INTRODUCTION OF MAXWELL'S EQUATIONS:

INTRODUCTION

In 1864 James Clerk Maxwell brought together and extended four basic laws in electromagnetism such as, Gauss's law in electrostatics, Gauss's law in magnetism, Ampere's law and Faraday's law. In fact entire theory of the electromagnetic field is condensed into these four laws. These laws govern the interaction of bodies which are magnetic or electrically charged or both.

Like Newton's laws in Mechanics, Maxwell's equations are the backbone of electrodynamics. Classical electromagnetic theory constituted by Maxwell's equations predicts that accelerated charges produce electromagnetic waves which propagates in space with a velocity equal to the velocity of light. It also showed that light waves are electromagnetic in character. In 1888 Heinrich Hertz produced and detected these waves.

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Ampere's Circuital Law :→

"The line integral of magnetic field intensity $\vec{H}$ about any closed path is exactly equal to the net current enclosed by that path. ie.

$$\oint \vec{H}\cdot d\vec{l} = I \left(= \int_s \vec{J}\cdot d\vec{s}\,;\ \text{when current density is not uniform}\right)$$

also represented as: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$

Displacement Current :→ (Read Resnick)

The concept of displacement current was first introduced by Maxwell's purely on theoretical ground. Maxwell postulated that it is not only the current in a conductor that produces a magnetic field, but a changing electric field in vacuum, or in a dielectric also produces a magnetic field. It means that a changing electric field is equivalent to a current which flows as long as the electric field is changing. This equivalent current produces the same magnetic effect as an ordinary current in a conductor. This equivalent current is known as "displacement current".

So $\oint \vec{B}\cdot d\vec{l} = \mu_0 (i + i_d)$

DERIVATION OF MAXWELL'S EQUATION IN DIFFERENTIAL FORM:

① **First eqn.**

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \text{(Differential form)}$$

Derivation

~~Proof~~: Gauss's law in electrostatics

$$\int_s \vec{E}\cdot ds = \frac{1}{\varepsilon_0}\cdot\int_v \rho\cdot dv \Rightarrow \int_s \vec{D}\cdot ds = \int_v \rho\cdot dv$$

Using Gauss divergence theorem:

$$\int_s \vec{D}\cdot ds = \int_v \vec{div}\cdot\vec{D}\, dv = \int_s \rho\cdot dv$$

$$\Rightarrow \int_v (\nabla\cdot\vec{D} - \rho)\, dv = 0 \quad \text{or} \quad \nabla\cdot\vec{D} - \rho = 0$$

$$\Rightarrow \boxed{\nabla\vec{D} = \rho}$$

Second eqn.

$$\nabla\cdot\vec{B} = 0$$

Since isolated magnetic poles have no physical significance Therefore the net magnetic flux of magnetic induction $\vec{B}$ through any closed Gaussian Surface is always zero.

i.e. $\int_s \vec{B}\cdot d\vec{s} = 0 \; (= \phi_B)$

using gauss divergence theorem

$$\int_s \vec{B}\cdot d\vec{s} = \int_v \nabla\cdot\vec{B}\cdot dV = 0 \Rightarrow \boxed{\nabla\cdot\vec{B} = 0}$$

Third eqn. According to Faraday's law of em induction, induced e.m.f. around a closed circuit is equal to the negative time rate of change of magnetic flux linked with the circuit.

$$e = -\partial\phi/\partial t$$

But $\phi = \int_s \vec{B}\cdot d\vec{s} \Rightarrow e = -\int_s \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s}$

Also $e = \int_c \vec{E}\cdot d\vec{l} \Rightarrow \int_c \vec{E}\cdot d\vec{l} = -\int_s \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s}$

using stoke's theorem

$$\int_c \vec{E}\cdot d\vec{l} = \int_s (\vec{\nabla}\times E)\cdot ds = -\int_s \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0}$$

Fourth eqn. From Ampere's circuital law

$$\int_c \vec{H} \cdot d\vec{l} = i \qquad \text{also } i = \int_s \vec{J} \cdot d\vec{s}$$

$$\Rightarrow \int_c \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot d\vec{s}$$

Use stoke's theorem $\Rightarrow \int_s (\vec{\nabla} \times \vec{H}) ds = \int_s \vec{J} \cdot d\vec{s}$

$$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J} \qquad \longleftarrow (66)$$

(It can be shown that this is valid for only static charge and insufficient for time varying fields. So we have to add a correction factor for displacement current.)

Shown: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \nabla \cdot \vec{J}$ (Took divergence)

But div. of curl is zero $\Rightarrow \nabla \cdot \vec{J} = 0$

From continuity eqn. $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \frac{\partial \rho}{\partial t} = 0$

Hence (66) to be true, Charge should be static only.

