Equational Logical Frameworks as Locally Cartesian Closed Categories

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1 Overview

1.1 Terminology

We follow [Har21] and use the word "class" for what is usually called "type" and the word "object" for what is usually called "term". This is to distinguish between the meta-level class in the logical framework and types internal to the type theory defined in the logical framework. To not confuse this with the categorical notion of object, we will write $\mathcal{C}$-object for an object in a category $\mathcal{C}$.

1.2 Overview of this note

In section 2, we present the syntax for an *equational logical framework* (LF for short). In section 3, we set out the main categorical definitions that are relevant to the discussion. In section 4, we describe the category of contexts for the LF. The main goal is examine this category of context via definitions in the literature. In particular I will identify two subcategories $\mathcal{S}_\Pi$ and $\mathcal{S}_{\Pi,=}$ of the category of contexts, and identify which definitions these satisfy.

1.3 Overview of definitions in the literature

There are many closely related definitions that are useful in analyzing the category of contexts. Let us denote with $\mathcal{C}$ the category of contexts for our LF, and fix two subcategories $\mathcal{S}_\Pi$ and $\mathcal{S}_{\Pi,=}$ which we define later.

- $\mathcal{C}$ is *not finitely complete*, in particular $\mathcal{C}$ does not have all pullbacks.
- Because $\mathcal{C}$ is not finitely complete, $\mathcal{C}$ is *not* a *representable map category* as in Uemura [Uem19]. However, by equivalently adding an equality class for objects in classes or taking the finite completion $\overline{\mathcal{C}}$, we get $(\overline{\mathcal{C}}, \mathcal{S}_\Pi)$ and $(\overline{\mathcal{C}}, \mathcal{S}_\Pi)$ which are both representable map categories.

$$\frac{\Gamma \vdash O_0 \ : \ K \quad \Gamma \vdash O_1 \ : \ K}{\Gamma \vdash O_0 =_K O_1 \ \mathsf{cls}} \ \textsc{eq-cls}$$

- Because $\mathcal{C}$ is not finitely complete, $\mathcal{C}$ is not locally Cartesian closed. Furthermore, adding an equality class/taking the finite completion $\overline{\mathcal{C}}$ still does not give a locally Cartesian closed category due to lack of impredicative dependent function classes/Πs for all $\overline{\mathcal{C}}$-morphisms. Then adding an impredicative dependent function class (and equality classes for objects in classes) is equivalent to taking the local Cartesian closure as in Gratzer [GS21],

$$\frac{\Gamma \vdash K_0 \ \mathsf{cls} \quad \Gamma, X : K_0 \vdash K_1 \ \mathsf{cls}}{\Gamma \vdash \{X \ : \ K_0\} \, K_1 \ \mathsf{cls}} \ \textsc{inpredicative-pi}$$

- Both $\mathcal{S}_\Pi$ and $\mathcal{S}_{\Pi,=}$ are *display structures* (hence sets of *display maps*) as in Taylor [Tay99]. Related to the first point about representable map categories, $\mathcal{S}_\Pi$ and $\mathcal{S}_{\Pi,=}$ are locally Cartesian closed.
- Both $\mathcal{S}_\Pi$ is similar in spirit to a universe (originally universe in a topos) as in Streicher [SD04]. The main differences are 1. that monos are not really related to our syntax; they correspond to "propositions" in the sense of bundles with subsingleton fibers. And 2. that our syntax has no dependent pair class (Σ-type) or unit type, which correspond to composition and identity respectively in the presence of El, a display map classifying context extension by a sort.

For most definitions there will be a strict/structure version as well as a non-strict/proposition version, due to coherence issues for pullback. For example, display structures in contrast to classes of displays. To tackle such issues for general semantics, we can use for example categories with families [CCD20], categories with attributes [Jac93], or natural models [Awo17]. One can verify that the category of contexts $\mathcal{C}$ can be used in any of these semantics.

1.4 Acknowledgements

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2 An Equational Logical Framework

$$\frac{}{\bullet \vdash \mathsf{ctx}}\ \textsc{ctx-emp} \qquad \frac{\Gamma \vdash K\ \mathsf{cls}}{\Gamma, X : K \vdash \mathsf{ctx}}\ \textsc{ctx-ext} \qquad \frac{}{X : K \in \Gamma, X : K}\ \textsc{ctx-hd} \qquad \frac{X_0 : K_0 \in \Gamma}{X_0 : K_0 \in \Gamma, X_1 : K_1}\ \textsc{ctx-tl}$$

Figure 1: Context formation

$$\frac{\Gamma \vdash \mathsf{ctx}}{\Gamma \vdash \mathsf{Sort}\ \mathsf{cls}}\ \textsc{sort-cls} \qquad \frac{\Gamma \vdash S : \mathsf{Sort}}{\Gamma \vdash S\ \mathsf{cls}}\ \textsc{incl-cls} \qquad \frac{\Gamma \vdash S : \mathsf{Sort} \quad \Gamma, X : S \vdash K\ \mathsf{cls}}{\Gamma \vdash \{X : S\}\, K\ \mathsf{cls}}\ \textsc{pi-cls}$$

Figure 2: Class formation

$$\frac{\Gamma \vdash S_0 : \mathsf{Sort} \quad \Gamma, X : S_0 \vdash S_1 : \mathsf{Sort}}{\Gamma \vdash \{X : S_0\}\, S_1 : \mathsf{Sort}}\ \textsc{pi-sort} \qquad \frac{\Gamma \vdash S : \mathsf{Sort} \quad \Gamma \vdash O_0 : S \quad \Gamma \vdash O_1 : S}{\Gamma \vdash O_0 =_S O_1 : \mathsf{Sort}}\ \textsc{eq-sort}$$

Figure 3: Sort formation

In this section we recall the equational logical framework, as described in [Har21] (with some minor changes). This LF can be used to define a wide variety of type theories, including dependent type theories such as Martin Löf type theory or ModTT [SH21]. Within the LF we define type theories by asserting the rules of the type theory via a collection of generators, called a signature and denoted Σ. We state only state the syntax and judgment rules and refer to [Har21] for details and examples.

variable	X	
expression	$O, S, K, e ::=$	$X \mid \mathsf{Sort} \mid \{X : S\}\, e \mid O_0 =_S O_1 \mid \mathsf{self} \mid [X : S]\, O \mid O_0\, O_1$
context	$\Gamma, \Delta ::=$	$\cdot \mid \Gamma, X : K$

O indicates expressions representing objects (though just an expression at this point), similarly S represents sorts and K represents classes. The agnostic expression symbol e represents sorts or a classes.

