Model Theory

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0.0.1 Monster model

Notation. *We write $\Sigma(\ldots, r)$ to mean the signature with an extra relation symbol thrown in with the rest.*

Definition – κ-monster [1]

Let κ be a cardinal and r a relation symbol not in the signature Σ. We say a Σ-structure $\mathcal{M}$ is a κ-monster (also called big or splendid by Hodges) if it satisfies: for any subset $A \subseteq \mathcal{M}$ of cardinality $< \kappa$ and any $\Sigma(A, r)$ structure $\mathcal{N}$ that such that $\mathcal{N}$ and $\mathcal{M}$ are elementarily equivalent *in* Σ, there is an interpretation of $\mathcal{M}$ as a $\Sigma(A, r)$-structure such that $\mathcal{N}$ and $\mathcal{M}$ are elementarily equivalent *in* $\Sigma(r)$.

$$\mathcal{M} \equiv_\Sigma \mathcal{N} \rightsquigarrow \mathcal{M} \equiv_{\Sigma(r)} \mathcal{N}$$

We want monsters to have the properties of being saturated, homogeneous and universal, which we define soon. We show that monsters are saturated and that saturated structures are homogeneous and universal.

$$\begin{tikzcd}
 & & \text{homogeneous} \\
\text{monster} \arrow[r] & \text{saturated} \arrow[ur] \arrow[dr] & \\
 & & \text{universal}
\end{tikzcd}$$

Exercise. *Let ϕ be a Σ-formula with n free variables. There exists a Σ-formula that means 'there exists a unique tuple $x = (x_1, \ldots, x_n)$ such that $\phi(x)$'. We write this as*

$$\exists! x, \phi(x)$$

Lemma – Monsters are saturated

Let the Σ-structure $\mathcal{M}$ be a κ-monster. Then it is κ-saturated.

Proof. Let A be a subset of $\mathcal{M}$ with cardinality $< \kappa$. Let $p \in S_n(\Theta_{\mathcal{M}}(A))$ be a maximal n-type. Since p is realised by $b \in \mathcal{N}^n$ in some elementary $\Sigma(A)$-extension $\mathcal{N}$ we can make $\mathcal{N}$ a $\Sigma(A, r)$-structure that interprets r as the singleton set $\{b\}$. We have $\mathcal{M} \equiv_\Sigma \mathcal{N}$ as $\mathcal{N}$ is an elementary extension. Since $\mathcal{M}$ is a κ-monster and $\mathcal{M} \equiv_\Sigma \mathcal{N}$ we have an interpretation of r in $\mathcal{M}$ such that $\mathcal{M} \equiv_{\Sigma(A,r)} \mathcal{N}$. Hence

$$\mathcal{N} \vDash_{\Sigma(A,r)} \exists! x, r(x) \Rightarrow \mathcal{M} \vDash_{\Sigma(A,r)} \exists! x, r(x)$$

Taking this unique element $a \in \mathcal{M}^n$ we see that

$$p = \mathrm{tp}_A^{\mathcal{N}}(b) = \mathrm{tp}_A^{\mathcal{N}}(\iota(a)) = a_A^{\mathcal{M}}$$

The last equality using the fact that $\mathcal{N}$ is an elementary $\Sigma(A)$-extension of $\mathcal{M}$. (Note that we might not have that it is an elementary $\Sigma(A, r)$-extension) Hence p is realised in $\mathcal{M}$ and $\mathcal{M}$ is κ-saturated. $\square$

Definition – κ-saturation (strong κ-homogeneity)

Let κ be a cardinal and $\mathcal{M}$ be a Σ-structure. $\mathcal{M}$ is κ-saturated if for subset $A \subseteq \mathcal{M}$ such that $|A| < \kappa$ every $n \in \mathbb{N}$ and every $p \in S_n(\mathrm{Th}_{\mathcal{M}}(A))$, p is realised in $\mathcal{M}$.

