Университет ИТМО
Факультет ПИиКТ

МАТЕМАТИЧЕСКИЙ АНАЛИЗ

I СЕМЕСТР

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Содержание

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1.1 Бином Ньютона

Определение 1.1 (Binomial coefficients). $C_n^k = \frac{n!}{k!(n-k)!}, n \in \mathbb{N}, k \in \mathbb{N} \setminus \{0\}, k \leqslant n$

Упражнение. *Вывести.*

Свойство 1.1.1.

1. $C_n^0 = C_n^n = 1$ (trivial)
2. $C_n^1 = C_n^{n-1} = n$
3. $C_n^k = C_n^{n-k}$
4. $C_n^k + C_n^{k+1} = C_{n+1}^{k+1}$

   Упражнение. Proof using Pascal's triangle (trivial).

   Доказательство. $C_n^k + C_n^{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!}{k!(n-k-1)!} \cdot \left(\frac{1}{n-k} + \frac{1}{k+1}\right) = $
   $= \frac{(n+1)!}{(k+1)! \cdot (n-k)!} = C_{n+1}^{k+1}$ □

Теорема 1.1 (Binomial theorem). $\forall a,b \in \mathbb{R}, \forall n \in \mathbb{N} : (a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$

Доказательство. Proving using induction.

- ▷ $n = 1 : (a+b)^1 = C_1^0 a^1 + C_1^1 b^1 = a + b$
- ▷ let $n = k : (a+b)^k = \sum_{m=0}^{k} C_k^m a^m b^{k-m}$
- ▷ Transition: $(a+b)^m \cdot (a+b) = (\sum_{m=0}^{k} C_k^m a^m b^{k-m}) \cdot (a+b) = C_m^0 a^{m+1} b^0 + \dots + C_m^m a^1 b^m + C_m^0 a^m b^1 + \dots + C_m^m a^0 b^{m+1} =^{\text{using property 1.4}} C_{m+1}^0 a^{m_1} b^0 + (C_m^1 + C_m^0) a^m b^1 + (C_m^2 + C_m^1) a^{m-1} b^2 + \dots + (C_m^{k+1} + C_m^k)(a^{m-k} b^{k+1}) + \dots + (C_m^m + C_m^{m-1}) a^1 b^m + C_{m+1}^{m+1} = \sum_{k=0}^{m+1} C_{m+1}^k a^k b^{m+1-k}$

□

1.2 Defining intervals on $\mathbb{R}$

Определение 1.2. отрезок: $[a,b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$

interval: $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

semi-interval: $(a, b], [a, b) = \{x \in \mathbb{R} : a < x \leqslant b\}, \{x \in \mathbb{R} : a \leqslant x < b\}$

луч: $(-\infty, a), (-\infty, a], [b, +\infty), (b, +\infty)$

Определение 1.3 (Окрестность точки x_0)**.** $x_0 \in (a,b) = U(x_0)$ (including $(-\infty, a), (b, +\infty)$)

ε-neighbourhood: $(x_0 - \varepsilon, x_0 + \varepsilon) = U_\varepsilon(x_0)$

Определение 1.4 (ε-neighbourhood for $\overline{\mathbb{R}}$)**.** ▷ $+\infty : (a; +\infty) = U(+\infty)$

▷ $-\infty : (-\infty, a) = U(-\infty)$

▷ $\infty = U(+\infty) \cup U(-\infty)$

▷ $U_\varepsilon(+\infty) = (\frac{1}{\varepsilon}; +\infty); U_\varepsilon(-\infty) = (-\infty, -\frac{1}{\varepsilon})$

1.3 Absolute value

Определение 1.5. $\forall x \in \mathbb{R} : |x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x \leqslant 0 \end{cases}$

Свойство 1.3.1.