The judgments we define are

$\Gamma \vdash \mathsf{ctx}$	Γ is a context
$X : K \in \Gamma$	$X : K$ appears in context Γ
$\Gamma \vdash K\ \mathsf{cls}$	K is a class in context Γ
$\Gamma \vdash O : K$	O is an object of class K in context Γ
$\Gamma \vdash K_0 = K_1\ \mathsf{cls}$	K_0, K_1 are equal classes in context Γ
$\Gamma \vdash O_0 = O_1 : K$	O_0, O_1 are equal objects of class K in context Γ

The legal judgments given by the closure under the judgment rules in figures 1, 2, 3, 4, 5.

Note that unlike in [Har21] we have EQ-SORT creating a sort rather than a class. This will ensure one of our URMs is an LCCC.

$$\frac{\Gamma \vdash \mathsf{ctx} \quad X : K \in \Gamma}{\Gamma \vdash X \ : \ K} \ \textsc{var-obj} \qquad \frac{\Gamma \vdash S \ : \ \mathsf{Sort} \quad \Gamma, X : S \vdash O \ : \ K}{\Gamma \vdash [X \ : \ S]\, O \ : \ \{X \ : \ S\}\, K} \ \textsc{pi-lam-obj}$$

$$\frac{\Gamma \vdash O_0 \ : \ \{X \ : \ S\}\, K \quad \Gamma \vdash O_1 \ : \ S}{\Gamma \vdash O_0\, O_1 \ : \ [O_1/X]K} \ \textsc{pi-app-obj} \qquad \frac{\Gamma \vdash O \ : \ K}{\Gamma \vdash \mathsf{self} \ : \ O =_K O} \ \textsc{eq-self-obj}$$

$$\frac{\Gamma \vdash O \ : \ K_0 \quad \Gamma \vdash K_0 = K_1 \ \mathsf{cls}}{\Gamma \vdash O \ : \ K_1} \ \textsc{je-obj}$$

Figure 4: Object formation

Judgmental equality for classes $\Gamma \vdash K_0 = K_1$ cls and objects $\Gamma \vdash O_0 = O_1 \ : \ K$ are defined to be congruences, i.e. equivalence relations that respect formation of classes, sorts, and objects. Furthermore, we have β and η rules for PI.

Finally, signatures are defined as contexts $\Sigma \vdash$ ctx, and the type theory generated by a signature Σ consists of legal judgments over Σ. Note that before introducing a signature we cannot make any sorts, nor can we make any interesting objects. The only rules for their formation are via PI-SORT and EQ-SORT, which require more sorts and objects as premises. It is the signature that populates Sort with sorts and populates those sorts with objects. Given a signature Σ, we write $\lambda^{\Pi\mathsf{Eq}}[\Sigma]$ to denote the generated type theory.

$$\begin{array}{lcl}
\Gamma \vdash_\Sigma \mathsf{ctx} & := & \Sigma, \Gamma \vdash \mathsf{ctx} \\
\Gamma \vdash_\Sigma K \ \mathsf{cls} & := & \Sigma, \Gamma \vdash K \ \mathsf{cls} \\
\Gamma \vdash_\Sigma O \ : \ K & := & \Sigma, \Gamma \vdash O \ : \ K \\
\Gamma \vdash_\Sigma K_0 = K_1 \ \mathsf{cls} & := & \Sigma, \Gamma \vdash K_0 = K_1 \ \mathsf{cls} \\
\Gamma \vdash_\Sigma O_0 = O_1 \ : \ K & := & \Sigma, \Gamma \vdash O_0 = O_1 \ : \ K
\end{array}$$

3 Categorical definitions

Definition – Cartesian closed category (CCC)
We say a category $\mathcal{C}$ is cartesian closed when

- $\mathcal{C}$ has finite products
- For each $\mathcal{C}$-object A, the product with A functor $\times A : \mathcal{C} \to \mathcal{C}$ has a right adjoint.

We denote the right adjoint by $[A, -]$ and call it internal hom.

Definition – Locally cartesian closed category (LCCC)
We say a category $\mathcal{C}$ is locally cartesian closed when either of the following equivalent definitions hold

1. Every slice category $\mathcal{C}/A$ over a $\mathcal{C}$-object A is cartesian closed.
2. $\mathcal{C}$ has pullbacks, and for each morphism $f \ : \ B \to A$ in $\mathcal{C}$, the base change (pullback along f) $f^* : \mathcal{C}/A \to \mathcal{C}/B$ has a right adjoint.

We denote the right adjoint to base change by Π_f.

We do not show that these two definitions are equivalent. The idea is that existence of pullbacks correspond to existence of products in slices and Πs correspond to internal homs in slices.

$$\frac{\Gamma \vdash K \ \mathsf{cls}}{\Gamma \vdash K = K \ \mathsf{cls}} \ \textsc{cls-rfl} \qquad \frac{\Gamma \vdash K_0 = K_2 \ \mathsf{cls} \qquad \Gamma \vdash K_1 = K_2 \ \mathsf{cls}}{\Gamma \vdash K_0 = K_1 \ \mathsf{cls}} \ \textsc{cls-st} \qquad \frac{\Gamma \vdash O \ : \ K}{\Gamma \vdash O = O \ : \ K} \ \textsc{obj-rfl}$$

$$\frac{\Gamma \vdash O_0 = O_2 \ : \ K \qquad \Gamma \vdash O_1 = O_2 \ : \ K}{\Gamma \vdash O_0 = O_1 \ : \ K} \ \textsc{obj-st} \qquad \frac{\Gamma \vdash O_0 = O_1 \ : \ K_0 \qquad \Gamma \vdash K_0 = K_1 \ \mathsf{cls}}{\Gamma \vdash O_0 = O_1 \ : \ K_1} \ \textsc{obj-je-cls}$$

$$\frac{\Gamma, X : S \vdash O_0 \ : \ K \qquad \Gamma \vdash O_1 \ : \ S}{\Gamma \vdash ([X \ : \ S] \, O_0) O_1 = [O_1/X] O_0 \ : \ [O_1/X] K} \ \textsc{app-lam} \qquad \frac{\Gamma \vdash O \ : \ \{X \ : \ S\} \, K}{\Gamma \vdash O = [X \ : \ S] \, (OX) \ : \ \{X \ : \ S\} \, K} \ \textsc{lam-app}$$

$$\frac{\Gamma \vdash O_0 \ : \ O_1 =_S O_2}{\Gamma \vdash O_1 = O_2 \ : \ S} \ \textsc{reflection} \qquad \frac{\Gamma \vdash O_0 \ : \ O_1 =_S O_2 \qquad \Gamma \vdash O_3 \ : \ O_1 =_S O_2}{\Gamma \vdash O_0 = O_3 \ : \ O_1 =_S O_2} \ \textsc{unicity}$$