Equivalently $\mathcal{M}$ is κ-saturated if for any $A \subseteq \mathcal{N}$ in any Σ-structure $\mathcal{N}$ satisfying $|A| < \kappa$ either of the equivalent following hold:

- For any partial elementary embedding $f : A \to \mathcal{M}$ and any $b \in \mathcal{N}$, f can be extended to a partial elementary Σ-embedding $A \cup \{b\} \to \mathcal{M}$.
- If $\mathcal{M} \equiv_{\Sigma(A)} \mathcal{N}$ then for any $b \in \mathcal{N}$ we have $\mathcal{M} \equiv_{\Sigma(A,c)} \mathcal{N}$ for some constant symbol c that is interpreted as b in $\mathcal{N}$.

The latter two definitions are equivalent due to the fact that $\mathcal{M} \equiv_{\Sigma(A)} \mathcal{N}$ if and only if there is a way to interpret symbols from A in $\mathcal{M}$ such that

$$\mathcal{M} \vDash_{\Sigma(A)} \phi \Leftrightarrow \mathcal{N} \vDash_{\Sigma(A)} \phi$$

for any $\Sigma(A)$-sentence ϕ, which is equivalent to the existance of a partial embedding $A \to \mathcal{M}$. The strong version implies the weak version by taking $\mathcal{N}$ to be $\mathcal{M}$. It remains to prove the equivalence with the first definition:

Proof. ($\Rightarrow$) Let $f : A \to \mathcal{M}$ be a partial elementary embedding, where $A \subseteq \mathcal{N}$ and $|A| < \kappa$; let $b \in \mathcal{N}$. We show that $\mathcal{M} \equiv_{\Sigma(A,c)} \mathcal{N}$ for a constant symbol c that is interpreted as b in $\mathcal{N}$ which gives a way to extend f over $A \cup \{b\}$.

By amalgamation there exists $\mathcal{L}$ and commuting elementary $\Sigma(A)$-embeddings $\iota_{\mathcal{M}}, \iota_{\mathcal{N}}$into $\mathcal{L}$ from $\mathcal{M}$ and $\mathcal{N}$. First we show that $\mathrm{tp}_A^{\mathcal{L}}(\iota_{\mathcal{N}}(b))$ is consistent with $\mathrm{Th}_{\mathcal{M}}(A)$. It suffices to show finite consistency: let Δ be a finite subset of $\mathrm{Th}_{\mathcal{M}}(A)$. The formula

$$\exists x, \bigwedge_{\psi \in \Delta} \psi(x)$$

is satisfied in $\mathcal{L}$ by taking $\iota_{\mathcal{N}}(b)$ hence is satisfied in $\mathcal{M}$ as $\iota_{\mathcal{M}}$ is $\Sigma(A)$-elementary. Thus $\mathrm{tp}_A^{\mathcal{L}}(\iota_{\mathcal{N}}(b))$ is consistent with $\mathrm{Th}_{\mathcal{M}}(A)$.

Hence, as $\mathcal{M}$ is κ-saturated we can find $a \in \mathcal{M}$ such that

$$\mathrm{tp}_A^{\mathcal{L}}(\iota_{\mathcal{N}}(b)) = \mathrm{tp}_A^{\mathcal{M}}(a)$$

Add a constant symbol c to the signature such that $c^{\mathcal{N}} = b$ and $c^{\mathcal{M}} = a$. Then $x = c$ is in both types and so $\iota_{\mathcal{N}}(b) = \iota_{\mathcal{M}}(a)$. Finally we have

$$\mathcal{N} \equiv_{\Sigma(A,c)} \mathcal{L} \equiv_{\Sigma(A,c)} \mathcal{M}$$

since $\mathcal{N} \to \mathcal{L}$ and $\mathcal{M} \to \mathcal{M}$ are elementary.

