an expected utility is used to find an equilibrium
- In a Bayesian game setting, there are three meaningful notions of expected utility:
  1. ex post,
  2. ex interim
  3. ex ante.

	his own type	**Others' types**
ex post	Know	Know
ex interim	Know	Don't know
ex ante	Don't know	Don't know

Ex post expected utility

Definition (ex post expected utility)

Agent i's ex post expected utility in a Bayesian game (N, A, Θ, p, u), where the agents' strategies are given by s and the agent's types are given by θ, is defined as

$$EU_i(s,\theta) = \sum_{a \in A} \underbrace{\left(\prod_{j \in N} s_j(a_j|\theta_j)\right)}_{\text{Probability of selecting a joint action } a \text{ given } \theta} u_i(a,\theta)$$

- The types $\theta = (\theta_1, \dots, \theta_n)$ of all agents are given (known)
  - In Bayesian games, no agent will know the others' types
- the only uncertainty concerns the other agents' mixed strategies

$$a \in A = \{(U,L),(U,R),(D,L),(D,R)\}$$

$$\begin{aligned} EU_1\left(s,(\theta_{1,1},\theta_{2,1})\right) &= s_1(U|\theta_{1,1})s_2(L|\theta_{2,1})u_1\left((U,L),(\theta_{1,1},\theta_{2,1})\right) \\ &+ s_1(U|\theta_{1,1})s_2(R|\theta_{2,1})u_1\left((U,R),(\theta_{1,1},\theta_{2,1})\right) \\ &+ s_1(D|\theta_{1,1})s_2(L|\theta_{2,1})u_1\left((D,L),(\theta_{1,1},\theta_{2,1})\right) \\ &+ s_1(D|\theta_{1,1})s_2(R|\theta_{2,1})u_1\left((D,R),(\theta_{1,1},\theta_{2,1})\right) \end{aligned}$$

a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Ex interim expected utility

Definition (ex interim expected utility)

Agent i's ex interim expected utility in a Bayesian game (N, A, Θ, p, u), where i's type is θ_i and where the agents' strategies are given by the mixed strategy profile s, is defined as

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} | \theta_i) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j | \theta_j) \right) u_i(a, \theta_{-i}, \theta_i)$$

Or equivalently as

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} | \theta_i) \underbrace{EU_i(s, (\theta_{-i}, \theta_i))}_{\text{ex post expected utility}}$$

- Agent i must consider every assignment of types to the other agents θ_{-i} and every pure action profile $a \in A$ in order to evaluate his utility function $u_i(a, \theta_{-i}, \theta_i)$

- Because uncertainty over mixed strategies was already handled in the ex post case, we can also write ex interim expected utility as a weighted sum of $EU_i(s, \theta)$ terms.

Ex interim expected utility

$$EU_i(s,\theta_i) = \sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a\in A}\left(\prod_{j\in N} s_j(a_j|\theta_j)\right) u_i(a,\theta_{-i},\theta_i)$$
$$= \sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_{-i}|\theta_i) EU_i(s,(\theta_{-i},\theta_i))$$

$$EU_1(s,\theta_{1,1}) = p(\theta_{2,1}|\theta_{1,1}) EU_1\left(s,(\theta_{2,1},\theta_{1,1})\right)$$
$$+p(\theta_{2,2}|\theta_{1,1}) EU_1\left(s,(\theta_{2,2},\theta_{1,1})\right)$$

a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Ex ante expected utility

Definition (ex ante expected utility)

Agent i's ex ante expected utility in a Bayesian game (N, A, Θ, p, u), where the agents' strategies are given by the mixed strategy profile s, is defined as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta)$$

or equivalently as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \underbrace{EU_i(s, \theta)}_{\text{ex post expected utility}}$$

or again equivalently as

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) \underbrace{EU_i(s, \theta_i)}_{\text{ex interim expected utility}}$$

$$\because EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) EU_i(s, (\theta_{-i}, \theta_i))$$

Ex ante expected utility

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta)$$

$$= \sum_{\theta \in \Theta} p(\theta) EU_i(s, \theta)$$

$$EU_1(s) = p\left(\left(\theta_{1,1}, \theta_{2,1},\right)\right) EU_1\left(s, \left(\theta_{2,1}, \theta_{1,1}\right)\right)$$

$$+p\left(\left(\theta_{1,1}, \theta_{2,2}\right)\right) EU_1\left(s, \left(\theta_{1,1}, \theta_{2,2}\right)\right)$$

$$+p\left(\left(\theta_{1,2}, \theta_{2,1}\right)\right) EU_1\left(s, \left(\theta_{1,2}, \theta_{2,1}\right)\right)$$

$$+p\left(\left(\theta_{1,2}, \theta_{2,2}\right)\right) EU_1\left(s, \left(\theta_{1,2}, \theta_{2,2}\right)\right)$$

a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Bayesian Nash equilibrium

Definition (Best response in a Bayesian game)

The set of agent i's best responses to mixed-strategy profile s_{-i} are given by

$$BR_i(s_{-i}) = \underset{s_i' \in S_i}{\operatorname{argmax}}\, EU_i(s_i', s_{-i})$$

- Note that BR_i is a set because there may be many strategies for i that yield the same expected utility

- Because $EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s, \theta_i)$

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} | \theta_i) EU_i\big(s, (\theta_{-i}, \theta_i)\big)$$

$$EU_i\big(s, (\theta_{-i}, \theta_i)\big) = \sum_{a \in A} \left(\prod_{j \in N} s_j\big(a_j | \theta_j\big) \right) u_i(a, \theta)$$

Bayesian Nash equilibrium

Definition (Best response in a Bayesian game)

The set of agent i's best responses to mixed-strategy profile s_{-i} are given by

$$BR_i(s_{-i}) = \underset{s_i' \in S_i}{\operatorname{argmax}}\, EU_i(s_i', s_{-i})$$

- Note that BR_i is a set because there may be many strategies for i that yield the same expected utility

Why best response is defined in terms of ex-ante expected utility?