$$\frac{\Gamma \vdash S_0 = S_1 \ : \ \mathsf{Sort}}{\Gamma \vdash S_0 = S_1 \ \mathsf{cls}} \ \textsc{je-incl-cls} \qquad \frac{\Gamma \vdash S_0 = S_1 \ : \ \mathsf{Sort} \quad \Gamma, X : S_0 \vdash K_0 = K_1 \ \mathsf{cls}}{\Gamma \vdash \{X \ : \ S_0\} \, K_0 = \{X \ : \ S_1\} \, K_1 \ \mathsf{cls}} \ \textsc{je-pi-cls}$$

$$\frac{\Gamma \vdash S_0 = S_1 \ : \ \mathsf{Sort} \quad \Gamma, X : S_0 \vdash S_2 = S_3 \ : \ \mathsf{Sort}}{\Gamma \vdash \{X \ : \ S_0\} \, S_2 = \{X \ : \ S_1\} \, S_3 \ : \ \mathsf{Sort}} \ \textsc{je-pi-sort}$$

$$\frac{\Gamma \vdash S_0 = S_1 \ : \ \mathsf{Sort} \quad \Gamma \vdash O_0 = O_1 \ : \ S_0 \quad \Gamma \vdash O_2 = O_3 \ : \ S_1}{\Gamma \vdash O_0 =_{S_0} O_2 = O_1 =_{S_0} O_3 \ : \ \mathsf{Sort}} \ \textsc{je-eq-sort}$$

$$\frac{\Gamma \vdash S_0 = S_1 \ : \ \mathsf{Sort} \quad \Gamma, X : S_0 \vdash O_0 = O_1 \ : \ K}{\Gamma \vdash [X \ : \ S_0] \, O = [X \ : \ S_1] \, O_1 \ : \ \{X \ : \ S_0\} \, K} \ \textsc{je-pi-lam-obj}$$

$$\frac{\Gamma \vdash O_0 = O_1 \ : \ \{X \ : \ S\} \, K \quad \Gamma \vdash O_2 = O_3 \ : \ S}{\Gamma \vdash O_0 \, O_2 = O_1 \, O_3 \ : \ [O_2/X] K} \ \textsc{je-pi-app-obj}$$

Figure 5: Judgmental equality

$$B \xrightarrow{f} A \qquad \begin{array}{ccc} & \mathcal{C}/A & \\ & {\scriptstyle f^*}\downarrow \dashv \uparrow{\scriptstyle \Pi_f} & \\ \times f & \mathcal{C}/B & -^f \\ & {\scriptstyle \Sigma_f}\downarrow \dashv \uparrow{\scriptstyle f^*} & \\ & \mathcal{C}/A & \end{array}$$

In the above Σ_f is the functor that composes the map from the slice with f.

> **Definition – Display**
> Let $\mathcal{S}$ be a set of morphisms in category $\mathcal{C}$. We say $\mathcal{S}$ is a *set of display maps* when for any compatible pair of morphisms $f : A \to B$ in $\mathcal{C}$ and $g : C \to B$ in $\mathcal{S}$ the pullback exists and $f^*g \in \mathcal{S}$.
>
> We say $\mathcal{S}$ is a *display structure* when we have an operation that takes a pair of morphisms $f : A \to B$ in $\mathcal{C}$ and $g : C \to B$ in $\mathcal{S}$ and returns a pullback such that $f^*g \in \mathcal{S}$.

The following definition is similar in spirit to both the definition of *universes in a topos* from [SD04] and also *representable map categories* from Uemura [Uem19].

> **Definition – Universe of representable maps**
> Let $\mathcal{C}$ be a category, and $\mathcal{S}$ a set of morphisms in $\mathcal{C}$. We may denote morphisms in $\mathcal{S}$ by $S : \Gamma, S \to \Gamma$ or $\sigma : \Delta \to \Gamma$, though this is just notation. We say $\mathcal{S}$ is an universe of representable maps (URM) when
>
> 1. $\mathcal{S}$ is stable under $\mathcal{C}$-pullbacks (strictly), i.e. if $\delta : \Delta \to \Gamma$ is in $\mathcal{C}$ and $S : \Gamma, S \to \Gamma$ is in $\mathcal{S}$ and we have a pair of morphisms $S^*\delta : \Delta, \delta S \to \Gamma, S$ in $\mathcal{C}$ and $\delta^* S : \Delta, \delta S \to \Delta$ in $\mathcal{S}$ such that
>
> $$\begin{array}{ccc} \Delta, \delta^* S & \xrightarrow{S^*\delta} & \Gamma, S \\ {\scriptstyle \delta^* S}\downarrow & \lrcorner & \downarrow{\scriptstyle S} \\ \Delta & \xrightarrow[\delta]{} & \Gamma \end{array}$$
>
> Furthermore, this preserves identities and compositions from $\mathcal{C}$ strictly, so that
>
> $$\mathbb{1}_\Gamma^* S = S \text{ and } S^* \mathbb{1}_\Gamma = \mathbb{1}_{\Gamma,S}$$
>
> and
>
> $$\varepsilon^*(\delta^* S) = (\delta \circ \varepsilon)^* S \text{ and } (S^*\delta)^* \varepsilon = S^*(\delta \circ \varepsilon)$$
>
> from which it follows that $S^* : \mathcal{C}/\Gamma \to \mathcal{C}/\Gamma, S$ is a functor.
>
> 2. $\mathcal{S}$ is closed under identities, i.e. if $\sigma : \Delta \to \Gamma$ is in $\mathcal{S}$ then so are $\mathbb{1}_\Delta$ and $\mathbb{1}_\Gamma$.
> 3. $\mathcal{S}$ is closed under composition (aka Σs), i.e. if
>
> $$\Gamma, S, T \xrightarrow[\in\mathcal{S}]{T} \Gamma, S \xrightarrow[\in\mathcal{S}]{S} \Gamma \qquad \text{then} \qquad \Sigma_S T := S \circ T \in \mathcal{S}$$
>
> 4. $\mathcal{S}$ is closed under Πs, i.e. if $S : \Gamma, S \to \Gamma$ is in $\mathcal{S}$ then the pullback functor $S^* : \mathcal{C}/\Gamma \to \mathcal{C}/\Gamma, S$ has a right adjoint $\Pi_S : \mathcal{C}/\Gamma, S \to \mathcal{C}/\Gamma$, and
>
> $$\Gamma, S, T \xrightarrow[\in\mathcal{S}]{T} \Gamma, S \xrightarrow[\in\mathcal{S}]{S} \Gamma \qquad \text{then} \qquad \Pi_S T \in \mathcal{S}$$

> Note that the first four conditions combined result in $\mathcal{S}$ forming a subcategory of $\mathcal{C}$ that is an LCCC. We can think of a URM as the smallest LCCC stable under $\mathcal{C}$-pullbacks, generated by some maps. The main difference of this definition with that of a representable map category is dropping the condition that $\mathcal{C}$ is finitely complete. We address the differences with the definition of a universe in a topos later. Note that in particular any URM is a display structure.

