Model Theory

JLH

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Chapter 1

Pure Model Theory

1.1 Basics

These first two sections follow Marker's book on Model Theory [5] with more emphasis on where things are happening, i.e. what signature we are working in, and some more general statements such as working with embeddings rather than subsets.

1.1.1 Signatures

Definition – First order language
We assume we have a tuple $\mathcal{L} = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathcal{V}, \{\neg\,, \vee\,, \forall, =, \top\})$ such that

- $|\mathcal{C}|, |\mathcal{F}|, |\mathcal{R}|$ each sufficiently large (say they have cardinality $\aleph_5$ or something).
- $|\mathcal{V}| = \aleph_0$. We index $\mathcal{V} = \{v_0, v_1, \dots\}$ using $\mathbb{N}$.
- $\mathcal{C}$,$\mathcal{F}$,$\mathcal{R}$, $\mathcal{V}$, $\{\neg\,, \vee\,, \forall, =\}$ do not overlap.

We call $\mathcal{L}$ the language and only really use it to get symbols to work with. Whenever we introduce new symbols to create larger signatures, we are pulling them out of this box.

Definition – Signature
In a language $\mathcal{L}$, a tuple $\Sigma = (C, F, n_\star, R, m_\star)$ is a signature when

- $C \subseteq \mathcal{C}$. We call C the set of constant symbols.
- $F \subseteq \mathcal{F}$ and $n_\star : F \to \mathbb{N}$, which we call the function arity. We call F the set of function symbols.
- $R \subseteq \mathcal{R}$ and $m_\star : R \to \mathbb{N}$, which we call the relation arity. We call T the set of relation symbols.

Given a signature Σ, we may refer to its constant, function and relation symbol sets as $\Sigma_{\text{con}}, \Sigma_{\text{fun}}, \Sigma_{\text{rel}}$. We will always denote function arity using $n_\star$ and relation arity using $m_\star$.

Definition – Σ-terms
Given Σ a signature, its set of terms Σ_{ter} is inductively defined using three constructors:

| If $c \in \Sigma_{\text{con}}$ then $c \in \Sigma_{\text{ter}}$.

| If $v_i \in \mathcal{V}$, v_i is a term.

| Given $f \in \Sigma_{\text{fun}}$ and $t \in (\Sigma_{\text{ter}})^{n_f}$ then there is the string $f(t) \in \Sigma_{\text{ter}}$.

The terms v_i are called variables and will be referred to as elements of Σ_{var} instead of $\mathcal{V}$.

Proposition – Terms have finitely many variables
For any term $t \in \Sigma_{\text{ter}}$ there exists a finite subset $S \subseteq \mathbb{N}$ such that for all $i \in \mathbb{N}$, $i \in S$ if and only if v_i occurs in t.

Proof. If there exists a finite subset T of $\mathbb{N}$ such that for all $i \notin T$, v_i does not occur in t, then we can take the intersection of all such sets and still have a finite set S.

We first prove existence of such a T using the inductive definition of t.

- If $t = c$ a constant symbol, then $T = \varnothing$ satisfies the above.
- If $t = v_i$ a variable symbol, then $T = \{i\}$ satisfies the above.
- If $t = f(t_0, \dots, t_{n_f})$, then by our induction hypothesis we have a T_i satisfying the condition for each t_i. Then $\cup_i T_i$ satisfies the condition for t.

Since one such T exists, we can obtain a finite set S by intersecting all sets satisfying the condition on T. Let $i \in \mathbb{N}$. Suppose v_i occurs in t, then i is in any T with the above property. Hence $i \in S$. Hence S also satisfies the above condition. Suppose v_i does not occur in t. Then $S \setminus \{i\}$ also satisfies the above condition, hence $S \subseteq S \setminus \{i\}$ by minimality of S. Thus $i \notin S$. □

Definition – Σ-formula, free variable
Given Σ a signature, its set of formulas Σ_{for} is inductively defined:

| $\top$ is an element of Σ_{for}.

| Given $t, s \in \Sigma_{\text{ter}}$, there is the string $(t = s) \in \Sigma_{\text{for}}$

| Given $r \in \Sigma_{\text{rel}}, t \in (\Sigma_{\text{ter}})^{m_r}$, there is the string $r(t) \in \Sigma_{\text{for}}$

| Given $\phi \in \Sigma_{\text{for}}$, the string $(\neg\,\phi) \in \Sigma_{\text{for}}$

| Given $\phi, \psi \in \Sigma_{\text{for}}$, the string $(\phi \vee \psi) \in \Sigma_{\text{for}}$

| Given $\phi \in \Sigma_{\text{for}}$ and $v_i \in \Sigma_{\text{var}}$, we take the replace all occurrences of v_i with an unused symbol such as z in the string $(\forall v_i, \phi)$ and call this a new element of Σ_{for}.

Shorthand for some strings in Σ_{for} include

- $\bot := \neg\,\top$
- $\phi \wedge \psi := \neg\,(\neg\,\phi \vee \neg\,\psi)$
- $\phi \to \psi := (\neg\,\phi) \vee \psi$
- $\exists v, \phi := \neg\,(\forall v, \neg\,\phi)$

The symbol z is meant to be a 'bounded variable', and will not be considered when we want to evaluate variables in formulas. Variables that we do want to evaluate we call 'free variables'.

Remark. There are two different uses of the symbol '=' from now on, and context will allow us to tell them apart. Similarly, logical symbols might be used in our 'higher' language and will not be confused with symbols from formulas.

Proposition – Formulas have finitely many *free* variables
Given $\phi \in \Sigma_{\text{for}}$, there exists a finite $S \subseteq \mathbb{N}$ such that for all $i \in \mathbb{N}$, $i \in S$ if and only if v_i occurs freely in ϕ.

Proof. Similarly to the previous proposition, we only need to show that there exists a T such that for all $i \notin T$, v_i does not occur in ϕ. We can do this by using the inductive definition of ϕ:

- If ϕ is $\top$ then it has no variables.
- If ϕ is $t = s$, then we have S_t, S_s indexing the variables of t and s by the previous proposition, and so we can pick $T = S_t \cup S_s$.
- If ϕ is $r(t_0, \ldots, t_{m_r})$, then for each t_i we have S_i indexing the variables of t_i. Hence we can pick $T = \cup_i S_i$.
- If ϕ is $\neg\, \psi$, then by the induction hypothesis we have T satisfying the above conditions for ψ. Pick this T for ϕ.
- If ϕ is $\psi \vee \chi$ then by the induction hypothesis we have T_ψ, T_χ satisfying the above conditions for ψ and χ. We take T to be the union of indexing sets for ψ and χ.
- If ϕ is $\forall v_i, \psi$ with v_i substituted for z, then by the induction hypothesis we have T_ψ satisfying the above conditions for ψ. Take $T = T_\psi \setminus \{v_i\}$.

□

NOTATION (SUBSTITUTING TERMS FOR VARIABLES). *If a formula has a free variable v_i then to remind ourselves of the variable we can write $\phi = \phi(v_i)$ to mean the same thing.*

If $t \in (\Sigma_{\text{ter}})^S$, then we write $\phi(t)$ to mean ϕ with t_i substituted for each v_i. We can show by induction on terms and formulas that this is still a formula.

EXAMPLE (THE EMPTY SIGNATURE AND THEORY). $\Sigma_\varnothing = (\varnothing, \varnothing, n_\star, \varnothing, m_\star)$ *is the empty signature. (We pick the empty functions for $n_\star, m_\star$.) The empty $\Sigma_\varnothing$-theory is $\varnothing$. Notice that any set is a model of $\Sigma_\varnothing$.*

Note that in set theory the carrier set of models are indeed sets, hence any set is a model of the empty theory.

Definition – Σ-structure, interpretation
Given a signature Σ, a set M and a tuple $\star_\Sigma^\mathcal{M} = \left(\star_{\Sigma_{\text{con}}}^\mathcal{M}, \star_{\Sigma_{\text{fun}}}^\mathcal{M}, \star_{\Sigma_{\text{rel}}}^\mathcal{M}\right)$, we say that $\mathcal{M} := (M, \star_\Sigma^\mathcal{M})$ is a Σ-structure when

- $\star_{\Sigma_{\text{con}}}^\mathcal{M} : \Sigma_{\text{con}} \to M$
- $\star_{\Sigma_{\text{fun}}}^\mathcal{M} : \Sigma_{\text{fun}} \to (M^{n_\star} \to M)$
- $\star_{\Sigma_{\text{rel}}}^\mathcal{M} : \Sigma_{\text{rel}} \to \mathcal{P}(M^{m_\star})$

The latter two are dependant types since the powers $n_\star, m_\star$ depend on the function and relation symbols given. The class of Σ-structures is denoted Σ_{str}. Given only the Σ-structure $\mathcal{M}$, we call its underlying carrier set M as $\mathcal{M}_{\text{car}}$.

We can think of $\star_\Sigma^\mathcal{M}$ as a single entity interpreting constant symbols as elements of $\mathcal{M}_{\text{car}}$, function symbols as $n_\star$-ary functions on $\mathcal{M}_{\text{car}}$, and relation symbols as $m_\star$-ary relations on $\mathcal{M}_{\text{car}}$. When the context is clear, given $c \in \Sigma_{\text{con}}, f \in \Sigma_{\text{fun}}, r \in \Sigma_{\text{rel}}$, we may write $c_{\Sigma_{\text{con}}}^\mathcal{M} = c_\Sigma^\mathcal{M} = c^\mathcal{M}, f_{\Sigma_{\text{fun}}}^\mathcal{M} = f_\Sigma^\mathcal{M} = f^\mathcal{M}, r_{\Sigma_{\text{rel}}}^\mathcal{M} = r_\Sigma^\mathcal{M} = r^\mathcal{M}$. We call these interpretations.

Definition – Interpretation of terms

Given a signature Σ, a Σ-structure $\mathcal{M}$ and a Σ-term t, There exists a unique induced map $t_T^{\mathcal{M}} : \mathcal{M}_{\text{car}}^S \to \mathcal{M}_{\text{car}}$, such that this commutes with the interpretation of constants and functions†. Here S is the unique set indexing the variables of t. We then refer to this map as *the* interpretation of the term t. This in turn defines a dependant Π-type

$$\star_T^{\mathcal{M}} : \Sigma_{\text{ter}} \to (\mathcal{M}_{\text{car}}^{S_*} \to \mathcal{M}_{\text{car}})$$

†Precisely: For all constant symbols c, $\iota(c)_T^{\mathcal{M}} = c_{\Sigma_{\text{con}}}^{\mathcal{M}}$, where ι is the inclusion $\Sigma_{\text{con}} \to \Sigma_{\text{ter}}$. For all $f \in \Sigma_{\text{fun}}, s \in \Sigma_{\text{ter}}$, $(f(s))_T^{\mathcal{M}}(\star) = f_{\Sigma_{\text{fun}}}^{\mathcal{M}}(s_T^{\mathcal{M}}(\star))$.

Proof. Given a signature Σ, a Σ-structure $\mathcal{M} = (M, \star_\Sigma^{\mathcal{M}})$ and a Σ-term t, there exists a finite $S \subseteq \mathbb{N}$ indexing the variables of t. To define a map $t_T^{\mathcal{M}} : M^S \to M$ for each t we use the inductive definition of $t \in \Sigma_{\text{ter}}$. If M is empty we define $t_T^{\mathcal{M}}$ as the empty function. Otherwise let $a \in M^S$:

- If $t = c \in \Sigma_{\text{con}}$ then define $t_T^{\mathcal{M}} : a \mapsto c_C^{\mathcal{M}}$, the constant map. This type checks since $S = \varnothing$ therefore $t_T^{\mathcal{M}} : M^0 \to M$.
- If $t = v_i \in \Sigma_{\text{var}}$ then define $t_T^{\mathcal{M}} : a \mapsto a$, the identity. This type checks since $|S| = 1$.
- If $t = f(s)$ for some $f \in \Sigma_{\text{fun}}$ and $s \in (\Sigma_{\text{var}})^{n_f}$ then define $t_T^{\mathcal{M}} : a \mapsto f_F^{\mathcal{M}}(s_T^{\mathcal{M}}(a))$. This type checks since s has the same number of variables as t.

By definition, this map commutes with the interpretation of constants and functions. Furthermore, it is unique since it is constructed to satisfy the commuting properties. □

Where there is no ambiguity, we write $t_T^{\mathcal{M}} = t^{\mathcal{M}}$. Furthermore, if we have a tuple $t \in (\Sigma_{\text{ter}})^k$, then we write $t_T^{\mathcal{M}} := ({t_0}_T^{\mathcal{M}}, \cdots, {t_k}_T^{\mathcal{M}})$

Definition – Satisfaction

Given Σ a signature, $\mathcal{M} \in \Sigma_{\text{str}}$, and a $\phi \in \Sigma_{\text{for}}$, there exists a subset $S \subseteq \mathbb{N}$ indexing the free variables of ϕ. Given $a \in (\mathcal{M}_{\text{car}})^S$, we want to define $\mathcal{M} \vDash_\Sigma \phi(a)$. If $\mathcal{M}_{\text{car}}$ is empty then we write $\mathcal{M} \vDash_\Sigma \phi(a)$. Otherwise, using the inductive definition of ϕ:

- If ϕ is $\top$ then $\mathcal{M} \vDash_\Sigma \phi$. (We can omit the a when there are no free variables.)
- If ϕ is $t = s$ then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)$.
- If ϕ is $r(t)$, where $r \in \Sigma_{\text{rel}}$ and $t \in (\Sigma_{\text{ter}})^{m_r}$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $t^{\mathcal{M}}(a) \in r^{\mathcal{M}}$.
- If ϕ is the string $\neg\psi$ for some $\psi \in \Sigma_{\text{for}}$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $\mathcal{M} \nvDash_\Sigma \psi(a)$
- If ϕ is the string $(\psi \vee \chi)$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $\mathcal{M} \vDash_\Sigma \psi(a)$ or $\mathcal{M} \vDash_\Sigma \chi(a)$.
- If ϕ is the string $(\forall v, \psi(a)) \in \Sigma_{\text{for}}$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ if for any $b \in \mathcal{M}_{\text{car}}$, $\mathcal{M} \vDash_\Sigma \psi(a)(b)$.

We say $\mathcal{M}$ satisfies $\phi(a)$.

Remark. Note that S is empty when there are no free variables. Note also by convention we have a rather nasty possible case of

$$\varnothing \vDash_\Sigma \exists v, v = v$$

which is intuitively wrong.

Remark. Any Σ-structure $\mathcal{M}$ satisfies $\top$ by casing on if the structure is empty or not.

1.1.2 Theories and Models

Definition – Σ-sentence, Σ-theory

We say $\phi \in \Sigma_{\text{for}}$ is a Σ-sentence when the set indexing the free variables of ϕ is the empty set.

T is an Σ-theory when it is a subset of Σ_{for} such that all elements of T are Σ-sentences. We denote the set of Σ-theories as Σ_{the}

Definition – Σ-Model, consistent, finitely consistent

Given an Σ-structure $\mathcal{M}$ and Σ-theory T, we write $\mathcal{M} \vDash_\Sigma T$ and say $\mathcal{M}$ *is a* Σ*-model of* T when for all $\phi \in T$ we have $\mathcal{M} \vDash_\Sigma \phi$.

A Σ-theory T is consistent if there exists $\mathcal{M} \in \Sigma_{\text{str}}$ such that $\mathcal{M} \vDash_\Sigma T$ and $\mathcal{M}_{\text{car}} \neq \varnothing$. It is finitely consistent if all finite subsets of T are consistent.

Let ϕ be Σ-formula with variables indexed by S. Then given $c \in (\Sigma_{\text{con}})^S$ we write $\phi(c)$ to mean the formula with its variables substituted for the constant symbols. In this case $\mathcal{M} \vDash_\Sigma \phi(c)$ would be the same thing as $\mathcal{M} \vDash_\Sigma \phi(c^{\mathcal{M}})$. Similarly we write $t(c)$ for terms.

Note that if ϕ is a Σ-formula with one variable v, and $\mathcal{M} \vDash_\Sigma \phi(a)$ for some $a \in \mathcal{M}_{\text{car}}$, we might not be able to say that there is a $c \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma \phi(c)$.

Definition – Consequence

Given a Σ-theory T and a Σ-sentence ϕ, we say ϕ is a consequence of T and say $T \vDash_\Sigma \phi$ when for all models $\mathcal{M}$ of T, we have $\mathcal{M} \vDash_\Sigma \phi$.

Remark. We have to be a bit careful when we go from something like $\mathcal{M} \vDash_\Sigma \phi(a)$ to deducing something about T. This is because there might not exist a Σ-constant c such that $c^{\mathcal{M}} = a$, it only makes sense to write $T \vDash_\Sigma \phi$ if ϕ is a *sentence*.

Exercise (Empty model). *Show that for any Σ-formula ϕ the Σ-structure $\mathcal{N} = (\varnothing, \varnothing)$ (if it is indeed a Σ-structure) satisfies $\mathcal{N} \vDash_\Sigma \phi$ and $\mathcal{N} \vDash_\Sigma \neg\phi$. Does this lead to a contradiction?*

Proposition – Inconsistent theory

Given T a Σ-theory, show that T not consistent if and only if there exists a Σ-sentence ϕ such that $T \vDash_\Sigma \phi$ and $T \vDash_\Sigma \neg\phi$.

Proof. ($\Rightarrow$) Take ϕ to be the Σ-sentence $\top$. Since the only model of T is the empty Σ-structure and $\varnothing \vDash_\Sigma \top$ and $\varnothing \vDash_\Sigma \bot$, $T \vDash_\Sigma \top$ and $T \vDash_\Sigma \bot$. ($\Leftarrow$) Suppose T is consistent. Let $\mathcal{M}$ be the non-empty Σ-model of T. Then $\mathcal{M} \vDash_\Sigma \phi$ and $\mathcal{M} \vDash_\Sigma \neg\phi$. Then since $\mathcal{M}$ is non-empty this implies $\mathcal{M} \vDash_\Sigma \phi$ and $\mathcal{M} \nvDash_\Sigma \phi$, a contradiction. $\square$

Thus the definition of consistent is thus intuitively 'T does not lead to a contradiction'.

1.1.3 The Compactness Theorem

Read ahead for the statement of the Compactness Theorem. The first two parts of the theorem are easy to prove. This chapter focuses on proving the final part.

Definition – Witness property

Given a signature Σ and a Σ-theory T, we say that Σ satisfies the witness property when for any Σ-

formula ϕ with exactly one free variable v, there exists $c \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma (\exists v, \phi(v)) \to \phi(c)$.

This says that for any Σ-model $\mathcal{M}$ of T, if there exists an element of the model such that $\phi(m)$ is true then there exists a constant symbol c of Σ such that $\phi(c^{\mathcal{M}})$ is true.

Definition – Maximal theory

A Σ theory T is Σ-maximal if for any Σ-formula ϕ, if ϕ is a Σ-sentences then $\phi \in T$ or $\neg\,\phi \in T$.

Proposition – Maximum property

Given a Σ-maximal and finitely consistent theory T and a Σ-sentence ϕ,

1. $T \vDash_\Sigma \phi$ if and only if $\phi \in T$.
2. $T \vDash_\Sigma \neg\,\phi$ if and only if $\phi \notin T$.
3. $T \vDash_\Sigma \phi$ if and only if $T \nvDash_\Sigma \neg\,\phi$
4. $T \nvDash_\Sigma \phi$ if and only if $T \vDash_\Sigma \neg\,\phi$

Proof.

1. $(\Rightarrow)$ Suppose $T \vDash_\Sigma \phi$. Since T is Σ-maximal, we have $\phi \in T$ or $\neg\,\phi \in T$. If $\neg\,\phi \in T$ then we have a finite subset $\{\phi, \neg\,\phi\} \subseteq T$ which is not consistent, thus the second case is false.

   $(\Leftarrow)$ Suppose $\phi \in T$. Case on $T \vDash_\Sigma \phi$ or $T \nvDash_\Sigma \phi$. If $T \nvDash_\Sigma \phi$ then there exists $\mathcal{N}$ a Σ-model of T such that $N \nvDash_\Sigma \phi$. But $\mathcal{N} \vDash_\Sigma \phi$ since $\phi \in T$. Thus the second case is false.
2. $(\Rightarrow)$ If $\neg\,\phi \in T$ then case on $\phi \in T$ or $\phi \notin T$. In the first case we have a finite subset $\{\phi, \neg\,\phi\} \subseteq T$ which is not consistent, thus the first case is false.

   $(\Leftarrow)$ If $\phi \notin T$ then since T is Σ-maximal, $\phi \in T$ or $\neg\,\phi \in T$. The first case is false.
3. $T \vDash_\Sigma \phi \Leftrightarrow \phi \in T \Leftrightarrow \neg\,\phi \notin T \Leftrightarrow T \nvDash_\Sigma \neg\,\phi$
4. $T \nvDash_\Sigma \phi \Leftrightarrow \phi \notin T \Leftrightarrow \neg\,\phi \in T \Leftrightarrow T \vDash_\Sigma \neg\,\phi$

$\square$

Proposition – Henkin construction

Let Σ be a signature. Let κ be a cardinal such that $|\Sigma_{\text{con}}| \leq \kappa$ If Σ-theory T

- has the witness property,
- is Σ-maximal,
- is finitely consistent,

then T has a non-empty Σ-model $\mathcal{M}$ such that $|\mathcal{M}| \leq \kappa$.

Proof. **The Σ-structure**: Consider quotienting Σ_{con} by the equivalence relation $c \sim d := T \vDash_\Sigma c = d$. (Check that this is an equivalence relation.) Let $\pi : \Sigma_{\text{con}} \to \Sigma_{\text{con}}/\sim$ This can define a Σ-structure $\mathcal{M}$ in the following way:

1. We let the carrier set be the image of the quotient $\pi(\Sigma)$.

   We let the constant symbols be interpreted as their equivalence classes: $\star^{\mathcal{M}}_{\Sigma_{\text{con}}} = \pi$. We have the right cardinality for $\mathcal{M}$:[1] $|\mathcal{M}| \leq |\Sigma_{\text{con}}| \leq \kappa$

[1]From this point onwards when it is obvious what we mean we write $\mathcal{M}$ for $\mathcal{M}_{\text{car}}$.

2. To interpret functions we must use the witness property. Given $f \in \Sigma_{\text{fun}}$ and $\pi(c) \in \mathcal{M}^{n_f}$ (valid as π is surjective) we consider the Σ-formula $\exists v, f(c) = v$. By the witness property, there exists a $d \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma (\exists v, f(c) = v) \to (f(c) = d)$. Let $\mathcal{N} \vDash_\Sigma T$ have non-empty carrier set, then unfolding the definition of satisfaction we get two cases: for any $a \in \mathcal{N}$, $f^{\mathcal{N}}(c^{\mathcal{N}}) \neq a$ (f has no image) or $f^{\mathcal{N}}(c^{\mathcal{N}}) = d^{\mathcal{N}}$. The first case is false: $\mathcal{N}$ is non-empty by assumption, thus f must have an image $a \in \mathcal{N}$. Thus we have a $d \in \Sigma_{\text{con}}$ such that for any $\mathcal{N} \vDash_\Sigma T$, $\mathcal{N} \vDash_\Sigma f(c) = d$, i.e. $T \vDash_\Sigma f(c) = d$. We define $f^{\mathcal{M}}$ to map $\pi(c) \mapsto \pi(d)$.

   To show that f is well-defined (in the case where $\Sigma_{\text{con}} \neq \varnothing$), take $c_0 \in (\Sigma_{\text{con}})^{n_f}$ such that $\pi(c) = \pi(c_0)$ and suppose there is $d_0 \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma f(c_0) = d_0$. Suppose $\mathcal{N} \vDash_\Sigma T$.

$$d_0^{\mathcal{N}} = f^{\mathcal{N}}(c_0^{\mathcal{N}}) = f^{\mathcal{N}}(c_1^{\mathcal{N}}) = d_1^{\mathcal{N}}$$

   Hence $T \vDash_\Sigma d_0 = d_1$ and so $\pi(d_0) = \pi(d_1)$.

3. Let $r \in \Sigma_{\text{rel}}$. We define $r_\Sigma^{\mathcal{M}} := \left\{ \pi(c) \,|\, c \in (\Sigma_{\text{con}})^{\mathcal{M}} \wedge T \vDash_\Sigma r(c) \right\}$

Hence $\mathcal{M}$ is a Σ-structure. We want to show that $\mathcal{M}$ is a Σ-model of T.

Terms: To make our lives easier we first prove a **useful claim**: If $t \in \Sigma_{\text{ter}}$ with variables indexed by S, $d \in \Sigma_{\text{con}}$ and c is in $(\Sigma_{\text{con}})^S$ then $T \vDash_\Sigma t(c) = d$ if and only if $t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}}$. i.e. $\mathcal{M} \vDash_\Sigma t(c) = d$. Case on what t is:

- Case $t \in \Sigma_{\text{con}}$: ($\Rightarrow$) Suppose $T \vDash_\Sigma t(c) = d$, then $T \vDash_\Sigma t = d$ since there are no variables. Then by definition of $\star_\Sigma^{\mathcal{M}}$, $t^{\mathcal{M}} = d^{\mathcal{M}}$ ($\Leftarrow$) Suppose $t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}}$ then $c^{\mathcal{M}} = t(c)^{\mathcal{M}} = d^{\mathcal{M}}$ hence by definition of $\star_\Sigma^{\mathcal{M}}$, $T \vDash_\Sigma c = d$. Taking any model $\mathcal{N}$ of T, we immediately have $t^{\mathcal{N}}(c^{\mathcal{N}}) = d^{\mathcal{N}}$.

- Suppose $t \in \Sigma_{\text{var}}$, then it suffices to show that $T \vDash_\Sigma c = d$ if and only if $c^{\mathcal{M}} = d^{\mathcal{M}}$, which we have already done.

- With the induction hypothesis, suppose $t = f(s)$, where $f \in \Sigma_{\text{fun}}$ and $s \in (\Sigma_{\text{ter}})^{n_f}$. ($\Rightarrow$) If we can find $e = (e_1, \dots e_{n_f}) \in (\Sigma_{\text{con}})^{n_f}$ such that each $T \vDash_\Sigma s_i(c) = e_i$ and $f(e)^{\mathcal{M}} = d^{\mathcal{M}}$, then we have $s_i^{\mathcal{M}}(c^{\mathcal{M}}) = e_i^{\mathcal{M}}$ and so

$$\begin{aligned} t^{\mathcal{M}}(c^{\mathcal{M}}) &= (f(s))^{\mathcal{M}}(c^{\mathcal{M}}) \\ &= f^{\mathcal{M}}(s^{\mathcal{M}}(c^{\mathcal{M}})) \\ &= f^{\mathcal{M}}(e^{\mathcal{M}}) \\ &= f(e)^{\mathcal{M}} = d^{\mathcal{M}} \end{aligned}$$

  Indeed, using the witness property n_f times (induction) we can construct e. Suppose $e_1, \dots, e_{i-1} \in \Sigma_{\text{con}}$ such that for each $j < i$, they satisfy

$$T \vDash_\Sigma \exists x_{j+1}, \dots, \exists x_{n_f}, f(\dots, e_{j-1}, e_j, x_{i+1}, \dots) = d \wedge e_j = s_j(c)$$

  Let ϕ_i be the single variable formulas $\exists x_{i+1}, \dots, \exists x_{n_f}, f(\dots, e_{i-1}, v, x_{i+1}, \dots) = d \wedge v = s_i(c)$, with variable v. Then by the witness property, there exists an $e_i \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma \exists v, \phi_i(v) \to \phi_i(e_i)$. Suppose $\mathcal{N} \vDash_\Sigma T$, then by deconstructing $\exists v, \phi_i(v) \to \phi_i(e_i)$ we have a 'for all' case that will turn out to be false and a second case $\exists x_{i+1}, \dots, \exists x_{n_f}, f(\dots, e_{i-1}, e_i, x_{i+1}, \dots) = d \wedge e_i = s_i(c)$. By assumption if $\mathcal{N} \vDash_\Sigma T$, then $t^{\mathcal{N}}(c^{\mathcal{N}}) = d^{\mathcal{N}}$, hence $f^{\mathcal{N}}(s_1^{\mathcal{N}}(c), \dots, s_{n_f}^{\mathcal{N}}(c)) = d^{\mathcal{N}}$. Therefore the first case is false, since we gave the existance of elements of $\mathcal{N}$ satisfying the formula. The second case gives us $T \vDash_\Sigma s(c) = e$ and $f(e)^{\mathcal{M}} = d^{\mathcal{M}}$.

  ($\Leftarrow$) $t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}} \Rightarrow f^{\mathcal{M}}(s^{\mathcal{M}}(c^{\mathcal{M}})) = d^{\mathcal{M}} \Rightarrow (f(s(c)))^{\mathcal{M}} = d^{\mathcal{M}}$. Note that $(s(c))^{\mathcal{M}} = e^{\mathcal{M}}$ for some $e \in \Sigma_{\text{con}}$ since π is surjective. By the induction hypothesis $T \vDash_\Sigma s(c) = e$. Hence $f^{\mathcal{M}}(e^{\mathcal{M}}) = d^{\mathcal{M}}$ and by the way $\mathcal{M}$ interprets functions, $T \vDash_\Sigma f(e) = d$. If $\mathcal{N} \vDash_\Sigma T$ then $(t(c))^{\mathcal{N}} = (f(s(c)))^{\mathcal{N}} = f^{\mathcal{N}}(e^{\mathcal{N}}) = d^{\mathcal{N}}$.

Thus $T \vDash_\Sigma t(c) = d \Leftrightarrow t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}}$.

Formulas: Lastly, we show that $\mathcal{M} \vDash_\Sigma T$. Since for all $\phi \in T$ we have $T \vDash_\Sigma \phi$, it suffices to show that for all Σ-sentences $\phi, T \vDash_\Sigma \phi$ implies $\mathcal{M} \vDash_\Sigma \phi$. We prove a stronger statement which will be needed for the induction: for all ϕ with its variables indexed by S and $c \in (\Sigma_{\text{con}})^S$,

$$T \vDash_\Sigma \phi(c) \Leftrightarrow \mathcal{M} \vDash_\Sigma \phi(c^\mathcal{M})$$

We case on what ϕ is:

- Case ϕ is $\top$: ($\Rightarrow$) All models satisfy $\top$ hence $\mathcal{M} \vDash_\Sigma \top$. ($\Leftarrow$) Let $\mathcal{N}$ be a model of T then $\mathcal{N} \vDash_\Sigma \top$ since all models satisfy $\top$.

- Case ϕ is $t = s$: ($\Rightarrow$) Apply the witness property to $t(c) = v \wedge s(c) = v$, obtaining $d \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma (\exists v, (t(c) = v \wedge s(c) = v)) \to (t(c) = d \wedge s(c) = d)$ If $T \vDash_\Sigma t = s$ then by deconstructing $(\exists v, (t(c) = v \wedge s(c) = v)) \to (t(c) = d \wedge s(c) = d)$ we can show that $T \vDash_\Sigma t(c) = d$ and $T \vDash_\Sigma t(c) = d$ Using the useful claim from before, we obtain $t^\mathcal{M}(c^\mathcal{M}) = d^\mathcal{M} = s^\mathcal{M}(c^\mathcal{M})$ Hence $\mathcal{M} \vDash_\Sigma t(c^\mathcal{M}) = s(c^\mathcal{M})$.

  ($\Leftarrow$) If $\mathcal{M} \vDash_\Sigma t(c^\mathcal{M}) = s(c^\mathcal{M})$ then since π is surjective, there exists a $d \in \Sigma_{\text{con}}$ such that $t^\mathcal{M}(c^\mathcal{M}) s^\mathcal{M}(c^\mathcal{M}) = d^\mathcal{M}$. Using the useful claim we obtain $T \vDash_\Sigma t(c) = d$ and $T \vDash_\Sigma s(c) = d$. It follows that $T \vDash_\Sigma t(c) = s(c)$.

- Case ϕ is $r(t)$: ($\Rightarrow$) Suppose $T \vDash_\Sigma r(t(c))$. By induction, apply the witness property m_r times to the formulas
$$\exists x_{i+1}, \dots, \exists x_{m_r}, r(\dots, e_{i-1}, v, x_{i+1}, \dots) \wedge v = t_i(c)$$
  each time obtaining $e_i \in \Sigma_{\text{con}}$ satisfying the formula. The result is $T \vDash_\Sigma r(e)$ and $T \vDash_\Sigma e = t(c)$. Using the useful claim this implies $t^\mathcal{M}(c^\mathcal{M}) = e^\mathcal{M} \in r^\mathcal{M}$, and hence $\mathcal{M} \vDash_\Sigma r(t(c))$. ($\Leftarrow$) Suppose $\mathcal{M} \vDash_\Sigma r(t(c))$. Since π is surjective, there exists $e \in \Sigma_{\text{con}}$ such that $e^\mathcal{M} = t^\mathcal{M}(c^\mathcal{M}) \in r^\mathcal{M}$. Using the useful claim we obtain $T \vDash_\Sigma t(c) = e$. Then the way $\mathcal{M}$ interprets relations, $T \vDash_\Sigma r(e)$. It follows that $T \vDash_\Sigma r(t(c))$.

- Case ϕ is $\neg\chi$: Using the maximal property of T for the first $\Leftrightarrow$ and the induction hypothesis for the second $\Leftrightarrow$ we have
$$T \vDash_\Sigma \neg\chi(c) \;\Leftrightarrow\; T \nvDash_\Sigma \chi(c) \;\Leftrightarrow\; \mathcal{M} \nvDash_\Sigma \chi(c) \;\Leftrightarrow\; \mathcal{M} \vDash_\Sigma \neg\chi(c)$$

- Case ϕ is $\chi_0 \vee \chi_1$
$$\begin{aligned} \mathcal{M} \vDash_\Sigma \chi_0(c^\mathcal{M}) \vee \chi_1(c^\mathcal{M}) &\Leftrightarrow \mathcal{M} \vDash_\Sigma \chi_0(c^\mathcal{M}) \text{ or } \mathcal{M} \vDash_\Sigma \chi_1(c^\mathcal{M}) \\ &\Leftrightarrow T \vDash_\Sigma \chi_0(c) \text{ or } T \vDash_\Sigma \chi_1(c) \quad \text{by the induction hypothesis} \end{aligned}$$
  Hence it suffices to show that $T \vDash_\Sigma \chi_0(c)$ or $T \vDash_\Sigma \chi_1(c) \;\Leftrightarrow\; T \vDash_\Sigma \chi_0(c) \vee \chi_1(c)$. ($\Rightarrow$) We show the contrapositive. Suppose $T \nvDash_\Sigma \chi_0(c) \vee \chi_1(c)$, then there exists $\mathcal{N}$ a Σ-model of T such that $\mathcal{N} \nvDash_\Sigma \chi_0(c^\mathcal{N}) \vee \chi_1(c^\mathcal{N})$. We can show that such an $\mathcal{N}$ satisfies $\mathcal{N} \nvDash_\Sigma \chi_0(c^\mathcal{N})$ and $\mathcal{N} \nvDash_\Sigma \chi_1(c^\mathcal{N})$ Hence $T \nvDash_\Sigma \chi_0(c)$ and $T \nvDash_\Sigma \chi_1(c)$

  ($\Leftarrow$) Suppose $T \vDash_\Sigma \chi_0(c)$ or $T \vDash_\Sigma \chi_1(c)$. For $\mathcal{N}$ a Σ-model of T,
$$\mathcal{N} \vDash_\Sigma \chi_0(c^\mathcal{N}) \text{ or } \mathcal{N} \vDash_\Sigma \chi_1(c^\mathcal{N}) \Rightarrow \mathcal{N} \vDash_\Sigma \chi_0(c^\mathcal{N}) \vee \chi_1(c^\mathcal{N})$$
  Thus $T \vDash_\Sigma \chi_0(c) \vee \chi_1(c)$.

- Case ϕ is $\forall v, \chi(v)$ ($\Rightarrow$) Let $d \in \mathcal{M}$, then since π surjective $\exists e \in \Sigma_{\text{con}}$ such that $\pi(e) = d$. Hence from $T \vDash_\Sigma \forall v, \chi(c, v)$ we can show that $T \vDash_\Sigma \chi(c, e)$ hence by induction $\mathcal{M} \vDash_\Sigma \chi(c^\mathcal{M}, d)$. Thus $\mathcal{M} \vDash_\Sigma \forall v, \chi(c^\mathcal{M}, v)$.

  ($\Leftarrow$) We show the contrapositive. If $T \nvDash_\Sigma \forall v, \chi(c)(v)$, then by the maximal property of T, $T \vDash_\Sigma \exists v, \neg\chi(c, v)$. Applying the witness property to $\neg\chi(c, v)$, we have that there is $e \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma (\exists v, \neg\chi(c)(v)) \to (\neg\chi(c)(v))$. Thus $T \vDash_\Sigma \neg\chi(c)(v)$, thus $T \nvDash_\Sigma \chi(c)(v)$ by the maximal property of T, thus $\mathcal{M} \nvDash_\Sigma \chi(c^\mathcal{M})(v)$ by the induction hypothesis. Hence $\mathcal{M} \nvDash_\Sigma \forall v, \chi(c^\mathcal{M})(v)$.

Thus $T \vDash_\Sigma \phi \Leftrightarrow \mathcal{M} \vDash_\Sigma \phi$ and we are done. Notice that the having the if and only if part of the statement was important when using the induction hypothesis. Furthermore, if $\mathcal{M}$ were empty then for the Σ-sentence phi which we pick to be $\forall v, v = v$ we have both $\mathcal{M} \vDash_\Sigma \phi$ and $\mathcal{M} \vDash_\Sigma \neg\phi$ thus $T \vDash_\Sigma \phi$ and $T \vDash_\Sigma \neg\phi$. Hence $\phi \in T$ and $\phi \notin T$ a contradiction. Hence $\mathcal{M}$ is non-empty. $\square$

NOTATION (ORDERING SIGNATURES). *We write* $\Sigma \leq \Sigma(*)$ *for two signatures if* $\Sigma_{\text{con}} \subseteq \Sigma(*)_{\text{con}}$, $\Sigma_{\text{fun}} \subseteq \Sigma(*)_{\text{fun}}$ *and* $\Sigma_{\text{rel}} \subseteq \Sigma(*)_{\text{rel}}$.

