Университет ИТМО
Факультет ПИиКТ

ЛИНЕЙНАЯ АЛГЕБРА

I СЕМЕСТР

Лектор: *Дмитрий Валерьевич Карпов*

Автор: *Александр Калиев*

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1.1 Ring and field

Let K be a set, we call it's elements *numbers*. Two operations are also defined:

$+ : K \times K \mapsto K$

$\cdot : K \times K \mapsto K$.

Properties:

1. (Ассоциативность $+$): $\forall a,b,c \in K : (a+b)+c = a+(b+c)$
2. (Commutativity $+$): $\forall a,b \in K : a+b = b+a$
3. Zero: $\exists 0 \in K : a+0 = a$
4. (Inverse element for $+$): $\forall a \in K \exists (-a) \in K : a+(-a) = 0$
5. (Distributivity): $\forall a,b,c \in K : a \cdot (b \cdot c) = (a+b) \cdot c \; \& \; a \cdot (b+c) = a \cdot b + a \cdot c$
6. (Associativity $\cdot$): (ab)c = a(bc)
7. (Commutativity $\cdot$): ab = ba
8. (Neutral element $\cdot$): $\exists 1 : \forall a \in K : 1 \cdot a = a$.
9. (Inverse element for $\cdot$):
$$\forall a \in K \setminus \{0\} \, \exists (a)^{-1} \in K : a \cdot (a)^{-1} = (a)^{-1} \cdot a = 1$$

- ▷ $1-6 \Rightarrow K - ring$
- ▷ $1-7 \Rightarrow K - commutative\ ring$
- ▷ $1-6 \;\&\; 8 \Rightarrow K - ring\ with\ 1$
- ▷ $1-6, 8, 9 \Rightarrow K - body$
- ▷ $1-9 \Rightarrow K - field$

Свойство 1.1.1. Zero is the only one.

Доказательство. Let there be 0_1 and 0_2. Then:

$$0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2.$$

□

Свойство 1.1.2. $\forall a \in K$, the reverse element for $+$ is the only one.

Доказательство. Let there be 2 reverse elements for $a \in K$: b_1 & b_2. Then:

$$b_1 = b_1 + 0 = b_1 + (a + b_2) = (b_1 + a) + b_2 = 0 + b_2 = b_2$$

□

Свойство 1.1.3. $\forall a \in K : -(-a) = a$

Доказательство. $a = a + ((-a) + (-(-a))) = (a + (-a)) + (-(-a)) = (-(-a))$ □

Свойство 1.1.4. No more than 1 unit in a ring.

Доказательство. Let there be 1_1 & 1_2. Then:

$$1_1 = 1_1 \cdot 1_2 = 1_2.$$

□

Определение 1.1. Let K be a ring with 1. An element $a \in K$ is reversible, if $\exists a^{-1} \in K$

◁ in the fiels, all elements except 0 are reversible.

Свойство 1.1.5. Let K be a ring with 1. Then, $\forall a \in K \exists$ no more than 1 reverse element for $\cdot$.

Доказательство. Let there be 2 reverse elements: b_1 & b_2. Then:

$$b_1 = b_1 \cdot 1 = b_1 \cdot (a \cdot b_2) = (b_1 \cdot a) \cdot b_2 = 1 \cdot b_2 = b_2$$

□

Свойство 1.1.6. Let K be a ring with 1. Then, $\forall$ reversible $a \in K : (a^{-1})^{-1} = a$

Доказательство. $a = a \cdot 1 = a \cdot (a^{-1} \cdot (a^{-1})^{-1}) = (a \cdot a^{-1}) \cdot (a^{-1})^{-1} = 1 \cdot (a^{-1})^{-1} = (a^{-1})^{-1}$ □

Свойство 1.1.7. -0 = 0

Доказательство. Follows from the $0 + 0 = 0$. □

Свойство 1.1.8. If K is a ring with 1, then $1^{-1} = 1$

Доказательство. Follows from $1 \cdot 1 = 1$ □

Определение 1.2. ▷ Substraction – addition a reverse element for $+$:

$$a - b := a + (-b).$$

▷ Division on a reversible element b is a multiplication by b^{-1}:

$$\frac{a}{b} := a \cdot b^{-1}.$$

1.2 Sub-field and sub-ring

Определение 1.3.

- ▷ Let $K \subset L$ (both are rings with the same operations). Then K is a **sub-ring** of L, and L is an **supra-ring** of K.
- ▷ Let $K \subset L$ (both are fields with the same operations). Then K is a **sub-field** of L; L is a **supra-field** of K.

