Coulomb's law

- SI standard quantity of a charge is a Coulomb (C)
  - 1 C = 6.0 x 10^{18} electrons
  - 1.6 x 10^{-19} C = 1 electron (-) or
  - 1 Proton (+) or 1 "elementary" charge
- Electric Force : (one of the 4 major forces) varies inversely with the distance between the charge
  - $F \alpha \frac{1}{d^2}$

Coulomb's law:

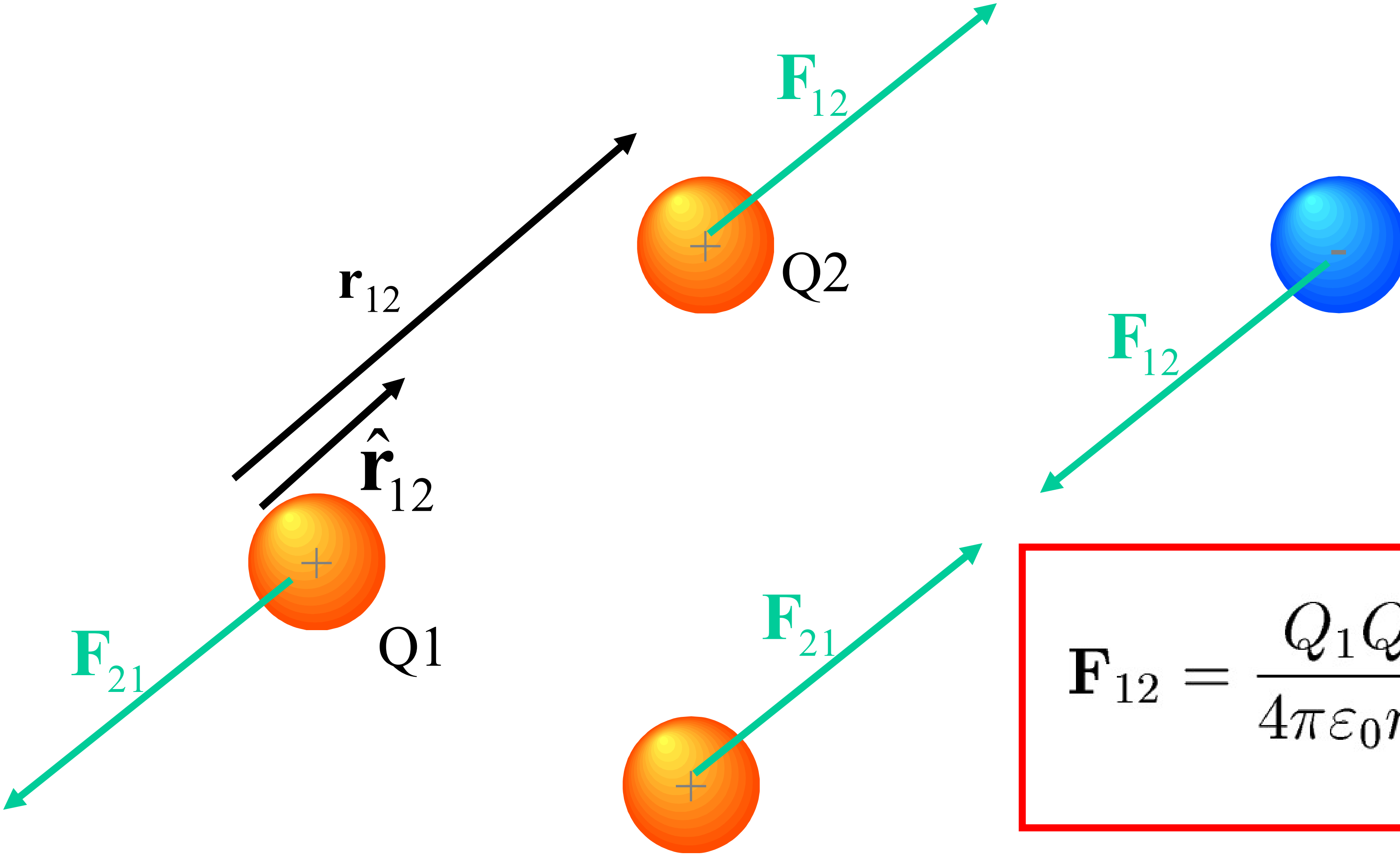
Electrical force is proportional to the product of the electrical charge and inversely proportional to the square of the distance. This is known as Coulomb's law. Mathematically,

$$F = k\frac{q_1 q_2}{d^2}$$

where,

- **F** is the force,
- **k** is a constant and has the value of **9.00 x 10^9 Newton•meters^2/coulomb^2 (9.00 x 10^9 N•m^2/C^2),**
- $\mathbf{q_1}$ represents the electrical charge of object 1 and $\mathbf{q_2}$ represents the electrical charge of object 2, and
- **d** is the distance between the two objects.

Vector form of Coulomb's Law



$$\boxed{\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r_{12}^2}\hat{\mathbf{r}}_{12},}$$

The Electric Field

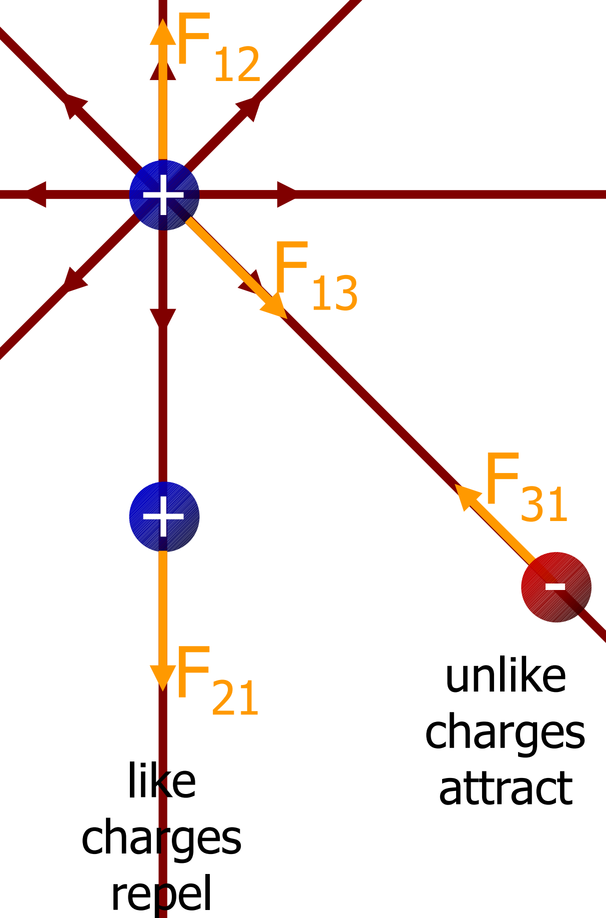
Coulomb's Law (demonstrated in 1785) shows that charged particles exert forces on each other over great distances.

How does a charged particle "know" another one is “there?”

Faraday, beginning in the 1830's, was the leader in developing the idea of the electric field. Here's the idea:

- A charged particle emanates a "field" into all space.

- Another charged particle senses the field, and "knows" that the first one is there.



We define the electric field by the force it exerts on a test charge q_0:

$$\boxed{\vec{E} = \frac{\vec{F}_0}{q_0}}$$

By convention the direction of the electric field is the direction of the force exerted on a POSITIVE test charge. The absence of absolute value signs around q_0 means you must include the sign of q_0 in your work.

If the test charge is "too big" it perturbs the electric field, so the “correct” definition is

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}_0}{q_0}$$

You won’t be required to use this version of the equation.

Any time you know the electric field, you can use this equation to calculate the force on a charged particle in that electric field. $\vec{F} = q\vec{E}$

The units of electric field are Newtons/Coulomb.

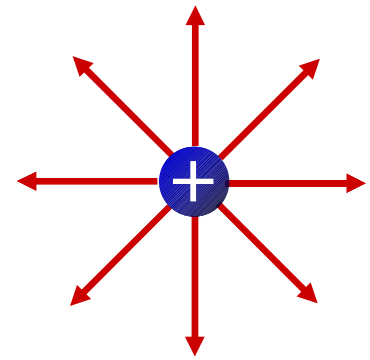
$$\left[\vec{E}\right] = \frac{\left[\vec{F}_0\right]}{\left[q_0\right]} = \frac{N}{C}$$

Later you will learn that the units of electric field can also be expressed as volts/meter:

$$\left[E\right] = \frac{N}{C} = \frac{V}{m}$$

The electric field exists independent of whether there is a charged particle around to “feel” it.

Remember: the electric field direction is the direction a + charge would feel a force.



A + charge would be repelled by another + charge.

Therefore the direction of the electric field is away from positive (and towards negative).

Summarizing:

Charge distributed along a line:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \hat{r}'\frac{\lambda\, dx}{r'^2}.$$

Charge distributed over a surface:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_S \hat{r}'\frac{\sigma\, dS}{r'^2}.$$

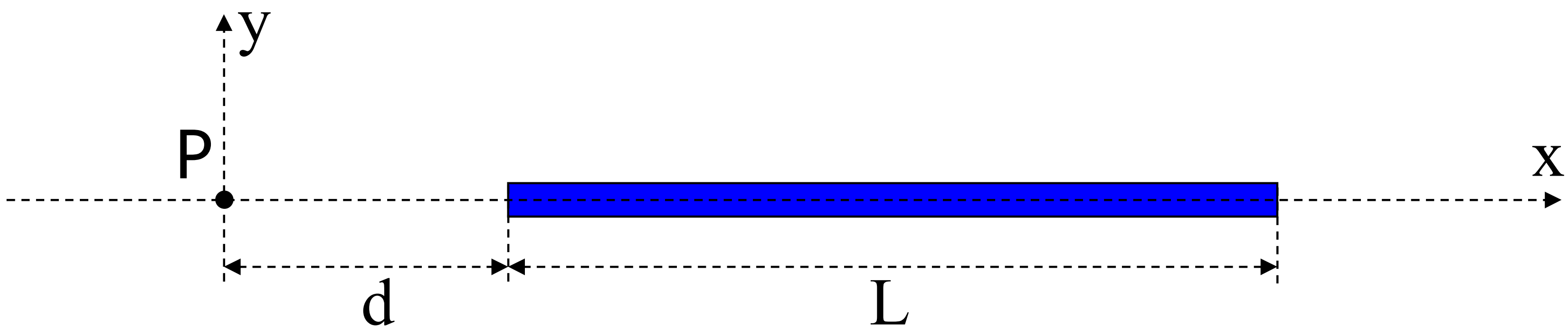
Charge distributed inside a volume:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_V \hat{r}'\frac{\rho\, dV}{r'^2}.$$

If the charge distribution is uniform, then λ, σ, and ρ can be taken outside the integrals.

The Electric Field
Due to a Continuous Charge Distribution
Can be found using Coulomb’s Law.
However, It is much easier using
Gauss’s law.

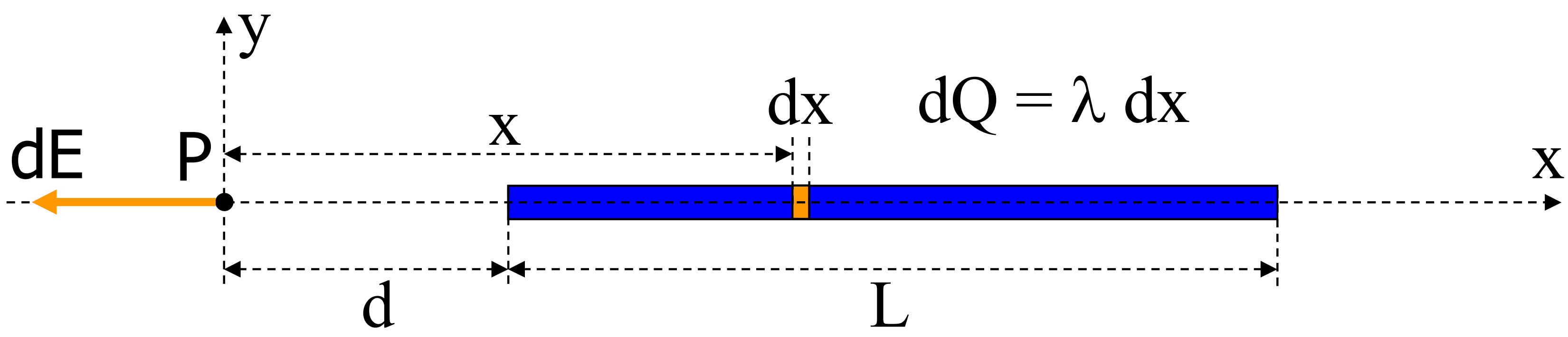
Example: A rod of length L has a uniform charge per unit length λ and a total charge Q. Calculate the electric field at a point P along the axis of the rod at a distance d from one end using Coulomb's law.



Let's put the origin at P. The linear charge density and Q are related by

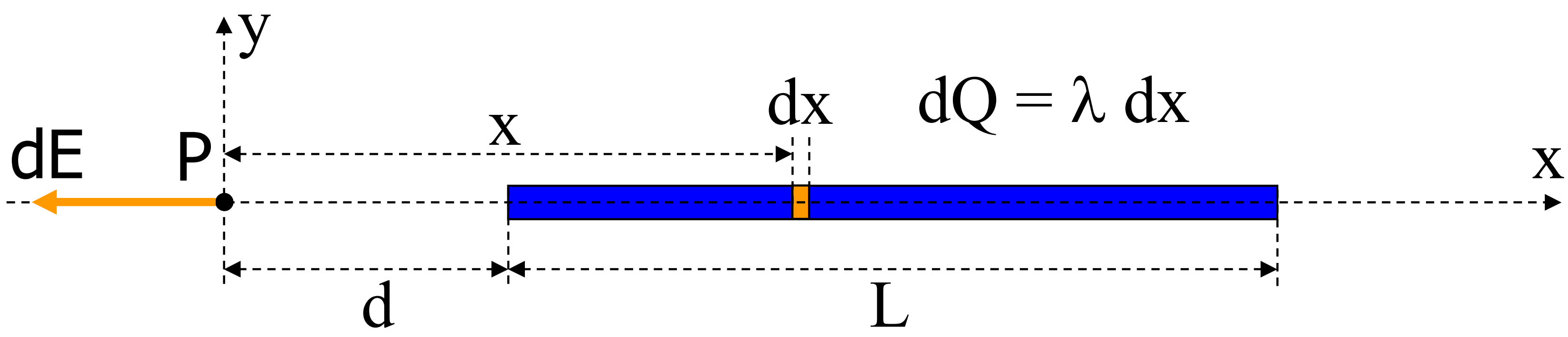
$$\lambda = \frac{Q}{L} \quad \text{and} \quad Q = \lambda L$$

Let's assume Q is positive.



The electric field points away from the rod. By symmetry, the electric field on the axis of the rod has no y-component. dE from the charge on an infinitesimal length dx of rod is

$$dE = k\frac{dq}{x^2} = k\frac{\lambda\, dx}{x^2}$$



$$\vec{E} = \int_d^{d+L} d\vec{E}_x = -k\int_d^{d+L} \frac{\lambda\ dx}{x^2}\hat{i} = -k\lambda\int_d^{d+L} \frac{dx}{x^2}\hat{i} = -k\lambda\left(-\frac{1}{x}\right)_d^{d+L}\hat{i}$$

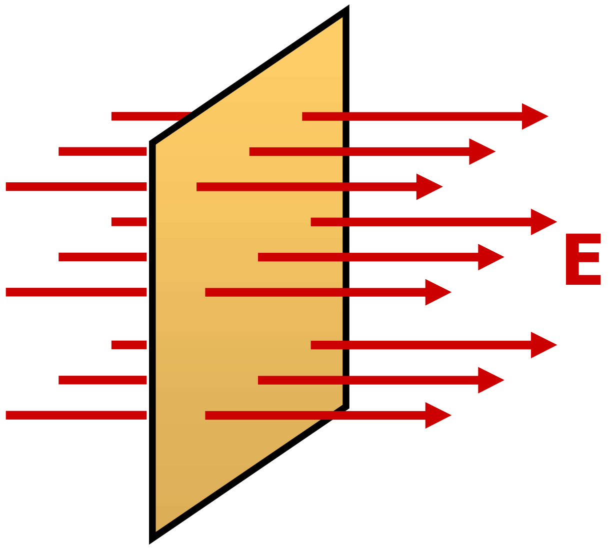
$$\vec{E} = -k\lambda\left(-\frac{1}{d+L} + \frac{1}{d}\right)\hat{i} = -k\lambda\left(\frac{-d+d+L}{d(d+L)}\right)\hat{i} = -k\frac{\lambda\ L}{d(d+L)}\hat{i} = -\frac{kQ}{d(d+L)}\hat{i}$$

Gauss’ Law

Electric Flux

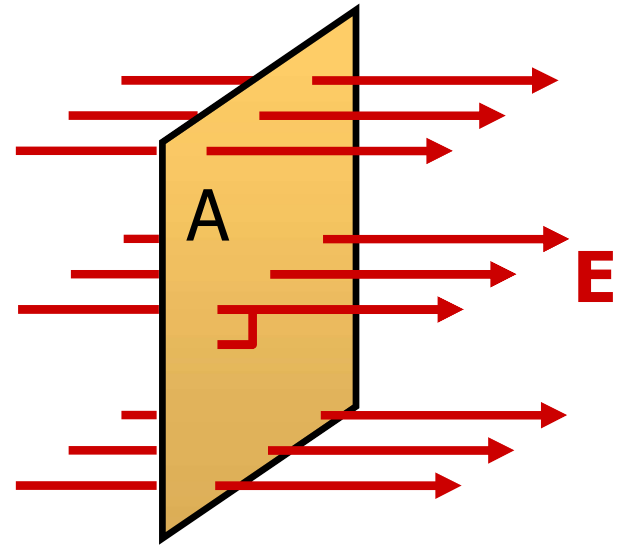
We have used electric field lines to visualize electric fields and indicate their strength.

We are now going to count the number of electric field lines passing through a surface, and use this count to determine the electric field.

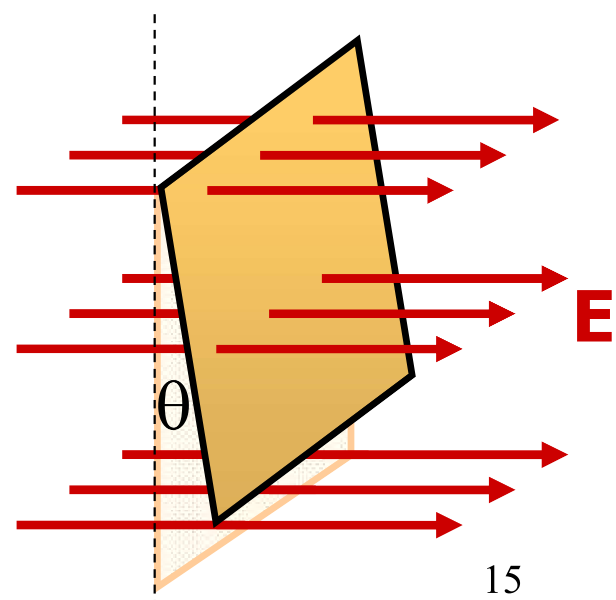


The **electric flux** passing through a surface is the number of electric field lines that pass through it.