Proposition – Moving models down signatures
Given two signatures such that $\Sigma \leq \Sigma(*)$ and $\mathcal{N}$ a $\Sigma(*)$-structure we can make $\mathcal{M}$ a Σ-structure such that

1. $\mathcal{M}_{\text{car}} = \mathcal{N}_{\text{car}}$
2. They have the same interpretation on Σ.
3. For any Σ-formula ϕ with free variables indexed by S and any $a \in \mathcal{M}^S$
$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$
4. If T is a Σ-theory and $T(*)$ is a $\Sigma(*)$-theory such that $T \subseteq T(*)$ and $\mathcal{N}$ a $\Sigma(*)$-model of $T(*)$, then $\mathcal{M}$ is a Σ-model of T.

Technically the new structure is not $\mathcal{N}$, but for convenience we will write $\mathcal{N}$ to mean either of the two and let subscripts involving Σ and $\Sigma(*)$ describe which one we mean.

Proof. We let the carrier set be the same and define $\star_\Sigma^{\mathcal{M}}$ by restriction:

- $\star_{\Sigma_{\text{con}}}^{\mathcal{M}}$ is the restriction of $\star_{\Sigma(*)_{\text{con}}}^{\mathcal{N}}$ to Σ_{con}
- $\star_{\Sigma_{\text{fun}}}^{\mathcal{M}}$ is the restriction of $\star_{\Sigma(*)_{\text{fun}}}^{\mathcal{N}}$ to Σ_{fun}
- $\star_{\Sigma_{\text{rel}}}^{\mathcal{M}}$ is the restriction of $\star_{\Sigma(*)_{\text{rel}}}^{\mathcal{N}}$ to Σ_{rel}

We will need that for any Σ-term t with variables indexed by S, the interpretation of terms is equal: $t_\Sigma^{\mathcal{M}} = t_{\Sigma(*)}^{\mathcal{N}}$. Indeed:

- If t is a constant then $t_\Sigma^{\mathcal{M}} = c_\Sigma^{\mathcal{M}} = c_{\Sigma(*)}^{\mathcal{N}} = t_{\Sigma(*)}^{\mathcal{N}}$
- If t is a variable then $t_\Sigma^{\mathcal{M}} = \text{id}_{\mathcal{M}} = \text{id}_{\mathcal{N}} t_{\Sigma(*)}^{\mathcal{N}}$
- If t is $f(s)$ then $t_\Sigma^{\mathcal{M}} = f_\Sigma^{\mathcal{M}}(s_\Sigma^{\mathcal{M}}) = f_{\Sigma(*)}^{\mathcal{N}}(s_{\Sigma(*)}^{\mathcal{N}}) = t_{\Sigma(*)}^{\mathcal{N}}$

Let ϕ be a Σ-formula with variables indexed by $S \subseteq \mathbb{N}$. Let a be in $\mathcal{M}^S$. Case on ϕ to show that $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$:

- If ϕ is $\top$ then both satisfy ϕ.
- If ϕ is $t = s$ then
$$\begin{aligned} \mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow t_\Sigma^{\mathcal{M}} = s_\Sigma^{\mathcal{M}} \\ &\Leftrightarrow t_{\Sigma(*)}^{\mathcal{M}} = s_{\Sigma(*)}^{\mathcal{M}} && \text{interpretation of terms are equal} \\ &\Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a) \end{aligned}$$
- If ϕ is $r(t)$ then
$$\begin{aligned} \mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow t_\Sigma^{\mathcal{M}}(a) \in r_\Sigma^{\mathcal{M}} \\ &\Leftrightarrow t_{\Sigma(*)}^{\mathcal{N}}(a) \in r_{\Sigma(*)}^{\mathcal{N}} && \text{by how we defined } r_\Sigma^{\mathcal{M}} \text{ and} \\ & && \text{interpretation of terms are equal} \\ &\Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a) \end{aligned}$$

- If ϕ is $\neg\psi$ then

$$\begin{aligned} \mathcal{M} \models_\Sigma \phi(a) &\Leftrightarrow \mathcal{M} \not\models_\Sigma \psi(a) \\ &\Leftrightarrow \mathcal{N} \not\models_{\Sigma(*)} \psi(a) && \text{by the induction hypothesis} \\ &\Leftrightarrow \mathcal{N} \models_{\Sigma(*)} \phi(a) \end{aligned}$$

- If ϕ is $\psi \vee \chi$ then

$$\begin{aligned} \mathcal{M} \models_\Sigma \phi(a) &\Leftrightarrow \mathcal{M} \models_\Sigma \psi(a) \text{ or } \mathcal{M} \models_\Sigma \chi(a) \\ &\Leftrightarrow \mathcal{N} \models_{\Sigma(*)} \psi(a) \text{ or } \mathcal{N} \models_{\Sigma(*)} \chi(a) && \text{by the induction hypothesis} \\ &\Leftrightarrow \mathcal{N} \models_{\Sigma(*)} \phi(a) \end{aligned}$$

- If ϕ is $\forall v, \psi$ then

$$\begin{aligned} \mathcal{M} \models_\Sigma \phi(a) &\Leftrightarrow \forall b \in \mathcal{M}, \mathcal{M} \models_\Sigma \psi(a, b) \\ &\Leftrightarrow \forall b \in \mathcal{M}, \mathcal{N} \models_{\Sigma(*)} \psi(a, b) && \text{by the induction hypothesis} \\ &\Leftrightarrow \forall b \in \mathcal{N}, \mathcal{N} \models_{\Sigma(*)} \psi(a, b) && \text{by the induction hypothesis} \\ &\Leftrightarrow \mathcal{N} \models_{\Sigma(*)} \phi(a) \end{aligned}$$

Hence $\mathcal{M} \models_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \models_{\Sigma(*)} \phi(a)$.

Suppose $T \subseteq T(*)$ are respectively Σ and $\Sigma(*)$-theories and $\mathcal{N} \models_{\Sigma(*)} T(*)$. If $\phi \in T \subseteq T(*)$ then by the previous part, $\mathcal{N} \models_{\Sigma(*)} \phi$ implies $\mathcal{M} \models_\Sigma \phi$ (ϕ is a sentence thus has no variables.) Hence $\mathcal{M} \models_\Sigma T$. □

> **Proposition – Moving models and theories up signatures**
> Suppose $\Sigma \leq \Sigma(*)$.
>
> 1. Suppose $\mathcal{M}$ is a Σ-model of Σ-theory T. Then if there exists $\mathcal{M}(*)$ a $\Sigma(*)$-structure whose carrier set is the same as $\mathcal{M}$ and whose interpretation agrees with $\star_\Sigma^\mathcal{M}$ on constants, functions and relations of Σ, then $\mathcal{M}(*)$ is a $\Sigma(*)$-model of T.
> 2. Suppose T is a Σ-theory and ϕ is a Σ-sentence such that $T \models_\Sigma \phi$. Then $T \models_{\Sigma(*)} \phi$.

Proof.

1. Suppose $\mathcal{M} \models_\Sigma T$. Let $\phi \in T$. To show that $\mathcal{M}(*) \models_{\Sigma(*)} \phi$ we first we prove a useful claim: if $t \in \Sigma_{\text{ter}}$ with no variables then $t_{\Sigma(*)}^{\mathcal{M}(*)} = t_\Sigma^\mathcal{M}$. Case on what t is:
   - If t is a constant symbol c in Σ_{con}, then since $\star_{\Sigma(*)}^{\mathcal{M}(*)} = \star_\Sigma^\mathcal{M}$ on Σ,
   $$t_{\Sigma(*)}^{\mathcal{M}(*)} = c_{\Sigma(*)}^{\mathcal{M}(*)} = c_\Sigma^\mathcal{M} = t_\Sigma^\mathcal{M}$$
   - If t is a variable then it has one variable, thus false.
   - If t is $f(s)$ then since $\star_{\Sigma(*)}^{\mathcal{M}(*)} = \star_\Sigma^\mathcal{M}$ on Σ,
   $$t_{\Sigma(*)}^{\mathcal{M}(*)} = f(s)_{\Sigma(*)}^{\mathcal{M}(*)} = f_{\Sigma(*)}^{\mathcal{M}(*)}(s_{\Sigma(*)}^{\mathcal{M}(*)}) = f_\Sigma^\mathcal{M}(s_\Sigma^\mathcal{M}) = f(s)_\Sigma^\mathcal{M} = t_\Sigma^\mathcal{M}$$

   Case on what ϕ is (ϕ has no variables):
   - If ϕ is $\top$ then it is satisfied.
   - If ϕ is $t = s$, then by the claim above,
   $$t_{\Sigma(*)}^{\mathcal{M}(*)} = t_\Sigma^\mathcal{M} = s_\Sigma^\mathcal{M} = s_{\Sigma(*)}^{\mathcal{M}(*)}$$

- If ϕ is $r(t)$, then by the claim above and the fact that relations are interpreted the same way,
$$t_{\Sigma(*)}^{\mathcal{M}(*)} = t_{\Sigma}^{\mathcal{M}} \in r_{\Sigma}^{\mathcal{M}} = r_{\Sigma(*)}^{\mathcal{M}(*)}$$
- If ϕ is $\neg\psi$ then using the induction hypothesis on ψ,
$$\mathcal{M} \models_{\Sigma} \phi \;\Leftrightarrow\; \mathcal{M} \not\models_{\Sigma} \psi \;\Leftrightarrow\; \mathcal{M}(*) \not\models_{\Sigma(*)} \psi \;\Leftrightarrow\; \mathcal{M}(*) \models_{\Sigma(*)} \phi$$
- If ϕ is $\psi \vee \chi$ then using the induction hypothesis on ψ and χ,
$$\mathcal{M} \models_{\Sigma} \phi \;\Leftrightarrow\; \mathcal{M} \models_{\Sigma} \psi \text{ or } \mathcal{M} \models_{\Sigma} \chi \qquad \Leftrightarrow\; \mathcal{M}(*) \models_{\Sigma(*)} \psi \text{ or } \mathcal{M}(*) \models_{\Sigma(*)} \chi \Leftrightarrow\; \mathcal{M}(*) \models_{\Sigma(*)} \phi$$
- If ϕ is $\forall v, \psi(v)$ and $a \in \mathcal{M}(*) = \mathcal{M}$ then using the induction hypothesis on ψ, $\mathcal{M} \models_{\Sigma} \psi(a) \Rightarrow \mathcal{M}(*) \models_{\Sigma(*)} \psi(a)$. Hence $\mathcal{M}(*) \models_{\Sigma(*)} \phi$.

Thus $\mathcal{M}(*)$ is a $\Sigma(*)$-model of T.

2. Suppose $T \models_{\Sigma} \phi$. If $\mathcal{M} \models_{\Sigma(*)} T$ then by moving $\mathcal{M}$ down to Σ, we have that $\mathcal{M} \models_{\Sigma} T$. Hence $T \models_{\Sigma(*)} \phi$.

□

Again, if we have constructed such a $\mathcal{M}(*)$ from $\mathcal{M}$ we tend to just refer to it as $\mathcal{M}$ and let subscripts involving Σ and $\Sigma(*)$ describe which one we mean.

Proposition – Giving Theories the Witness Property

Suppose $\Sigma(0)$-theory $T(0)$ is finitely consistent. Then there exists a signature $\Sigma(*)$ and $\Sigma(*)$-theory $T(*)$ such that

1. $\Sigma(0)_{\text{con}} \subseteq \Sigma(*)_{\text{con}}$ and they share the same function and relation symbols.
2. $|\Sigma(*)_{\text{con}}| = |\Sigma(0)_{\text{con}}| + \aleph_0$
3. $T(0) \subseteq T(*)$
4. $T(*)$ is finitely consistent
5. Any $\Sigma(*)$-theory T' such that $T(*) \subseteq T'$ has the witness property

Proof. We want to define $\Sigma(i), T(i)$, for each $i \in \mathbb{N}$. By induction, we assume we have $\Sigma(i)$ a signature and $T(i)$ a Σ-theory such that

1. $\Sigma(0)_{\text{con}} \subseteq \Sigma(i)_{\text{con}}$ and they share the same function and relation symbols.
2. $|\Sigma(i)_{\text{con}}| = |\Sigma(0)_{\text{con}}| + \aleph_0$
3. $T(0) \subseteq T(i)$
4. $T(i)$ is finitely consistent

Let
$$W(i) := \{\phi \in \Sigma(i)_{\text{for}} \mid \phi \text{ has exactly one free variable}\}$$
We construct $\Sigma(i+1)$ by adding constant symbols and keeping the same function and relation symbols of $\Sigma(i)$:
$$\Sigma(i+1)_{\text{con}} := \Sigma(i)_{\text{con}} \cup \{c(\phi) \mid \phi \in W(i)\}$$
We create a witness formula for each formula $\phi \in W$:
$$w : W(i) \to \Sigma(i+1)_{\text{for}}, \phi \mapsto ((\exists v, \phi(v)) \to \phi(c_{\phi}))$$
Then let
$$T(i+1) := T(i) \cup w(W(i))$$

Certainly, $T(i+1)$ is a $\Sigma(i+1)$-theory such that $T(0) \subseteq T(i+1)$, $\Sigma(0)_{\mathrm{con}} \subseteq \Sigma(i+1)_{\mathrm{con}}$ and the function and relation symobls are unchanged. Since $W(i)$ is countibly infinite, $|\Sigma(i+1)_{\mathrm{con}}| = |\Sigma(i)_{\mathrm{con}}| + \aleph_0 = |\Sigma(0)_{\mathrm{con}}| + \aleph_0$. We need to check that $T(i+1)$ is finitely consistent. Take a finite subset of $T(i+1)$. It is a union of two finite sets $\Delta_T \subseteq T(i)$ and $\Delta_w \subseteq w(W(i))$. Since $T(i)$ is consistent there exists a non-empty $\mathcal{M}(i)$ that is a $\Sigma(i)$-model of $T(i)$. Let $\mathcal{M}(i+1)$ be defined to have carrier set $\mathcal{M}(i)$. Let $b \in \mathcal{M}(i)$ be some arbitrary constant symbol. To define interpretation for $\mathcal{M}(i+1)$, let $c \in \Sigma(i+1)_{\mathrm{con}}$:

$$c_{\Sigma(i+1)}^{\mathcal{M}(i+1)} := \begin{cases} c_{\Sigma(i)}^{\mathcal{M}(i)} & \text{when} \quad c \in \Sigma(i)_{\mathrm{con}} \\ a & \text{when} \quad c = c_\phi \text{ and } \exists a \in \mathcal{M}(i), \mathcal{M}(i) \vDash_{\Sigma(i)} \phi(a) \\ b & \text{when} \quad c = c_\phi \text{ and } \forall a \in \mathcal{M}(i), \mathcal{M}(i) \nvDash_{\Sigma(i)} \phi(a) \end{cases}$$

Then $\mathcal{M}$ is a well defined $\Sigma(i+1)$-structure. We check is is a $\Sigma(i+1)$-model of $\Delta_T \cup \Delta_w$. Since $\star_{\Sigma(i+1)}^{\mathcal{M}(i+1)}$ agrees with $\star_{\Sigma(i)}^{\mathcal{M}(i)}$ for constants, functions and relations from $\mathcal{M}(i)$, and $\mathcal{M}(i) \neq$ it can be made into a $\Sigma(*)$-model of Δ_T by interpreting the new constant symbols as the element of the non-empty carrier set. If $\psi \in \Delta_w$ then it is $(\exists v, \phi(v)) \to \phi(c_\phi)$ for some $\phi \in W(i)$. Either there exists $a \in \mathcal{M}(i)$ such that $\mathcal{M}(i) \vDash_{\Sigma(i)} \phi(a)$ or not. Hence there exists $a \in \mathcal{M}(i+1)$ such that $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \phi(a)$ or not. In both cases we have a proof of $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \psi$. Thus the induction is complete.

Let $\Sigma(*)$ be the signature such that its function and relations are the same as $\Sigma(0)$ and $\Sigma(*)_{\mathrm{con}} = \bigcup_{i \in \mathbb{N}} \Sigma(i)_{\mathrm{con}}$. Then

$$|\Sigma(*)_{\mathrm{con}}| = |\bigcup_{i \in \mathbb{N}} \Sigma(i)_{\mathrm{con}}| = \aleph_0 \times (\aleph_0 + \Sigma(0)_{\mathrm{con}}) = \aleph_0 + \Sigma(0)_{\mathrm{con}}$$

Let $T(*) = \bigcup_{i \in \mathbb{N}} T(i)$. Any finite subset of $T(*)$ is a subset of some $T(i)$, hence has a non-empty $\Sigma(i)$-model $\mathcal{M}$. Checking the relevant conditions for moving models up signatures, we have $\mathcal{M}(*)$ a non-empty $\Sigma(*)$-model of the finite subset (by interpreting the new constant symbols as the element of the non-empty carrier set.). Hence $T(*)$ is consistent.

If T' is a $\Sigma(*)$-theory such that $T(*) \subseteq T'$, and ϕ is a $\Sigma(*)$-formula of exactly one variable. There exists an $i \in \mathbb{N}$ such that $\phi \in \Sigma(i)_{\mathrm{for}}$. Since $c_\phi \in \Sigma(i+1)$ satisfies $T(i) \vDash_{\Sigma(i+1)} (\exists v, \phi(v)) \to \phi(c_\phi)$, by moving the logical consequence up to $\Sigma(*)$, we have $T(i) \vDash_{\Sigma(*)} (\exists v, \phi(v)) \to \phi(c_\phi)$. If $\mathcal{N}$ is a $\Sigma(*)$-model of T' then it is a $\Sigma(*)$-model of $T(i)$, then $\mathcal{N} \vDash_{\Sigma(*)} (\exists v, \phi(v)) \to \phi(c_\phi)$. Hence $T' \vDash_{\Sigma(*)} (\exists v, \phi(v)) \to \phi(c_\phi)$, satisfying the witness property. □

Lemma – Adding Formulas to Consistent Theories

If T is a finitely consistent Σ-theory and ϕ is a Σ-sentence then at least one of $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely consistent.

Proof. We show that for any finite $\Delta \subseteq T \cup \{\phi\}$ and for any finite $\Delta_\neg T \cup \{\neg\phi\}$, there Δ is consistent or $\Delta_\neg$ is consistent. (The two 'for all's come before the 'or', but one can use the negation to show that this is equivalent to what we want to prove) $(\Delta \setminus \{\phi\}) \cup (\Delta_\neg \setminus \{\neg\phi\}) \subseteq T$ hence is finitely consistent. Let $\mathcal{M}$ be the non-empty model of $(\Delta \setminus \{\phi\}) \cup (\Delta_\neg \setminus \{\neg\phi\})$. Case on whether $\mathcal{M} \vDash_\Sigma \phi$ or not. In the first case $\mathcal{M} \vDash_\Sigma \Delta$ and in the second $\mathcal{M} \vDash_\Sigma \Delta_\neg$. Hence $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely consistent. □

Exercise. *Find a signature Σ, a consistent Σ-theory T and Σ-sentence ϕ such that $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are both consistent? What is the smallest example of this (comparing cardinality of Σ_{con})?*

Proposition – Extending a finitely consistent theory to a maximal theory

Given a finitely consistent Σ-theory $T(0)$ there exists a Σ-theory $T(*)$ such that

1. $T(0) \subseteq T(*)$
2. $T(*)$ is finitely consistent.

3. $T(*)$ is Σ-maximal.

Proof. We use Zorn's Lemma. Let

$$Z := \{T \in \Sigma_{\text{the}} \mid T \text{ finitely consistent and } T(0) \subseteq T\}$$

be ordered by inclusion. Let $T(0) \subseteq T(1) \subseteq \cdots$ be a chain. Then $\cup_{i\in\mathbb{N}} T(i) \in \Sigma_{\text{the}}$ and any finite subset of $\cup_{i\in\mathbb{N}} T(i) \in \Sigma_{\text{the}}$ is a subset of some $T(i)$, hence is finitely consistent. Thus by Zorn's Lemma we have that there exists a $T(*) \in Z$ that is maximal (in the order theory sense). Since $T(*)$ is in Z, we have that it is a superset of T and that it is finitely consistent.

To show that it is Σ-maximal we take a Σ-sentence ϕ. By the previous result, $T(*) \cup \{\phi\}$ or $T(*) \cup \{\neg\phi\}$ is finitely consistent. Hence $T(*) \cup \{\phi\} = T(*)$ or $T(*) \cup \{\neg\phi\} = T(*)$, and $\phi \in T(*)$ or $\neg\phi \in T(*)$. □

Notation (Cardinalities a signatures and structures). *Given a signature Σ, we write $|\Sigma| := |\Sigma_{\text{con}}| + |\Sigma_{\text{fun}}| + |\Sigma_{\text{rel}}|$ and call this the cardinality of the signature Σ.*

This is not so relevant in yet, but it becomes useful later on, so it is used in the Compactness Theorem.

Proposition – The compactness theorem

If T is a Σ-theory, then the following are equivalent:

1. T is finitely consistent.
2. T is consistent
3. For any infinite cardinal κ such that $|\Sigma| \leq \kappa$, there exists a non-empty Σ-model of T with cardinality $\leq \kappa$.

Proof. 3. implies 2. and 2. implies 1. are both obvious. A proof of 1. implies 3. follows.

Suppose an $\Sigma(0)$-theory $T(0)$ is finitely consistent. Let κ be an infinite cardinal such that $|\Sigma(0)| \leq \kappa$. Then $|\Sigma(0)_{\text{con}}| \leq |\Sigma(0)| \leq \kappa$. We wish to find a Σ-model of T with cardinality $\leq \kappa$. We have shown that there exists a signature $\Sigma(1)$ and $\Sigma(1)$-theory $T(1)$ such that

1. $T(0) \subseteq T(1)$
2. $\Sigma(0)_{\text{con}} \subseteq \Sigma(1)_{\text{con}}$ and they share the same function and relation symbols.
3. $|\Sigma(1)_{\text{con}}| = |\Sigma(0)_{\text{con}}| + \aleph_0$
4. $T(1)$ is finitely consistent.
5. Any $\Sigma(1)$-theory T such that $T(1) \subseteq T$ has the witness property.

Since finite subsets of $T(1)$ are consistent there exists a $\Sigma(1)$-theory $T(2)$ such that

6. $T(1) \subseteq T(2)$
7. $T(2)$ is finitely consistent.
8. $T(2)$ is $\Sigma(1)$-maximal.

Furthermore, $T(2)$ has the witness property due to point 5. Since $T(2)$ has the witness property, is $\Sigma(1)$-maximal and finitely consistent, $T(2)$ has a non-empty $\Sigma(1)$-model $\mathcal{M}$ such that $|\mathcal{M}| \leq \kappa$ by Henkin Construction. $\mathcal{M} \vDash_{\Sigma(1)} T(0)$ since $T(0) \subseteq T(1) \subseteq T(2)$. We can move $\mathcal{M}$ down to $\Sigma(0)$, obtaining $\mathcal{M} \vDash_{\Sigma(0)} T(0)$. Thus $T(0)$ has a non-empty model $\mathcal{M}$ with the desired cardinality. □

1.2 Important Lemmas

1.2.1 Morphisms

Definition – Σ-morphism, Σ-embedding, Σ-isomorphism

Given Σ a signature, $\mathcal{M}, \mathcal{N}$ both Σ-structures, $A \subseteq \mathcal{M}_{\text{car}}$ and $\iota : A \to \mathcal{N}_{\text{car}}$, we call ι a partial Σ-morphism from $\mathcal{M}$ to $\mathcal{N}$ when

- For all $c \in C$ (such that $c^{\mathcal{M}} \in A$),
$$\iota(c^{\mathcal{M}}) = c^{\mathcal{N}}$$
- For all $f \in F$ and all $a \in M^{n_f}$ (such that $f^{\mathcal{M}}(a) \in A$),
$$\iota \circ f^{\mathcal{M}}(a) = f^{\mathcal{N}} \circ \iota(a)$$
- For all $r \in R$, for all $a \in M^{m_r} \cap A^{m_r}$,
$$a \in r^{\mathcal{M}} \Rightarrow \iota(a) \in r^{\mathcal{N}}$$

If in addition for relations we have

$$a \in r^{\mathcal{M}} \Leftarrow \iota(a) \in r^{\mathcal{N}} \quad \text{and} \quad \iota \text{ is injective,}$$

then ι is called a partial Σ-embedding (the word extension is often used interchangably with embedding).

In the case that $A = \mathcal{M}_{\text{car}}$ we write $\iota : \mathcal{M} \to \mathcal{N}$ and call ι a Σ-morphism; in this case if it is both an embedding and a bijection then we say $\mathcal{M}$ and $\mathcal{N}$ are Σ-isomorphic and write $\mathcal{M} \cong_\Sigma \mathcal{N}$.

Usually the notion of isomorphism here will work out to be the same as isomorphism in the usual algebraic setting. For example in the language of groups, preserving interpretation of constant symbols implies the identity is sent to the identity and preserving interpretation of function symbols implies the multiplication is preserved.

Definition – Elementary Embedding

A partial Σ-embedding $\iota : A \to \mathcal{N}$ (for $A \subseteq \mathcal{M}$) is elementary if for any Σ-formula ϕ with variables indexed by S and $a \in A^S$,
$$\mathcal{M} \vDash_\Sigma \phi(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

Proposition – Σ-morphisms commute with interpretation of terms

Given a Σ-morphism $\iota : \mathcal{M} \to \mathcal{N}$, we have that for any Σ-term t with variables indexed by S and $a \in \mathcal{M}^S$,
$$\iota(t^{\mathcal{M}}(a)) = t^{\mathcal{N}}(\iota(a))$$

Proof. We case on what t is:

- If $t = c \in \Sigma_{\text{con}}$ then
$$\iota(t^{\mathcal{M}}(a)) = \iota(c^{\mathcal{M}}) = c^{\mathcal{N}} = t^{\mathcal{N}}(\iota(a))$$
- If $t = v \in \Sigma_{\text{var}}$ then
$$\iota(t^{\mathcal{M}}(a)) = \iota(v^{\mathcal{M}}(a)) = \iota(a) = v^{\mathcal{N}}(\iota(a)) = t^{\mathcal{N}}(\iota(a))$$

- If $t = f(s)$ then

$$\iota(t^{\mathcal{M}}(a)) = \iota \circ f^{\mathcal{M}}(s^{\mathcal{M}}(a)) = f^{\mathcal{N}} \circ \iota(s^{\mathcal{M}}(a)) = f^{\mathcal{N}} \circ s^{\mathcal{N}}(\iota(a)) = t^{\mathcal{N}}(\iota(a))$$

  Using the induciton hypothesis in the penultimate step.

$\square$

It is worth knowing that the set of Σ-structures form a category:

Definition – The category of Σ-structures
Given a signature Σ, we denote the category of Σ-structures $(\Sigma_{\text{str}}, \text{Mor}(\star,\star), \circ)$ as $\mathbb{M}\mathbf{od}(\Sigma)$, where

$$\text{Mor}(\mathcal{M},\mathcal{N}) = \{\iota : \mathcal{M} \to \mathcal{N} \mid \iota \text{ is a } \Sigma - \text{morphisms}\}$$

Clearly for any $\mathcal{M}$, the identity exists and is a Σ-morphism.

We refuse to check associativity of $\circ$, but do prove that composition of morphisms are morphisms. Furthermore, composition of embeddings are embeddings and composition of elementary embeddings are elementary. Thus we could also define morphisms between objects to be embeddings or elementary embeddings and obtain a subcategory.

Proof. Let $\iota_1 : \mathcal{M}_0 \to \mathcal{M}_1$ and $\iota_2 : \mathcal{M}_1 \to \mathcal{M}_2$ be Σ-morphisms. We show that the composition is a Σ-morphism:

- If $c \in \Sigma_{\text{con}}$ then
$$\iota_1 \circ \iota_0(c^{\mathcal{M}_0}) = \iota_1(c^{\mathcal{M}_1}) = c^{\mathcal{M}_2}$$

- If $f \in \Sigma_{\text{fun}}$ and $a \in {\mathcal{M}_0}^{n_f}$ then

$$\iota_1 \circ \iota_0 \circ f^{\mathcal{M}_0}(a) = \iota_1 \circ f^{\mathcal{M}_1} \circ \iota_0(a) = f^{\mathcal{M}_2} \circ \iota_1 \circ \iota_0(a)$$

- If $r \in \Sigma_{\text{rel}}$ and $a \in {\mathcal{M}_0}^{m_f}$ then

$$a \in r^{\mathcal{M}_0} \Rightarrow \iota_1(a) \in r^{\mathcal{M}_1} \Rightarrow \iota_2 \circ \iota_1(a) \in r^{\mathcal{M}_2}$$

To show that embeddings compose to be embeddings we note that the composition of injective functions is injective and if $r \in \Sigma_{\text{rel}}$ and $a \in {\mathcal{M}_0}^{m_f}$ then

$$a \in r^{\mathcal{M}_0} \Leftrightarrow \iota_1(a) \in r^{\mathcal{M}_1} \Leftrightarrow \iota_2 \circ \iota_1(a) \in r^{\mathcal{M}_2}$$

To show that composition of elementary embeddings are elementary, let $\phi \in \Sigma_{\text{for}}$ and a in $\mathcal{M}_0$ be chosen suitably. Then
$$\mathcal{M}_0 \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M}_1 \vDash_\Sigma \phi(\iota_1(a)) \Leftrightarrow \vDash_\Sigma \phi(\iota_2 \circ \iota_1(a))$$

$\square$

Definition – Atomic Formula
Given Σ a signature, its set of atomic Σ-formulas is inductively defined

| Given $t, s \in \Sigma_{\text{ter}}$, the string $(t = s)$ is an atomic Σ-formula.

| Given $r \in \Sigma_{\text{rel}}, t \in (\Sigma_{\text{ter}})^{m_r}$, the string $r(t)$ is an atomic Σ-formula.

This is just the subset of Σ_{for} where the formulas are not built inductively.

Definition – Quantifier Free Formula
The set of quantifier free Σ-formulas is defined inductively:

| Given ϕ an atomic Σ-formula, ϕ is a quantifier free Σ-formula.

| Given ϕ a quantifier free Σ-formula, the string $\neg\phi$ is a quantifier free Σ-formula.

| Given ϕ, ψ both quantifier free Σ-formulas, the string $\phi \vee \psi$ is a quantifier free Σ-formula.

Notice quantifier free Σ-formulas are indeed Σ-formulas

Proposition – Embeddings Preserve Satisfaction of Quantifier Free Formulas
Given $\iota : \mathcal{M} \to \mathcal{N}$ a Σ-embedding and ϕ a Σ-formula,

1. If ϕ is $\top$ then it is satisfied by both.
2. If ϕ is $t = s$ then $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$.
3. If ϕ is $r(s)$ then $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$.
4. If ϕ is $\neg\chi$ and $\mathcal{M} \vDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi(\iota(a))$ then
$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$
5. If ϕ is $\chi_0 \vee \chi_1$ and $\mathcal{M} \vDash_\Sigma \chi_i(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_i(\iota(a))$ then
$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

Each of these will be useful in the induction steps of technical proofs.

Furthermore, from the above, we can then immediately conclude (by induction on ϕ) that if ϕ is a quantifier free Σ-formula,
$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

Proof.

1. Trivial.
2. If ϕ is $t = s$ then
$$\begin{aligned} \mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a) \\ &\Leftrightarrow \iota(t^{\mathcal{M}}(a)) = \iota(s^{\mathcal{M}}(a)) && \text{by injectivity} \\ &\Leftrightarrow t^{\mathcal{N}}(\iota(a)) = s^{\mathcal{N}}(\iota(a)) && \text{morphisms commute with} \\ &\Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a)) && \text{interpretation of terms} \end{aligned}$$
3. If ϕ is $r(s)$ then
$$\begin{aligned} \mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow a \in r^{\mathcal{M}} \\ &\Leftrightarrow \iota(a) \in r^{\mathcal{N}} && \text{embeddings property} \\ &\Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a)) \end{aligned}$$
4. If ϕ is $\neg\chi$ and $\mathcal{M} \vDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi(\iota(a))$ then
$$\begin{aligned} \mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow \mathcal{M} \nvDash_\Sigma \chi(a) \\ &\Leftrightarrow \mathcal{N} \nvDash_\Sigma \chi(\iota(a)) \\ &\Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a)) \end{aligned}$$

5. If ϕ is $\chi_0 \vee \chi_1$ and $\mathcal{M} \vDash_\Sigma \chi_i(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_i(\iota(a))$

$$\begin{aligned}\mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow \mathcal{M} \vDash_\Sigma \chi_0(a) \text{ or } \mathcal{M} \vDash_\Sigma \chi_1(a)\\ &\Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_0(\iota(a)) \text{ or } \mathcal{N} \vDash_\Sigma \chi_1(\iota(a))\\ &\Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))\end{aligned}$$

□

Definition – Universal Formula, Universal Sentence

The set of universal Σ-formulas is defined with two constructors:

| If ϕ is a quantifier free Σ-formula then it is a universal Σ-formula.

| If ϕ is a universal Σ-formula then $\forall v, \phi(v)$ is a universal Σ-formula.

Universal Σ-sentences are Σ-formulas with no free variables which we denote by Σ_{uni}.

Proposition – Embeddings preserve satisfaction of universal formulas downwards

Given $\iota : \mathcal{M} \to \mathcal{N}$ a Σ-embedding and ϕ a universal Σ-formula with variables indexed by S (χ is quantifier free). For any $a \in \mathcal{M}^S$ Then

$$\mathcal{N} \vDash_\Sigma \phi(\iota(a)) \quad \Rightarrow \quad \mathcal{M} \vDash_\Sigma \phi(a)$$

By taking the contrapositive we can show that embeddings preserve satisfaction of 'existential' Σ-formulas upwards.

Proof. We induct on ϕ:

- If ϕ is a quantifier free then since embeddings preserve satisfaction of quantifier free formulas, $\mathcal{N} \vDash_\Sigma \phi(\iota(a)) \Rightarrow \mathcal{M} \vDash_\Sigma \phi(a)$.
- If ϕ is $\forall v_i, \psi$ with S indexing the variables of ψ. Let $T := S \setminus \{i\}$ Assuming the inductive hypothesis: for any $a \in \mathcal{M}^T$ and $b \in \mathcal{M}$,

$$\mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b)) \quad \Rightarrow \quad \mathcal{M} \vDash_\Sigma \psi(a, b)$$

  Then since T indexes the variables of ϕ it is clear that for any $a \in \mathcal{M}^T$

$$\begin{aligned}&\mathcal{N} \vDash_\Sigma \phi(\iota(a))\\ &\Rightarrow \forall b \in \mathcal{M}, \mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b))\\ &\Rightarrow \forall b \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \psi(\iota(a), \iota(b)) && \text{by the induction}\\ &\Rightarrow \mathcal{M} \vDash_\Sigma \phi(a)\end{aligned}$$

□

Proposition – Isomorphisms are Elementary

If two Σ-structures $\mathcal{M}$ and $\mathcal{N}$ are Σ-isomorphic then the isomorphism is elementary.

Proof. Let $\iota : \mathcal{M} \to \mathcal{N}$ be a Σ-isomorphism. We case on what ϕ is:

- If ϕ is quantifier free, then each case is follows from applying embeddings preserve satisfaction of quantifier free formulas.

- If ϕ is $\forall v, \chi(v)$ then ($\Rightarrow$) Let $b \in \mathcal{N}_{\text{con}}$ then $\iota^{-1}(b) \in \mathcal{M}_{\text{con}}$ is well defined by surjectivity. Hence $\mathcal{M} \vDash_\Sigma \chi(\iota^{-1}(b), a)$ and so $\mathcal{N} \vDash_\Sigma \chi(b, \iota(a))$ by the induction hypothesis. Hence $\mathcal{N} \vDash_\Sigma \phi(\iota(a))$ ($\Leftarrow$) This follows from applying embeddings preserve satisfaction of universal formulas downwards.

□

1.2.2 Vaught's Completeness Test

Read ahead to the statement of Vaught's Completeness Test. We aim to prove it in this section.

> **Definition – Finitely modelled, infinitely modelled**
>
> A Σ-theory T is finitely modelled when there exists a Σ-model of T with finite carrier set.
>
> A Σ-theory T is finitely modelled when there exists a Σ-model of T with infinite carrier set.

Finitely modelled is *not* the same as finitely consistent.

> **Proposition – Infinite Modelled Theories Have Arbitrary Large Models**
>
> Given Σ a signature, T a Σ-theory that is infinitely modelled, and a cardinal κ such that $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$, there exists $\mathcal{M}$ a Σ-model of T such that $\kappa = |\mathcal{M}|$.

Proof. Enrich only the signature's constant symbols to create $\Sigma(*)$ a signature such that $\Sigma(*)_{\text{con}} = \Sigma_{\text{con}} \cup \{c_\alpha \,|\, \alpha \in \kappa\}$. Let $T(*) = T \cup \{c_\alpha \neq c_\beta \,|\, \alpha, \beta \in \kappa \wedge \alpha \neq \beta\}$ be a $\Sigma(*)$-theory.

Using the compactness theorem, it suffices to show that $T(*)$ is finitely consistent. Take a finite subset of $T(*)$. This is the union of a finite subset $\Delta_T \subseteq T$, and a finite subset of $\Delta_\kappa \subseteq \{c_\alpha \neq c_\beta \,|\, \alpha, \beta \in \kappa \wedge \alpha \neq \beta\}$. Let $\mathcal{M}$ be the Σ-model of T with infinite cardinality. We want to make $\mathcal{M}$ a $\Sigma(*)$-model of $\Delta_T \cup \Delta_\kappa$ by interpreting the new symbols of $\{c_\alpha \,|\, \alpha \in \kappa\}$ in a sensible way.

Since Δ_κ is finite, we can find a finite subset $I \subset \kappa$ that indexes the constant symbols appearing in Δ_κ. Since $\mathcal{M}$ is infinite and I is finite, we can find distinct elements of $\mathcal{M}$ to interpret the elements of $\{c_\alpha \,|\, \alpha \in I\}$. Interpret the rest of the new constant symbols however, for example let them all be sent to the same element, then $\mathcal{M} \vDash_{\Sigma*} \Delta_T \cup \Delta_\kappa$. Hence $T*$ is consistent.