1. $|x| = |-x|$
2. $|x|^2 = x^2$
3. $|x| \geqslant 0; |x| = 0 \Leftrightarrow x = 0$
4. $|xy| = |x||y|$
5. $\frac{|x|}{|y|} = \left|\frac{x}{y}\right|$
6. $-|x| \leqslant x \leqslant |x|$
7. $|x + y| \leqslant |x| + |y|$

   Доказательство. $(x + y)^2 \leqslant x^2 + y^2 + 2|xy| \Leftrightarrow 2xy \leqslant |2xy|$ □

8. $|x| \leqslant a \Leftrightarrow -a \leqslant x \leqslant a$
9. $|x| \geqslant b \Leftrightarrow x \leqslant -b \vee x \geqslant b$
10. $|x - y| \geqslant ||x| - |y||$

1.4 Bounds of the set in $\mathbb{R}$

Let $X \subset \mathbb{R}$

Определение 1.6. We say that X is *bounded above*, if $\exists M \in \mathbb{R} : x \leqslant M \forall x \in X$ (M is upper bound)

We say that X is *bouded below*, if $\exists m \in \mathbb{R} : m \leqslant x, \forall x \in X$ (m is lower bound)

We assume that X is **bounded**, if its *bounded both below and above.*

Пример. Let $X = [0, 1)$

Определение 1.7 (min and max element)**.** max. element: $x_{\max} = \max X : x_{\max} \in X, \forall x \in X : x \leqslant x_{\max}$

Замечание. X doesn't have max. element.

Доказательство. Let $M = \max X$. Then, $\exists M_0 \in \frac{M+1}{2} > M$!? □

Определение 1.8 (supremum and infremum of the set)**.** $S \in \mathbb{R}$ is called *an exact upper bound*, (or a *supremum of* X), if $S =$ lowest upper bound

We denote is as $\sup X = S = \min \{M : x \leqslant M \forall x \in X\}$

If X is not bounded above, then $\sup X = +\infty$

$s \in \mathbb{R}$ is called *an exact lower bound*, (or an *infremum* of X), if $S =$ highest lower bound. We denote it as $s = \inf X = \max \{m : x \geqslant m, \forall x \in X\}$

Замечание. $X = \{x \in \mathbb{Q} : x^2 < 2\} \Rightarrow \sup X$ is undefined.

Лемма 1.1. *X is bounded* $\Leftrightarrow \exists c \in \mathbb{R} : |x| \leqslant c, \forall x \in X$

Упражнение. *Proof.*

2 2023-09-22

2.1

Let $X \subset \mathbb{R}$

Замечание. If X is not bounded, then $\sup X = +\infty$ & $\inf X = -\infty$. Let $i = [0; 1)$. We will proof that there is a supremum of i.

Лемма 2.1. *If* $\exists \max X$, *then* $\sup X = \max X$.

If $\exists \min X$, *then* $\inf X = \min X$.

Implication only!

Доказательство.
- ▷ Obviously, let $M = \max X$. Then M is upper bound by definition of max. Let there be $M' < M$ such that M' is an upper bound. Then it's not an upper bound by definition.
- ▷ Same story for $\min X$

□

Лемма 2.2 (different definition of supremum and infremum)**.** $M = \sup X \Leftrightarrow M : \forall x \in X : x \leqslant M$ & $\forall \varepsilon > 0 \exists x \in X : x > M - \varepsilon$

$m = \inf X \Leftrightarrow m : \forall x \in X : x \geqslant m$ & $\forall \varepsilon > 0 \exists x \in X : x > m + \varepsilon$

Доказательство. By definition. $\square$

Теорема 2.1 (Exact bound principle). *$\forall X : X$ is upper bounded $\Rightarrow \exists \sup X$. Same for $\forall X : X$ is lower bounded $\Rightarrow \exists \inf X$*

Доказательство. If the set is upper bounded, then $\exists$ an upper bound. Let B be a set of upper bounds: $B = \{M \in \mathbb{R} : x \leqslant M, x \in X\}$. Then $\forall M, x : x \leqslant M$. By continuity axiom, $\exists c \in \mathbb{R} : x \leqslant c \leqslant M \forall x \in X, M \in B$. Let's proof that c is a supremum of X. c is an upper bound of X and it's lower that every other upper bounds in M. Then, $c = \sup X$ $\square$

Замечание. Even if X is not upper bounded. Then $\forall X \neq \varnothing \exists c \in \overline{\mathbb{R}} : c = \sup X$