Because electric field lines are drawn arbitrarily, we **quantify** electric flux like this: Φ_E=EA, except that…

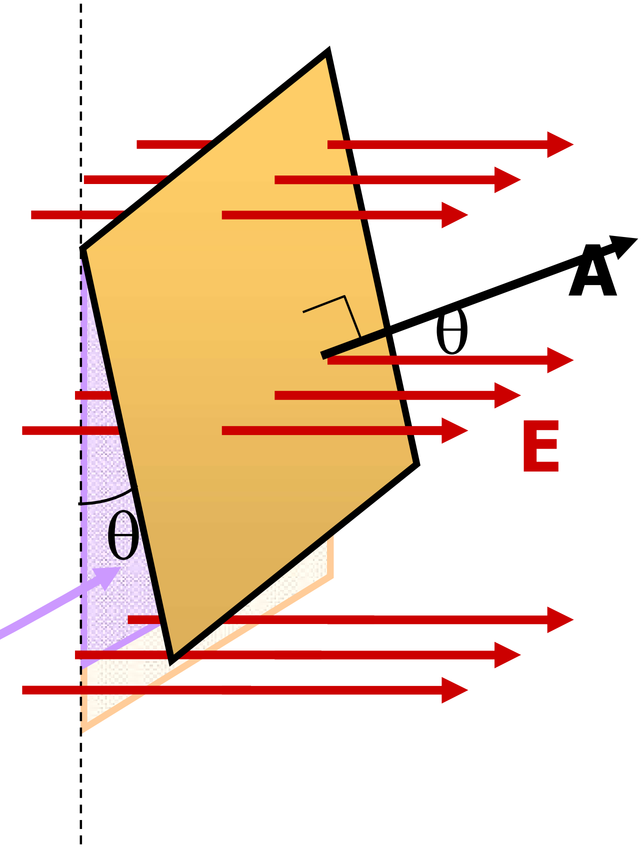


If the surface is tilted, fewer lines cut the surface.



Later we'll learn about magnetic flux, which is why I will use the subscript E on electric flux.

We define $\vec{A}$ to be a vector having a magnitude equal to the area of the surface, in a direction normal to the surface.



The "amount of surface" perpendicular to the electric field is $A\cos\theta$.

Because $\vec{A}$ is perpendicular to the surface, the amount of $\vec{A}$ parallel to the electric field is $A\cos\theta$.

$$A_{\parallel} = A\cos\theta \quad \text{so} \quad \Phi_E = EA_{\parallel} = EA\cos\theta.$$

$$\Phi_E = \vec{E}\cdot\vec{A}$$

Gauss' Law

Mathematically, we express the idea two slides back as

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

Gauss' Law

What is it's differential form?

We will find that Gauss law gives a simple way to calculate electric fields for charge distributions that exhibit a high degree of symmetry…

Strategy for Solving Gauss' Law Problems

- Select a Gaussian surface with symmetry that matches the charge distribution.

- Draw the Gaussian surface so that the electric field is either constant or zero at all points on the Gaussian surface.

- Use symmetry to determine the direction of $\vec{E}$ on the Gaussian surface.

- Evaluate the surface integral (electric flux).

- Determine the charge inside the Gaussian surface.

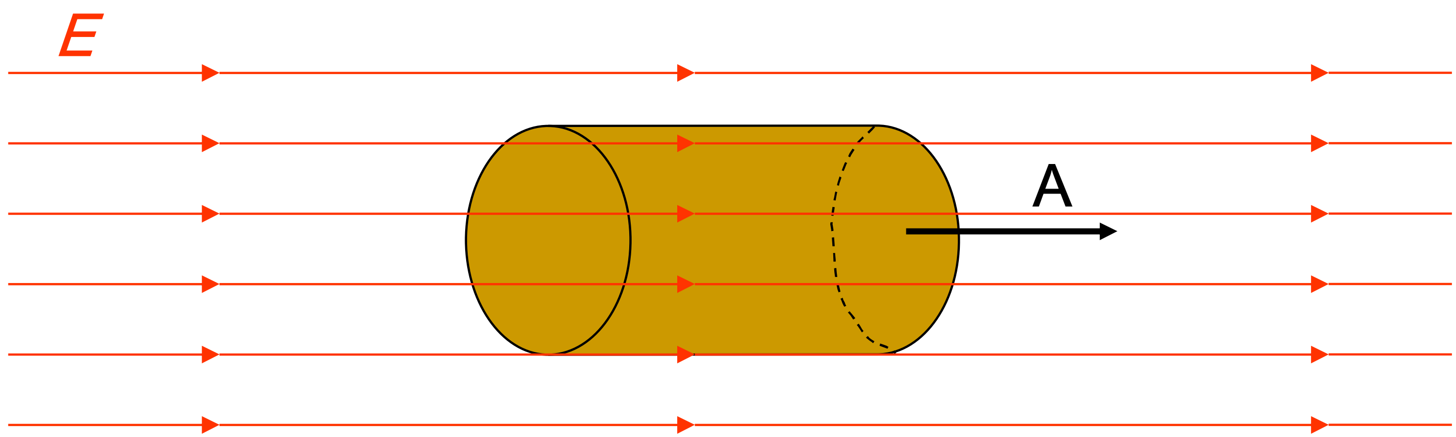
- Solve for $\vec{E}$.

Next few slides will explain the use of Gauss' law with the help of following 10 worked examples:

1. **Flux through a cylinder**
2. **E due to charges spread over a long line**
3. **E due to a point charge**
4. **E due to a sphere**
5. **E due to sheet**
6. **E due to two sheets**
7. **E due to hollow shell**
8. **E due to hollow cylinder**

Worked Example 1

Compute the electric flux from a cylinder with an axis parallel to the electric field direction.

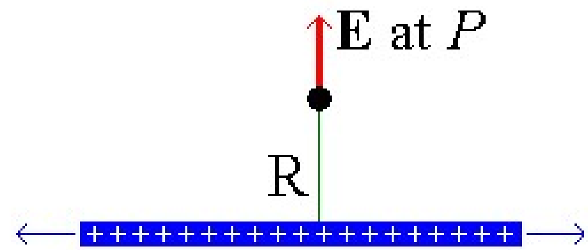


The flux through the curved surface is zero since E is perpendicular to *d*A there. For the ends, the surfaces are perpendicular to E, and E and A are parallel. Thus the flux through the left end (*into* the cylinder) is –*EA*, while the flux through right end (*out* of the cylinder) is +*EA*. Hence the net flux from the cylinder is zero.

Worked Example 2
Field due to a long line charge

1) Understand the geometry

Pick a point P for evaluation of $\vec{\mathbf{E}}$.



2) Understand the symmetry

The only variable on which **E** may depend is R, the distance from the line of charge. There is no angular dependence because the line is cylindrically symmetric, *i.e.* it does not matter at which angle about the line you view it. There is no axial dependence because the line of charge is infinitely long, *i.e.* there is no preferred position along the line.

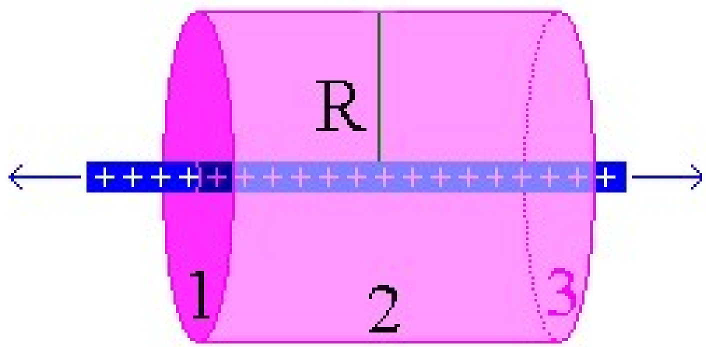
Similar arguments determine that the direction of **E** is directly away from the line, perpendicular to it at all locations. There may be no azimuthally or axial components because the line is infinite and has no extent in the transverse directions.

Worked Example 2 contd….

3) Construct the Gaussian surface

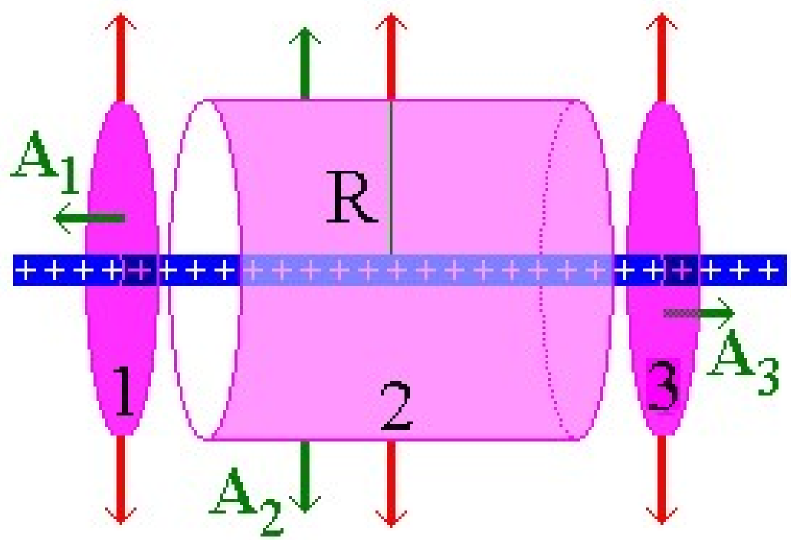
A cylindrical tube is chosen to match the symmetry of the line of charge, centered about it, with a radius equal to the distance between the point of evaluation and the line of charge. The electric field anywhere on this surface has the same direction as the infinitesimal area vector and has a constant value everywhere on the surface. This surface is denoted by 2.

Since a gaussian surface must be closed, the tube is then capped with flat endcaps, 1 and 3. The electric field does not have the same value at all points on the endcaps, but the field vector is perpendicular to the area vector so there is no flux through the endcaps.



Worked Example 2 contd….

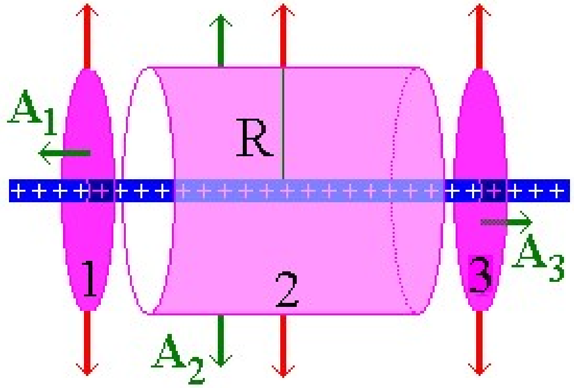
4) Examine the Gaussian surface



The flux through the endcap 1 is zero since $\mathbf{A}_1$ is perpendicular to $\mathbf{E}$ everywhere on this surface. Similarly, the flux through surface 3 is zero.

Since the electric field is everywhere constant on surface 2 and points in the same direction as the surface vector at any point, the flux through surface 2 will be the field magnitude E times the area of the tube wall.

Cylindrical Symmetry



5) Evaluate the electric flux through the Gaussian surface

$$\oint \mathbf{E}\cdot d\mathbf{A} = \underbrace{\cancel{\int_1 \mathbf{E}\cdot d\mathbf{A}_1}}_{0} + \int_2 \mathbf{E}\cdot d\mathbf{A}_2 + \underbrace{\cancel{\int_3 \mathbf{E}\cdot d\mathbf{A}_3}}_{0}$$

$$= \mathrm{E}\cdot(\text{area of tube wall})$$

$$= \mathrm{E}\cdot 2\pi \mathrm{R\,L}$$

6) Evaluate the charge enclosed by the Gaussian surface

$$\frac{q_{enc}}{\varepsilon_0} = \frac{\lambda\,\mathrm{L}}{\varepsilon_0}$$

Worked Example 2 contd….

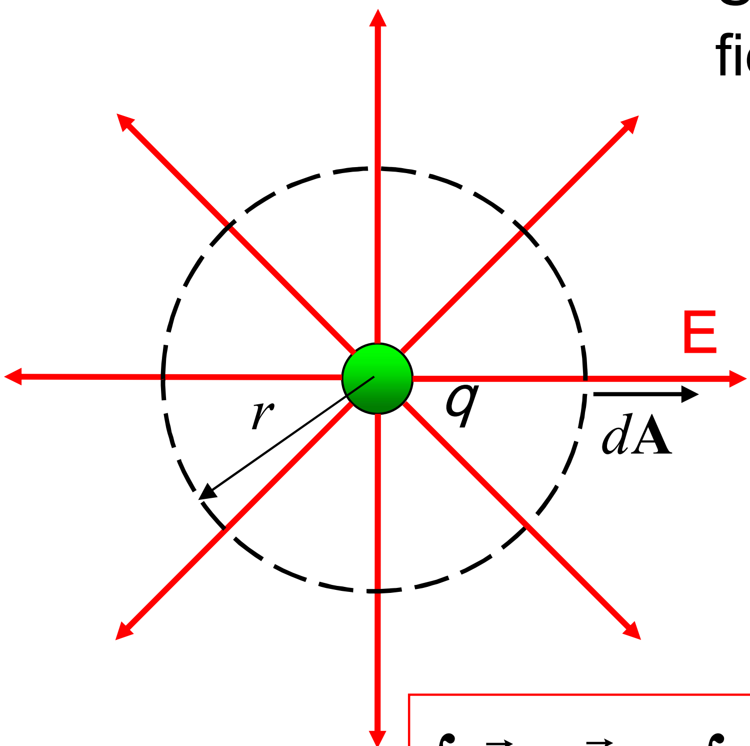
7) Apply Gauss's law for the result!

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0}$$

$$E \cdot 2\pi R L = \frac{\lambda L}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 R}$$

Worked Example 3



Starting with Gauss's law, calculate the electric field due to an isolated point charge *q*.

We choose a Gaussian surface that is a sphere of radius *r* centered on the point charge. I have chosen the charge to be positive so the field is radial outward by symmetry and therefore everywhere perpendicular to the Gaussian surface.

$$\boxed{\vec{E}\cdot d\vec{A} = E\,dA}$$

Gauss's law then gives:

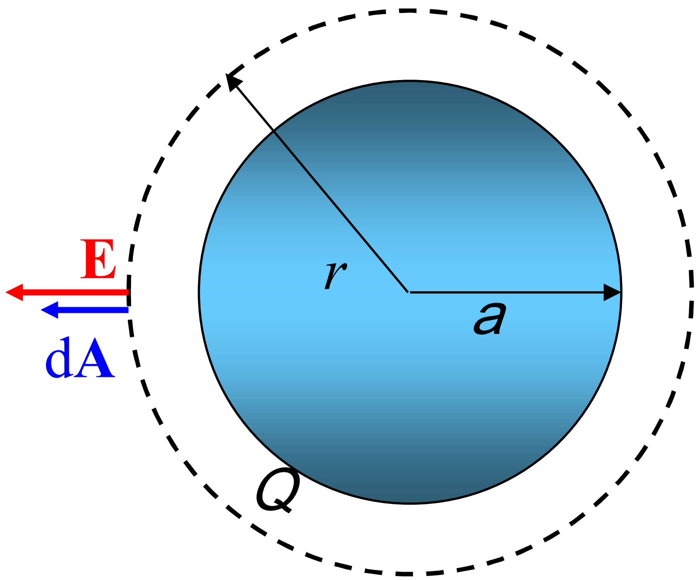
$$\boxed{\oint \vec{E}\cdot d\vec{A} = \oint E\,dA = \frac{Q_{in}}{\varepsilon_0} = \frac{q}{\varepsilon_0}}$$

Symmetry tells us that the field is constant on the Gaussian surface.

$$\boxed{\oint E\,dA = E\oint dA = E\left(4\pi r^2\right) = \frac{q}{\varepsilon_0} \quad \text{so} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = k_e\frac{q}{r^2}}$$

Worked Example 4

An insulating sphere of radius *a* has a uniform charge density ρ and a total positive charge *Q*. Calculate the electric field outside the sphere.



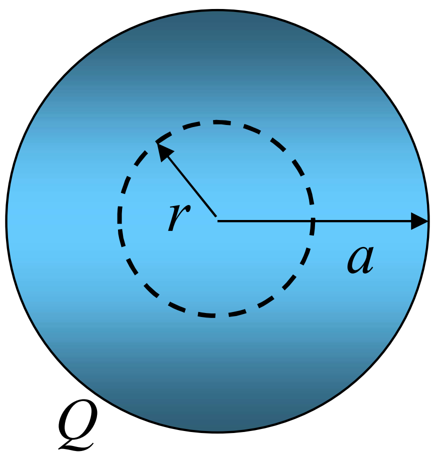
Since the charge distribution is spherically symmetric we select a spherical Gaussian surface of radius *r* > *a* centered on the charged sphere. Since the charged sphere has a positive charge, the field will be directed radially outward. On the Gaussian sphere E is always parallel to dA, and is constant.

❶ $$\text{Left side: } \oint \vec{E} \cdot d\vec{A} = \oint E\, dA = E \oint dA = E\left(4\pi r^2\right)$$

❷ $$\text{Right side: } \frac{Q_{in}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

❸ $$E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_0} \quad \text{or} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = k_e \frac{Q}{r^2}$$

Worked Example 4 cont'd



Find the electric field at a point inside the sphere.

Now we select a spherical Gaussian surface with radius $r < a$. Again the symmetry of the charge distribution allows us to simply evaluate the left side of Gauss's law just as before.