Using the third equivalence of $T(*)$ being consistent, there exists $\mathcal{M}$ a $\Sigma(*)$-model of $T(*)$ with $|\mathcal{M}| \leq \kappa$. If $|\mathcal{M}| < \kappa$ then there would be c_α, c_β that are interpreted as equal, hence $\mathcal{M} \vDash_{\Sigma(*)} c_\alpha = c_\beta$ and $\mathcal{M} \nvDash_{\Sigma(*)} c_\alpha = c_\beta$, a contradiction. Thus $|\mathcal{M}| = \kappa$. Move $\mathcal{M}$ down a signature to make it a Σ-model of T. This doesn't change the cardinality of $\mathcal{M}$, so we have a Σ-model of T with cardinality κ. □

> **Definition – Complete**
>
> A Σ-theory T is complete when for any Σ-formula ϕ, $T \vDash_\Sigma \phi$ or $T \vDash_\Sigma \neg\phi$.

Note that if we have completeness and compactness, then for any Σ-sentence ϕ, *either* $T \vDash_\Sigma \phi$ or $T \vDash_\Sigma \neg\phi$.

> **Proposition – Not a consequence is consistent**
>
> Let T be a Σ-theory and ϕ is a Σ-sentence then $T \nvDash_\Sigma \phi$ if and only if $T \cup \{\neg\phi\}$ is consistent. Furthermore, $T \nvDash_\Sigma \neg\phi$ if and only if $T \cup \{\phi\}$ is consistent.

Proof. For the first statement: ($\Rightarrow$) Unfolding $T \nvDash_\Sigma \phi$, we have that there exists a Σ-model $\mathcal{M}$ of T such that $\mathcal{M} \nvDash_\Sigma \phi$. If $\mathcal{M}$ were empty, then we also have $\mathcal{M} \vDash_\Sigma \phi$, a contradiction. Hence $\mathcal{M}$ is non-empty, hence $\mathcal{M} \vDash_\Sigma \neg\phi$ and we are done. The backward proof is straightward.

For the second statement, apply the first to $\neg\phi$ and obtain $T \nvDash_\Sigma \neg\phi$ if and only if $T \cup \{\neg\neg\phi\}$ is consistent. Note that for any non-empty Σ-structure $\mathcal{M}$, $\mathcal{M} \vDash_\Sigma \neg\neg\phi$ if and only if $\mathcal{M} \vDash_\Sigma \phi$. This completes the proof.

□

Proposition – Vaught's Completeness Test
Suppose that Σ-theory T is consistent, not finitely modelled, and κ-categorical for some cardinal satisfying $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$. Then T is complete.

Proof. Suppose not: If T is not complete then there exists Σ-formula ϕ such that $T \nvDash_\Sigma \phi$ and $T \nvDash_\Sigma \neg\phi$. These imply $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ are respectively consistent by the previous proposition. Let $\mathcal{M}_\neg$ and $\mathcal{M}$ be models of $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ respectively. Then each are models of T so they are infinite and so $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ are infinitely modelled.

Since we have κ such that $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$, there exists $\mathcal{N}_\neg, \mathcal{N}$ respectively Σ-models of $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ such that $\kappa = |\mathcal{N}_\neg| = |\mathcal{N}|$. Since T is κ-categorical $\mathcal{N}$ and $\mathcal{N}_\neg$ are isomorphic by an elementary Σ-embedding. As ϕ has no free variables this implies that $\mathcal{N} \vDash_\Sigma \phi$ and $\mathcal{N} \vDash_\Sigma \neg\phi$. $\mathcal{N}$ is non-empty so this is a contradiction. □

1.2.3 Elementary embeddings and diagrams of models

Proposition – Tarski-Vaught Elementary Embedding Test
Let $\iota : \mathcal{M} \to \mathcal{N}$ be a Σ-embedding, then the following are equivalent:

1. ι is elementary
2. For any $\phi \in \Sigma_{\text{for}}$ with free variables indexed by S, any $i \in S$ and any $a \in (\mathcal{M})^{S\setminus\{i\}}$,
$$\forall b \in \mathcal{M}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b)) \quad \Rightarrow \quad \forall c \in \mathcal{N}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), c),$$
which we call the Tarski-Vaught condition. Note that if $i \in S$ then a 'fills in' all but one of the free variables of ϕ, leaving only one free variable for substitution. (S can be empty, but then this doesn't say very much.)
3. For any $\phi \in \Sigma_{\text{for}}$ with free variables indexed by S, any $i \in S$ and any $a \in (\mathcal{M})^{S\setminus\{i\}}$,
$$\exists c \in \mathcal{N}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), c) \quad \Rightarrow \quad \exists b \in \mathcal{M}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b))$$
This is essentially the contrapositive of the previous statement, and is included because it is more commonly version of the statement.

Proof. We only show the first two statements are equivalent and leave the third as an exercise. ($\Rightarrow$) First show that $\mathcal{M} \vDash_\Sigma \forall v, \phi(a, v)$. Let $b \in \mathcal{M}$, then by assumption $\mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b))$, which is implies $\mathcal{M} \vDash_\Sigma \phi(a, b)$ as ι is an elementary embedding. Thus we indeed have $\mathcal{M} \vDash_\Sigma \forall v, \phi(a, v)$ which in turn implies $\mathcal{N} \vDash_\Sigma \forall v, \phi(\iota(a), v)$ and we are done.

($\Leftarrow$) We case on what ϕ is, though most of the work was already done before.

- If ϕ is quantifier free, then each case follows from applying embeddings preserve satisfaction of quantifier free formulas.
- The backwards implication follows from applying embeddings preserve satisfaction of universal formulas downwards.

  For the forwards implication we use the Tarski-Vaught condition (so far ι just needed to be a Σ-embedding)

$$\begin{aligned}
\mathcal{M} \vDash_\Sigma \forall v, \psi(a, v) &\Rightarrow \forall b \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \psi(a, b) \\
&\Rightarrow \forall b \in \mathcal{M}, \mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b)) && \text{by the induction hypothesis} \\
&\Rightarrow \forall c \in \mathcal{N}, \mathcal{N} \vDash_\Sigma \psi(\iota(a), c) && \text{by the Tarski-Vaught condition} \\
&\Rightarrow \mathcal{N} \vDash_\Sigma \phi
\end{aligned}$$

□

Proposition – Moving Morphisms Down Signatures

Suppose $\Sigma \leq \Sigma(*)$. If $\iota : \mathcal{M} \to \mathcal{N}$ is a $\Sigma(*)$-morphism then

1. ι can be made into a Σ-morphism.
2. If ι is an embedding then it remains an embedding.
3. If ι is an elementary embedding then it remains elementary.

Proof. 1. Move $\mathcal{M}$ and $\mathcal{N}$ down to being Σ structures (by picking $T(*) = T = \varnothing$). We show that the same set morphism $\iota : \mathcal{M} \to \mathcal{N}$ is a Σ-morphism.

- If $c \in \Sigma_{\text{con}}$ then since moving structures down signatures preserves interpretation on the lower signature, and since ι is a $\Sigma(*)$ embedding,
$$\iota(c^{\mathcal{M}}_{\Sigma}) = \iota(c^{\mathcal{M}}_{\Sigma(*)}) = c^{\mathcal{N}}_{\Sigma(*)} = c^{\mathcal{N}}_{\Sigma}$$
- If $f \in \Sigma_{\text{fun}}$ and $a \in (\mathcal{M})^{n_f}$ then similarly
$$\iota \circ f^{\mathcal{M}}_{\Sigma}(a) = \iota \circ f^{\mathcal{M}}_{\Sigma(*)}(a) = f^{\mathcal{N}}_{\Sigma(*)}(\iota(a)) = f^{\mathcal{N}}_{\Sigma}(\iota(a))$$
- If $r \in \Sigma_{\text{rel}}$ and $a \in (\mathcal{M})^{m_r}$ then
$$a \in r^{\mathcal{M}}_{\Sigma}(a) = r^{\mathcal{M}}_{\Sigma(*)} \Rightarrow \iota(a) \in r^{\mathcal{N}}_{\Sigma(*)} = r^{\mathcal{N}}_{\Sigma}$$

2. If we also have that it is an embedding in $\Sigma(*)$, then injectivity is preserved as it is a property of set morphisms. Given $r \in \Sigma_{\text{rel}}$ and $a \in (\mathcal{M})^{m_r}$,
$$\iota(a) \in r^{\mathcal{N}}_{\Sigma} = r^{\mathcal{N}}_{\Sigma(*)} \Rightarrow a \in r^{\mathcal{M}}_{\Sigma(*)}(a) = r^{\mathcal{M}}_{\Sigma}$$

3. If we also have that ι is elementary in $\Sigma(*)$ then we use the Tarski-Vaught Test: let $\phi \in \Sigma_{\text{for}}$ have free variables indexed by S, let $i \in S$ and let $a \in (\mathcal{M})^{S \setminus \{i\}}$. Then due to the construction in moving $\mathcal{M}$ and $\mathcal{N}$ down a signature we have that for any $b \in \mathcal{N}$,
$$\mathcal{N} \vDash_{\Sigma} \phi(\iota(a), \iota(b)) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(\iota(a), \iota(b))$$
and similarly for $\mathcal{M}$. Hence
$$\begin{aligned} &\forall b \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma} \phi(\iota(a), \iota(b)) \\ \Rightarrow &\forall b \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma(*)} \phi(\iota(a), \iota(b)) \\ \Rightarrow &\forall c \in \mathcal{M}, \mathcal{M} \vDash_{\Sigma(*)} \phi(a, c) && \iota \text{ is elementary in } \Sigma(*) \\ \Rightarrow &\forall c \in \mathcal{M}, \mathcal{M} \vDash_{\Sigma} \phi(a, c) \end{aligned}$$
Hence ι is elementary in Σ.

□

Notation. *Let A be a set and Σ be a signature, enriching only the constant symbols of Σ we can create a signature $\Sigma(A)$ such that*
$$\Sigma(A)_{\text{con}} := \Sigma_{\text{con}} \cup \{c_a \mid a \in A\}$$
In the case that $A = \mathcal{M}$ for some model $\mathcal{M}$ then we write $\Sigma(\mathcal{M}) := \Sigma(A)$.

Definition – $\Sigma(\mathcal{M})$, Diagram and the Elementary Diagram of a Structure

Let $\mathcal{M}$ be a Σ-structure, we move $\mathcal{M}$ up to the signature $\Sigma(\mathcal{M})$ by interpreting each new constant symbol c_a as a. ($\mathcal{M}$ satisfies the conditions of our lemma for moving models up signatures by choosing $T = \varnothing$). Thus we may treat $\mathcal{M}$ as a $\Sigma(\mathcal{M})$ structure. We define the atomic diagram of $\mathcal{M}$ over Σ:

| If ϕ is an atomic $\Sigma(\mathcal{M})$-sentence such that $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi$, then $\phi \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$.

| If $\phi \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$ then $\neg\phi \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$.

We define the elementary diagram of $\mathcal{M}$ over Σ:

| If $\phi \in \Sigma(\mathcal{M})_{\text{for}}$ and ϕ is a $\Sigma(\mathcal{M})$-sentence and $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi$, then $\phi \in \mathrm{ElDiag}(\Sigma, \mathcal{M})$.

Proposition – Models of the elementary diagram correspond to elementary extensions

Given $\mathcal{M}$ a Σ-structure and $\mathcal{N}$ a $\Sigma(\mathcal{M})$-structure such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{AtDiag}(\Sigma, \mathcal{M})$, we can make $\mathcal{N}$ into a Σ-structure and find a Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$. Furthermore $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$ then the embedding is elementary.

Conversely, given an elementary Σ-embedding from $\mathcal{M}$ into a Σ-structure $\mathcal{N}$, we can move $\mathcal{N}$ up to being a $\Sigma(\mathcal{M})$ structure such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$. Note that we don't show the converse for the atomic case.

Proof. ($\Rightarrow$) Suppose $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{AtDiag}(\Sigma, \mathcal{M})$. Firstly we work in $\Sigma(\mathcal{M})$ to define the embedding: move $\mathcal{M}$ up a signature by taking the same interpretation as used in the definition of $\Sigma(\mathcal{M})$:

$$\star^{\mathcal{M}}_{\Sigma(\mathcal{M})} : c_a \mapsto a$$

and preserving the same interpretation for symbols of Σ. This makes $\star^{\mathcal{M}}_{\Sigma(\mathcal{M})_{\text{con}}}$ surjective. Thus we write elements of $\mathcal{M}$ as $c^{\mathcal{M}}_{\Sigma(\mathcal{M})}$, for some $c \in \Sigma(\mathcal{M})_{\text{con}}$

Next we define the $\Sigma(\mathcal{M})$-morphism $\iota : \mathcal{M} \to \mathcal{N}$ such that $\iota : c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \to c^{\mathcal{N}}_{\Sigma(\mathcal{M})}$. To check that ι is well defined, take $c, d \in \Sigma(\mathcal{M})_{\text{con}}$ such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} = d^{\mathcal{M}}_{\Sigma(\mathcal{M})}$.

$$\begin{aligned}
&\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} c = d \\
&\Rightarrow c = d \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\
&\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} c = d \\
&\Rightarrow c^{\mathcal{N}}_{\Sigma(\mathcal{M})} = d^{\mathcal{N}}_{\Sigma(\mathcal{M})}
\end{aligned}$$

Thus ι is well defined. In fact doing 'not' gives us injectivity in the same way: Take $c, d \in \Sigma(\mathcal{M})_{\text{con}}$ such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \neq d^{\mathcal{M}}_{\Sigma(\mathcal{M})}$.

$$\begin{aligned}
&\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} c \neq d \\
&\Rightarrow c \neq d \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\
&\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} c \neq d \\
&\Rightarrow c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \neq d^{\mathcal{N}}_{\Sigma(\mathcal{M})}
\end{aligned}$$

Thus ι is injective. To check that ι is a $\Sigma(\mathcal{M})$-morphism, we check interpretation of functions and relations. Let $f \in \Sigma(\mathcal{M})_{\text{fun}} = \Sigma_{\text{fun}}$ and $c \in (\Sigma(\mathcal{M})_{\text{con}})^{n_f}$. $\star^{\mathcal{M}}_{\Sigma(\mathcal{M})_{\text{con}}}$ is surjective thus we can find $d \in \Sigma\mathcal{M}_{\text{con}}$ such that $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} f(c) = d$. Hence $f(c) = d \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$. Hence $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} f(c) = d$.

$$\begin{aligned}
\iota \circ f^{\mathcal{M}}_{\Sigma(\mathcal{M})}(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) &= \iota(d^{\mathcal{M}}_{\Sigma(\mathcal{M})}) \\
&= d^{\mathcal{N}}_{\Sigma(\mathcal{M})} \\
&= f^{\mathcal{N}}_{\Sigma(\mathcal{M})}(c^{\mathcal{N}}_{\Sigma(\mathcal{M})}) \\
&= f^{\mathcal{N}}_{\Sigma(\mathcal{M})} \circ \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})})
\end{aligned}$$

Let $r \in \Sigma(\mathcal{M})_{\text{rel}} = \Sigma_{\text{rel}}$ and $c \in (\Sigma(\mathcal{M})_{\text{con}})^{m_r}$.

$$\begin{aligned} c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \in r^{\mathcal{M}}_{\Sigma(\mathcal{M})} &\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow r(c) \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \in r^{\mathcal{N}}_{\Sigma(\mathcal{N})} \end{aligned}$$

To show that ι is an embedding it remains to show the backward implication for relations. Let $r \in \Sigma(\mathcal{M})_{\text{rel}} = \Sigma_{\text{rel}}$ and $c \in (\Sigma(\mathcal{M})_{\text{con}})^{m_r}$.

$$\begin{aligned} c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \notin r^{\mathcal{M}}_{\Sigma(\mathcal{M})} &\Rightarrow \mathcal{M} \nvDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow \neg\, r(c) \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \nvDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \notin r^{\mathcal{N}}_{\Sigma(\mathcal{N})} \end{aligned}$$

Assume furthermore that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$. We show that this embedding is elementary. Let ϕ be a $\Sigma(\mathcal{M})$-formula with variables indexed by S and $a \in (\mathcal{M})^S$. Let $c \in (\Sigma_{\text{con}})^S$ be such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} = a$.

$$\begin{aligned} \mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi(a) &\Rightarrow \phi(c) \in \mathrm{ElDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi(c) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi(\iota(a)) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{M} \nvDash_{\Sigma(\mathcal{M})} \phi(a) &\Rightarrow \phi(c) \notin \mathrm{ElDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \nvDash_{\Sigma(\mathcal{M})} \phi(c) \\ &\Rightarrow \mathcal{N} \nvDash_{\Sigma(\mathcal{M})} \phi(\iota(a)) \end{aligned}$$

Hence ι is an elementary embedding. Moving $\iota : \mathcal{M} \to \mathcal{N}$ down to being a Σ-morphism of Σ-structures completes the proof. ($\Leftarrow$) Sketch: Suppose $\iota : \mathcal{M} \to \mathcal{N}$ is an elementary embedding. Make both $\mathcal{M}$ and $\mathcal{N}$ $\Sigma(\mathcal{M})$-structures by $\star^{\mathcal{M}}_{\Sigma(\mathcal{M})} : c_a \to a$ and $\star^{\mathcal{N}}_{\Sigma(\mathcal{M})} : c_a \to \iota(a)$, where $a \in \mathcal{M}$. Show that ι is still an elementary embedding when moved up to $\Sigma(\mathcal{M})$. Then for any $\phi \in \mathrm{ElDiag}(\Sigma, \mathcal{M})$, $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi$ and so by the embedding being elementary $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi$. Hence $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$. □

1.2.4 Universal axiomatization

Definition – Axiomatization

A Σ-theory A is an axiomatization of a Σ-theory T if for all Σ-structures $\mathcal{M}$,

$$\mathcal{M} \vDash_\Sigma T \Leftrightarrow \mathcal{M} \vDash_\Sigma A$$

Definition – Universal Theory, Universal Axiomatization

If A is a subset of Σ_{uni} then it is called a universal theory.

A Σ-theory A is a universal axiomatization of a Σ-theory T if it is a universal theory that is an axiomatization of T.

Lemma – Lemma on constants

Suppose $\Sigma_{\text{con}} \subseteq \Sigma(*)_{\text{con}}$, $T \in \Sigma_{\text{the}}$, $\phi \in \Sigma_{\text{for}}$ with variables indexed by S. Suppose there exists a list of constant symbols not from Σ, i.e. $c \in (\Sigma(*)_{\text{con}} \setminus \Sigma_{\text{con}})^S$ such that $T \vDash_{\Sigma(*)} \phi(c)$. Then

$$T \vDash_\Sigma \forall v, \phi(v)$$

Proof. We prove the contrapositive. Suppose $T \nvDash_\Sigma \forall v, \phi(v)$ then there exists $\mathcal{M}$ a Σ-model of T such that $\mathcal{M} \nvDash_\Sigma \forall v, \phi(v)$. Thus there exists $a \in \mathcal{M}^S$ such that $\mathcal{M} \nvDash_\Sigma \phi(a)$.

Let b be the element of $\mathcal{M}$. We move $\mathcal{M}$ up a signature by extending the interpretation to the new constant symbols. Suppose $d \in \Sigma(*)_{\text{con}} \setminus \Sigma_{\text{con}}$,

$$d^{\mathcal{M}}_{\Sigma(\mathcal{M})} = \begin{cases} a_i & \text{, if } \exists i \in S, d = c_i, \\ b & \text{, otherwise} \end{cases}$$

Then $\mathcal{M}$ is a $\Sigma(*)$-model of T such that $\mathcal{M} \nvDash_{\Sigma(*)} \phi(a)$, which by construction is equivalent to $\mathcal{M} \nvDash_{\Sigma(*)} \phi(c)$. □

Notation. *Universal consequences of T Let T be a Σ-theory, then*

$$T_\forall := \{\phi \in \Sigma_{\text{uni}} \,|\, T \vDash_\Sigma \phi\}$$

is called the set of universal consequences of T.

Proposition – Universal axiomatizations make substructures models

T a Σ-theory has a universal axiomatization if and only if for any Σ-model $\mathcal{N}$ of T and any Σ-embedding $\mathcal{M} \to \mathcal{N}$, $\mathcal{M}$ is a Σ-model of T.

Proof. ($\Rightarrow$) Suppose A is a universal axiomatization of T, $\mathcal{N}$ is a Σ-model of T and $\mathcal{M} \to \mathcal{N}$ is a Σ-embedding. $\mathcal{N} \vDash_\Sigma T$ if and only if $\mathcal{N} \vDash_\Sigma A$ by definition of A. $\mathcal{N} \vDash_\Sigma A$ if and only if $\mathcal{M} \vDash_\Sigma A$ since embeddings preserve the satisfaction of quantifier free formulas Finally $\mathcal{M} \vDash_{Si} A$ if and only if $\mathcal{M} \vDash_\Sigma T$ by definition of A.

($\Leftarrow$) We show that $T_\forall$ is a universal axiomatization of T. Let $\mathcal{M} \vDash_\Sigma T$ and let $\phi \in T_\forall$. Then by definition of $T_\forall$, $T \vDash_\Sigma \phi$. Hence $\mathcal{M} \vDash_\Sigma \phi$ and any Σ-model of T is a Σ-model of $T_\forall$.

Suppose $\mathcal{M} \vDash_\Sigma T_\forall$. We first show that $T \cup \text{AtDiag}(\Sigma, \mathcal{M})$ is consistent. By the compactness theorem it suffices to show that for any subset Δ of $\text{AtDiag}(\Sigma, \mathcal{M})$, $T \cup \Delta$ is consistent. Write $\Delta = \{\psi_1, \ldots, \psi_n\}$. Let $\psi = \bigwedge_{1 \le i \le n} \psi_i$. We can find S that indexes the constant symbols in $\Sigma(\mathcal{M})_{\text{con}} \setminus \Sigma_{\text{con}}$ that appear in ψ (in the same way as we made indexing sets of the variables). Then we can create $\phi \in \Sigma_{\text{for}}$ with variables indexed by S such that $\phi(c) = \psi$, where c is a list of constant symbols in $\Sigma(\mathcal{M})_{\text{con}} \setminus \Sigma_{\text{con}}$ indexed by S. Since $\Delta \subseteq \text{AtDiag}(\Sigma, \mathcal{M})$ we have $\forall i, \mathcal{M} \vDash_\Sigma \psi_i$. Hence $\mathcal{M} \vDash_\Sigma \phi(c)$. We can then show that $\mathcal{M} \vDash_\Sigma \exists v, \phi(v)$ and so $\mathcal{M} \nvDash_\Sigma \forall v, \neg \phi(v)$.

Since each ψ_i is from the the atomic diagram of $\mathcal{M}$ they are all quantifier free. Thus ϕ is a quantifier free Σ-formula and $\forall v, \neg \phi(v) \in \Sigma_{\text{uni}}$. Hence $T \nvDash_\Sigma \forall v, \neg \phi(v)$ by the definition of $T_\forall$. By the lemma on constants this implies that $T \nvDash_{\Sigma(\mathcal{M})} \neg \phi(a)$. Hence there exists a non-empty $\Sigma(\mathcal{M})$-model of $T \cup \phi(a)$ Then it follows that this is also a $\Sigma(\mathcal{M})$-model of $T \cup \Delta$. Thus $T \cup \Delta$ is consistent so $T \cup \text{AtDiag}(\Sigma, \mathcal{M})$ is consistent.

Thus there exists a non-empty Σ-model $\mathcal{N}$ of $T \cup \text{AtDiag}(\Sigma, \mathcal{M})$. This is a model of $\text{AtDiag}(\Sigma, \mathcal{M})$ thus by there is a $\Sigma(\mathcal{M})$-embedding $\mathcal{M} \to \mathcal{N}$ there is a $\Sigma(\mathcal{M})$-embedding $\mathcal{M} \to \mathcal{N}$. We make this a Σ-embedding, hence using the theorem's hypothesis $\mathcal{M}$ is a Σ-model of T. □

The following result has doesn't come up at all until much later, but is included here as another demonstration of the lemma on constants in use. It appears as an exercise in the second chapter of Marker's book [5].

Corollary – Amalgamation

Let $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures, and suppose we have partial elementary Σ-embeddings $\iota_{\mathcal{M}} : A \to \mathcal{M}$ and $\iota_{\mathcal{N}} : A \to \mathcal{N}$, for $\varnothing \neq A \subseteq \mathcal{A}$. Then there exists a common elementary extension $\mathcal{L}$ of $\mathcal{M}$ and $\mathcal{N}$ such that the following commutes:

$$\begin{tikzcd} \mathcal{M} \arrow[r] & \mathcal{L} \\ A \arrow[u, "\iota_{\mathcal{M}}"] \arrow[r, "\iota_{\mathcal{N}}"] & \mathcal{N} \arrow[u] \end{tikzcd}$$

$\mathcal{L}$ is the 'amalgamation' of $\mathcal{M}$ and $\mathcal{N}$.

Proof. We show first that the theory $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \mathrm{ElDiag}(\Sigma, \mathcal{N})$ is consistent as a $\Sigma(\mathcal{M}, \mathcal{N})$-theory, where $\Sigma(\mathcal{M}, \mathcal{N})_{\mathrm{con}}$ is defined to be

$$\{c_a \mid a \in A\} \cup \{c_a \mid a \in \mathcal{M} \setminus \iota_{\mathcal{M}}(A)\} \cup \{c_a \mid a \in \mathcal{N} \setminus \iota_{\mathcal{N}}(A)\}$$

where terms and formulas from $\Sigma(A), \Sigma(\mathcal{M}), \Sigma(\mathcal{N})$ are interpreted in the natural way: the constants $c_{\iota_{\mathcal{M}}(a)} \mapsto c_a$ (similarly for $\mathcal{N}$). For the rest of the proof we identify $\Sigma(\mathcal{M})_{\mathrm{con}}$ with $\Sigma(A)_{\mathrm{con}} \cup \{c_a \mid a \in \mathcal{M} \setminus \iota_{\mathcal{M}}(A)\}$ (similarly with $\mathcal{N}$).

By the compactness theorem it suffices to show that for any finite subset $\Delta \subseteq \mathrm{ElDiag}(\Sigma, \mathcal{N})$, $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \Delta$ is consistent. Let ϕ be the $\Sigma(\mathcal{M})$-formula and $a \in \mathcal{N}^\star$ be such that [1]

$$\phi(a) = \bigwedge_{\psi \in \Delta} \psi$$

$\phi(a)$ is naturally a $\Sigma(\mathcal{M}, \mathcal{N})$-sentence such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M}, \mathcal{N})} \phi(a)$.

Suppose for a contradiction $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \Delta$ is inconsistent. Then any $\Sigma(\mathcal{M}, \mathcal{N})$-model of $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ is not a model of Δ, which implies it does not satisfy $\phi(a)$. Hence

$$\mathrm{ElDiag}(\Sigma, \mathcal{M}) \vDash_{\Sigma(\mathcal{M}, \mathcal{N})} \neg \phi(a)$$

By the lemma on constants applied to $\Sigma(\mathcal{M}) \leq \Sigma(\mathcal{M}, \mathcal{N})$, $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ and $a \in \Sigma(\mathcal{M}, \mathcal{N})_{\mathrm{con}} \setminus \Sigma(\mathcal{M})_{\mathrm{con}}$ we have

$$\mathrm{ElDiag}(\Sigma, \mathcal{M}) \vDash_{\Sigma(\mathcal{M})} \forall v, \neg \phi(v)$$

Noting that $\mathcal{M}$ is a Σ-model of its elementary diagram, and moving $\mathcal{M}$ down a signature we have that

$$\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \forall v, \neg \phi(v) \Rightarrow \mathcal{M} \vDash_{\Sigma(A)} \forall v, \neg \phi(v) \Rightarrow$$

Since $A \to \mathcal{M}$ and $A \to \mathcal{N}$ are partial elementary Σ-embeddings (and thus naturally $\Sigma(A)$-embeddings) we have that $\mathcal{A} \vDash_{\Sigma(A)} \forall v, \neg \phi(v)$ and so $\mathcal{N} \vDash_{\Sigma(A)} \forall v, \neg \phi(v)$. Move this up to $\Sigma(\mathcal{M}, \mathcal{N})$ we have a contradiction, by choosing v to be a: $\mathcal{N} \vDash_{\Sigma(\mathcal{M}, \mathcal{N})} \neg \phi(a)$, but we remarked before that $\mathcal{N} \vDash_{\Sigma(\mathcal{M}, \mathcal{N})} \phi(a)$.

Hence $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \mathrm{ElDiag}(\Sigma, \mathcal{N})$ is consistent as a $\Sigma(\mathcal{M}, \mathcal{N})$-theory. Let $\mathcal{L}$ be a $\Sigma(\mathcal{M}, \mathcal{N})$-model of this (and naturally a $\Sigma(\mathcal{M})$ or a $\Sigma(\mathcal{N})$ structure). Then there exist elementary $\Sigma(\mathcal{M})$ and $\Sigma(\mathcal{N})$-extensions $\lambda_{\mathcal{M}} : \mathcal{M} \to \mathcal{L}$ and $\lambda_{\mathcal{N}} : \mathcal{N} \to \mathcal{L}$ such that $\lambda_{\mathcal{M}}(c_{m_{\Sigma(\mathcal{M})}}^{\mathcal{M}}) = c_{m_{\Sigma(\mathcal{M})}}^{\mathcal{L}}$ for each constant symbol c_m for $m \in \mathcal{M} \setminus \iota_{\mathcal{M}}(A)$ and $\lambda_{\mathcal{M}}(c_{a_{\Sigma(\mathcal{M})}}^{\mathcal{M}}) = c_{a_{\Sigma(\mathcal{M})}}^{\mathcal{L}}$ for each constant symbol c_a for $a \in A$ (similarly with $\mathcal{N}$).

Naturally, we can move everything down to $\Sigma(A)$. Thus for any $a \in A$

$$\lambda_{\mathcal{M}} \circ \iota_{\mathcal{M}}(a) = \lambda_{\mathcal{M}}(c_{a_{\Sigma(A)}}^{\mathcal{M}}) = \lambda_{\mathcal{M}}(c_{a_{\Sigma(\mathcal{M})}}^{\mathcal{M}}) = c_{a_{\Sigma(\mathcal{M})}}^{\mathcal{L}} = c_{a_{\Sigma(A)}}^{\mathcal{L}}$$

By symmetry we have $\lambda_{\mathcal{M}} \circ \iota_{\mathcal{M}}(a) = c_{a_{\Sigma(A)}}^{\mathcal{L}} = \lambda_{\mathcal{N}} \circ \iota_{\mathcal{N}}(a)$. □

[1] Take out all the finitely many constants appearing from $\mathcal{N} \setminus \iota_{\mathcal{N}}(A)$ in Δ and make them into a tuple a, replacing them with free variables. What remains is a finite set of $\Sigma(A)$-formulas, which are naturally also $\Sigma(\mathcal{M})$-formulas. We take the 'and' of all of them to be ϕ.

1.2.5 The Löwenheim-Skolem Theorems

The results in this subsection aren't used until much later.It is worth skipping for now, but the material can be covered with the foundations made so far.

Proposition – Upward Löwenheim-Skolem Theorem

If $\mathcal{M}$ is an infinite Σ-structure and κ a cardinal such that $|\mathcal{M}| + |\Sigma_{\text{con}}| \leq \kappa$, there exists a Σ-structure $\mathcal{N}$ with cardinality κ as well as an elementary Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$.

Proof. Make $\mathcal{M}$ a $\Sigma(\mathcal{M})$ structure as in the definition of $\Sigma(\mathcal{M})$, then clearly $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \text{ElDiag}(\Sigma, \mathcal{M})$. Thus $\text{ElDiag}(\Sigma, \mathcal{M})$ is a $\Sigma(\mathcal{M})$-theory with an infinite model $\mathcal{M}$, hence there exists a $\Sigma(\mathcal{M})$-structure $\mathcal{N}$ of cardinality κ such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \text{ElDiag}(\Sigma, \mathcal{M})$. Hence we can make $\mathcal{N}$ a Σ-structure and find a Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$. □

Definition – Skolem Functions

We say that a Σ-theory T *has built in Skolem functions* when for any Σ-formula ϕ that is not a sentence, with free variables indexed by S, there exists $f \in \Sigma_{\text{fun}}$ such that $n_f = k$ and

$$T \vDash_\Sigma \bigforall_{i \in S} w_i, (\exists v, \phi(v, w) \to \phi(f(w), w)),$$

Note that w can be length 0, in which case f has arity 0 and so would be interpreted as a constant map. We would have

$$T \vDash_\Sigma \exists v, \phi(v) \to \phi(f)$$

Proposition – Skolemization

Let $T(0)$ be a $\Sigma(0)$-theory, then there exists T a Σ theory such that

1. $|\Sigma_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$
2. $\Sigma(0)_{\text{fun}} \subseteq \Sigma_{\text{fun}}$, and they share the same constant and relation symbols
3. $T(0) \subseteq T$
4. T has built in Skolem functions
5. All models of $T(0)$ can be moved up to being models of T with interpretations agreeing on Σ.

We call T the Skolemization of $T(0)$.

Proof. Similarly to the Witness Property proof, we define $\Sigma(i), T(i)$ for each $i \in \mathbb{N}$. Suppose by induction that we have $T(i) \in \Sigma_{\text{the}}$, such that

1. $|\Sigma(i)_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$
2. $\Sigma(0)_{\text{fun}} \subseteq \Sigma(i)_{\text{fun}}$ and they share the same constant and relation symbols
3. $T(0) \subseteq T(i)$
4. All models of $T(0)$ can be moved up to being models of $T(i)$ with interpretations agreeing on Σ

Then define $\Sigma(i+1)$ such that only the function symbols are enriched:

$$\Sigma(i+1)_{\text{fun}} := \Sigma(i)_{\text{fun}} \cup \{f_\phi \mid \phi \in \Sigma(i)_{\text{for}} \text{ and } \phi \text{ is not a sentence}\}$$

extending the arity $n_\star$ to by having $n_{f_\phi} = |S| - 1$, where S is indexes the free variables of ϕ. There are countably infinite $\Sigma(i)$-formulas, thus $|\Sigma(i)_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$.

Define $\Psi : \Sigma(i)_{\text{for}} \to \Sigma(i+1)_{\text{for}}$ mapping

$$\phi \mapsto \forall w, (\exists v, \phi(v,w)) \to (\phi(f_\phi(w), w)),$$

where w is a list of variables of the suitable length. We then define

$$T(i+1) := T(i) \cup \Psi(\Sigma(i)_{\text{for}})$$

which is a $\Sigma(i+1)$-theory because the image of Ψ has only Σ-sentences. Note that $T(0) \subseteq T(i) \subseteq T(i+1)$.

Let $\mathcal{M}(0)$ be a $\Sigma(0)$-model of $T(0)$, then we have $\mathcal{M}(i)$ a $\Sigma(i)$-model of $T(i)$ with the same carrier set and same interpretation on $\Sigma(0)$ as $\mathcal{M}(0)$. Let $\mathcal{M}(i+1)$ have the same carrier set as $\mathcal{M}(0)$. To extend interpretation to $\Sigma(i+1)$, we first deal with the case where $\mathcal{M}(i)$ is empty by simply interpreting all new function symbols as the empty function. Otherwise we have a $c \in \mathcal{M}(0)$. For $f_\phi \in \Psi(\Sigma(i)_{\text{for}})$ define

$$f_\phi^{\mathcal{M}(i+1)} : \mathcal{M}(i+1)^{n_{f_\phi}} \to \mathcal{M}(i+1)$$
$$a \mapsto \begin{cases} b & \text{, if } \exists b \in \mathcal{M}, \mathcal{M}(i) \vDash_{\Sigma(i)} \phi(b,a) \\ c & \text{, otherwise} \end{cases}$$

Then by construction, $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \Psi(\Sigma(i)_{\text{for}})$. By checking the conditions on moving $\mathcal{M}(i)$ up to $\mathcal{M}(i+1)$, we can also conclude $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} T(i)$. Hence $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} T(i+1)$.

Let $\Sigma(*)$ be the signature such that its constants and relations agree with $\Sigma(0)$ and $\Sigma(*)_{\text{fun}} = \bigcup_{i\in\mathbb{N}} \Sigma(i)_{\text{fun}}$. Then

$$|\Sigma(*)_{\text{fun}}| = |\bigcup_{i\in\mathbb{N}} \Sigma(i)_{\text{fun}}| = \aleph_0 \times (\aleph_0 + \Sigma(0)_{\text{fun}}) = \aleph_0 + \Sigma(0)_{\text{fun}}$$

Let $T(*) = \bigcup_{i\in\mathbb{N}} T(i)$. We show that $T(*)$ has built in Skolem functions. Let ϕ be a $\Sigma(*)$-formula that is not a Σ-sentence. Then $\phi \in \Sigma(i)_{\text{for}}$ for some $i \in \mathbb{N}$. Thus $\Psi(\phi) \in T(i+1) \subseteq T(*)$, hence

$$T(*) \vDash_{\Sigma(*)} \forall w, (\exists v, \phi(v,w)) \to (\phi(f_\phi(w), w))$$

Thus $T(*)$ has built in Skolem functions.

If $\mathcal{M} \vDash_\Sigma T$ then let $\mathcal{M}(*) = \mathcal{M}$ and define the interpretation such that for all $i \in \mathbb{N}$, and $f \in T(i)$, $f_{\Sigma(*)}^{\mathcal{M}(*)} = f_{\Sigma(i)}^{\mathcal{M}(i)}$. Since all interpretations agree upon intersection this is well-defined. To show that $\mathcal{M}(*)$ is a $\Sigma(*)$-model of $T(*)$, let ϕ be in $T(*)$; there is some $i \in \mathbb{N}$ such that $\phi \in T(i)$. Using our lifted $\mathcal{M}(i)$ from before we have $\mathcal{M}(i) \vDash_{\Sigma(i)} \phi$. By checking the conditions on moving $\mathcal{M}(i)$ up to $\mathcal{M}(*)$, we can also conclude $\mathcal{M}(*) \vDash_{\Sigma(*)} \phi$ (by taking the $\Sigma(*)$ theory $\{\phi\}$). Hence $\mathcal{M}(*) \vDash_{\Sigma(*)} T(*)$. □

Definition – Theory of a Structure

We define the theory of a Σ-structure $\mathcal{M}$ to be

$$\text{Th}_{\mathcal{M}} := \{\phi \in \Sigma_{\text{for}} \mid \phi \text{ is a } \Sigma\text{-sentence and } \mathcal{M} \vDash_\Sigma \phi\}$$

Proposition – Downward Löwenheim-Skolem Theorem

Let $\mathcal{N}$ be a $\Sigma(0)$-structure and $M(0) \subseteq \mathcal{N}$. Then there exists a $\Sigma(0)$-structure $\mathcal{M}$ such that

- $M(0) \subseteq \mathcal{M} \subseteq \mathcal{N}$
- $|\mathcal{M}| \leq |M(0)| + |\Sigma(0)_{\text{fun}}| + \aleph_0$
- The inclusion $\subseteq : \mathcal{M} \to \mathcal{N}$ is an elementary embedding.

