Университет ИТМО
Факультет ПИиКТ

МАТЕМАТИЧЕСКИЙ АНАЛИЗ

I СЕМЕСТР

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Проект на GitHub

осень 2023

Содержание

1 2023-09-04

1.1 Intro

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The whole semester is splitted by 3 parts, each of them contains:

1. control test
2. theory test (on GeoLin)
3. hometask
4. colloquium

Summary: 100 p.

The course is linked to A. Boytsev's course.

1.2 Logical symbolic

Определение 1.1. A statement is a sentence that is either **true** or **false**.

- ▷ $\forall$ – for all
- ▷ $\exists$ – exists
- ▷ ! – single
- ▷ $\sqsupset$ – let

Пример. $\forall a \in \mathbb{N}\ \exists!\ b \in \mathbb{N} : a + b = 0$

$A \Rightarrow B, A \Leftarrow B, A \Leftrightarrow B$

$$A = "a \vdots 6" B = "a \vdots 3" A \implies B$$

$\wedge$ – conjunction, $\vee$ – dis junction.

$$A \iff B \vee C$$

Лемма 1.1. *A is true $\iff \neg A$ is false*

1.3 Sets

Определение 1.2. A set is a group of an objects, with defined rule which can define, is object in a set or not

Парадокс Рассела (множество множеств, которое не содержит себя в качестве элемента)

$a \in A$

$b \notin A \iff \neg(b \in A)$

Определение 1.3. $A \subset B \iff \forall a \in A \implies a \in B$

Определение 1.4. $A = B \iff A \subset B \wedge B \subset A$

1.4 Operations between sets

1. Union – $A \cup B = \{x : x \in A \vee x \in B\}$
2. Intersection – $A \cap B = \{x : x \in A \wedge x \in B\}$

Let A be a set of indexes. $\alpha \in A$. $\alpha \mapsto G_\alpha$. We want do define a union of n sets.

Определение 1.5.

$$\bigcup_{\alpha \in A} G_\alpha = \{x : \exists \alpha : x \in G_\alpha\} = G_{A_1} \cup G_{A_2} \cup G_{A_3} \cup \dots \cup G_{A_n}$$

Определение 1.6.

$$\bigcap_{\alpha \in A} G_\alpha = \{x : \forall \alpha : x \in G_\alpha\} = G_{A_1} \cap G_{A_2} \cap G_{A_3} \cap \dots \cap G_{A_n}$$

Определение 1.7.

$$A \setminus B = \{x : x \in A \wedge x \notin B\}$$

We define U is an universal set: $\forall x : x \in U$.

$U \setminus A = A^c$ – complement ion.

$A \times B$ – desert multiplication of sets. $A \times B = \{(x, y) : x \in A, y \in B\}$

Лемма 1.2 (Свойства операций)**.** *$\forall A, B, C$:*

1. *$A \cup B = B \cup A$ (коммутативность)*
2. *$A \cap B = B \cap A$ (коммутативность)*
3. *$A \cup (B \cup C) = (A \cup B) \cup C$ (ассоциативность)*
4. *$A \cap (B \cap C) = (A \cap B) \cap C$ (ассоциативность)*

5. $A \cup A = A \cup \varnothing = A$
6. $A \cap \varnothing = \varnothing$
7. $A \cap A = A$
8. $A \cup A^c = U$
9. $A \cap A^c = \varnothing$
10. $(A^c)^c = A$

Упражнение. *Доказать верхние 10 свойств. Доказательство достаточно тривиально.*

Замечание. Доказательство следует из определения.