- Note that $EU_i(s_i', s_{-i}) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s_i', s_{-i}, \theta_i)$
- $EU_i(s_i', s_{-i}, \theta_i)$ does not depend on strategies that i would play if his type were not θ_i
- Thus, we are in fact performing independent maximization of i's ex interim expected utilities conditioned on each type that he could have
- That is, for each $\theta_i \in \Theta_i$, the following equations should be satisfied

$$EU_i(s_i, s_{-i}, \theta_i) \geq EU_i(s_i', s_{-i}, \theta_i) \quad \forall s_i' \in S_i$$

Bayesian Nash equilibrium

Definition (Bayes-Nash equilibrium)

A Bayes-Nash equilibrium is a mixed-strategy profile s that satisfies $\forall i \; s_i \in BR_i(s_{-i})$

- This is exactly the same definition we gave for the Nash equilibrium: each agent plays a best response to the strategies of the other players
- Thus, for every player $i \in N$, the following relationship should be satisfied

for each $\theta_i \in \Theta_i$, the following equations should be satisfied

$$EU_i(s_i, s_{-i}, \theta_i) \geq EU_i(s_i', s_{-i}, \theta_i) \quad \forall s_i' \in S_i$$

for every player $i \in N$, for each $\theta_i \in \Theta_i$

$$\sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) EU_i\big(s_i, s_{-i}, (\theta_{-i}, \theta_i)\big) \geq \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) EU_i\big(s_i', s_{-i}, (\theta_{-i}, \theta_i)\big) \, \forall s_i' \in S_i$$

Bayesian Nash equilibrium

Definition (Bayes-Nash equilibrium)

A Bayes-Nash equilibrium is a mixed-strategy profile s that satisfies $\forall i$ $s_i \in BR_i(s_{-i})$

- This is exactly the same definition we gave for the Nash equilibrium: each agent plays a best response to the strategies of the other players
- Thus, for every player $i \in N$, the following relationship should be satisfied

for each $\theta_i \in \Theta_i$, the following equations should be satisfied

$$EU_i(s_i, s_{-i}, \theta_i) \geq EU_i(s_i', s_{-i}, \theta_i) \quad \forall s_i' \in S_i$$

for every player $i \in N$, for each $\theta_i \in \Theta_i$

$$\sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) EU_i\big(s_i, s_{-i}, (\theta_{-i}, \theta_i)\big) \geq \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) EU_i\big(s_i', s_{-i}, (\theta_{-i}, \theta_i)\big) \, \forall s_i' \in S_i$$

When s_i and s_{-i} are pure strategies

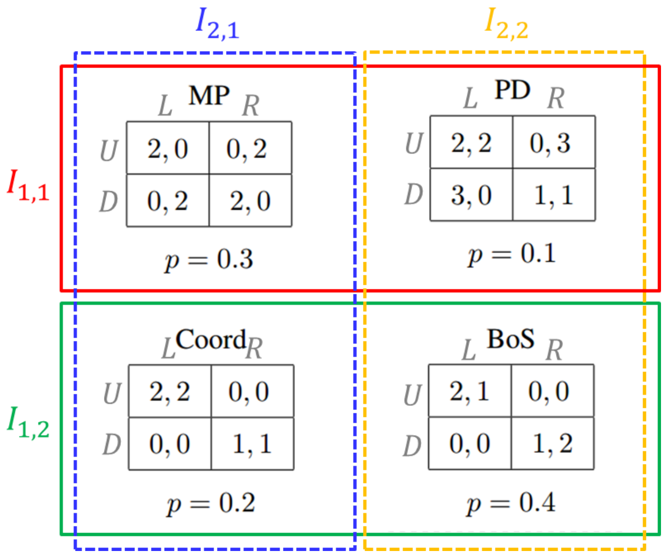
$$\Rightarrow \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) u_i(s_i(\theta_i), s_{-i}(\theta_{-i}); \theta_i) \geq \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) u_i(s_i'(\theta_i), s_{-i}(\theta_{-i}); \theta_i) \, \forall s_i' \in S_i$$

$$\Rightarrow \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) u_i(s_i(\theta_i), s_{-i}(\theta_{-i}); \theta_i) \geq \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) u_i(a_i, s_{-i}(\theta_{-i}); \theta_i) \, \forall a_i \in A_i$$

Computing Bayesian Nash equilibrium

- As we did with extensive-form games, we can construct a normal-form representation that corresponds to a given Bayesian game.
- Each agent i's payoff given a pure-strategy profile s is his ex ante expected utility under s
- The Bayes–Nash equilibria of a Bayesian game are precisely the Nash equilibria of its induced normal form
- This fact allows us to note that Nash's theorem applies directly to Bayesian games, and hence that Bayes–Nash equilibria always exist.

Computing Bayesian Nash equilibrium

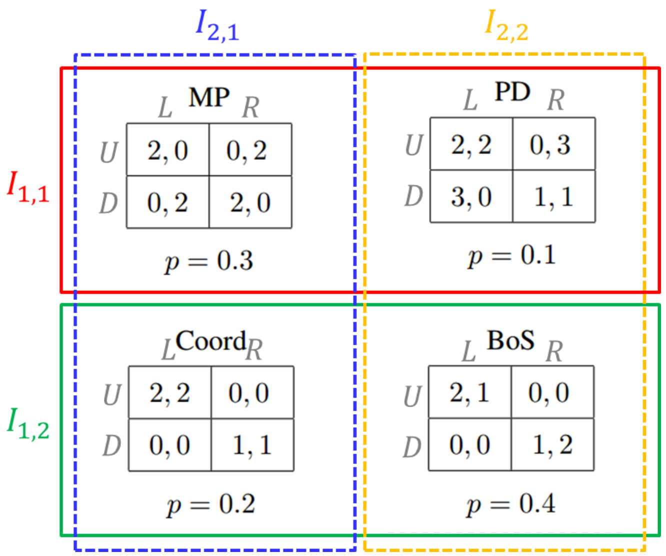


a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

- Each agent has four possible pure strategies (two types and two actions).
  - Give player 1's type $\theta_{1,1}$: we can select U or D
  - Give player 1's type $\theta_{1,2}$: we can select U or D

    $\left.\right\} s_1 = \big(s_1(\theta_{1,1}), s_1(\theta_{1,2})\big)$
  - Then player 1's four strategies in the Bayesian game can be labeled UU, UD, DU, and DD
  - Similarly, we can denote the strategies of player 2 by RR, RL, LR, and LL.