❶
$$\boxed{\text{Left side: } \oint \vec{E}\cdot d\vec{A} = \oint E\,dA = E\oint dA = E\left(4\pi r^2\right)}$$

The charge inside the Gaussian sphere is no longer Q. If we call the Gaussian sphere volume V' then

❷
$$\boxed{\text{Right side: } Q_{in} = \rho V' = \rho \frac{4}{3}\pi r^3}$$

❸
$$\boxed{E\left(4\pi r^2\right) = \frac{Q_{in}}{\varepsilon_0} = \frac{4\rho\pi r^3}{3\varepsilon_0}}$$

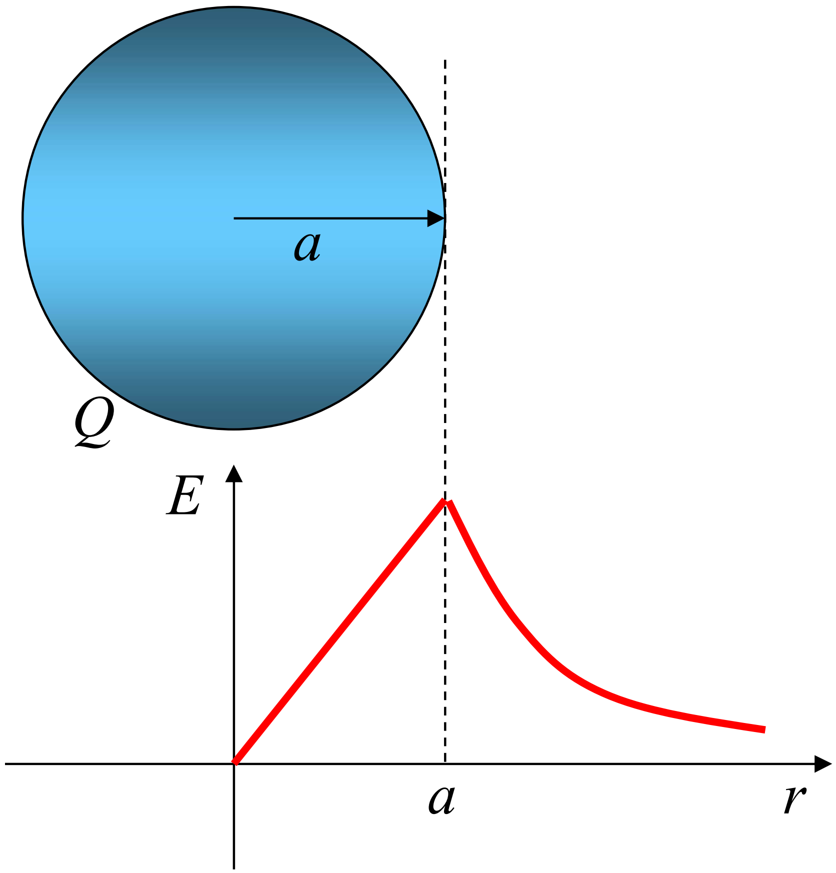
❹
$$\boxed{E = \frac{4\rho\pi r^3}{3\varepsilon_0\left(4\pi r^2\right)} = \frac{\rho}{3\varepsilon_0} r \;\; \text{but} \;\; \rho = \frac{Q}{\frac{4}{3}\pi a^3} \;\; \text{so} \;\; E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a^3} r = k_e \frac{Q}{a^3} r}$$

Worked Example 4 cont'd

We found for $r > a$, $E = k_e \dfrac{Q}{r^2}$

and for $r < a$, $E = \dfrac{k_e Q}{a^3} r$

Let's plot this:



Worked Example 5

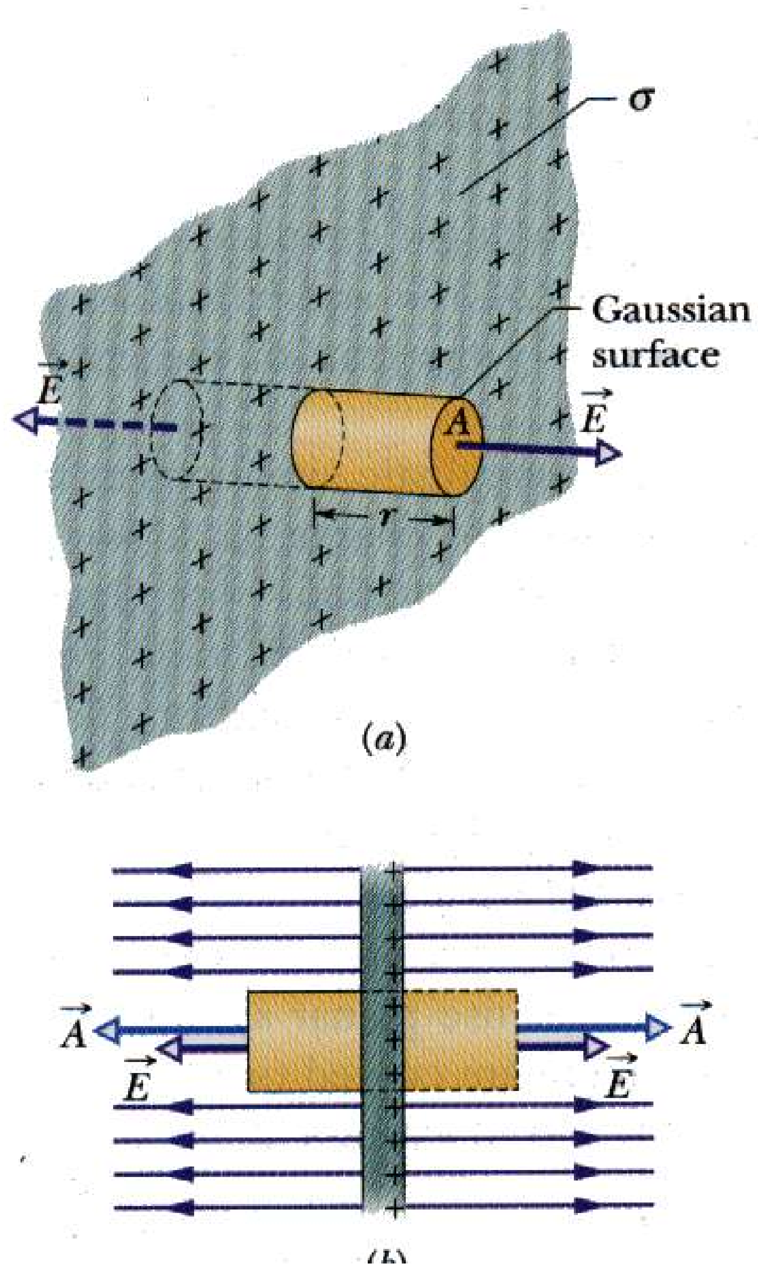
E due to a sheet

Two different views of a very large thin plastic sheet, uniformly charged on one side with surface density of σ. A closed cylindrical Gaussian surface passes throught the sheet and is perpendicular.

From symmetry , E must be perpedicular to the surface and endcaps. The surface charge is positive so E emanates from the surface. Since the field lines do not piece the curved surface there is no flux.

$$\epsilon_0 \Phi = \epsilon_0 (EA + EA) = \sigma A$$

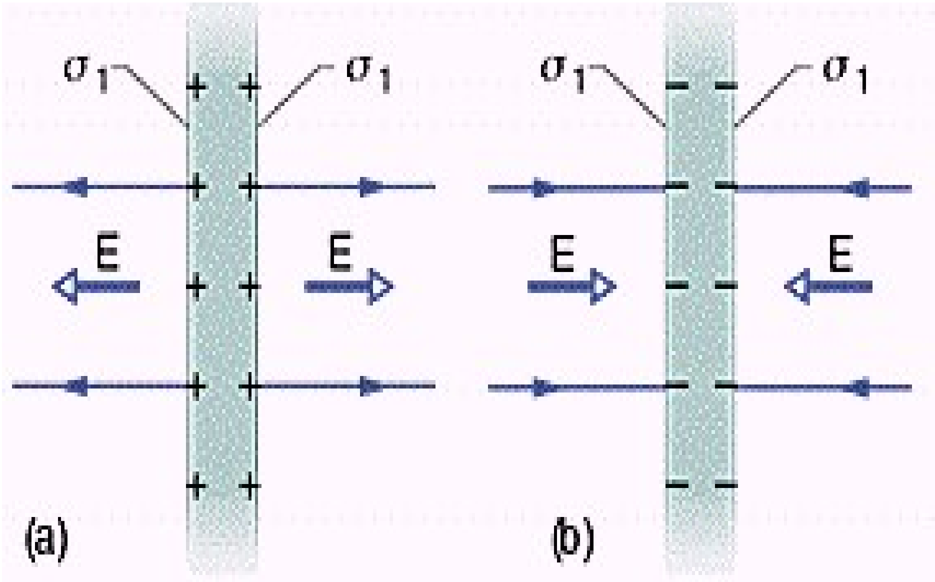
$$E = \frac{\sigma}{2\epsilon_0}$$



What happens if we magically make the sheet a conductor ?? Will the E field change ?

Worked Example 5 cont'd

Figure (a) show a cross-section of a thin, infinite conducting plate with excess positive charge. We know the this excess charge lies on the surface of the plate. If there is no external electric field to force the charge into some particular distribution, it will spread out onto the two surfaces with an uniform charge density of σ_1. From the previous slide we know this charge sets up an electric field $\boldsymbol{E = \sigma_1 / \varepsilon_0}$ which points

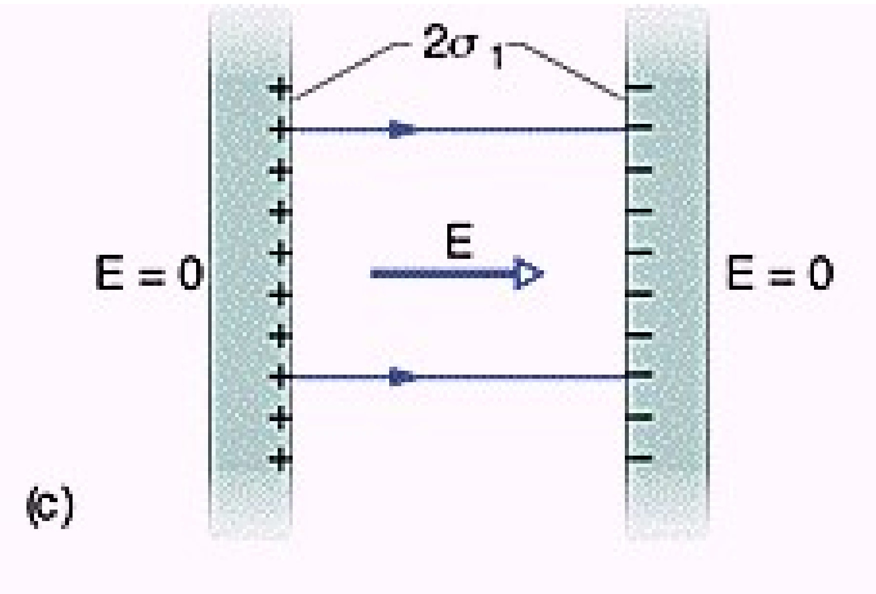


away from the plate. Figure (b) is a identical plate with negative charge. In this case E points inward.

Worked Example 6

E due to two sheets

Suppose we move the two plates in the figures (a) and (b) close together and parallel. Since the plates are conductors, the excess charge on one plate attracts the excess charge on the other and all the excess charge moves into the inner faces of the plates. The new surface charge density has now doubled,



therefore the electric field between the plates is

$$E = \frac{2\sigma_1}{\epsilon_0}$$

The field points away from positive charged plate and toward the negative plate. Since no excess is left on the outer faces, the electric field outside is zero.

Worked Example 7

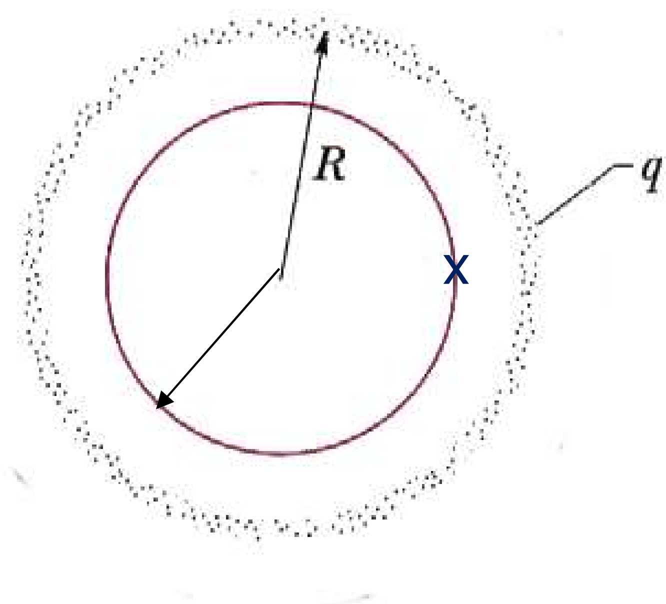
E due to hollow spherical shell

1st choose an Gaussian surface inside the shell that coincides with the point x.

Since fields lines can not cross, the spherical symmetry requires E field to be in the radially direction and uniform on the Gaussian surface, therefore Gauss' Law

$$\epsilon_0 \oint E \cdot dA = \epsilon_0 E \oint dA = \epsilon_0 E(4\pi r^2) = q_{enc}$$

$$q_{enc} = 0 \rightarrow E = 0$$

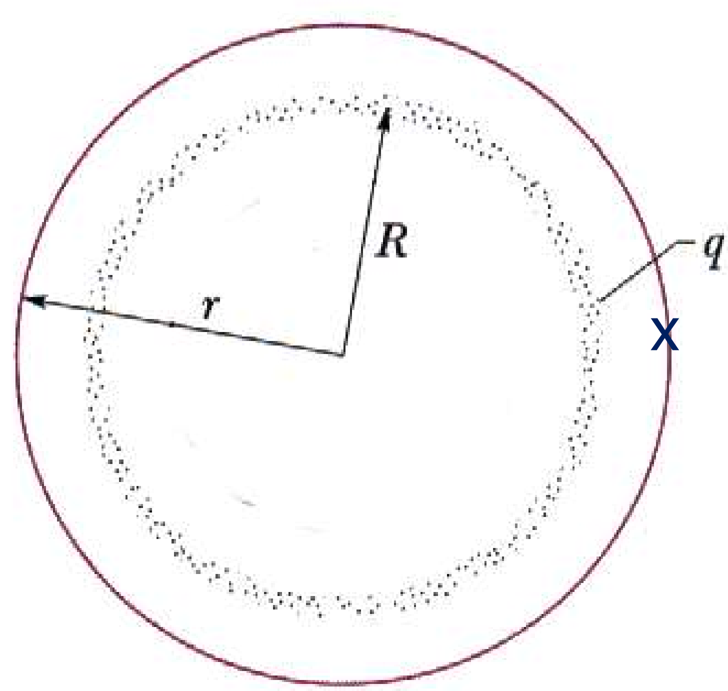


Worked Example 7 cont'd

Now let's consider the electric field at a point outside the charged sphere.

We define a Gaussian surface around the sphere for which r>R.

Again since fields lines can not cross, the spherical symmetry requires E field to be in the radially direction and uniform on the Gaussian surface, therefore Gauss' Law becomes



$$\epsilon_0 \oint E \cdot dA = \epsilon_0 E \oint dA = \epsilon_0 E(4\pi r^2) = q_{enc}$$

$$E = \frac{q}{4\pi\epsilon_0 r^2}$$

Worked Example 7 cont’d

Using Gauss’ Law we were able to prove two shell theorems .

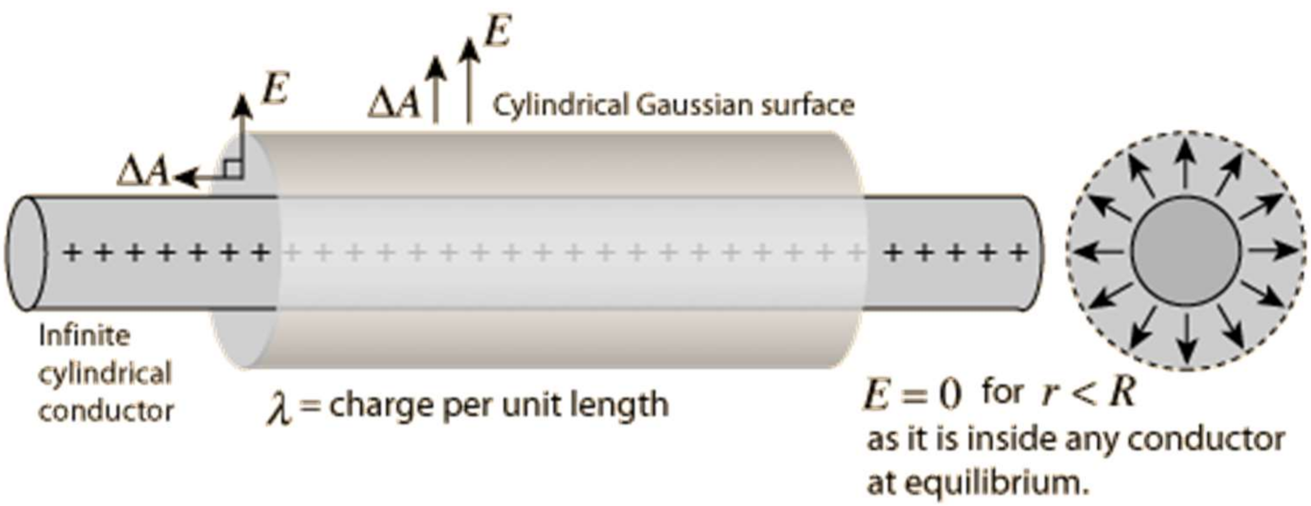
A shell of uniform charge attracts and repels a charged particle that is outside the shell as if all shell’s charge were concentrated at the center.

If a charged particle is located inside a shell of uniform charge, there is no net electric force on the particle from the shell.

Worked Example 8

E due to long cylinder

Considering a Gaussian surface in the form of a cylinder at radius r > R, the electric field has the same magnitude at every point of the cylinder and is directed outward. The electric flux is then just the electric field times the area of the cylinder.



$$\Phi = E2\pi rL = \frac{\lambda L}{\varepsilon_0}$$

For $r \geq R$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

This expression is a good approximation for the field close to a long conducting cylinder.

29.5 ELECTRIC POTENTIAL ENERGY

Consider a system of charges. The charges of the system exert electric forces on each other. If the position of one or more charges is changed, work may be done by these electric forces. We define *change in electric potential energy* of the system as negative of the work done by the electric forces as the configuration of the system changes.

Consider a system of two charges q_1 and q_2. Suppose, the charge q_1 is fixed at a point A and the charge q_2 is taken from a point B to a point C along the line ABC (figure 29.7).

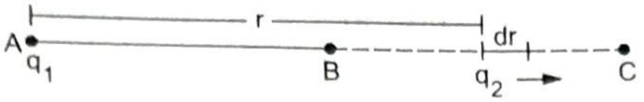


Figure 29.7

Let the distance $AB = r_1$ and the distance $AC = r_2$.

Consider a small displacement of the charge q_2 in which its distance from q_1 changes from r to $r + dr$. The electric force on the charge q_2 is

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \text{ towards } \overrightarrow{AB}.$$

The work done by this force in the small displacement dr is

$$dW = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} dr.$$

The total work done as the charge q_2 moves from B to C is

$$W = \int_{r_1}^{r_2} \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} dr = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

No work is done by the electric force on the charge q_1 as it is kept fixed. The change in potential energy $U(r_2) - U(r_1)$ is, therefore,

$$U(r_2) - U(r_1) = -W = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \quad \ldots \quad (29.5)$$

We choose the potential energy of the two-charge system to be zero when they have infinite separation (that means when they are widely separated). This means $U(\infty) = 0$. The potential energy when the separation is r is

$$U(r) = U(r) - U(\infty)$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{\infty}\right) = \frac{q_1 q_2}{4\pi\varepsilon_0 r}. \quad \ldots \quad (29.6)$$

The above equation is derived by assuming that one of the charges is fixed and the other is displaced. However, the potential energy depends essentially on the separation between the charges and is independent of the spatial location of the charged particles. Equations (29.5) and (29.6) are, therefore, general.