Теорема 1.1 (великая теорема Ферма)**.** $\forall x,y,z \in \mathbb{Z} : x,y,z > 2 : x^n + y^n = z^n$ *не имеет решений.*

2 2023-09-10 (NAL)

(not a lecture)

2.1 De Morgan laws

Утверждение 2.1 (De Morgan laws)**.**

$$A \setminus \bigcup_{i \in I} X_i = \bigcap_{i \in I} (A \setminus X_i)$$

$$A \setminus \bigcap_{i \in I} X_i = \bigcup_{i \in I} (A \setminus X_i)$$

Доказательство. Let's proof the first formula. Using the definition:

$$\begin{aligned} A \setminus \bigcup_{i \in I} X_\alpha &= A \setminus \ \{x \in U : \exists i \in I : x \in X_i\} \\ &= \{x : x \in A \ \& \ \forall i \in I : x \notin X_i\} \\ &= \{x : \forall i \in I : x \in A \ \& \ x \notin X_i\} \\ &= \bigcap_{i \in I} (A \setminus X_i) \end{aligned}$$

Similarly, proving the second formula, but with a little bit different approach:

$$\begin{aligned} A \setminus \bigcap_{i\in I} X_i &= A \setminus \{x \in U : X_1 \cap X_2 \cap \dots \cap X_n\} \\ &= \{x \in U : x \in A \wedge x \notin (X_1 \cap X_2 \cap \dots \cap X_n)\} \\ &= \{\} \end{aligned}$$

It's enough for x to not be in any of X_i (this statement is trivial). Then, the set is: $\{x \in U : \exists i \in I : x \in A \wedge$

This is equal to:

$$\bigcup_{i\in I} (A \setminus X_i)$$

□

2.2 Distribution laws

Утверждение 2.2 (Distribution)**.**

$$Y \cap \bigcup_{i\in I} X_i = \bigcup_{i\in I} (Y \cap X_i)$$

$$Y \cup \bigcap_{i\in I} X_i = \bigcap_{i\in I} (Y \cup X_i)$$

Доказательство. Proving the first law:

$$\begin{aligned} Y \cap \bigcup_{i\in I} X_i &= \{x \in U : x \in Y \wedge x \in (X_1 \cup X_2 \cup \dots \cup X_N)\} \\ &= \{x \in U : x \in Y \wedge \exists i \in I : x \in X_i\} \\ &= \{x \in U : \exists i \in I : x \in Y \cap X_i\} \\ &= \bigcup_{i\in I} (Y \cap X_i) \end{aligned}$$

Similarly, proving the second law:

$$\begin{aligned} Y \cup \bigcap_{i\in I} X_i = \bigcap_{i\in I} (Y \cup X_i) \\ &= \{x \in U : x \in Y \vee \forall i \in I : x \in X_i\} \\ &= \{x \in U : \forall i \in I : x \in (Y \cup X_i)\} \\ &= \bigcap_{i\in I} (Y \cup X_i) \end{aligned}$$

□

2.3 Injection, surjection and biection

Определение 2.1 (mapping)**.** A mapping is a rule $f : \forall x \in X \exists ! y \in Y : f(x) = y$.

Определение 2.2 (injection)**.** A mapping $f : X \mapsto Y$ is called **an injection**, if $\forall x_1, x_2 \in X : x_1 \neq x_2 \wedge f(x_1) \neq f(x_2)$

Определение 2.3 (surjection)**.** A mapping $f : X \mapsto Y$ is called **a surjection**, if $\forall y \in Y : \exists x \in X : f(x) = y$

Определение 2.4 (biection)**.** We call f a biection if f is both an injection and a surjection.

2.4 Properties of images and prototypes

We define $A, B \in X, \quad A', B' \in Y$.

Определение 2.5 (an image)**.** $f^{-1}(Y) = \{x \in X : f(x) \in Y\}$

1. $A \subset B \Rightarrow f(A) \subset f(B)$. It's obvious.
2. $f(A \cup B) = f(A) \cup f(B)$.

   Доказательство. Let $y \in f(A \cup B) \Rightarrow \exists x \in A \cup B : f(x) = y \Rightarrow x \in A \vee x \in B \Rightarrow f(x) \in f(A) \vee f(x) \in f(B) \Rightarrow f(x) \in f(A) \cup f(B)$. □

3. $f(A \cap B) = f(A) \cap f(B)$.