Computing Bayesian Nash equilibrium



a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

- We can compute the expected payoff for each profile of strategies
- For example, player 2's ex ante expected utility under the strategy profile (UU, LL) is calculated as follows:

$$
\begin{aligned}
u_2(UU, LL) &= \sum_{\theta \in \Theta} p(\theta) u_2(U, L, \theta) \\
&= p(\theta_{1,1}, \theta_{2,1}) u_2(U, L, \theta_{1,1}, \theta_{2,1}) + p(\theta_{1,1}, \theta_{2,2}) u_2(U, L, \theta_{1,1}, \theta_{2,2}) + \\
&\quad p(\theta_{1,2}, \theta_{2,1}) u_2(U, L, \theta_{1,2}, \theta_{2,1}) + p(\theta_{1,2}, \theta_{2,2}) u_2(U, L, \theta_{1,2}, \theta_{2,2}) \\
&= 0.3(0) + 0.1(2) + 0.2(2) + 0.4(1) = 1.
\end{aligned}
$$

- We can convert the Bayesian game into an induced normal form game, each payoff is ex ante expected value

a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Bayesian game

	LL	LR	RL	RR
UU	2, 1	1, 0.7	1, 1.2	0, 0.9
UD	0.8, 0.2	1, 1.1	0.4, 1	0.6, 1.9
DU	1.5, 1.4	0.5, 1.1	1.7, 0.4	0.7, 0.1
DD	0.3, 0.6	0.5, 1.5	1.1, 0.2	1.3, 1.1

Induced normal form game

Computing Bayesian Nash equilibrium

- Given a particular signal, the agent can compute the posterior probabilities and re-compute the expected utility of any given strategy vector
- Once the row agent gets the signal $\theta_{1,1}$ he can update the expected payoffs and compute the new game shown bellow

a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Bayesian game

Bayesian updating

	LL	LR	RL	RR
UU	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
UD	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
DU	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25
DD	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25

Induced normal form game

Computing Bayesian Nash equilibrium

- Given a particular signal, the agent can compute the posterior probabilities and re-compute the expected utility of any given strategy vector
- Once the row agent gets the signal $\theta_{1,1}$ he can update the expected payoffs and compute the new game shown bellow

a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Bayesian game

Bayesian updating

	LL	LR	RL	RR
UU	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
UD	2, 0.5	1.5, 0.75	0.5, 2	0, 2.25
DU	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25
DD	0.75, 1.5	0.25, 1.75	2.25, 0	1.75, 0.25

Induced normal form game

- The induced normal form game can be used to find the best response for player 1 given his signal $\theta_{1,1}$
- However, it cannot be used for Bayesian Nash equilibrium, because the signal is not common knowledge

Example: Inefficient trade and adverse selection

- Market allocate goods to the people who value them the most
  - If a good is misallocated so that some people who have it value it less than people who do not
  - Market pressures will cause the price of that good to increase until the current owners will prefer to sell it rather than hold on to it
  - The people who value it more will be willing to pay that price

- It is important to understand the extent to which these arguments stand or fall in the face of incomplete information, when some people are better informed about the value of goods than others

Example: Inefficient trade and adverse selection

- Imagine a scenario in which player 1 owns an orange grove
  - The yields of fruit depends on the quality of the soli and other local conditions
  - There are three types of land representing soli quality $\theta_1 \in \Theta_1 = \{L, M, H\}$
  - Only player 1 knows the quality of the land
  - The value for player 1 of owning land of type θ_1 is given as

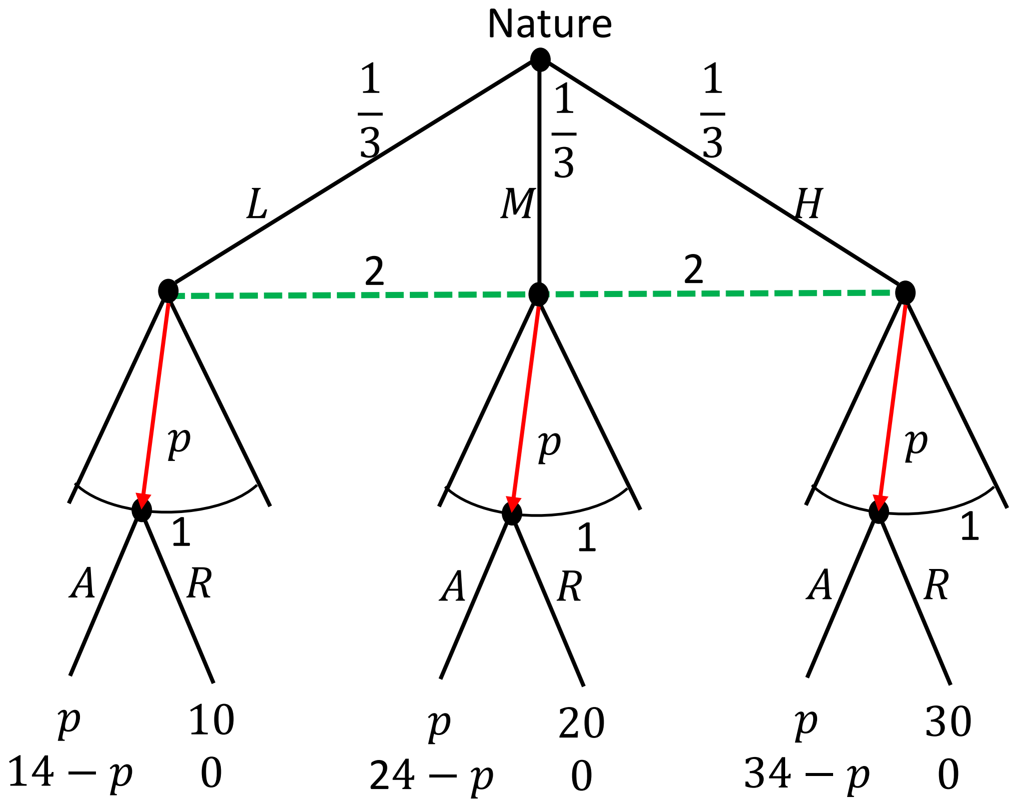
$$v_1(\theta_1) = \begin{cases} 10 & \text{if } \theta_1 = L \\ 20 & \text{if } \theta_1 = M \\ 30 & \text{if } \theta_1 = H \end{cases}$$