Лемма 1.1. *Let L be a ring, $K \subset L$. Conditions:*

1. *Closedness of* $+ : \forall a,b \in K : \quad a+b \in K$
2. *Closedness of* $\cdot : \forall a,b \in K \quad a \cdot b \in K$
3. *Existence of reverse element for* $+$
   $\forall a \in K \quad \exists -a \in K$
4. *Existence of reverse element for* $\cdot$
   $\forall a \in K, a \neq 0, \quad \exists a^{-1} \in K.$

Then K is a field, then, it's a sub-field of L.

Доказательство.

- ▷ By Lemma 1, K – commutative sub-ring of L.
- ▷ It remains to check the existence of 1 in K.
  Consider any non-zero element $a \in K$. Then $a^{-1} \in K$, and that means, that $a \cdot a^{-1} = 1 \in K$.

□

1.3 Homomorphism

Определение 1.4. ◁ Let K, L be a rings. Then a relation $f : K \mapsto L$ is called **homomorphism**, if $\forall a,b \in K$:

$$f(a+b) = f(a) + f(b) \;\&\; f(ab) = f(a)f(b)$$

A kernel of homomorphism f is denoted as $\operatorname{Ker} f = \{x \in K : f(x) = 0\}$ An image of homomorphism f is denoted as $\operatorname{Im} f = \{y \in L : \exists x \in K : f(x) = y\}$.

Свойство 1.3.1. If $f : K \mapsto L$ is homomorphism, then $f(0_K) = 0_L$.

Доказательство. $f(0_K) = f(0_K + 0_k) = f(0_K) + f(0_K)$. Substracting from left and right side $f(0_k)$, we get $f(0_K) = 0_L$ □

Лемма 1.2. *Let K, L be rings, $f : K \mapsto L$ – homomorphism of rings. Then:*

- ▷ $\operatorname{Ker} f$ *is a sub-ring of* K.
- ▷ $\operatorname{Im} f$ *is a sub-ring of* L.

Доказательство. It's enough to check conditions from Lemma 1.

1. ▷ Let a,$b \in \operatorname{Ker} f$. Then $f(a + b) = f(a) + f(b) = 0 + 0 = 0 \Rightarrow a + b \in \operatorname{Ker} f$.
   ▷ $f(ab) = f(a)f(b) = 0 \cdot 0 = 0 \Rightarrow ab \in \operatorname{Ker} f$.
   ▷ $f(-a) = -f(a) = -0_L = 0_L$.
2. ▷ Let $y, y' \in \operatorname{Im} f$, and $x, x' \in K$ are such that $f(x) = y$ & $f(x') = y'$.
   ▷ Then $y + y' = f(x) + f(x') = f(x + x') \in \operatorname{Im} f$ & $y \cdot y' = f(x) \cdot f(x') \in \operatorname{Im} f$.
   ▷ $-y = -f(x) = f(-x) \in \operatorname{Im} f$.

□

1.4 Homomorphism types

- ▷ Let $f : K \mapsto L$ – homomorphism of rings.
- ▷ If f is an injection, then f is **monomorphism**
- ▷ If f is a surjection ($\operatorname{Im} f = L$), then f is an **epimorphism**
- ▷ **If f is a biection, then f is isomorphism**
- ▷ **Isomorphism = monomorphism + epimorphism.**

Лемма 1.3. *Let $f : K \mapsto L$ be a homomorphism of rings. Then f is monomorphism if and only if* $\operatorname{Ker} f = \{0\}$.

Доказательство. $\Rightarrow$

- ▷ If f is monomorphism, then f is an injection.
- ▷ Let $a \in \operatorname{Ker} f$. From $f(a) = 0 = f(0)$ implies, that $a = 0$ (because of the injection f).

$\Leftarrow$

- ▷ Let $f(a) = f(b)$. Then $f(a - b) = f(a) - f(b) = 0$.

- ▷ That means that $a - b \in \operatorname{Ker} f = \{0\}$, from this $a = b$. In conclusion, f is an injection, and that means f is monomorphism.

□

Лемма 1.4. *Let $f : K \mapsto L$ be an isomorphism of rings. Then $f^{-1} : L \mapsto K$ is an isomorphism of rings.*

Доказательство.

