Equational Logical Frameworks as Locally Cartesian Closed Categories

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1 Overview

- Introduce LF with equality. Closely follow [Har21]
- Introduce the categorical way of looking at the type formers (motivational)
- Construct category of contexts formally
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In the first section, we present an *equational logical framework* as in [Har21], which can be used to define dependent type theories such as Martin Löf type theory or ModTT [SH21].

1.1 Notation

We follow [Har21] and use the word "class" for what is usually called "type" and the word "object" for what is usually called "term". This is to distinguish between the meta-level class in the logical framework and types internal to the type theory defined in the logical framework. To not confuse this with the categorical notion of object, we will write $\mathcal{C}$-object for an object in a category $\mathcal{C}$.

2 An Equational Logical Framework

In this section we recall the equational logical framework, as described in [Har21] (with some minor changes). We want this to be minimalistic but sufficiently expressive so that we can define type theories by asserting the rules the type theory via a collection of generators, called a signature and denoted Σ. We state the syntax and judgment rules for convenience and refer to [Har21] for the details.

$$
\begin{array}{lll}
\textit{variable} & X & \\
\textit{expression} & O, S, K, e & ::= \quad X \mid \mathsf{Sort} \mid \{X \,:\, S\}\, e \mid O_0 =_S O_1 \mid \mathsf{self} \mid [X \,:\, S]\, O \mid O_0\, O_1 \\
\textit{context} & \Gamma, \Delta & ::= \quad \cdot \mid \Gamma, X : K
\end{array}
$$

O indicates this is meant to represent an object (though they are all expressions at this point), similarly S for sorts and K for classes. The agnostic expression symbol e should represent a sort or a class.

The judgments we define are

$\Gamma \vdash \mathsf{ctx}$	Γ is a context
$X : K \in \Gamma$	$X : K$ appears in context Γ
$\Gamma \vdash K\ \mathsf{cls}$	K is a class in context Γ
$\Gamma \vdash O \,:\, K$	O is an object of class K in context Γ
$\Gamma \vdash K_0 = K_1\ \mathsf{cls}$	K_0, K_1 are equal classes in context Γ
$\Gamma \vdash O_0 = O_1 \,:\, K$	O_0, O_1 are equal objects of class K in context Γ

The legal judgments given by the closure under the judgment rules in figures 1, 2, 3, 4, and those for judgmental equality (not included).

$$
\frac{}{\bullet \vdash \mathsf{ctx}}\ \textsc{ctx-emp} \qquad \frac{\Gamma \vdash K\ \mathsf{cls}}{\Gamma, X : K \vdash \mathsf{ctx}}\ \textsc{ctx-ext} \qquad \frac{}{X : K \in \Gamma, X : K}\ \textsc{ctx-hd} \qquad \frac{X_0 : K_0 \in \Gamma}{X_0 : K_0 \in \Gamma, X_1 : K_1}\ \textsc{ctx-tl}
$$

Figure 1: Context formation

$$
\frac{\Gamma \vdash \mathsf{ctx}}{\Gamma \vdash \mathsf{Sort}\ \mathsf{cls}}\ \textsc{sort-cls} \qquad \frac{\Gamma \vdash S \,:\, \mathsf{Sort}}{\Gamma \vdash S\ \mathsf{cls}}\ \textsc{incl-cls} \qquad \frac{\Gamma \vdash S \,:\, \mathsf{Sort} \quad \Gamma, X : S \vdash K\ \mathsf{cls}}{\Gamma \vdash \{X \,:\, S\}\, K\ \mathsf{cls}}\ \textsc{pi-cls}
$$

$$
\frac{\Gamma \vdash S \,:\, \mathsf{Sort} \quad \Gamma \vdash O_0 \,:\, S \quad \Gamma \vdash O_1 \,:\, S}{\Gamma \vdash O_0 =_S O_1\ \mathsf{cls}}\ \textsc{eq-cls}
$$

Figure 2: Class formation

Judgmental equality for classes $\Gamma \vdash K_0 = K_1\ \mathsf{cls}$ and objects $\Gamma \vdash O_0 = O_1 \,:\, K$ are defined to be congruences, i.e. equivalence relations that respect formation of classes, sorts, and objects. Furthermore, we have β and η rules for PI. See [Har21] for details.