($\Leftarrow$) If $p \in S_n(\mathrm{Th}_{\mathcal{M}}(A))$ then it is realised in some elementary Σ-extension $\iota : \mathcal{M} \to \mathcal{N}$ by $b \in \mathcal{N}$. Hence we have a partial elementary Σ-embedding $\iota^{-1} : \iota(\mathcal{M}) \to \mathcal{M}$ that can be extended to have domain $\iota^{-1} \cup \{b\}$ by assumption. Hence we have the image of b under this map as the element of $\mathcal{M}$ realising the type p. $\square$

A special case of the second definition of saturation comes in the form of homogeneity.

Definition – κ-homogeneity

Let κ be a cardinal and $\mathcal{M}$ a Σ-structure. $\mathcal{M}$ is κ-homogeneous if for any $A \subseteq \mathcal{M}$, any partial elementary embedding $f : A \to \mathcal{M}$ and any $b \in \mathcal{M}$, if $|A| < \kappa$ then f can be extended to a partial elementary embedding $A \cup \{b\} \to \mathcal{M}$.

Definition – κ-universality

Let T be a Σ-theory and let $\mathcal{M} \vDash_{\Sigma} T$. Then $\mathcal{M}$ is a κ-universal Σ-model of T if any $\mathcal{N} \vDash_{\Sigma} T$ such that $|\mathcal{N}| < \kappa$ has an elementary embedding into $\mathcal{M}$.

We replace κ with κ^+ when the $<$ is improved to $\leq$.

Lemma – Saturated structures are universal [2]
Let $\mathcal{M}$ be a κ-saturated Σ-structure. Then $\mathcal{M}$ is a κ^+-universal Σ-structure.

Proof. Let $\mathcal{N}_\beta$ be a Σ-structure such that $|\mathcal{N}_\beta| = \beta \le \kappa$. Then enumerate $\mathcal{N}_\beta = \{n_\gamma \,|\, \gamma \le \beta\}$ and define nested subsets $\mathcal{N}_\alpha := \{n_\gamma \,|\, \gamma \le \alpha\}$ for each $\alpha \le \beta$. We then use transfinite induction to define partial elementary embeddings $f_\alpha : \mathcal{N}_\alpha \to \mathcal{M}$ for $\alpha \le \beta$ such that they mutually agree upon restriction.

If α is a limit ordinal and for each $\gamma \le \alpha$ we have a well-defined partial elementary embedding $f_\gamma : \mathcal{N}_\gamma \to \mathcal{M}$ that agree upon restriction then $f_\alpha := \bigcup_{\gamma<\alpha} f_\gamma$ is well-defined.

To define $f_{\alpha+1}$ using f_α we first define

$$\Gamma = \{\phi(v, f_\alpha(a)) \in \Sigma(f_\alpha(\mathcal{N}_\alpha))_{\text{for}} \,|\, \exists a \in \mathcal{N}^\star, \mathcal{N} \vDash_\Sigma \phi(n_\alpha, a))\}$$

Γ is finitely consistent with $\mathrm{Th}_{\mathcal{M}}(f_\alpha(\mathcal{N}_\alpha))$: Gor any finite subset $\Delta \subseteq \Gamma$, we can take the conjuction,

$$\exists v, \bigwedge_{\phi(v, f_\alpha(a)) \in \Delta} \phi(v, a)$$

Note that this is satisfied by $\mathcal{N}$ therefore by the partial elementary Σ-embedding f_α:

$$\mathcal{M} \vDash_\Sigma \exists v, \bigwedge_{\phi(v, f_\alpha(a)) \in \Delta} \phi(v, f_\alpha(a))$$

Hence $\mathcal{M} \vDash_\Sigma \Delta(b)$ for some $b \in \mathcal{M}$. By compactness for types Γ is consistent with $\mathrm{Th}_{\mathcal{M}}(f_\alpha(\mathcal{N}_\alpha))$. This implies that it can be extended to a maximal 1-type $p \in S_1(\mathrm{Th}_{\mathcal{M}}(\mathcal{N}_\alpha))$. By κ-saturation, $\Gamma(v)$ is realised by some $b \in \mathcal{M}$. Define $f_{\alpha+1} : n_\alpha \mapsto b$ and agreeing with f_α otherwise. By definition $f_{\alpha+1}$ is a partial elementary Σ-embedding.