- ▷ It's enough to proof that f^{-1} is homomorphism (because relation that is reverse to biection is a biection).
- ▷ Consider any $a,b \in L$.
- ▷ Let $w = f^{-1}(a+b) - f^{-1}(a) - f^{-1}(b)$. Because of f is a biection, we have:
$$f(w) = f(f^{-1}(a+b)) - f(f^{-1}(a)) - f(f^{-1}(b)) = a + b - a - b = 0$$
- ▷ From $(f(w) = 0 = f(0)\ )$ and because of f is a biection, we implie that $w = 0$.
- ▷ Therefore, $f^{-1}(a+b) = f^{-1}(a) + f^{-1}(b)$
- ▷ Let $z = f^{-1}(ab) - f(f^{-1}(a)) \cdot f(f^{-1}(b)) = ab - ab = 0$.
- ▷ From $f(z) = 0 = f(0)$ and because of f is a biection, we implie that $z = 0$

Therefore, $f^{-1}(ab) = f^{-1}(a) \cdot f^{-1}(b)$. □

1.5 Isomorphic rings

Определение 1.5. If $\exists f : K \mapsto L$ (f – isomorphism), then we say that K, L are isomorphic. Denotion: $K \simeq L$.

Теорема 1.1. *$\simeq$ is a relation of equality on the set of all rings.*

Доказательство.

- ▷ Reflexivity is obvious: $\mathrm{id} : K \mapsto K$ $(\mathrm{id}(x) = x \quad \forall x \in K)$ is obviously an isomorphism
- ▷ Symmetry is proven in Lemma 5.
- ▷ Let's prove transitivity: let K,L,M be rings, $K \simeq L$ & $L \simeq M$.
- ▷ Then there are isomorphisms $f : K \mapsto L$ & $g : L \mapsto M$. Let's prove that $g \cdot f : K \mapsto M$ (set up by rule $gf(a) := g(f(a))$) is also an isomorphism.

- ▷ Composition of these biections is obviously a biection.
- ▷ Checking that gf is homomorphism of rings:

$$gf(a+b) = g(f(a+b)) = g(f(a)+f(b)) = g(f(a)) + g(f(b)) = gf(a) + gf(b)$$

$$gf(ab) = g(f(ab)) = g(f(a)\cdot f(b)) = g(f(a))\cdot g(f(b)) = gf(a)\cdot gf(b)$$

$\square$

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2.1 Complex numbers

Определение 2.1.

- ▷ A set of *complex numbers* contains sorted pairs of real numbers:

$$\mathbb{C} = \{(a,b) : a,b \in \mathbb{R}\}$$

- ▷ Addition: $(a,b) + (a',b') := (a+a', b+b')$
- ▷ Multiplication: $(a,b)\cdot(a',b') := (aa' - bb', ab' + ba')$.

Определение 2.2.

- ▷ Let $z = (a,b) \in \mathbb{C}$
- ▷ A **real part** of z is denoted as $\mathrm{Re}(z) := a$.
- ▷ An **imaginary part** of z is denoted as $\mathrm{Im}(z)$
- ▷ Complex conjugation: $\overline{z} := (a, -b)$
- ▷ Norm of z is denoted as $N(z) := a^2 + b^2$
- ▷ Module of z is denoted as $|z| := \sqrt{N(z)} = \sqrt{a^2+b^2}$
- ▷ Obviously, $\overline{\overline{z}} = z$.

Теорема 2.1. $\mathbb{C}$ *is a field.*

Доказательство. ▷ (1) and (2) because addition in $\mathbb{C}$ is componented, so associativity and commutativity are inherited from $\mathbb{R}$.

- ▷ (3) Zero in $\mathbb{C}$ is $0 := (0,0)$.

- (4) Reverse element for $+$. For $z = (a,b)$ set $-z := (-a, -b)$.
- (7) Commutativity of multiplication:

$$(a,b) \cdot (a',b') = (aa' - bb', ab' + ba') = (a'a - b'b, a'b + b'a) = (a', b') \cdot (a,b)$$

- (5) It's enough to check one distributivity (because multiplication is commutative):

$$\begin{aligned}(a,b) \cdot ((c_1, d_1) + (c_2, d_2)) = (a,b) \cdot (c_1 + c_2, d_1 + d_2) = \\ (ac_1 + ac_2 - bd_1 - bd_2, ad_1 + ad_2 + bc_1 + bc_2) = \\ (ac_1 - bd_1,ad_1 + bc_1) + (ac_2 - bd_2,ad_2 + bc_2) = (a,b) \cdot (c_1, d_1) + (a,b) \cdot (c_2,d_2)\end{aligned}$$

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$\square$