2.2 Archimed's axiom

Лемма 2.3. *Let $X \subset \mathbb{N}, X \neq \varnothing, X$ is bounded. Then, the maximum exists.*

Доказательство. $\exists M = \sup X = k \in \mathbb{R}$ for $\varepsilon = 1$. Then $\exists x \in X : k - 1 < x \leqslant k$. Then $x \in \mathbb{N}$. Proving that $x = k$. $k < x + 1$ then $\forall y \in X : y \leqslant x$, because of $y \leqslant k < x + 1 \Rightarrow y < x + 1 \Rightarrow y \leqslant x \Rightarrow x = \max X$. $\square$

Следствие.
1. $\mathbb{N}$ is not bounded above.
2. $\mathbb{Z}$ is bounded nor below and above.
3. $X \subset \mathbb{Z}$ if X is bounded below then $\exists \min X$; if X is bounded above the $\exists \max X$

Теорема 2.2 (Archimed's axiom). *Let $x \in R, x > 0$. Then $\forall y \in \mathbb{R} \exists k \in \mathbb{Z} : (k-1)x \leqslant y \leqslant kx$.*

Interpretation: we can fill a segment of length y with a segments of length x.

Доказательство. Consider $T = \left\{t \in \mathbb{Z} : \frac{y}{x} \leqslant t\right\}$. $T \neq \varnothing$ & is bounded below. Then $\exists k = \min T : \frac{y}{x} < k \Rightarrow y < kx \Rightarrow k-1 \leqslant \frac{y}{x}$, cuz if it's false then $k-1 \in T$ but $k-1 < k = \min T$!? $\square$

Следствие.
1. $\forall \varepsilon > 0 \exists n \in \mathbb{N} : 0 < \frac{1}{n} < \varepsilon$

   Доказательство. $y = 1, x = \varepsilon \Rightarrow \exists n : 1 < n\varepsilon$ $\square$

2. If $x \geqslant 0$ and $\forall \varepsilon > 0 x < \varepsilon \Rightarrow x = 0$

   Доказательство. $0 \leqslant x < \varepsilon$. Let there be $x > 0 \Rightarrow \varepsilon = \frac{x}{2}$ the statement is *false.* $\square$

3. $\forall x \in \mathbb{R} \exists ! k = [x] \in \mathbb{Z} : k \leqslant x < k + 1$

   Доказательство. $x = 1, \varepsilon = x$ $\square$

Лемма 2.4 (density of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$). *Let there be $a < b$. Then on $(a, b) \exists q \in \mathbb{Q}, j \in \mathbb{I}$*

Доказательство. $\exists n \in \mathbb{N} : \frac{1}{n} < b - a, [na] \leqslant na < [na] + 1 \Rightarrow a < \frac{[na]+1}{n} = q \leqslant \frac{na+1}{n} < b$

$\sqrt{2} \in \mathbb{I}$. Consider $q \in (a - \sqrt{2}, b - \sqrt{2})$. Then, $q + \sqrt{2} \in (a, b)$. Из-за замкнутости $q + \sqrt{2} \in \mathbb{I}$. Let $q + \sqrt{2} \in \mathbb{Q}$. Then, $q + \sqrt{2} - q = \sqrt{2} \in \mathbb{Q}$!? $\square$

2.3 Canthor's theorem about line segments

Let $I_n = [a_n, b_n], a_n \leqslant b_n$

We assume that $\cdots \subset I_n \subset I_{n-1} \subset \cdots \subset I_2 \subset I_1$

Теорема 2.3. *If there is this system of line segments, then:*

- ▷ $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$(*i.e.* $\exists c \in \bigcap I_n$
- ▷ *If* $\forall \varepsilon > 0 \exists I_n : b_n - a_n < \varepsilon$, *then* $\bigcap I_n = \{c\}$, *i.e.* $\exists! c$

Доказательство. ▷ $A = \{a_n\}, B = \{b_n\}, a_n \leqslant b_m \Rightarrow_{\text{continuity A.}} \exists c : a_n \leqslant c \leqslant b_n \Rightarrow c \in I_n \forall n \in \mathbb{N}$

▷ Let $c_1, c_2 \in I_n$ $\forall n$. Let $c_1 < c_2 \Rightarrow$ for $\varepsilon = c_2 - c_1 \exists (a_n b_n) : b_n - a_n < \varepsilon$!?

