**Университет ИТМО**
**Факультет ПИиКТ**

# МАТЕМАТИЧЕСКИЙ АНАЛИЗ

I СЕМЕСТР

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*Проект на GitHub*

осень 2023

# Содержание

# 1 2023-09-04

## 1.1 Intro

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The whole semester is splitted by 3 parts, each of them contains:

1. control test
2. theory test (on GeoLin)
3. hometask
4. colloquium

**Summary: 100 p.**

The course is linked to A. Boytsev's course.

## 1.2 Logical symbolic

**Определение 1.1.** A statement is a sentence that is either **true** or **false**.

- $\forall$ – for all
- $\exists$ – exists
- ! – single
- $\sqsupset$ – let

**Пример.** $\forall a \in \mathbb{N}\ \exists!\ b \in \mathbb{N} : a + b = 0$

$A \Rightarrow B, A \Leftarrow B, A \Leftrightarrow B$

$$A = "a \vdots 6" B = "a \vdots 3" A \implies B$$

$\wedge$ – conjunction, $\vee$ – dis junction.

$$A \iff B \vee C$$

**Лемма 1.1.** *$A$ is true $\iff$ $\neg$ $A$ is false*

## 1.3 Sets

**Определение 1.2.** A set is a group of an objects, with defined rule which can define, is object in a set or not

Парадокс Рассела (множество множеств, которое не содержит себя в качестве элемента)

$a \in A$

$b \notin A \iff \neg(b \in A)$

**Определение 1.3.** $A \subset B \iff \forall a \in A \implies a \in B$

**Определение 1.4.** $A = B \iff A \subset B \land B \subset A$

## 1.4 Operations between sets

1. Union – $A \cup B = \{x : x \in A \lor x \in B\}$
2. Intersection – $A \cap B = \{x : x \in A \land x \in B\}$

Let $A$ be a set of indexes. $\alpha \in A$. $\alpha \mapsto G_\alpha$. We want do define a union of $n$ sets.

**Определение 1.5.**

$$\bigcup_{\alpha \in A} G_\alpha = \{x : \exists \alpha : x \in G_\alpha\} = G_{A_1} \cup G_{A_2} \cup G_{A_3} \cup \dots \cup G_{A_n}$$

**Определение 1.6.**

$$\bigcap_{\alpha \in A} G_\alpha = \{x : \forall \alpha : x \in G_\alpha\} = G_{A_1} \cap G_{A_2} \cap G_{A_3} \cap \dots \cap G_{A_n}$$

**Определение 1.7.**

$$A \setminus B = \{x : x \in A \land x \notin B\}$$

We define $U$ is an universal set: $\forall x : x \in U$.

$U \setminus A = A^c$ – complement ion.

$A \times B$ – desert multiplication of sets. $A \times B = \{(x, y) : x \in A, y \in B\}$

**Лемма 1.2** (Свойства операций)**.** *$\forall A, B, C$:*

1. *$A \cup B = B \cup A$ (коммутативность)*
2. *$A \cap B = B \cap A$ (коммутативность)*
3. *$A \cup (B \cup C) = (A \cup B) \cup C$ (ассоциативность)*
4. *$A \cap (B \cap C) = (A \cap B) \cap C$ (ассоциативность)*

5. $A \cup A = A \cup \varnothing = A$
6. $A \cap \varnothing = \varnothing$
7. $A \cap A = A$
8. $A \cup A^c = U$
9. $A \cap A^c = \varnothing$
10. $(A^c)^c = A$

**Упражнение.** *Доказать верхние 10 свойств. Доказательство достаточно тривиально.*

**Замечание.** Доказательство следует из определения.

**Теорема 1.1** (великая теорема Ферма)**.** $\forall x,y,z \in \mathbb{Z} : x,y,z > 2 : x^n + y^n = z^n$ *не имеет решений.*

# 2 2023-09-10 (NAL)

(not a lecture)

## 2.1 De Morgan laws

**Утверждение 2.1** (De Morgan laws)**.**

$$A \setminus \bigcup_{i \in I} X_i = \bigcap_{i \in I} (A \setminus X_i)$$

$$A \setminus \bigcap_{i \in I} X_i = \bigcup_{i \in I} (A \setminus X_i)$$

*Доказательство.* Let's proof the first formula. Using the definition:

$$\begin{aligned} A \setminus \bigcup_{i \in I} X_\alpha &= A \setminus \ \{x \in U : \exists i \in I : x \in X_i\} \\ &= \{x : x \in A \ \& \ \forall i \in I : x \notin X_i\} \\ &= \{x : \forall i \in I : x \in A \ \& \ x \notin X_i\} \\ &= \bigcap_{i \in I} (A \setminus X_i) \end{aligned}$$

Similarly, proving the second formula, but with a little bit different approach:

$$\begin{aligned} A \setminus \bigcap_{i\in I} X_i &= A \setminus \{x \in U : X_1 \cap X_2 \cap \dots \cap X_n\} \\ &= \{x \in U : x \in A \land x \notin (X_1 \cap X_2 \cap \dots \cap X_n)\} \\ &= \{\} \end{aligned}$$

