Университет ИТМО
Факультет ПИиКТ

ДИСКРЕТНАЯ МАТЕМАТИКА

I СЕМЕСТР

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1 2023-09-05

1.1 Введение

Пастор – прочие элементы дискретной математики, в большинстве – комбинаторика

Карпов – теория графов

1.2 Sets and relations

Reading materials (about sets theory)

1. Senpinsky (med)
2. Vilenkin (easy)
3. Yekh (hard, 6th semester)

Slides

Определение 1.1 (Sets). множество — это группа объектов (неформально)

- objects of a set are its elements
- any object can be an element of a set
- $x \in Y$ means «x is an element of a set Y»
- $\varnothing$ is a set that has 0 elements
- it's really hard to define set **formally**
  - formally, sets are also elements of another sets
  - other objects have to be interpreted using any sets

We are learning **naive sets theory**

1.3 Defining a set

- Перечисление

  Пример. $Y = \{1,3,7,19,2021\}$

- Defining a set using condition

  $Y = \{x \in X \mid \textit{condition} \}$

Пример. $Y = \left\{ n \in Z | n \vdots 2 \right\}$

- If all elements of Y are in X, then Y is an subset of X. $Y \subset X$

  Замечание. $\mathcal{P}(X)$ – множество всех подмножеств множества X

  Замечание. We can also use $Y = \{x|condition\}$, but we call it a class!

$$X = \{1, 2\} \, Y = \{0, \{1, 2\}, \{3\}\} \, x \in Y, \ but \neg(x \subset Y)$$

1.4 Проблемы и парадоксы теории множеств

- Can a set be an item of itself?
- Russel's paradox (1901): let $Y = \{x|x \notin x\}$
  - is $Y \in Y$?
  - this question is neither true of false.
- in formal sets theory, $x \in x$ is forbidden (regularity axiom)
- class Y in Russel's paradox is not a set.
- class of every possible set (U) is not a set

1.5 Operations between sets

Let A, B be sets. Then:

- $A \cap B =^{def} \{x \in A | x \in B\}$ – intersection
- $A \cup B =^{def} \{x | (x \in A \vee x \in B)\}$ – union
- $A \backslash B =^{def} \{x \in A | x \notin B\}$ – difference
- $A \triangle B =^{def} \{(A \cup B) \backslash (A \cap B)\}$ – symmetrical difference

◁ Complemention. Let U be an *universum*, a set that contains every set. then

◁ $A =^{def} \{x \in U | x \notin A\}$

Упорядоченные пары элементов x y

$$(x,y) = \{(x), (x,y), (y)\}$$

1.6 Dechart multiplication

Определение 1.2. $A \times B =^{def} \{(x, y) \mid (x \in A \& y \in B)\}$

Пример. let $X = \{1,2,3\}, Y = \{1,2,3,4,5\}$. Then we got a matrix:

$$\begin{pmatrix} (x_1, y_1) & (x_1, y_2) & ... & (x_1, y_5) \\ (x_2, y_1) & (x_2, y_2) & ... & (x_2, y_5) \\ (x_3, y_1) & (x_3, y_2) & ... & (x_3, y_5) \end{pmatrix}$$

Аналогично мы можем определить умножение для n множеств. (можно сделать индукцией)

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1,x_2, \ldots, x_n | (x_1 \in A_1 \& \ldots \& x_n \in A_n))\}$$

We assume that $A^1 = A$

If it's not a multiset, then $\{3, 3\} = \{3\}$

1.7 Binary relations

Let $R \subset X \times Y$

If $X = Y$, then R is a binary relation on X.

Lets call a pair $(x,y) : x \in X \& y \in Y$, *appropriate*

An appropriate pair (x,y) is written: xRy.

Замечание. Binary relation is either *true* or *false* (R is a set of pairs, for which the statement $x \in X \& y \in Y$ is true)

Замечание. Binary relation can be interpreted as an oriented graph: elements of a set is it's vertices, and edges is drawn only if xRy.

1.8 Binary and n-ary relations

Similarly, $R \subset X_1 \times \cdots \times X_n$. If $X_1 = X_2 = \cdots = X_n$, then R is an n-ary relation on X.