Equation (29.6) gives the electric potential energy of a pair of charges. If there are three charges q_1, q_2 and q_3, there are three pairs. Similarly for an N-particle system, the potential energy of the system is equal to the sum of the potential energies of all the pairs of the charged particles.

29.6 ELECTRIC POTENTIAL

The electric field in a region of space is described by assigning a vector quantity $\vec{E}$ at each point. The same field can also be described by assigning a scalar quantity V at each point. We now define this scalar quantity known as *electric potential*.

Suppose, a test charge q is moved in an electric field from a point A to a point B while all the other charges in question remain fixed. If the electric potential energy changes by $U_B - U_A$ due to this displacement, we define the *potential difference* between the point A and the point B as

$$V_B - V_A = \frac{U_B - U_A}{q}. \qquad \text{... (29.7)}$$

Conversely, if a charge q is taken through a potential difference $V_B - V_A$, the electric potential energy is increased by $U_B - U_A = q(V_B - V_A)$. This equation defines potential difference between any two points in an electric field. We can define absolute electric potential at any point by choosing a reference point P and saying that the potential at this point is zero. The electric potential at a point A is then given by (equation 29.7)

$$V_A = V_A - V_P = \frac{U_A - U_P}{q}. \qquad \text{... (29.8)}$$

So, *the potential at a point A is equal to the change in electric potential energy per unit test charge when it is moved from the reference point to the point A.*

Suppose, the test charge is moved in an electric field without changing its kinetic energy. The total work done on the charge should be zero from the work–energy theorem. If W_{ext} and W_{el} be the work done by the external agent and by the electric field as the charge moves, we have,

$$W_{ext} + W_{el} = 0$$

or, $$W_{ext} = -W_{el} = \Delta U,$$

where ΔU is the change in electric potential energy. Using this equation and equation (29.8), the potential at a point A may also be defined as follows:

The potential at a point A is equal to the work done per unit test charge by an external agent in moving the test charge from the reference point to the point A (without changing its kinetic energy).

The choice of reference point is purely ours. Generally, a point widely separated from all charges in question is taken as the reference point. Such a point is assumed to be at infinity.

As potential energy is a scalar quantity, potential is also a scalar quantity. Thus, if V_1 is the potential at a given point due to a charge q_1 and V_2 is the potential at the same point due to a charge q_2, the potential due to both the charges is $V_1 + V_2$.

29.7 ELECTRIC POTENTIAL DUE TO A POINT CHARGE

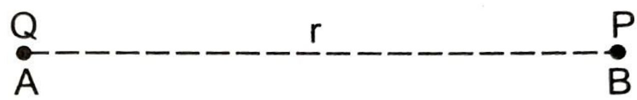


Figure 29.8

Consider a point charge Q placed at a point A (figure 29.8). We have to find the electric potential at a point P where $AP = r$. Let us take the reference point at $r = \infty$. Suppose, a test charge q is moved from $r = \infty$ to the point P. The change in electric potential energy of the system is, from equation (29.6),

$$U_P - U_\infty = \frac{Qq}{4\pi\varepsilon_0 r}.$$

The potential at P is, from equation (29.8),

$$V_P = \frac{U_P - U_\infty}{q} = \frac{Q}{4\pi\varepsilon_0 r}. \quad \ldots \quad (29.9)$$

The electric potential due to a system of charges may be obtained by finding potentials due to the individual charges using equation (29.9) and then adding them. Thus,

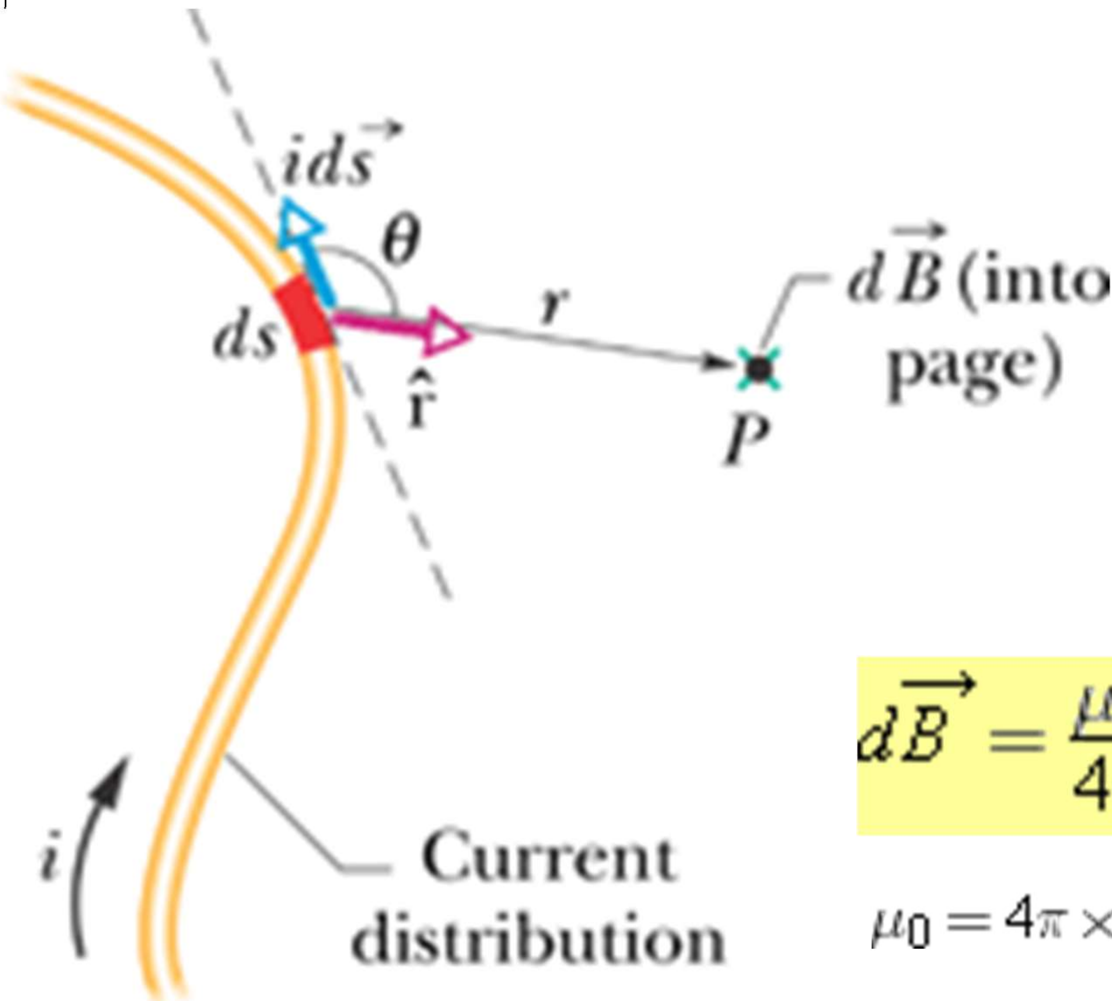
$$V = \frac{1}{4\pi\varepsilon_0} \sum \frac{Q_i}{r_i}.$$

Biot-Savart Law

At any point P the magnitude of the magnetic field intensity produced by a differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired; also, the magnitude of the field is inversely proportional to the square of the distance from the filament to the point P. The constant of proportionality is $1/4\pi$.

MATHEMATICALLY:

Biot-Savart Law

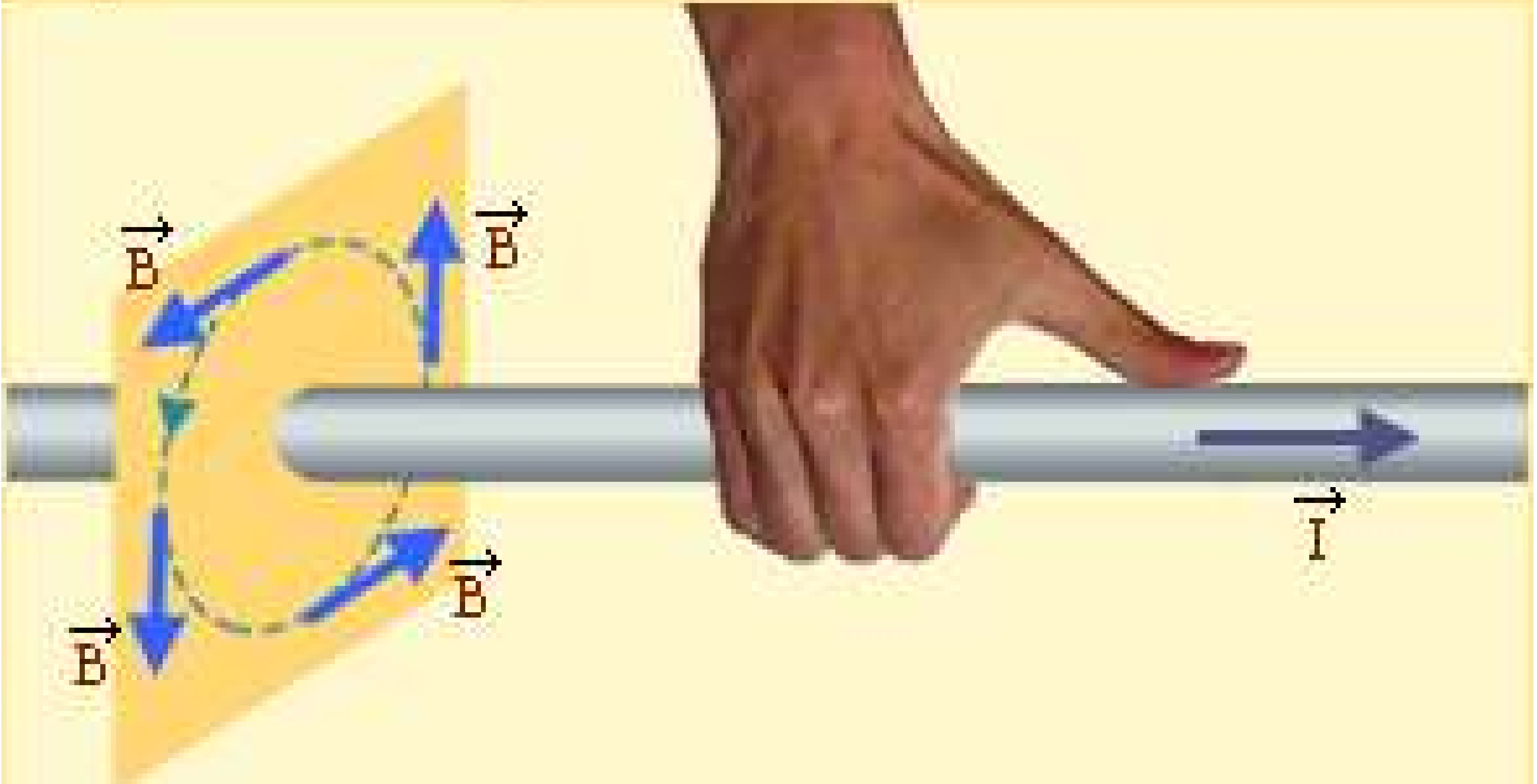


$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s} \times \hat{r}}{r^2} \quad \text{(Biot–Savart law)}.$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T \cdot m / A} \approx 1.26 \times 10^{-6}\,\mathrm{T \cdot m / A}.$$

So, the magnetic field "circulates" around the wire

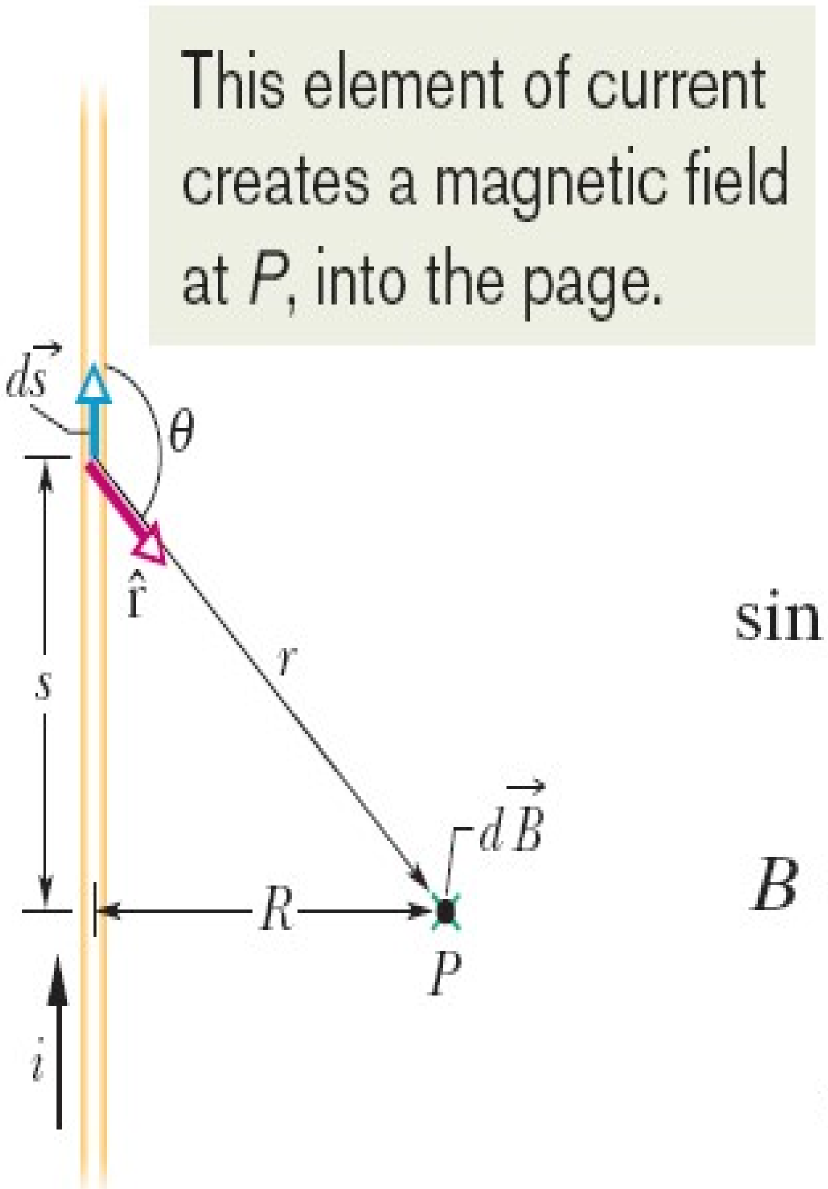
right-hand rule



To apply the right-hand rule for determining the direction of the magnetic field produced by a steady current, imagine gripping the wire with your right hand so that your thumb points in the direction of current flow. Your fingers naturally curl in the direction in which the magnetic field circulates.

Applications of Biot Savart law

Application 1:
Magnetic field due to a long straight wire



$$dB = \frac{\mu_0}{4\pi} \frac{i\, ds \sin\theta}{r^2}.$$

$$B = 2\int_0^\infty dB = \frac{\mu_0 i}{2\pi} \int_0^\infty \frac{\sin\theta\, ds}{r^2}.$$

$$r = \sqrt{s^2 + R^2}$$

$$\sin\theta = \sin(\pi - \theta) = \frac{R}{\sqrt{s^2 + R^2}}.$$

$$B = \frac{\mu_0 i}{2\pi} \int_0^\infty \frac{R\, ds}{(s^2 + R^2)^{3/2}}$$

$$= \frac{\mu_0 i}{2\pi R} \left[\frac{s}{(s^2 + R^2)^{1/2}} \right]_0^\infty = \frac{\mu_0 i}{2\pi R}$$

Application 2:
Magnetic field from a circular current loop at point P

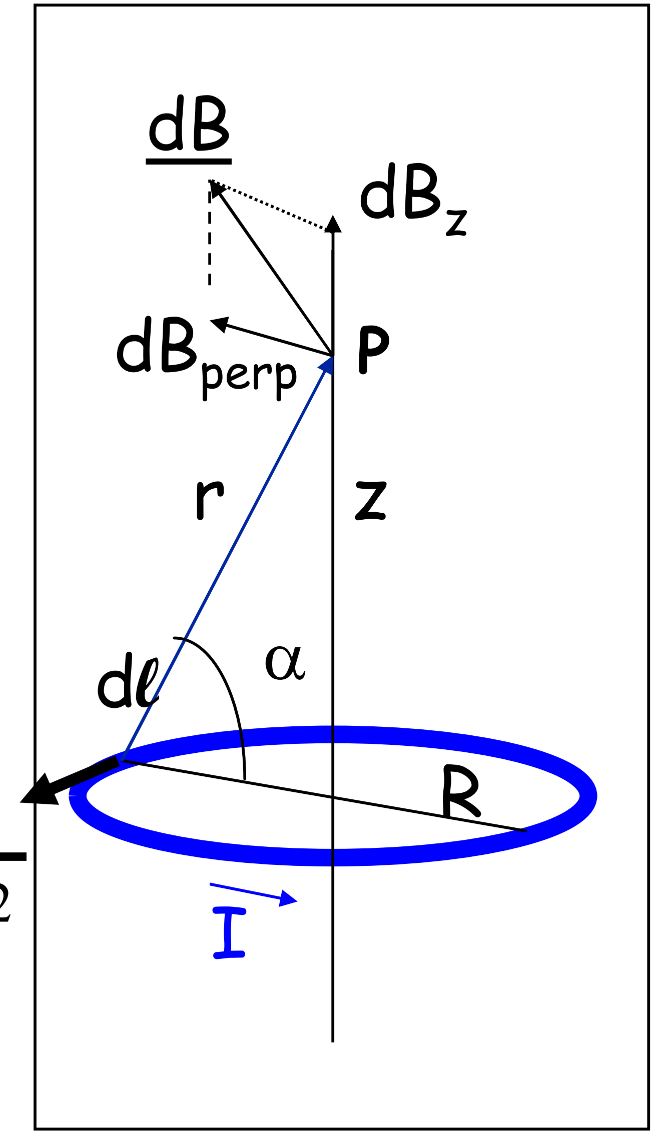
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \hat{\mathbf{r}}}{r^2}$$

$$dB = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}$$

Only z component is nonzero.