It's enough for $x$ to not be in any of $X_i$ (this statement is trivial). Then, the set is: $\{x \in U : \exists i \in I : x \in A \land$

This is equal to:

$$\bigcup_{i\in I}(A \setminus X_i)$$

□

## 2.2 Distribution laws

**Утверждение 2.2** (Distribution)**.**

$$Y \cap \bigcup_{i\in I} X_i = \bigcup_{i\in I}(Y \cap X_i)$$

$$Y \cup \bigcap_{i\in I} X_i = \bigcap_{i\in I}(Y \cup X_i)$$

*Доказательство.* Proving the first law:

$$\begin{aligned} Y \cap \bigcup_{i\in I} X_i &= \{x \in U : x \in Y \land x \in (X_1 \cup X_2 \cup \dots \cup X_N)\} \\ &= \{x \in U : x \in Y \land \exists i \in I : x \in X_i\} \\ &= \{x \in U : \exists i \in I : x \in Y \cap X_i\} \\ &= \bigcup_{i\in I}(Y \cap X_i) \end{aligned}$$

Similarly, proving the second law:

$$\begin{aligned} Y \cup \bigcap_{i\in I} X_i = \bigcap_{i\in I}(Y \cup X_i) \\ &= \{x \in U : x \in Y \lor \forall i \in I : x \in X_i\} \\ &= \{x \in U : \forall i \in I : x \in (Y \cup X_i)\} \\ &= \bigcap_{i\in I}(Y \cup X_i) \end{aligned}$$

□

## 2.3 Injection, surjection and biection

**Определение 2.1** (mapping)**.** A mapping is a rule $f : \forall x \in X \exists! y \in Y : f(x) = y$.

**Определение 2.2** (injection)**.** A mapping $f : X \mapsto Y$ is called **an injection**, if $\forall x_1, x_2 \in X : x_1 \neq x_2 \wedge f(x_1) \neq f(x_2)$

**Определение 2.3** (surjection)**.** A mapping $f : X \mapsto Y$ is called **a surjection**, if $\forall y \in Y : \exists x \in X : f(x) = y$

**Определение 2.4** (biection)**.** We call $f$ a biection if $f$ is both an injection and a surjection.

## 2.4 Properties of images and prototypes

We define $A$,$B \in X$, $\quad A', B' \in Y$.

**Определение 2.5** (an image)**.** $f^{-1}(Y) = \{x \in X : f(x) \in Y\}$

1. $A \subset B \Rightarrow f(A) \subset f(B)$. It's obvious.
2. $f(A \cup B) = f(A) \cup f(B)$.

   *Доказательство.* Let $y \in f(A \cup B) \Rightarrow \exists x \in A \cup B : f(x) = y \Rightarrow x \in A \vee x \in B \Rightarrow f(x) \in f(A) \vee f(x) \in f(B) \Rightarrow f(x) \in f(A) \cup f(B)$. $\square$

3. $f(A \cap B) = f(A) \cap f(B)$.

   *Доказательство.* Let $y \in f(A \cap B) \Rightarrow \exists x \in A \cap B : f(x) = y \Rightarrow f(x) \in f(A) \wedge f(x) \in f(B) \Rightarrow y \in A \wedge y \in B \Rightarrow f(x) \in A \wedge f(x) \in B \Rightarrow f(A \cap B) = f(A) \cap f(B)$ $\square$

4. $A' \subset B' \Rightarrow f^{-1}(A') \subset f^{-1}(B')$. Obviously, true.
5. $f^{-1}(A' \cup B') = f^{-1}(A') \cup f^{-1}(B')$.

   *Доказательство.* Let $x \in f^{-1}(A' \cup B') \Rightarrow y \in A' \vee y \in B' \Rightarrow x \in f^{-1}(A') \vee x \in f^{-1}(B') \Rightarrow f^{-1}(A' \cup B') \in f^{-1}(A) \cup f^{-1}(B)$ $\square$

6. $f^{-1}(A' \cap B') = f^{-1}(A') \cap f^{-1}(B')$

Let $f : X \mapsto Y$ be a biection. Then:

**Определение 2.6** (reverse map)**.** $f^{-1} : Y \mapsto X$ is called **reverse map** if $\forall y \in Y \exists! x \in X : f^{-1}(y) = x$

### 2.5 Superposition of mapping

**Теорема 2.1** (associativity)**.** $f \circ (g \circ h) = (f \circ g) \circ h$

*Доказательство.* Left side: $f \circ g(h) = f(g(h))$. Right side: $f(g) \circ h = f(g(h))$ $\square$

## 3 2023-09-11

$\triangleleft$ talked about mappings (and will be in the 1st semester)

### 3.1 Defining $\mathbb{R}$

Мы выбираем *аксиоматический* подход.