Пример.
1. Equality $(a = b)$ – binary relation or $\mathbb{R}$
2. divisibility $(a \vdots b)$ – binary relation on $\mathbb{Z}$
3. let $G = (V, E)$, then:
   - смежность графа – на V.
   - инцидентность – между V, E.
4. A,B,C form a line – 3-ary relation on a plain

1.9 Properties of relations

Определение 1.3. Binary relation is called:

- reflexive, if xRx is true $\forall x \in X$
- irreflexive, if xRx is false $\forall x \in X$
- symmetrical, if $xRy \Rightarrow yRx$
- antisymmetrical, if $xRy \& yRx \Rightarrow x = y$.
- transitive, if $xRy, yRz \Rightarrow xRz$

Определение 1.4. Binary relation is relation of identity if it's reflexive, symmetrical and transitive

Замечание. Relation of identity splits the set to **identity classes**, so for any 2 elements of 1 class are equal, and 2 elements from different classes aren't

Examples:

- $a = b$
- $a||b$
- $a \sim b$
- division of polygons by amount of vertices.

2 2023-09-12

2.1 Order relation

- **Определение 2.1.** Binary relation $\prec$ on X is called *a relation of частичного order*, if it's antisymmetrical ond transitive
- If $\prec$ is irreflexive, then it's called an relation of strict order
- If it's reflexive – it's called an unstrict relation of particular order
  - As usual, for unstrict relation we use $\geqslant and \leqslant$.
- A set is particularly sorted if the order relation is defined.
  - Formally, a set of sorted order is sorted pair $(<, X)$, whwre X is a set and $<$ is order relation.
  - In particularly sorted set some pairs are *uncompareable*. Then can $\exists a,b \in X$, such that every expression $a = b, b < a, a < b$ is false.

2.2 Relation of linear order

Определение 2.2. Binary relation $<$ on X is called anj relation of linear order, if it's a relation of particular order and $\forall a,b \in X : a = b \vee a < b \vee b < a$.

In this case, a pair $(X, <)$ is called linearly sorted set

Пример.
1. $a < b$ (on $\mathbb{R}$)
2. $a \vdots b$ (on $\mathbb{N}$))
3. let X be a set. Then, $A \subset B$ is a particular orderly relation on $\mathcal{P}(X)$

2.3 Mappings and functions

Not formally, *a mapping* from X to Y is a rule f such that: $\forall x \in X \exists! y \in Y : f(x) = y$

Определение 2.3.
- ▷ A binary relation $f \subset X \times Y$ is *mapping* from $X \mapsto Y$, if $\forall x \in X \exists! y \in Y$, such that only one pair $(x, y) \in f$ exists.
- ▷ Notation: $f : X \mapsto Y$
- ▷ Second element of pair is denoted as $f(x)$ and it's called an image of element x for mapping f.
- ▷ if $y = f(x)$, then x is prototype of y
- ▷ Different to the image, a prototype is not guaranteed to exist, and prototype can be not the only one

2.4 Injection, surjection and biection

Определение 2.4. A mapping $f : X \mapsto Y$ is called:

- ▷ an injection, if $\forall x_1, x_2 \in X : x_1 \neq x_2 : f(x_1) \neq f(x_2)$
- ▷ a surjection, if $\forall y \in Y \exists x \in X : f(x) = y$.
- ▷ a biection, if it's an injection and a surjection.

Замечание.
- ▷ A biection is a one-to-one correspondence between $X, Y : \forall x \in X \exists! y \in Y \wedge \forall y \in Y \exists! x \in X$.
- ▷ in particular, if X, Y are not endless sets and $\exists$ biection , then $|X| = |Y|$

2.5 Composition of relations

Определение 2.5. A composition of mappings $f : X \mapsto Y$ & $g : Y \mapsto Z$ is mapping $g \circ f : X \mapsto Z$, that is defined by formula $(g \circ f)(x) = g(f(x))$

A mapping is called **reversible**.

▷

Определение 2.6. A mapping $g : Y \mapsto X$ is called reverse mapping, if both $f \circ g$ and $g \circ f$ are equal

▷ Then, $g(f(x)) = x \forall x \in X$, and $f(g(y)) = y \forall y \in Y$.