   Доказательство. Let $y \in f(A \cap B) \Rightarrow \exists x \in A \cap B : f(x) = y \Rightarrow f(x) \in f(A) \wedge f(x) \in f(B) \Rightarrow y \in A \wedge y \in B \Rightarrow f(x) \in A \wedge f(x) \in B \Rightarrow f(A \cap B) = f(A) \cap f(B)$ □

4. $A' \subset B' \Rightarrow f^{-1}(A') \subset f^{-1}(B')$. Obviously, true.
5. $f^{-1}(A' \cup B') = f^{-1}(A') \cup f^{-1}(B')$.

   Доказательство. Let $x \in f^{-1}(A' \cup B') \Rightarrow y \in A' \vee y \in B' \Rightarrow x \in f^{-1}(A') \vee x \in f^{-1}(B') \Rightarrow f^{-1}(A' \cup B') \in f^{-1}(A) \cup f^{-1}(B)$ □

6. $f^{-1}(A' \cap B') = f^{-1}(A') \cap f^{-1}(B')$

Let $f : X \mapsto Y$ be a biection. Then:

Определение 2.6 (reverse map)**.** $f^{-1} : Y \mapsto X$ is called **reverse map** if $\forall y \in Y \exists ! x \in X : f^{-1}(y) = x$

2.5 Superposition of mapping

Теорема 2.1 (associativity). $f \circ (g \circ h) = (f \circ g) \circ h$

Доказательство. Left side: $f \circ g(h) = f(g(h))$. Right side: $f(g) \circ h = f(g(h))$ $\square$

3 2023-09-11

$\triangleleft$ talked about mappings (and will be in the 1st semester)

3.1 Defining $\mathbb{R}$

Мы выбираем *аксиоматический* подход.

Определение 3.1 ($\mathbb{R}$). We call a set an $\mathbb{R}$ if:

- ▷ Addition

  def $"+" : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is satisfied:

  1. (commutativity): $a + b = b + a$
  2. (associativity): $a + (b + c) = (a + b) + c$
  3. $\exists 0 : \forall a + 0 = a$. We call 0 a **neutral** element.
  4. $\forall a \in \mathbb{R} : \exists(-a) : a + (-a) = 0$

- ▷ Multiplication

  def $" \cdot " : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is satisfied:

  1. (commutativity): $a \cdot b = b \cdot a$
  2. (associativity): $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
  3. $\exists 1 \neq 0 : \forall a \in A : a \cdot 1 = a$
  4. $\forall a \in A : \exists a^{-1} \in A : a \cdot a^{-1} = 1$

- ▷ (distributivity): $\forall a,b,c \in \mathbb{R} : a \cdot (b + c) = a \cdot b + a \cdot c \ \& \ (a + b) \cdot c = a \cdot c + b \cdot c$

- ▷ (axioms of order) $\forall a,b \in \mathbb{R}$ mapping of order $\leqslant$ set if:

  1. $x \leqslant x$
  2. $(x \leqslant y \wedge y \leqslant x) \Rightarrow x = y$
  3. (transitivity) $x \leqslant y \wedge y \leqslant z \Rightarrow x \leqslant z$
  4. $\forall x,y \in \mathbb{R} : x \leqslant y \vee y \leqslant x$

- ▷ (Connection between $\leqslant, +$) $\forall x,y,z \in \mathbb{R} : x \leqslant y \Rightarrow x + z \leqslant y + z$ (this is not implied by previous conditions)

- ▷ (Connection betwen $\cdot$ and $\leqslant$): $0 \leqslant x \wedge 0 \leqslant y \Rightarrow 0 \leqslant x \cdot y$
- ▷ (Axiom of continuity (completeness)): Let $X, Y \subset \mathbb{R} : \forall x \in X : \forall y \in Y : x \leqslant y$. Then $\exists c \in \mathbb{R} : x \leqslant c \leqslant y$

  Пример (This axiom doesn't work on $\mathbb{Q}$)**.** Let $X = \{x \in \mathbb{Q} : x \cdot x \leqslant 2\}, Y = \{y \in \mathbb{Q} : y \cdot y \geqslant 2\}$. Then, $\exists! a \notin \mathbb{Q} \quad (a = \sqrt{2})$: satisfies this axiom.