Finally, signatures are defined as contexts $\Sigma \vdash \mathsf{ctx}$, and the type theory generated by a signature Σ consists of legal judgments over Σ. Note that before introducing a signature we cannot make any sorts, nor can we make any interesting objects. The only rules for their formation are via PI-SORT and EQ-SORT, which require

$$\frac{\Gamma \vdash S_0 \ : \ \mathsf{Sort} \quad \Gamma, X : S_0 \vdash S_1 \ : \ \mathsf{Sort}}{\Gamma \vdash \{X \ : \ S_0\}\, S_1 \ : \ \mathsf{Sort}} \ \textsc{pi-sort} \qquad \frac{\Gamma \vdash S \ : \ \mathsf{Sort} \quad \Gamma \vdash O_0 \ : \ S \quad \Gamma \vdash O_1 \ : \ S}{\Gamma \vdash O_0 =_S O_1 \ : \ \mathsf{Sort}} \ \textsc{eq-sort}$$

Figure 3: Sort formation

$$\frac{\Gamma \vdash \mathsf{ctx} \quad X : K \in \Gamma}{\Gamma \vdash X \ : \ K} \ \textsc{var-obj} \qquad \frac{\Gamma \vdash S \ : \ \mathsf{Sort} \quad \Gamma, X : S \vdash O \ : \ K}{\Gamma \vdash [X \ : \ S]\, O \ : \ \{X \ : \ S\}\, K} \ \textsc{pi-lam-obj}$$

$$\frac{\Gamma \vdash O_0 \ : \ \{X \ : \ S\}\, K \quad \Gamma \vdash O_1 \ : \ S}{\Gamma \vdash O_0\, O_1 \ : \ [O_1/X]K} \ \textsc{pi-app-obj} \qquad \frac{\Gamma \vdash O \ : \ S}{\Gamma \vdash \mathsf{self} \ : \ O =_S O} \ \textsc{eq-self-obj}$$

$$\frac{\Gamma \vdash O \ : \ K_0 \quad \Gamma \vdash K_0 = K_1 \ \mathsf{cls}}{\Gamma \vdash O \ : \ K_1} \ \textsc{id-obj}$$

Figure 4: Object formation

more sorts and objects as premises. We have added EQ-SORT in addition to the rules from [Har21], since we will need this for the LCCC structure. It is the signature that populates Sort with sorts and populates those sorts with objects. Given a signature Σ, we write $\lambda^{\Pi\mathrm{Eq}}[\Sigma]$ to denote the generated type theory.

$$\begin{aligned}
&\Sigma\Gamma \vdash \mathsf{ctx} \\
&\Sigma\Gamma \vdash K \ \mathsf{cls} \\
&\Sigma\Gamma \vdash O \ : \ K \\
&\Sigma\Gamma \vdash K_0 = K_1 \ \mathsf{cls} \\
&\Sigma\Gamma \vdash O_0 = O_1 \ : \ K
\end{aligned}$$

3 Categorical definitions

Definition – Cartesian closed category (CCC)
We say a category $\mathcal{C}$ is cartesian closed when

- $\mathcal{C}$ has finite products
- For each $\mathcal{C}$-object A, the product with A functor $\times A : \mathcal{C} \to \mathcal{C}$ has a right adjoint.

We denote the right adjoint by $[A, -]$ and call it internal hom.

Definition – Locally cartesian closed category (LCCC)
We say a category $\mathcal{C}$ is locally cartesian closed when either of the equivalent definitions hold

1. Every slice category $\mathcal{C}/A$ over a $\mathcal{C}$-object A is cartesian closed.
2. $\mathcal{C}$ has pullbacks, and for each morphism $f : B \to A$ in $\mathcal{C}$, the base change (pullback along f)

> $f^* : \mathcal{C}/A \to \mathcal{C}/B$ has a right adjoint.
>
> We denote the right adjoint to base change by Π_f.