4 The category of contexts

I order to find categorical semantics for a type theory it is useful to study the category of contexts. In this section we construct the category of contexts. We will call this category $\mathcal{C}$ for the rest of this section.

We begin by recalling $\lambda^{\times\to}$, simply typed λ-calculus with only unit, product and function classes. And use intuition of its categorical semantics to motivate those for logical frameworks. Recall that $\lambda^{\times\to}$ can be interpreted in a CCC $\mathcal{D}$ by taking classes K as $\mathcal{D}$-objects $[\![K]\!]$, contexts $X_1 : K_1, \ldots, X_n : K_n$ as products $[\![K_1]\!] \times \cdots \times [\![K_n]\!]$, and objects $\Gamma \vdash K : O$ as morphisms $[\![O]\!] : [\![\Gamma]\!] \to [\![K]\!]$. This seems quite natural, since product classes become products, and function classes become internal homs in the semantics. If we are to immitate this for $\lambda^{\Pi\mathsf{Eq}}[\Sigma]$, we would only be able to talk about *closed classes* $\bullet \vdash K$ cls, since in $\lambda^{\times\to}$ we don't have dependent classes $\Gamma \vdash K$ cls. However, in the presence of dependent pairs (aka Σ-types) we could view a context $\Gamma = X_1 : K_1, \ldots, X_n : K_n$ as a closed class $\Sigma_\Gamma := \Sigma_{X_1:K_1} \cdots \Sigma_{X_{n-1}:K_{n-1}} K_n$. Then any dependent class can be viewed as a context, which can be viewed as a closed class.

4.1 Contexts and dependent classes

Instead of including dependent pairs, let us view dependent classes $\Gamma \vdash K$ cls as extended contexts $\Gamma, X : K$, and take *contexts* (up to judgmental equality) as $\mathcal{C}$-objects, and *as a consequence* dependent classes as $\mathcal{C}$-objects.

$$\Gamma \vdash \mathsf{ctx} \qquad\qquad [\![\Gamma]\!] \in \mathcal{C}$$

It is then natural to define morphisms of contexts (aka a substitution) (up to judgmental equality), denoted $\Gamma \vdash \sigma : \Delta$, consisting of the data of objects

$$\begin{array}{l} \Gamma \vdash \sigma_1 : K_1 \\ \Gamma \vdash \sigma_2 : [\sigma_1/X_1]K_2 \\ \vdots \\ \Gamma \vdash \sigma_n : [\sigma_{n-1}/X_{n-1}] \cdots [\sigma_1/X_1]K_n \end{array} \qquad \text{when } \Delta = X_1 : K_1, \cdots, X_n : K_n$$

Such a substitution corresponds to a closed object of the closed class $\Sigma_\Gamma \to \Sigma_\Delta$ in the presence of dependent pairs.

$$\Gamma \vdash \sigma : \Delta \qquad\qquad [\![\sigma]\!] : [\![\Gamma]\!] \to [\![\Delta]\!]$$

The empty context gives a $\mathcal{C}$-object $[\![\bullet]\!]$, and for any context Γ there is exactly one substitution $\Gamma \vdash \sigma : \bullet$, given by the data of no objects. This makes $[\![\bullet]\!]$ the terminal object.

$$\frac{}{\bullet \vdash \mathsf{ctx}}\ \textsc{ctx-emp} \qquad\qquad \mathbf{1}_{\mathcal{C}} \in \mathcal{C}$$

For context extension, given $\Gamma \vdash K\ \mathsf{cls}$, we obtain a substitution $\Gamma, X : K \vdash \sigma\ :\ \Gamma$ by $\Gamma, X : K \vdash X_i\ :\ K_i$ for each $X_i : K_i \in \Gamma$. Then let's take the interpretation of K to be a $\mathcal{C}/[\![\Gamma]\!]$-object, where $\mathcal{C}/[\![\Gamma]\!]$ is the slice category.

$$[\![\Gamma \vdash K\ \mathsf{cls}]\!] := [\![\Gamma, X : K]\!] \text{ and } [\![\Gamma, X : K \vdash \sigma\ :\ \Gamma]\!]$$

We can informally identify use the slice-object $[\![\Gamma \vdash K\ \mathsf{cls}]\!]$ to denote the underlying morphism.

$$\frac{\Gamma \vdash K\ \mathsf{cls}}{\Gamma, X : K \vdash \mathsf{ctx}}\ \textsc{ctx-ext} \qquad\qquad [\![\Gamma, X : K]\!] \xrightarrow{[\![\Gamma \vdash K\ \mathsf{cls}]\!]} [\![\Gamma]\!]$$

4.2 Judgmental Equality

Technically we should we taking $\mathcal{C}$-objects to be the set of contexts quotiented by the equivalence relation induced by judgmental equality on classes, i.e. $X_1 : K_1, \cdots, X_n : K_n \sim Y_1 : K'_1, \cdots, Y_n : K'_n$ if and only if

$$\begin{aligned} \bullet &\vdash K_1 = K'_1\ \mathsf{cls} \\ X_1 : K_1 &\vdash K_2 = K'_2\ \mathsf{cls} \\ &\vdots \\ X_1 : K_1, \cdots, X_{n-1} : K_{n-1} &\vdash K_n = K'_n\ \mathsf{cls} \end{aligned}$$

We always denote an equivalence class using a representative $[\![\Gamma]\!]$, and omit the quotient-related proofs.

We should also be taking $\mathcal{C}$-morphisms from $[\![\Gamma]\!]$ to $[\![\Delta]\!]$ to be substitutions quotiented by the equivalence relation induced by judgmental equality on terms, i.e. $\Gamma \vdash \sigma\ :\ \Delta \sim E \vdash \rho\ :\ Z$ if and only if $[\![\Gamma]\!] = [\![E]\!]$ and $[\![\Delta]\!] = [\![Z]\!]$ and

$$\begin{aligned} &\Gamma \vdash \sigma_1 = \rho_1\ :\ K_1 \\ &\Gamma \vdash \sigma_2 = \rho_2\ :\ [\sigma_1/X_1]K_2 \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad \text{when } \Delta = X_1 : K_1, \cdots, X_n : K_n \\ &\Gamma \vdash \sigma_n = \rho_n\ :\ [\sigma_{n-1}/X_{n-1}] \cdots [\sigma_1/X_1]K_n \end{aligned}$$

Likewise, we always denote an equivalence class using a representative $[\![\sigma]\!]$, and omit the quotient-related proofs.