Замечание. Definition of $\mathbb{R}$ just contains the conditions that satisfy the **field**.

3.2 Corrolaries

Следствие (Corrolaries on Axioms 1–3)**.**

1. $\exists!0, \exists!1$.

   Доказательство для 0. Let there be $0_1, 0_2$. Then:

   $$0_1 = 0_1 + 0_2 = 0_2$$

   □

2. $\exists!(-x)\forall x$
3. $\forall x \neq 0 \exists! x^{-1}$

   Доказательство. Let there be $-x_1$ and $-x_2$. Then:

   $$(-x_1) = (-x_1) + (x + (-x_2)) = (x + (-x_1)) + (-x_2) = (-x_2)$$

   □

4. $\forall a,b \in \mathbb{R}$ an equality $x + a = b$ is set. Then there is only one solution $x = b + (-a)$.
5. $x \cdot a = b (a,b \in \mathbb{R})$. Then, $\exists! x = b \cdot a^{-1}$
6. $\forall x : x \cdot 0 = 0$

   Доказательство. $x \cdot 0 = x \cdot (0 + 0) = 0 \cdot x + 0 \cdot x = 0 \Rightarrow 0 = x \cdot 0$ □

7. $x \cdot y = 0 \Leftrightarrow x = 0 \vee y = 0$

   Доказательство. $\Leftarrow$ is proven.
   $\Rightarrow$: $x \neq 0 \Rightarrow \exists x^{-1} : x \cdot y \cdot x^{-1} = 0 \Rightarrow y = 0$. Proof for y is similar. □

8. $-x = -1 \cdot x$

Доказательство. $-1 \cdot x + x = -1 \cdot x + 1 \cdot x = x(-1+1) = x \cdot 0 = 0$ □

9. $-1 \cdot (-x) = x$. Proof is trivial based on previous)
10. $(-x) \cdot (-x) = x \cdot x$. Proof is also trivial.

Определение 3.2.

$$x \leqslant y \Leftrightarrow \geqslant x$$

$$x < y \Leftrightarrow x \leqslant y \land x \neq y$$

$$x > y \Leftrightarrow y \geqslant x \land y \neq x$$

Следствие (Corrolaries on axioms 4 – 6)**.**

1. $\forall x, y \in \mathbb{R}$: the only one statement is true
   - ▷ $x < y$
   - ▷ $x = y$
   - ▷ $x > y$
2. $x < y \land y \leqslant z \Rightarrow x < z$
3. ...
4. $x > 0 \Leftrightarrow -x < 0$. The proof is obvious.
5. $x < 0 \land y < 0 \Rightarrow xy > 0$
6. Can add to strict inequality.
7. $x \leqslant y \land z \leqslant w \Rightarrow x + z \leqslant y + w$
8. $0 < x \land 0 < y \Rightarrow 0 < xy$
9. $0 < x \land y < z \Rightarrow xz < yz$
10. $1 > 0$

    Доказательство. Let $1 \leqslant 0 \Rightarrow 1 < 0 \Rightarrow 1 \cdot 1 > 0!?$. Then, $1 > 0$. □

4 2023-09-15

4.1 Expanding $\mathbb{R}$

Определение 4.1 ($\overline{\mathbb{R}}$)**.** $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

Свойство 4.1.1. $\forall x \in \mathbb{R}$:

- ▷ $x + (+\infty) = +\infty := x + \infty$
- ▷ $x + (-\infty) = -\infty := x - \infty$
- ▷ $x \cdot (\pm\infty) = \begin{cases} \pm\infty, & \text{if } x > 0 \\ \mp\infty, & \text{if } x < 0 \\ undefined, & \text{if } x = 0 \end{cases}$
- ▷ $\frac{x}{\pm\infty} = 0$
- ▷ $\frac{\pm\infty}{x} = \begin{cases} \pm\infty, & \text{if } x > 0 \\ \mp\infty, & \text{if } x < 0 \end{cases}$