$$dB_z = dB\cos\alpha = \frac{\mu_0}{4\pi}\frac{Idl\cos\alpha}{r^2}$$

$$r = \sqrt{R^2 + z^2}, \cos\alpha = R\Big/\sqrt{R^2 + z^2}$$

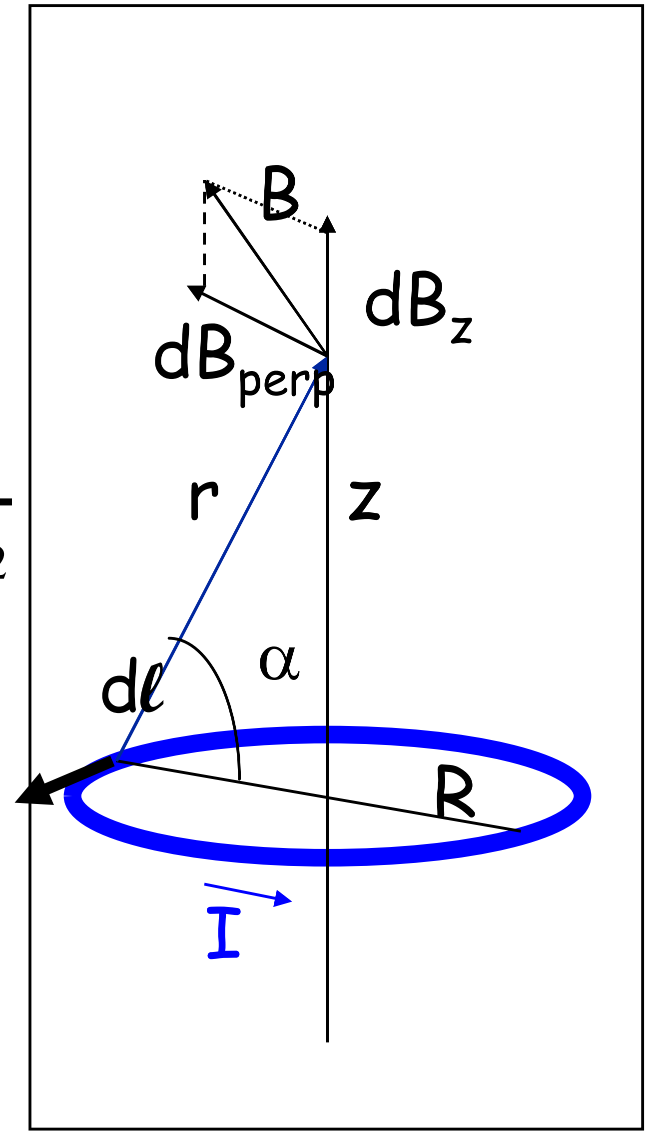


Magnetic field from a circular current loop

$$B = \int dB_z = \int \frac{\mu_0}{4\pi} \frac{IR}{(R^2 + z^2)^{3/2}} dl$$

$$B = \frac{\mu_0}{4\pi} \frac{IR}{(R^2 + z^2)^{3/2}} \int dl =$$

$$B = \frac{\mu_0}{4\pi} \frac{IR}{(R^2 + z^2)^{3/2}} 2\pi R = \frac{\mu_0}{2} \frac{IR^2}{(R^2 + z^2)^{3/2}}$$



At the center of the loop $B = \frac{\mu_0 I}{2R}$

At distance z on axis from the loop, z>>R $B = \frac{\mu_0 IR^2}{2z^3}$

Magnetic field in terms of dipole moment

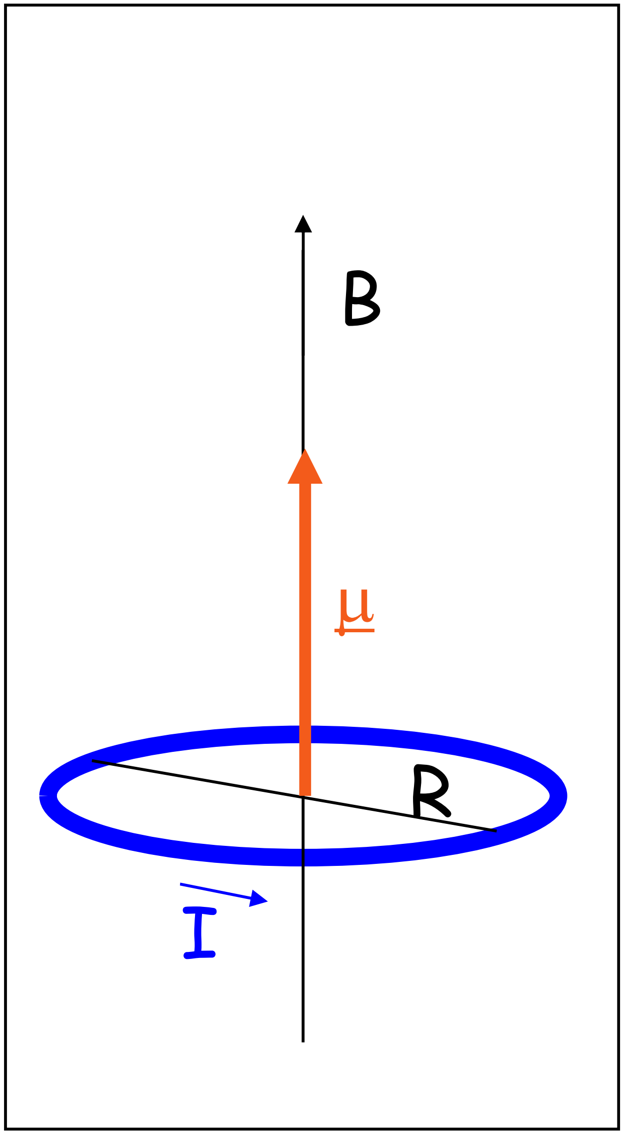
Far away on the axis,

$$B = \frac{\mu_0 I R^2}{2z^3}$$

The magnetic dipole moment of the loop is defined as m = IA =IπR^2.

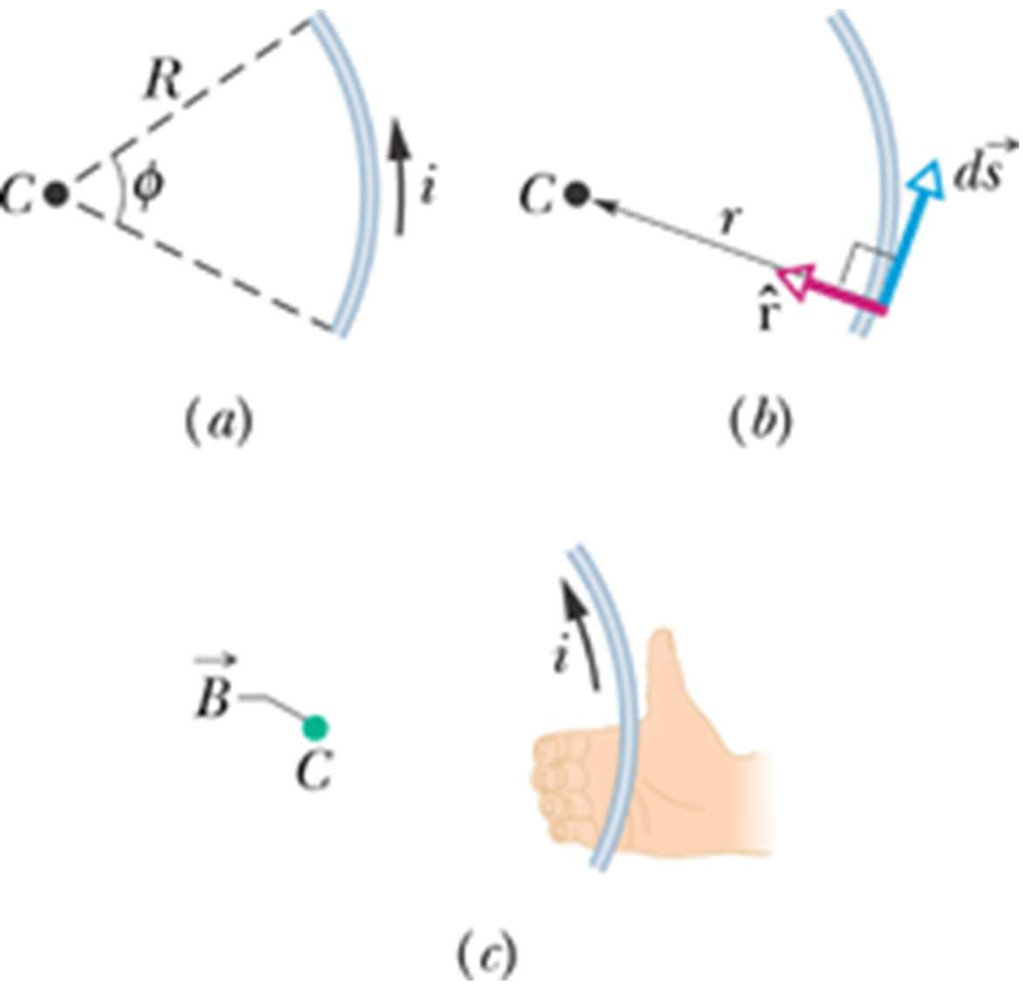
The direction is given by the right hand rule: with fingers closed in
the direction of the current flow, the thumb points along m.

$$B = \frac{\mu_0 m}{2\pi z^3}$$



Special case: Magnetic Field Due to a Current in a Circular Arc of Wire

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin 90^\circ}{R^2} = \frac{\mu_0}{4\pi}\frac{i\,ds}{R^2}\,.$$

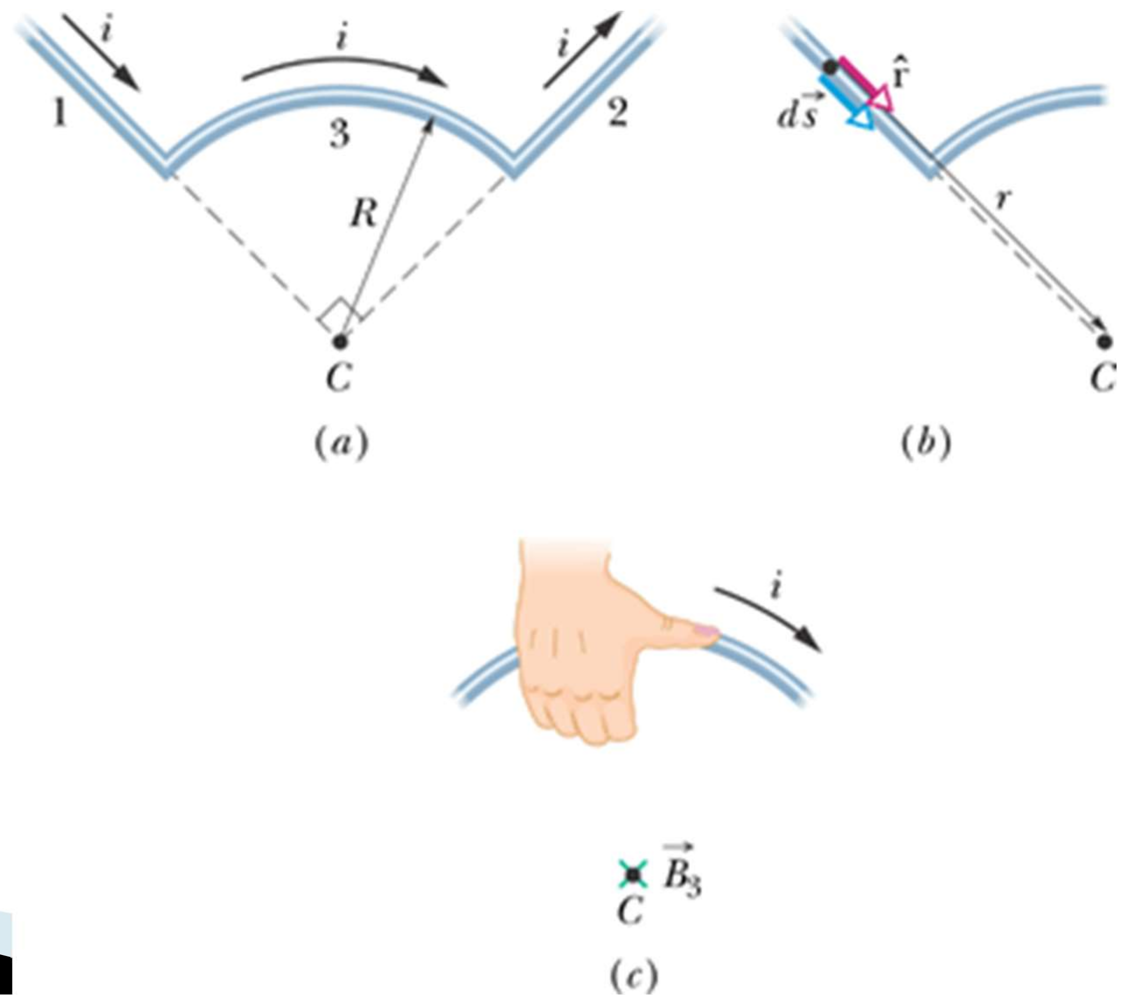


$$B = \int dB = \int_0^{\phi}\frac{\mu_0}{4\pi}\frac{iR\,d\phi}{R^2} = \frac{\mu_0 i}{4\pi R}\int_0^{\phi} d\phi\,.$$

$$B = \frac{\mu_0 i \phi}{4\pi R} \quad \text{(at center of circular arc)}\,.$$

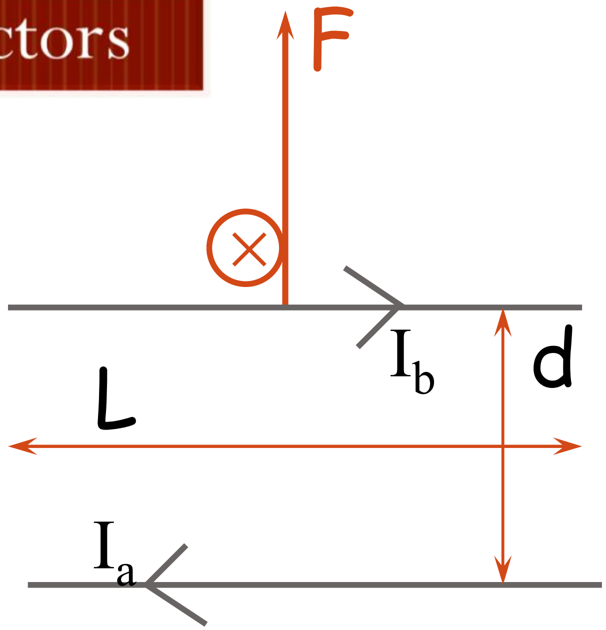
Sample Problem

The wire in Fig. 'a' carries a current i and consists of a circular arc of radius R and central angle $\pi / 2$ rad, and two straight sections whose extensions intersect the center C of the arc. What magnetic field $\vec{B}$ does the current produce at C?



Force on 2 Parallel Current-Carrying Conductors

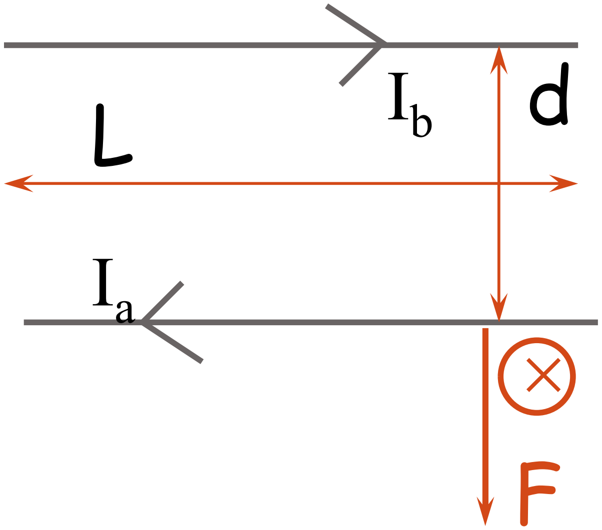
- Calculate force on length L of wire b due to field of wire a:



The field at b due to a is given by:

$$B_a = \frac{\mu_0 I_a}{2\pi d} \Rightarrow$$ Force on b =

$$\boxed{\vec{F}_b = I_b \vec{L} \times \vec{B}_a = \frac{\mu_0 I_a I_b L}{2\pi d}}$$



$$B_b = \frac{\mu_0 I_b}{2\pi d} \Rightarrow$$ Force on a =

$$\boxed{\vec{F}_a = I_a \vec{L} \times \vec{B}_b = \frac{\mu_0 I_a I_b L}{2\pi d}}$$

Definition of Ampere:

$$\vec{F}_a = I_a \vec{L} \times \vec{B}_b = \frac{\mu_0 I_a I_b L}{2\pi d}$$

- This expression is used to define a standard 1 Ampere current:

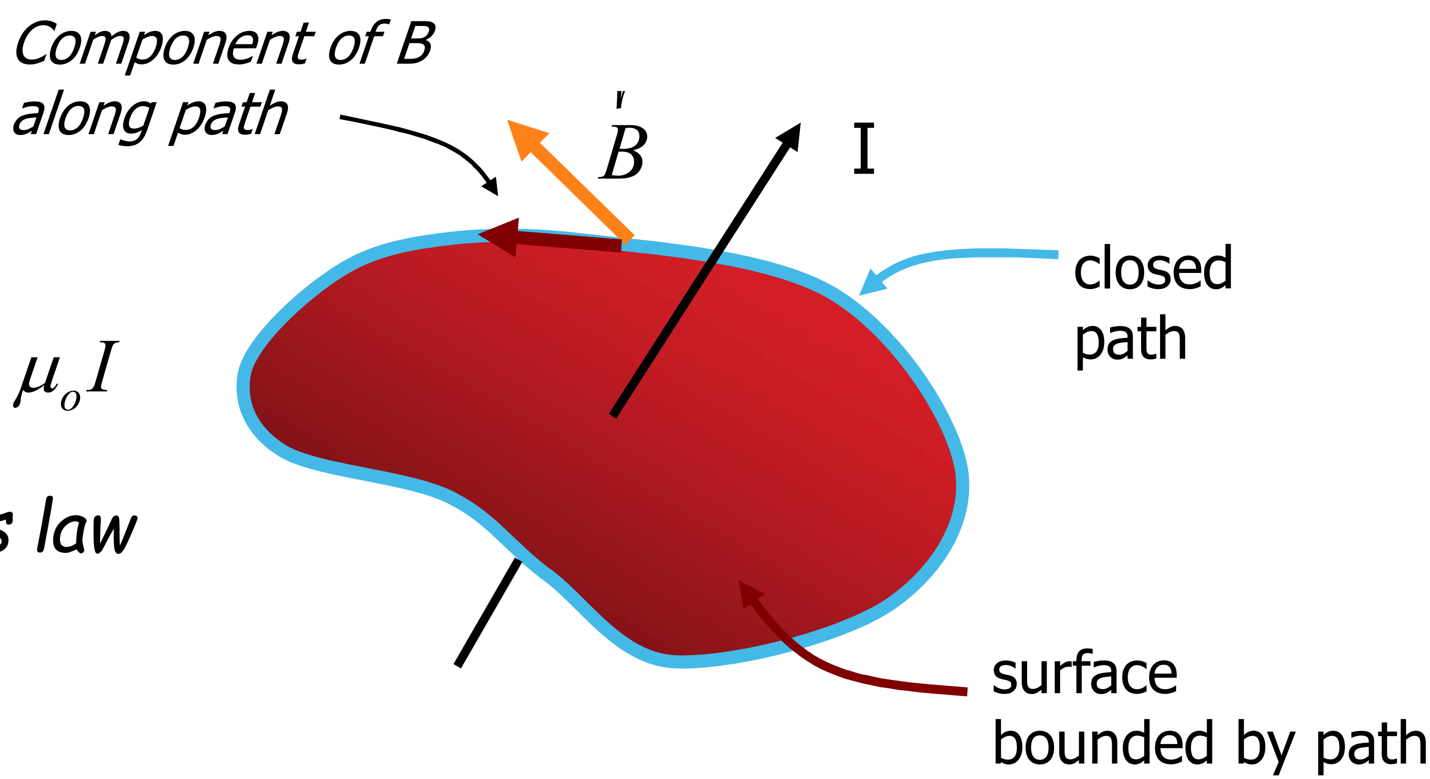
"The constant current which will produce an attractive force of 2×10^{-7} Newton per metre of length between two straight, parallel conductors of infinite length and negligible circular cross section placed one metre apart in a vacuum."

