Университет ИТМО
Факультет ПИиКТ

МАТЕМАТИЧЕСКИЙ АНАЛИЗ

I СЕМЕСТР

Лектор: *Екатерина Станиславовна Трифанова*

Автор: *Александр Калиев*
Проект на GitHub

осень 2023

Содержание

1 2023-11-03

Consider $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. $y_n=\ln(n+1)$

Лемма 1.1. $\exists \lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)=\gamma\approx 0{,}577$

Доказательство. Let $\gamma_n=1+\cdots+\frac{1}{n}-\ln n$. Proving by Weierstrass. Consider $\gamma_{n+1}-\gamma_n=\frac{1}{n+1}-\ln(n+1)+\ln n=\frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right)<0\Rightarrow\gamma_n\downarrow$.

Consider $\overline{\gamma_n}=1+\cdots+\frac{1}{n}-\ln(n+1).\gamma_n-\gamma_{n+1}=\frac{1}{n}-\ln\left(1+\frac{1}{n}\right)>0\Rightarrow\overline{\gamma_n}\uparrow$. But $\overline{\gamma_n}=\gamma_{n+1}-\frac{1}{n+1}<\gamma_{n+1}\Rightarrow$ by Weierstrass $\exists\lim\limits_{n\to\infty}\gamma_n,\overline{\gamma_n}$ and $\lim\limits_{n\to\infty}\gamma_n=\lim\limits_{n\to\infty}\overline{\gamma_n}$ □

Замечание.

$$1+\frac{1}{2}+\cdots+\frac{1}{n}=\ln n+\gamma+\frac{\theta}{n},\theta\in[0,1]$$

Доказательство. Consider $0\leqslant\gamma_n-\gamma\leqslant\gamma_n-\overline{\gamma_{n-1}}=\frac{1}{n}$. Then $0\leqslant R\leqslant\frac{1}{n}\Rightarrow\exists\theta\in[0,1]:R=\frac{\theta}{n}$ □

1.1 CH. 2. FUNCTION LIMIT

1.2 Definitions

Let $f:\mathbb{R}\supset E\to\mathbb{R},x_0$ is a limit point $,a_n:\mathbb{N}\to\mathbb{R}$.

Определение 1.1 (Cauchy)**.** $\lim\limits_{x\to x_0}f(x)=A\in\mathbb{R},x_0\in\mathbb{R}\Leftrightarrow\forall\varepsilon>0\exists\delta(\varepsilon)>0:\forall x\in E,0<|x-x_0|<\delta\Rightarrow|f(x)-A|<\varepsilon$

Определение 1.2. $\forall\varepsilon>0\exists\delta>0:\forall x\in E\cap\overset{\circ}{U}_\delta(x_0):f(x)\in U_\varepsilon(A)$. Works for $x_0,A\in\overline{\mathbb{R}}$

Пример. $\lim\limits_{x\to-\infty}f(x)=+\infty\Leftrightarrow\forall\varepsilon>0\exists\delta>0:\forall x\in E,x<-\frac{1}{\delta}\Rightarrow f(x)>\frac{1}{\varepsilon}$

Замечание. $\lim\limits_{x\to x_0}f(x)=\infty\Leftrightarrow\lim\limits_{x\to x_0}|f(x)|=+\infty$

Определение 1.3 (topological)**.** $\lim\limits_{x\to x_0}f(x)=A,x_0,A\in\mathbb{R}\Leftrightarrow\forall V(A)\exists\overset{\circ}{U}(x_0):Im\left(\overset{\circ}{U}(x_0)\cap E\right)\subset V(A)$

Упражнение. *Докажите равносильность.*

Определение 1.4 (ГЕЙне)**.** $\lim\limits_{x \to x_0} f(x) = A; x_0, A \in \overline{\mathbb{R}} \Leftrightarrow \forall x_n : x_n \in E, x_n \neq x_0, x_n \to x_0 \Rightarrow \lim\limits_{n \to \infty} f(x_n) = A$