- Imagine that player 2 is a soybean grower who is considering the purchase of this land for production
- Player 2's monetary-equivalent values are given by

$$v_2(\theta_1) = \begin{cases} 14 & \text{if } \theta_1 = L \\ 24 & \text{if } \theta_1 = M \\ 34 & \text{if } \theta_1 = H \end{cases}$$

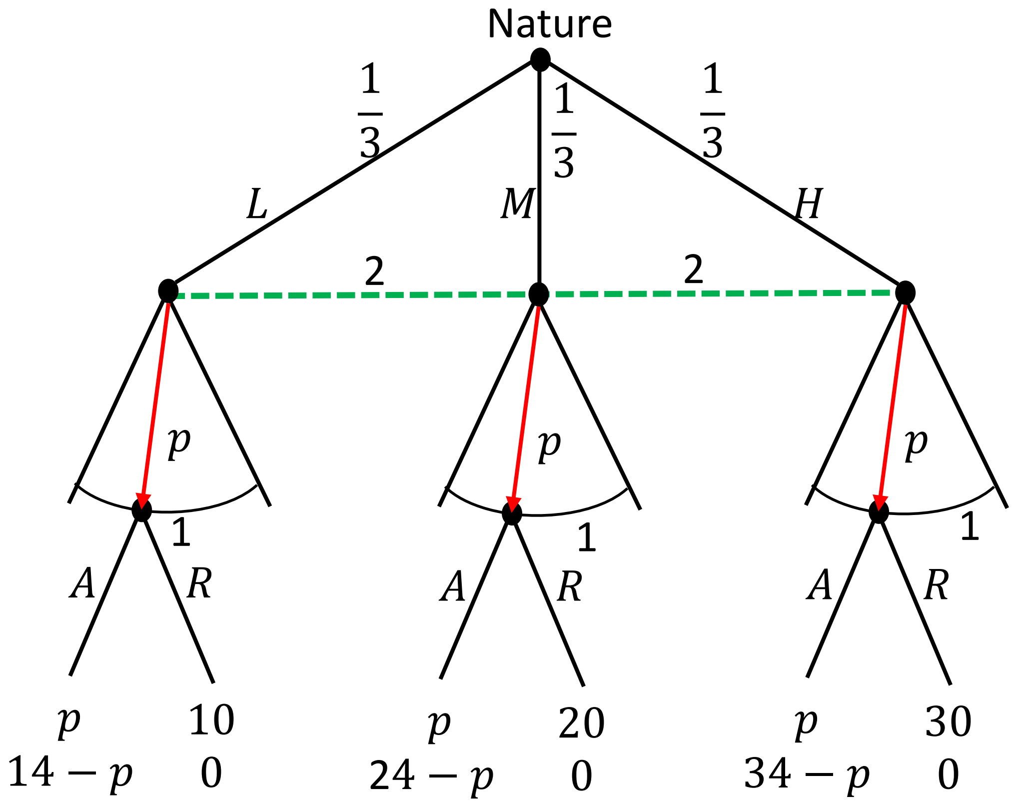
  - Player 2 knows only that the quality is distributed equally among the three options
  - i.e., $P(\theta_1 = L) = P(\theta_1 = M) = P(\theta_1 = H) = 1/3$