Ampere's law and applications

Sum up component of B around path Equals current through surface.

$$\oint \mathbf{B} \bullet dl = \mu_o I$$

- *Ampere's law*



Ampere's Law

$$\boxed{\sum_{\text{path}} \mathbf{B} \bullet \mathbf{d}l = \mu_0 I}$$

- No Different Physics from Biot-Savart Law
- Useful in cases where there is a high degree of symmetry
- Can be compared with Coulomb's Law and Gauss's Law in electrostatics in functionality.

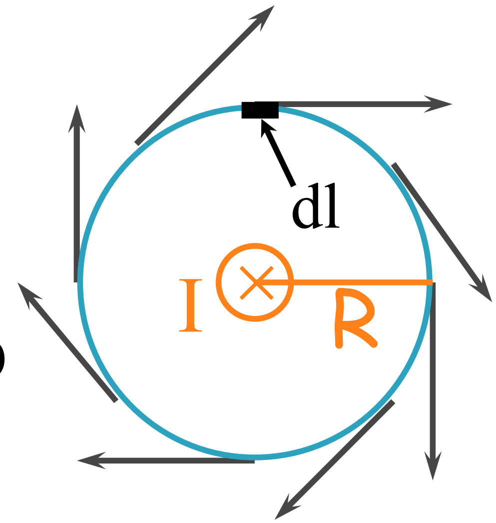
- Assignment 2: Prove Ampere's law !

APPLICATIONS OF AMPERE'S LAW

Magnetic Field of ∞ Straight Wire

Calculate field at distance R from wire using
Ampere's Law:

$$\oint B \bullet dl = \mu_0 I$$

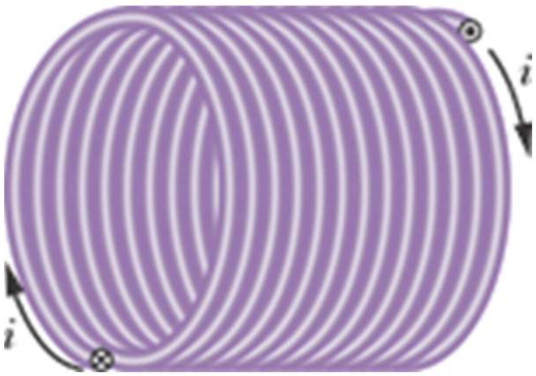


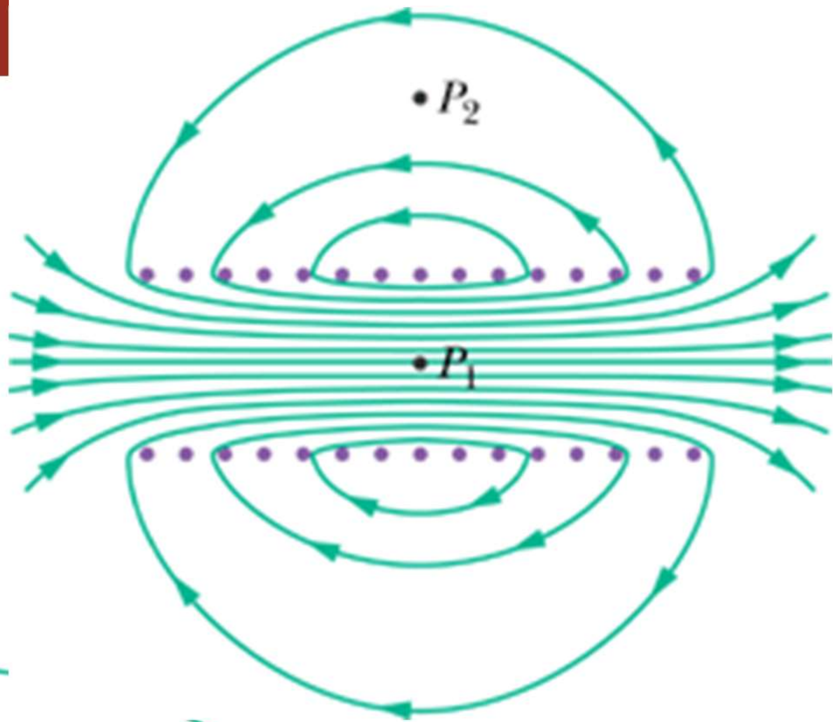
- Choose loop to be circle of radius R centered on the wire in a plane ⊥ to wire.

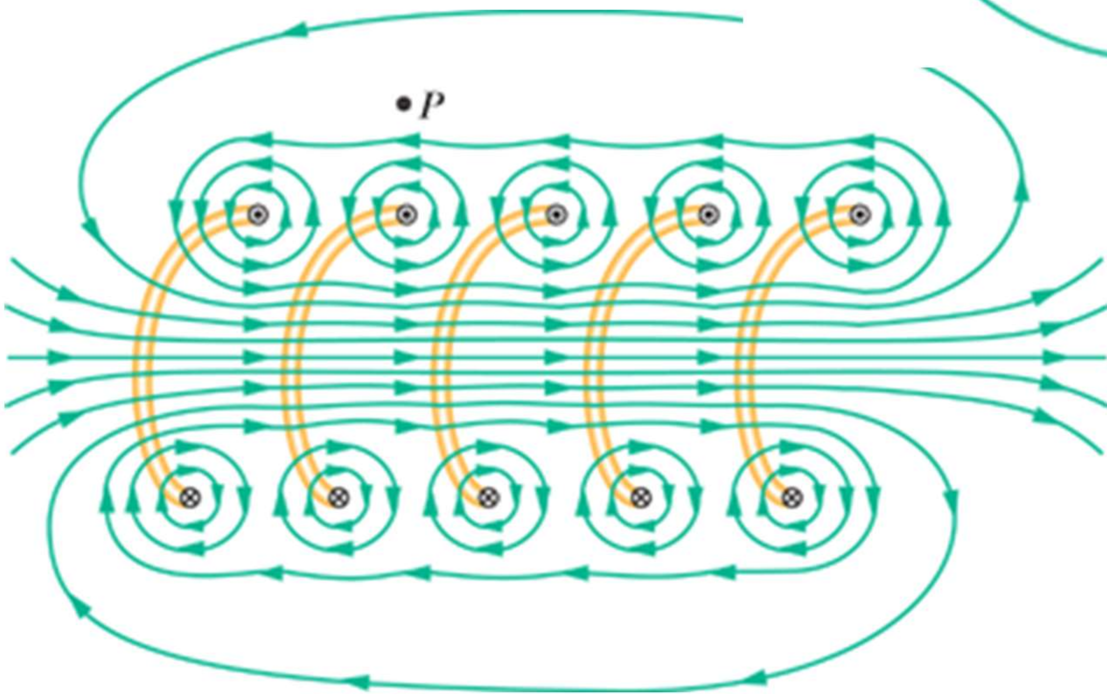
  - Evaluate line integral in Ampere's Law: $\oint \vec{B} \bullet d\vec{l} = B(2\pi R)$
  - Current enclosed by path = I
  - Apply Ampere's Law: $2\pi RB = \mu_0 I \Rightarrow$ $\boxed{B = \dfrac{\mu_0 I}{2\pi R}}$

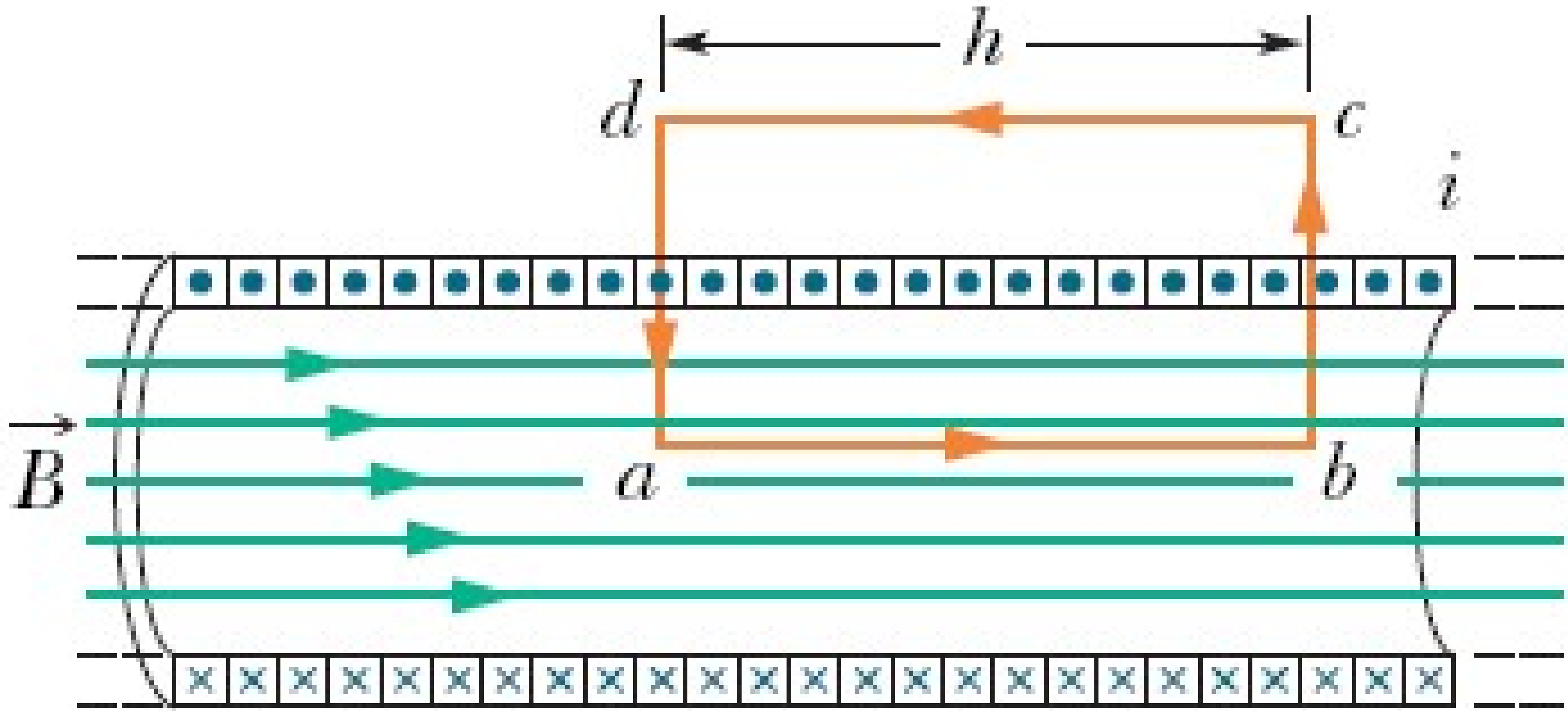
    - Ampere's Law simplifies the calculation thanks to symmetry of the current! (axial/cylindrical)

Magnetic Field of a Solenoid









h
d
c
i
$\vec{B}$
a
b

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}},$$

$$\oint \vec{B} \cdot d\vec{s} = \int_a^b \vec{B} \cdot d\vec{s} + \int_b^c \vec{B} \cdot d\vec{s} + \int_c^d \vec{B} \cdot d\vec{s} + \int_d^a \vec{B} \cdot d\vec{s}.$$

$$i_{\text{enc}} = i(nh).$$

Here n be the number of turns per unit length of the solenoid

$$Bh = \mu_0 inh$$

$$B = \mu_0 in \qquad \text{(ideal solenoid).}$$

Faraday's Law of Electromagnetic Induction

It was Michael Faraday who was able to link the induced current with a **changing** magnetic flux.

He stated that

The induced emf in a closed loop equals the negative time rate of change of the magnetic flux through the loop

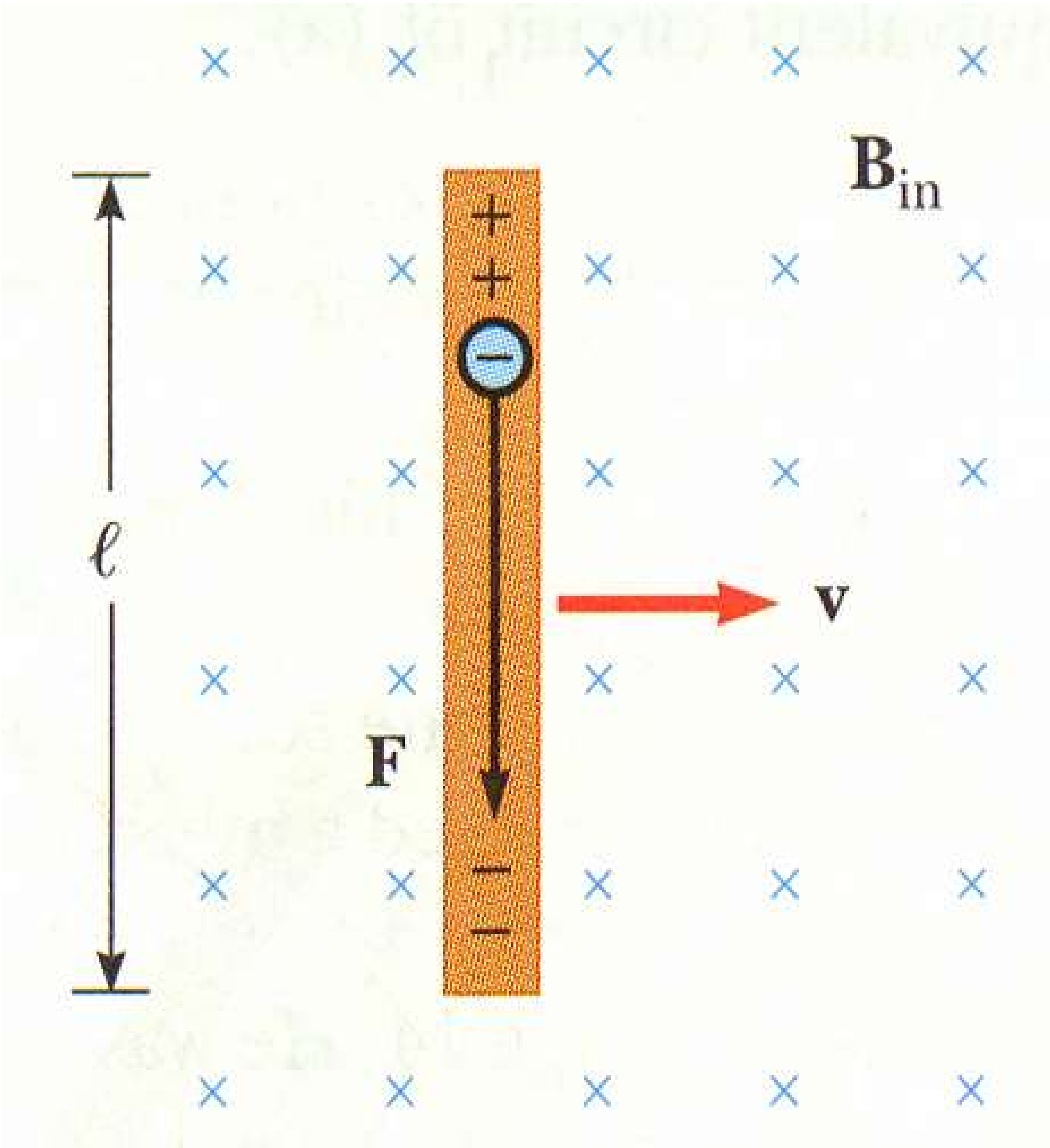
$$\varepsilon = -\frac{d\Phi_B}{dt}$$

The induced emf *opposes* the change that is occurring

LET'S UNDERSTAND FARADAY'S LAW
Basic Terminology

- **Electromotive Force ($\mathcal{E}$,E, V)**
  - known as **emf**, potential difference, or voltage
  - unit is volt [V]
  - "force" which causes electrons to move from one location to another
  - operates like a pump that moves charges (predominantly electrons) through "pressure" (= voltage)

Faraday's Experiment: Motional emf



Apply the Lorentz Force equation:

$$F = qE - qvB = 0$$

$$qE = qvB$$

$$E = vB$$

$$E\ell = \varepsilon = vB\ell$$

$$\boxed{\varepsilon = B\ell v}$$

Faraday's Experiment

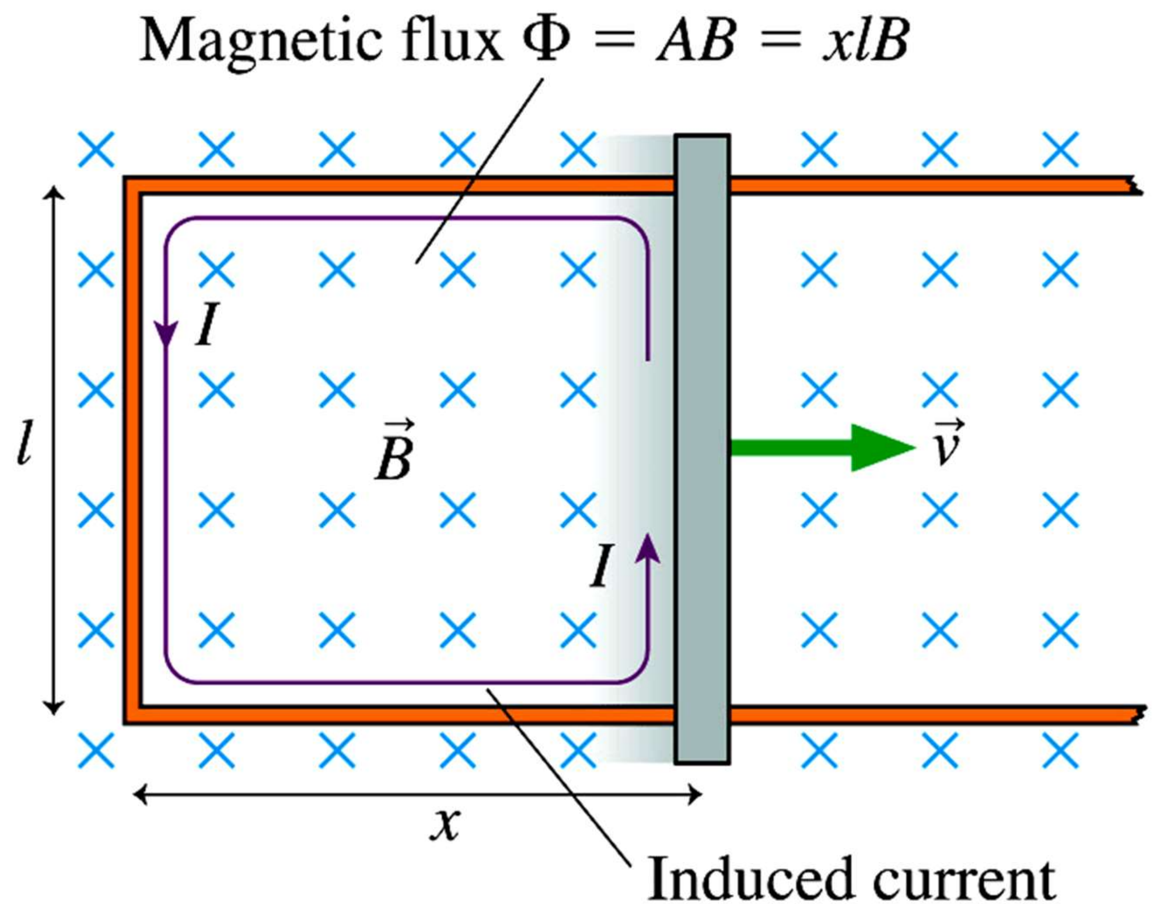
Consider the loop shown:

$$\frac{d\Phi_m}{dt} = \frac{d}{dt} Blx = Bl\frac{dx}{dt}$$

And from last slide …

$$\mathrm{E} = Blv = Bl\frac{dx}{dt}$$

$$\boxed{\text{Therefore, } \mathrm{E} = \left|\frac{d\Phi_m}{dt}\right|}$$



<u>CONCLUSION</u>: to produce emf one should make ANY change in a magnetic flux with time!