Proof. We first take the Skolemization of $\mathrm{Th}_{\mathcal{N}}$ and call the new signature and theories Σ and T. Since $\mathcal{N} \vDash_{\Sigma(0)} \mathrm{Th}_{\mathcal{N}}$, we can move it up to being a Σ-structure so that $\mathcal{N} \vDash_{\Sigma} T$.

We want to create the carrier set of $\mathcal{M}$, it has to be big enough so that interpreted functions are closed on $\mathcal{M}$. Given $M(i)$ such that $|M(i)| \leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0$, we inductively define $M(i+1)$:

$$M(i+1) := M(i) \cup \{ f_\Sigma^{\mathcal{N}}(a) \mid f \in \Sigma_{\text{fun}} \wedge a \in M(i)^{n_f} \}$$

Then

$$\begin{aligned}
|M(i+1)| &\leq |M(i)| + |\Sigma_{\text{fun}}| \times |M(i)^{n_f}| \\
&\leq |M(i)| + |\Sigma_{\text{fun}}| \times (|M(i)| + \aleph_0) \\
&\leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0 + |\Sigma_{\text{fun}}| \times (|M(0)| + |\Sigma_{\text{fun}}| + \aleph_0) \\
&\leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0
\end{aligned}$$

Then $\mathcal{M} := \bigcup_i M(i)$ and $|\mathcal{M}| \leq |M(i)| \times \aleph_0 = |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0 \leq |M(0)| + |\Sigma(0)_{\text{fun}}| + \aleph_0$.

We first interpret function symbols, which will give us a way to interpret constant symbols. For $f \in \Sigma_{\text{fun}}$ and $a \in (\mathcal{M})^{n_f}$, define $f_\Sigma^{\mathcal{M}}(a) = f_\Sigma^{\mathcal{N}}(a)$. This is well-defined as there exists $i \in \mathbb{N}$ such that $a \in (M(i))^{n_f}$,

$$f_\Sigma^{\mathcal{M}}(a) \in M(i+1) \subseteq \mathcal{M}$$

Then to interpret constant symbols, we consider for each $c \in \Sigma_{\text{con}}$ the formula $v = c$. Since T has built in Skolem functions and $\mathcal{N} \vDash_\Sigma T$, there exists f with arity $n_f = 0$ such $\mathcal{N} \vDash_\Sigma (\exists v, v = c) \to f = c$. Since $\mathcal{N} \vDash_\Sigma \exists v, v = c$, we have $f_\Sigma^{\mathcal{N}} = c_\Sigma^{\mathcal{N}}$. Since $f_\Sigma^{\mathcal{N}} = f_\Sigma^{\mathcal{M}} : (\mathcal{M})^0 \to \mathcal{M}$ we can define $c_\Sigma^{\mathcal{M}} = c_\Sigma^{\mathcal{N}} = f_\Sigma^{\mathcal{N}} = f_\Sigma^{\mathcal{M}} \in \mathcal{M}$.

Lastly define the interpretation of relations as $r_\Sigma^{\mathcal{M}} = (\mathcal{M})^{m_r} \cap r_\Sigma^{\mathcal{N}}$.

By construction the inclusion $\subseteq$ is a Σ-embedding. We check that it is elementary using the third equivalent condition in the Tarski Vaught Test: let $\phi \in \Sigma_{\text{for}}$ with free variables indexed by S, $i \in S$ and $a \in (\mathcal{M})^{S \setminus \{i\}}$. Suppose $\exists c \in \mathcal{N}, \mathcal{N} \vDash_\Sigma \phi(a, c)$. T has built in Skolem functions, and $\mathcal{N} \vDash_\Sigma T$. Hence there exists $f \in \Sigma_{\text{fun}}$ such that

$$\mathcal{N} \vDash_\Sigma (\exists v, \phi(a, v)) \to \phi(a, f(a))$$

We can deduce $\mathcal{N} \vDash_\Sigma \phi(a, f(a))$. Noting that $f_\Sigma^{\mathcal{M}}(a) = f_\Sigma^{\mathcal{N}}(a)$ completes the Tarski Vaught Test. Hence $\subseteq$ is an elementary Σ-embedding.

We move $\subseteq : \mathcal{M} \to \mathcal{N}$ down a signature since by Skolemization we have $\Sigma(0) \leq \Sigma$. Then $\subseteq : \mathcal{M} \to \mathcal{N}$ is an elementary Σ-embedding. □

1.3 Types

This section mainly follows material from Tent and Ziegler's book [4].

1.3.1 Types on theories

Definition – $F(\Sigma, n)$ and formulas consistent with a theory
Let $v_1, \ldots v_n$ be variables, T be a Σ-theory. Let $F(\Sigma, n)$ be the set of Σ-formulas with at most $v_1, \ldots v_n$ as their free variables. For any $c \in \Sigma_{\text{con}}^n$, $p \subseteq F(\Sigma, n)$ we write

$$p(c) = \{ \phi(c) \mid \phi \in p \}$$

and if $\mathcal{M}$ is a Σ-structure with $a \in \mathcal{M}^n$ we write

$$\mathcal{M} \vDash_\Sigma p(a)$$

to mean for every $\phi \in p$, $\mathcal{M} \vDash_\Sigma \phi(a)$.

We say $p \subseteq F(\Sigma, n)$ is consistent with T if there exists a non-empty $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma p(a)$.

We say $p \subseteq F(\Sigma, n)$ is finitely consistent with T if for any finite subset $\Delta \subseteq p$, Δ is consistent with T.

We say p is a maximal if for any $\phi \in F(\Sigma, n)$, $\phi \in p$ or $\neg\phi \in p$.

Lemma – Equivalent definition of consistency

Let T be a Σ-theory and p be a subset of $F(\Sigma, n)$. Let $c_1, \dots, c_n$ be new constant symbols and let $\Sigma(c)$ be the signature with these added constant symbols. The following are equivalent:

- $T \cup p(c)$ is consistent in $\Sigma(c)$. (Where $p(c)$ is the formulas of p with the variables substituted by $c_1, \dots, c_n$.)
- p is consistent with T.

Proof. ($\Rightarrow$) Suppose we have a non-empty $\Sigma(c)$-structure $\mathcal{M} \vDash_{\Sigma(c)} T \cup p(c)$. Then by taking the images of the interpretation of each c_i in $\mathcal{M}$ we obtain $a = c^{\mathcal{M}} \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_{\Sigma(c)} p(a)$. Moving this down to Σ preserves satisfaction of $p(a)$ as elements of $p(a)$ are Σ-formulas with values in $\mathcal{M}$ (and T for the same reason):

$$\mathcal{M} \vDash_\Sigma T \cup p(a)$$

and we have what we want.

($\Leftarrow$) Suppose we have $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma p(a)$. We can make $\mathcal{M}$ a $\Sigma(c)$-structure such that everything from Σ is interpreted in the same way and each constant symbol c_i is interpreted as a_i. Thus $\mathcal{M} \vDash_{\Sigma(c)} T$ and for any $\phi(c) \in p(c)$,

$$\mathcal{M} \vDash_\Sigma \phi(a) \Rightarrow \mathcal{M} \vDash_{\Sigma(c)} \phi(a) \Rightarrow \mathcal{M} \vDash_{\Sigma(c)} \phi(c)$$

as c is interpreted as a. Hence $\mathcal{M} \vDash_{\Sigma(c)} T \cup p(c)$ and $T \cup p(c)$ is consistent in $\Sigma(c)$. □

Lemma – Compactness for types

Let T be a Σ-theory and p be a subset of $F(\Sigma, n)$. Let $c_1, \dots, c_n$ be new constant symbols and let $\Sigma(c)$ be the signature with these added constant symbols. The following are equivalent:

- $T \cup p(c)$ is consistent in $\Sigma(c)$. (Where $p(c)$ is the formulas of p with the variables substituted by $c_1, \dots, c_n$.)
- p is consistent with T.
- p is finitely consistent with T.

Proof. We use 1. $\Leftrightarrow$ 2. to show 2. $\Leftrightarrow$ 3.

(2. $\Leftrightarrow$ 3.)

$$\begin{aligned}
&p \text{ consistent with } T\\
&\Leftrightarrow T \cup p(c) \text{ consistent in } \Sigma(c) \quad \text{by (1. } \Leftrightarrow \text{ 2.)}\\
&\Leftrightarrow \text{for any finite } \Delta(c) \subseteq p(c), T \cup \Delta(c) \text{ consistent in } \Sigma(c) \quad \text{by compactness}\\
&\Leftrightarrow \text{for any finite } \Delta \subseteq p, T \cup \Delta(c) \text{ consistent in } \Sigma(c)\\
&\Leftrightarrow \text{for any finite } \Delta \subseteq p, \Delta \text{ consistent with } T \quad \text{by (1. } \Leftrightarrow \text{ 2.)}
\end{aligned}$$

□

Definition – Types on a theory

Let T be a Σ-theory. Any subset $p \subseteq F(\Sigma, n)$ that is consistent with T is called an n-type on T.

Definition – Stone space of a theory

Let T be a Σ-theory. Let the stone space of T, $S_n(T)$ be the set of all maximal n-types on T. (The signature of the n-types of the on T is implicit, given by the signature of T.) We give a topology on $S_n(T)$ by specifying an open basis; $U \subseteq S_n(T)$ is an element of the basis when there exists $\phi \in F(\Sigma, n)$ such that

$$U = [\phi]_T := \{p \in S_n(T) \mid \phi \in p\}$$

Proposition – Extending to maximal n-types (Zorn)

Any n-type can be extended to a maximal n-type.

Proof. Let T be a theory and p be a n-type. Order by inclusion the set

$$Z = \{q \in S_n(T) \mid q \text{ is an } n\text{-type and } p \subseteq q\}$$

This is non-empty as it contains p. Let $p_0 \subseteq p_1 \subseteq \ldots$ be a chain in Z. Then $m = \bigcup_{i \in \mathbb{N}} p_i$ is finitely consistent with T (by taking large enough i) and so is consistent with T. By Zorn we have the existence of a maximal element q in Z. To show that q is a maximal n-type let $\phi \in F(\Sigma, n)$. As q is consistent with T there exists a Σ-structure $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma q(a)$. In the case that $\mathcal{M} \vDash_\Sigma \phi(a)$ we have $q \cup \{\phi\}$ is consistent with T and so by maximality $\phi \in q$. In the other case $q \cup \{\neg \phi\}$ is consistent and so $\neg \phi \in q$. $\square$

Proposition – Basic facts about the basis elements

Let T be a Σ-theory, $\phi, \psi \in F(\Sigma, n)$.

- $(\neg \phi) \in p$ if and only if $p \notin [\psi]_T$.
- $[\phi]_T = [\psi]_T$ if and only if ϕ and ψ are equivalent modulo T.

  The basis elements are closed under Boolean operations
- $[\top]_T = S_n(T)$
- $[\neg \phi]_T = S_n(T) \setminus [\phi]_T$
- $[\phi \vee \psi]_T = [\phi]_T \cup [\psi]_T$
- $[\bot]_T = \varnothing$
- $[\phi \wedge \psi]_T = [\phi]_T \cap [\psi]_T$

Proof.

- Suppose $(\neg \phi) \in p$. Then if $p \in [\phi]_T$ then since p is consistent with T there exists a non-empty model $\mathcal{M}$ and a from $\mathcal{M}$ such that $\mathcal{M} \vDash_\Sigma \phi(a)$ and $\mathcal{M} \nvDash_\Sigma \phi(a)$, a contradiction. For the other direction, $p \notin [\psi]_T$ and so $\psi \notin p$ and by maximality $\neg \phi \in p$.
- $(\Rightarrow)$ Suppose for a contradiction $T \nvDash_\Sigma \forall v, (\phi \Leftrightarrow \psi)$. then there exists $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma \phi \wedge \neg \psi$ or $\mathcal{M} \vDash_\Sigma (\neg \phi) \wedge \psi$. In the first case we have that $\{\phi, \neg \psi\}$ is consistent with T and so can be extended to a maximal n-type p. Thus $p \in [\phi]_T = [\psi]_T$ and $p \notin [\psi]_T$, a contradiction. $(\Leftarrow)$ Suppose $T \vDash_\Sigma \forall v, (\phi \Leftrightarrow \psi)$. Let $p \in [\phi]_T$. It suffices to show that $p \in [\psi]_T$. Since p is consistent with T there exists a Σ-structure $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma p(a)$. By assumption $\mathcal{M} \vDash_\Sigma (\phi \Leftrightarrow \psi)(a)$ and $p \in [\phi]_T$ so $\mathcal{M} \vDash_\Sigma \psi(a)$. Suppose $p \notin \psi$, then $\neg \psi \in p$ hence we have a contradiction.

- For any maximal n-type p, either $\top$ or $\bot$ is in p and in the latter case we have a contradiction as p is consistent with T.
- $[\neg\,\phi]_T = S_n(T) \setminus [\phi]_T$ follows from the first point.
- $p \in [\phi \,\vee\, \psi]_T$ if and only if $(\phi \,\vee\, \psi) \in p$. Suppose $\phi \notin p$ then by maximality $(\neg\,\phi) \in p$ and so $p \in [\psi]_T$. In the other case $p \in [\phi]_T$. For the other direction $p \in [\phi]_T \cup [\psi]_T$ implies $\phi \in p$ or $\psi \in p$. In the first case we have $\mathcal{M} \vDash_\Sigma T$ such that $\mathcal{M} \vDash_\Sigma \phi$. Then $\mathcal{M} \vDash_\Sigma \phi \,\vee\, \psi$ and so $(\phi \,\vee\, \psi) \in p$.

We omit the last two parts. □

Proposition – Properties of the Stone space

Let T be a theory.

- Elements of the basis of $S_n(T)$ are clopen.
- $S_n(T)$ is Hausdorff.
- $S_n(T)$ is compact.

Proof.

- By maximality of each p the complement of U is also in the open basis:

$$\{p \in S_n(T) \,|\, \phi \notin p\} = \{p \in S_n(T) \,|\, (\neg\,\phi) \in p\}$$

  Hence each element of the basis is clopen.

- Let $p, q \in S_n(T)$ and suppose $p \neq q$. By maximality and the fact that Σ_{for} is non-empty we can assume without loss of generality that there is $\phi \in p \setminus q$. Again by maximality $(\neg\,\phi) \in q$, and so $p \in [\phi], q \in [\neg\,\phi]_T$. These opens are disjoint: if $r \in [\phi]_T \cap [\neg\,\phi]_T$ then as r is consistent with T, there exists a non-empty $\mathcal{M} \vDash_\Sigma T$ such that $\mathcal{M} \vDash_\Sigma \phi$ and $\mathcal{M} \vDash_\Sigma (\neg\,\phi)$ a contradiction.

- Let C be a collection of closed sets with finite intersection property. Then each closed set can be written as an intersection of basis elements (a finite union of closed sets is still a basis element since $[\phi] \cup [\psi]_T = [\phi \,\vee\, \psi]_T$):

$$C = \left\{ \bigcap_{\phi \in \alpha} [\phi]_T \,\middle|\, \alpha \in I \right\}$$

  Let

$$\Gamma = \{\phi \,|\, \phi \in \alpha \in I\} \quad \text{and} \quad [\Gamma]_T = \{[\phi]_T \,|\, \phi \in \alpha \in I\}$$

  Then the intersection any finite subset of $[\Gamma]_T$ is non-empty as it contains a finite intersection of elements in C. Thus for any finite subset $\Delta \subseteq \Gamma$ there exists $p \in S_n(T)$ such that $\Delta \subseteq p$, and as p is consistent with T so is Δ. Hence Γ is finitely consistent with T and by Γ is consistent with T. Extending Γ to a maximal n-type q gives us $\phi \in q$ for every $\phi \in \Gamma$. Hence for all $\alpha \in I$ and for all $\phi \in \alpha$, $p \in [\phi]_T$ and the intersection of C is non-empty.

□

Stone space is meant to have a geometric interpretation as $\mathrm{Spec}(\mathcal{M}[x_1, \ldots, x_n])$ when $\mathcal{M}$ is an algebraically closed field. We will this in a few results.

1.3.2 Types on structures

Definition – Realisation

Let $\mathcal{M}$ be a Σ-structure and $A \subseteq \mathcal{M}$. Let p be a subset of $F(\Sigma(A), n)$ (we will often be considering the n-types on $\mathrm{ElDiag}(\Sigma, \mathcal{M})$, a special case of this where $A = \mathcal{M}$). Let $\mathcal{N}$ be a $\Sigma(A)$-structure.

- p is realised in $\mathcal{N}$ by $a \in \mathcal{N}^n$ over A if

$$\mathcal{N} \vDash_{\Sigma(A)} p(a)$$

We also just say p is realised in $\mathcal{N}$. If p is not realised in $\mathcal{N}$ then we say $\mathcal{N}$ omits p.

- p is finitely realised in $\mathcal{N}$ over A if for each finite subset $\Delta \subseteq p$ there exists $a \in \mathcal{N}^n$ such that Δ is realised in $\mathcal{M}$ by a.

Lemma – Finite realisation and embeddings

Let $\mathcal{M}$ be a non-empty Σ-structure, A a subset of $\mathcal{M}$ and p a subset of $F(\Sigma(A), n)$. Then the following are equivalent

- p is consistent with $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ (i.e. it is an n-type over $\mathrm{ElDiag}(\Sigma, \mathcal{M})$).
- There exists an elementary embedding $\mathcal{M} \to \mathcal{N}$ and a $b \in \mathcal{N}^n$ such that p is realised by b in $\mathcal{N}$.
- There exists an elementary embedding $\mathcal{M} \to \mathcal{N}$ such that p is finitely realised in $\mathcal{N}$.
- p is finitely realised in $\mathcal{M}$.

The elementary embeddings can be seen as both Σ-embeddings or $\Sigma(A)$-embeddings for any subset $A \subseteq \mathcal{M}$.

Proof. $(1. \Rightarrow 2.)$ If there exists a non-empty $\mathcal{N}$ and a $b \in \mathcal{N}^n$ such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$ and $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} p(b)$. Then since models of the elementary diagram correspond to elementary extensions, there exists an elementary $\Sigma(\mathcal{M})$-embedding $\mathcal{M} \to \mathcal{N}$ and $b \in \mathcal{N}^n$ such that p is realised by b in $\mathcal{N}$. (This can be moved down to being a $\Sigma(A)$-embedding for any subset $A \subseteq \mathcal{M}$.)

$(2. \Rightarrow 3.)$ Let $\Delta \subseteq p$ be finite. Then for the same embedding into $\mathcal{N}$ we can see that Δ is realised by b in $\mathcal{N}$.

$(3. \Rightarrow 4.)$ Let $\Delta \subseteq p$ be finite. Then by assumption there exists an elementary $\Sigma(A)$-embedding $\iota : \mathcal{M} \to \mathcal{N}$ and $b \in \mathcal{N}^n$ such that $\mathcal{N} \vDash_{\Sigma(A)} \Delta(b)$. Choose the constant symbols $c_1, \dots, c_m$ from $\Sigma(\mathcal{M})$ such that $c_i^{\mathcal{N}} = b_i$ for each i. We take $a = (c_1^{\mathcal{M}}, \dots, c_n^{\mathcal{M}})$ to realise Δ. Note that since embeddings commute with interpretation of constants, $\iota(a) = \iota(c^{\mathcal{M}}) = c^{\mathcal{N}} = b$ and since the embedding is elementary

$$\mathcal{N} \vDash_{\Sigma(A)} \phi(b) \Rightarrow \mathcal{N} \vDash_{\Sigma(A)} \phi(\iota(a)) \Rightarrow \mathcal{M} \vDash_{\Sigma(A)\phi(a)}$$

$(4. \Rightarrow 1.)$ By compactness for types, it suffices to show that p is finitely consistent with the elementary diagram. Let $\Delta \subseteq p$ be finite. Then by assumption there is $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_{\Sigma(A)(a)} \Delta(a)$ and so $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \Delta(a)$. Clearly $\mathcal{M}$ is a model of its elementary diagram. Hence we have satisfied the conditions for 2. □

Definition – Type of an element

Let $\mathcal{M}$ be a Σ-structure, $A \subseteq \mathcal{M}$ and $a \in \mathcal{M}^n$ Then

$$\mathrm{tp}^{\mathcal{M}}_{A,n}(a) := \left\{\phi \in F(\Sigma(A), n) \,|\, \mathcal{M} \vDash_{\Sigma(A)} \phi(a)\right\}$$

is the type of a in $\mathcal{M}$ over A. One can verify that this is a maximal n-type over T if $\mathcal{M}$ is a non-empty model of T.

Proposition

A type p of a Σ-theory T is realised by a in an extension $\mathcal{N}$ if and only if $p = \mathrm{tp}(a)^{\mathcal{N}}_{\varnothing,n}$. Any element of the Stone space is the type of an element.

Proof. If p is realised by a in $\mathcal{N}$ then $\mathcal{N} \vDash_\Sigma p(a)$ hence $p \subseteq \mathrm{tp}^{\mathcal{M}}_{\varnothing,n}(a)$ and by maximality of p they are equal.

Any element of the Stone space is realised in some Σ-structure due to consistency hence it is the type of an element by the above. □

Proposition – All maximal realised n-types are types of an element
Let $\mathcal{M}$ be a non-empty Σ-structure, A a subset of $\mathcal{M}$ and p a subset of $F(\Sigma(A), n)$. Let $a \in \mathcal{M}^n$. Then

- p is a maximal n-type over $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ that is realised by $a \in \mathcal{M}^n$ if and only if $p = \mathrm{tp}^{\mathcal{M}}_{A,n}(a)$.
- If $\mathcal{M} \subseteq \mathcal{N}$ is an elementary embedding then

$$\mathrm{tp}^{\mathcal{M}}_{A,n}(a) = \mathrm{tp}^{\mathcal{N}}_{A,n}(a)$$

Proof.

- ($\Rightarrow$) As p is realised by a, $p \subseteq \mathrm{tp}^{\mathcal{M}}_{A,n}(a)$. By maximality of p any formula in $\mathrm{tp}^{\mathcal{M}}_{A,n}(a)$ is either in p or its negation is in p. If its negation is in $p \subseteq \mathrm{tp}^{\mathcal{M}}_{A,n}(a)$ we have a contradiction as this would imply $\mathcal{M} \vDash_{\Sigma(A)} \phi(a)$ and $\mathcal{M} \nvDash_{\Sigma(A)} \phi(a)$. ($\Leftarrow$) If $p = \mathrm{tp}^{\mathcal{M}}_{A,n}(a)$ then clearly p is realised by a and so it is consistent with $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ thus it is an n-type over $\mathrm{ElDiag}(\Sigma, \mathcal{M})$. For any $\phi \in F(\Sigma(A), n)$, $\mathcal{M} \vDash_{\Sigma(A)} \phi(a)$ or $\mathcal{M} \nvDash_{\Sigma(A)} \phi(a)$. Hence ϕ or $\neg\phi$ is in p and so it is maximal.
- $$\phi \in \mathrm{tp}^{\mathcal{M}}_{A,n}(a) \Leftrightarrow \mathcal{M} \vDash_{\Sigma(A)} \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(A)} \phi(a) \Leftrightarrow \phi \in \mathrm{tp}^{\mathcal{N}}_{A,n}(a)$$

□

1.3.3 Quantifier free types

Quantifier free types become important when we start making back and forth constructions. All the results carry through, thus there isn't much to say until they are relevant. Thus this section is worth skipping for now.

Definition – $QFF(\Sigma, n)$ and formulas consistent with a theory
We define $QFF(\Sigma, n)$ to be the subset of $F(\Sigma, n)$ of quantifier free formulas.

For subsets $p \subseteq QFF(\Sigma, n) \subseteq F(\Sigma, n)$ the definitions of consistency carry through and we can apply compactness for types on these subsets too. The definition of maximality is the same but restricted to $QFF(\Sigma, n)$.

Let T be a Σ-theory. Any subset $p \subseteq QFF(\Sigma, n)$ that is consistent with T is called a quantifier free n-type on T. Note that any quantifier free n-type is an n-type.

Definition – Quantifier free Stone space of a theory
Let T be a Σ-theory. Let the stone space of T, $S^{\mathrm{qf}}_n(T)$ be the set of all maximal quantifier free n-types on T. We give the same topology: $U \subseteq S_n(T)$ is an element of the basis when there exists $\phi \in QFF(\Sigma, n)$ such that

$$U = [\phi]^{\mathrm{qf}}_T := \left\{p \in S^{\mathrm{qf}}_n(T) \,|\, \phi \in p\right\}$$

Proposition – Extending to maximal quantifier free n-types
Any quantifier free n-type on a theory can be extended to a maximal quantifier free n-type.

Proof. Any quantifier free n-type is an n-type, hence can be extended to a maximal n-type. The intersection of a maximal n-type with QFF is a maximal quantifier free n-type. This intersection extends the quantifier free n-type and we are done. □

Proposition – Properties of the Stone space
Let T be a Σ-theory, $\phi, \psi \in QFF(\Sigma, n)$. Elementary properties:

- $(\neg\,\phi) \in p$ if and only if $p \notin [\psi]^{\mathrm{qf}}_T$.
- $[\phi]_T = [\psi]^{\mathrm{qf}}_T$ if and only if ϕ and ψ are equivalent modulo T.
- $[\top]^{\mathrm{qf}}_T = S^{\mathrm{qf}}_n(T)$
- $[\neg\,\phi]^{\mathrm{qf}}_T = S_n(T) \setminus [\phi]^{\mathrm{qf}}_T$
- $[\phi \,\vee\, \psi]^{\mathrm{qf}}_T = [\phi]^{\mathrm{qf}}_T \cup [\psi]^{\mathrm{qf}}_T$
- $[\bot]^{\mathrm{qf}}_T = \varnothing$
- $[\phi \,\wedge\, \psi]^{\mathrm{qf}}_T = [\phi]^{\mathrm{qf}}_T \cap [\psi]^{\mathrm{qf}}_T$

Topological properties:

- Elements of the basis of $S^{\mathrm{qf}}_n(T)$ are clopen.
- $S^{\mathrm{qf}}_n(T)$ is Hausdorff.
- $S^{\mathrm{qf}}_n(T)$ is compact.

Proof. The proofs of these are exactly the same as before, some of them can even be bypassed by using the previous results. □

Definition – Quantifier free type of an element
Let $\mathcal{M}$ be a Σ-structure, $A \subseteq \mathcal{M}$ and $a \in \mathcal{M}^n$ Then

$$\mathrm{qftp}^{\mathcal{M}}_{A,\,n}(a) := \left\{\phi \in QFF(\Sigma(A), n) \,|\, \mathcal{M} \vDash_{\Sigma(A)} \phi(a)\right\}$$

is the quantifier free type of a in $\mathcal{M}$ over A. One can verify that this is a maximal quantifier free n-type over T if $\mathcal{M}$ is a non-empty model of T. We will often drop parts of the subscripts and superscripts when it is clear. In fact the n can be deduced by the length of a and serves only to explicitely spell things out.

1.4 Quantifier elimination and model completeness

Written whilst following section on algebraically closed fields.

1.4.1 Quantifier elimination

Definition – Equivalence modulo a theory
We say two Σ-formulas ϕ and ψ with free variables indexed by S are equivalent modulo a Σ-theory T if

$$T \vDash_{\Sigma} \forall v, (\phi \Leftrightarrow \psi)$$

where $v = (v_i)_{i \in S}$.

Definition – Quantifier elimination

Let T be a Σ-theory and ϕ a Σ-formula. We say the quantifiers of ϕ can be eliminated if there exists a quantifier free Σ-formula ψ that is equivalent to ϕ modulo T. We say ϕ is reduced to ψ.

We say T has quantifier elimination if the quantifiers of any Σ-formula can be eliminated.

Lemma – Deduction

Let T be a Σ-theory, Δ a finite Σ-theory and ψ a Σ-sentence. Then $T \cup \Delta \models_\Sigma \psi$ if and only if

$$T \models_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$$

Proof. We first case on if Δ is empty or not. If it is empty then $T \cup \Delta \models_\Sigma \psi$ if and only if $T \models_\Sigma \psi$ if and only if $T \models_\Sigma \top \to \psi$ if and only if

$$T \models_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$$

($\Rightarrow$) Suppose $\mathcal{M} \models_\Sigma T$ then we need to show $\mathcal{M} \models_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$. Indeed, suppose $\mathcal{M} \models_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right)$ then by induction $\mathcal{M} \models_\Sigma T \cup \Delta$ and so by assumption that $T \cup \Delta \models_\Sigma \psi$ we have $\mathcal{M} \models_\Sigma \psi$. Hence $\mathcal{M} \models_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$.

($\Leftarrow$) Suppose $\mathcal{M} \models_\Sigma T \cup \Delta$ then $\mathcal{M} \models_\Sigma T$ thus by assumption that $T \models_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$ we have $\mathcal{M} \models_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$. By induction $\mathcal{M} \models_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right)$ thus we have $\mathcal{M} \models_\Sigma \psi$. □

Lemma – Proofs are finite

Suppose T is a Σ-theory and ϕ a Σ-sentence such that $T \models_\Sigma \phi$. Then there exists a finite subset Δ of T such that $\Delta \models_\Sigma \phi$.

Proof. We show the contrapositive. Suppose for all finite subsets Δ of T, $\Delta \not\models_\Sigma \phi$, then $\Delta \cup \{\phi\}$ is consistent and by compactness $T \cup \{\phi\}$ is consistent. Hence $T \not\models_\Sigma \phi$. □

Proposition – Eliminating quantifiers of a formula

Let Σ be a signature such that $\Sigma_{\text{con}} \neq \varnothing$. Suppose T is a Σ-theory and ϕ is a Σ-formula with free-variables $v = (v_1, \dots, v_n)$. Then the quantifiers of ϕ can be eliminated if and only if the following holds: for any two Σ-models $\mathcal{M}, \mathcal{N}$ of T and any Σ-structure $\mathcal{A}$ that with Σ-embeddings into both $\mathcal{M}$ and $\mathcal{N}$ $(\iota_\mathcal{M}, \iota_\mathcal{N})$, if $a \in \mathcal{A}^n$ then

$$\mathcal{M} \models_\Sigma \phi(\iota_\mathcal{M}(a)) \Leftrightarrow \mathcal{N} \models_\Sigma \phi(\iota_\mathcal{N}(a))$$

Proof. ($\Rightarrow$) Let $a \in \mathcal{A}^n$. By assumption there exists $\psi \in \Sigma_{\text{for}}$ such that $T \models_\Sigma \forall v, (\phi(v) \leftrightarrow \psi(v))$ Then $\mathcal{M} \models_\Sigma \phi(\iota_\mathcal{M}(a))$ if and only if $\mathcal{M} \models_\Sigma \psi(\iota_\mathcal{M}(a))$ if and only if $\mathcal{A} \models_\Sigma \psi(a)$, since embeddings preserve the satisfaction of quantifier free formulas. Similarly, this is if and only if $\mathcal{N} \models_\Sigma \phi(\iota_\mathcal{N}(a))$.

($\Leftarrow$) Let

$$\Gamma := \{\psi \text{ quantifier free } \Sigma_{\text{for}} \,|\, T \models_\Sigma \forall v, (\phi \to \psi)\}$$

and let $\Sigma(*)$ be such that $\Sigma(*)_{\text{con}} = \Sigma_{\text{con}} \cup \{d_1, \dots, d_n\}$ for some new constant symbols d_i (indexed according to the free-variables of ϕ). We claim that $T \cup \{\psi(d) \,|\, \psi \in \Gamma\} \models_{\Sigma(*)} \phi(d)$. We first look at how this would complete the proof. If it is true then as proofs are finite we have a finite subsets $\Delta \subseteq \Gamma$ such that $T \cup \{\psi(d) \,|\, \psi \in \Delta\} \vDash_{\Sigma(*)} \phi(d)$. By deduction we have

$$T \vDash_{\Sigma(*)} \left(\bigwedge_{\psi \in \Delta} \psi(d) \right) \to \phi(d)$$

and by the lemma on constants

$$T \vDash_{\Sigma} \forall v, \left(\bigwedge_{\psi \in \Delta} \psi(v) \right) \to \phi(v)$$

where $\left(\bigwedge_{\psi\in\Delta} \psi(v)\right)$ is quantifier free. By the definition of Δ we have the other implication as well:

$$T \vDash_{\Sigma} \forall v, \left(\bigwedge_{\psi \in \Delta} \psi(v) \right) \leftrightarrow \phi(v)$$

hence the result.

Suppose for a contradiction $T \cup \{\psi(d) \,|\, \psi \in \Gamma\} \nvDash_{\Sigma(*)} \phi(d)$. Then there exists a non-empty model $\mathcal{M}$ of $T \cup \{\psi(d) \,|\, \psi \in \Gamma\}$ such that $\mathcal{M} \nvDash_{\Sigma} \phi(d)$.

Suppose for a second contradiction that the $\Sigma(*)(\mathcal{M})$-theory $T \cup \mathrm{AtDiag}(\Sigma(*), \mathcal{M}) \cup \{\phi(d)\}$ is inconsistent. Then by compactness some subset $T \cup \Delta \cup \{\phi(d)\}$ is inconsistent, where $\Delta \subseteq \mathrm{AtDiag}(\Sigma(*), \mathcal{M})$ is finite. This implies $T \cup \Delta \vDash_{\Sigma(*)(\mathcal{M})} \neg\phi(d)$. Hence by deduction we have

$$T \vDash_{\Sigma(*)(\mathcal{M})} \left(\bigwedge_{\psi(d) \in \Delta} \psi(d) \right) \to \neg\phi(d)$$

By the lemma on constants applied to $\Sigma_{\text{con}} \subseteq \Sigma(*)(\mathcal{M})_{\text{con}}$

$$T \vDash_{\Sigma} \forall v, \left[\left(\bigwedge_{\psi(d) \in \Delta} \psi(v) \right) \to \neg\phi(v) \right]$$

Taking the contrapositive,

$$T \vDash_{\Sigma} \forall v, \left[\phi(v) \to \left(\bigvee_{\psi(d) \in \Delta} \neg\psi(v) \right) \right]$$

Hence $\bigvee_{\psi(d)\in\Delta} \neg\psi(v) \in \Gamma$ and so $\mathcal{M} \vDash_{\Sigma(*)} \bigvee_{\psi(d)\in\Delta} \neg\psi(v)$ by definition of $\mathcal{M}$. However each $\Delta \subseteq \mathrm{AtDiag}(\Sigma(*), \mathcal{M})$ and so $\mathcal{M} \vDash_{\Sigma(*)} \bigwedge_{\psi(d)\in\Delta} \psi(v)$, a contradiction. Thus there exists a non-empty model

$$\mathcal{N} \vDash_{\Sigma(*)(\mathcal{M})} T \cup \mathrm{AtDiag}(\Sigma(*), \mathcal{M}) \cup \{\phi(d)\}$$

Since $\mathcal{N} \vDash_{\Sigma(*)(\mathcal{M})} \mathrm{AtDiag}(\Sigma(*), \mathcal{M})$ there exists a $\Sigma(*)(\mathcal{M})$ morphism $\iota : \mathcal{M} \to \mathcal{N}$. Move this morphism down to Σ, then by assumption with $\mathcal{A} := \mathcal{M}$, for any sentence χ

$$\mathcal{M} \vDash_{\Sigma} \chi \Leftrightarrow \mathcal{N} \vDash_{\Sigma} \chi$$

Since $\mathcal{N} \vDash_{\Sigma(*)(\mathcal{M})} \phi(d)$ by the lemma on constants $\mathcal{N} \vDash_{\Sigma} \forall v, \phi(v)$ and so $\mathcal{M} \vDash_{\Sigma} \forall v, \phi(v)$. Which is a contradiction because $\mathcal{M} \vDash_{\Sigma(*)} \neg\phi(d)$ and so by the lemma on constants $\mathcal{M} \vDash_{\Sigma} \forall v, \neg\phi(v)$. (We have $\mathcal{M} \vDash_{\Sigma} \phi(c^{\mathcal{M}}, \ldots c^{\mathcal{M}})$ and $\mathcal{M} \nvDash_{\Sigma} \phi(c^{\mathcal{M}}, \ldots c^{\mathcal{M}})$.) $\square$

Lemma – Sufficient condition for quantifier elimination
Let T be a Σ-theory and suppose for any quantifier free Σ-formula ψ with at least one free variable w, the quantifier of $\forall w, \psi(w)$ can be eliminated. Then T has quantifier elimination.

Proof. Induct on what ϕ is.

- If ϕ is $\top$, an equality or a relation then it is already quantifier free.
- If ϕ is $\neg\chi$ and there exists a quantifier free Σ-formula ψ such that $T \vDash_\Sigma \forall v, \chi \leftrightarrow \psi$. Then $T \vDash_\Sigma \forall v, \neg\chi \leftrightarrow \neg\psi$. Hence ϕ can be reduced to $\neg\psi$ which is quantifier free.
- If ϕ is $\chi_0 \vee \chi_1$ and there exist respective reductions of these ψ_0 and ψ_1 then ϕ reduces to $\psi_0 \vee \psi_1$ which is quantifier free.
- If ϕ is $\forall w, \chi(w)$ and there exists quantifier free ψ such that

  $$T \vDash_\Sigma \forall w, \bigforall_{v \in S} v, (\chi \leftrightarrow \psi)$$

  where S indexes the rest of the free variables in χ and ψ. Then we can show that

  $$T \vDash_\Sigma \bigforall_{v \in S} (\phi \leftrightarrow (\forall w, \psi))$$

  By assumption there exists ω a quantifier free Σ-formula such that

  $$T \vDash_\Sigma \bigforall_{v \in S} (\omega \leftrightarrow (\forall w, \psi))$$

  Hence ϕ can be reduced to ω.