We do not show that these two definitions are equivalent. The idea is that pullbacks correspond to products in slices and Πs correspond to internal homs in slices.

4 Intuition for considering LCCCs

We want to find categorical semantics for our type theory. This amounts to one of two equivalent things

- View each aspect of the type theory in a category $\mathcal{C}$
- Construct a syntactic category (which will be the "category of contexts") and take semantics to be anything that imitates it (in a Platonic form sense).

For this section we refer to $\lambda^{\times\to}$, a simply typed λ-calculus with only unit, product and function classes. And use intuition of its categorical semantics to motivate those for logical frameworks.

Recall $\lambda^{\times\to}$ can be interpreted in a CCC $\mathcal{C}$ by taking classes K as $\mathcal{C}$-objects $[\![K]\!]$, contexts $X_1 : K_1, \ldots, X_n : K_n$ as products $[\![K_1]\!] \times \cdots \times [\![K_n]\!]$, and objects $\Gamma \vdash K \ : \ O$ as morphisms $[\![O]\!] : [\![\Gamma]\!] \to [\![K]\!]$. This seems quite natural, since product classes become products, and function classes become internal homs in the semantics. If we are to immitate this for $\lambda^{\Pi\mathsf{Eq}}[\Sigma]$, we would only be able to talk about *closed classes* $\bullet \vdash K$ cls, since in $\lambda^{\times\to}$ we don't have dependent classes $\Gamma \vdash K$ cls. However, in the presence of dependent pairs (aka Σ-types) we could view a context $\Gamma = X_1 : K_1, \ldots, X_n : K_n$ as a closed class $\Sigma_\Gamma := \Sigma_{X_1:K_1} \cdots \Sigma_{X_{n-1}:K_{n-1}} K_n$. Then any dependent class can be viewed as a context, which can be viewed as a closed class.

4.1 Contexts and dependent classes

Instead of including dependent pairs, let us view dependent classes $\Gamma \vdash K$ cls as extended contexts $\Gamma, X : K$, and first interpret *contexts* as $\mathcal{C}$-objects, and *as a consequence* dependent classes as $\mathcal{C}$-objects.

$$\Gamma \vdash \mathsf{ctx} \qquad\qquad [\![\Gamma]\!] \in \mathcal{C}$$

It is then natural to define morphisms of contexts (aka a substitution), denoted $\Gamma \vdash \sigma \ : \ \Delta$, consisting of the data of objects

$$\begin{array}{ll} \Gamma \vdash \sigma_1 \ : \ K_1 & \\ \Gamma \vdash \sigma_2 \ : \ [\sigma_1/X_1]K_2 & \\ \vdots & \text{when } \Delta = X_1 : K_1, \cdots, X_n : K_n \\ \Gamma \vdash \sigma_n \ : \ [\sigma_{n-1}/X_{n-1}] \cdots [\sigma_1/X_1]K_n & \end{array}$$

Such a substitution corresponds to a closed object of the closed class $\Sigma_\Gamma \to \Sigma_\Delta$ in the presence of dependent pairs.

$$\Gamma \vdash \sigma \ : \ \Delta \qquad\qquad [\![\sigma]\!] : [\![\Gamma]\!] \to [\![\Delta]\!]$$

The empty context gives an object $[\![\bullet]\!]$, and for any context Γ there is exactly one substitution $\Gamma \vdash \sigma : \bullet$, given by the data of no objects. This makes $[\![\bullet]\!]$ the terminal object.

$$\frac{}{\bullet \vdash \mathsf{ctx}}\ \textsc{ctx-emp} \qquad\qquad \mathbf{1}_{\mathcal{C}} \in \mathcal{C}$$

For context extension, given $\Gamma \vdash K\ \mathsf{cls}$, we obtain a substitution $\Gamma, X : K \vdash \sigma : \Gamma$ by $\Gamma, X : K \vdash X_i : K_i$ for each $X_i : K_i \in \Gamma$. Then let's take the interpretation of K to be a $\mathcal{C}/[\![\Gamma]\!]$-object, where $\mathcal{C}/[\![\Gamma]\!]$ is the slice category.