Polarity of the Induced Emf

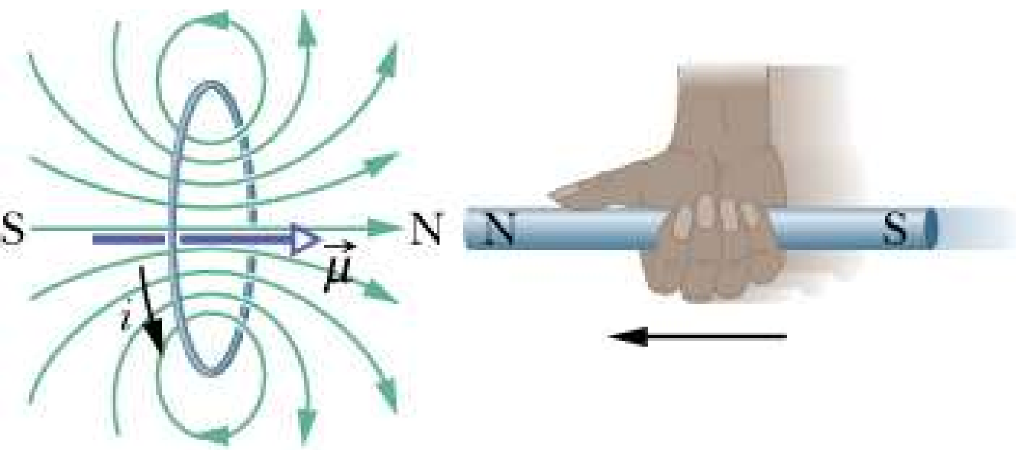
The polarity (direction) of the induced emf is determined by **Lenz's law**.

Including Lenz's law, the Complete Faraday's law becomes:

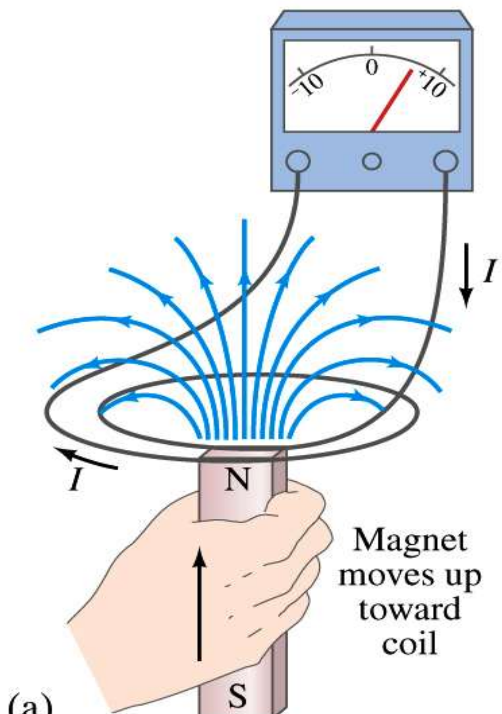
$$\varepsilon = -\frac{d\Phi_B}{dt}$$

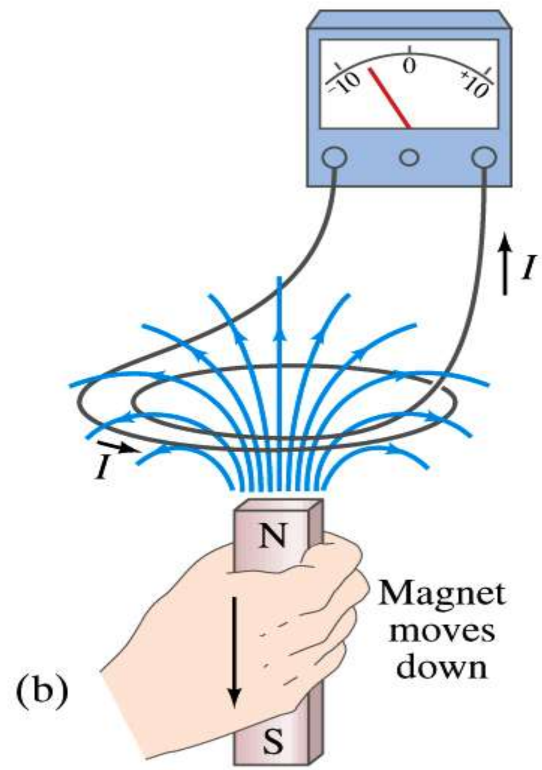
Let's talk about Lenz's law:

LENZ'S Law

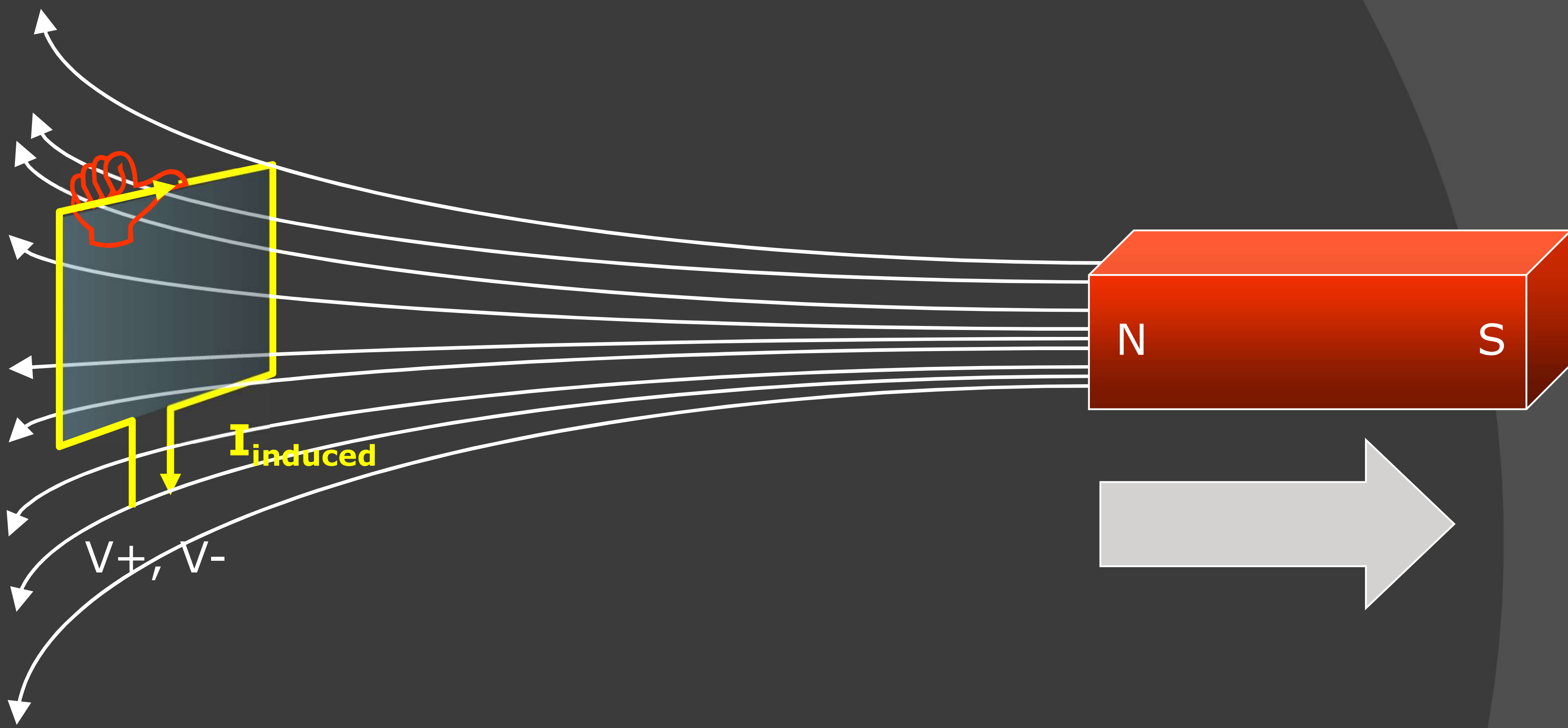


The direction of the emf induced by changing flux will produce a current that generates a magnetic field opposing the flux change that produced it.



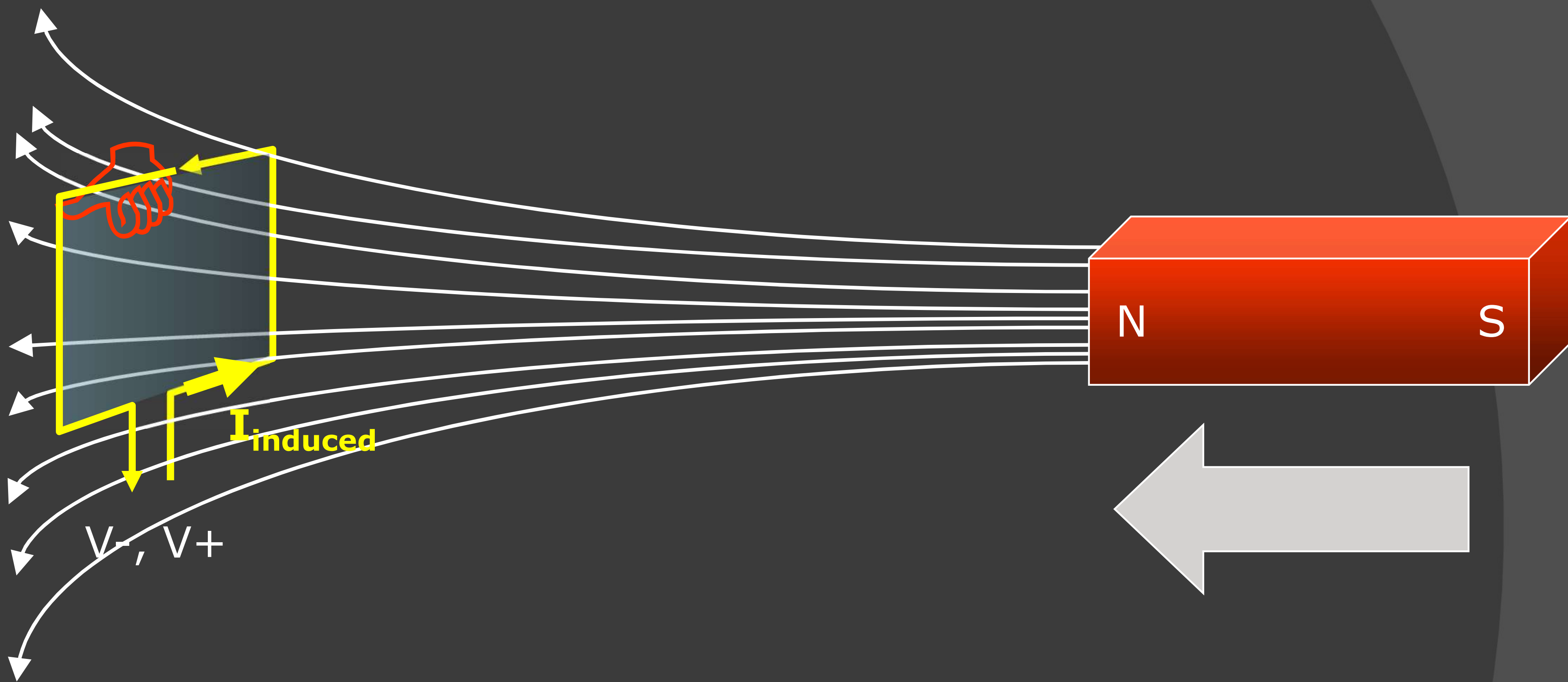


Another view of Lenz's Law



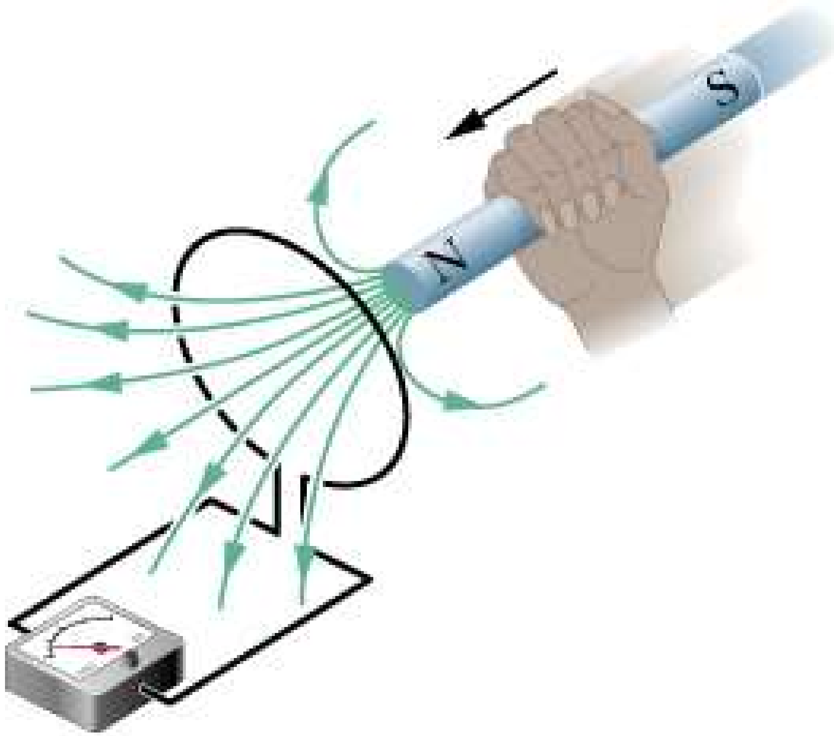
Lenz's Law: emf appears and current flows that creates a magnetic field that opposes the change – in this case an decrease – hence the negative sign in Faraday's Law.

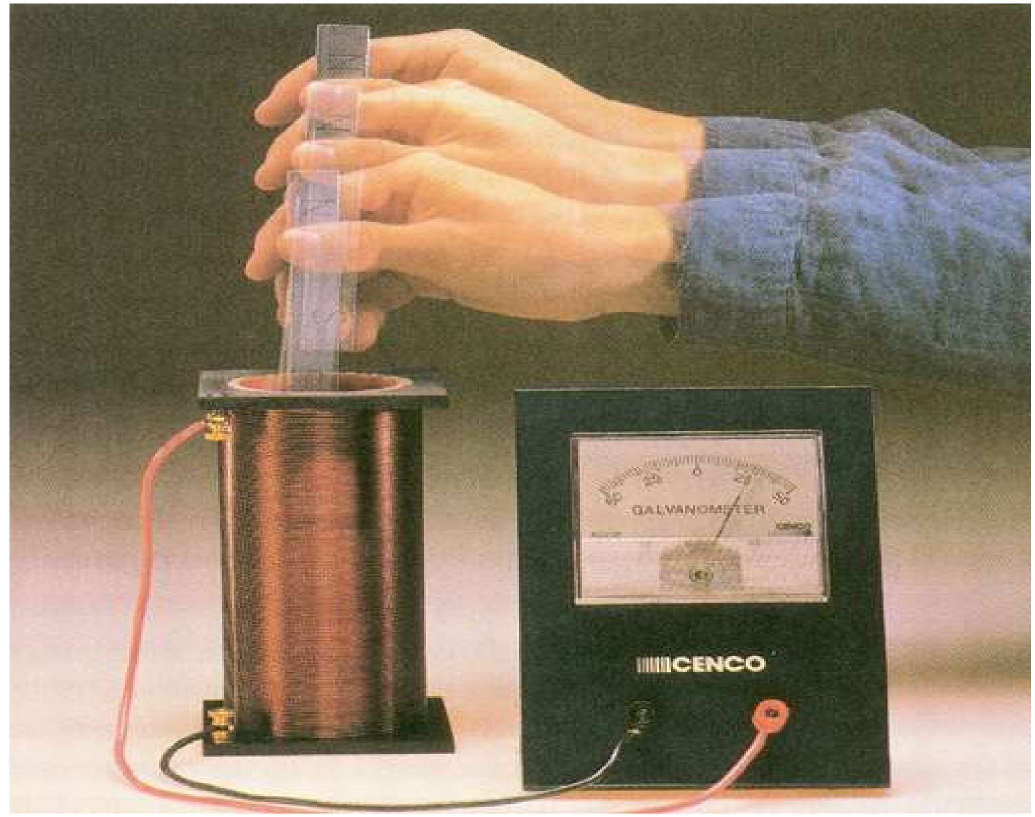
Lenz's Law



Lenz's Law: **emf appears and current flows that creates a magnetic field that opposes the change – in this case an increase – hence the negative sign in Faraday's Law.**

Faraday's Law for a Single/ N Loops





$$E = \varepsilon = -\frac{d\Phi}{dt}$$

$$E = \varepsilon = -N\frac{d\Phi}{dt}$$

More about Lenz's Law

Claim: Direction of induced current must be so as to oppose the change; otherwise conservation of energy would be violated.

- ***Why???***
  - If current reinforced the change, then the change would get bigger and that would in turn induce a larger current which would increase the change, etc..