□

Corollary – Improvement: Sufficient condition for quantifier elimination
If T be a Σ-theory if for any quanfier free Σ-formula ϕ with at least one free variable w (index the rest by S), for any $\mathcal{M}, \mathcal{N}$ Σ-models of T, for any Σ-structure $\mathcal{A}$ that embeds into $\mathcal{M}$ and $\mathcal{N}$ (via $\iota_\mathcal{M}, \iota_\mathcal{N}$) and any $a \in (\mathcal{A})^S$,

$$\mathcal{M} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{M}(a)) \Rightarrow \mathcal{N} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{N}(a))$$

then T has quantifier elimination.

Equivalently we can use the statement

$$\mathcal{M} \vDash_\Sigma \exists w, \phi(\iota_\mathcal{M}(a)) \Rightarrow \mathcal{N} \vDash_\Sigma \exists w, \phi(\iota_\mathcal{N}(a))$$

by negating ϕ.

Proof. To show that T has quantifier elimination it suffices to show that for any quanfier free Σ-formula ϕ with at least one free variable w (index the rest by S), the quantifiers of $\forall w, \phi$ can be eliminated. This is true if and only if for any $\mathcal{M}, \mathcal{N}$ Σ-models of T, for any Σ-structure $\mathcal{A}$ that injects into $\mathcal{M}$ and $\mathcal{N}$ (via $\iota_\mathcal{M}, \iota_\mathcal{N}$) and any $a \in (\mathcal{A})^S$,

$$\mathcal{M} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{M}(a)) \Rightarrow \mathcal{N} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{N}(a))$$

By symmetry of $\mathcal{M}$ and $\mathcal{N}$ we only require one implication. Hence the proposition. □

Remark. For quantifier elimination it also suffices to show that for any Σ-models $\mathcal{M}$ of T, for any Σ-structure $\mathcal{A}$ that embeds into $\mathcal{M}$ (via $\iota_{\mathcal{M}}$) and any $a \in (\mathcal{A})^S$,

$$\mathcal{M} \vDash_\Sigma \forall w, \phi(\iota(a)) \Rightarrow \mathcal{A} \vDash_\Sigma \forall w, \phi(a)$$

since embeddings preserve satisfaction of universal formulas downwards. Equivalently we can use

$$\mathcal{A} \vDash_\Sigma \exists w, \phi(a) \Rightarrow \mathcal{M} \vDash_\Sigma \exists w, \phi(\iota(a))$$

1.4.2 Back and Forth

'Back and forth' is a technique used to determine elementary equivalence of models, quantifier elimination of theories and completeness of theories. This section draws together work from Poizat [10], OLP [8], and Pillay [9]. It is motivated by the example at the end, which should be looked at first.

Definition – Substructure generated by a subset
Let $\mathcal{M}$ be a Σ-structure. Let $A \subseteq \mathcal{M}$. Then the following are equal:

- The set $\langle A \rangle$ defined inductively: $A \subseteq \langle A \rangle$; if $c \in \Sigma_{\text{con}}$ then $c^{\mathcal{M}} \in \langle A \rangle$; if $f \in \Sigma_{\text{fun}}$ and $\alpha \in \langle A \rangle^{n_f}$ then $f^{\mathcal{M}}(\alpha) \in \langle A \rangle$.
- $\bigcap \{\mathcal{N} \text{ substructure of } \mathcal{M} \mid A \subseteq \mathcal{N}\}$

and define a substructure of $\mathcal{M}$. We say it is the 'substructure of $\mathcal{M}$ generated by A'.

We say a substructure is finitely generated if there exists a finite set A such that it is equal to $\langle A \rangle$.

Proof. We note show that $\langle A \rangle$ is a substructure of $\mathcal{M}$ containing A: It contains the interpretations of constant symbols from $\mathcal{M}$. By definition $f^{\langle A \rangle} := f^{\mathcal{M}}$ is well defined. Each relation r is naturally interpreted as the intersection of relations on $\mathcal{M}$ intersected with $\langle A \rangle^{m_r}$. Hence $\bigcap \mathcal{N} \subseteq \langle A \rangle$.

For the other direction note that if $a \in \langle A \rangle$ then it is in A, $c^{\mathcal{M}}$ or $f^{\mathcal{M}}(\alpha)$ for some $\alpha \in \langle A \rangle^{n_f}$. If it is in A then we are done. Any substructure of $\mathcal{M}$ contains the $c^{\mathcal{M}}$ for each constant symbol hence the first case is fine. Any substructure of $\mathcal{M}$ is closed under $f^{\mathcal{M}}$ and by induction $\alpha \in \mathcal{N}^{n_f}$ for any substructure $\mathcal{N}$. Hence $f(\alpha) \in \mathcal{N}$ for any substructure. Thus $\langle A \rangle \subseteq \bigcap \mathcal{N}$ and we are done. $\square$

Proposition – Image of generators are generators of the image
The image of a substructure generated by a subset is a substructure generated by the image of a set. In particular, a finitely generated substructure has finitely generated image under a Σ-morphism given by the image of the generators.

Proof. Let $\iota : \langle A \rangle \to \mathcal{N}$ be a Σ-morphism. We show that $\langle \iota(A) \rangle = \iota(\langle A \rangle)$. If $b \in \langle \iota(A) \rangle$ then $b = c^{\mathcal{N}}$ or $b = f^{\mathcal{N}}(\iota(\alpha))$ for $al \in \langle A \rangle$. Hence $b = c^{\mathcal{N}} = \iota(c^{\mathcal{M}}) \in \iota(\langle A \rangle)$ or

$$b = f^{\mathcal{N}}(\iota(\alpha)) = \iota(f^{\mathcal{M}}(\alpha)) \in \iota(\langle A \rangle)$$

Thus $\langle \iota(A) \rangle \subseteq \iota(\langle A \rangle)$. The other direction is similar. $\square$

Definition – Partial isomorphisms
Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. A partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$ is a Σ-isomorphism p with finitely generated domain in of $\mathcal{M}$ and codomain in $\mathcal{N}$.

Proposition – Equivalent definition of partial isomorphism
Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. Let $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$. The following are equivalent:

- There exists a partial isomorphism $p : \langle a \rangle \to \langle b \rangle$ such that $p(a) = b$.
- $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$

Proof. ($\Rightarrow$) We induct on terms to show that $t^{\mathcal{M}}(a) = t^{\langle a \rangle}(a)$ for each term t:

- If t is a constant symbol or a variable then by definition of the the substructure interpretation they are equal.
- If t is $f(s)$ and we have the inductive hypothesis $s^{\mathcal{M}}(a) = s^{\langle a \rangle}(a)$ then by definition of the substructure interpretation
$$t^{\mathcal{M}}(a) = f^{\mathcal{M}}(s^{\mathcal{M}}(a)) = f^{\mathcal{M}}(s^{\langle a \rangle}(a)) = f^{\langle a \rangle}(s^{\langle a \rangle}(a)) = t^{\langle a \rangle}(a)$$

Let ϕ be a quantifier free Σ-formula with up to n variables. We show by induction on ϕ that

$$\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow \langle a \rangle \vDash_{\Sigma} \phi(a)$$

- If ϕ is $\top$ it is trivial.
- If ϕ is $t = s$ then it is clear that
$$t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a) \Leftrightarrow t^{\langle a \rangle}(a) = s^{\langle a \rangle}(a)$$
by what we showed for terms.
- If ϕ is $r(t)$ then
$$(a_{i_1}, \ldots, a_{i_m}) \in r^{\mathcal{M}} \Leftrightarrow (a_{i_1}, \ldots, a_{i_m}) \in r^{\mathcal{M}} \cap \langle a \rangle = r^{\langle a \rangle}$$
- If ϕ is $\neg\psi$ or $\psi \vee \chi$ then it is clear by induction.

As p is an Σ-isomorphism, for any quantifier free Σ-formula with up to n variables,

$$\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow \langle a \rangle \vDash_{\Sigma} \phi(a) \Leftrightarrow \langle b \rangle \vDash_{\Sigma} \phi(b) \mathcal{N} \vDash_{\Sigma} \phi(b)$$

($\Leftarrow$) Suppose $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$. We define $p : \langle a \rangle \to \mathcal{N}$ by the following: if $\alpha \in \langle a \rangle$ then one can write α as a term t evaluated at a: $\alpha = t^{\mathcal{M}}(a)$; p maps a to $t^{\mathcal{N}}(b)$. To show that p is well-defined, note that if two terms t and s are such that $t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)$ then $t = s$ is a formula in $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$ and so $t^{\mathcal{N}}(b) = s^{\mathcal{N}}(b)$. it is injective because if two terms t and s are such that $t^{\mathcal{N}}(b) = s^{\mathcal{N}}(b)$ then $t = s$ is a formula in $\mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b) = \mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a)$ and so $t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)$.

By definition p commutes with the interpretation of constant symbols, function symbols, and relations. Furthermore, for each i, $p(a_i) = b_i$ by taking the term to be a variable and evaluating at a_i. The image of p is $\langle b \rangle$ as the image of a is b. Hence it is a partial isomorphism $\langle a \rangle \to \langle b \rangle$ such that $p(a) = b$. $\square$

Proposition – Basic facts about partial isomorphisms
Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures.

- The inverse of a partial isomorphism is a partial isomorphism.
- The restriction of a partial isomorphism is a partial isomorphism.
- The composition of partial isomorphisms is a partial isomorphism.

Definition – Partially isomorphic structures

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. A partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$ is said to have the back and forth property if

- (Forth) For each $a \in \mathcal{M}$ there exists a partial isomorphism q such that q extends p and $a \in \operatorname{dom} p$.
- (Back) For each $p \in I$ there exists a partial isomorphism q such that q extends p and $b \in \operatorname{codom} q$.

We say $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent when all partial isomorphisms from $\mathcal{M}$ to $\mathcal{N}$ have the back and forth property.

Proposition – Equivalent definition of back and forth property

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. Let $p : \langle a \rangle \to \langle b \rangle$ for $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ be a partial isomorphism such that $p(a) = b$. It has the back and forth property if and only if the two conditions hold

- (Forth) For any $\alpha \in \mathcal{M}$, there exists $\beta \in \mathcal{N}$ such that $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a, \alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b, \beta)$
- (Back) For any $\beta \in \mathcal{N}$, there exists $\alpha \in \mathcal{M}$ such that $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a, \alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b, \beta)$

Proof. ($\Rightarrow$) Suppose p has the back and forth property. We only show 'forth' as the 'back' case is similar. Let $\alpha \in \mathcal{M}$. By 'forth' there exists q a partial isomorphism extending p such that $\alpha \in \mathrm{dom}(q)$. By restriction and the fact that the image of generators generates the image, there exists $\beta \in \mathcal{N}$ such that

$$q\big|_{\langle a,\alpha\rangle \to \langle b,\beta\rangle}$$

is a local isomorphism. Using the the equivalent definition we obtain $\mathrm{qftp}(a, \alpha) = \mathrm{qftp}(b, \beta)$.

($\Leftarrow$) We show that p has the 'forth' property. Let $\alpha \in \mathcal{M}$. By assumption there exists $\beta \in \mathcal{M}$ such that

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a, \alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b, \beta)$$

Thus there exists $q : \langle a, \alpha\rangle \to \langle b, \beta\rangle$ such that $q(a) = b$ and $q(\alpha) = \beta$. Hence p is extended by q with α in its domain. $\square$

Proposition – Quantifier elimination for types

Let T be a Σ-theory. T has quantifier elimination if and only if for any $n \in \mathbb{N}$, any two non-empty Σ-models of T and any $a \in \mathcal{M}^n, b \in \mathcal{N}^n$, if

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

then

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

Proof. ($\Rightarrow$) Let $\phi \in \mathrm{tp}(a)$. By quantifier elimination there exists quantifier free ψ such that they are equivalent modulo T. Then $\mathcal{M} \vDash_\Sigma \psi(a)$ and $\psi \in \mathrm{qftp}(a) = \mathrm{qftp}(b)$. Thus $\mathcal{N} \vDash_\Sigma \psi(b)$ and by equivalence modulo T $\mathcal{N} \vDash_\Sigma \phi(b)$. Hence $\phi \in \mathrm{tp}(b)$. The other inclusion is similar.

($\Leftarrow$) Let $n \in \mathbb{N}$. Define a map $f : S_n(T) \to S^{\mathrm{qf}}_n(T)$ that takes a maximal n-type p to $p \cap QFF(\Sigma, n)$. It is well-defined as the image is indeed a maximal n-type. It is a surjection as any quantifier free maximal n type is an n-type and therefore can be extended to a maximal n-type. To show injectivity we note that any two elements of $S_n(T)$ can be written as types of elements $\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a)$ and $\mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$. If their images are equal then

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

thus by assumption they are equal.

To show that f is continuous we show that elements of the clopen basis have clopen preimage. Let $[\phi]_T^{\mathrm{qf}}$ be in the clopen basis of $S_n^{\mathrm{qf}}(T)$. Then $p \in [\phi]_T$ if and only if $\phi \in p$ if and only if $\phi \in f(p)$ if and only if $f(p) \in [\phi]_T^{\mathrm{qf}}$. Hence the preimage is $[\phi]_T$ which is clopen.

A continuous bijection between Hausdorff compact spaces is a homeomorphism. Hence for any $\phi \in F(\Sigma, n)$ the image of the clopen set generated by ϕ is clopen: there exists $\psi \in QFF(\Sigma, n)$ such that $[\phi]_T = f^{-1}[\psi]_T^{\mathrm{qf}} = [\psi]_T$. $[\phi]_T = [\psi]_T$ if and only if they are equivalent modulo T. Thus we can eliminate quantifiers for any $\phi \in F(\Sigma, n)$ for any n. Thus T has quantifier elimination. □

Lemma – Back and forth equivalence implies quantifier elimination for types

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. If $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent and $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ are such that

$$\mathrm{qftp}_{\varnothing}^{\mathcal{M}}(a) = \mathrm{qftp}_{\varnothing}^{\mathcal{N}}(b)$$

then

$$\mathrm{tp}_{\varnothing}^{\mathcal{M}}(a) = \mathrm{tp}_{\varnothing}^{\mathcal{N}}(b)$$

Proof. Let $\phi \in F(\Sigma, n)$. If ϕ is quantifier free then $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(b)$. By induction on formulas it suffices to show that if ϕ is the formula $\forall v, \psi$ and for any $\alpha \in \mathcal{M}$ there exists $\beta \in \mathcal{N}$ such that $\mathcal{M} \vDash_\Sigma \psi(a, \alpha) \Leftrightarrow \mathcal{N} \vDash_\Sigma \psi(b, \beta)$, then we have $\mathcal{M} \vDash_\Sigma \forall v, \psi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \forall v, \psi(b)$.

By the equivalent definition of partial isomorphisms, there exists $p : \langle a \rangle \to \langle b \rangle$ a partial isomorphism in p such that $p(a) = b$. Suppose $\mathcal{M} \vDash_\Sigma \forall v, \psi(a)$ and let $\beta \in \mathcal{N}$, then $\mathcal{M} \vDash_\Sigma \forall v, \psi(a, \alpha)$. By 'back' in the equivalent definition of the back and forth property there exists $\alpha \in \mathcal{M}$ such that $\mathrm{qftp}_{\varnothing}^{\mathcal{M}}(a, \alpha) = \mathrm{qftp}_{\varnothing}^{\mathcal{N}}(b, \beta)$ Hence $\mathcal{N} \vDash_\Sigma \forall v, \psi(a, \alpha)$. The other direction is similar. □

Definition – Elementary equivalence

Let $\mathcal{M}$, $\mathcal{N}$ be Σ-structures. They are elementarily equivalent if for any Σ-sentence ϕ, $\mathcal{M} \vDash_\Sigma \phi$ if and only if $\mathcal{N} \vDash_\Sigma \phi$. We write $\mathcal{M} \equiv_\Sigma \mathcal{N}$.

Corollary – Back and forth implies elementary equivalence

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. If $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent then they are elementarily equivalent.

Proof. Let ϕ be a quantifier free Σ-formula with 0 variables, i.e. a quantifier free sentence. As the empty set is a partial isomorphism. Thus by the equivalent definition of a partial isomorphism,

$$\mathrm{qftp}_{\varnothing,0}^{\mathcal{M}}(\varnothing) = \mathrm{qftp}_{\varnothing,0}^{\mathcal{N}}(\varnothing)$$

By the fact that back and forth equivalence implies quantifier elimination for types,

$$\mathrm{tp}_{\varnothing,0}^{\mathcal{M}}(\varnothing) = \mathrm{tp}_{\varnothing,0}^{\mathcal{N}}(\varnothing)$$

Thus for any Σ-sentence ϕ, $\mathcal{M} \vDash_\Sigma \phi$ if and only if $\phi \in \mathrm{tp}_{\varnothing,0}^{\mathcal{M}}(\varnothing) = \mathrm{tp}_{\varnothing,0}^{\mathcal{N}}(\varnothing)$ if and only if $\mathcal{N} \vDash_\Sigma \phi$. □

Definition – ω-saturation

Let $\mathcal{M}$ be a Σ-structure. $\mathcal{M}$ is ω-saturated if for every finite subset $A \subseteq \mathcal{M}$, every $n \in \mathbb{N}$ and every $p \in S_n(\mathrm{Th}_{\mathcal{M}}(A))$, p is realised in $\mathcal{M}$.

See the general version κ-saturated here.

Proposition – ∞-equivalence
Let $\mathcal{M}$ and $\mathcal{N}$ be ω-saturated Σ-structures. If $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ satisfy

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,n}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,n}(b)$$

then

- (Forth) For any $\alpha \in \mathcal{M}$ there exists $\beta \in \mathcal{N}$ such that

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,n+1}(a,\alpha) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,n+1}(b,\beta)$$

- (Back) For any $\beta \in \mathcal{N}$ there exists $\alpha \in \mathcal{M}$ such that

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,n+1}(a,\alpha) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,n+1}(b,\beta)$$

If this property holds for any pair a, b related by a partial isomorphism we say $\mathcal{M}$ and $\mathcal{N}$ are ∞-equivalent.

Proof. Let $\alpha \in \mathcal{M}$ and consider

$$p(a,v) := \mathrm{tp}^{\mathcal{M}}_{a,1}(\alpha) \in S_1(\mathrm{Th}_{\mathcal{M}}(a))$$

Any formula in $p(a,v)$ can be written as a Σ-formula $\phi(w,v)$ with variables w replaced with elements of a (v represents a single variable to be replaced by α). Let

$$p(w,v) := \{\phi(w,v) \mid \phi(a,v) \in p(a,v)\}$$

We claim that

$$p(b,v) := \{\phi(b,v) \mid \phi \in p(w,v)\} \in S_1(\mathrm{Th}_{\mathcal{N}}(b))$$

To this end, we note that it is indeed a maximal subset of $F(\Sigma(b),1)$ since for any $\phi(b) \in F(\Sigma(b),1)$

$$\phi(a) \in p(a,v) \text{ or } \neg\,\phi(a) \in p(a) \Rightarrow \phi(b) \in p(b,v) \text{ or } \neg\,\phi(b) \in p(b)$$

We just need to show that it is consistent with $\mathrm{Th}_{\mathcal{N}}(b)$.

By compactness for types and noting that $\mathcal{N}$ is a $\Sigma(b)$-model of $\mathrm{Th}_{\mathcal{N}}(b)$, it suffices to show that for any finite subset $\Delta(w,v) \subseteq p(w,v)$ there exists $\beta \in \mathcal{N}^m$ such that $\mathcal{N} \vDash_{\Sigma(b)} \Delta(b,\beta)$.

$$\begin{aligned}
&\mathcal{M} \vDash_{\Sigma(a)} \bigwedge_{\phi\in\Delta} \phi(a,\alpha) \\
\Rightarrow &\mathcal{M} \vDash_{\Sigma} \exists v, \bigwedge_{\phi\in\Delta} \phi(a,v) \\
\Rightarrow &\left(\exists v, \bigwedge_{\phi\in\Delta} \phi(a,v)\right) \in \mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b) \\
\Rightarrow &\mathcal{N} \vDash_{\Sigma} \exists v, \bigwedge_{\phi\in\Delta} \phi(b,v) \\
\Rightarrow &\exists \beta \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma} \bigwedge_{\phi\in\Delta} \phi(b,\beta) \\
\Rightarrow &\exists \beta \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma(b)} \Delta(b,\beta)
\end{aligned}$$

Thus $p(b,v) \in S_1(\mathrm{Th}_{\mathcal{N}}(b))$ and since $\mathcal{N}$ is ω-saturated $p(b,v)$ is realised in $\mathcal{N}$ by some β. Thus by maximality, $p(b,v) = \mathrm{tp}^{\mathcal{N}}_{b,1}(\beta)$.

Finally, for $\phi(v,w) \in F(\Sigma, n+1)$

$$\begin{aligned}
&\phi(v,w) \in \mathrm{tp}^{\mathcal{M}}_{\varnothing}(a,\alpha) \Leftrightarrow && \mathcal{M} \vDash_{\Sigma} \phi(a,\alpha) \Leftrightarrow \mathcal{M} \vDash_{\Sigma(a)} \phi(a,\alpha)\\
\Leftrightarrow &\phi(a,v) \in \mathrm{tp}^{\mathcal{M}}_{a,1}(\alpha) = p(a,v)\\
\Leftrightarrow &\phi(b,v) \in p(b,v) = \mathrm{tp}^{\mathcal{M}}_{b,1}(\beta)\\
\Leftrightarrow &\mathcal{N} \vDash_{\Sigma(b)} \phi(b,\beta) \Leftrightarrow \mathcal{N} \vDash_{\Sigma} \phi(b,\beta)\\
\Leftrightarrow &\phi(w,v) \in \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b,\beta)
\end{aligned}$$

□

Proposition – Back and forth method for showing quantifier elimination

Let T be a Σ-theory. If T has quantifier elimination then for any two ω-saturated Σ-models of T are back and forth equivalent.

If any two Σ-models of T are back and forth equivalent then T has quantifier elimination. †

† We could also phrase this as T has quantifier elimination if and only if any two ω-saturated Σ-models of T are back and forth equivalent, but the saturation requirement becomes redundant in one direction.

Proof. ($\Rightarrow$) Let p be a partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$. By the equivalent definition of partial isomorphisms there exists $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ such that $p(a) = b$ and

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

By quantifier elimination for types

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

The models are ω-saturated, hence by ∞-equivalence for any $\alpha \in \mathcal{M}$ there exists $\beta \in \mathcal{N}$ such that

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a,\alpha) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b,\beta)$$

Taking only the quantifier free elements, we obtain

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a,\alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b,\beta)$$

and by the equivalent definition of the back and forth property we have that p has the back and forth property.

($\Leftarrow$) Let $n \in \mathbb{N}$, $\mathcal{M}$ and $\mathcal{N}$ be models of T, $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$. By quantifier elimination for types it suffices to show that if

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

then

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

This is satisfied as any two models of T are back and forth equivalent. □

Lemma – Equivalent definition of completeness

Let T be a Σ-theory. T is complete if and only if all non-empty models of T are elementarily equivalent.†

† Similarly to before, 'non-empty' is redundant in the backwards direction.

Proof. ($\Rightarrow$) Any two non-models of T will satisfy the formulas of T, which is a maximal set of Σ-sentences. Since we are assuming that they are non-empty we have that T is consistent hence for any formula either it or its negation is in T. Hence they satisfy the same formulas.

($\Leftarrow$) If ϕ is a Σ-sentence then suppose for a contradiction

$$T \not\vDash_\Sigma \phi \text{ and } T \not\vDash_\Sigma \neg\phi$$

Then there exist non-empty models of T such that $\mathcal{M} \not\vDash_\Sigma \phi$ and $\mathcal{N} \not\vDash_\Sigma \neg\phi$. By assumption they are elementarily equivalent and so $\mathcal{M} \vDash_\Sigma \neg\phi$ implies $\mathcal{N} \vDash_\Sigma \neg\phi$, a contradiction. □

Corollary – Back and forth condition for completeness
Let T be a Σ-theory. If any two models of T are back and forth equivalent then T is complete.

Proof. If any two models are back and forth equivalent then any two non-empty models are elementarily equivalent (the non-empty is redundant information). Hence T is complete. □

We end this section with a nice example of all of this in action.

EXAMPLE (INFINITE INFINITE EQUIVALENCE CLASSES).

$$\Sigma_{\mathrm{ER}} := (\varnothing, \varnothing, n_f, \{E\}, m_r)$$

where $m_E = 2$, is the signature of equivalence relations. We write for variables x and y, we write $x \sim y$ as notation for $E(x, y)$ The theory of equivalence relations ER *is set set containing the following formulas:*

Reflexivity - $\forall x, x \sim x$
Symmetry - $\forall x \forall y, x \sim y \to y \sim x$
Transitivity - $\forall x \forall y \forall z, (x \sim y \wedge y \sim z) \to x \sim z$

For $n \in \mathbb{N}_{>1}$ define

$$\phi_n := \exists_{i=1}^{n} x_i, \bigwedge_{i<j} x_i \nsim x_j$$
$$\psi_n := \forall x, \exists_{i=1}^{n} x_i, \bigwedge_{i=1}^{n} (x \sim x_i) \wedge \bigwedge_{i<j} (x_i \neq x_j)$$

Show that the theory $T = \mathrm{ER} \cup \phi_n, \psi_{n1<i}$ has quantifier elimination and is complete. (You may wonder if it is indeed a theory and what nasty induction must be done to show that its formulas can be constructed.)

Proof. We first define the projection into the quotient: if $\mathcal{M} \vDash_{\Sigma_{\mathrm{ER}}} T$ and $a \in \mathcal{M}$ then

$$\pi_{\mathcal{M}}(a) := \{b \in \mathcal{M} \mid \mathcal{M} \vDash_{\Sigma_{\mathrm{ER}}} a \sim b\}$$

If $A \subseteq \mathcal{M}$ we write $\pi_{\mathcal{M}}(A)$ to be the image

$$\{\pi_{\mathcal{M}}(a) \mid \exists a \in A\}$$

Note that the quotient is $\pi_{\mathcal{M}}(\mathcal{M})$.

Let $\mathcal{M}, \mathcal{N}$ be Σ_{ER}-models of T and let p be a partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$. By the back and forth condition for quantifier elimination and the back and forth condition for completeness it suffices to show that p has the back and forth property.

We only show 'forth'. Let $\alpha \in \mathcal{M}$. Suppose $\pi_{\mathcal{M}}(\alpha) \cap \operatorname{dom} p$ is empty. We can show that $\pi_{\mathcal{N}}(\mathcal{N})$ is infinite whilst $\pi_{\mathcal{N}}(\operatorname{codom} p)$ is finite, hence there exists $\beta \in \mathcal{N}$ such that $\pi(\beta) \in \pi_{\mathcal{N}}(\mathcal{N}) \setminus \pi_{\mathcal{N}}(\operatorname{codom} p)$ is non-empty. Then define $q : \operatorname{dom} p \cup \{\alpha\} \to \operatorname{codom} p \cup \{\beta\}$ to agree with p on its domain and send α to β. Note that the domain and codomain of q are substructures as the language only contains a relation symbol (thus all subsets are substructures). We show that q is an isomorphism. It is clearly bijective, and to be an embedding it just needs to preserve interpretation of the relation. Let $a, b \in \operatorname{dom} q$, if both are in $\operatorname{dom} p$ then as p is a partial isomorphism

$$a \sim^{\mathcal{M}} b \Leftrightarrow p(a) \sim^{\mathcal{N}} p(b) \Leftrightarrow q(a) \sim^{\mathcal{N}} q(b)$$

Otherwise WLOG $a = \alpha$. If $b = \alpha$ then it is clear. If $b \in \operatorname{dom} p$ then by assumption $b \notin \pi_{\mathcal{M}}(\alpha) = \pi(a)$ hence $\neg\, a \sim^{\mathcal{M}} b$. By construction

$$q(a) = q(\alpha) = \beta \Rightarrow \pi_{\mathcal{N}}(q(a)) \notin \pi_{\mathcal{N}}(\operatorname{codom} p) \quad \text{and} \quad q(b) = p(b) \in \operatorname{codom} p$$

hence $\neg\, q(a) \sim^{\mathcal{N}} q(b)$. Thus q is a local isomorphism extending p.

Suppose $\pi_{\mathcal{M}}(\alpha) \cap \operatorname{dom} p$ is non-empty, i.e. there exists $a \in \operatorname{dom} p$ such that $\alpha \sim^{\mathcal{M}} a$ We can show that $\pi_{\mathcal{N}}(p(a))$ is infinite and $\operatorname{codom} p$ is finite hence there exists $\beta \in \pi_{\mathcal{N}}(p(a)) \setminus \operatorname{codom} p$. Then define $q : \operatorname{dom} p \cup \{\alpha\} \to \operatorname{codom} p \cup \{\beta\}$ to agree with p on its domain and send α to β. Again p is clearly a bijection on substructures, and we show that the relation is preserved. Let $b, c \in \operatorname{dom} q$. If $b, c \in \operatorname{dom} p$ then it is clear as p is an isomorphism, it is also clear if $b, c = \alpha$. Otherwise WLOG $c = \alpha$ and $b \in \operatorname{dom} p$. Then $c = \alpha \sim^{\mathcal{M}} a$ and by construction of β

$$q(c) = q(\alpha) = \beta \sim^{\mathcal{N}} p(a)$$

Noting $a \sim^{\mathcal{M}} b$ if and only if $p(a) \sim^{\mathcal{N}} p(b)$ as p is a partial isomorphism thus $c \sim^{\mathcal{M}} a \sim^{\mathcal{M}} b$ if and only if $q(c) \sim^{\mathcal{N}} p(a) \sim^{\mathcal{N}} p(b) = q(b)$. Hence q is a local isomorphism extending p. Thus p has the 'forth' property (and similarly the 'back' property). ☐

Proposition – Countable back and forth equivalent structures are isomorphic

Let $\mathcal{M}$ and $\mathcal{N}$ be countably infinite Σ-structures. If $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Proof. Write $\mathcal{M} = \{a_i\}_{i \in \mathbb{N}}$ and $\mathcal{N} = \{b_i\}_{i \in \mathbb{N}}$. Inductively define partial isomorphisms p_n for $n \in \mathbb{N}$:

- Take p_0 to be the empty function.
- If $n+1$ is odd then ensure $a_{n/2}$ is in the domain: by the 'forth' property of p there exists p_{n+1} extending p_n such that $a_{n/2} \in \operatorname{dom}(p_{n+1})$.
- If $n+1$ is even then ensure $b_{(n+1)/2}$ is in the codomain: by the 'back' property of p there exists p_{n+1} extending p_n such that $b_{(n-1)/2} \in \operatorname{codom}(p_{n+1})$.

We claim that p, the union of the partial isomorphisms p_n for each $n \in \mathbb{N}$, is an isomorphism. Note that it is well-defined and has image $\mathcal{N}$ as the p_i are nested and for any $a_i \in \mathcal{M}$ and $b_i \in \mathcal{N}$, $a_i \in \operatorname{dom}(p_{2i+1})$ and $b_i \in \operatorname{dom}(p_{2i+2})$. It is injective: if $a_i, a_j \in \mathcal{M}$ and $p(a_i) = p(a_j)$ then $p_{2i+2}(a_i) = p_{2i+2}(a_j)$ and so $a_i = a_j$ as p_{2i+2} is a partial isomorphism. One can show that it is an Σ-embedding. ☐

1.4.3 Model completeness

Definition – Model Completeness

We say a Σ-theory T is model complete when given two Σ-models of T and a Σ-embedding $\iota : \mathcal{M} \to \mathcal{N}$, the embedding is elementary.

Remark. Any Σ-theory T with quantifier elimination is model complete. If ϕ is a Σ-formula and $a \in (\mathcal{M})^S$. Then given two Σ-models of T and a Σ-embedding $\iota : \mathcal{M} \to \mathcal{N}$ we can take ψ a quantifier free formula such that $T \vDash_\Sigma \forall v, \phi \leftrightarrow \psi$. Since embeddings preserve satisfaction of quantifier free formulas

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \vDash_\Sigma \psi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \psi(\iota(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

Thus the extension is elementary.

1.5 Morley Rank

1.5.1 Saturation

Proposition – Elementarily equivalent structures have a common extension [10]
Let $\mathcal{N}_i$ be Σ-structures, for each $i \in I$ a non-empty set. Suppose that for each $i, j \in I$, $\mathcal{N}_i$ is elementarily equivalent to $\mathcal{N}_j$. Then there exists $\mathcal{M}$ a Σ-structure and elementary Σ-embeddings $\iota_i : \mathcal{N}_i \to \mathcal{M}$ for each $i \in I$.

Proof. We show that $\bigcup_{i\in I} \mathrm{ElDiag}(\Sigma, \mathcal{N}_i)$ is consistent as a $\Sigma(*) := \Sigma(\bigcup_{i\in I}(\mathcal{N}_i))$ theory. This would give us elementary $\Sigma(*)$-embeddings from each $\mathcal{N}_i$ into some non-empty model $\mathcal{M}$, and we would then be done (by moving this embedding down to Σ).

By compactness it suffices to show that each finite subset of $\bigcup_{i\in I} \mathrm{ElDiag}(\Sigma, \mathcal{N}_i)$ is consistent. The finite subset can be written in the form

$$\bigcup_{i\in S} \Delta_i$$

where $S \subseteq I$ is a finite subset and each $\Delta_i \subseteq \mathrm{ElDiag}(\Sigma, \mathcal{N}_i)$ is finite. Let 0 denote the element of I (non-empty). We show that

$$\mathcal{N}_0 \vDash_{\Sigma(*)} \bigcup_{i\in S} \Delta_i$$

By definition of the elementary diagram, each $\mathcal{N}_i$ is a $\Sigma(\mathcal{N}_i)$-model of Δ_i. We can write Δ_i as $\Gamma_i(a_i)$ where Γ_i is a set of Σ-formulas with at most n variables and $a_i \in {\mathcal{N}_i}^n$. Then using elementary equivalence

$$\begin{aligned}
&\mathcal{N}_i \vDash_{\Sigma(\mathcal{N}_i)} \bigwedge_{\phi\in\Delta_i} \phi \Rightarrow \mathcal{N}_i \vDash_{\Sigma} \bigwedge_{\phi\in\Gamma_i} \phi(a_i) \\
\Rightarrow &\mathcal{N}_i \vDash_{\Sigma} \exists_{j=1}^{n} v_j, \bigwedge_{\phi\in\Gamma_i} \phi(v) \Rightarrow \mathcal{N}_0 \vDash_{\Sigma} \exists_{j=1}^{n} v_j, \bigwedge_{\phi\in\Gamma_i} \phi(v) \\
\Rightarrow &\exists b_i \in {\mathcal{N}_0}^n, \mathcal{N}_0 \vDash_{\Sigma} \bigwedge_{\phi\in\Gamma_i} \phi(b_i)
\end{aligned}$$

We proceed to use this fact to define interpretation on $\mathcal{N}_0$ as a $\Sigma(*)$ structure.

If $\mathcal{N}_0$ were empty then any sentence is satisfied by it, hence by elementary equivalence every $\mathcal{N}_i$ satisfies all sentences. Hence all the structures are empty and we can find a common extension, namely the empty set. Otherwise let $c \in \mathcal{N}_0$. We only need to update interpretation of constant symbols from $\mathcal{N}_i$ for $i \neq 0$. Let $d \in \mathcal{N}_i$. If d appears in the tuple a_i from above, we interpret it as the corresponding term in the tuple b_i such that $a_i^{\mathcal{N}_0} = b_i$. Otherwise intperpret d as c. Clearly for each $i \in S$ and each $\phi \in \Gamma_i$

$$\mathcal{N}_0 \vDash_{\Sigma} \phi(b_i) \Rightarrow \mathcal{N}_0 \vDash_{\Sigma(*)} \phi(b_i) \Rightarrow \mathcal{N}_0 \vDash_{\Sigma(*)} \phi(a_i)$$

and so

$$\mathcal{N}_0 \vDash_{\Sigma(*)} \bigcup_{i\in S} \Gamma_i(a_i) \Rightarrow \mathcal{N}_0 \vDash_{\Sigma(*)} \bigcup_{i\in S} \Delta_i$$

$\square$

Definition – More general version of the theory of a structure
Let $\mathcal{M}$ be a Σ-structure and let $A \subseteq \mathcal{M}$. Move $\mathcal{M}$ up to $\Sigma(A)$ in the obvious way. The theory of $\mathcal{M}$ over A is defined by

$$\mathrm{Th}_{\mathcal{M}}(A) := \{\phi \in \Sigma(A)\text{-sentences} \,|\, \mathcal{M} \vDash_{\Sigma(A)} \phi\}$$

Note that if $\mathcal{M}$ is non-empty then $\mathrm{Th}_{\mathcal{M}}(A)$ is a consistent and complete $\Sigma(A)$-theory as it is modelled by $\mathcal{M}$ and any formula is either satisfied by $\mathcal{M}$ or not. Note also that the theory of $\mathcal{M}$ over $\mathcal{M}$ is the elementary diagram.

Lemma – Types over $\mathrm{Th}_{\mathcal{M}}(A)$ are realised in extensions

Let $\mathcal{M}$ be a Σ-structure, $A \subseteq \mathcal{M}$ and $p \in S_n(\mathrm{Th}_{\mathcal{M}}(A))$. Then there exists an elementary $\Sigma(A)$-embedding $\mathcal{M} \to \mathcal{N}$ such that p is realised in $\mathcal{N}$.