Hence $f_\beta : \mathcal{N}_\beta \to \mathcal{M}$ is an elementary Σ-embedding. $\square$

Definition – Monster model
Let T be an infinitely modelled Σ-theory. Then for any infinite cardinal κ there exists a 'Monster model' of T: a Σ-model $\mathbb{M}_{\kappa^+}$ such that

- $\kappa \le |\mathbb{M}_{\kappa^+}|$
- $\mathbb{M}_{\kappa^+}$ is κ-saturated
- For any $\mathcal{N} \vDash_\Sigma T$ such that $|\mathcal{N}| < \kappa$ there is an elementary Σ-embedding $\mathcal{N} \to \mathbb{M}_{\kappa^+}$. For any model $\mathcal{M}$ of T and elementary Σ-embeddings $\mathcal{M} \to \mathbb{M}$ and $\mathcal{M} \to \mathcal{N}$ the following commutes

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[dr] & \mathcal{N} \arrow[d] \\ & \mathbb{M} \end{tikzcd}$$

Proof. $\square$

Definition – Algebraic, algebraic closure
Let $\mathcal{M}$ be a Σ-structure and let D be a subset of $\mathcal{M}$ (often strongly minimalstrongly minimal). Let A be a subset of D, $a \in \mathcal{M}$ is algebraic over A if a belongs to a finite $\Sigma(A)$-definable set . Define the algebraic closure of A over D to be

$$\mathrm{acl}_D(A) := \{a \in D \,|\, a \text{ is algebraic over } A\}$$

We drop the subscript D when it is sufficiently obvious.

Lemma – Some definable sets

Let $\mathcal{M}$ be a Σ-structure and let $B, C \subseteq \mathcal{M}$ be a Σ-definable set. Let $\phi(x)$ be a Σ-formula with n free variables. For $b \in B^n$, let $\psi(x,b)$ be a $\Sigma(B)$-formula with m free variables ($\psi(x,y)$ is a Σ-formula with $n+m$ free variables).

Then the following sets are definable by a Σ-formula:

- The intersection of B and C, the union of B and C the complement of B.
- The set of $b \in B^n$ that satisfy $\phi(x)$:

$$\{b \in B^n \,|\, \mathcal{M} \vDash_\Sigma \phi(b)\}$$

- The elements $b \in \mathcal{M}^n$ such that $\psi(x,b)$ defines a set of at most k elements.
- The elements of $\mathcal{M}^n$ such that $\psi(x,b)$ defines a set of at least k elements.
- The elements of $\mathcal{M}^n$ such that $\psi(x,b)$ defines a set of cardinality k.

$$\{b \in \mathcal{M}^n \mid |\psi(\mathcal{M}, b)| = k\}$$

  and even $\{b \in B^n \mid |\psi(\mathcal{M}, b)| = k\}$ by taking the intersection of two definable sets.

We will become lazier when dealing with definable sets as we gain an idea of what should and should not be definable.

Proof.