$$[\![\Gamma \vdash K\ \mathsf{cls}]\!] := [\![\Gamma, X : K]\!] \text{ and } [\![\Gamma, X : K \vdash \sigma : \Gamma]\!]$$

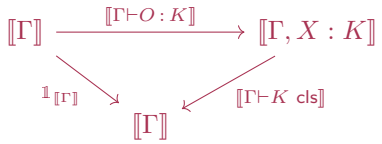
We can informally identify use the slice-object $[\![\Gamma \vdash K\ \mathsf{cls}]\!]$ to denote the underlying morphism.

$$\frac{\Gamma \vdash K\ \mathsf{cls}}{\Gamma, X : K \vdash \mathsf{ctx}}\ \textsc{ctx-ext} \qquad\qquad [\![\Gamma, X : K]\!] \xrightarrow{[\![\Gamma \vdash K\ \mathsf{cls}]\!]} [\![\Gamma]\!]$$

4.2 Objects

In the $\lambda^{\times\to}$ case we interpreted objects $\Gamma \vdash K : O$ as morphisms $[\![O]\!] : [\![\Gamma]\!] \to [\![K]\!]$ in $\mathcal{C}$. Since the classes are now dependent, and the object is situated in some context, we should upgrade this to a morphism in the slice $\mathcal{C}/[\![\Gamma]\!]$.

$$\Gamma \vdash O : K$$



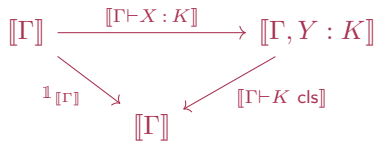
We can call $[\![\Gamma \vdash O : K]\!]$ a section of the bundle $[\![\Gamma \vdash K\ \mathsf{cls}]\!]$ in the sense that for any collection of objects $O_1, \cdots, O_n$ in the classes from Γ, there is a closed class

$$\bullet \vdash [O_n/X_n] \cdots [O_1/X_1]K\ \mathsf{cls}$$

A section of the bundle then picks out for any such collection of objects, an object

$$\bullet \vdash [O_n/X_n] \cdots [O_1/X_1]O : [O_n/X_n] \cdots [O_1/X_1]K$$

$$\frac{\Gamma \vdash \mathsf{ctx} \quad X : K \in \Gamma}{\Gamma \vdash X : K}\ \textsc{var-obj}$$



We may also write $[\![K]\!]$ to mean $[\![\Gamma \vdash K\ \mathsf{cls}]\!]$ for brevity.

4.3 Substitution

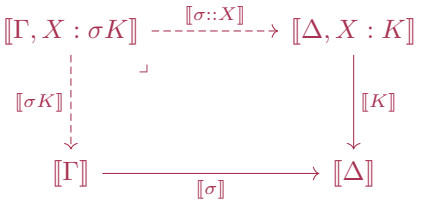
Context morphisms (aka substitutions) are dependent tuples of objects, just as contexts are dependent tuples of classes. They can be composed, which we can view as sequential substitution. They can induce substitutions on dependent classes, which we will show corresponds to pullback of bundles. They can induce substitution on objects (more generally substitution on substitutions), which we will show corresponds to pullback of sections (more generally existence of all pullbacks).

First let us suppose $\Gamma \vdash \sigma \,:\, \Delta$ is a substitution and $\Delta \vdash K$ cls. Then we can form $\Gamma \vdash \sigma K$ cls, where

$$\sigma K := [\sigma_n/X_n] \cdots [\sigma_1/X_1]K \quad \text{and} \quad \Delta = X_1 : K_1, \cdots, X_n : K_n$$

This automatically gives us a substitution $\Gamma, X : \sigma K \vdash \sigma :: X \,:\, \Delta, X : K$. This says we have a pullback

$$\frac{\Gamma \vdash \sigma \,:\, \Delta \qquad \Delta \vdash K \text{ cls}}{\Gamma \vdash \sigma K \text{ cls}}$$



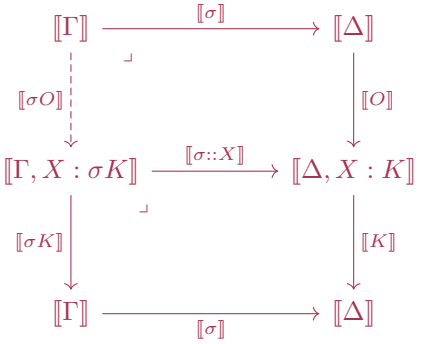
To check the diagram commutes, we use a notion of judgmental equality between substitutions, which we can define component-wise by judgmental equality between objects. For the universal property we need to apply (syntactic) substitution on objects.