Proof. By definition of the Stone space, p is consistent with $\mathrm{Th}_{\mathcal{M}}(A)$ and so there exists $\mathcal{R}$ a $\Sigma(A)$-model of $\mathrm{Th}_{\mathcal{M}}(A)$ that realises p. Since $\mathrm{Th}_{\mathcal{M}}(A)$ is complete, and both $\mathcal{M}$ and $\mathcal{R}$ are $\Sigma(A)$-models of it, $\mathcal{M}$ and $\mathcal{R}$ are elementarily equivalent. Hence they have a common elementary extension $\mathcal{N}$. Since $\mathcal{R}$ realises p and the extension $\mathcal{R} \to \mathcal{N}$ is elementary $\mathcal{N}$ realises p. Hence we have an elementary $\Sigma(A)$-extension of $\mathcal{M}$ that realises p. □

Lemma – Types are preserved downwards in elementary embeddings

Suppose $\iota : \mathcal{M} \to \mathcal{N}$ is an elementary Σ-embedding and $A \subseteq \mathcal{M}$ is a finite subset. Consider $p \in F(\Sigma(\iota(A)), n)$. For convinience we write this as $q(A)$, where $q \in F(\Sigma, n+m)$, m is the cardinality of A. Then

$$q(\iota(A)) \in S_n(\mathrm{Th}_{\mathcal{N}}(\iota(A))) \Rightarrow q(A) \in S_n(\mathrm{Th}_{\mathcal{M}}(A))$$

Proof. Suppose $q(\iota(A)) \in S_n(\mathrm{Th}_{\mathcal{N}}(\iota(A)))$. Then there exists $\mathcal{R}$ a $\Sigma(\iota(A))$-model of $\mathrm{Th}_{\mathcal{N}}(\iota(A))$ that realises $q(\iota(A))$. It suffices to show that $\mathcal{R} \vDash_{\Sigma(A)} \mathrm{Th}_{\mathcal{M}}(A)$ and realises $q(A)$. Take the interpretation of a constant symbol $a \in A$ as the $\iota(a)^{\mathcal{R}}$, the interpretation in $\Sigma(\iota(A))$. Then for any $\phi \in \Sigma_{\text{for}}$ (with m free variables) and $a \in \mathcal{R}^m$

$$\mathcal{R} \vDash_{\Sigma(A)} \phi(A)(a) \Leftrightarrow \mathcal{R} \vDash_{\Sigma(\iota(A))} \phi(\iota(A))(a)$$

Let $\phi(A) \in \mathrm{Th}_{\mathcal{M}}(A)$ such that $\phi \in \Sigma_{\text{for}}$. Then $\mathcal{M} \vDash_{\Sigma(A)} \phi(A)$ hence $\mathcal{M} \vDash_{\Sigma} \phi(A)$ and as the embedding is elementary $\mathcal{N} \vDash_{\Sigma} \phi(\iota(A))$ and $\mathcal{N} \vDash_{\Sigma(\iota(A))} \phi(\iota(A))$. Hence $\phi(\iota(A)) \in \mathrm{Th}_{\mathcal{N}}(A)$ and $\mathcal{R} \vDash_{\Sigma(\iota(A))} \phi(\iota(A))$ and $\mathcal{R} \vDash_{\Sigma(A)} \phi(A)$. Thus $\mathcal{R} \vDash_{\Sigma(A)} \mathrm{Th}_{\mathcal{M}}(A)$

There exists $a \in \mathcal{R}^n$ such that $\mathcal{R} \vDash_{\Sigma(\iota(A))} q(\iota(A))(a)$. If $\phi \in q$ then $\mathcal{R} \vDash_{\Sigma(\iota(A))} \phi(\iota(A))(a)$ and $\mathcal{R} \vDash_{\Sigma(A)} q(A)(a)$. □

Definition – Embedding Chain, Elementary Chain [5]

Given $(I, \leq)$ a non-empty linear order and a functor $M : I \to \mathbb{M}\mathbf{od}(\Sigma)$ that sends each $\alpha \in I$ to a Σ-structure $\mathcal{M}(\alpha)$ and each $\alpha \leq \beta$ in I to a Σ-embedding $\uparrow_\alpha^\beta : \mathcal{M}(\alpha) \to \mathcal{M}(\beta)$, called a lift. Then we call the functor M an embedding chain of Σ-structures.

Furthermore if M only results in elementary Σ-embeddings then M is an elementary chain.

Proposition – ω-saturated elementary extensions [10]

Every Σ-structure $\mathcal{M}$ has an ω-saturated elementary extension.

Proof. Let $\mathcal{M}$ be a Σ-structure. We will inductively create an elementary chain of Σ-structures $\mathcal{M} = \mathcal{M}(0) \to \mathcal{M}(1) \to \cdots$ such that for each $\alpha \in \mathbb{N}$, every finite subset $A \subseteq \mathcal{M}(\alpha)$, every $n \in \mathbb{N}$ and every $p(A) \in S_n(\mathrm{Th}_{\mathcal{M}(\alpha)}(A))$, $p(\uparrow_\alpha^{\alpha+1} A)$ is realised in $\mathcal{M}(\alpha+1)$. Taking the direct limit gives us a Σ-structure $\mathcal{N}$ and elementary Σ-embeddings $\iota_\alpha : \mathcal{M}(\alpha) \to \mathcal{N}$ for each $\alpha \in \mathbb{N}$ that commute with the lifts from the chain. In particular we will have that $\mathcal{N}$ is an elementary Σ-extension of $\mathcal{M}$. For any finite subset $A \subseteq \mathcal{N}$, any $n \in \mathbb{N}$ and any $p(A) \in S_n(\mathrm{Th}_{\mathcal{N}}(A))$, since A is finite there exists $\alpha \in \mathbb{N}$ such that ${\iota_\alpha}^{-1}(A) \subseteq \mathcal{M}(\alpha)$ bijects with A. As types are preserved downwards in elementary embeddings $p(A) \in S_n(\mathrm{Th}_{\mathcal{N}}(A))$ implies $p(\iota_\alpha^{-1} A) \in$

$S_n(\mathrm{Th}_{\mathcal{M}(\alpha)}(\iota_\alpha^{-1}A))$. By construction we see that p is realised in $\mathcal{M}(\alpha+1)$ thus p is realised in $\mathcal{N}$ as $\iota_{\alpha+1}$ is elementary. This $\mathcal{N}$ is ω-saturated.

Furthermore, if $A \subseteq \mathcal{M}$, $n \in \mathbb{N}$ and $p(A) \in S_n(\mathrm{Th}_{\mathcal{M}})(A)$, then $p(\uparrow_0^1 A)$ is realised in $\mathcal{M}(1)$. Hence $p(\iota_0 A)$ is realised in $\mathcal{N}$ as ι_1 is elementary.

It suffices to build $\mathcal{M}(1)$ from $\mathcal{M}$. Let $A \subseteq \mathcal{M}$, $n \in \mathbb{N}$ and $p_{A,n} \in S_n(\mathrm{Th}_{\mathcal{M}}(A))$. $p_{A,n}$ is realised in some $\Sigma(A)$-extension $\mathcal{M}(p_{A,n})$. As $\mathcal{M}(p_{A,n})$ extends $\mathcal{M}$, it is naturally a $\Sigma(\mathcal{M})$-model of $\mathrm{ElDiag}(\Sigma, \mathcal{M})$. We create such a $\Sigma(\mathcal{M})$-structure for each A, n and p_{A_n}. Consider the set

$$\{\mathcal{M}(p_{A,n}) \mid A \subseteq \mathcal{M}, n \in \mathbb{N}, p_{A,n} \in S_n(\mathrm{Th}_{\mathcal{M}}(A))\}$$

Any two structures in the set are elementarily equivalent since they would both model $\mathrm{ElDiag}(\Sigma, \mathcal{M}) = \mathrm{Th}_{\mathcal{M}}(\mathcal{M})$ which is a complete $\Sigma(\mathcal{M})$-theory. Thus there exists a common $\Sigma(\mathcal{M})$-extension $\mathcal{M}(1)$ of all the structures in the set. In particular there exists an elementary $\Sigma(\mathcal{M})$-embedding $\uparrow_0^1 \colon \mathcal{M} \to \mathcal{M}(1)$ (for the existance of one intermediate structure take $A = \varnothing, n = 0$ and extend $\mathrm{Th}_{\mathcal{M}}(\varnothing)$ to a maximal Σ-theory).

$$\begin{tikzcd}
& & & \mathcal{M}(1) & & & \\
\mathcal{M}(\dots) \arrow[urrr] & \mathcal{M}(\dots) \arrow[urr] & \mathcal{M}(\dots) \arrow[ur] & \mathcal{M}(p_{A,n}) \arrow[u] & \mathcal{M}(\dots) \arrow[ul] & \mathcal{M}(\dots) \arrow[ull] & \mathcal{M}(\dots) \arrow[ulll] \\
& & & \mathcal{M} \arrow[ulll] \arrow[ull] \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] \arrow[urrr] & & &
\end{tikzcd}$$

To show that $\mathcal{M}(1)$ has the desired property let $A \subseteq \mathcal{M}$, $n \in \mathbb{N}$ and $p \in S_n(\mathrm{Th}_{\mathcal{M}}(A))$. By design there exists $\mathcal{M}(p)$ that realises p. Since $\mathcal{M}(1)$ is an elementary $\Sigma(\mathcal{M})$-extension of $\mathcal{M}(p)$ (we can turn this into a $\Sigma(A)$-embedding for things to make sense) we have that p is realised in $\mathcal{M}(1)$. $\square$

Lemma – Disjunctive normal form

Let ϕ be a quantifier free Σ-formula with variables indexed by S. Then there exist atomic Σ-formulas f_{ij} such that for any Σ-structure $\mathcal{M}$

$$\mathcal{M} \vDash_\Sigma \forall_{s \in S} v_s, \phi(v) \leftrightarrow \bigvee_{i \in I} \bigwedge_{j \in J_i} f_{ij}(v)$$

An immediate improvement to this is that there exist Σ-formulas f_{ij}, g_{ij} of the form $s = t$ or $r(t)$ (equality of terms or a relation) such that for any Σ-structure $\mathcal{M}$

$$\mathcal{M} \vDash_\Sigma \forall_{s \in S} v_s, \phi(v) \leftrightarrow \bigvee_{i \in I} \left(\bigwedge_{j \in J_{i0}} f_{ij}(v) \wedge \bigwedge_{j \in J_{i1}} \neg g_{ij}(v) \right)$$

Proof. We induct on ϕ:

- If ϕ is $\top$ then we take a single 'or' $I = \{0\}$ and the empty 'and' for J_0.
- If ϕ is $t = s$ then let $I = J_0 = \{0\}$ and take f_{00} to be $t = s$.
- If ϕ is $r(t)$ then let $I = J_0 = \{0\}$ and take f_{00} to be $r(t)$.
- If ϕ is $\neg\psi$ and there exist the relevant things for ψ that satisfy

$$\mathcal{M} \vDash_\Sigma \forall_{s \in S} v_s, \psi(v) \leftrightarrow \bigvee_{i \in I} \bigwedge_{j \in J_i} f_{ij}(v)$$

for all Σ-structures $\mathcal{M}$ and for all $a \in \mathcal{M}^S$. We can negate and rearrange this statement to give us what we want for ϕ. The gist of the induction goes as follows:

$$
\begin{aligned}
&\neg \bigvee_{i\in I}\bigwedge_{j\in J_i} f_{ij}(v) \leftrightarrow \bigwedge_{i\in I}\bigvee_{j\in J_i} \neg f_{ij}(v) \\
\leftrightarrow &\left(\bigvee_{j\in J_0} \neg f_{0j}\right) \wedge \left(\bigwedge_{0\neq i\in I}\bigvee_{j\in J_i} \neg f_{ij}(v)\right) \\
\leftrightarrow &\bigvee_{k_0\in J_0}\left[\neg f_{0k_0} \wedge \bigwedge_{0\neq i\in I}\bigvee_{j\in J_i} \neg f_{ij}(v)\right] \text{ (induction on } J_0) \\
\leftrightarrow &\bigvee_{k_0\in J_0}\left[\neg f_{0k_0} \wedge \bigvee_{k_1\in J_1}\left(\neg f_{1k_1} \wedge \bigwedge_{i\neq 0,1}\bigvee_{j\in J_i} \neg f_{ij}(v)\right)\right] \\
\leftrightarrow &\bigvee_{k_0\in J_0}\bigvee_{k_1\in J_1}\left(\neg f_{0k_0} \wedge \neg f_{1k_1} \wedge \bigwedge_{i\neq 0,1}\bigvee_{j\in J_i} \neg f_{ij}(v)\right) \\
\leftrightarrow &\ldots \text{ induction on } I \leftrightarrow \bigvee_{k_0\in J_0} \cdots \bigvee\left[\bigwedge g_{ij}\right]
\end{aligned}
$$

- If ϕ is $\chi_0 \vee \chi_1$ and there exist the relevant things for χ_0 and χ_1, then we can simply take the or of the two formulas found and obtain what we want.

$\square$

Lemma – Type of an element from the model is isolated

Let T be a complete Σ-theory and $\mathcal{M}$ be a non-empty Σ-model of T. Any type over $\mathcal{M}$ is of the form[†] $\mathrm{tp}^{\mathcal{N}}_{\varnothing,1}(a)$, where $\mathcal{N}$ is an ω-saturated elementary extension of $\mathcal{M}$ and $a \in \mathcal{N}$. Then a is in the image of $\mathcal{M}$ if and only if $\mathrm{tp}(a)$ is an isolated point in $S_1(\mathcal{M})$. Thus $\mathcal{M}$ bijects with the isolated points of $S_1(\mathcal{M})$ by taking $c \mapsto \{\mathrm{tp}(io(c))\}$ and the derived set is

$$S_1(\mathcal{M})' = S_1(\mathcal{M}) \setminus \bigcup_{c\in\mathcal{M}} [x = c] = \bigcap_{c\in\mathcal{M}} [x \neq c]$$

[†]Here we have $\varnothing$ in $\mathrm{tp}^{\mathcal{N}}_{\varnothing,1}(a)$ as we are working in the theory $\mathrm{ElDiag}(\Sigma, \mathcal{M})$, which is in the language $\Sigma_{\mathrm{ER}}(\mathcal{M})$, thus we don't need to add symbols from $\mathcal{M}$.

Proof. ($\Rightarrow$) Suppose a is in the image of $\mathcal{M}$. Then we have $c \in \mathcal{M}$ such that $a = c^{\mathcal{N}}$. Hence $x = c$ is an element of $\mathrm{tp}(a)$ and so $\mathrm{tp}(a) \in [x = c]$. If $q \in [x = c]$ then as q is realised by some $b \in \mathcal{N}$, $b = c^{\mathcal{N}} = a$, which implies $q = \mathrm{tp}(a)$. Thus the open set $[x = c]$ is a singleton and the point is isolated. Thus $c \mapsto \mathrm{tp}(\iota(c))$ has image a subset of the isolated points. It is injective since for $c, d \in \mathcal{M}$, if $\mathrm{tp}(\iota(c)) = \mathrm{tp}(\iota(d))$ then $x = c \in \mathrm{tp}(\iota(c)) = \mathrm{tp}(\iota(d))$ and so $\iota(d) = \iota(c)$ which implies $c = d$ as ι is injective. ($\Leftarrow$) On the other hand, suppose there is an isolated point p in $S_1(\mathcal{M})$, then $\{p\}$ is open. This open set is equal to $\bigcup_{\phi\in I}[\phi]$ as a finite intersection of clopen sets is clopen, hence $\psi \in I$ such that $p \in [\psi]$, whence $[\psi] = \{p\}$. As $\psi \in p$ (which is consistent with T) we have that $\exists x, \psi(x)$ is realised in some model of T (any elementary extension of $\mathcal{M}$ is a model of T). Since T is complete this implies $T \vDash_{\Sigma_{\mathrm{ER}}} \exists x, \psi(x)$ In particular $\mathcal{M}$ is a model of T and so there exists $a \in \mathcal{M}$ such that $\mathcal{M} \vDash_{\Sigma_{\mathrm{ER}}} \psi(a)$. Thus $\mathcal{N} \vDash_{\Sigma_{\mathrm{ER}}} \psi(\iota(a))$ so $\psi \in \mathrm{tp}(\iota(a))$ and $\mathrm{tp}(\iota(a)) \in [\psi]$. Since there is only one element in this set $p = \mathrm{tp}(\iota(a))$. Thus $c \mapsto \{\mathrm{tp}(\iota(c))\}$ is surjective. $\square$

Example (Infinite infinite equivalence classes revisited). *Consider again the theory T of infinite infinite equivalence classes, and $\mathcal{M}$ a non-empty Σ_{ER}-model of T. We wish to classify all the types over $\mathcal{M}$, i.e. the* *elements of $S_1(\mathcal{M}) := S_1(\mathrm{ElDiag}(\Sigma, \mathcal{M}))$. We know that any type over $\mathcal{M}$ is of the form $\mathrm{tp}^{\mathcal{N}}_{\varnothing,1}(a)$, where $\mathcal{N}$ is the ω-saturated elementary extension of $\mathcal{M}$ and $a \in \mathcal{N}$. We show that there are three cases:*

1. *The element a is in the image of $\mathcal{M}$ if and only if $\mathrm{tp}(a)$ is an isolated point in $S_1(\mathcal{M})$. Thus $\mathcal{M}$ bijects with the isolated points of $S_1(\mathcal{M})$ by taking $c \mapsto \{\mathrm{tp}(io(c))\}$.*
2. *The element a is not in the image but it equivalent under the relation to something in the image if and only if $\mathrm{tp}(a)$ is isolated in $S_1(\mathcal{M})'$, the derived set of the Stone space.*
3. *Otherwise it is an isolated point in $S_1(\mathcal{M})''$ the second derived set. There is exactly one such type.*

$$S_1(\mathcal{M}) = \{\mathrm{tp}(a) \mid a \in \mathcal{M}\} \sqcup \{\mathrm{tp}(a) \mid a \notin \mathcal{M} \wedge \exists c \in \mathcal{M}, a \sim c\} \sqcup \{\mathrm{tp}(a) \mid \forall c \in \mathcal{M}, a \nsim c\}$$

Hence $|\mathcal{M}| \leq |\mathcal{M}| + |\mathcal{M}| + 1 \leq |S_1(\mathcal{M})|$. By (1.), $\mathcal{M}$ bijects with a subset of $S_1(\mathcal{M})$ and so $|\mathcal{M}| = |S_1(\mathcal{M})|$.

Proof. (1.) This is covered in the lemma.

(2.) Suppose a is not in the image of $\mathcal{M}$ but is equivalent to an element of $c \in \mathcal{M}$. Then

$$\mathrm{tp}(a) \in \bigcap_{d \in \mathcal{M}} [x \neq d] \cap [x \sim c]$$

Notice that we cannot make the intersection a formula because this would then be quantifying over $\mathcal{N}$ instead of $\mathcal{M}$, also this intersection is infinite as $\mathcal{M}$ is infinite. To show that this is the only element of this set, suppose $\mathrm{tp}(b)$ also satisfies

$$(\forall d \in \mathcal{M}, x \neq d \in \mathrm{tp}(b)) \text{ and } x \sim c \in \mathrm{tp}(b)$$

then let $\phi \in QFF(\Sigma(\mathcal{M}), 1)$. We case on ϕ to show that $\phi \in \mathrm{tp}(a) \Leftrightarrow \phi \in \mathrm{tp}(b)$. By quantifier elimination of T and the types version of quantifier elimination it suffices to show that $\mathrm{qftp}(a) = \mathrm{qftp}(b)$. (for each step we only show $\phi \in \mathrm{qftp}(a) \Rightarrow \phi \in \mathrm{qftp}(b)$ the other direction is the same)

- If $\phi = \top$ then it is trivial.
- If ϕ is $r = s$ then as the only $\Sigma_{\mathrm{ER}}(\mathcal{M})$-terms that exist are constants $d_i \in \mathcal{M}$ or variables x_i, we case on r and s. If both are constants d_0, d_1 then since ϕ is in $\mathrm{qftp}(a)$, $d_0^{\mathcal{N}} = d_1^{\mathcal{N}}$ and so ϕ is in $\mathrm{qftp}(b)$. If one is a constant d and the other a variable x then $d^{\mathcal{N}} = x^{\mathcal{N}}(a) = a$ which is a contradiction as $x \neq d \in \mathrm{qftp}(a)$. If both are variables then as ϕ has at most one variable it is $x = x$ which is clearly in $\mathrm{tp}(b)$.
- If ϕ is $r \sim s$ then we again case on what the terms are. If they are constants d_0, d_1 we have $d_0^{\mathcal{N}} \sim^{\mathcal{N}} d_1^{\mathcal{N}}$ and so ϕ is in $\mathrm{tp}(b)$. If one is a constant d and the other a variable x then by the fact that $x \neq d \in \mathrm{qftp}(a) \cap \mathrm{qftp}(b)$

  $$d^{\mathcal{N}} \sim^{\mathcal{N}} x^{\mathcal{N}}(a) = a \sim^{\mathcal{N}} c \sim^{\mathcal{N}} b$$

  Hence $d \sim x \in \mathrm{qftp}(b)$. If both are variables then as above it is $x \sim x$ which is in $\mathrm{tp}(b)$.
- If ϕ is $\neg\psi$ then $\mathcal{N} \vDash_{\Sigma_{\mathrm{ER}}(\mathcal{M})} \phi(a)$ implies $\mathcal{N} \nvDash_{\Sigma_{\mathrm{ER}}(\mathcal{M})} \psi(a)$ which by the induction hypothesis tells us $\mathcal{N} \nvDash_{\Sigma_{\mathrm{ER}}(\mathcal{M})} \psi(b)$, whence $\phi(b) \in \mathrm{qftp}(b)$.
- If ϕ is $\psi \vee \chi$ then $\mathcal{N} \vDash_{\Sigma_{\mathrm{ER}}(\mathcal{M})} \psi(a)$ or $\mathcal{N} \vDash_{\Sigma_{\mathrm{ER}}(\mathcal{M})} \chi(a)$ and by the induction hypothesis $\mathcal{N} \vDash_{\Sigma_{\mathrm{ER}}(\mathcal{M})} \psi(b)$ or $\mathcal{N} \vDash_{\Sigma_{\mathrm{ER}}(\mathcal{M})} \chi(b)$ hence $\phi \in \mathrm{qftp}(b)$.

Thus (though a is not unique) $\mathrm{tp}(a)$ is the unique type satisfying this characterisation. Hence the intersection of the derived set and $[x \sim c]$ is $\{\mathrm{tp}(a)\}$, as any p in the derived set is (as we proved above) *not* due to some element from $\mathcal{M}$. Thus these are isolated in the derived set.

We prove a lemma: if ψ is a formula of one variable then $[\psi] \cap S_1(\mathcal{M})' = [x \sim d] \cap S_1(\mathcal{M})'$ for some $d \in \mathcal{M}$ or $S_1(\mathcal{M})' \subseteq [\psi]$. As T has quantifier elimination we can assume ψ is quantifier free. The disjunctive normal form of ψ gives us

$$[\psi] = \bigcup_{i \in I} \bigcap_{j \in J_i} [f_{ij}(v)]$$

There exists an i such that $\bigcap_{j\in J_i}[f_{ij}(v)]$ contains p hence

$$[\psi]\cap S_1(\mathcal{M})' = \bigcap_{j\in J_i}[f_{ij}(v)] \cap \bigcap_{c\in\mathcal{M}}[x\neq c]$$

If f_{ij} is of the form $r=s$ then if r and s are both constant symbols or both variable symbols it is true and we can remove it, otherwise it is $x=c$ for some $c\in\mathcal{M}$, which implies that p contains $x=c$ and $x\neq c$, a contradiction. If f_{ij} is of the form $r\neq s$ then if r and s are both constant symbols or both variable symbols it is either true and we can remove it, or false and we have a contradiction; otherwise it is $x\neq c$ for some $c\in\mathcal{M}$, which is already given in the second intersection and we can remove it. If f_{ij} is $r\nsim s$ then $r\neq s$ is also in p so by the previous point we can remove it. Thus the only remaining case is when the f_{ij} are $r\sim s$, again we can remove the cases where r and s are both constants or variables and we are left with $x\sim c$ for some $c\in\mathcal{M}$. If there is more than one of these we can remove them as they either contradict one another (via transitivity) or they are redundant information and can be removed. Hence we are left with either $[\psi]\cap S_1(\mathcal{M})' = [x\sim d]\cap S_1(\mathcal{M})'$ for some $d\in\mathcal{M}$ or $[\psi]\cap S_1(\mathcal{M})' = S_1(\mathcal{M})'$. In the second case $S_1(\mathcal{M})'\subseteq[\psi]$.

Using the lemma we show that for any isolated point $\mathrm{tp}(a)$ in the derived set $S_1(\mathcal{M})'$, there exists $d\in\mathcal{M}$ such that $a\sim d$. If it is isolated

$$\{\mathrm{tp}(a)\} = \bigcup_{\phi\in I}[\phi]\cap S_1(\mathcal{M})'$$

which implies there is some ψ such that

$$\{\mathrm{tp}(a)\} = [\psi]\cap S_1(\mathcal{M})'$$

By the lemma, either $\{\mathrm{tp}(a)\} = [x\sim d]\cap\bigcap_{c\in\mathcal{M}}[x\neq c]$ for some $d\in\mathcal{M}$ or $\{\mathrm{tp}(a)\}=\bigcap_{c\in\mathcal{M}}[x\neq c]$. In the second case we have a contradiction as for two elements d,ein distinct equivalence classes of $\mathcal{M}$ $\bigcap_{d\in\mathcal{M}}[x\neq d]\cap[x\sim c]$ and $\bigcap_{e\in\mathcal{M}}[x\neq e]\cap[x\sim c]$ contain distinct types and are both subsets of $\bigcap_{c\in\mathcal{M}}[x\neq c]$, which is a singleton set. Thus there exists $d\in\mathcal{M}$ such that $x\sim d\in\mathrm{tp}(a)$.

(3.) There is the final case where a is not equivalent to anything from $\mathcal{M}$ (therefore not in the image). By the above two cases we have that $\mathrm{tp}(a)$ is not isolated in $S_1(\mathcal{M})$ or $S_1(\mathcal{M})'$. We show that there is exactly one such type, which implies $S_1(\mathcal{M})''$ consists of one isolated point. $\mathcal{M}$ has infinitely many equivalence classes and so the set $\Gamma := \{x\nsim c \mid c\in\mathcal{M}\}$ is finitely realised in $\mathcal{M}$, hence consistent with the elementary diagram of $\mathcal{M}$, which implies it can be extended to a maximal type. Hence there exists a type $\mathrm{tp}(a)$ such that a is not equivalent to any thing from $\mathcal{M}$. There is only one: Suppose $\mathrm{tp}(a)$ and $\mathrm{tp}(b)$ both satisfy the above. Let $\phi\in\mathrm{tp}(a)$. By the lemma $[\psi]\cap S_1(\mathcal{M})' = [x\sim d]\cap S_1(\mathcal{M})'$ for some $d\in\mathcal{M}$ or $S_1(\mathcal{M})'\subseteq[\psi]$. The first case is false since it implies $\mathrm{tp}(a)\in[\psi]\cap S_1(\mathcal{M})'\subseteq[x\sim d]$ but $x\nsim d$ is in $\mathrm{tp}(a)$ by assumption. Thus $\mathrm{tp}(b)\in S_1(\mathcal{M})'\subseteq[\psi]$ and so $\psi\in\mathrm{tp}(b)$. The other direction is the same and so $\mathrm{tp}(a)=\mathrm{tp}(b)$ and this point is unique. $\square$

1.5.2 Morley Rank

This subsection follows Marker's [5] material again.

Definition – Definable

Let $\mathcal{M}$ be a Σ-structure and $A\subseteq\mathcal{M}$. We say $X\subseteq\mathcal{M}^0$ is $\Sigma(A)$-definable if X is non-empty. In the non-degenerate case: $X\subseteq\mathcal{M}^n$ is $\Sigma(A)$-definable over $\mathcal{M}$ if there exists a $\Sigma(A)$-formula in n free variables such that

$$X = \{a\in\mathcal{M}^n \mid \mathcal{M}\vDash_\Sigma \phi(a)\}$$

For a $\Sigma(A)$-formula ϕ with n free variables we use $\phi(\mathcal{M})$ to denote $\{a\in\mathcal{M}^n \mid \mathcal{M}\vDash_{\Sigma(\mathcal{M})}\phi(a)\}$, the set defined by ϕ.

Note that if ϕ is a sentence then $\phi(\mathcal{M}) := \{\varnothing\}$ when ϕ is satisfied by $\mathcal{M}$ and it is empty otherwise. For now we only concern outselves with the case where $A = \mathcal{M}$.

Definition – Morley rank with respect to a structure (not necessarily saturated)
Let $\mathcal{M}$ be a Σ-structure, let α be an ordinal. By transfinite induction of α we define what it means for any $\Sigma(\mathcal{M})$-formula ϕ to satisfy $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$:

- If $\phi(\mathcal{M})$ is non-empty then 0 is in R.
- If α is a non-zero limit ordinal and for each $\beta < \alpha$ and ϕ, $\beta \leq \mathrm{MR}^{\mathcal{M}}(\phi)$ then $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$.
- Suppose $\alpha + 1$ is a successor ordinal. If there exists for each $n \in \mathbb{N}$ a $\Sigma(\mathcal{M})$-formula ψ_n satisfying $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\psi_n)$ such that the $\psi_n(\mathcal{M})$ are pairwise disjoint subsets of $\phi(\mathcal{M})$, then $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$. †

We then define $\mathrm{MR}^{\mathcal{M}}(\phi) \in \{-\infty, \infty\} \cup \mathrm{Ord}$, the Morley rank of ϕ:

- If for each ordinal α, $\alpha \not\leq \mathrm{MR}^{\mathcal{M}}(\phi)$ then we take $\mathrm{MR}^{\mathcal{M}}(\phi) := -\infty$.
- If for all ordinals α, $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$ then we take $\mathrm{MR}^{\mathcal{M}}(\phi) := \infty$.
- If there exists an ordinal α such that $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$ but $\alpha + 1 \not\leq \mathrm{MR}^{\mathcal{M}}(\phi)$, then $\mathrm{MR}^{\mathcal{M}}(\phi) := \alpha$.

†The relations $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\psi_n)$ is already defined by induction. Each ψ_n has the same number of variables as ϕ since we require $\psi_n(\mathcal{M}) \subseteq \phi(\mathcal{M})$.

Proof. To show that the second part of the definition is well-defined we need a lemma: By induction on α we show that if $\beta \leq \alpha$, then for any $\Sigma(\mathcal{M})$-formula ϕ, $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$ implies $\beta \leq \mathrm{MR}^{\mathcal{M}}(\phi)$.

- If $\alpha = 0$ then it is vaccuously true.
- Suppose α is a non-zero limit ordinal. If $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$, $\beta < \alpha$ and $\phi \in \Sigma(\mathcal{M})_{\mathrm{for}}$ then by definition of $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$ for non-zero limit ordinals we have $\beta \leq \mathrm{MR}^{\mathcal{M}}(\phi)$.
- Suppose α satisfies the condition. We show that $\alpha + 1$ satisfies the condition as well. Let ϕ be a $\Sigma(\mathcal{M})$-formula. Suppose $\alpha + 1 \leq \mathrm{MR}^{\mathcal{M}}(\phi)$. If $\beta \leq \alpha + 1$ then either $\beta = \alpha$ or $\beta < \alpha$. It suffices to show the first case - $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$ - as by the induction hypothesis this implies that for any $\beta < \alpha$ we also have $\beta \leq \mathrm{MR}^{\mathcal{M}}(\phi)$, which covers the second case. By definition we have $\Sigma(\mathcal{M})$-formulas ψ_n for each natural n with $\psi_n(\mathcal{M})$ disjoint subset of $\phi(\mathcal{M})$ and $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\psi_n)$. This is not quite what we want so we show $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$ by induction on α once again.
  - If $\alpha = 0$ then since $\psi_0(\mathcal{M}) \subseteq \phi(\mathcal{M})$ and $\psi_0(\mathcal{M}) \neq \varnothing$ by the fact that $0 \leq \mathrm{MR}^{\mathcal{M}}(\psi_n)$, we have that $\phi(\mathcal{M}) \neq \varnothing$ and so $0 \leq \mathrm{MR}^{\mathcal{M}}(\phi)$.
  - If α is a non-zero ordinal and all $\beta < \alpha$ satisfy $\beta \leq \mathrm{MR}^{\mathcal{M}}(\phi)$ then clearly $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$.
  - If $\alpha = \beta + 1$ then by the original induction hypothesis, we have for each $n \in \mathbb{N}$,
  $$\alpha \leq \mathrm{MR}^{\mathcal{M}}(\psi_n) \Rightarrow \beta \leq \mathrm{MR}^{\mathcal{M}}(\psi_n)$$
  Thus $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$.

To show then that Morley rank is well-defined: if the first and second cases don't hold, we can find the minimal element $\alpha \in \mathrm{Ord}$ such that $\alpha \not\leq \mathrm{MR}^{\mathcal{M}}(\phi)$ as Ord is well-ordered. This is not a limit ordinal, since if β less than it satisfy $\beta \leq \mathrm{MR}^{\mathcal{M}}(\phi)$ and it it a limit ordinal then $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi)$, a contradiction. Thus it must be a successor and so we can find its predecessor. This is unique: suppose for a contradiction that $\alpha < \beta$ satisfy
$$\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi), \alpha + 1 \not\leq \mathrm{MR}^{\mathcal{M}}(\phi), \beta \leq \mathrm{MR}^{\mathcal{M}}(\phi), \beta + 1 \not\leq \mathrm{MR}^{\mathcal{M}}(\phi)$$
then $\alpha + 1 < \beta$ or $\alpha + 1 = \beta$, so using the lemma above for the first case, we have in either case $\alpha + 1 \leq \mathrm{MR}^{\mathcal{M}}(\phi)$, which is a contradiction. Thus $\alpha = \beta$ and this ordinal is unique. ☐

Remark. We tacitely order $\{-\infty, \infty\} \cup \mathrm{Ord}$ by the usual ordering on Ord together with $-\infty \leq$ everything $\leq \infty$. We can then compare the Morley rank of two formulas by $\leq$.

> **Lemma – Morley rank for elementary extensions between saturated structures**
> If $\mathcal{M}$ and $\mathcal{N}$ are both ω saturated Σ-structures and $\mathcal{M} \to \mathcal{N}$ is an elementary Σ-extension, then for any $\Sigma(\mathcal{M})$-formula ϕ,
> $$\mathrm{MR}^{\mathcal{M}}(\phi) = \mathrm{MR}^{\mathcal{N}}(\phi)$$
> (where in the second case ϕ is considered to be a $\Sigma(\mathcal{N})$-formula).

Proof. Again it suffices to show by induction that for each $\alpha \in \mathrm{Ord}$, given $\mathcal{M}$ and $\mathcal{N}$ and ϕ as above, we have

$$\alpha \leq \mathrm{MR}^{\mathcal{M}}(\phi) \quad \Leftrightarrow \quad \alpha \leq \mathrm{MR}^{\mathcal{N}}(\phi)$$

The 0 case:

$$\begin{aligned} & 0 \leq \mathrm{MR}^{\mathcal{M}}(\phi) \\ \Leftrightarrow & \mathcal{M} \vDash_{\Sigma(\mathcal{M})} \exists v, \phi(v) \\ \Leftrightarrow & \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \exists v, \phi(v) \\ \Leftrightarrow & 0 \leq \mathrm{MR}^{\mathcal{N}}(\phi) \end{aligned}$$

The non-zero limit ordinal case is standard.

Successor: With the induction hypothesis for α, suppose $\alpha + 1 \leq \mathrm{MR}^{\mathcal{M}}(\phi)$ then there exist ψ_n for each $n \in \mathbb{N}$ such that $\alpha \leq \mathrm{MR}^{\mathcal{M}}(\psi_n)$, $\psi_n(\mathcal{M}) \subseteq \phi(\mathcal{M})$ and these sets are pairwise disjoint. By the induction hypothesis we have for each n that $\alpha \leq \mathrm{MR}^{\mathcal{N}}(\psi_n)$. Since the embedding is elementary

$$\begin{aligned} & \psi_n(\mathcal{M}) \subseteq \phi(\mathcal{M}) \\ \Rightarrow & \mathcal{M} \vDash_{\Sigma} \forall v, \psi_n \to \phi \\ \Rightarrow & \mathcal{N} \vDash_{\Sigma} \forall v, \psi_n \to \phi \\ \Rightarrow & \psi_n(\mathcal{N}) \subseteq \phi(\mathcal{N}) \end{aligned}$$

Similarly

$$\begin{aligned} & \psi_n(\mathcal{M}) \cap \psi_l(\mathcal{M}) = \varnothing \\ \Rightarrow & \mathcal{M} \vDash_{\Sigma} \forall v, \neg(\psi_n \wedge \phi_{n+1}) \\ \Rightarrow & \mathcal{N} \vDash_{\Sigma} \forall v, \neg(\psi_n \wedge \phi_{n+1}) \\ \Rightarrow & \psi_n(\mathcal{N}) \cap \psi_l(\mathcal{N}) = \varnothing \end{aligned}$$

Hence $\alpha + 1 \leq \mathrm{MR}^{\mathcal{N}}(\phi)$.