Example: Inefficient trade and adverse selection



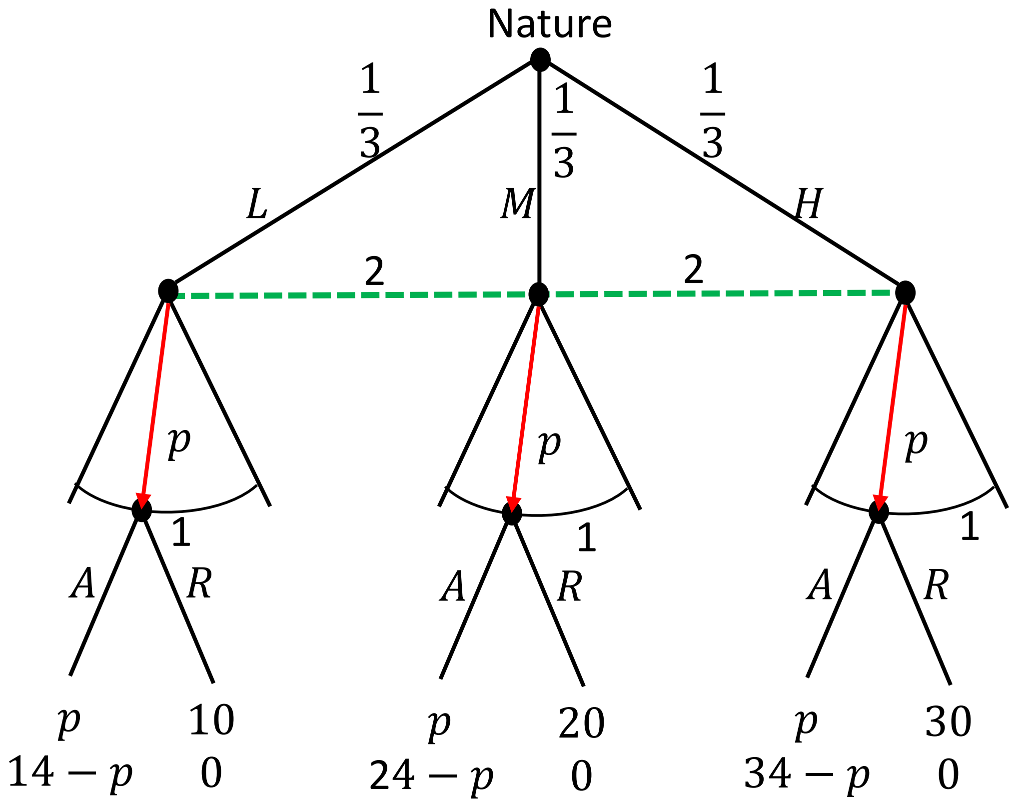
- Consider the following game
  - Player 2 makes a take-it-or-leave-it price offer to player 1
  - Player 1 accept (A) or reject (R) the offer
  - The game ends with either a transfer of land for the suggested price or no transfer
- A strategy for player 2 is a single price offer $p \geq 0$
- A Pure strategy for player 1 is a mapping from the offered price and his type space Θ_1 to a response, $s_1\colon [0,\infty) \times \Theta_1 \to \{A, R\}$

Example: Inefficient trade and adverse selection



- Trade can occur in a Bayesian Nash equilibrium only if it involves the lowest type of player 1 trading. Furthermore any price $p^* \in [10,14]$ can be supported as a Bayesian Nash equilibrium
- Why?

Example: Inefficient trade and adverse selection



- Trade can occur in a Bayesian Nash equilibrium only if it involves the lowest type of player 1 trading. Furthermore any price $p^* \in [10,14]$ can be supported as a Bayesian Nash equilibrium
- Why?

$$s_1(\theta_1) = \begin{cases} A \ \text{ if and only if } \ p \geq p^* \text{ and otherwise when } \ \theta_1 = L \\ A \ \text{ if and only if } \ p \geq 20 \text{ and otherwise when } \ \theta_1 = M \\ A \ \text{ if and only if } \ p \geq 30 \text{ and otherwise when } \ \theta_1 = H \end{cases}$$

Auctions and competitive biding

Motivations

Motivations

- Many economic transactions are conducted through auctions
    - Treasury bills
    - Foreign exchange
    - Publicly owned companies
    - Mineral rights
    - Airwave spectrum rights
    - artwork
    - antiques
    - cars
    - houses
    - Government contracts

- Also can be thought of as auctions
    - Takeover battles
    - Queues
    - Wars of attrition
    - Lobbying contests

Types of Auctions

- **Open bid auctions**
  - **Ascending-bid Auction**
    - ✓ Known as English auction
    - ✓ Price is raised until only one bidder remains, who wins and pays the final price
  - **Descending-bid Auction**
    - ✓ Known as Dutch auction
    - ✓ Price is lowered until someone accepts, who wins and pays the current price
- **Sealed bid auctions**
  - **First price auction**
    - ✓ Highest bidder wins and pays his bid
  - **Second price auction**
    - ✓ Highest bidder wins and pays the second highest bid

Open Bid		Sealed Bid
English Auction	$\sim$	Second Price
Dutch Auction	$\sim$	First Price

Types of Auctions

- **Auctions also differ with respect to the valuation of the bidders**

  - Private value auctions **(IPV)**
    - ✓ Each bidder knows only his own value
    - ✓ Valuations are independent across bidders
    - ✓ Bidders have beliefs over other bidder's values
    - ✓ Risk neutral bidders
      - ➢ If the winner's value is θ_i and pays p, her payoff is $\theta_i - p$
    - ✓ Artwork, antiques, memorabilia
    - ✓ $u_i(a_1, a_2, \ldots, a_n; \theta_i)$

  - Common value auctions
    - ✓ Actual value of the object is the same for everyone
    - ✓ Bidders have different private information about that value
    - ✓ Oil field auctions, company takeovers
    - ✓ $u_i(a_1, a_2, \ldots, a_n; \theta_1, \theta_2, \ldots, \theta_n)$

Second Price Sealed Auctions

- Set of players $N = \{1,2,\dots,n\}$
- Type set $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$, $\underline{\theta}_i \geq 0$
- Actions set $A_i = \mathbb{R}_+$
- Beliefs
  - ✓ Opponent's valuations are independent draws from a distribution function F
  - ✓ F is strictly increasing and continuous
- Strategy for player i is a function $s_i\colon \Theta_i \to A_i$
- Payoff function

$$u_i(a_i, a_{-i}; \theta_i) = \begin{cases} \theta_i - a_j^*, & \text{if } a_j \leq a_i \text{ for all } j \neq i \text{ and } a_j^* \equiv \max_{j \neq i}\{a_j\} \\ 0, & \text{if } a_j > a_i \text{ for some } j \neq i \end{cases}$$

a_j^* is the price paid by the winner if the bid profile is a

- Payoff of player i with valuation θ_i as a function of his own bid a_i and the strategies used by the other players $s_j(\cdot), j \neq i$:

$$E_{\theta_{-i}}[u_i(a_i, s_{-i}(\theta_{-i}); \theta_i)|\theta_i] = \Pr\{i \text{ wins and pays } p\} \times (\theta_i - p) + \Pr\{i \text{ loses}\} \times 0$$