**Определение 3.1** ($\mathbb{R}$)**.** We call a set an $\mathbb{R}$ if:

- $\triangleright$ Addition

  def $"+" : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is satisfied:

  1. (commutativity): $a + b = b + a$
  2. (associativity): $a + (b + c) = (a + b) + c$
  3. $\exists 0 : \forall a + 0 = a$. We call 0 a **neutral** element.
  4. $\forall a \in \mathbb{R} : \exists(-a) : a + (-a) = 0$

- $\triangleright$ Multiplication

  def $" \cdot " : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is satisfied:

  1. (commutativity): $a \cdot b = b \cdot a$
  2. (associativity): $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
  3. $\exists 1 \neq 0 : \forall a \in A : a \cdot 1 = a$
  4. $\forall a \in A : \exists a^{-1} \in A : a \cdot a^{-1} = 1$

- $\triangleright$ (distributivity): $\forall a,b,c \in \mathbb{R} : a \cdot (b + c) = a \cdot b + a \cdot c \;\&\; (a + b) \cdot c = a \cdot c + b \cdot c$

- $\triangleright$ (axioms of order) $\forall a,b \in \mathbb{R}$ mapping of order $\leqslant$ set if:

  1. $x \leqslant x$
  2. $(x \leqslant y \wedge y \leqslant x) \Rightarrow x = y$
  3. (transitivity) $x \leqslant y \wedge y \leqslant z \Rightarrow x \leqslant z$
  4. $\forall x,y \in \mathbb{R} : x \leqslant y \vee y \leqslant x$

- $\triangleright$ (Connection between $\leqslant, +$) $\forall x,y,z \in \mathbb{R} : x \leqslant y \Rightarrow x + z \leqslant y + z$ (this is not implied by previous conditions)

- (Connection betwen $\cdot$ and $\leqslant$): $0 \leqslant x \wedge 0 \leqslant y \Rightarrow 0 \leqslant x \cdot y$
- (Axiom of continuity (completeness)): Let $X, Y \subset \mathbb{R} : \forall x \in X : \forall y \in Y : x \leqslant y$. Then $\exists c \in \mathbb{R} : x \leqslant c \leqslant y$

  **Пример** (This axiom doesn't work on $\mathbb{Q}$)**.** Let $X = \{x \in \mathbb{Q} : x \cdot x \leqslant 2\}, Y = \{y \in \mathbb{Q} : y \cdot y \geqslant 2\}$. Then, $\exists! a \notin \mathbb{Q} \quad (a = \sqrt{2})$ : satisfies this axiom.

**Замечание.** Definition of $\mathbb{R}$ just contains the conditions that satisfy the **field**.

## 3.2 Corrolaries

**Следствие** (Corrolaries on Axioms 1–3)**.**

1. $\exists! 0, \exists! 1$.

   *Доказательство для 0.* Let there be $0_1, 0_2$. Then:

   $$0_1 = 0_1 + 0_2 = 0_2$$

   □

2. $\exists! (-x) \forall x$
3. $\forall x \neq 0 \exists! x^{-1}$

   *Доказательство.* Let there be $-x_1$ and $-x_2$. Then:

   $$(-x_1) = (-x_1) + (x + (-x_2)) = (x + (-x_1)) + (-x_2) = (-x_2)$$

   □

4. $\forall a,b \in \mathbb{R}$ an equality $x + a = b$ is set. Then there is only one solution $x = b + (-a)$.
5. $x \cdot a = b (a,b \in \mathbb{R})$. Then, $\exists! x = b \cdot a^{-1}$
6. $\forall x : x \cdot 0 = 0$

   *Доказательство.* $x \cdot 0 = x \cdot (0 + 0) = 0 \cdot x + 0 \cdot x = 0 \Rightarrow 0 = x \cdot 0$ □

7. $x \cdot y = 0 \Leftrightarrow x = 0 \vee y = 0$

   *Доказательство.* $\Leftarrow$ is proven.
   $\Rightarrow$: $x \neq 0 \Rightarrow \exists x^{-1} : x \cdot y \cdot x^{-1} = 0 \Rightarrow y = 0$. Proof for $y$ is similar. □

8. $-x = -1 \cdot x$

*Доказательство.* $-1 \cdot x + x = -1 \cdot x + 1 \cdot x = x(-1+1) = x \cdot 0 = 0$ □

9. $-1 \cdot (-x) = x$. Proof is trivial based on previous)
10. $(-x) \cdot (-x) = x \cdot x$. Proof is also trivial.