Now suppose $\Delta \vdash O \,:\, K$. Then we can form $\Gamma \vdash \sigma O \,:\, \sigma K$, where

$$\sigma O := [\sigma_n/X_n] \cdots [\sigma_1/X_1]O$$

This says our pullback diagram extends

$$\frac{\Gamma \vdash \sigma \,:\, \Delta \qquad \Delta \vdash O \,:\, K}{\Gamma \vdash \sigma O \,:\, \sigma K}$$



Now we can use substitution on classes and objects to define composition of context morphisms. Given two substitutions $\Gamma_0 \vdash \sigma \,:\, \Gamma_1$ and $\Gamma_1 \vdash \rho \,:\, \Gamma_2$ we define the composition $\Gamma_0 \vdash \sigma \gg \rho \,:\, \Gamma_2$ by giving

$$\begin{aligned}
&\Gamma_0 \vdash \sigma\rho_1 \,:\, \sigma K_1 \\
&\Gamma_0 \vdash \sigma\rho_2 \,:\, \sigma[\rho_1/X_1]K_2 \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \text{where } \Gamma_2 = X_1 : K_1, \cdots, X_n : K_n \\
&\Gamma_0 \vdash \sigma\rho_n \,:\, \sigma[\rho_{n-1}/X_{n-1}] \cdots [\rho_1/X_1]K_n
\end{aligned}$$

So we have

$$\Gamma_0 \vdash \sigma \gg \rho : \Gamma_2 \qquad\qquad [\![\rho]\!] \circ [\![\sigma]\!] : [\![\Gamma_0]\!] \to [\![\Gamma_2]\!]$$

There is a techincal issue here about substitution not working out in a general LCCC, because of pullback only being unique up to isomorphism. We delay this issue until later, and assume strict pullbacks, i.e. all pullbacks in $\mathcal{C}$ of the same diagram are equal.

4.4 Equality classes

In the previous subsection we showed that the semantics category $\mathcal{C}$ pullbacks of context morphisms along context extensions/classes. By (assume strict) pullback pasting [**?**] we even have pullback of context morphisms along a finite composition of context extensions. Do we have pullback along any substitution? The answer is yes, given the presence of an equality class.

$$\begin{array}{ccc} [\![\Delta]\!] \times_{[\![\Gamma]\!]} [\![E]\!] & \xrightarrow{\pi_E} & [\![E]\!] \\ \downarrow{\scriptstyle \pi_\Delta} & \lrcorner & \downarrow{\scriptstyle [\![\varepsilon]\!]} \\ [\![\Delta]\!] & \xrightarrow[[\![\delta]\!]]{} & [\![\Gamma]\!] \end{array}$$

Again imagining Δ and E as closed classes, the pullback should be "pairs of objects in $\Delta \times E$" that are equal upon their substitutions into the objects given by δ and ε. In set-theoretic notation (since we are inspired by pullback in **Set**)

$$\{(X, Y) \in \Delta \times E \mid \delta(X) = \varepsilon(Y)\}$$

Translating this idea into a context means that we need

$$\begin{aligned} [\![\Delta]\!] \times_{[\![\Gamma]\!]} [\![E]\!] &= [\![X : \Delta, Y : E, H : \delta =_\Gamma \varepsilon]\!] \\ \pi_\Delta &= [\![X]\!] \\ \pi_E &= [\![Y]\!] \end{aligned}$$

In detail: if $\Delta = X_1 : \Delta_1, \cdots, X_n : \Delta_n$, $E = Y_1 : E_1, \cdots, Y_m : E_m$ and $\Gamma = Z_1 : \Gamma_1, \cdots, Z_l : \Gamma_l$ then the context is

$$X_1 : \Delta_1, \cdots, X_n : \Delta_n, Y_1 : E_1, \cdots, Y_m : E_m, H_1 : \delta_1 =_{\Gamma_1} \varepsilon_1, \cdots, H_k : \delta_l =_{\Gamma_l} \varepsilon_l$$

and the context morphisms are those that return the relevant variables from this context.