Second Price Sealed Auctions

Proposition

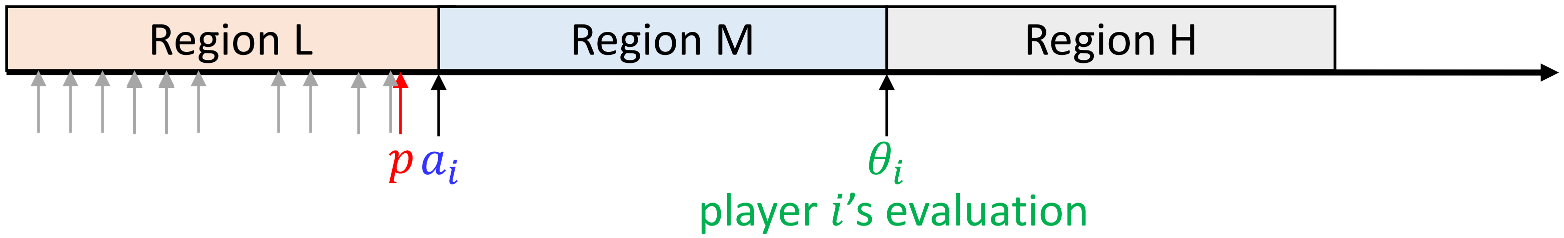
In the second-price sealed-bid auction, each player has a weakly dominant strategy, which is to bid his true valuation. That is, $s_i(\theta_i) = \theta_i$ for all $i \in N$ is a Bayesian Nash equilibrium in weakly dominant strategies

Region L	Region M	Region H

- There are three possible cases of interest with respect to the other $n - 1$ bids:

Proposition

In the second-price sealed-bid auction, each player has a weakly dominant strategy, which is to bid his true valuation. That is, $s_i(\theta_i) = \theta_i$ for all $i \in N$ is a Bayesian Nash equilibrium in weakly dominant strategies



- There are three possible cases of interest with respect to the other $n-1$ bids:

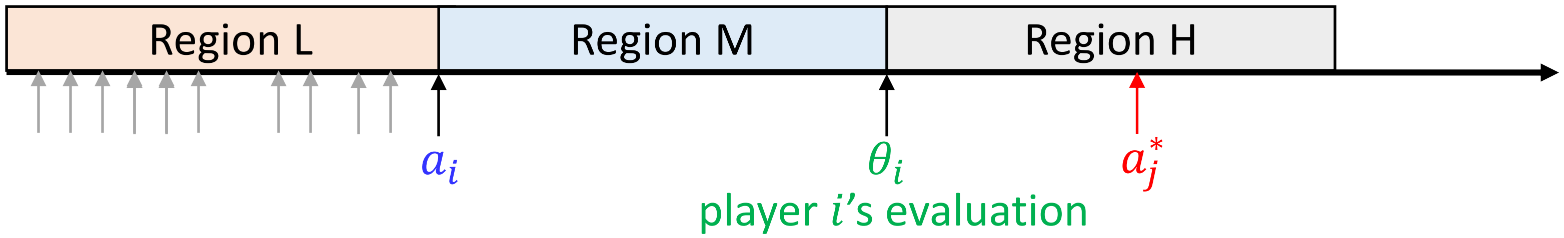
Case 1:

- Player i is the highest bidder a_i, in which case i wins and pays price $p < a_i$.
- This corresponds to the situation in which all the other $n-1$ bids are in region L, including the second highest bid p.
- If instead of bidding a_i, player i would have bid θ_i then he would still win and paly the same price, so in case 1, bidding his valuation is as good as bidding a_i

Second Price Sealed Auctions

Proposition

In the second-price sealed-bid auction, each player has a weakly dominant strategy, which is to bid his true valuation. That is, $s_i(\theta_i) = \theta_i$ for all $i \in N$ is a Bayesian Nash equilibrium in weakly dominant strategies



- There are three possible cases of interest with respect to the other $n-1$ bids:

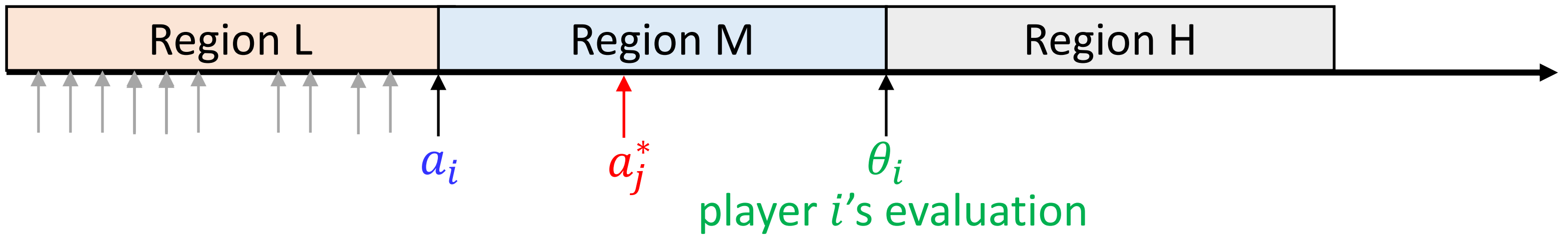
Case 2:

- The highest bidder j bids $a_j^* > \theta_i$, in which case i loses
- This corresponds to the situation in which the winning bid is in region H
- If instead of bidding a_i, player i would have bid θ_i then he would still loses to a_j^*, so in case 2 bidding his valuation is as good as bidding a_i