**Определение 3.2.**

$$x \leqslant y \Leftrightarrow \geqslant x$$

$$x < y \Leftrightarrow x \leqslant y \wedge x \neq y$$

$$x > y \Leftrightarrow y \geqslant x \wedge y \neq x$$

**Следствие** (Corrolaries on axioms 4 – 6)**.**

1. $\forall x, y \in \mathbb{R}$: the only one statement is true
   - $x < y$
   - $x = y$
   - $x > y$
2. $x < y \wedge y \leqslant z \Rightarrow x < z$
3. ...
4. $x > 0 \Leftrightarrow -x < 0$. The proof is obvious.
5. $x < 0 \wedge y < 0 \Rightarrow xy > 0$
6. Can add to strict inequality.
7. $x \leqslant y \wedge z \leqslant w \Rightarrow x + z \leqslant y + w$
8. $0 < x \wedge 0 < y \Rightarrow 0 < xy$
9. $0 < x \wedge y < z \Rightarrow xz < yz$
10. $1 > 0$

*Доказательство.* Let $1 \leqslant 0 \Rightarrow 1 < 0 \Rightarrow 1 \cdot 1 > 0!?$. Then, $1 > 0$. □

# 4 2023-09-15

## 4.1 Expanding $\mathbb{R}$

**Определение 4.1** ($\overline{\mathbb{R}}$)**.** $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

*Свойство* 4.1.1. $\forall x \in \mathbb{R}$:

- $x + (+\infty) = +\infty := x + \infty$
- $x + (-\infty) = -\infty := x - \infty$
- $x \cdot (\pm\infty) = \begin{cases} \pm\infty, & \text{if } x > 0 \\ \mp\infty, & \text{if } x < 0 \\ undefined, & \text{if } x = 0 \end{cases}$
- $\frac{x}{\pm\infty} = 0$
- $\frac{\pm\infty}{x} = \begin{cases} \pm\infty, & \text{if } x > 0 \\ \mp\infty, & \text{if } x < 0 \end{cases}$

$(+\infty) + (+\infty) = +\infty$
$(-\infty) + (-\infty) = -\infty$
$(+\infty) \cdot (+\infty) = (-\infty) \cdot (-\infty) = +\infty$
$(+\infty) \cdot (-\infty) = (-\infty) \cdot (+\infty) = -\infty$

$\forall x : -\infty < x < +\infty$

Actions undefined in $\mathbb{R}$:

- $0 \cdot (\pm\infty)$
- $(+\infty) + (-\infty)$
- $1^\infty$
- $\frac{\pm\infty}{\pm\infty}$
- $\frac{0}{0}$
- $0^0$

## 4.2 Defining $\mathbb{N}$

**Определение 4.2** (Inductive set)**.** A set $X \subset \mathbb{R}$ is *inductive*, if $\forall x \in X : x + 1 \in X$

**Лемма 4.1.** *Let $X_1, X_2, \ldots, X_n$ be inductive sets. Then, $X_1 \cap X_2 \cap \ldots X_n$ is also inductive.*

*Доказательство.* Trivially proof the $x \mapsto x+1$ $\square$

**Определение 4.3.** $\mathbb{N}$ is an intersection of every inductive sets: $\forall i : 1 \in A_i$

**Замечание.** $\mathbb{N}$ is minimal inductive set, that contains 1.

**Теорема 4.1** (Math. induction principle). *Let $X \subset \mathbb{N}, 1 \in X, X$ is inductive. Then, $\mathbb{N} = X$*

**Упражнение.** *Proof that $\forall n > -1, n \in \mathbb{N}, x \in \mathbb{R} : (1+x)^n \geqslant 1 + nx$*

## 4.3 Properties of $n \in \mathbb{N}$

**Лемма 4.2.** $\forall a,b \in \mathbb{N} : a + b \in \mathbb{N}, ab \in \mathbb{N}$

**Замечание.** Proof using math. induction.

**Определение 4.4** ($\mathbb{Z}$). $\mathbb{Z} := \mathbb{N} \cup \{0\} \cup \{x : -x \in \mathbb{N}\}$

**Определение 4.5** ($\mathbb{Q}$). $\mathbb{Q} := \left\{\frac{m}{n} := m \cdot n^{-1}, m \in \mathbb{Z}, n \in \mathbb{N}\right\}$

**Теорема 4.2** (Existence of irrational number). *A set $\mathbb{R} \setminus \mathbb{Q} = \mathbb{I}$ is not empty.*

Let's proof that $\sqrt{2}$ is irrational.

*Доказательство.* Plan:

1. Prove that $\exists c \in \mathbb{R} : c^2 = 2$.
2. Prove that $c$ is irrational.

2. Let $c = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}$. Then $c^2 \cdot n^2 = m^2 \Rightarrow 2n^2 = m^2 !?$

1. Using axiom of continuity. Let $X = \{x \in \mathbb{R}_{x>0} : x^2 < 2\}, Y = \{y \in \mathbb{R}_{y>0} : y^2 > 2\}$. Then $x \leqslant y \Rightarrow \exists c \in \mathbb{R} : x \leqslant c \leqslant y \forall x \in X, y \in Y$

   Proving that $c \notin X$. Let $c \in X$, i.e. $c^2 < 2$. Consider $c + \frac{2-c^2}{3c} = c + \frac{\Delta}{3c} = \xi$
   $(c + \frac{\Delta}{3c})^2 = c^2 + \frac{2}{3}\Delta + \frac{\Delta \cdot \Delta}{9c^2} \leqslant c^2 + (\frac{2}{3} + \frac{1}{3})\Delta = 2 \Rightarrow \xi \in X,$ but $\xi > c!? \Rightarrow c \notin X$. Similarly, we proof for $Y$.