Hence, to include the time varying fields, replace $\vec{J}$ by $\vec{J} + \vec{J}_d$. Hence (66) should be

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d$$

$$\Rightarrow \nabla \cdot (\vec{\nabla} \times \vec{H}) = \nabla \cdot (\vec{J} + \vec{J}_d) = 0$$

or $\nabla \cdot \vec{J} = -\nabla \cdot J_d$

$\therefore$ from continuity eqn.: $\nabla \vec{J}_d = \frac{\partial \rho}{\partial t}$

$$\Rightarrow \nabla \vec{J}_d = \frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = \nabla \cdot \left(\frac{\partial \vec{D}}{\partial t}\right) \quad \{\text{from I eqn. of maxwell}\}$$

$$\Rightarrow \vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

$$\therefore \boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

Note:- In free space, $\rho = \vec{J} = 0$

So can convert equations in free space.

ELECTROMAGNETIC WAVES IN FREE SPACE:

In free space, $\rho = 0$, $\vec{J} = 0$, $\sigma = 0$, $\mu = 1$, $\varepsilon = 1$

$D = \varepsilon_0 E$, $B = \mu_0 H$.

Take curl of Maxwell's III eqn.

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

From maxwell's IV eqn.

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial}{\partial t}\left(\frac{\partial D}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

Note: Now, Maxwell's eqn. are $\nabla \cdot \vec{D} = 0$, $\nabla \cdot \vec{B} = 0$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}.$$

$$\Rightarrow \nabla \cdot \nabla \vec{E} - \nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

From I eqn., $\nabla \cdot \vec{D} = 0$ or $\nabla \cdot E = 0$

$$\Rightarrow \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Similarily get $\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}$

wave equations for $\vec{E}$ and $\vec{H}$ in free space.

General wave eqn. is $\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$

Comparing $\frac{1}{v^2} = \mu_0 \varepsilon_0 \Rightarrow v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \sqrt{\frac{4\pi}{\mu_0 4\pi \varepsilon_0}}$

$$\Rightarrow v = \sqrt{\frac{4\pi \times 9 \times 10^9}{4\pi \times 10^{-7}}} = 3 \times 10^8 \text{ m/sec}.$$

Solution

Now take eqn. for $\vec{E}$, it is now

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

The plane wave solu. of this maybe written in the well known form as

$$\vec{E}(r, t) = E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

where $\vec{E_0}$ is complex amplitude of electric fields. $\vec{k}$ is wave propagation vector: $\vec{k} = k \cdot \hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi \nu}{c}\hat{n} = \frac{\omega}{c}\hat{n}$

$\hat{n}$ is the unit vector in the direction of propagation of electromagnetic wave.

SHOWING THE TRANSVERSE NATURE ......

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Now, take Maxwell's I eqn. $\nabla \cdot \vec{E} = 0$

$$\nabla \cdot \vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Now $\vec{k}\cdot\vec{r} = (\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}x + \hat{j}y + \hat{k}z) = k_x x + k_y y + k_z z$

$$\therefore \nabla \cdot \vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left[(\hat{i}\cdot E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i(k_x x + k_y y + k_z z - \omega t)} \right]$$

$$= \frac{\partial}{\partial x}\left[E_{0x}\, e^{i(k_x x + k_y y + k_z z - \omega t)} \right] + \frac{\partial}{\partial y}[\quad] + \frac{\partial}{\partial z}[\quad]$$

$$= E_{0x} \cdot e^{i(k_x x + k_y y + k_z z - \omega t)} \cdot (ik_x) + \cdots + \cdots$$

$$= (E_{0x}\, ik_x + E_{0y}\, ik_y + E_{0z}\, ik_z)\, e^{i(k_x x + k_y y + k_z z) - i\omega t}$$

$$= i(E_{0x} k_x + E_{0y} k_y + E_{0z} k_z)\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

$$= i\,\vec{k}\cdot\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \Rightarrow \boxed{\nabla \cdot \vec{E} = i(\vec{k}\cdot\vec{E})} = 0$$

$$\Rightarrow i(\vec{k}\cdot\vec{E}) = 0$$

Similarily can find $\vec{k}\cdot\vec{H} = 0$

$$\Rightarrow \vec{k}\cdot\vec{E} = 0$$

That means both $\vec{H}$ & $\vec{E}$ are perpendicular to direction of motion of wave.

Similarily later two Maxwell's eqn. with same solution can give $\vec{k}\times\vec{E} = \mu_0 \omega \vec{H}$ and $\vec{k}\times\vec{H} = -\varepsilon_0 \omega \vec{E}$ which clearly show that $\vec{E}, \vec{H}, \vec{k}$ are mutually perpendicular and thus the transverse nature of light establishes.

NOTE: Treatment for the electromagnetic radiation in non conducting/ dielectric material will be similar to the free space except the values of permittivity and permeability will now have certain values for the particular medium.