Теорема 1.1 (iff)**.** *1. K to G is obv.*

Доказательство. Let $x_n \in E, x_n \neq x_0, x_n \to x_0$. For $\delta > 0 \exists n_0 : \forall n \geqslant n_0 : x_n \in U_\delta(x_0) \overset{x_n \neq x_0}{\Rightarrow} x_n \in \overset{\circ}{U}_\delta(x_0) \cap E \overset{Cauchy\ def.}{\Rightarrow} f(x_n) \in U_\varepsilon(A) \Rightarrow \lim\limits_{n \to \infty} f(x_n) = A$ □

2. G to K.

Доказательство. Let $\forall x_n : x_n \in E x_n \neq x_0, x_n \to x_0 \Rightarrow f(x_n) \to A$. let Cauchy be false then $\exists \varepsilon > 0 : \forall \delta > 0 \exists x \in E \cap \overset{\circ}{U}_\delta(x_0) : f(x) \notin U_\varepsilon(A)$. Let $\delta = \dfrac{1}{k}, k \in \mathbb{N} \Rightarrow \exists x_k \in E \cap \overset{\circ}{U}_\delta(y_k) : f(x_k) \notin U_\varepsilon(A)$. Consider $x_k : x_k \in E, x_k \neq x_0 0 < |x_k - x_0| < \dfrac{1}{k} \Rightarrow x_k \to x_0 \Rightarrow f(x_k) \to A$ □

2 2023-11-06

2.1 Limit properties

Теорема 2.1 (Properties)**.** *Let* $\lim\limits_{x \to x_0} f(x) = A, x_0 \in \overline{\mathbb{R}}$. *Then*

1. *For* $A \in \overline{\mathbb{R}}$ *a limit is the only one*

   Доказательство. By Geine definition □

2. *For* $A \in \mathbb{R} : \exists \overset{\circ}{U}(x_0) : f(x)$ *is bounded in* $\overset{\circ}{U}(x_0)$

3. *For* $A \in \overline{R} \setminus \{0\} \exists \overset{\circ}{U}(x_0) : f(x) \cdot A > 0 \forall x \in \overset{\circ}{U}_\delta(x_0)$

   Замечание. Take $\varepsilon = \dfrac{|A|}{2}$.

Теорема 2.2 (Arithmetic properties)**.** *Let* $f,g : \mathbb{R} \supset E \to \mathbb{R}, x_0$ *is LP of* $E, x_0 \in \overline{\mathbb{R}}, \lim\limits_{x \to x_0} f(x) = A, \lim\limits_{x \to x_0} g(x) = B, A,B \in \overline{\mathbb{R}}$. *Then*

1. $\lim\limits_{x \to x_0} (f + g)(x) = A + B$
2. $\lim\limits_{x \to x_0} (fg(x)) = A \cdot B$
3. *for* $g(x) \neq 0 : \lim\limits_{x \to x_0} \dfrac{f}{g}(x) = \dfrac{A}{B}$

Доказательство. By Geine definition. □

Теорема 2.3 (Inequalities). *Let $f,g : E \to \mathbb{R}, x_0$ is LP of E. Same for the previous one + $A < B$. Then $\exists \overset{\circ}{U}(x_0) : f(x) < g(x) \forall x \in \overset{\circ}{U}(x_0)$*

Доказательство. Take $\varepsilon = \dfrac{B - A}{3} > 0$.

$$\exists \delta_1 : x \in \overset{\circ}{U}_{\delta_1}(x_0) \Rightarrow f(x) < A + \varepsilon$$
$$\exists \delta_2 : x \in \overset{\circ}{U}_{\delta_2}(x_0) \Rightarrow g(x) > B - \varepsilon$$
$$\text{Let } \delta = \min(\delta_1, \delta_2) : f(x) < g(x)$$

□

Следствие (Предельный переход). Let $f(x) \leqslant g(x)$ in $\overset{\circ}{U}(x_0)$ & $\lim f(x) = A, \lim g(x) = B$. Then $A \leqslant B$