To show the universal property, given cone $[\![Z]\!]$ we use that judgmental equalities $Z \vdash \zeta \gg \delta = \zeta \gg \varepsilon : \Gamma$ lift to $Z \vdash \mathsf{self} : \zeta \gg \delta =_\Gamma \zeta \gg \varepsilon$ for existence (this fact follows from judgmental equality being a congruence); we use UNICITY for uniqueness.

4.5 Dependent function classes

In the $\lambda^{\times\to}$ case, λ-abstraction and function application provide the adjunction isomorphism in the semantics

$$\mathcal{C}([\![\Gamma \times K_0]\!], [\![K_1]\!]) \underset{\mathsf{ap}}{\overset{\lambda}{\rightleftarrows}} \mathcal{C}([\![\Gamma]\!], [\![K_0 \to K_1]\!])$$

Let us try to consider what happens when function classes are replaced with dependent function classes. Instead of morphisms in the ambient category, we need morphisms in slices, since we are concerned with objects $\Gamma \vdash O : \{X : S\}\, K$.

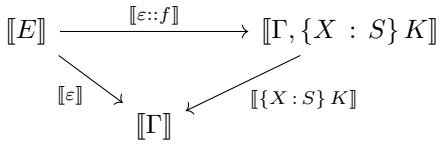
$$\mathcal{C}/?\,(?, [\![K]\!]) \underset{\text{PI-APP-OBJ}}{\overset{\text{PI-LAM-OBJ}}{\rightleftarrows}} \mathcal{C}/[\![\Gamma]\!](\mathbb{1}_{[\![\Gamma]\!]}, [\![\{X : S\}\, K]\!])$$

The introduction rule PI-LAM-OBJ fill in the missing details. We only use a special case of PI-APP-OBJ here.

$$\frac{\Gamma \vdash S : \mathsf{Sort} \quad \Gamma, X : S \vdash O : K}{\Gamma \vdash [X : S]\, O : \{X : S\}\, K}\ \text{PI-LAM-OBJ} \qquad \frac{\Gamma, X : S \vdash O : \{X : S\}\, K \quad \Gamma, X : S \vdash X : S}{\Gamma, X : S \vdash O\, X : K}\ \text{PI-APP-OBJ}$$

$$\mathcal{C}/[\![\Gamma, X : S]\!]\,(\mathbb{1}_{\Gamma, X:S}, [\![K]\!]) \underset{\text{PI-APP-OBJ}}{\overset{\text{PI-LAM-OBJ}}{\rightleftarrows}} \mathcal{C}/[\![\Gamma]\!](\mathbb{1}_{[\![\Gamma]\!]}, [\![\{X : S\}\, K]\!])$$

The β and η rules for PI state that the maps back and forth form a bijection. It is not yet clear what adjunction the above is a special case of. Following the moves from before and our categorical intuition, we first replace $\mathbb{1}_{[\![\Gamma]\!]}$ with a more general $\mathcal{C}/[\![\Gamma]\!]$-object, a context morphism $[\![E \vdash \varepsilon : \Gamma]\!]$. The object is then replaced with a morphism in the slice $[\![\varepsilon :: f]\!] \in \mathcal{C}/[\![\Gamma]\!]([\![\varepsilon]\!], [\![\{X : S\}\, K]\!])$.