2.6 Reverse cryteria

Теорема 2.1. *A relation $f : X \mapsto Y$ is reversible $\Leftrightarrow$ f is a biection.*

Доказательство. $\Leftarrow$: $\forall y \in Y$ we denote $f^{-1}(y)$ – the only one prototype of y.

Then, $f^{-1} : Y \mapsto X$ – is reverse mapping to f.

$\Rightarrow$: Let f^{-1} is reverse mapping to f.

f is a injection, because of $f(x) = f(y) \Rightarrow x = f^{-1}(f(x)) = y = f^{-1}(f(y)) \Rightarrow x = y$

f is a surjection, because $\forall y \in Y$ we have: $y = f^{-1}(f(y))$ □

2.7 finite sets

▷ Let X be a finite set. A number of it's elements we denote as $|X|$.

▷ We already know that $|X| = |Y| \Leftrightarrow$ we can set a biection between X, Y.

Лемма 2.1. *If $|X| = m$ and $|Y| = n$, then $|X \times Y| = mn$*

Доказательство. Every m elements are in n pairs with Y set. □

Следствие. if $|X_i| = m$, where $i \in [1..k]$, then $|X_1 \times X_2 \times \cdots \times X_k| = m_1 \cdot m_2 \cdot \cdots \cdot m_k$.

Упражнение. *Proof using induction*

2.8 Finite sets, a number of subsets

Теорема 2.2. *If* $|X| = m$, *then* $\mathcal{P}(X) = 2^m$

Доказательство. Trivial. □

Замечание. We have literally built a bisection between $\mathcal{P}, \{0{,}1\}^m$

◁ $A \subset X$ corresponds to $(a_1, \dots, a_m) \in \{0,1\}^m$, where:

$$a_i = \begin{cases} 1, & \text{if } x_i \in A \\ 0, & \text{if } x_i \notin A \end{cases}$$

2.9 Finite sets: a number of relations

Теорема 2.3. *Let* $|X| = k, |Y| = n$, *then*

1. *A number of mappings is* n^k
2. *A number of injections* $f : X \mapsto Y$ *is* $n(n-1)\dots(n-k+1)$

Доказательство. 1. $\forall x \in X$ we can choose an image by only n choices

2. An image x_1 can be chosen by n choices. Then – $(n-1)$, and so on.

□

Замечание. For $n \geqslant K$ we have: $n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$

Упражнение. *What is the number of surjections from* X *to* Y*?*

2.10 Finite sets: permutations and размещения

Определение 2.7 (Permutation)**.** A permutation is any biection $\sigma : X \mapsto X$.

Следствие. If $|X| = n$, then $n!$ is number of permutations.

Определение 2.8. ▷ A number of injections $f : [1..k] \mapsto [1..n]$ is called an accomodation, from n elements on k and denoted A_n^k.

▷ A number of mappings $f : [1..k] \mapsto [1..n]$is called a number of accomodations with repetitons and denoted as $\overline{A}_n^k$

1. A_n^k

2.11 Countable sets

Определение 2.9. A set is called **countable** if $\exists f : X \mapsto \mathbb{N}$ (biection)

Замечание. ▷ That means, that we can numerate X using natural numbers.

▷ It's elements can be written as: $X = \{x_1,x_2,\dots\}$, where $x_k = f^{-1}(k)$

Пример. ▷ $2\mathbb{N} = \{2n|n \in \mathbb{N}\}$ is a set of every even number

▷ $\mathbb{Z}$ is a set of every целых чисел

2.12 Countability of multiplication

Теорема 2.4. $\mathbb{N} \times \mathbb{N}$ *is countable.*

Доказательство. A function $f(x,y) = \frac{(x+y-1)(x+y-2)}{2} + y$ – biection from $\mathbb{N} \times N$ to $\mathbb{N}$. □

Замечание.

▷ This function is naming a cells of infinite table «by diagonals»

▷ Another example of biection: $g(x.y) = 2^{x-1}(2y-1)$

Следствие. Let $X_1, X_2, \dots, X_n$ be countable sets. Then $X_1 \times X_2 \times X_3 \times \dots \times X_n$ is countable too.

Доказательство. Proof using induction. □

Теорема 2.5. *An infinite subset of countable set is countable.*

Доказательство. Let X be a countable set and $A \subset X$ is it's infinite subset.