$\Rightarrow \exists c \in \mathbb{R} : c^2 = 2 \Rightarrow |\mathbb{I}| \neq 0$ $\square$

# 5 2023-09-18

## 5.1 Бином Ньютона

**Определение 5.1** (Binomial coefficients). $C_n^k = \frac{n!}{k!(n-k)!}, n \in \mathbb{N}, k \in \mathbb{N} \setminus \{0\}, k \leqslant n$

**Упражнение.** *Вывести.*

*Свойство* 5.1.1.

1. $C_n^0 = C_n^n = 1$ (trivial)
2. $C_n^1 = C_n^{n-1} = n$
3. $C_n^k = C_n^{n-k}$
4. $C_n^k + C_n^{k+1} = C_{n+1}^{k+1}$

   *Упражнение. Proof using Pascal's triangle (trivial).*

   *Доказательство.* $C_n^k + C_n^{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!}{k!(n-k-1)!} \cdot \left(\frac{1}{n-k} + \frac{1}{k+1}\right) = = \frac{(n+1)!}{(k+1)! \cdot (n-k)!} = C_{n+1}^{k+1}$ □

**Теорема 5.1** (Binomial theorem). $\forall a,b \in \mathbb{R}, \forall n \in \mathbb{N} : (a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$

*Доказательство.* Proving using induction.

- ▷ $n = 1 : (a+b)^1 = C_1^0 a^1 + C_1^1 b^1 = a + b$
- ▷ let $n = k : (a+b)^k = \sum_{m=0}^{k} C_k^m a^m b^{k-m}$
- ▷ Transition: $(a+b)^m \cdot (a+b) = (\sum_{m=0}^{k} C_k^m a^m b^{k-m}) \cdot (a+b) = C_m^0 a^{m+1} b^0 + \cdots + C_m^m a^1 b^m + C_m^0 a^m b^1 + \cdots + C_m^m a^0 b^{m+1} =^{\text{using property 1.4}} C_{m+1}^0 a^{m_1} b^0 + (C_m^1 + C_m^0) a^m b^1 + (C_m^2 + C_m^1) a^{m-1} b^2 + \cdots + (C_m^{k+1} + C_m^k)(a^{m-k} b^{k+1}) + \cdots + (C_m^m + C_m^{m-1}) a^1 b^m + C_{m+1}^{m+1} = \sum_{k=0}^{m+1} C_{m+1}^k a^k b^{m+1-k}$

□

## 5.2 Defining intervals on $\mathbb{R}$

**Определение 5.2.** отрезок: $[a,b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$

interval: $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

semi-interval: $(a,b], [a,b) = \{x \in \mathbb{R} : a < x \leqslant b\}, \{x \in \mathbb{R} : a \leqslant x < b\}$

луч: $(-\infty, a), (-\infty, a], [b, +\infty), (b, +\infty)$

**Определение 5.3** (Окрестность точки $x_0$)**.** $x_0 \in (a,b) = U(x_0)$ (including $(-\infty, a), (b, +\infty)$)

$\varepsilon$-neighbourhood: $(x_0 - \varepsilon, x_0 + \varepsilon) = U_\varepsilon(x_0)$

**Определение 5.4** ($\varepsilon$-neighbourhood for $\overline{\mathbb{R}}$)**.** ▷ $+\infty : (a; +\infty) = U(+\infty)$

▷ $-\infty : (-\infty, a) = U(-\infty)$

▷ $\infty = U(+\infty) \cup U(-\infty)$

▷ $U_\varepsilon(+\infty) = (\frac{1}{\varepsilon}; +\infty); U_\varepsilon(-\infty) = (-\infty, -\frac{1}{\varepsilon})$

## 5.3 Absolute value

**Определение 5.5.** $\forall x \in \mathbb{R} : |x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x \leqslant 0 \end{cases}$

*Свойство* 5.3.1.

1. $|x| = |-x|$
2. $|x|^2 = x^2$
3. $|x| \geqslant 0; |x| = 0 \Leftrightarrow x = 0$
4. $|xy| = |x||y|$
5. $\frac{|x|}{|y|} = \left|\frac{x}{y}\right|$
6. $-|x| \leqslant x \leqslant |x|$
7. $|x + y| \leqslant |x| + |y|$

   *Доказательство.* $(x+y)^2 \leqslant x^2 + y^2 + 2|xy| \Leftrightarrow 2xy \leqslant |2xy|$ □

8. $|x| \leqslant a \Leftrightarrow -a \leqslant x \leqslant a$
9. $|x| \geqslant b \Leftrightarrow x \leqslant -b \vee x \geqslant b$
10. $|x - y| \geqslant ||x| - |y||$