Доказательство. Обратное следствие из теоремы. □

Теорема 2.4 (милиционеры))))))))))))))))). *Let $f,g,h : E \to \mathbb{R}, x_0$ is LP $E, x_0 \in \overline{\mathbb{R}}$ & $f(x) \leqslant g(x) \leqslant h(x)$ for $\forall x \in \overset{\circ}{U}(x_0)$ & $\lim f(x) = \lim h(x)$. Then $\lim g(x) = \lim f(x)$*

Доказательство. By Geine definition. □

Определение 2.1. Let $f : E \to \mathbb{R}$. f is increasing $\Leftrightarrow \forall x_1, x_2 \in E : x_1 < x_2 \Rightarrow f(x_1) \leqslant f(x_2)$
Strictly $\uparrow \Leftrightarrow f(x_1) < f(x_2)$

Same for (strictly) decreasing ☺.

Теорема 2.5 (Weierstrass). *Let $f : E \to \mathbb{R}, s = \sup E \in \overline{\mathbb{R}}, s$ is LP of E, let $f \uparrow$ on E. Then $\lim\limits_{x\to s} f(x) = \sup\limits_{x\in E} f(x)$ and $\lim\limits_{x\to s} f(x) \in \mathbb{R} \Leftrightarrow f(x)$ is limited above on E*

Доказательство. Let f is increasing. Then $\exists \sup\limits_{x\in E} f(x) = A \in \mathbb{R}$. $\forall \varepsilon > 0 \exists x_1 \in E : A - \varepsilon < f(x_1) \leqslant A \Rightarrow \forall x > x_1 : A - \varepsilon < f(x_1) \leqslant f(x) \leqslant A \Rightarrow \lim\limits_{x\to s} f(x) = A$ □

Теорема 2.6 (Cauchy cryteria). *Let $f : E \to \mathbb{R}, x_0$ is LP of E. Then $\exists \lim f(x) \in \mathbb{R} \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 : \forall x_1, x_2 \in \overset{\circ}{U}_\delta(x_0) \cap E \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$*

Доказательство. $\boxed{\Rightarrow}$ By Cauchy definition, $\forall \varepsilon > 0 \exists \delta > 0 : x \in \overset{\circ}{U}_\delta(x_0) \cap E \Rightarrow |f(x) - A| < \dfrac{\varepsilon}{2}$. Take $x_1, x_2 \in \overset{\circ}{U}_\delta(x_0) \cap E \Rightarrow |f(x_1) - f(x_2)| \leqslant |f(x_1) - A| + |f(x_2) - A| < \varepsilon$

$\boxed{\Leftarrow}$. By Geine definition, let $x_n : \lim\limits_{n\to\infty} f(x_n) = A$. $x_n \to x_0 \Rightarrow$ for $\delta \exists n_0 : \forall n \geqslant n_0 : x_n \in \overset{\circ}{U}_\delta(x_0)$. Then $|f(x_{n+p}) - f(x_n)| < \varepsilon \forall n \geqslant n_0, p \in \mathbb{N} \Rightarrow f(x_n)$ is fundamental. Then $\exists \lim f(x_n) = A \in \mathbb{R}$

Задача. *Do limits have the same value?*

Proving that A is the same.

Take $x'_n, x''_n : f(x'_n) \to A', f(x''_n) \to A''$. Consider $y_n : x'_1, x''_1, x'_2, x''_2, \dots$. $f(y_n)$ doesnt have a limit if $A' \neq A''$ $\square$

3 2023-11-10

3.1 infinitesimal and infinity large functions

Let all functions be in a set of LP. Then $f : E \to \mathbb{R}$.

Определение 3.1. If $\lim\limits_{x \to x_0} \alpha(x) = 0$ then $\alpha(x)$ is infinitesimal function in $x_0 (x \to x_0)$

If $\lim\limits_{x \to x_0} \beta(x) = \infty$, then $\beta(x)$ is infinitely large in x_0.