4.3 Objects

In the $\lambda^{\times\to}$ case we interpreted objects $\Gamma \vdash O\ :\ K$ as morphisms $[\![O]\!] : [\![\Gamma]\!] \to [\![K]\!]$. Since the classes are now dependent, and the object is situated in some context, we should upgrade this to a morphism in the slice $\mathcal{C}/[\![\Gamma]\!]$.

$$\Gamma \vdash O\ :\ K \qquad\qquad \begin{array}{ccc} [\![\Gamma]\!] & \xrightarrow{[\![\Gamma \vdash O : K]\!]} & [\![\Gamma, X : K]\!] \\ & {\scriptstyle \mathbb{1}_{[\![\Gamma]\!]}} \searrow \quad \swarrow {\scriptstyle [\![\Gamma \vdash K\ \mathsf{cls}]\!]} & \\ & [\![\Gamma]\!] & \end{array}$$

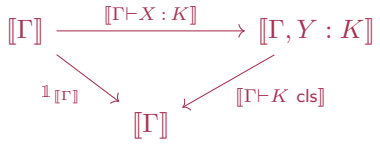
We can call $[\![\Gamma \vdash O \ : \ K]\!]$ a *section* of the *bundle* $[\![\Gamma \vdash K \ \mathsf{cls}]\!]$. This is a bundle in the sense that for any collection of objects $O_1, \cdots, O_n$ in the classes from Γ, there is a closed class

$$\bullet \vdash [O_n/X_n] \cdots [O_1/X_1]K \ \mathsf{cls}$$

which is the *fiber* over that collection of objects. A section of the bundle then picks out for any such collection of objects, an object in the *fiber*

$$\bullet \vdash [O_n/X_n] \cdots [O_1/X_1]O \ : \ [O_n/X_n] \cdots [O_1/X_1]K$$

$$\frac{\Gamma \vdash \mathsf{ctx} \quad X : K \in \Gamma}{\Gamma \vdash X \ : \ K} \ \textsc{var-obj}$$



We may also write $[\![K]\!]$ to mean $[\![\Gamma \vdash K \ \mathsf{cls}]\!]$ for brevity.

4.4 Substitution action on classes and objects

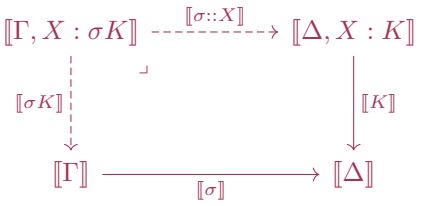
Context morphisms (aka substitutions) are dependent tuples of objects, just as contexts are dependent tuples of classes. They can be composed, which we can view as sequential substitution. They have an action on dependent classes, which we will show corresponds to pullback of bundles. They have an action on objects (more generally action on other substitutions), which we will show corresponds to pullback of sections (more generally existence of all pullbacks).

First let us suppose $\Gamma \vdash \sigma \ : \ \Delta$ is a substitution and $\Delta \vdash K \ \mathsf{cls}$. Then we can form $\Gamma \vdash \sigma K \ \mathsf{cls}$, where

$$\sigma K := [\sigma_n/X_n] \cdots [\sigma_1/X_1]K \quad \text{and} \quad \Delta = X_1 : K_1, \cdots, X_n : K_n$$

This automatically gives us a substitution $\Gamma, X : \sigma K \vdash \sigma :: X \ : \ \Delta, X : K$. This says we have a pullback

$$\frac{\Gamma \vdash \sigma \ : \ \Delta \qquad \Delta \vdash K \ \mathsf{cls}}{\Gamma \vdash \sigma K \ \mathsf{cls}}$$



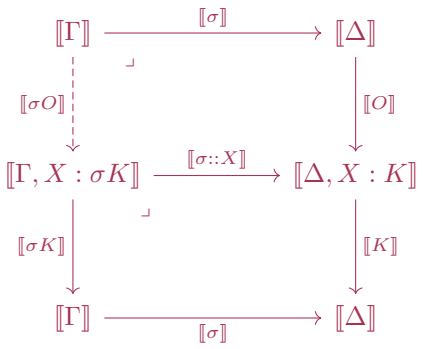
To check the diagram commutes, we need to check judgmental equality component-wise (opening up the equivalence class definition). For the universal property we need to apply (syntactic) substitution on objects.

Now suppose $\Delta \vdash O \ : \ K$. Then we can form $\Gamma \vdash \sigma O \ : \ \sigma K$, where

$$\sigma O := [\sigma_n/X_n] \cdots [\sigma_1/X_1]O$$

This says our pullback diagram extends

$$\frac{\Gamma \vdash \sigma \,:\, \Delta \qquad \Delta \vdash O \,:\, K}{\Gamma \vdash \sigma O \,:\, \sigma K}$$



Now we can use substitution on classes and objects to define composition of context morphisms. Given two substitutions $\Gamma_0 \vdash \sigma \,:\, \Gamma_1$ and $\Gamma_1 \vdash \rho \,:\, \Gamma_2$ we define the composition $\Gamma_0 \vdash \sigma \gg \rho \,:\, \Gamma_2$ by giving

$$\begin{array}{l} \Gamma_0 \vdash \sigma\rho_1 \,:\, \sigma K_1 \\ \Gamma_0 \vdash \sigma\rho_2 \,:\, \sigma[\rho_1/X_1]K_2 \\ \vdots \\ \Gamma_0 \vdash \sigma\rho_n \,:\, \sigma[\rho_{n-1}/X_{n-1}]\cdots[\rho_1/X_1]K_n \end{array} \qquad \text{where } \Gamma_2 = X_1 : K_1, \cdots, X_n : K_n$$

So we have

$$\Gamma_0 \vdash \sigma \gg \rho \,:\, \Gamma_2 \qquad\qquad \llbracket\rho\rrbracket \circ \llbracket\sigma\rrbracket : \llbracket\Gamma_0\rrbracket \to \llbracket\Gamma_2\rrbracket$$

There is a techincal issue here about substitution not working out in a general LCCC, because of pullback only being unique up to isomorphism. However, in the syntax category we have "strict pullbacks".