## 5.4 Bounds of the set in $\mathbb{R}$

Let $X \subset \mathbb{R}$

**Определение 5.6.** We say that $X$ is *bounded above*, if $\exists M \in \mathbb{R} : x \leqslant M \forall x \in X$ ($M$ is upper bound)

We say that $X$ is *bouded below*, if $\exists m \in \mathbb{R} : m \leqslant x, \forall x \in X$ ($m$ is lower bound)

We assume that $X$ is **bounded**, if its *bounded both below and above.*

**Пример.** Let $X = [0, 1)$

**Определение 5.7** (min and max element)**.** max. element: $x_{\max} = \max X : x_{\max} \in X, \forall x \in X : x \leqslant x_{\max}$

**Замечание.** $X$ doesn't have max. element.

*Доказательство.* Let $M = \max X$. Then, $\exists M_0 \in \frac{M+1}{2} > M$!? □

**Определение 5.8** (supremum and infremum of the set)**.** $S \in \mathbb{R}$ is called *an exact upper bound*, (or a *supremum of* $X$), if $S =$ lowest upper bound

We denote is as $\sup X = S = \min \{M : x \leqslant M \forall x \in X\}$

If $X$ is not bounded above, then $\sup X = +\infty$

$s \in \mathbb{R}$ is called *an exact lower bound*, (or an *infremum* of X), if $S =$ highest lower bound. We denote it as $s = \inf X = \max \{m : x \geqslant m, \forall x \in X\}$

**Замечание.** $X = \{x \in \mathbb{Q} : x^2 < 2\} \Rightarrow \sup X$ is undefined.

**Лемма 5.1.** *$X$ is bounded* $\Leftrightarrow \exists c \in \mathbb{R} : |x| \leqslant c, \forall x \in X$

**Упражнение.** *Proof.*

# 6 2023-09-22

## 6.1

Let $X \subset \mathbb{R}$

**Замечание.** If $X$ is not bounded, then $\sup X = +\infty$ & $\inf X = -\infty$. Let $i = [0; 1)$. We will proof that there is a supremum of $i$.

**Лемма 6.1.** *If* $\exists \max X$*, then* $\sup X = \max X$.

*If* $\exists \min X$*, then* $\inf X = \min X$.

***Implication only!***

*Доказательство.* ▷ Obviously, let $M = \max X$. Then $M$ is upper bound by definition of max. Let there be $M' < M$ such that $M'$ is an upper bound. Then it's not an upper bound by definition.

▷ Same story for $\min X$

□

**Лемма 6.2** (different definition of supremum and infremum)**.** $M = \sup X \Leftrightarrow M : \forall x \in X : x \leqslant M$ & $\forall \varepsilon > 0 \exists x \in X : x > M - \varepsilon$

$m = \inf X \Leftrightarrow m : \forall x \in X : x \geqslant m$ & $\forall \varepsilon > 0 \exists x \in X : x > m + \varepsilon$

*Доказательство.* By definition. □

**Теорема 6.1** (Exact bound principle). *$\forall X : X$ is upper bounded $\Rightarrow \exists \sup X$. Same for $\forall X : X$ is lower bounded $\Rightarrow \exists \inf X$*

*Доказательство.* If the set is upper bounded, then $\exists$ an upper bound. Let $B$ be a set of upper bounds: $B = \{M \in \mathbb{R} : x \leqslant M, x \in X\}$. Then $\forall M, x : x \leqslant M$. By continuity axiom, $\exists c \in \mathbb{R} : x \leqslant c \leqslant M \forall x \in X, M \in B$. Let's proof that $c$ is a supremum of $X$. $c$ is an upper bound of $X$ and it's lower that every other upper bounds in $M$. Then, $c = \sup X$ □

**Замечание.** Even if $X$ is not upper bounded. Then $\forall X \neq \varnothing \exists c \in \overline{\mathbb{R}} : c = \sup X$

## 6.2 Archimed's axiom

**Лемма 6.3.** *Let $X \subset \mathbb{N}, X \neq \varnothing, X$ is bounded. Then, the maximum exists.*

*Доказательство.* $\exists M = \sup X = k \in \mathbb{R}$ for $\varepsilon = 1$. Then $\exists x \in X : k - 1 < x \leqslant k$. Then $x \in \mathbb{N}$. Proving that $x = k$. $k < x + 1$ then $\forall y \in X : y \leqslant x$, because of $y \leqslant k < x + 1 \Rightarrow y < x + 1 \Rightarrow y \leqslant x \Rightarrow x = \max X$. □

**Следствие.**
1. $\mathbb{N}$ is not bounded above.
2. $\mathbb{Z}$ is bounded nor below and above.
3. $X \subset \mathbb{Z}$ if $X$ is bounded below then $\exists \min X$; if $X$ is bounded above the $\exists \max X$

**Теорема 6.2** (Archimed's axiom). *Let $x \in R, x > 0$. Then $\forall y \in \mathbb{R} \exists k \in \mathbb{Z} : (k-1)x \leqslant y \leqslant kx$.*