▷ Consider biection $f : X \mapsto \mathbb{N}$

▷ Then $g(x) =^{def} |\{a \in A | f(a) \leqslant f(x)\}|$– biection from A to $\mathbb{N}$

□

2.13 No more that a countable set

Определение 2.10. ▷ X is *no more than countable*, if X is either finite or countable

▷ X is uncountable, if it's neither finite or countable.

Теорема 2.6. *Let $X \neq \varnothing$. Then:*

1. *X is no more than countable.*
2. *$\exists f : X \mapsto \mathbb{N}$ (injection)*
3. *$\exists g : \mathbb{N} \mapsto X$ (surjection)*

Доказательство. $1 \Rightarrow 3$. Let X be no more than countable.

▷ If X is infinite, then it's countable.

- Then $\exists f : X \mapsto \mathbb{N}$ (injection)
- Then, $\exists f^{-1} : \mathbb{N} \mapsto X$ (surjection)

if X is finite, then $|X| = n$

▷ Then, a biection exists: $f : X \mapsto [1..n]$

▷ Let $g(y) = \begin{cases} f^{-1}(y), y \leqslant n \\ f^{-1}(n), y > n \end{cases}$

It's easy to see that $g(y)$ is a surjection.

$3 \Rightarrow 2$. Let $g : \mathbb{N} \mapsto X$ – surjection.

▷ $\forall x \in X$ it has a prototype.

▷ Select the least prototype.

▷ Let $f(x) = min \{y \in \mathbb{N} | g(y) = x\}$

▷ Easy to see, that $f : X \mapsto \mathbb{N}$ is an injection.
$2 \Rightarrow 1$. Let $f : X \mapsto \mathbb{N}$ be an injection.

- Consider a set $f(X) = \{f(x) | x \in X\}$
- Then $f : X \mapsto f(X)$ is a biection
- Because of $f(X) \subset \mathbb{N}$, an $f(X)$ is no more than countable.
- If $f(X)$ is finite, the X is finite too.
- If $f(X)$ is countable, then X is countable too.

□

Следствие. If $f : X \mapsto Y$ is an injection and Y is countable, then X is no more than countable.

Доказательство. ▷ Let $g : Y \mapsto \mathbb{N}$ be a biection

▷ Then $g \circ f : X \mapsto \mathbb{N}$ – injection

□

Следствие. If $g : Y \mapsto X$ is a surjection and Y is countable, then X is no more than countable.

Доказательство. ▷ Let $f : \mathbb{N} \mapsto Y$ – biection

▷ Then $g \circ f : \mathbb{N} \mapsto X$ – surjection.

□

Теорема 2.7. $\mathbb{Q}$ *is countable.*

Доказательство. Consider a mapping $g : \mathbb{Z} \times \mathbb{N} \mapsto \mathbb{Q}$, set by formula $g(a,b) = \frac{a}{b}$

▷ Obviously, g is a surjection.

▷ Then, $\mathbb{Q}$ is no more than countable.

▷ But $\mathbb{Q}$ is infinite.

▷ Then, $\mathbb{Q}$ is countable.

□

Теорема 2.8. *A union of one no more than countable set of no more than countable sets is no more, than countable*

Замечание. That means if we are given infinite последовательность of sets $A_1, A_2, \ldots$, each of them is no more than countable, then a set $B = \bigcup_i A_i$ is also no more than countable.

2.14 A union of no-more-than-countable sets

Доказательство. Let $f_i : A_i \mapsto \mathbb{N}$ be an injection.

▷ $\forall x \in B$ let $s(x) = min\,\{n \in \mathbb{N} | x \in A_n\}$

▷ Consider a mapping $h : B \mapsto \mathbb{N} \times \mathbb{N}$, defined by a formula

$$h(x) = (s(x), f_{s(x)}(x)$$

▷ Obviously, h is an injection.

- ▷ Then, B is no-more-than-countable.

$\square$

Замечание (Will be on practice)**.** In particular, an union of any finite or countable set of countable sets is always countable.

Определение 2.11. ▷ a real number α is called **algebraic**, if α is a root of non-zero polynomial, with $\mathbb{Q}$ coefficients

- ▷ Else, α is transcendent.

Упражнение. *A set of all algebraic numbers is countable.*