- This is clear.
- Since B is Σ-definable we can take $\chi(x)$ as the Σ-formula defining B and consider the Σ-formula

$$\phi(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{n} \chi(x_i)$$

  Clearly this defines $\{b \in B^n \,|\, \mathcal{M} \vDash_\Sigma \phi(b)\}$.
- To make $\{b \in \mathcal{M}^n \mid |\psi(\mathcal{M}, b)| \leq k\}$ we take the Σ-formula $\chi(x)$:

$$\chi(x) = \mathop{\forall}_{i=1}^{k+1} x_i, \bigwedge_{i=1}^{k+1} \psi(x_i, y) \to \bigvee_{i \neq j}^{x_i = x_j}$$

  where potentially x_i represents m variables, which we can quantify over as it is finite.
- To make $\{b \in \mathcal{M}^n \,|\, k \leq |\psi(\mathcal{M}, b)|\}$ we take the Σ-formula $\chi(x)$:

$$\chi(x) = \mathop{\exists}_{i=1}^{k} x_i, x_i \neq x_j$$

$\square$

Definition – Minimal, strongly minimal [2]

Let $\mathcal{M}$ be a Σ-structure. Let D be an infinite Σ-definable subset of $\mathcal{M}^n$. D is minimal in $\mathcal{M}$ if any Σ-definable subset of D if finite or cofinite. D is strongly minimal if it is minimal in $\mathcal{N}$ for any elementary extension $\mathcal{N}$ of $\mathcal{M}$. A Σ-theory T is strongly minimal if any Σmodel of T is strongly minimal (note that any Σ-structure is definable by the formula $v = v$).

Proposition – Algebraic closure is a pregeometry
Let $\mathcal{M}$ be a Σ-structure. Let D be a minimal subset of $\mathcal{M}$. Then $(D, \mathrm{acl}_D(\star)$ is almost a pregeometry.

[**fandom**]. Preserves order: *if* $A \subseteq B \subseteq D$ *then* $\mathrm{acl}_D(A) \subseteq \mathrm{acl}_D(B)$. Let $a \in \mathrm{acl}_D(A)$. Then there exists a finite $\Sigma(A)$-definable set containing a. Any $\Sigma(A)$-formula is naturally a $\Sigma(B)$-formula thus $a \in \mathrm{acl}_D(B)$.

Idempotence: *for any* $A \subseteq D$, $\mathrm{acl}_D(A) = \mathrm{acl}_D(\mathrm{acl}_D(A))$. ($\Rightarrow$) We first show that for any subset $A \subseteq D$, $A \subseteq \mathrm{acl}_D(A)$. Let $a \in A$ then $a = x$ is a $\Sigma(A)$-formula that is defines a finite set. Thus $a \in \mathrm{acl}_D(A)$. Directly we have the corollary $\mathrm{acl}_D(A) \subseteq \mathrm{acl}_D(\mathrm{acl}_D(A))$.

($\Leftarrow$) We show that $\mathrm{acl}_D(\mathrm{acl}_D(A)) \subseteq \mathrm{acl}_D(A)$. Let $a \in \mathrm{acl}_D(\mathrm{acl}_D(A))$. Then there exists $\phi(x, v) = \phi(x, v_0, \ldots, v_n)$ a Σ-formula and $b_0, \ldots, b_n \in \mathrm{acl}_D(A)$ such that $\phi(x, b)$ defines a finite subset of $\mathcal{M}$ containing a. Let k be the finite cardinality of $\phi(\mathcal{M}, b)$. There exists a Σ-formula $\psi(v)$ that defines the set $\{b \in B \mid |\phi(\mathcal{M}, b)| \leq n\}$

$$\phi'(x, v) := \phi(x, v) \wedge \psi(v)$$

We have that $a \in \phi(\mathcal{M}, b) = \phi'(\mathcal{M}, b)$ and for any $c \in \mathcal{M}^n$, $\phi'(\mathcal{M}, c)$ is finite.