*Interpretation: we can fill a segment of length $y$ with a segments of length $x$.*

*Доказательство.* Consider $T = \left\{t \in \mathbb{Z} : \frac{y}{x} \leqslant t\right\}$. $T \neq \varnothing$ & is bounded below. Then $\exists k = \min T : \frac{y}{x} < k \Rightarrow y < kx \Rightarrow k-1 \leqslant \frac{y}{x}$, cuz if it's false then $k-1 \in T$ but $k-1 < k = \min T$!? □

**Следствие.**
1. $\forall \varepsilon > 0 \exists n \in \mathbb{N} : 0 < \frac{1}{n} < \varepsilon$

   *Доказательство.* $y = 1, x = \varepsilon \Rightarrow \exists n : 1 < n\varepsilon$ □

2. If $x \geqslant 0$ and $\forall \varepsilon > 0 x < \varepsilon \Rightarrow x = 0$

   *Доказательство.* $0 \leqslant x < \varepsilon$. Let there be $x > 0 \Rightarrow \varepsilon = \frac{x}{2}$ the statement is *false*. □

3. $\forall x \in \mathbb{R} \exists ! k = [x] \in \mathbb{Z} : k \leqslant x < k + 1$

   *Доказательство.* $x = 1, \varepsilon = x$ □

**Лемма 6.4** (density of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$). *Let there be $a < b$. Then on $(a, b) \exists q \in \mathbb{Q}, j \in \mathbb{I}$*

*Доказательство.* $\exists n \in \mathbb{N} : \frac{1}{n} < b - a, [na] \leqslant na < [na] + 1 \Rightarrow a < \frac{[na]+1}{n} = q \leqslant \frac{na+1}{n} < b$

$\sqrt{2} \in \mathbb{I}$. Consider $q \in (a - \sqrt{2}, b - \sqrt{2})$. Then, $q + \sqrt{2} \in (a, b)$. Из-за замкнутости $q + \sqrt{2} \in \mathbb{I}$. Let $q + \sqrt{2} \in \mathbb{Q}$. Then, $q + \sqrt{2} - q = \sqrt{2} \in \mathbb{Q}$!? □

### 6.3 Canthor's theorem about line segments

Let $I_n = [a_n, b_n], a_n \leqslant b_n$

We assume that $\dots \subset I_n \subset I_{n-1} \subset \dots \subset I_2 \subset I_1$

**Теорема 6.3.** *If there is this system of line segments, then:*

- ▷ $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$(*i.e.* $\exists c \in \bigcap I_n$
- ▷ *If* $\forall \varepsilon > 0 \exists I_n : b_n - a_n < \varepsilon$, *then* $\bigcap I_n = \{c\}$, *i.e.* $\exists! c$

*Доказательство.* ▷ $A = \{a_n\}, B = \{b_n\}, a_n \leqslant b_m \Rightarrow_{\text{continuity A.}} \exists c : a_n \leqslant c \leqslant b_n \Rightarrow c \in I_n \forall n \in \mathbb{N}$

- ▷ Let $c_1, c_2 \in I_n$ $\forall n$. Let $c_1 < c_2 \Rightarrow$ for $\varepsilon = c_2 - c_1 \exists (a_n b_n) : b_n - a_n < \varepsilon$!?

□

## 7 2023-09-25

### 7.1 Borrel-Lebeg lemma

Consider $G_\alpha$ as a sets.

$G_\alpha$ is forming a cover of a set $X$ if $X \subset \bigcup_\alpha G_\alpha$

**Лемма 7.1** (Borrel-Lebeg lemma (GBL lemma))**.** *From any coverage of a segment by interval we can choose a finite one.*

*Доказательство.* Consider $[a_0, b_0]$. Let there be no option to choose a finite coverage from $\bigcup_\alpha G_\alpha$. Split the segment by half and we get $[a_1, b_1]$ - a cover where we cant choose a finite covering. $\dots [a_2, b_2] \subset [a_1, b_1] \subset [a_0, b_0] . b_n - a_n = \frac{b_0 - a_0}{2^n} \underset{2^n > n}{<} \frac{b_0 - a_0}{n} < \frac{1}{\varepsilon} \Rightarrow \forall \varepsilon > 0 \exists n \Rightarrow \exists! c : c \in [a_i, b_i]$

$\exists G = (k, l) : c \in G \Rightarrow \exists n_0 : [a_{n_0}, b_{n_0}] \subset (k, l)$!?

**Замечание.** Basically proving Canthor's theorem and getting !? for this.

□

**Лемма 7.2** (limit point)**.**

**Определение 7.1.** A dot $x_0$ is called **limited** point of this set $E$ if $\forall \overset{\circ}{U}(x_0) \cap E \neq \varnothing$, i.e. $U(x_0) \cap E$ is infinite.