Лемма 3.1. *Let $\alpha(x)$ be infinitesimal and $\beta(x)$ is infinitely large. Then*

1. $\lim\limits_{x \to x_0} \dfrac{1}{\beta(x)} = 0$
2. $\lim\limits_{x \to x_0} \dfrac{1}{\alpha(x) = \infty}$, *if* $\alpha(x) \neq 0$

Лемма 3.2. *Let α_1, α_2 be infinitesimal for $x \to x_0$. Then*

1. $\alpha_1(x) \pm \alpha_2(x)$ *is infinitesimal.*
2. $\alpha_1(x) \cdot \beta(x)$ *is infinitesimal if* $\beta(x)$ *is limited in* $\overset{\circ}{U}(x_0)$

Упражнение. *Proof the 1st statement*

Доказательство 2 пункта. Let $m \leqslant \beta(x) \leqslant M$, then $|\beta(x)| < C \Rightarrow 0 \leqslant |\alpha_1(x) \cdot \beta(x)| \leqslant C \cdot |\alpha_1(x)| \to 0$ $\square$

3.2 One sided limits

Let $x_0 \in \mathbb{R}$.

Определение 3.2. $U^+(x_0) = U(x_0) \cap \{x : x > x_0\}$ $U^-(x_0) = U(x_0) \cap \{x : x < x_0\}$

Определение 3.3. Let x_0 be a LP of $U^{\pm} \cap E, f : E \to \mathbb{R}$. Then $\lim\limits_{x \to x_0 \pm 0} = A \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 : \forall x \in U^{\pm}_{\delta}(x_0) \cap E \Rightarrow f(x) \in U_{\varepsilon}(A)$

Теорема 3.1 (Limit cryteria (one sided limits)). *x_0 Is LP of $U^+ \cap E, U^- \cap E$. Then $\lim\limits_{x \to x_0} f(x) = A \in \overline{\mathbb{R}} \Leftrightarrow \lim\limits_{x \to x_0+0} f(x) = \lim\limits_{x \to x_0-0} f(x) = A$*

Доказательство. $\boxed{\Rightarrow}$. $\forall \varepsilon \exists \delta : \forall x \in \overset{\circ}{U}_\delta(x_0) \cap E \Rightarrow f(x) \in U_\varepsilon(A) \Rightarrow$ for $x \in U^+_\delta(x_0) \cap E$ it's true too.

$\boxed{\Leftarrow}$. for $\varepsilon > 0 \exists \delta_1 > 0 : \forall x \in E : x_0 < x < x_0 + \delta_1 \Rightarrow f(x) \in U_\varepsilon(A)$ & $\exists \delta_2 > 0 : \forall x \in E : x_0 - \delta_2 < x < x_0 \Rightarrow f(x) \in U_\varepsilon(A)$ Then $\delta = \min\{\delta_1, \delta_2\} : x \in \overset{\circ}{U}_\delta(x_0) \cap E \Rightarrow f(x) \in U_\delta(A)$ $\square$

Теорема 3.2 (In terms of infinitesimal functions). *$\lim\limits_{x \to x_0} f(x) = A \in \mathbb{R} \Leftrightarrow f(x) = A + \alpha(x)$, where $\alpha(x) \underset{x \to x_0}{\to} 0$*

3.3 Continuity

Let $f : E \mapsto \mathbb{R}, x_0 \in E$.

Определение 3.4. f is continuous in $x_0 \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 : \forall x \in E \cap U_\delta(x_0) \Rightarrow f(x) \in U_\delta(f(x_0)) \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 : \forall x \in E : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \Leftrightarrow \forall V(f(x_0)) \exists U(x_0) : f(U(x_0) \cap E) \subset V(f(x_0))$

Теорема 3.3. *Let $f : E \to \mathbb{R}, x_0 \in E$. If*

1. *x_0 is LP of E, then f is continuous in $x_0 \Leftrightarrow \lim\limits_{x \to x_0} f(x) = f(x_0)$*
2. *x_0 is not a LP of E ($\Rightarrow x_0$ is isolated . Then f is continuous in x_0)*