Second Price Sealed Auctions

Proposition

In the second-price sealed-bid auction, each player has a weakly dominant strategy, which is to bid his true valuation. That is, $s_i(\theta_i) = \theta_i$ for all $i \in N$ is a Bayesian Nash equilibrium in weakly dominant strategies



- There are three possible cases of interest with respect to the other $n-1$ bids:

Case 3:

- The highest bidder j bids $a_i < a_j^* < \theta_i$, so that the highest bid is in region M and i does not win
- If player i would have bid θ_i, he would have won the auction and received a payoff of

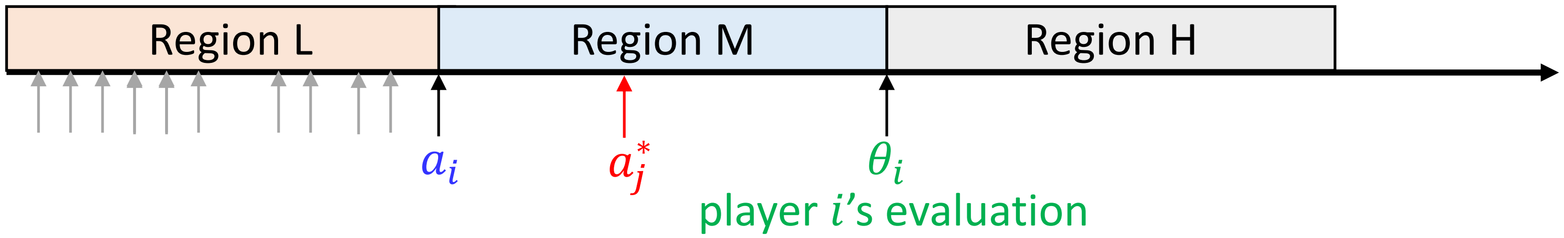
$$u_i = \theta_i - a_j^*$$

making this a profitable deviation, so in case 3, bidding his valuation θ_i is strictly better than bidding a_i

Second Price Sealed Auctions

Proposition

In the second-price sealed-bid auction, each player has a weakly dominant strategy, which is to bid his true valuation. That is, $s_i(\theta_i) = \theta_i$ for all $i \in N$ is a Bayesian Nash equilibrium in weakly dominant strategies



- There are three possible cases of interest with respect to the other $n-1$ bids:
- Since **cases 1-3** cover all the relevant situations, we conclude that bidding θ_i weakly dominates any lower bid because it is never worse and sometimes better
- A similar argument shows that a bid $a_i > \theta_i$ will also be weakly dominated by bidding θ_i

Notes:

- The fact that every player has a weakly dominant strategy, $s_i(\theta_i) = \theta_i$, implies that each player bidding his valuation is a Bayesian Nash equilibrium in weakly dominant strategies

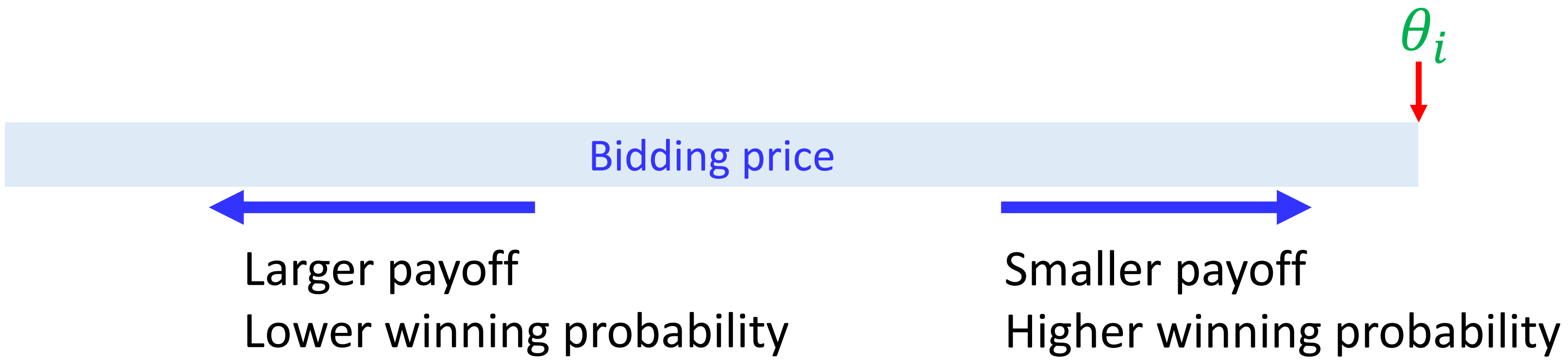
- This result is **noteworthy** not only because its simple prescription, but also

  - ✓ In the private value setting, bidders in a second-price sealed-bid auction do not care about the probability distribution over their opponents' type
    - ✓ The assumption of common knowledge of the distribution of types can be relaxed
    - ✓ We can apply this result even when we have no idea about their opponents' value

  - ✓ Even if types are correlated but values are private, then it is a weakly dominant strategy to bid truthfully

  - ✓ In the private value setting, the outcome of a second-price sealed-bid auction is Pareto optimal because the person who values the good most will be the one who gets it

- Highest bidder wins and pays his bid
  - Would you bid your value?
  - What happens if you bid less than your values?