For the other direction assume the induction hypothesis and suppose $\alpha + 1 \leq \mathrm{MR}^{\mathcal{N}}(\phi)$. First replace ϕ with a Σ-formula $\phi(v, a)$ at some $a \in \mathcal{M}^i$. For each $n \in \mathbb{N}$ there exist Σ-formulas ψ_n and $c_n \in \mathbb{N}^{i_n}$ such that $\alpha \leq \mathrm{MR}^{\mathcal{N}}(\psi_n(v, c_n))$ and $\psi_n(\mathcal{N}, c_n)$ are disjoint subset of $\phi(\mathcal{N}, a)$. Since the extension is elementary $\mathrm{tp}^{\mathcal{M}}(a) = \mathrm{tp}^{\mathcal{N}}(a)$ and by ∞-equivalence for each n there exist $d_0 \in \mathcal{M}^{i_0}, \dots, d_n \in \mathcal{M}^{i_n}$ such that

$$\mathrm{tp}^{\mathcal{N}}(a, c_0, \dots, c_n) = \mathrm{tp}^{\mathcal{M}}(a, d_0, \dots, d_n) = \mathrm{tp}^{\mathcal{N}}(a, d_0, \dots, d_n)$$

Similarly to before, for each $n \in \mathbb{N}$

$$\begin{aligned} & \psi_n(\mathcal{N}, c_n) \subseteq \phi(\mathcal{N}, a) \\ \Leftrightarrow & \mathcal{N} \vDash_{\Sigma} \forall v, \psi_n(v, c_n) \to \phi(v, a) \\ \Leftrightarrow & \forall v, \psi_n(v, c_n) \to \phi(v, a) \in \mathrm{tp}^{\mathcal{N}}(a, c_n) = \mathrm{tp}^{\mathcal{M}}(a, d_n) \\ \Leftrightarrow & \mathcal{M} \vDash_{\Sigma} \forall v, \psi_n(v, d_n) \to \phi(v, a) \\ \Leftrightarrow & \psi_n(\mathcal{M}, d_n) \subseteq \phi(\mathcal{M}, a) \end{aligned}$$

They are disjoint:

$$\begin{aligned}
&\psi_n(\mathcal{N}, c_n) \cap \psi_l(\mathcal{N}, c_l) = \varnothing \\
\Leftrightarrow &\mathcal{N} \not\models_\Sigma \exists v, \psi_n(v, c_n) \wedge \psi_l(v, c_l) \\
\Leftrightarrow &\exists v, \psi_n(v, w_n) \wedge \psi_l(v, w_l) \notin \mathrm{tp}(a, c_n, c_l) = \mathrm{tp}(b, d_n, d_l) \\
\Leftrightarrow &\mathcal{M} \not\models_\Sigma \exists v, \psi_n(v, d_n) \wedge \psi_l(v, d_l) \\
\Leftrightarrow &\psi_n(\mathcal{M}, d_n) \cap \psi_l(\mathcal{M}, d_l) = \varnothing
\end{aligned}$$

Thus the induction is complete. □

Corollary – Saturated elementary extensions give equal Morley rank

Let $\mathcal{A}, \mathcal{M}, \mathcal{N}$ be Σ-structures such that $\mathcal{M}$ and $\mathcal{N}$ are ω-saturated elementary Σ-extensions of $\mathcal{A}$. Then for any $\Sigma(\mathcal{A})$-formula ϕ,

$$\mathrm{MR}^{\mathcal{M}}(\phi) = \mathrm{MR}^{\mathcal{N}}(\phi)$$

Proof. Let $\mathcal{L}$ be the amalgamation of $\mathcal{M}$ and $\mathcal{N}$, and let Ω be an ω-saturated elementary extension of $\mathcal{L}$. The composition of elementary extensions is elementary and so $\mathcal{M} \to \Omega$ and $\mathcal{N} \to \Omega$ is elementary. Hence by the previous lemma we have that

$$\mathrm{MR}^{\mathcal{M}}(\phi) = \mathrm{MR}^{\Omega}(\phi) = \mathrm{MR}^{\mathcal{N}}(\phi)$$

□

Proposition – Subset implies inequality of Morley rank

Let ϕ and ψ be two $\Sigma(\mathcal{M})$-formulas and $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. If $\psi(\mathcal{N}) \subseteq \phi(\mathcal{N})$ then

$$\mathrm{MR}^{\mathcal{N}}(\psi) \leq \mathrm{MR}^{\mathcal{N}}(\phi)$$

Hence if X is defined by two formulas then the Morley rank of the two formulas in some $\mathcal{N}$ is the same.

$$\phi(\mathcal{N}) = X = \psi(\mathcal{N}) \quad \Rightarrow \quad \mathrm{MR}^{\mathcal{N}}(\phi) = \mathrm{MR}^{\mathcal{N}}(\psi)$$

Proof. Induct on $\alpha \in \{-\infty\} \cup \mathrm{Ord}$ to show that

$$\alpha \leq \mathrm{MR}^{\mathcal{N}}(\psi) \Rightarrow \alpha \leq \mathrm{MR}^{\mathcal{N}}(\phi)$$

The $-\infty$ case is clear. The 0 case follows from noting that if $0 \leq \mathrm{MR}^{\mathcal{N}}(\psi)$ then $\varnothing \neq \psi(\mathcal{N}) \subseteq \phi(\mathcal{N})$. The non-zero limit case is the same as usual. The successor case follows from noting that the ψ_i given for each $i \in \mathbb{N}$ define subsets of $\psi(\mathcal{N})$ and therefore subsets of $\phi(\mathcal{N})$.

Now that we have that it is true for all ordinals the ∞ case also follows.

The 'hence' uses anti-symmetry of the ordering on $\{-\infty, \infty\} \cup \mathrm{Ord}$. □

Definition – Morley rank in saturated extensions

Let $\mathcal{M}$ be a Σ-structure and ϕ a $\Sigma(\mathcal{M})$-formula. Then $\mathrm{MR}(\phi) := \mathrm{MR}^{\mathcal{N}}(\phi)$ for $\mathcal{N}$ any ω-saturated extension of $\mathcal{M}$. This is unique with respect to the choice of $\mathcal{N}$ since saturated elementary extensions give equal Morley rank. Suppose $X = \phi(\mathcal{M})$ is the subset of $\mathcal{M}^n$ defined by ϕ. Then we write $\mathrm{MR}(X) = \mathrm{MR}(\phi)$ to be the Morley rank of the $\Sigma(\mathcal{M})$-definable set. It is unique with respect to the choice of ϕ since equal $\Sigma(\mathcal{M})$-definable sets implies equal Morley rank.

Note that for an ω-saturated structure $\mathcal{M}$ and ϕ a $\Sigma(\mathcal{M})$-formula

$$\mathrm{MR}(\phi) = \mathrm{MR}^{\mathcal{M}}(\phi)$$

since $\mathcal{M}$ is an ω-saturated extension of itself.

Proposition – Basic facts about Morley rank of a definable set
Let X and Y be $\Sigma(\mathcal{M})$-definable sets in $\mathcal{M}^n$ a Σ-structure.

1. If $X \subseteq Y$ then $\mathrm{MR}(X) \leq \mathrm{MR}(Y)$.
2. $\mathrm{MR}(X) = 0$ if and only if X is finite and non-empty.
3. $\mathrm{MR}(X \cup Y) = \max\{\mathrm{MR}(X), \mathrm{MR}(Y)\}$.

Proof. (1.) Follows from the previous lemma.

(2.) First note that $0 \leq \mathrm{MR}(X)$ if and only if the formula defining X is satisfied if and only if X is non-empty. It remains to show that $1 \not\leq \mathrm{MR}(X)$ if and only if X is finite. ($\Rightarrow$) If X were infinite, we can let ϕ be the formula defining X and define formulas $\phi \wedge x = a$ for each $a \in X$, hence having infinitely many satisfiable formulas corresponding to disjoint subsets of X. Hence $1 \leq \mathrm{MR}(X)$, a contradiction.

($\Leftarrow$) If $1 \leq \mathrm{MR}(X)$ then we would have infinitely many disjoint subsets of X, which implies X is infinite, a contradiction.

(3.) It suffices to show that for all X and Y $\Sigma(\mathcal{M})$-definable subsets of $\mathcal{M}$ if $\mathrm{MR}(X) \leq \mathrm{MR}(Y)$ then $\mathrm{MR}(Y) = \mathrm{MR}(X \cup Y)$. By the first part we have that $\mathrm{MR}(Y) \leq \mathrm{MR}(X \cup Y)$, thus we need only to show the other inequality by induction on $\alpha \in \{-\infty\} \cup \mathrm{Ord}$:

$$\alpha \leq \mathrm{MR}(X \cup Y) \Rightarrow \alpha \leq \mathrm{MR}(Y)$$

For $\alpha = -\infty$ then it is clear. For $\alpha = 0$ suppose $0 \leq \mathrm{MR}(X \cup Y)$. Then $X \cup Y$ is non empty. Suppose for a contradiction Y is empty then X must be non-empty and we have that $\mathrm{MR}(Y) < \mathrm{MR}(X)$, a contradiction. Hence $0 \leq \mathrm{MR}(Y)$.

The non-zero limit ordinal case is trivial as usual. Suppose it is true for α and suppose $\alpha + 1 \leq \mathrm{MR}(X \cup Y)$. Then there exist disjoint subsets $S_i \subseteq X \cup Y$ such that $\alpha \leq \mathrm{MR}(S_i)$. Consider the subsets $S_i \cap X \subseteq X$ and $S_i \cap Y \subseteq Y$. If infinitely many $S_i \cap Y$ satisfy $\alpha \leq \mathrm{MR}(S_i \cap Y)$ then we are done. On the other hand if only finitely many do so then by induction

$$\alpha \leq \mathrm{MR}(S_i) = \max\{\mathrm{MR}(S_i \cap X), \mathrm{MR}(S_i \cap Y)\}$$

Hence infinitely many satisfy $\alpha \leq \mathrm{MR}(S_i \cap X)$. Thus $\alpha + 1 \leq \mathrm{MR}(X) \leq \mathrm{MR}(Y)$ and the induction is complete.

As usual the ∞ case follows from the result given by the induction. $\square$

Naturally parts 2. and 3. also have analogues in the formula version of Morley rank.

1.5.3 Morley degree

NOTATION. *Let A be a set. Then $A^{<\omega}$ is the set*

$$A^{<\omega} := \bigsqcup_{n \in \mathbb{N}} A^n$$

To be explicit we mean a disjoint union of the sets of maps (a.k.a n-tuples) $n \to A$.

For $a = (a_1, \dots, a_n) \in A^n \subseteq A^{<\omega}$ and $b \in A$ we write $a; b$ to mean a with b appended to it:

$$(a_1, \dots, a_n, b) \in A^{n+1} \subseteq A^{<\omega}$$

The following lemma has been adapted to be suitable without pre-requisites. For the general version see Proof Wiki [11].

Lemma – König's Tree

Partially order $2^{<\omega}$ by $t \leq s$ if and only if s restricts to the map t on the domain domain of t. Let $T \subseteq 2^{<\omega}$ be such that

- For any $s \in T$ and $t \in 2^{<\omega}$ if $t \leq s$ then $t \in T$. We call this property T being a 'binary tree'.
- T is countably infinite.

Then there exists an 'infinite branch' $b \in 2^{\mathbb{N}}$ (i.e. $b : \mathbb{N} \to 2$) such that for each $n \in \mathbb{N}$ the restriction $b\big|_n$ is in T.

Proof. By induction we construct for each $n \in \mathbb{N}$, $t_n \in 2^n$ such that

- $t_n \in T$
- for each $i < n$, $t_{i+1} \in D(t_i)$
- for each $i \leq n$, $D(t_i)$ is infinite

We take t_0 to be the empty map. T is non-empty (it is countably infinite), the empty map is in $2^{\varnothing}$, and anything is equal to the empty map upon restriction to $\varnothing$. Thus applying the binary tree property we have $t_0 \in T$. For the same reason $D(t_0) = T$ is infinite.

Assuming by induction that we have defined $t_0, \dots, t_n$ we assume for a contradiction that both $D(t_n; 0)$ and $D(t_n; 1)$ are finite. Then clearly

$$D(t_n) \subseteq \{t_n; 0\,,\, t_n; 1\} \cup D(t_n; 0) \cup D(t_n; 1)$$

and so $D(t_n)$ is finite, a contradiction. Hence we pick one that has infinitely many descendents to be t_{n+1}. Taking any descendent and restricting its domain to 2^{n+1} we see that $t_{n+1} \in T$; restricting t_{n+1} to 2^n we obtain t_n. Hence the induction is complete.

We take $b = \bigcup_{n \in \mathbb{N}} t_n$ to be the infinite branch. By construction each finite restriction lies in T. □

Definition – α-minimality

Let $\mathbb{M}$ be an ω-saturated Σ-structure and α an ordinal. We say a $\Sigma(\mathbb{M})$-formula ϕ is α-minimal if

$$\mathrm{MR}(\phi) = \alpha \text{ and there does not exist } \psi \in \Sigma(\mathbb{M})_{\text{for}} \text{ such that } \mathrm{MR}(\phi \wedge \psi) = \mathrm{MR}(\phi \wedge \neg\psi) = \alpha$$

The following lemma gives us the existence of Morley degree, a way to specify how many disjoint subsets of a definable set we can produce. This also gives us a way to split up any formula into α-minimal formulas which are easier to work with.

Lemma – Morley degree decomposition [5]

Let $\mathbb{M}$ be an ω-saturated Σ-structure. Let ϕ be a $\Sigma(\mathbb{M})$-formula. Suppose that $\mathrm{MR}(\phi) = \alpha \in \mathrm{Ord}$. Then there exist α-minimal formulas $\psi_1, \dots, \psi_d$ defining disjoint subsets of $\phi(\mathbb{M})$ each with $\mathrm{MR}(X_i) = \alpha$. Futhermore if $X_1, \dots, X_n$ are pairwise disjoint subsets of $\phi(\mathbb{M})$ each with $\mathrm{MR}(X_i) = \alpha$ then $n \leq d$.

Proof. We build a binary tree $T \subseteq 2^{<\omega}$ and apply König's tree lemma. Inductively we define for each $n \in \mathbb{N}$ a finite 'binary tree' $T_n \subseteq 2^{\leq n}$ and $\Phi_n : T_n \to \Sigma(\mathbb{M})_{\text{for}}$ such that

- for each $i < n$, $T_{i+1} \cap 2^{\leq i} = T_i$
- for each $i < n$, $\Phi_{i+1}\big|_{2^{\leq i}} = \Phi_i$
- for each $t \in T_n$, $\mathrm{MR}(\Phi_n(t)) = \alpha$

We take $T_0 = \{\varnothing\}$ and have $\Phi_n : \varnothing \mapsto \phi$.

By induction, supposing that we have defined T_i and Φ_i for each $i \leq n$, we define T_{n+1} as

$$\begin{aligned} T_{n+1} = T_n &\cup \{t;0 \,|\, t \in T_n \text{ is } \alpha - \text{minimal}\} \\ &\cup \{t;1 \,|\, t \in T_n \text{ is } \alpha - \text{minimal}\} \end{aligned}$$

Then we find (using the axiom of choice) for each $t;0 \in T_{n+1}$ a $\Sigma(\mathbb{M})$-formula χ_{n+1} such that

$$\mathrm{MR}(\Phi_n(t) \wedge \chi_{n+1}) = \mathrm{MR}(\Phi_n(t) \wedge \neg\chi_{n+1}) = \alpha$$

We define Φ_{n+1} such that it agrees upon restriction with Φ_n and for each $t \in T_n \cap 2^n$,

$$t;0 \mapsto \Phi_{n+1}(t) \wedge \chi_{n+1} \text{ and } t;1 \mapsto \Phi_{n+1}(t) \wedge \neg\chi_{n+1}$$

By assumption

$$\mathrm{MR}(\Phi_{n+1}(t;0)) = \mathrm{MR}(\Phi_n(t) \wedge \chi_{n+1}) = \alpha$$

Hence Φ_{n+1} satisfies the conditions we required. It then makes sense to define

$$T := \bigcup_{n\in\mathbb{N}} T_n \quad \text{and} \quad \Phi := \bigcup_{n\in\mathbb{N}} \Phi_n : T \to \Sigma(\mathbb{M})_{\text{for}}$$

T is a binary tree: if $s \in T \cap 2^n$ and we have $t < s \in T$ then $0 < n$ and so it must be that s is created due to the induction step. Thus we have $s\big|_{s^{n-1}} \in T$ and by induction $t = s\big|_{s^m} \in T$, where m is power of 2 which t lies in.

Suppose for a contradiction T is infinite. Then

$$|T| \leq \left|2^{<\omega}\right| = \left|\bigsqcup_{n\in\mathbb{N}} 2^n\right| = \aleph_0$$

Hence T is countably infinite and we can apply König's tree lemma, giving us an 'infinite brach' $b : \mathbb{N} \to 2$ such that for each $n \in \mathbb{N}$ we have $b\big|_n \in T_n \subseteq T$. From this we obtain for each $n \in \mathbb{N}$ a formula $\Phi_n(b\big|_n)$ and define a $\Sigma(\mathbb{M})$-formula

$$\psi_n := \Phi_n(b\big|_n) \wedge \neg\Phi_{n+1}(b\big|_{n+1})$$

These will define infinitely many disjoint subsets of $\phi(\mathbb{M})$ with Morley rank α. Without loss of generality $b\big|_{n+1} = b\big|_n;0$, hence for each n:

$$\begin{aligned} \psi_n(\mathbb{M}) &= \left(\Phi_n(b\big|_n) \wedge \neg\Phi_{n+1}(b\big|_{n+1})\right)(\mathbb{M}) \\ &= \left(\Phi_n(b\big|_n) \wedge (\neg\Phi_n(b\big|_n) \vee \neg\chi_{n+1})\right)(\mathbb{M}) \\ &= \left(\Phi_n(b\big|_n) \wedge \neg\chi_{n+1}\right)(\mathbb{M}) \end{aligned}$$

Computing the Morley rank is then straightforward:

$$\mathrm{MR}(\psi_n) = \mathrm{MR}(\Phi_n(b\big|_n) \wedge \neg\chi_{n+1}) = \alpha$$

which follows from the definition of χ_{n+1}. Furthermore to see that any two distinct ψ_n define disjoint subsets of ϕ, let $n < m$, then

$$\Phi_n(b\big|_n)(\mathbb{M}) \subseteq \Phi_{n+1}(b\big|_{n+1})(\mathbb{M}) \subseteq \ldots \Phi_m(b\big|_m)(\mathbb{M})$$

Hence $\neg\,\Phi_m(b\big|_m)(\mathbb{M}) \subseteq \neg\,\Phi_n(b\big|_n)(\mathbb{M})$ and

$$\psi_m(\mathbb{M}) \subseteq (\neg\,\Phi_{m+1}(b\big|_{m+1}))(\mathbb{M}) \subseteq (\neg\,\Phi_n(b\big|_n))(\mathbb{M})$$

the last of which does not intersect with $\psi_n(\mathbb{M})$. Hence they are disjoint. They are subsets of $\phi(\mathbb{M})$ since they are subset of $\Phi_n(b\big|_n)$ which by induction are subset of $\phi(\mathbb{M})$. Hence ϕ has Morley rank $\alpha + 1$ which is a contradiction. Thus T is finite.

We consider the set of terminal nodes $N = \{t \in T \mid \forall s \in T, t \not< s\}$. Since T is finite N is also finite; we take d to be the cardinality of N. Consider $\Phi(N)$, the image of N. There are three points to note about this:

- For each $t \in N$, $\Phi(t)$ is α-minimal.
- The members of $\Phi(N)$ define pairwise disjoint subsets of $\phi(\mathbb{M})$.
- Thus $\phi(\mathbb{M})$ is equal to a disjoint union $\left(\bigvee_{t\in N} \Phi(t)\right)(\mathbb{M}) = \bigsqcup_{t\in\mathbb{N}} \Phi(t)(\mathbb{M})$

We complete the proof using these points first, then come back and prove them.

Let $\Sigma(\mathbb{M})$-formulas $\omega_1, \dots, \omega_n$ have Morley rank α and define pairwise disjoint subsets of $\phi(\mathbb{M})$. We need to show that $n \leq d$. Suppose for a contradiction $d < n$. Suppose for another contradiction that for all $i \leq n$ there exists $t \in N$ such that $\alpha \leq \mathrm{MR}(\Phi(t) \wedge \omega_i)$ (hence $\alpha = \mathrm{MR}(\Phi(t) \wedge \omega_i)$). Then there exist $1 \leq i < j \leq n$ such that

$$\mathrm{MR}(\omega_i \wedge \Phi(t)) = \mathrm{MR}(\omega_j \wedge \Phi(t)) = \alpha$$

Then

$$\alpha = \mathrm{MR}(\omega_j \wedge \Phi(t)) \leq \mathrm{MR}((\neg\omega_i) \wedge \Phi(t)) \leq \mathrm{MR}(\Phi(t)) = \alpha$$

This implies $\Phi(t) \notin N$, a contradiction. Then there must be some $i \leq n$ such that for all $t \in N$ $\mathrm{MR}(\Phi(t) \wedge \omega_i) < \alpha$. However

$$\omega_i(\mathbb{M}) = (\omega_i \wedge \phi)(\mathbb{M}) = \left(\omega_i \wedge \bigvee_{t\in N} \Phi(t)\right)(\mathbb{M}) = \left(\bigvee_{t\in N} \omega_i \wedge \Phi(t)\right)(\mathbb{M})$$

Hence $\mathrm{MR}(\omega_i) = \max_{t\in\mathbb{N}} \mathrm{MR}(\omega_i \wedge \Phi(t)) < \alpha$, a contradiction. Hence $n \leq d$.

We show the three facts from before. The first is clear from the definition of $\mathbb{N}$. Let s and t be distinct elements of N. Let $n \in \mathbb{N}$ be the maximal natural such that $s\big|_n = t\big|_n$. If $s \in 2^n$ then $s < t$ which is a contradiction. Thus $s, t \notin 2^n$. As n is maximal, $s\big|_{n+1} \neq t\big|_{n+1}$ and so $\Phi(s\big|_{n+1})(\mathbb{M})$ is disjoint with $\Phi(t\big|_{n+1})(\mathbb{M})$. $\Phi(s)(\mathbb{M}) \subseteq \Phi(s\big|_{n+1})(\mathbb{M})$ and similarly with t hence $\Phi(s)(\mathbb{M})$ and $\Phi(t)(\mathbb{M})$ are disjoint.

Clearly $\left(\bigvee_{t\in N} \Phi(t)\right)(\mathbb{M}) \subseteq \phi(\mathbb{M})$. For the other direction note that for any $t \in T$ such that $\exists s \in T, t < s$,

$$\Phi(t)(\mathbb{M}) = \Phi(t;0)(\mathbb{M}) \sqcup \Phi(t;1)(\mathbb{M})$$

Hence each element of T (in 2^n) is either in N or defines a set that is a disjoint union due two elements in $2^{n+1} \cap T$.As T is finite we can find the maximal $n \in \mathbb{N}$ such that $\exists t \in 2^n \cap T, a \in \Phi(t)(\mathbb{M})$. If $t \in N$ then we are done. Suppose $t \notin N$ then $a \in \Phi(t;0)(\mathbb{M}) \sqcup \Phi(t;1)(\mathbb{M})$ and so $\exists s \in 2^{n+1} \cap T, a \in \Phi(s)(\mathbb{M})$, contradicting maximality of t. Hence $a \in \bigsqcup_{t\in N} \Phi(t)(\mathbb{M})$ and

$$\phi(\mathbb{M}) = \left(\bigvee_{t\in N} \Phi(t)\right)(\mathbb{M}) = \bigsqcup_{t\in\mathbb{N}} \Phi(t)(\mathbb{M})$$

□

Definition – Morley degree

Let ϕ be a $\Sigma(\mathcal{M})$-formula for a Σ-structure $\mathcal{M}$. We natually interpret ϕ as a $\Sigma(\mathbb{M})$-formula where $\mathbb{M}$ is an ω-saturated elementary Σ-extension of $\mathcal{M}$. We define the Morley degree of ϕ, which we write as $\mathrm{m.deg}(\phi)$: If $\mathrm{MR}(\phi) \in \mathrm{Ord}$ then $\mathrm{MR}(\phi)$ is the minimal $d \in \mathbb{N}$ such that if $X_1, \dots, X_n$ are pairwise disjoint subsets of $\phi(\mathbb{M})$ each with $\mathrm{MR}(X_i) = \mathrm{MR}(\phi)$ then $n \leq d$. The existence of such a d comes from the lemma. If $\mathrm{MR}(\phi) = -\infty$ then $\mathrm{m.deg}(\phi) := -\infty$. If $\mathrm{MR}(\phi) = \infty$ then $\mathrm{m.deg}(\phi) := \infty$.

We also write $\mathrm{m.deg}(\phi(\mathcal{M})) := \mathrm{m.deg}\,(\phi)$ for the Morley degree of a $\Sigma(\mathcal{M})$-definable subset of $\mathcal{M}$.

Note that Morley degree for formulas with ordinal Morley rank will always be natural numbers greater than 0, since the formula itself defines a (trivially pairwise disjoint) subset.

1.5.4 Morley rank and degree for types

Definition – Morley rank and degree for types
Let $\mathcal{M}$ be a Σ-structure with $A \subseteq \mathcal{M}$. Let $p \in S_n(\mathrm{Th}_{\mathcal{M}}(A))$. Then we define the Morley rank of p to be

$$\mathrm{MR}(p) := \min \{\mathrm{MR}(\phi) \mid \phi \in p \text{ with exactly } n \text{ free variables}\}$$

which is well-defined since ordinals are well-ordered.

Note that for any p (by choice) there exists $\phi \in p$ with exactly n free variables such that $\mathrm{MR}(\phi) = \mathrm{MR}(p)$, we call ϕ the rank representative.

We also define Morley degree of p: If $\mathrm{MR}(p) = -\infty$ then $\mathrm{m.deg}(p) = -\infty$ and if $\mathrm{MR}(p) = \infty$ then $\mathrm{m.deg}(p) = \infty$. Otherwise we take

$$\mathrm{m.deg}(p) := \min \{\mathrm{m.deg}(\phi) \mid \phi \in p \text{ and } \mathrm{MR}(\phi) = \mathrm{MR}(p) \text{ and } \phi \text{ has exactly } n \text{ free variables}\}$$

Lemma – α-minimal rank representative of a type
Suppose $\mathbb{M}$ is an ω-saturated strongly minimal Σ-structure, $A \subseteq \mathbb{M}$ and $p \in S_n(\mathrm{Th}_{\mathbb{M}}(A))$. Let $\alpha = \mathrm{MR}(p)$. Then there exists an α-minimal rank representative ϕ of p.

Proof. Suppose no such ϕ exists, then starting with any rank representative ϕ_0 and using maximality of p there exists $\psi_0 \in p$ such that

$$\mathrm{MR}(\phi_0 \wedge \psi_0) = \mathrm{MR}(\phi_0 \wedge \neg\psi_0) = \alpha$$

We define ϕ_1 as $\phi_0 \wedge \psi_0$ which by assumption is not α-minimal and proceed by induction to create ϕ_i and ψ_i. Now $\phi_i \wedge \neg\psi_i$ are infinitely many formulas with Morley rank α defining disjoint subsets of ϕ_0, so $\alpha + 1 \leq \mathrm{MR}(\phi_0)$, a contradiction. □

The following lemma tells us we can also go the other way. Given a formula we can find a type with equal Morley rank containing the formula. This uses the Morley degree decomposition of a formula.

Lemma – Formulas are represented by types [4]
Let $A \subseteq \mathbb{M}$, for $\mathbb{M}$ an ω-saturated Σ-structure. If ϕ is a $\Sigma(A)$-formula with $\mathrm{MR}(\phi) \in \mathrm{Ord}$, then there exists a type $p \in S_n(\mathrm{Th}_{\mathbb{M}}(A))$ such that $\mathrm{MR}(\phi) = \mathrm{MR}(p)$ and $\phi \in p$.

Hence

$$\mathrm{MR}(\phi) = \max \{\mathrm{MR}(q) \mid \phi \in q \in S_n(\mathrm{Th}_{\mathbb{M}}(A))\}$$

Proof. We first show that this holds when ϕ is α-minimal, where $\alpha := \mathrm{MR}(\phi)$. We take

$$p := \{\psi \in F(\Sigma(A), n) \mid \mathrm{MR}(\phi \wedge \neg\psi)\} < \alpha$$

as our n-type. To show that it is finitely consistent with $\mathrm{Th}_{\mathbb{M}}(A)$ let Δ be a finite subset of p, it suffices to show that the conjunction $\Psi := \bigwedge_{\psi\in\Delta} \psi$ has Morley rank at least α.

$$\mathrm{MR}(\phi \wedge \neg\Psi) = \mathrm{MR}(\bigvee_{\psi\in\Delta} \phi \wedge \neg\psi) = \max_{\psi\in\Delta}(\mathrm{MR}(\phi \wedge \neg\psi)) < \alpha$$

Hence the conjunction Φ is in p. Supposing $\mathrm{MR}(\Psi) < \alpha$ implies $\mathrm{MR}(\phi \wedge \Psi) < \alpha$ and

$$\mathrm{MR}(\phi) = \max(\mathrm{MR}(\phi \wedge \Psi), \mathrm{MR}(\phi \wedge \neg \Psi)) < \alpha$$

which is a contradiction. Hence $\alpha \leq \mathrm{MR}(\Psi)$ and so p is finitely consistent. To show that it is maximal, suppose $\psi \notin p$. Then

$$\alpha \leq \mathrm{MR}(\phi \wedge \neg \psi) \leq \mathrm{MR}(\phi) = \alpha$$

and by α-minimality of ϕ we cannot have $\alpha \leq \mathrm{MR}(\phi \wedge \psi)$. Hence $\neg \psi \in p$.

Clearly $\phi \in p$, hence $\mathrm{MR}(p) \leq \mathrm{MR}(\phi)$. Furthermore if $\psi \in p$ then $\alpha \not\leq \mathrm{MR}(\phi \wedge \neg \psi)$ hence

$$\mathrm{MR}(\phi) = \max(\mathrm{MR}(\phi \wedge \psi), \mathrm{MR}(\phi \wedge \neg \psi)) = \mathrm{MR}(\phi \wedge \psi) \leq \mathrm{MR}(\psi)$$

hence $\mathrm{MR}(p) = \mathrm{MR}(\psi)$.

Now we can do the case without the α-minimal hypothesis. Write $\alpha \in \mathrm{Ord}$ for the Morley rank and $d \in \mathbb{N}$ the Morley degree of ϕ. There exist α-minimal formulas $\psi_1, \ldots, \psi_d$ that partition $\phi(\mathbb{M})$, each with $\mathrm{MR}(\psi_i) = \alpha$. By the first part we have for ψ_1 a maximal type $p \in S_n(\mathrm{Th}_{\mathbb{M}}(A))$ such that $\psi_1 \in p$ and $\mathrm{MR}(p) = \mathrm{MR}(\psi_1) = \alpha$. In fact $\phi \in p$ as well since $\psi_1(\mathbb{M}) \subseteq \phi(\mathbb{M})$ so we have found the type that we required.

The 'hence' follows immediately. □

1.5.5 Constructable sets

Definition – Constructable

Let $\mathcal{M}$ be a Σ-structure. The set of constructable subsets of $\mathcal{M}^n$ are defined by:

| If ϕ is an atomic $\Sigma(\mathcal{M})$-formula with up to n free variables then $\phi(\mathcal{M}) \subseteq K^n$ is constructable.

| If $X \subseteq \mathcal{M}^n$ is constructable then $\mathcal{M}^n \setminus X$ is constructable.

| If $X, Y \subseteq \mathcal{M}^n$ are constructable then $X \cup Y$ is constructable.

Thus these are 'finite boolean combinations' of sets $\Sigma(\mathcal{M})$-definable by atomic formulas, i.e. sets $\Sigma(\mathcal{M})$-definable by a quantifier free formula.

Proposition – Constructable is definable

Let T be a Σ-theory with quantifier elimination and $\mathcal{M}$ be a Σ-model of T. Then subsets of $\mathcal{M}^n$ are constructable if and only if they are $\Sigma(\mathcal{M})$-definable by a quantifier free formula if and only if they are $\Sigma(\mathcal{M})$-definable.

Proof. We only show the first equivalence as the second is trivial. ($\Rightarrow$) Suppose $X \subseteq \mathcal{M}^n$ is constructable. Then we induct on X:

- If X is defined by an atomic formula then there is nothing to show.
- If X is $\mathcal{M}^n \setminus Y$ and Y is constructable and by induction Y is $\Sigma(\mathcal{M})$-definable then we take the negation of its defining formula.
- If X is $Y \cup Z$, both constructable and by induction Y, Z are both $\Sigma(\mathcal{M})$-definable then we take the 'or' of their defining formulas.

($\Leftarrow$) Suppose X is $\Sigma(\mathcal{M})$-definable by the quantifier free formula $\phi(v, b)$ with $b \in \mathcal{M}^m$ for some m. We induct on what ϕ is:

- If ϕ is atomic then X is constructable.
- If ϕ is $\neg \psi$ and $\psi(\mathcal{M}, b)$ is constructable then $X = \mathcal{M}^n \setminus \psi(\mathcal{M}, b)$ hence it is constructable.

- If ϕ is $\psi \vee \chi$ then X is the union of the sets $\psi(\mathcal{M}, b)$ and $\chi(\mathcal{M}, b)$ which are constructable by induction. The union of constructable is constructable.

$\square$

Chapter 2

Appendix

2.1 Direct limits

Proposition – Direct Limit of Chains [5]
If $M : I \to \mathbb{M}\mathbf{od}(\Sigma)$ is an embedding chain of Σ-structures then there exists $\mathcal{N} \in \Sigma_{\text{str}}$ such that for all $\alpha \in I$ there is a Σ-embedding $\iota_\alpha : \mathcal{M}(\alpha) \to \mathcal{N}$ and for any $\alpha \leq \beta$ in I, the diagram

$$\begin{tikzcd} \mathcal{M}(\alpha) \arrow[r, "\uparrow_\alpha^\beta"] \arrow[dr, "\iota_\alpha"'] & \mathcal{M}(\beta) \arrow[d, "\iota_\beta"] \\ & \mathcal{N} \end{tikzcd}$$

commutes. Furthermore if M were elementary then each ι_α is elementary by the same construction.

Proof. We consider the disjoint union $\bigsqcup_{\alpha \in I} \mathcal{M}(\alpha)$. For $a, b \in \bigsqcup_{\alpha \in I} \mathcal{M}(\alpha)$ we want to define $a \sim b$. There exist $\alpha, \beta \in I$ such that $a \in \mathcal{M}(\alpha)$ and $b \in \mathcal{M}(\beta)$. Since I is a linear order, $\alpha \leq \beta$ or vice versa. Then we say $a \sim b$ if $\uparrow_\alpha^\beta (a) = b$ or vice versa. To show that the relation is transitive we use that fact that M is a functor so any $\alpha \leq \beta \leq \gamma$ in I, $\uparrow_\beta^\gamma \circ \uparrow_\alpha^\beta = \uparrow_\alpha^\gamma$.

We define the carrier set as the quotient:

$$\mathcal{N}_{\text{car}} := \bigsqcup_{\alpha \in I} \mathcal{M}(\alpha) / \sim$$

Then for each $\alpha \in I$ there is an induced map of sets $\iota_\alpha : \mathcal{M}(\alpha) \to \mathcal{N}$ sending $a \mapsto [a]$:

$$\begin{tikzcd} \mathcal{M}(\alpha) \arrow[r, "\subseteq"] \arrow[dr, "\iota_\alpha"'] & \bigsqcup \arrow[d] \\ & \bigsqcup / \sim \end{tikzcd}$$

We immeditately have that it commutes with lifts:

$$\begin{tikzcd} \mathcal{M}(\alpha) \arrow[r, "\uparrow_\alpha^\beta"] \arrow[dr, "\iota_\alpha"'] & \mathcal{M}(\beta) \arrow[d, "\iota_\beta"] \\ & \mathcal{N} \end{tikzcd}$$

let $\alpha \leq \beta$ in I and let $a \in \mathcal{M}(\alpha)$, then

$$\iota_\beta \circ \uparrow_\alpha^\beta (a) = [\uparrow_\alpha^\beta (a)] = [a] = \iota_\alpha(a)$$

since $\uparrow_\alpha^\beta \sim a$. Furthermore, for any $\alpha \in I$, ι_α is injective: let $a, b \in \mathcal{M}(\alpha)$ such that $\iota_\alpha(a) = \iota_\alpha(b)$, then by definition of ι_α we have $a \sim b$. Note that $\uparrow_\alpha^\alpha$ is the identity, hence

$$a \sim b \quad \Rightarrow \quad a = \uparrow_\alpha^\alpha (b) = b$$

We define interpretation for $\mathcal{N}$ such that it commutes with ι_α for each α. We note that I is non-empty and take $\alpha \in I$.

| For $c \in \Sigma_{\text{con}}$, $c^{\mathcal{N}} := \iota_\alpha(c^{\mathcal{M}(\alpha)})$

| For $f \in \Sigma_{\text{fun}}$ define $f^{\mathcal{N}} : \mathcal{N}^{n_f} \to \mathcal{N}$ such that for $a \in \mathcal{N}^{n_f}$ there exists a $\beta \in I$ (the maximum element of a finite totally ordered set) and $b \in \mathcal{M}(\beta)^{n_f}$ such that $a = \iota_\beta(b)$. Then have $f^{\mathcal{N}} : a \mapsto \iota_b e(f^{\mathcal{M}(\beta)}(b))$. To check that $f^{\mathcal{M}}$ is well defined, let $\gamma \in I$ and $c \in \mathcal{M}(\gamma)$ be such that $a = \iota_\gamma(c)$. WLOG $\beta \leq \gamma$. First note that $\iota_\gamma(c) = a = \iota_\beta(b) = \iota_\gamma \circ \uparrow_\beta^\gamma (b)$ since we showed that the $\iota_\star$ commute with lifts. Since ι_γ is injective, $c = \uparrow_\beta^\gamma (c)$. Hence

$$\iota_\beta \circ f^{\mathcal{M}(\beta)}(b) = \qquad \iota_\gamma \circ \uparrow_\beta^\gamma \circ f^{\mathcal{M}(\beta)}(b) = \iota_\gamma \circ f^{\mathcal{M}(\gamma)} \circ \uparrow_\beta^\gamma (b) = \qquad \iota_\gamma \circ f^{\mathcal{M}(\gamma)}(c)$$

Hence $f^{\mathcal{N}}$ is well defined.

| For $r \in \Sigma_{\text{rel}}$, define

$$r^{\mathcal{M}} = \bigcup_{\beta \in I} \iota_\beta(r^{\mathcal{M}(\beta)})$$

By the way we define interpretation it is clear that for any α, ι_α is a Σ-morphism. We already have that it is injective. To show that it is an embedding, take $a \in \mathcal{M}(\alpha)^{m_r}$ such that

$$io_\alpha(a) \in r^{\mathcal{M}} = \bigcup_{\beta \in I} \iota_\beta(r^{\mathcal{M}(\beta)})$$

There exists a β and a $b \in r^{\mathcal{M}(\beta)}$ such that $\iota_\alpha(a) = \iota_\beta(b)$. Case on $\alpha \leq \beta$ or $\beta \leq \alpha$: If $\alpha \leq \beta$ then

$$\iota_\beta \uparrow_\alpha^\beta (a) = \iota_\alpha(a) = \iota_\beta(b) \quad \Rightarrow \quad \uparrow_\alpha^\beta (a) = b$$

by injectivity of ι_β. Hence $\uparrow_\alpha^\beta (a) \in r^{\mathcal{M}(\beta)}$ so $a \in r^{\mathcal{M}(\alpha)}$ since $\uparrow_\alpha^\beta$ is a Σ-embedding. If $\beta \leq \alpha$ then we obtain $\uparrow_\beta^\alpha (b) = a$ in the same way. Therefore $b \in r^{\mathcal{M}(\beta)}$ implies $a = \uparrow_\beta^\alpha (b) \in r^{\mathcal{M}(\alpha)}$ as $\uparrow_\beta^\alpha$ is a Σ-morphism.