□

3 2023-09-25

3.1 Borrel-Lebeg lemma

Consider G_α as a sets.

G_α is forming a cover of a set X if $X \subset \bigcup_\alpha G_\alpha$

Лемма 3.1 (Borrel-Lebeg lemma (GBL lemma))**.** *From any coverage of a segment by interval we can choose a finite one.*

Доказательство. Consider $[a_0,b_0]$. Let there be no option to choose a finite coverage from $\bigcup_\alpha G_\alpha$. Split the segment by half and we get $[a_1, b_1]$ - a cover where we cant choose a finite covering. $\ldots [a_2, b_2] \subset [a_1,b_1] \subset [a_0, b_0] .b_n - a_n = \frac{b_0 - a_0}{2^n} \underset{2^n > n}{<} \frac{b_0 - a_0}{n} < \frac{1}{\varepsilon} \Rightarrow \forall \varepsilon > 0 \exists n \Rightarrow \exists! c : c \in [a_i, b_i]$

$\exists G = (k,l) : c \in G \Rightarrow \exists n_0 : [a_{n_0}, b_{n_0}] \subset (k,l)$!?

Замечание. Basically proving Canthor's theorem and getting !? for this.

□

Лемма 3.2 (limit point)**.**

Определение 3.1. A dot x_0 is called **limited** point of this set E if $\forall \overset{\circ}{U}(x_0) \cap E \neq \varnothing$, i.e. $U(x_0) \cap E$ is infinite.

Пример. Consider $[0, 1)$ A set of limit points $E' = [0,1]$

Пример. Consider $E = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$ $E' = \{0\}$

Определение 3.2. If $x_0 \in E$ & x_0 is not a limit point, then x_0 is **isolated** point of E, i.e. $\exists U(x_0) : \overset{\circ}{U}(x_0) \cap E = \varnothing$.

(Lemma) Let E is an infinite and bounded $E \subset \mathbb{R}$. Then $\exists x_0 : x_0$ is limit point of E.

Доказательство. E is bounded $\Rightarrow \exists\, [a{,}b] : E \subset [a{,}b]$. Let there be no limit points in $[a{,}b]$, i.e. $\forall x \in [a{,}b]$ is not limit point for E , i.e. there is a finite number of points in $E \in U(x)$

$\{U(x)\}$ is cover of a segment by intervals $\underset{BGL\ lemma}{\Rightarrow} \exists\, \{U(x_1), \ldots, U(x_n)\}$

$E \subset \bigcup_{i=1}^{n} U(x_i)$, but a subset contains a finite number of points from E !? □

Замечание. We can select limit points in $\overline{\mathbb{R}}$

e.g. $\mathbb{N}' = \{+\infty\}$

3.2 closedness of sets

Определение 3.3. A set E is close d (in $\mathbb{R}$) if it contains every it's limit point, i.e. $E' \subset E$.

An $\varnothing$ is closed *by definition.*

Пример. $E = [0, 1)$ isn't closed

$[0{,}1]$ is closed

$E = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\}$ is closed .

Лемма 3.3. *Let $E \subset \mathbb{R}$, E is closed and bounded above (below). Then $\exists \max E (\min E)$*

Доказательство. By exact bound principle $\exists M = \sup E \in \mathbb{R}$. Proving that $M \in E$. Let $M \notin E$. Then, we consider any neighbourhood $(\alpha, \beta) \ni M$. For $\varepsilon_1 = M - \alpha > 0 : \exists x_1 \in E \cap (\alpha, M)$. For $\varepsilon_2 = M - x_1$, for $\varepsilon_3 = M - x_2, \cdots \Rightarrow$ there is infinite amount of dots in E, i.e. M is a limit point of $E \Rightarrow M \in E$ □

Следствие. Any finite set has it's maximum and minimum. (it has no limit points).

Следствие. For any $(\alpha, \beta) \subset \mathbb{R}$: infinite number of numbers of $\mathbb{Q}$,