**Пример.** Consider $[0, 1)$ A set of limit points $E' = [0,1]$

**Пример.** Consider $E = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$ $E' = \{0\}$

**Определение 7.2.** If $x_0 \in E$ & $x_0$ is not a limit point, then $x_0$ is **isolated** point of $E$, i.e. $\exists U(x_0) : \overset{\circ}{U}(x_0) \cap E = \varnothing$.

*(Lemma) Let $E$ is an infinite and bounded $E \subset \mathbb{R}$. Then $\exists x_0 : x_0$ is limit point of $E$.*

*Доказательство.* $E$ is bounded $\Rightarrow \exists\,[a{,}b] : E \subset [a{,}b]$. Let there be no limit points in $[a{,}b]$, i.e. $\forall x \in [a{,}b]$ is not limit point for $E$ , i.e. there is a finite number of points in $E \in U(x)$

$\{U(x)\}$ is cover of a segment by intervals $\underset{BGL\ lemma}{\Rightarrow} \exists\,\{U(x_1), \ldots, U(x_n)\}$

$E \subset \bigcup_{i=1}^{n} U(x_i)$, but a subset contains a finite number of points from $E$ !? □

**Замечание.** We can select limit points in $\overline{\mathbb{R}}$

e.g. $\mathbb{N}' = \{+\infty\}$

## 7.2 closedness of sets

**Определение 7.3.** A set $E$ is close d (in $\mathbb{R}$ ) if it contains every it's limit point, i.e. $E' \subset E$.

An $\varnothing$ is closed *by definition.*

**Пример.** $E = [0, 1)$ isn't closed

$[0{,}1]$ is closed

$E = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\}$ is closed .

**Лемма 7.3.** *Let $E \subset \mathbb{R}$, $E$ is closed and bounded above (below). Then* $\exists \max E (\min E)$

*Доказательство.* By exact bound principle $\exists M = \sup E \in \mathbb{R}$. Proving that $M \in E$. Let $M \notin E$. Then, we consider any neighbourhood $(\alpha, \beta) \ni M$. For $\varepsilon_1 = M - \alpha > 0 : \exists x_1 \in E \cap (\alpha, M)$. For $\varepsilon_2 = M - x_1$, for $\varepsilon_3 = M - x_2, \cdots \Rightarrow$ there is infinite amount of dots in $E$, i.e. $M$ is a limit point of $E \Rightarrow M \in E$ □

**Следствие.** Any finite set has it's maximum and minimum. (it has no limit points).

**Следствие.** For any $(\alpha, \beta) \subset \mathbb{R}$: infinite number of numbers of $\mathbb{Q}, \mathbb{I}$

# 8 2023-09-29

## 8.1 cardinality of a set

**Определение 8.1.** Consider 2 sets $A, B$. We call them equal-powered, if a biection exists.

**Пример.** $\mathbb{N} \sim 2\mathbb{N}$

**Определение 8.2.** The class of equality that our element are in is called a cardinality of a set.

If a set is finite, then we assume it's cardinality as a number of elements.

A set is *countable* if a biection $f : A \mapsto \mathbb{N}$ exists. (i.e. we can enum a set)

### 8.1.1 Properties of countable sets

1. Any infinite set has a countable set inside.
2. Any infinite subset of countable set is countable.

   *Доказательство.* Consider $\mathbb{N}, A \subset \mathbb{N}, A$ is finite . Then A is bounded below $\Rightarrow \exists \min A = a_1$. A set is bounded and closed $\Rightarrow \exists \min(A \setminus \{a_1\}) = a_2$, and so on...

   Let $a \in A, \{x \in A : x < a\}$ is finite □

3. $\mathbb{N} \times \mathbb{N}$ is a countable set. Proof is trivial. (BY PASTOR A.V.)
4. $\mathbb{Z}$ is countable. Super trivial.
5. $\mathbb{Q}$ is countable. Proof is also trivial (BY PASTOR A.V.)
6. $A$,$B$ is NMTC $\Rightarrow A \cup B, A \cap B, A \times B$ is also NMTC
7. $A_1, A_2, \ldots, A_n$ are NMTC. Then $\bigcup_{i=0}^{n} A_i$ is NMTC. Proven by Pastor A. V.

### 8.1.2 Canthor's theorem

**Теорема 8.1.** [0,1] *is uncountable* .

*Доказательство.* Let there be a biection $[0,1] \mapsto \mathbb{N}$.Then $\exists [a_1,b_1] \subset [0,1] : x_1 \notin [a_1, b_1] ; \exists [a_2,b_2] \subset [0,1] : x_2 \notin [a_2, b_2] ; \ldots$. By Canthor's theorem $\exists c : c \in$ all segments $\Rightarrow c \neq x_i$ □

**Лемма 8.1.** *Continuums:* $\langle a,b\rangle , \langle a, +\infty), (-\infty, a\rangle , \mathbb{R}$

*Доказательство.*

1. $y(x) = a + x \cdot (b - a), x \in [0,1]$
2. $[0,1] \mapsto (0, 1] : \frac{1}{n} \mapsto \frac{1}{n+1}, n \geqslant 2$

□