  - Bidding less than your value is known as bid shading

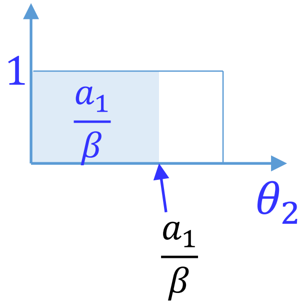
Bayesian Equilibrium for First Price Auctions

Simple scenario

(Simple assumptions)

- Only 2 bidders
- You are player 1 and your value is $\theta_1 > 0$
- You believe the other bidder's value is uniformly distributed over $[0,1]$
- You believe the other bidder uses strategy $s_1(\theta_2) = \beta\theta_2$: proportional to his vale θ_2
- Your expected payoff if you bid a_i

$$
\begin{aligned}
u_1 = (\theta_1 - a_1)\Pr(\text{you win}) &= (\theta_1 - a_1)\Pr(a_1 > \beta\theta_2) \\
&= (\theta_1 - a_1)\Pr\left(\theta_2 < \frac{a_1}{\beta}\right) \\
&= (\theta_1 - a_1)\frac{a_1}{\beta}
\end{aligned}
$$



- Maximizing implies first derivative with respect to a_1 equal to zero

$$
-\frac{a_1}{\beta} + \frac{(\theta_1 - a_1)}{\beta} = 0
$$

- Solving for a_1

$$
a_1 = \frac{\theta_1}{2}
$$

- Bidding half the value is a Bayesian equilibrium

Bayesian Equilibrium for First Price Auctions

Little bit completed

- n bidders
- You are player 1 and your value is $\theta_1 > 0$
- You believe the other bidder's value θ_i is uniformly distributed over $\theta_i \sim U[0,1]$
- You believe the other bidder uses strategy $s_i(\theta_i) = \beta\theta_i$
- Your expected payoff if you bid a_i

$$u_1 = (\theta_1 - a_1)\,\mathrm{Pr}(\text{you win}) = (\theta_1 - a_1)\,\mathrm{Pr}(a_1 > \beta\theta_2, a_1 > \beta\theta_3, \dots, a_1 > \beta\theta_n)$$
$$= (\theta_1 - a_1)\,\mathrm{Pr}\left(\theta_2 < \frac{a_1}{\beta}\right) \times \mathrm{Pr}\left(\theta_3 < \frac{a_1}{\beta}\right) \times \dots \times \mathrm{Pr}\left(\theta_n < \frac{a_1}{\beta}\right)$$
$$= (\theta_1 - a_1)\left(\frac{a_1}{\beta}\right)^{n-1}$$

- Maximizing implies first derivative with respect to a_1 equal to zero

$$-\left(\frac{a_1}{\beta}\right)^{n-1} + (n-1)\frac{(\theta_1 - a_1)}{\beta}\left(\frac{a_1}{\beta}\right)^{n-2} = 0$$

- Solving for a_1

$$a_1 = \frac{n-1}{n}\theta_1$$

Bayesian Equilibrium for First Price Auctions

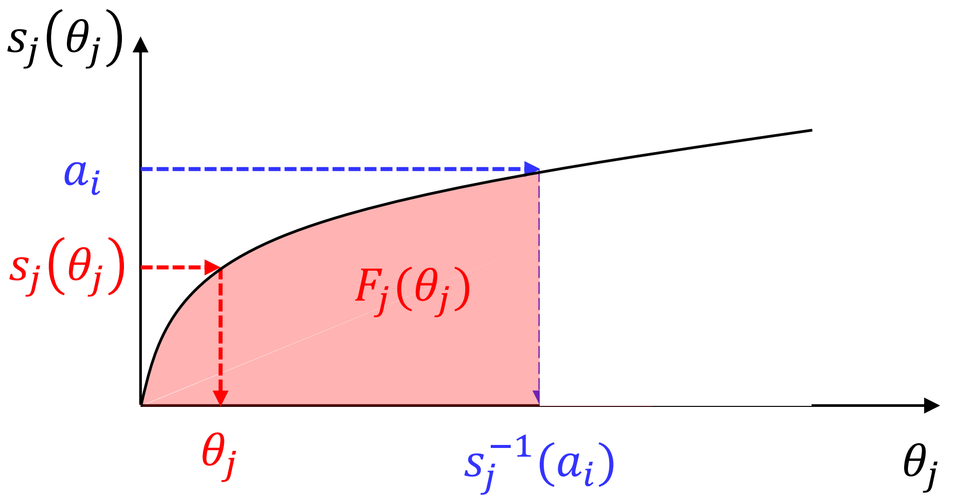
General setting

- **Assumption:** The higher a player's valuation, the higher is his bid, That is,

$$\text{if } \theta_j' > \theta_j'' \text{ then } s_j(\theta_j') > s_j(\theta_j'') \qquad \text{(more general assumption)}$$

- The probability that i's bid is higher than $j's$ bid can be simply computed as

$$\Pr\{s_j(\theta_j) < a_i\} = \Pr\{\theta_j < s_j^{-1}(a_i)\} = F_j\left(s_j^{-1}(a_i)\right)$$



Bayesian Equilibrium for First Price Auctions

General setting

- **Assumption:** The higher a player's valuation, the higher is his bid, That is,

$$\text{if } \theta_j' > \theta_j'' \text{ then } s_j(\theta_j') > s_j(\theta_j'') \qquad \text{(more general assumption)}$$

- The probability that i's bid is higher than $j's$ bid can be simply computed as

$$\Pr\{s_j(\theta_j) < a_i\} = \Pr\{\theta_j < s_j^{-1}(a_i)\} = F_j\left(s_j^{-1}(a_i)\right)$$

- Then, we can compute the expected payoff of player i is

$$E_{\theta_{-i}}[u_i(a_i, s_{-i}(\theta_{-i}); \theta_i)|\theta_i] = (\theta_1 - a_i)\Pr(\text{you win})$$
$$= (\theta_i - a_i) \times \prod_{j \neq i}\left[F_j\left(s_j^{-1}(a_i)\right)\right]$$

- We can compute the best action a_i as one that maximizes the expected payoff

- For n bidders, the Bayesian Nash equilibrium bid (strategy) function is

$$s(\theta) = \theta - \frac{\int_{\underline{\theta}}^{\overline{\theta}} [F(x)]^{n-1} dx}{[F(x)]^{n-1}}$$