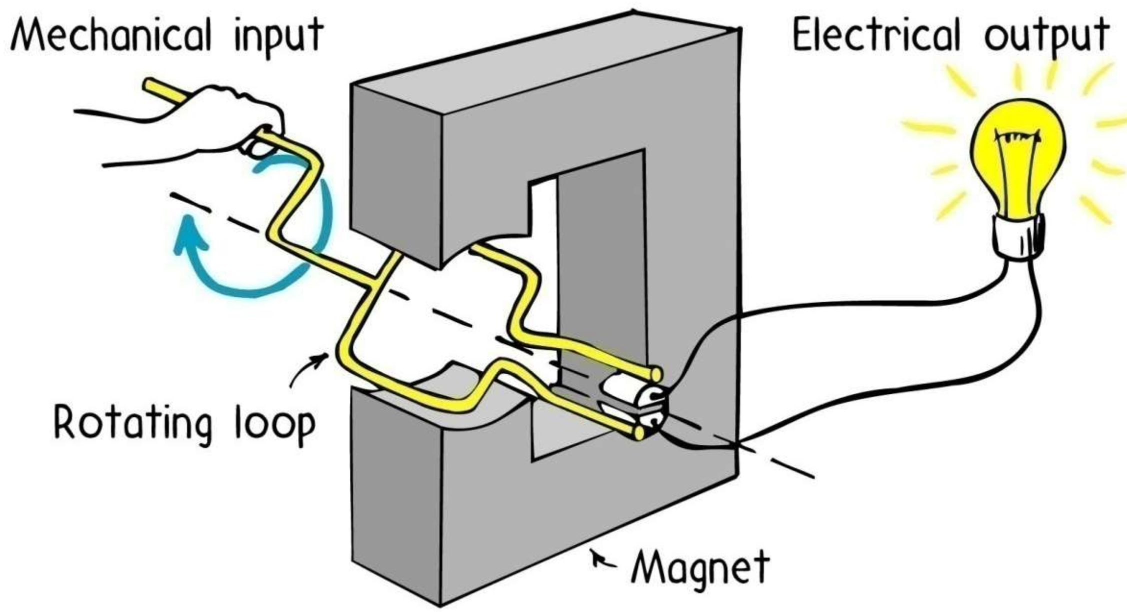
– ***No perpetual motion machine!***

<u>Conclusion:</u> Lenz's law results from energy conservation principle.

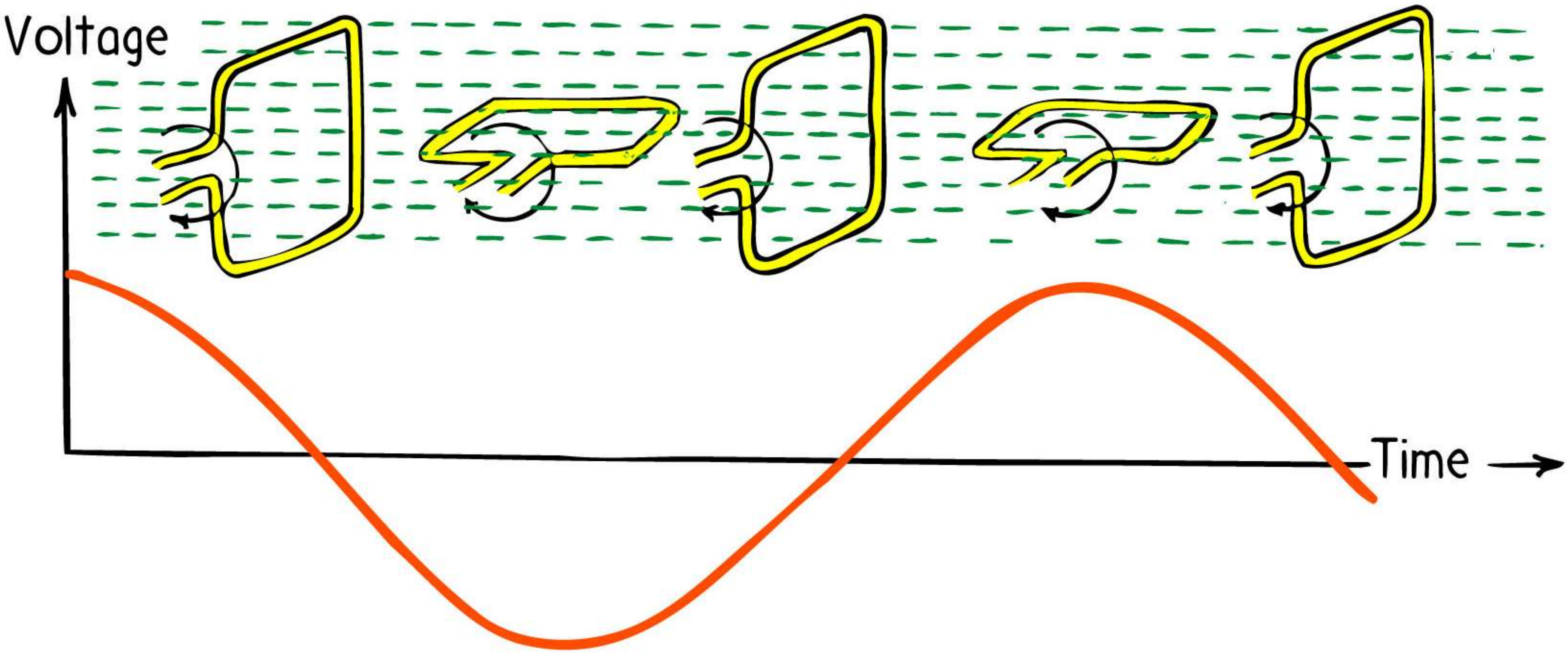
Applications of Magnetic Induction

- **AC Generator**

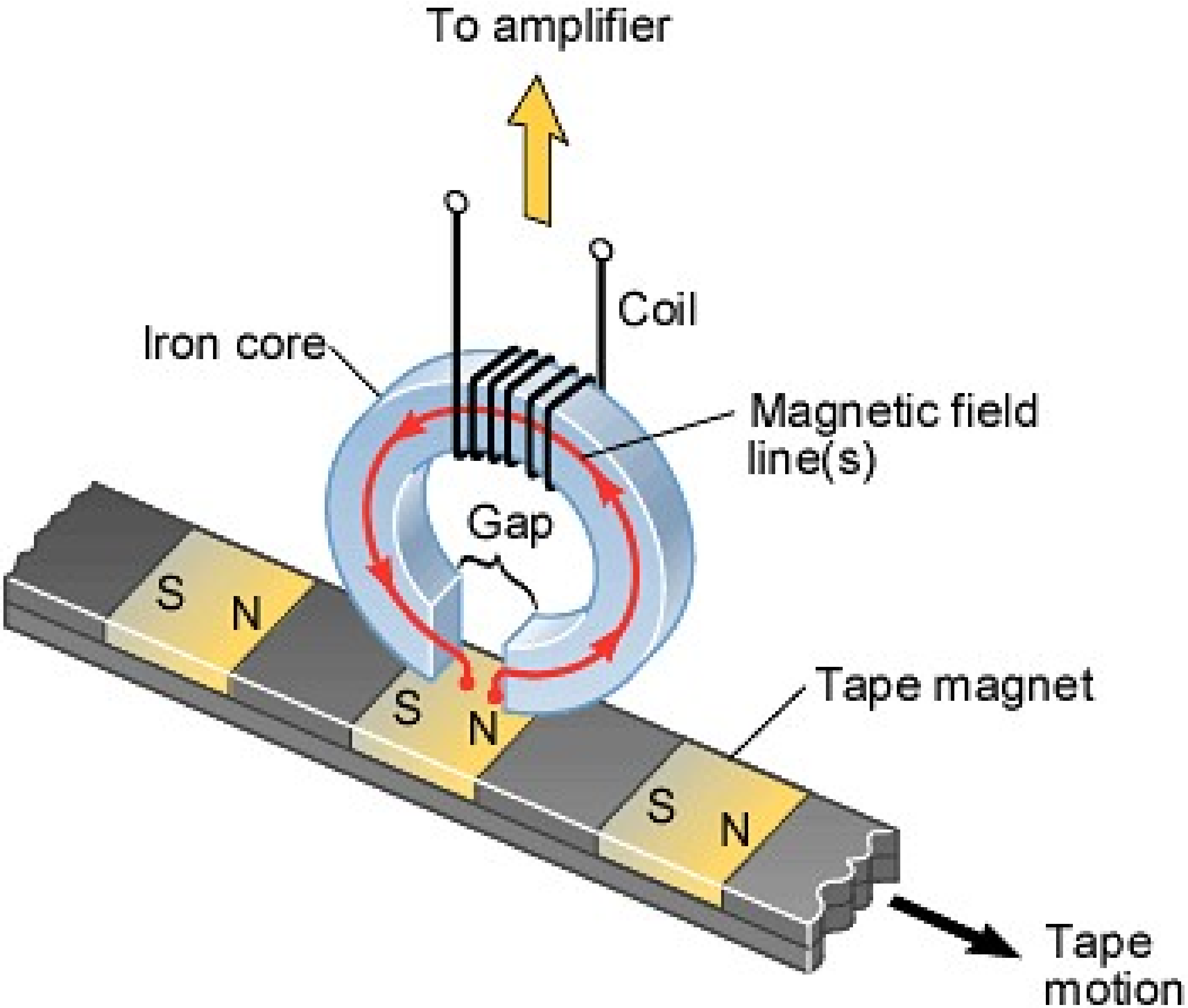
Water turns wheel
→ rotates magnet
→ changes flux
→ induces emf
→ drives current



Single-Phase Generator



The Magnetic Playback Head of a Tape Deck



- Tape / Hard Drive etc
  - Tiny coil responds to change in flux as the magnetic domains go by (encoding 0's or 1's).

  - Credit Card Reader
  - Must swipe card
    → generates changing flux
  - Faster swipe → bigger signal

INTRODUCTION OF MAXWELL'S EQUATIONS:

INTRODUCTION

In 1864 James Clerk Maxwell brought together and extended four basic laws in electromagnetism such as, Gauss's law in electrostatics, Gauss's law in magnetism, Ampere's law and Faraday's law. In fact entire theory of the electromagnetic field is condensed into these four laws. These laws govern the interaction of bodies which are magnetic or electrically charged or both.

Like Newton's laws in Mechanics, Maxwell's equations are the backbone of electrodynamics. Classical electromagnetic theory constituted by Maxwell's equations predicts that accelarated charges produce electromagnetic waves which propagates in space with a velocity equal to the velocity of light. It also showed that light waves are electromagnetic in character. In 1888 Heinrich Hertz produced and detected these waves.

3

Ampere's Circuital Law :→

"The line integral of magnetic field intensity $\vec{H}$ about any closed path is exactly equal to the net current enclosed by that path. ie.

$$\oint \vec{H}\cdot d\vec{l} = I \quad \left(= \int_s \vec{J}\cdot d\vec{s}\right); \text{ when current density is not uniform}$$

also represented as: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$

Displacement Current :→ (Read Resnick)

The concept of displacement current was first introduced by Maxwell's purely on theoretical ground. Maxwell postulated that it is not only the current in a conductor that produces a magnetic field, but a changing electric field in vacuum, or in a dielectric also produces a magnetic field. It means that a changing electric field is equivalent to a current which flows as long as the electric field is changing. This equivalent current produces the same magnetic effect as an ordinary current in a conductor. This equivalent current is known as "displacement current".

So $\oint \vec{B}\cdot d\vec{l} = \mu_0 (i + i_d)$

DERIVATION OF MAXWELL'S EQUATION IN DIFFERENTIAL FORM:

① First eqn.

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \text{(Differential form)}$$

Derivation
~~Proof~~: Gauss's law in electrostatics

$$\int_s \vec{E} \cdot ds = \frac{1}{\varepsilon_0} \cdot \int_v \rho \cdot dv \Rightarrow \int_s \vec{D} \cdot ds = \int_v \rho \cdot dv$$

Using Gauss divergence theorem:

$$\int_s \vec{D} \cdot ds = \int_v \vec{div} \cdot \vec{D} \, dv = \int_s \rho \cdot dv$$

$$\Rightarrow \int_v (\nabla \cdot \vec{D} - \rho) \, dv = 0 \quad \text{or} \quad \nabla \cdot \vec{D} - \rho = 0$$

$$\Rightarrow \boxed{\nabla \vec{D} = \rho}$$

Second eqn.

$$\nabla \cdot \vec{B} = 0$$

Since isolated magnetic poles have no physical significance Therefore the net magnetic flux of magnetic induction $\vec{B}$ through any closed Gaussian Surface is always zero.

i.e. $\int_s \vec{B} \cdot d\vec{s} = 0 \; (= \phi_B)$

using gauss divergence theorem

$$\int_s \vec{B} \cdot d\vec{s} = \int_v \nabla \cdot \vec{B} \cdot dV = 0 \Rightarrow \boxed{\nabla \cdot \vec{B} = 0}$$

Third eqn.

According to Faraday's law of em induction, induced e.m.f. around a closed circuit is equal to the negative time-rate of change of magnetic flux linked with the circuit.

$$e = -\partial \phi / \partial t$$

$$\text{But } \phi = \int_s \vec{B} \cdot d\vec{s} \Rightarrow e = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$\text{Also } e = \int_c \vec{E} \cdot d\vec{l} \Rightarrow \int_c \vec{E} \cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

using Stoke's theorem

$$\int_c \vec{E} \cdot d\vec{l} = \int_s (\vec{\nabla} \times E) \cdot ds = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0}$$

Fourth eqn. From Ampere's circuital law

$\int_c \vec{H} \cdot d\vec{l} = i$ also $i = \int_s \vec{J} \cdot d\vec{s}$

$\Rightarrow \int_c \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot d\vec{s}$

Use stoke's theorem $\Rightarrow \int_s (\vec{\nabla} \times \vec{H}) ds = \int_s \vec{J} \cdot d\vec{s}$

$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J}$ ← (66)

(It can be shown that this is valid for only static charge and insufficient for time varying fields. So we have to add a correction factor for displacement current.)

Shown: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \nabla \cdot \vec{J}$ (Took divergence)

But div. of curl is zero $\Rightarrow \nabla \cdot \vec{J} = 0$

From continuity eqn. $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \frac{\partial \rho}{\partial t} = 0$

Hence (66) to be true, Charge should be static only.

Hence, to include the time varying fields, replace $\vec{J}$ by $\vec{J} + \vec{J}_d$. Hence (66) Should be

$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d$

$\Rightarrow \nabla \cdot (\vec{\nabla} \times \vec{H}) = \nabla \cdot (\vec{J} + \vec{J}_d) = 0$

or $\nabla \cdot \vec{J} = -\nabla \cdot J_d$

$\therefore$ from continuity eqn.: $\nabla \vec{J}_d = \frac{\partial \rho}{\partial t}$

$\Rightarrow \nabla \vec{J}_d = \frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = \nabla \cdot \left(\frac{\partial \vec{D}}{\partial t}\right)$ {from I eqn. of maxwell}

$\Rightarrow \vec{J}_d = \frac{\partial \vec{D}}{\partial t}$

$$\therefore \boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

Note:- In free space, $\rho = \vec{J} = 0$

So can convert equations in free space.

ELECTROMAGNETIC WAVES IN FREE SPACE:

In free space, $\rho = 0$, $\vec{J} = 0$, $\sigma = 0$, $\mu = 1$, $\varepsilon = 1$

$D = \varepsilon_0 E$, $B = \mu_0 H$.

Take curl of Maxwell's III eqn.

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

From maxwell's IV eqn.

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial}{\partial t}\left(\frac{\partial D}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

Note: Now, Maxwell's eqn. are $\nabla \cdot \vec{D} = 0$, $\nabla \cdot \vec{B} = 0$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}.$$

$$\Rightarrow \nabla . \nabla \vec{E} - \nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

From I eqn., $\nabla \cdot \vec{D} = 0$ or $\nabla \cdot E = 0$

$$\Rightarrow \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Similarily get $\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}$

wave equations for $\vec{E}$ and $\vec{H}$ in free space.

General wave eqn. is $\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$

Comparing $\frac{1}{v^2} = \mu_0 \varepsilon_0 \Rightarrow v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \sqrt{\frac{4\pi}{\mu_0 4\pi \varepsilon_0}}$

$$\Rightarrow v = \sqrt{\frac{4\pi \times 9 \times 10^9}{4\pi \times 10^{-7}}} = 3 \times 10^8 \text{ m/sec}.$$

Solution Now take eqn. for $\vec{E}$, it is now

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

The plane wave solu. of this maybe written in the well known form as

$$\vec{E}(r, t) = E_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where $\vec{E}_0$ is complex amplitude of electric fields. $\vec{k}$ is wave propagation vector: $\vec{k} = k \cdot \hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi\nu}{c}\hat{n} = \frac{\omega}{c}\hat{n}$

$\hat{n}$ is the unit vector in the direction of propagation of electromagnetic wave.

Dr. Munendra Singh

SHOWING THE TRANSVERSE NATURE ......

Now, take Maxwell's I eqn. $\nabla\cdot\vec{E}=0$ (16)

$$\nabla\cdot\vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Now $\vec{k}\cdot\vec{r} = (\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}x + \hat{j}y + \hat{k}z) = k_x x + k_y y + k_z z$

$$\therefore \nabla\cdot\vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(\hat{i}\cdot E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i(k_x x + k_y y + k_z z - \omega t)}\right]$$

$$= \frac{\partial}{\partial x}\left[E_{0x}\, e^{i(k_x x + k_y y + k_z z - \omega t)}\right] + \frac{\partial}{\partial y}\left[\quad\right] + \frac{\partial}{\partial z}\left[\quad\right]$$

$$= E_{0x}\cdot e^{i(k_x x + k_y y + k_z z - \omega t)}\cdot(ik_x) + \cdots + \cdots$$

$$= \left(E_{0x}\, ik_x + E_{0y}\, ik_y + E_{0z}\, ik_z\right) e^{i(k_x x + k_y y + k_z z) - i\omega t}$$

$$= i\left(E_{0x} k_x + E_{0y} k_y + E_{0z} k_z\right) e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

$$= i\,\vec{k}\cdot\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow \boxed{\nabla\cdot\vec{E} = i(\vec{k}\cdot\vec{E})} = 0$$

$$\Rightarrow i(\vec{k}\cdot\vec{E}) = 0$$

Similarily can find $\vec{k}\cdot\vec{H}=0$

$$\Rightarrow \vec{k}\cdot\vec{E} = 0$$

That means both $\vec{H}$ & $\vec{E}$ are perpendicular to direction of motion of wave.

Similarily later two Maxwell's eqn. with same solution can give $\vec{k}\times\vec{E} = \mu_0\omega\vec{H}$ and $\vec{k}\times\vec{H} = -\epsilon_0\omega\vec{E}$ which clearly show that $\vec{E}, \vec{H}, \vec{k}$ are mutually perpendicular and thus the transverse nature of light establishes.

NOTE: Treatment for the electromagnetic radiation in non conducting/ dielectric material will be similar to the free space except the values of permittivity and permeability will now have certain values for the particular medium.