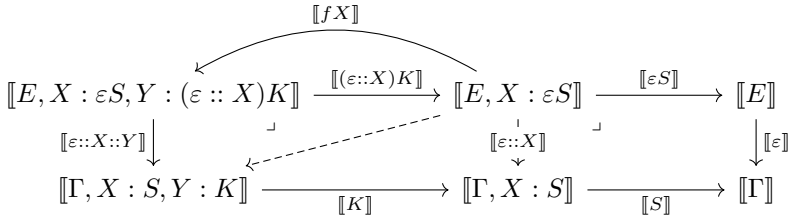


This in particular gives some object

$$E \vdash f : \{X : \varepsilon S\}\ ((\varepsilon :: X)K)$$

Weakening and applying PI-APP-OBJ we obtain

$$E, X : \varepsilon S \vdash f X : (\varepsilon :: X)K$$



The dashed arrow is then a morphism in the slice $\mathcal{C}/[\![\Gamma, X : S]\!]$. So we have (making the same argument the other way round and using β and η)

$$\mathcal{C}/[\![\Gamma, X : S]\!]\,([\![\varepsilon :: X]\!], [\![K]\!]) \underset{\text{PI-APP-OBJ}}{\overset{\text{PI-LAM-OBJ}}{\rightleftarrows}} \mathcal{C}/[\![\Gamma]\!]([\![\varepsilon]\!], [\![\{X : S\}\, K]\!])$$

This is closer to an adjunction: we can see that the left adjoint should be pullback along $[\![S]\!] : [\![\Gamma, X : S]\!] \to [\![\Gamma]\!]$. Could ask for this bijection to work for arbitrary pullback? The answer is *not quite*. We can replace the context extension $[\![\Gamma, X : S \to [\![\Gamma]\!]]\!]$ with a context morphism $[\![\Delta \vdash \delta \ : \ \Gamma]\!]$, but we will only be able to find a right adjoint for pullback along $[\![\delta]\!]$ when Δ *only contains sorts*. This is because PI-CLS restricts formation of function classes to classes dependent over sorts.

$$\frac{}{\bullet \vdash \mathsf{Sort}}\ \textsc{ctx-sort-emp} \qquad\qquad \frac{\Gamma \vdash \mathsf{Sort} \quad \Gamma \vdash S \ : \ \mathsf{Sort}}{\Gamma, X : S \vdash \mathsf{Sort}}\ \textsc{ctx-sort-ext}$$

We can also replace $[\![K]\!]$ with a slice over $[\![\Delta]\!]$ so that we have an adjunction

$$\mathcal{C}/[\![\Delta]\!]\,([\![\delta]\!]^*[\![\varepsilon]\!], [\![\zeta]\!]) \underset{\text{PI-APP-OBJ}}{\overset{\text{PI-LAM-OBJ}}{\rightleftarrows}} \mathcal{C}/[\![\Gamma]\!]([\![\varepsilon]\!], \Pi_{[\![\delta]\!]}[\![\zeta]\!])$$

Briefly and roughly: Write context $Z = V_1 : Z_1, \cdots, V_n : Z_n$. A slice morphism on the left consists of objects of class Z_i in context $E, \Delta, \varepsilon =_\Gamma \delta$. Repeated λ-abstraction gives objects of class $\{X \ : \ \Delta\}\,\{Y \ : \ \varepsilon =_\Gamma \delta\}\,Z_i$ in context E, on the right hand side. So we should define $\Pi_{[\![\delta]\!]}[\![\zeta]\!]$ such that its object part is context

$$[\![V_1 : \{X \ : \ \Delta\}\,\{Y \ : \ \varepsilon =_\Gamma \delta\}\,Z_1, \cdots, V_n : \{X \ : \ \Delta\}\,\{Y \ : \ \varepsilon =_\Gamma \delta\}\,Z_n]\!]$$

and its morphism part comes from each Z_i being a class in context Δ. The proof that it is an adjunction is in the same vein as before.

4.6 Universes

It is not the case that the category of contexts is a locally cartesian closed category. In particular, we have pullbacks but not Πs for arbitrary morphisms in the category. However, the full subcategory of sort-only contexts *is* locally cartesian closed. Note that although this subcategory of the syntactic category is an LCCC, it is not the case that locally cartesian closed categories are fit for semantic purposes due to the substitution/pullback issue addressed later.

References

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[SH21] Jonathan Sterling and Robert Harper. Logical relations as types: Proof-relevant parametricity for program modules. *Journal of the ACM*, 68(6):1–47, oct 2021.