4.5 Strict pullbacks

The previous construction of the pullback $\llbracket K\rrbracket \mapsto \llbracket\sigma\rrbracket \mapsto (\llbracket\sigma K\rrbracket, \llbracket\sigma :: X\rrbracket)$ defines a strictly equal diagram.

$$\llbracket\sigma\rrbracket^*\llbracket\rho\rrbracket^*\llbracket K\rrbracket = \llbracket\sigma(\rho K)\rrbracket = \llbracket(\sigma \gg \rho)K\rrbracket = (\llbracket\rho\rrbracket \circ \llbracket\sigma\rrbracket)^* \llbracket K\rrbracket$$

$$\begin{array}{ccccccc} \Gamma_0, (\sigma \gg \rho)K & -\!=\!\to & \Gamma_0, \sigma\rho K & \longrightarrow & \Gamma_1, \rho K & \longrightarrow & \Gamma_2, K \\ \downarrow{\scriptstyle (\sigma\gg\rho)K} & \lrcorner & \downarrow{\scriptstyle \sigma\rho K} & \lrcorner & \downarrow{\scriptstyle \rho K} & \lrcorner & \downarrow{\scriptstyle K} \\ \Gamma_0 & \xrightarrow{=} & \Gamma_0 & \xrightarrow[\sigma]{} & \Gamma_1 & \xrightarrow[\rho]{} & \Gamma_2 \end{array}$$

Note that in general categories (including LCCCs) the pullback along a composition is only necessarily isomorphic to double pullback.

4.6 Dependent function classes

In the $\lambda^{\times\to}$ case, λ-abstraction and function application provide the adjunction isomorphism in the semantics

$$\mathcal{C}([\![\Gamma \times K_0]\!], [\![K_1]\!]) \underset{\mathsf{ap}}{\overset{\lambda}{\rightleftarrows}} \mathcal{C}([\![\Gamma]\!], [\![K_0 \to K_1]\!])$$

Let us try to consider what happens when function classes are replaced with dependent function classes. Instead of morphisms in the ambient category, we need morphisms in slices, since we are concerned with objects $\Gamma \vdash O : \{X : S\}\, K$.

$$\mathcal{C}/?\,(?\,, [\![K]\!]) \underset{\text{PI-APP-OBJ}}{\overset{\text{PI-LAM-OBJ}}{\rightleftarrows}} \mathcal{C}/[\![\Gamma]\!](\mathbb{1}_{[\![\Gamma]\!]}, [\![\{X : S\}\, K]\!])$$

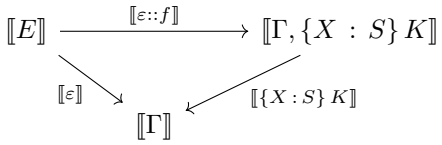
The introduction rule PI-LAM-OBJ fill in the missing details. We only use a special case of PI-APP-OBJ here.

$$\frac{\Gamma \vdash S : \mathsf{Sort} \quad \Gamma, X : S \vdash O : K}{\Gamma \vdash [X : S]\, O : \{X : S\}\, K}\ \textsc{pi-lam-obj} \qquad \frac{\Gamma, X : S \vdash O : \{X : S\}\, K \quad \Gamma, X : S \vdash X : S}{\Gamma, X : S \vdash O\, X : K}\ \textsc{pi-app-obj}$$

$$\mathcal{C}/[\![\Gamma, X : S]\!]\,(\mathbb{1}_{\Gamma, X:S}, [\![K]\!]) \underset{\text{PI-APP-OBJ}}{\overset{\text{PI-LAM-OBJ}}{\rightleftarrows}} \mathcal{C}/[\![\Gamma]\!](\mathbb{1}_{[\![\Gamma]\!]}, [\![\{X : S\}\, K]\!])$$

The β and η rules for PI state that the maps back and forth form a bijection. It is not yet clear what adjunction the above is a special case of.

Following the moves from before and our categorical intuition, we first replace $\mathbb{1}_{[\![\Gamma]\!]}$ with a more general $\mathcal{C}/[\![\Gamma]\!]$-object, a context morphism $[\![E \vdash \varepsilon : \Gamma]\!]$. The object is then replaced with a morphism in the slice $[\![\varepsilon :: f]\!] \in \mathcal{C}/[\![\Gamma]\!]([\![\varepsilon]\!], [\![\{X : S\}\, K]\!])$.

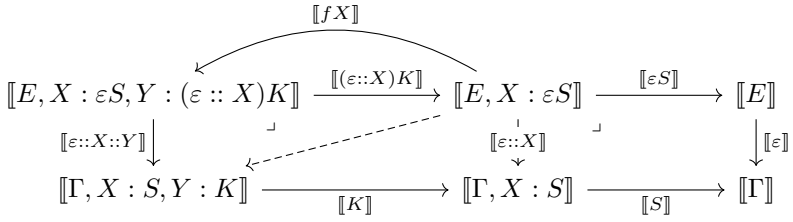


This in particular gives some object

$$E \vdash f : \{X : \varepsilon S\}\ ((\varepsilon :: X)K)$$

Weakening and applying PI-APP-OBJ we obtain

$$E, X : \varepsilon S \vdash fX : (\varepsilon :: X)K$$



The dashed arrow is then a morphism in the slice $\mathcal{C}/[\![\Gamma, X : S]\!]$. So we have (making the same argument the other way round and using β and η)

$$\mathcal{C}/[\![\Gamma, X : S]\!]\,([\![\varepsilon :: X]\!], [\![K]\!]) \underset{\text{PI-APP-OBJ}}{\overset{\text{PI-LAM-OBJ}}{\rightleftarrows}} \mathcal{C}/[\![\Gamma]\!]([\![\varepsilon]\!], [\![\{X \,:\, S\}\, K]\!])$$

This is closer to an adjunction: we can see that the left adjoint should be pullback along $[\![S]\!] : [\![\Gamma, X : S]\!] \to [\![\Gamma]\!]$.

$$[\![S]\!]^*[\![\varepsilon]\!] = [\![\varepsilon :: X]\!]$$

Now we make the right adjoint functorial as well. We replace $[\![K]\!]$ with a slice $[\![\delta]\!] : [\![\Delta]\!] \to [\![\Gamma, X : S]\!]$.

$$\mathcal{C}/[\![\Gamma, X : S]\!]\,([\![S]\!]^*[\![\varepsilon]\!], [\![\delta]\!]) \underset{\text{PI-APP-OBJ}}{\overset{\text{PI-LAM-OBJ}}{\rightleftarrows}} \mathcal{C}/[\![\Gamma]\!]([\![\varepsilon]\!], \Pi_{[\![S]\!]}[\![\delta]\!])$$