For each b_i appearing in b there exists a $\Sigma(A)$-formula $\psi_i(v_i)$ such that $b_i \in \psi_i(\mathcal{M})$ and this definable set is finite. Define the $\Sigma(A)$-formula

$$\phi''(x) := \coprod_{i=1}^{n} v_i, \phi'(x, v_0, \ldots, v_n) \wedge \bigwedge_{i=1}^{n} \psi_i(v_i)$$

Then $a \in \phi''(\mathcal{M})$ by taking the v_i to be b_i and

$$\begin{aligned} d \in \phi''(\mathcal{M}) &\Rightarrow \exists c \in \mathcal{M}^n, \mathcal{M} \vDash_\Sigma \phi'(d, c) \text{ and for each } i, \mathcal{M} \vDash_\Sigma \psi_i(c_i) \\ &\Rightarrow \text{there exist for each } i\ c_i \in \psi_i(\mathcal{M}), \mathcal{M} \vDash_\Sigma \phi'(d, c) \\ &\Rightarrow d \in \bigcup_{i=0}^{n} \bigcup_{c_i \in \psi_i(\mathcal{M})} \phi'(\mathcal{M}, c) \end{aligned}$$

The last expression is a finite union of finite sets which is finite. Hence $\phi''(\mathcal{M})$ is finite and $a \in \mathrm{acl}_D(A)$

Finite character: *if* $A \subseteq D$ *and* $a \in \mathrm{acl}_D(A)$ *then there exists a finite subset* $F \subseteq A$ *such that* $a \in \mathrm{acl}_D(F)$. Take the $\Sigma(A)$-formula defining the finite set containing a. Pick out the (finitely many) constant symbols from A, forming a finite subset $F \subseteq A$. Then $a \in \mathrm{acl}_D(F)$.

Exchange: *if* $A \subseteq D$ *and* $a, b \in D$ *such that* $a \in \mathrm{acl}_D(A, b)$ (*shorthand for* $A, \{a\}$) *then* $a \in \mathrm{acl}_D(A)$ *or* $b \in \mathrm{acl}_D(A, a)$. Since $a \in \mathrm{acl}(A, b)$ there exists a $\Sigma(A)$-formula $\phi(v, w)$ such that $a \in \phi(\mathcal{M}, b)$ and $\phi(\mathcal{M}, b)$ is finite - say it has cardinality n (if b does not appear in the formula then we immediately have $a \in \mathrm{acl}(A)$). There exists a $\Sigma(A)$-formula $\psi(w)$ defining the set

$$\psi(\mathcal{M}) = \{b' \in D \mid n = |\phi(\mathcal{M}, b')|\}$$

As $\psi(\mathcal{M}) \subseteq D$ and D is minimal, $\psi(\mathcal{M})$ is finite or cofinite. If it is finite then $b \in \psi(\mathcal{M})$ and so $b \in \mathrm{acl}(A) \subseteq \mathrm{acl}(A, a)$.

If it is $\psi(\mathcal{M})$ then consider the $\Sigma(A)$-formula $\phi(v, w) \wedge \psi(w)$. For each $a' \in D$ let $X(a')$ be the subset of D defined by $\phi(a', w) \wedge \psi(w)$. Consider $b \in X(a)$, and case on whether it is finite or cofinite. If it is finite then $b \in \mathrm{acl}(A, a)$ as $\phi(a, w) \wedge \psi(w)$ is a $\Sigma(A)$-formula defining a finite set.

If $X(a)$ is cofinite then let $m = |D \setminus X(a)| \in \mathbb{N}$. There exists a $\Sigma(A)$-formula $\chi(v)$ defining the set

$$\chi(\mathcal{M}) = \{a' \in D \mid m = |D \setminus X(a')|\}$$

If $\chi(\mathcal{M})$ is finite then $a \in \chi(\mathcal{M})$ and so $a \in \mathrm{acl}(A)$. If $\chi(\mathcal{M})$ is confinite then there exist $n+1$ distinct elements $a_i \in \chi(\mathcal{M})$ since D is infinite by definition. Take the (finite) intersection of the cofinite $X(a_i)$, producing a non-empty (infinite) set. Take

$$b' \in \bigcap_{i=1}^{n+1} X(a_i) = \bigcap_{i=1}^{n+1} \phi(a_i, \mathcal{M}) \cap \psi(\mathcal{M})$$

Then for each i, $\mathcal{M} \vDash_\Sigma \phi(a_i, b')$, hence $n+1 \leq |\phi(\mathcal{M}, b')|$. However $\mathcal{M} \vDash_\Sigma \psi(b')$ implies $n = |\phi(\mathcal{M}, b')|$, a contradiction. □

Remark. The definition of dimension for pregeometries thus carries through for subsets of D.