$(+\infty) + (+\infty) = +\infty$
$(-\infty) + (-\infty) = -\infty$
$(+\infty) \cdot (+\infty) = (-\infty) \cdot (-\infty) = +\infty$
$(+\infty) \cdot (-\infty) = (-\infty) \cdot (+\infty) = -\infty$

$\forall x : -\infty < x < +\infty$

Actions undefined in $\mathbb{R}$:

- ▷ $0 \cdot (\pm\infty)$
- ▷ $(+\infty) + (-\infty)$
- ▷ 1^∞
- ▷ $\frac{\pm\infty}{\pm\infty}$
- ▷ $\frac{0}{0}$
- ▷ 0^0

4.2 Defining $\mathbb{N}$

Определение 4.2 (Inductive set)**.** A set $X \subset \mathbb{R}$ is *inductive*, if $\forall x \in X : x + 1 \in X$

Лемма 4.1. *Let $X_1, X_2, \dots, X_n$ be inductive sets. Then, $X_1 \cap X_2 \cap \dots X_n$ is also inductive.*

Доказательство. Trivially proof the $x \mapsto x + 1$ $\square$

Определение 4.3. $\mathbb{N}$ is an intersection of every inductive sets: $\forall i : 1 \in A_i$

Замечание. $\mathbb{N}$ is minimal inductive set, that contains 1.

Теорема 4.1 (Math. induction principle)**.** *Let* $X \subset \mathbb{N}, 1 \in X, X$ *is inductive. Then,* $\mathbb{N} = X$

Упражнение. *Proof that* $\forall n > -1, n \in \mathbb{N}, x \in \mathbb{R} : (1 + x)^n \geqslant 1 + nx$

4.3 Properties of $n \in \mathbb{N}$

Лемма 4.2. $\forall a,b \in \mathbb{N} : a + b \in \mathbb{N}, ab \in \mathbb{N}$

Замечание. Proof using math. induction.

Определение 4.4 ($\mathbb{Z}$)**.** $\mathbb{Z} := \mathbb{N} \cup \{0\} \cup \{x : -x \in \mathbb{N}\}$

Определение 4.5 ($\mathbb{Q}$)**.** $\mathbb{Q} := \left\{\frac{m}{n} := m \cdot n^{-1}, m \in \mathbb{Z}, n \in \mathbb{N}\right\}$

Теорема 4.2 (Existence of irrational number)**.** *A set* $\mathbb{R} \setminus \mathbb{Q} = \mathbb{I}$ *is not empty.*

Let's proof that $\sqrt{2}$ is irrational.

Доказательство. Plan:

1. Prove that $\exists c \in \mathbb{R} : c^2 = 2$.
2. Prove that c is irrational.

2. Let $c = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}$. Then $c^2 \cdot n^2 = m^2 \Rightarrow 2n^2 = m^2 !?$

1. Using axiom of continuity. Let $X = \{x \in \mathbb{R}_{x>0} : x^2 < 2\}, Y = \{y \in \mathbb{R}_{y>0} : y^2 > 2\}$. Then $x \leqslant y \Rightarrow \exists c \in \mathbb{R} : x \leqslant c \leqslant y \forall x \in X, y \in Y$

   Proving that $c \notin X$. Let $c \in X$, i.e. $c^2 < 2$. Consider $c + \frac{2-c^2}{3c} = c + \frac{\Delta}{3c} = \xi$
   $(c + \frac{\Delta}{3c})^2 = c^2 + \frac{2}{3}\Delta + \frac{\Delta \cdot \Delta}{9c^2} \leqslant c^2 + (\frac{2}{3} + \frac{1}{3})\Delta = 2 \Rightarrow \xi \in X,$ but $\xi > c!? \Rightarrow c \notin X$. Similarly, we proof for Y.

$\Rightarrow \exists c \in \mathbb{R} : c^2 = 2 \Rightarrow |\mathbb{I}| \neq 0$ $\square$