Briefly and roughly: Write context $\Delta = V_1 : \Delta_1, \cdots, V_n : \Delta_n$. A slice morphism O on the left consists of objects $E, X : \varepsilon S \vdash O_i \,:\, [O/V]\Delta_i$. Then λ-abstraction gives objects $E \vdash [X \,:\, \varepsilon S]\, O_i \,:\, \{X \,:\, \varepsilon S\}\, [O/V]\Delta_i$ on the right hand side. So we should define $\Pi_{[\![\delta]\!]}[\![\zeta]\!]$ such that its $\mathcal{C}$-object part is context

$$[\![V_1 : \{X \,:\, \varepsilon S\}\, \Delta_0, \cdots, V_n : \{X \,:\, \varepsilon S\}\, \Delta_n]\!]$$

and its morphism part comes from each Z_i being a class in context $\Gamma, X : S$. The proof that it is an adjunction is in the same vein as before.

$$[\![S]\!]^* \dashv \Pi_{[\![S]\!]}$$

4.7 Sorts (pre-equality)

The category of contexts is *not* a locally cartesian closed category. In particular, we don't have pullbacks for arbitrary morphisms in the category. However, (even for the fragment of the LF without the equality class) we can find a subcategory that *is* locally cartesian closed, which essentially consists of context extentions, where the extensions only introduce sorts. Furthermore, this subcategory $\mathcal{S}_\Pi$ will be a universe of representable maps.

We know that for contexts extended with sorts $[\![S]\!] : [\![\Gamma, X : S]\!] \to [\![\Gamma]\!]$ we have the adjunction $[\![S]\!]^* \dashv \Pi_{[\![S]\!]}$. In particular $[\![\mathsf{El}]\!] : [\![\mathsf{El} : \mathsf{Sort}, X : \mathsf{El}]\!] \to [\![\mathsf{El} : \mathsf{Sort}]\!]$ is such an extension, and there is a correspondence between pullbacks of $[\![\mathsf{El}]\!]$ and sort-judgments

$$\Gamma \vdash S \,:\, \mathsf{Sort}$$

$$\begin{array}{ccc} [\![\Gamma, S]\!] & \longrightarrow & [\![\mathsf{El} : \mathsf{Sort}, X : \mathsf{El}]\!] \\ {\scriptstyle [\![S]\!]}\downarrow & \lrcorner & \downarrow{\scriptstyle [\![\mathsf{El}]\!]} \\ [\![\Gamma]\!] & \longrightarrow & [\![\mathsf{El} : \mathsf{Sort}]\!] \end{array}$$

Let us try to generate from $[\![\mathsf{El}]\!]$, the smallest LCCC containing it (supposing it exists[1]). So we throw El into $\mathcal{S}_\Pi$, throw in identities, take the closure under composition, and closure under pullback against $\mathcal{C}$-morphisms. After checking some examples, we guess that $\mathcal{S}_\Pi$ should be characterized as the set of compositions of sort-context-exentions, i.e. $[\![S_1]\!] \circ \cdots \circ [\![S_n]\!]$ (the identity on $[\![\Gamma]\!]$ when $n = 0$) whenever

$$\Gamma \vdash \mathsf{ctx} \qquad \Gamma \vdash S_1 \,:\, \mathsf{Sort} \qquad \Gamma, X_1 : S_1 \vdash S_2 \,:\, \mathsf{Sort} \qquad \cdots \qquad \Gamma, X_1 : S_1, \cdots, X_{n-1} : S_{n-1} \vdash S_n \,:\, \mathsf{Sort}$$

[1] This is reasonable given that we already have pullbacks and products for this class of maps.

To ensure these two characterizations of $\mathcal{S}_\Pi$ are equivalent it suffices to check that the second is stable under pullbacks, and closed under Πs.

To check stability under pullback along $[\![\sigma]\!] : [\![\Delta]\!] \to [\![\Gamma]\!]$, first recall that for any sort $[\![S]\!] : [\![\Gamma, X : S]\!] \to [\![\Gamma]\!]$, the pullback is the substition on the sort $[\![\sigma^* S]\!] : [\![\Delta, X : \sigma^* S]\!] \to [\![\Delta]\!]$ which is another sort. This means that the pullback of any composition $[\![S_1]\!] \circ \cdots \circ [\![S_n]\!]$ is just the composition of the pullbacks ("pasting"), and given that each piece of the pullback is a sort, the composition is in $\mathcal{S}_\Pi$. For the case when $n = 0$ note that the pullback of the identity is the identity.

Closure under Πs: by stability under pullback along general context substitutions, we know that for any morphism $g = [\![S_1]\!] \circ \cdots \circ [\![S_n]\!] \in \mathcal{S}_\Pi$, g^* is a well-defined functor. Furthermore, for each $[\![S_i]\!]$ we have an adjunction $[\![S_i]\!]^* \dashv \Pi_{[\![S_i]\!]}$, and the composition of adjunctions yields an adjunction

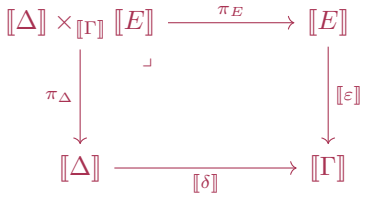
$$[\![S_n]\!]^* \circ \cdots \circ [\![S_1]\!]^* = [\![g]\!]^* \dashv \Pi_{[\![g]\!]} = \Pi_{[\![S_1]\!]} \circ \cdots \circ \Pi_{[\![S_n]\!]}$$

Thus the two charactizations of $\mathcal{S}_\Pi$ are equivalent and $\mathcal{S}_\Pi$ is a URM.

4.8 Equality classes

We showed that pullbacks of context morphisms along context extensions/classes corresponds to substitution on classes. Do we have pullback along any context morphism? The answer is *not in this type theory*: in order to construct the pullback, we need an equality class, but our equality classes are restricted to Sort. Let us temporarily throw in an extra rule for the sake of exposition, and use it to construct a pullback.[1]

$$\frac{\Gamma \vdash O_0 \ : \ K \quad \Gamma \vdash O_1 \ : \ K}{\Gamma \vdash O_0 =_K O_1 \ \mathsf{cls}} \ \textsc{eq-cls}$$



Again imagining Δ and E as closed classes, the pullback should be "pairs of objects in $\Delta \times E$" that are equal upon their substitutions into the objects given by δ and ε. In set-theoretic notation (since we are inspired by pullback in **Set**)

$$\{(X, Y) \in \Delta \times E \mid \delta(X) = \varepsilon(Y)\}$$

Translating this idea into a context means that we need

$$\begin{aligned} [\![\Delta]\!] \times_{[\![\Gamma]\!]} [\![E]\!] &= [\![X : \Delta, Y : E, H : \delta =_\Gamma \varepsilon]\!] \\ \pi_\Delta &= [\![X]\!] \\ \pi_E &= [\![Y]\!] \end{aligned}$$

In detail: if $\Delta = X_1 : \Delta_1, \cdots, X_n : \Delta_n$ and $E = Y_1 : E_1, \cdots, Y_m : E_m$ and $\Gamma = Z_1 : \Gamma_1, \cdots, Z_l : \Gamma_l$ then the context is

$$X_1 : \Delta_1, \cdots, X_n : \Delta_n, Y_1 : E_1, \cdots, Y_m : E_m, H_1 : \delta_1 =_{\Gamma_1} \varepsilon_1, \cdots, H_k : \delta_l =_{\Gamma_l} \varepsilon_l$$

and the context morphisms are those that return the relevant variables from this context.