We begin by looking at 1-types, as done in the example with infinite infinite equivalence classes.

> **Proposition – Classification of 1-types on a model of ACF_p**
> Let $\mathcal{N}$ be an ω-saturated extension of $\mathcal{M}$, a Σ_{RNG}-model of ACF_p. Any element of the Stone space $S_1(\mathrm{ElDiag}((,\Sigma)_{\mathrm{RNG}}, \mathcal{M}))$ is of the form $\mathrm{tp}^{\mathcal{N}}_{\varnothing,1}(a)$. There are two cases for what the types over $\mathcal{M}$ can be:
>
> - a is in the image of $\mathcal{M}$; equivalently the type is an isolated point.
> - a is not in the image of $\mathcal{M}$; equivalently its type is the unique non-isolated type.

Proof. The first part is covered in the lemma.

For the second part suppose a is not in the image of $\mathcal{M}$. It is not isolated by the first part. Note that for any $p \in \mathcal{M}[x] \setminus 0$, $\iota(p)(a) \neq 0$ because $\mathcal{M}$ is algebraically closed. (If $\iota(p)(a) = 0$ then we can keep factoring roots of the polynomial from $\mathcal{M}$ to find one that corresponds to a.) We show that any b not in the image gives rise to the same type as a.

Let $\phi \in \mathrm{tp}(a)$. As ACF_p has quantifier elimination we have equivalently $\phi \in \mathrm{qftp}(a)$. Then by the disjunctive normal form for formulas in Σ_{RNG} we have that

$$\mathcal{N} \vDash_{\Sigma_{\mathrm{RNG}}} \forall v, \phi(v) \leftrightarrow \bigvee_{i \in I} \left(\bigwedge_{j \in J_{i0}} p_{ij}(v) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(v) \neq 0 \right)$$

Thus there exists $i \in I$ such that

$$\mathcal{N} \vDash_{\Sigma_{\mathrm{RNG}}} \forall v, \phi(v) \Leftrightarrow \bigwedge_{j \in J_{i0}} p_{ij}(v) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(v) \neq 0$$

If J_{i0} contains a non-zero polynomial p_{ij} then we would have a contradiction by the above remark. Thus

$$\mathcal{N} \vDash_{\Sigma_{\mathrm{RNG}}} \forall v, \phi(v) \Leftrightarrow \bigwedge_{j \in J_{i1}} q_{ij}(v) \neq 0$$

Suppose for a contradiction that ϕ is not in $\mathrm{tp}(b)$, in which case $\mathcal{N} \nvDash_{\Sigma_{\mathrm{RNG}}} \phi(b)$ and $\mathcal{N} \vDash_{\Sigma_{\mathrm{RNG}}} \bigvee_{j \in J_{i1}} q_{ij}(b) = 0$. Hence either $\mathcal{N} \vDash_{\Sigma_{\mathrm{RNG}}} \perp$, which is a contradiction, or $\mathcal{N} \vDash_{\Sigma_{\mathrm{RNG}}} q_{ij}(b) = 0$ for some $j \in J_{i1}$. By the above remark q_{ij} must therefore be the zero polynomial, which also leads to a contradiction as

$$\mathcal{N} \vDash_{\Sigma_{\mathrm{RNG}}} q_{ij}(a) \neq 0$$

Hence $\mathrm{tp}(a) = \mathrm{tp}(b)$. □

Bibliography

[1] Wilfrid Hodges. *A Shorter Model Theory*. Cambridge University Press, 1997.
[2] David Marker. *Model Theory - an Introduction*. Springer.