To show the universal property, given cone $[\![Z]\!]$ we use that judgmental equalities $Z \vdash \zeta \gg \delta = \zeta \gg \varepsilon \ : \ \Gamma$ lift to $Z \vdash \mathsf{self} \ : \ \zeta \gg \delta =_\Gamma \zeta \gg \varepsilon$ for existence (this fact follows from judgmental equality being a congruence); we use UNICITY for uniqueness.

[1] This rule is included in Uemura's logical framework, and adding it corresponds to having representable map categories for semantics [Uem19].

Hence we conclude that adding the rule EQ-CLS then would mean that the whole context category has pullbacks.

4.9 Sorts (post-equality)

We want the corresponding semantics for the rule EQ-SORT. This should give us some URM that is closed under equality-formation. The situation is similar to EQ-CLS, except our assumption should be that we have

$$\begin{array}{ccc} [\![Y : E, X : \Delta, H : \delta =_\Gamma \varepsilon]\!] & \xrightarrow{\pi_E} & [\![E]\!] \\ \downarrow{\scriptstyle \pi_\Delta} & \lrcorner & \downarrow{\scriptstyle [\![\varepsilon]\!]} \\ [\![\Delta]\!] & \xrightarrow[[\![\delta]\!]]{} & [\![\Gamma]\!] \end{array}$$

when Γ only contains sorts. This guarantees that each equality class formed $\delta_i =_{\Gamma_i} \varepsilon_i$ is again a sort.

Now, ensuring Γ is sort-only, we naively try to add $[\![\delta]\!] : [\![\Delta]\!] \to [\![\Gamma]\!]$ into $\mathcal{S}_\Pi$, and try to make this a URM. Firstly this means that $\pi_E = [\![\varepsilon]\!]^* [\![\delta]\!]$ should be in this set. Although we can form the pullback along π_E against any $\mathcal{C}$-morphism (it is a series of context extensions), we do not get an adjunction $\pi_E^* \dashv \Pi_{\pi_E}$. Having the adjunction requires an impredicative dependent function type, i.e. one that has general classes as hypotheses.

However, if we require that *both* Δ *and* Γ are sort only, then π_E is a context extension of E extended with only sorts, so it is in $\mathcal{S}_\Pi$ already. So let us characterize our new candidate URM as

$$\mathcal{S}_{\Pi,=} := \mathcal{S}_\Pi \cup \{ [\![\Delta \vdash \delta \ : \ \Gamma]\!] \mid \Delta, \Gamma \text{ sort-only } \}$$

$\mathcal{S}_{\Pi,=}$ is stable under pullback, as noted above. It is clearly closed under identity. To show it is closed under composition, we case on which part of the union the parts came from; each case is easy. To show it is closed under Πs, we just need to generalize the adjunction from before:

$$\mathcal{C}/[\![\Delta]\!]\,([\![\delta]\!]^*[\![\varepsilon]\!], [\![\zeta]\!]) \underset{\text{PI-APP-OBJ}}{\overset{\text{PI-LAM-OBJ}}{\rightleftarrows}} \mathcal{C}/[\![\Gamma]\!]([\![\varepsilon]\!], \Pi_{[\![\delta]\!]}[\![\zeta]\!])$$

Briefly and roughly, if $[\![\zeta]\!] : [\![Z]\!] \to [\![\Delta]\!]$ for $Z = V_1 : Z_1, \cdots, V_n : Z_n$ then the $\mathcal{C}$-object part of $\Pi_{[\![\delta]\!]}[\![\zeta]\!]$ is

$$[\![V_1 : \{X \ : \ \Delta\}\,\{Y \ : \ \varepsilon =_\Gamma \delta\}\,Z_1, \cdots, V_n : \{X \ : \ \Delta\}\,\{Y \ : \ \varepsilon =_\Gamma \delta\}\,Z_n]\!]$$

This can be formed since Δ is sort-only and Γ is sort only (hence the equality classes are sorts). Furthermore, if $[\![\zeta]\!] \in \mathcal{S}_{\Pi,=}$ then either it is a sort-context-extension of sort-only Δ, hence Z is sort-only, or it Z is sort-only by assumption. Thus $\Pi_\delta[\![\zeta]\!]$ would be sort-only by PI-SORT. So $\mathcal{S}_{\Pi,=}$ is closed under Πs. This concludes that $\mathcal{S}_{\Pi,=}$ is a URM.

Note that $\mathcal{S}_\Pi$ as previously described is still a URM when we have equality as part of the syntax. So now we have two URMs $\mathcal{S}_\Pi \hookrightarrow \mathcal{S}_{\Pi,=}$.

4.10 Signatures

Given a signature Σ, we can simply take the slice over $[\![\Sigma]\!]$ in the semantics to obtain semantics for the type theory generated by Σ. Since a slice of a locally closed cartesian category is locally cartesian closed, $\mathcal{S}_\Pi/[\![\Sigma]\!]$ and $\mathcal{S}_{\Pi,=}/[\![\Sigma]\!]$ are still LCCC.

4.11 Classifying context extension

Let us briefly return to studying $\mathcal{S}_\Pi$. In the original definition of a universe in a topos [SD04], property 2 includes all monos from $\mathcal{C}$ (in particular identity morphisms) into the universe. Including all monos corresponds to including all proposition-context-extensions, i.e. $\Gamma, X : P$ where P is a proposition.

The original property 5 said that all morphisms of $\mathcal{S}_\Pi$ are pullbacks of a single morphism $\mathsf{El} : E \to U$. In our case this should be the map $[\![S]\!] : [\![S : \mathsf{Sort}, X : S]\!] \to [\![S : \mathsf{Sort}]\!]$. This would imply in particular that composition and identity are also certain pullbacks of (El). In the case of composition, this amounts to requiring dependent pair sorts (aka Σ-types) in the type theory. In the case of identity, this amounts to requiring a unit sort, i.e. a sort containing exactly one object.

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