Lastly we show that if the chain M were elementary, then for all $\alpha \in I$, ι_α is elementary. We prove the equivalent statement: for any $\phi \in \Sigma_{\text{for}}$ with free variables indexed by S, given $\alpha \in I$ and $a \in \mathcal{M}(\alpha)^S$,

$$\mathcal{M}(\alpha) \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota_\alpha(a))$$

by induction on ϕ. By the two lemmas 'embeddings preserve satisfaction of quantifier free formulas' and 'embeddings preserve satisfaction of universal formulas downwards' we only need to show that under the assumption of the inductive hypothesis,

$$\mathcal{M}(\alpha) \vDash_\Sigma \forall v, \psi(a, v) \ \Rightarrow \mathcal{N} \vDash_\Sigma \forall v, \psi(\iota_\alpha(a), v)$$

Suppose $\mathcal{M}(\alpha) \vDash_\Sigma \forall v, \psi(a, v)$ and let $c \in \mathcal{N}$. There exist $\beta \in I$ and $b \in \mathcal{M}(\beta)$ such that $\iota_\beta(b) = c$. Case on $\alpha \leq \beta$ or $\beta \leq \alpha$. If $\alpha \leq \beta$ then the lift $\uparrow_\alpha^\beta$ is elementary by assumption and so

$$\begin{aligned}
&\mathcal{M}(\beta) \vDash_\Sigma \forall v, \psi(\uparrow_\alpha^\beta (a), v) \\
\Rightarrow &\mathcal{M}(\beta) \vDash_\Sigma \psi(\uparrow_\alpha^\beta (a), b) \\
\Rightarrow &\mathcal{N} \vDash_\Sigma \psi(\iota_\beta \circ \uparrow_\alpha^\beta (a), \iota_\beta(b)) && \text{induction hypothesis} \\
\Rightarrow &\mathcal{N} \vDash_\Sigma \psi(\iota_\alpha(a), c)
\end{aligned}$$

Thus ι_α is indeed elementary. $\square$

2.2 Boolean Algebras, Ultrafilters and the Stone Space

2.2.1 Boolean Algebras

There is a very detailed wikipedia page [13] on Boolean algebras.

Definition – Partially ordered set
The signature of partially ordred sets Σ_{PO} consists of $(\varnothing, \varnothing, n_f, \{\leq\}, m_f)$, where $n_\leq = 2$. The theory of partially ordered sets PO consists of

| Reflexivity: $\forall x, x \leq x$ (this is just notation for $\leq(x,x)$)

| Antisymmetry: $\forall x \forall y, (x \leq y \wedge y \leq x) \to x = y$

| Transitivity: $\forall x \forall y \forall z, (x \leq y \wedge y \leq z) \to x \leq z$

Definition – Boolean algebra
The signature of Boolean algebras Σ_{BLN} consists of $(\{1,0\}, \{\leq, \cap, \cup, \setminus\}, n_f, \varnothing, m_f)$, where $n_\leq = 2$, $n_\cap = n_\cup = 2$ and $n_\setminus = 1$. The theory of Boolean algebras BLN consists of the theory of partially ordered sets† PO together with the formulas

| Assosiativity of adjunction: $\forall x \forall y \forall z, (x \cap y) \cap z = x \cap (y \cap z)$

| Assosiativity of disjunction: $\forall x \forall y \forall z, (x \cup y) \cup z = x \cup (y \cup z)$

| Identity for adjunction: $\forall x, x \cap 1 = x$

| Identity for disjunction: $\forall x, x \cup 0 = x$

| Commutativity of adjunction: $\forall x \forall y, x \cap y = y \cap x$

| Commutativity of disjunction: $\forall x \forall y, x \cup y = y \cup x$

| Distributivity of adjunction: $\forall x \forall y \forall z, x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$

| Distributivity of disjunction: $\forall x \forall y \forall z, x \cup (y \cap z) = (x \cup y) \cap x(\cup z)$

| Negation on adjunction: $\forall x, x \cap \setminus x = 0$

| Negation on disjunction: $\forall x, x \cup \setminus x = 1$

| Order on adjunction: $\forall x \forall y, (x \cap y) \leq x$

| Maximal property of adjunction: $\forall x \forall y \forall z, (x \leq y) \wedge (x \leq z) \to (x \leq y \cap z)$

| Order on disjunction: $\forall x \forall y, x \leq (x \cup y)$

| Minimal property of disjunction: $\forall x \forall y \forall z, (x \leq z) \wedge (y \leq z) \to (x \cup y \leq z)$

Often 'absorption' is also included, but it can be deduced from the other axioms. I have not used the usual logical symbols due to obvious clashes with our notation, and we will be using this in the context of sets anyway. If B is a Σ_{BLN}-model of BLN we call it a Boolean algebra.

†Often the ordering is defined afterwards as $a \leq b$ if and only if $a = a \cap b$.

Lemma – Facts about Boolean algebras
Let B be a Boolean algebra, let $\mathcal{F}$ be an ultrafilter on B, let $a, b \in B$ and let $f : B \to C$ be a morphism.

- $a \cap a = a$ and $a \cup a = a$.
- $a \cup 1 = 1$ and $a \cap 0 = 0$.

- If $a \cap b = 0$ and $a \cup b = 1$ then $a = \smallsetminus b$ (negations are unique.)
- $\smallsetminus(a \cup b) = (\smallsetminus a) \cap (\smallsetminus b)$ and its dual. (De Morgan)
- ($a \in \mathcal{F}$ or $b \in \mathcal{F}$) if and only if $a \cup b \in \mathcal{F}$.
- Morphisms are order preserving.
- Morphisms commute with negation.
- $a \cap b = 1$ if and only if $a = 1$ and $b = 1$. Similarly for $\cup$ with 0.
- If $a \cap b = 0$ then $b \leq \smallsetminus a$.

Proof.

- We prove only $a \cap a = a$ as the other has the same proof.

$$a \cap a = (a \cap a) \cup 0 = (a \cap a) \cup (a \cap \smallsetminus a) = a \cap (a \cup \smallsetminus a) = a \cap 1 = a$$

- Again, we prove only $a \cup 1 = 1$. Using the previous part,

$$a \cup 1 = a \cup (a \cup \smallsetminus a) = (a \cup a) \cup \smallsetminus a = a \cup \smallsetminus a = 1$$

-

$$\begin{aligned} a &= a \cup 0 = a \cup (b \cap \smallsetminus b) = (a \cup b) \cap (a \cup \smallsetminus b) = 1 \cap (a \cup \smallsetminus b) \\ &= 0 \cup (a \cup \smallsetminus b) = (b \cap \smallsetminus b) \cup (a \cup \smallsetminus b) = (b \cup a) \cup \smallsetminus b = 0 \cup \smallsetminus b \\ &= \smallsetminus b \end{aligned}$$

- By the previous part it suffices to show that

$$((\smallsetminus a) \cap (\smallsetminus b)) \cap (a \cup b) = 0 \text{ and } ((\smallsetminus a) \cup (\smallsetminus b)) \cup (a \cup b) = 1$$

  These are clear.

- ($\Rightarrow$) In the case that $a \in \mathcal{F}$ we have $a \leq a \cup b \in \mathcal{F}$ as $\mathcal{F}$ is under superset. The other case is the same. ($\Leftarrow$) Suppose $a \notin \mathcal{F}$ and $b \notin \mathcal{F}$ then $\smallsetminus a \in \mathcal{F}$ and $\smallsetminus b \in \mathcal{F}$ hence $\smallsetminus a \cap \smallsetminus b \in \mathcal{F}$ by closure under intersection. By the previous part this is equal to $\smallsetminus(a \cup b)$, which implies $(a \cup b) \notin \mathcal{F}$ as $\mathcal{F}$ is an ultrafilter.

- Suppose $b \leq a$. We show that $f(a) \leq f(b)$. By the minimal property of disjunction,

$$b \leq a \,\wedge\, a \leq a \Rightarrow b \cup a \leq a$$

  Clearly $a \leq b \cup a$ and so $a = b \cup a$ Hence $f(b) \leq f(b) \cup f(a) = f(b \cup a) = f(a)$.

- Suppose $f(a) = b$. Then $f(a) \wedge f(\smallsetminus a) = f(a \wedge \smallsetminus a) = f(0) = 0$ and $f(a) \vee f(\smallsetminus a) = f(a \vee \smallsetminus a) = f(1) = 1$. As negations are unique (shown above) this gives us that $f(\smallsetminus a) = \smallsetminus f(a)$.

- Suppose $a \cap b = 1$ then $1 \leq a \cap b \leq a$ hence $1 = a$ and similarly $1 = b$.

-

$$\begin{aligned} & a \cup \smallsetminus a = 1 \\ \Rightarrow & b \cap (a \cup \smallsetminus a) = b \cap 1 = b \\ \Rightarrow & (b \cap a) \cup (b \cap \smallsetminus a) = b \\ \Rightarrow & 0 \cup (b \cap \smallsetminus a) = b \\ \Rightarrow & b = b \cap \smallsetminus a \leq \smallsetminus a \end{aligned}$$

$\square$

Definition – Filters and ultrafilters on Boolean algebras
Let B be a Boolean algebra. A subset $\mathcal{F}$ of B is an filter on B if

- $1 \in \mathcal{F}$.
- For any two members $a, b \in \mathcal{F}$ the adjunction $a \cap b$ is in $\mathcal{F}$. (Closure under finite intersection.)
- If $a \in \mathcal{F}$ then any $b \in B$ such that $a \leq b$ is also a member of $\mathcal{F}$. (Closure under superset.)

We say a filter on B is proper if it does not contain 0. A proper filter $\mathcal{F}$ on B is an ultrafilter (maximal filter) when for any filter $\mathcal{G}$ on B, if $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G} = \mathcal{F}$ or $\mathcal{G} = B$.

Proposition – Equivalent definition of ultrafilter
Let B be a boolean algebra. Let $\mathcal{F}$ be a proper filter on B. The following are equivalent:

1. $\mathcal{F}$ is an ultrafilter over B.
2. If $a \cup b \in \mathcal{F}$ then $a \in \mathcal{F}$ or $b \in \mathcal{F}$. '$\mathcal{F}$ is prime'.
3. For any a of B, $a \in \mathcal{F}$ or $(\smallsetminus a) \in \mathcal{F}$.

For 2. the or is in fact 'exclusive or' since if both a and its negation were in $\mathcal{F}$ then $\mathcal{F}$ would not be proper.

Proof. $(1. \Rightarrow 2.)$ Suppose $a_0 \cup a_1 \in \mathcal{F}$.

$$\mathcal{G}_{a_0} = \{b \in B \,|\, \exists c \in \mathcal{F}, c \cap a_0 \leq b\}$$

Then clearly $\mathcal{G}_{a_0}$ is a filter containing $\mathcal{F}$ and a_0. Similarly have $\mathcal{G}_{a_1}$. Since $\mathcal{F}$ is an ultrafilter, we can case on whether $\mathcal{G}_{a_i}$ is $\mathcal{F}$ or B. If either is $\mathcal{F}$ then $a_i \in \mathcal{F}$ and we are done. If $\mathcal{G}_{a_0} = \mathcal{G}_{a_1} = B$ then both contain 0 and so there exist $c_i \in \mathcal{F}$ such that $c_i \cap a_i = 0$. Hence $c_i \leq \smallsetminus a_i$, and so $(\smallsetminus a_i) \in \mathcal{F}$. Thus $\smallsetminus(a_0 \cup a_1) = \smallsetminus a_0 \cap \smallsetminus a_1 \in \mathcal{F}$, and so $0 \in \mathcal{F}$, a contradiction.

$(2. \Rightarrow 3.)$ Let $a \in \mathcal{F}$. Then we have that $a \cup \smallsetminus a = 1$, which is by definition a member of $\mathcal{F}$. By assumption this implies that $a \in \mathcal{F}$ or $\smallsetminus a \in \mathcal{F}$.

$(3. \Rightarrow 1.)$ Let $\mathcal{G}$ be a proper filter such that $\mathcal{F} \subseteq \mathcal{G}$. It suffices to show that $\mathcal{G} = \mathcal{F}$. Then let $a \in \mathcal{G}$. Suppose $a \notin \mathcal{F}$. Then $\smallsetminus a \in \mathcal{F}$ and so $\smallsetminus a \in \mathcal{G}$. Thus $0 = (a \cap \smallsetminus a) \in \mathcal{G}$ thus $\mathcal{G}$ is not proper, a contradiction. Hence $\mathcal{G} = \mathcal{F}$. □

Proposition – Extending filters to ultrafilters

Definition
The category of Boolean algebras consists of Boolean algebras as the objects and for any two Boolean algebras B, C morphisms $f : B \to C$ such that for any $a, b \in B$

$$f(0) = 0, f(1) = 1, f(a \cup b) = f(a) \cup f(b), f(a \cap b) = f(a) \cap f(b)$$

Definition
The category of Stone spaces has 0-dimensional, compact and Hausdorff topological spaces as objects and continuous maps as morphisms.

Proposition – Contravariant functor from Boolean algebras to Stone spaces [4]
The map $S(\star)$ that sends a Boolean algebra B to the Stone space

$$S(B) := \{\mathcal{F} \subseteq B \mid \mathcal{F} \text{ is an ultrafilter}\}$$

and a Boolean algebra morphism $f : A \to B$ to a continuous map of Stone spaces:

$$S(f) := f^{-}1(\star) : S(B) \to S(A)$$

is a contravariant functor from Boolean algebras to the category of Stone spaces.

Proof. We give the topology on $S(B)$ the topology generated by the clopen sets: Let $b \in B$, then an element of the clopen basis is given by

$$[b] := \{\mathcal{F} \subseteq B \mid b \in \mathcal{F}\}$$

Thus $S(B)$ is 0-dimensional by definition. It is Hausdorff because if $\mathcal{F}, \mathcal{G} \in S(B)$ are not equal then without loss of generality there exists $b \in \mathcal{F}$ such that $b \notin \mathcal{G}$. As ultrafilters are proper, $[b] \cap [\neg b] = \varnothing$ and so we have obtained disjoint open sets such that $\mathcal{F} \in [b]$ and $\mathcal{G} \in [\neg b]$.

Next we show that it is compact. To do this, we will look at $S(B)$ as a subspace of the power set of B which is isomorphic to 2^B, the set of functions from B to 2. We endow $2 = \{0, 1\}$ with the discrete topology and note that it is compact. Then $2^B = \prod_{a \in B} 2$ has an induced product topology which is compact by Tychonoff's Theorem. The isomorphism from 2^B to the power set of B is given by

$$2^B \to \mathcal{P}(B) := f \mapsto f^{-1}(1)$$

We take the topology on $\mathcal{P}(B)$ induced by this isomorphism. We must show that

- $S(B)$ is the image of $\mathrm{Mor}(B, 2) := \left\{f \in 2^B \mid f \text{ is a Boolean algebra morphism}\right\}$ under this isomorphism.
- $\mathrm{Mor}(B, 2)$ is a closed subset of 2^B and therefore compact.
- The induced topology on $S(B)$ is the same as its original topology.

With these facts we see that $S(B)$ is also compact.

Let $f : B \to 2$ be a Boolean algebra morphism. We must show that $\mathcal{F} := f^{-1}(1)$ is an ultrafilter. Since $f(1) = 1$, $1 \in \mathcal{F}$. If $a, b \in \mathcal{F}$ then $f(a) = f(b) = 1$ so $f(a \cap b) = f(a) \cap f(b) = 1$ thus $\mathcal{F}$ is closed under intersection. If $a \subseteq b$ and $a \in \mathcal{F}$ then as f is order preserving $f(a) \subseteq f(b)$. It is a proper filter as $f(0) = 0 \neq 1$. To show it is an ultrafilter we use the equivalent definition: let $a \in \mathcal{P}(B)$. If $f(a) = 1$ then we are done, otherwise

$$f(\neg a) = \neg f(a) = \neg 0 = 1 \Rightarrow \neg a \in \mathcal{F}$$

and we are done. To show that this is a surjection we use the inverse:

$$\mathcal{F} \mapsto \left(a \to \begin{cases} 1, a \in \mathcal{F} \\ 0, a \notin \mathcal{F} \end{cases}\right)$$

To show that this is a morphism we note that $\mathcal{F}$ is proper and contains 1 thus $f(0) = 0$ and $f(1) = 1$. Also

$$\begin{aligned} & f(a \cap b) = 1 \Leftrightarrow a \cap b \in \mathcal{F} \\ & \Leftrightarrow a \in \mathcal{F} \text{ and } b \in \mathcal{F} \quad \text{by closure under finite intersection and superset} \\ & \Leftrightarrow f(a) = 1 \text{ and } f(b) = 1 \\ & \Leftrightarrow f(a) \cap f(b) = 1 \quad \text{as proven before} \end{aligned}$$

Hence $f(a \cap b) = f(a) \cap f(b)$. Lastly since

$$\neg f(a) = 1 \Leftrightarrow f(a) = 0 \Leftrightarrow a \notin \mathcal{F} \Leftrightarrow \neg a \in \mathcal{F} \Leftrightarrow f(\neg a) = 1$$

thus by De Morgan and uniqueness of negations we have

$$f(a \cup b) = f(\neg(\neg a \cap \neg b)) = \neg[\neg f(a) \cap \neg f(b)] = (\neg\neg f(a)) \cup \neg\neg f(b) = f(a) \cup f(b)$$

Thus the inverse map gives back a Boolean algebra morphism and $\mathrm{Mor}(B,2) \cong S(B)$ under the isomorphism.

To show that $\mathrm{Mor}(B,2)$ is a closed subset we write it as

$$\mathrm{Mor}(B,2) = \{f \mid f(0) = 0\} \cap \{f \mid f(1) = 1\} \cap \{f \mid f \text{ commutes with } \cap\} \cap \{f \mid f \text{ commutes with } \cup\}$$

and show that all of these four sets are closed. Call them $\min, \max, C_\cap, C_\cup$ respectively and for each $a \in B$ call the projection map $\pi_a : 2^B \to 2$ (these send $f \mapsto f(a)$ such that $\pi_a(f) = f(a)$) and note that by definition of the product topology each π_a is continuous; the closed sets of the product are generated by preimages of closed sets.

$$f(0) = 0 \Leftrightarrow \pi_0(f) = 0 \Leftrightarrow f \in \pi_0^{-1}(0)$$

Hence $\min = \pi_0^{-1}(0)$ is closed as $\{0\}$ is closed in the discrete topology on 2. Similarly $\max = \pi_1^{-1}(1)$ is closed.

$$C_\cap = \bigcap_{a,b \in B} \{f \mid f(a \cap b) = f(a) \cap f(b)\} = \bigcap_{a,b \in B} \{f \mid \pi^{-1}_{a\cap b}(f(a) \cap f(b))\}$$

Thus $C_\cap$ is an arbitrary intersection of preimages of closed sets since each $f(a) \cap f(b)$ is closed in the discrete toipology on 2, hence $C_\cap$ is closed. Similarly

$$C_\cup = \bigcap_{a,b \in B} \{f \mid \pi^{-1}_{a\cup b}(f(a) \cup f(b))\}$$

is closed and so $\mathrm{Mor}(B,2)$ is closed.

With regards to compactness it remains to show that the topologies on $S(B)$ are the same. It suffices to show that any (closed) basis element of each can be written as a closed set in the other. Let $[b]$ be an element of the basis for $S(B)$ under the Stone topology. Then this is the image of the closed subset $\pi_b^{-1}(1) \subseteq \mathrm{Mor}(B,2)$ under the isomorphism:

$$\mathrm{iso}(\pi_b^{-1}(1)) = \left\{f^{-1}(1) \mid f(b) = 1\right\} = \{\mathcal{F} \text{ ultrafilter} \mid b \in \mathcal{F}\}$$

Conversely, any element of the closed basis for $\mathrm{Mor}(B,2)$ is of the form $\pi_b^{-1}(X)$ where $b \in B$ and $X \subseteq 2$. Hence

$$\mathrm{iso}(\pi_b^{-1}(X)) = \left\{f^{-1}(1) \mid f(b) \in X\right\}$$

We can case on if $X = \varnothing, \{0\}, \{1\}, 2$ and deduce that respectively $\mathrm{iso}(\pi_b^{-1}(X))$ becomes $\varnothing, [\neg b], [b], S(B)$, all of which are closed in the Stone topology. Thus the topologies are the same under this isomorphism and hence $S(B)$ is compact.

To show that $S(\star)$ is a contravariant functor we need to check that the morphism map

$$S(f) := f^{-}1(\star) : S(B) \to S(A)$$

is a well-defined, respects the identity and composition. We show that $S(f)$ is continuous: it suffices that preimages of clopen elements are clopen. Let $[b] \subseteq S(A)$ be clopen.

$$\begin{aligned}
&S(f)^{-1}[b] \\
=&\left\{\mathcal{F} \in S(B) \mid f^{-1}(\mathcal{F}) \in [b]\right\} \\
=&\{\mathcal{F} \in S(B) \mid f(b) \in \mathcal{F}\} \\
=&[f(b)]
\end{aligned}$$

which is clopen. □

Proposition – Stone Duality

There is an equivalence between the category of Stone spaces and the category of Boolean algebras. Given by the functor $\mathcal{B}_\star$ sending any topological space X to the set of its clopen subsets (this is a basis of X as it is 0 dimensional):

$$\mathcal{B}_X := \{a \subseteq X \mid a \text{ is clopen}\}$$

and its inverse $S(\star)$.

Proof. Let X be a 0-dimensional compact Hausdorff topological space. There is an obvious Boolean algebra to take on

$$\mathcal{B}_X := \{a \subseteq X \mid a \text{ is clopen}\}$$

which is interpreting 0 to be $\varnothing$, 1 to be X, $\leq$ as $\subseteq$, adjunction as intersection, disjunction as union and negation to be taking the complement in X. One can check that this is a Boolean algebra.

We make this a contravariant functor by taking any continuous map $f : X \to Y$ to an induced map $f^\diamond := f^{-1}(\star) : \mathcal{B}_Y \to \mathcal{B}_X$. One can check that this is a well-defined functor.

Lastly we prove the equivalence of categories by giving natural transformations $S(\mathcal{B}_\star) \to \mathrm{id}_\star$ in the category of topological spaces and $\mathrm{id}_\star \to \mathcal{B}_{S(\star)}$ in the category of Stone spaces. □

2.2.2 Isolated points of the Stone space

Definition – Isolated point

Let X be a topological space and $x \in X$. We say x is isolated if $\{x\}$ is open.

Definition – Derived set

Let X be a topological space. The derived set of X is defined as

$$X' := X \setminus \{x \in X \mid x \text{ isolated}\}$$

Exercise (Equivalent definition of derived set). *Let U be a subspace of X a topological space. $x \in U$ is not isolated (in the subspace topology) if and only if for any open set O_x containing x, $O_x \cap U \setminus \{x\}$ is non-empty. (it is a limit point of U.)*

Definition – Atom

Let B be a Boolean algebra. We say $a \in B$ is an atom if it is non-zero and for any $b \in B$ if $b \leq a$ then $b = a$ or $b = 0$.

Definition – Principle filter

Let B be a Boolean algebra and $a \in B$. Suppose a is non-zero. Then principle filter of a is defined as

$$a^\uparrow := \{b \in B \mid a \leq b\}$$

One should check that this is a proper filter.

Proposition

Let $a \in B$ a Boolean algebra. Then $a^\uparrow$ is an ultrafilter if and only if a is atomic.

Proof. ($\Rightarrow$) Suppose $b \leq a$. As $a^{\uparrow}$ is ultrafilter either $b \in a^{\uparrow}$ or $\neg b \in a^{\uparrow}$. In the first case $a \leq b$ hence $a = b$. If $\neg b \in a^{\uparrow}$ then $b \leq a \leq \neg b$ and so

$$1 = b \cup \neg b \leq \neg b \cup \neg b = \neg b$$

and so $1 = \neg b$ and $b = 0$.

($\Leftarrow$) Suppose a is an atom. We show that for any $b \in B$, it is in $a^{\uparrow}$ or its negation is in $a^{\uparrow}$ Since $1 \in a^{\uparrow}$, $b \cup \neg b \in a^{\uparrow}$ and so $a \leq b \cup \neg b$. Thus

$$a = a \cap (b \cup \neg b) = (a \cap b) \cup (a \cap \neg b)$$

Since a is non-zero, either $a \cap b$ or $a \cap \neg b$ is non-zero. If $a \cap b \neq 0$ then together with the fact that $a \cap b \leq a$ we conclude that $a \cap b = a$ as a is atomic. Hence $a \leq b$ and $b \in a^{\uparrow}$. Similarly the other case results in $\neg b \in a^{\uparrow}$. □

Proposition – Correspondence between atoms and isolated points

Let $a \in B$ a Boolean algebra. If a is an atom if and only if $[a]$ is a singleton in $S(B)$. Hence $a^{\uparrow}$ is an isolated point in $S(B)$.

Proof. ($\Rightarrow$) If a is an atom then $[a] = \{a^{\uparrow}\}$ is the only ultrafilter containing a is the principle filter of a: Let $\mathcal{F}$ be an ultrafilter containing a. Then for any $b \in \mathcal{F}$, $a \cap b \in \mathcal{F}$ and non-zero. Therefore $a \cap b = a$ as a is an atom and $a \leq b$. Hence $b \in a^{\uparrow}$. By maximality $\mathcal{F} = a^{\uparrow}$. Hence $[a]$ is a singleton.

($\Leftarrow$) Suppose a is not atomic. Then there exists $b \leq a$ such that $b \neq 0$ and $b \neq a$.

$$\begin{aligned} b \cup (a \cap \neg b) &= (a \cap b) \cup (a \cap \neg b) \\ &= a \cup (b \cap \neg b) = a \cup 0 = a \end{aligned}$$

There exist ultrafilters extending the principle filters of b and $a \cap \neg b$. These are not equal since b and $\neg b$ cannot be in the same proper filter. Both filters contain a. Hence $[a]$ is not a singleton. □

2.2.3 Ultraproducts and Łos's Theorem

This section introduces ultrafilters and ultraproducts and uses Łos's Theorem to prove the compactness theorem. Łos's theorem appears as an exercise in Tent and Ziegler's book [4].

Definition – Filters on sets

Let X be a set. The power set of X is a Boolean algebra with 0 interpreted as $\varnothing$, 1 interpreted as X and $\leq$ interpreted as $\subseteq$. We say $\mathcal{F}$ is a filter on X if it is a filter on the power set. The definition translates to:

- $X \in \mathcal{F}$.
- For any two members of $\mathcal{F}$ their intersection is in $\mathcal{F}$.
- If $a \in \mathcal{F}$ then any b in the power set of X such that $a \subseteq b$ is also a member of $\mathcal{F}$.

Translating definitions over we have that a filter on X is proper if and only if it does not contain the empty set, if and only if the filter is not equal to the power set. Furthermore a proper filter $\mathcal{F}$ is an ultrafilter if and only if for any filter $\mathcal{G}$, if $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{F} = \mathcal{G}$ or $\mathcal{G}$ is the power set of X.

Definition – Ultraproduct

Let $\mathcal{F}$ be an ultrafilter on X. We define a relation on $\prod_{x \in X} x$ by

$$(a_x)_{x \in X} \sim (b_x)_{x \in X} := \{x \in X \mid a_x = b_x\} \in \mathcal{F}$$

This is an equivalence relation as

- $(a_x)_{x\in X} \sim (a_x)_{x\in X} \Leftrightarrow \{x \in X \mid a_x = a_x\} = X \in \mathcal{F}$
- Symmetry is due to symmetry of $=$.
- If $\{x \in X \mid a_x = b_x\} \in \mathcal{F}$ and $\{x \in X \mid b_x = c_x\} \in \mathcal{F}$ then $\{x \in X \mid a_x = b_x = c_x\}$ is their intersection and so is in $\mathcal{F}$. Thus its superset $\{x \in X \mid a_x = c_x\}$ is in $\mathcal{F}$.

We define the ultraproduct of X over $\mathcal{F}$:

$$\prod X/\mathcal{F} := \prod_{x\in X} x/\sim$$

Proposition – Equivalent definition of ultrafilter (translated to the power set)
Let X be a set. Let $\mathcal{F}$ be a proper filter on X. $\mathcal{F}$ is an ultrafilter over X if and only if for every subset $U \subseteq X$, *either* $U \in \mathcal{F}$ or $X \setminus U \in \mathcal{F}$.

Proof. Follows immediately from the equivalent definition of an ultrafilter. □

Proposition – Łos's Theorem
Let $\mathfrak{M} \subseteq \Sigma_{\text{str}}$ where Σ is a signature such that *each carrier set is non-empty*. Let

$$X = \{\mathcal{M} \mid \mathcal{M} \in \mathfrak{M}\}$$

Write $\mathcal{M} = \mathcal{M}$ and $\mathfrak{M} = X$ to make things look nicer. Suppose $\mathcal{F}$ is an ultrafilter on $\mathfrak{M}$ (i.e. an ultrafilter on the Boolean algebra $P(\mathfrak{M})$). Then we want to make $\mathcal{N} := \prod \mathfrak{M}/\mathcal{F}$ into a Σ-structure. Let π be the natural surjection $\prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M} \to \prod \mathfrak{M}/\mathcal{F}$. If $a = (a_1, \ldots, a_n) \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ then write $a_{\mathcal{M}} := ((a_1)_{\mathcal{M}}, \ldots, (a_n)_{\mathcal{M}})$.

- Constant symbols $c \in \Sigma_{\text{con}}$ are interpreted as

$$c^{\mathcal{N}} := \pi(c^{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}}$$

- Any function symbol $f \in \Sigma_{\text{fun}}$ is interpreted as the function

$$f^{\mathcal{N}} : \left(\prod \mathfrak{M}/\mathcal{F}\right)^n \to \prod \mathfrak{M}/\mathcal{F} := \pi\left((a_{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}}\right) \mapsto \pi(f^{\mathcal{M}}(a_{\mathcal{M}}))_{\mathcal{M}\in\mathfrak{M}}$$

  where $\pi\left((a_{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}}\right) = (\pi((a_i)_{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}})_{i=1}^n$.
- Any relation symbols $r \in \Sigma_{\text{rel}}$ is interpreted as the subset such that

$$\pi(a) \in r^{\mathcal{N}} \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \mid a_{\mathcal{M}} \in r^{\mathcal{M}}\} \in \mathcal{F}$$

  where $a = a_1, \ldots, a_m$ and $\pi(a) = (\pi(a_i))_{i=1}^m$.

Then for any Σ-formula ϕ with free variables indexed by S, If $a = (a_1, \ldots, a_n) \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ then

$$\mathcal{N} \vDash_\Sigma \phi(\pi(a)) \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_\Sigma \phi(a_{\mathcal{M}})\} \in \mathcal{F}$$

Proof. We show that the interpretation of functions is well defined. Let $a, b \in \left(\prod_{\mathcal{M}\in\mathfrak{M}}\right)^{n_f}$. Suppose for each

$i \in \{1, \dots, n_f\}$, $a_i \sim b_i \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$. Then

$$\begin{aligned}
&\text{for each } i, \{\mathcal{M} \in \mathfrak{M} \,|\, (a_i)_{\mathcal{M}} = (b_i)_{\mathcal{M}}\} \in \mathcal{F} \\
&\Rightarrow \left\{\mathcal{M} \in \mathfrak{M} \,|\, \bigwedge_{i=1}^{n} (a_i)_{\mathcal{M}} = (b_i)_{\mathcal{M}}\right\} \in \mathcal{F} \text{ by closure under finite adjunction} \\
&\Rightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, a_{\mathcal{M}} = b_{\mathcal{M}}\} \in \mathcal{F} \\
&\Rightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, f^{\mathcal{M}}(a_{\mathcal{M}}) = f^{\mathcal{M}}(b_{\mathcal{M}})\} \in \mathcal{F} \text{ by closure under superset} \\
&\Rightarrow \pi(f^{\mathcal{M}}(a_{\mathcal{M}})) = \pi(f^{\mathcal{M}}(b_{\mathcal{M}})) \text{ by definition of the quotient}
\end{aligned}$$

We use the following claim: If t is a term with variables S and $a \in (\prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M})^S$ then there exists $b \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ such that

$$t^{\mathcal{N}} \circ \pi(a) = \pi(b) \text{ and } \forall \mathcal{M} \in \mathfrak{M}, t^{\mathcal{M}}(a_{\mathcal{M}}) = b_{\mathcal{M}}$$

We prove this by induction on t:

- If t is a constant symbol c then pick $b := (c^{\mathcal{M}})_{\mathcal{M} \in \mathfrak{M}}$.
- If t is a variable then let $a \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ (only one varible), pick $b := a$.
- If t is a $f(s)$ then let $a \in (\prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M})^S$ by the inducition hypothesis there exists $c \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ such that

  $$s^{\mathcal{N}} \circ \pi(a) = \pi(c) \text{ and } \forall \mathcal{M} \in \mathfrak{M}, s^{\mathcal{M}}(a_{\mathcal{M}}) = c_{\mathcal{M}}$$

  Then we can take $b = (f^{\mathcal{M}}(c_{\mathcal{M}}))_{\mathcal{M} \in \mathfrak{M}}$. Thus

  $$t^{\mathcal{N}} \circ \pi(a) = f^{\mathcal{N}}(s^{\mathcal{N}} \circ (\pi(a))) = f^{\mathcal{N}}(\pi(c)) = \pi(f^{\mathcal{M}}(c_{\mathcal{M}}))_{\mathcal{M} \in \mathfrak{M}} = \pi(b)$$

  and for any $\mathcal{M} \in \mathfrak{M}$,

  $$t^{\mathcal{M}}(a_{\mathcal{M}}) = f^{\mathcal{M}} \circ s^{\mathcal{M}}(a_{\mathcal{M}}) = f^{\mathcal{M}}(c_{\mathcal{M}}) = b_{\mathcal{M}}$$

We now induct on ϕ to show that for any appropriate a,

$$\mathcal{N} \vDash_{\Sigma} \phi(\pi(a)) \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_{\Sigma} \phi(a_{\mathcal{M}})\} \in \mathcal{F}$$

- The case where ϕ is $\top$ is trivial (noting that anything models $\top$ and $\mathfrak{M} \in \mathcal{F}$).
- If ϕ is $s = t$ then it suffices to show that

  $$s^{\mathcal{N}} \circ \pi(a) = t^{\mathcal{N}} \circ \pi(a) \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_{\Sigma} t^{\mathcal{M}}(a_{\mathcal{M}}) = s^{\mathcal{M}}(a_{\mathcal{M}})\} \in \mathcal{F}$$

  ($\Rightarrow$) If for two terms s, t we have $s^{\mathcal{N}} \circ \pi(a) = t^{\mathcal{N}} \circ \pi(a)$ then by the claim there exists $b \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ such that

  $$\mathfrak{M} = \{\mathcal{M} \in \mathfrak{M} \,|\, s^{\mathcal{M}}(a_{\mathcal{M}}) = b_{\mathcal{M}} = t^{\mathcal{M}}(a_{\mathcal{M}})\} = \{\mathcal{M} \in \mathfrak{M} \,|\, t^{\mathcal{M}}(a_{\mathcal{M}}) = s^{\mathcal{M}}(a_{\mathcal{M}})\}$$

  which is therefore in the filter $\mathcal{F}$. ($\Leftarrow$) If for two terms s, t we have $s^{\mathcal{N}} \circ \pi(a) \neq t^{\mathcal{N}} \circ \pi(a)$ then by the claim there exist $b \neq c \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ such that

  $$\{\mathcal{M} \in \mathfrak{M} \,|\, t^{\mathcal{M}}(a_{\mathcal{M}}) = s^{\mathcal{M}}(a_{\mathcal{M}})\} = \{\mathcal{M} \in \mathfrak{M} \,|\, b_{\mathcal{M}} = c_{\mathcal{M}}\} = \varnothing$$

  which is not in the filter $\mathcal{F}$ as it is proper.
- If ϕ is $r(t)$ then by the claim we have $b \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ with the desired properties. It suffices to show that

  $$\pi(b) \in r^{\mathcal{N}} \Leftrightarrow \{\mathcal{M} \in \mathcal{M} \,|\, b_{\mathcal{M}} \in r^{\mathcal{M}}\} \in \mathcal{F}$$

  This follows from our definition of interpretation of relation symbols.

- If ϕ is $\neg\psi$ then $\mathcal{N} \vDash_\Sigma \phi(\pi(a))$ if and only if $\{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M})\} \notin \mathcal{F}$ by induction. This holds if and only if its complement is in the filter: $\{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \nvDash_\Sigma \psi(a_\mathcal{M})\} \in \mathcal{F}$ which is if and only if $\{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \phi(a_\mathcal{M})\} \notin \mathcal{F}$
- Without loss of generality we can use $\wedge$ instead of $\vee$ to make things simpler (replacing this comes down to dealing with a couple of $\neg$ statements). If ϕ is $\psi \wedge \chi$ then one direction follows filters being closed under intersection:

$$\begin{aligned}
&\mathcal{N} \vDash_\Sigma \phi(\pi(a))\\
&\Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M})\} \in \mathcal{F} \text{ and } \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \chi(a_\mathcal{M})\} \in \mathcal{F}\\
&\Rightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M})\} \cap \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \chi(a_\mathcal{M})\} \in \mathcal{F}\\
&\Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M}) \wedge \chi(a_\mathcal{M})\} \in \mathcal{F}
\end{aligned}$$

  To make second implication a double implication we note that each of the two sets

$$\{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M})\} \in \mathcal{F} \text{ and } \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \chi(a_\mathcal{M})\} \in \mathcal{F}$$

  are supersets of the intersection which is in $\mathcal{F}$.
- Without loss of generality we can use $\exists$ instead of $\forall$ to make things simpler. ($\Rightarrow$) Suppose $\mathcal{N} \vDash_\Sigma \exists v, \psi(\pi(a), v)$. Then there exists $b \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ such that $\mathcal{N} \vDash_\Sigma \psi(\pi(a), \pi(b))$. Then by induction

$$\{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M}, b_\mathcal{M})\} \in \mathcal{F}$$

  This is a subset of

$$\{\mathcal{M} \in \mathfrak{M} \,|\, \exists c \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M}, c)\} = \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \exists v, \psi(a_\mathcal{M}, v)\}$$

  ($\Leftarrow$) Suppose $Y := \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \exists v, \psi(a_\mathcal{M}, v)\} \in \mathcal{F}$. Then by the axiom of choice we have for each $\mathcal{M}$

$$\begin{cases} b_\mathcal{M} \in \mathcal{M}, \mathcal{M} \vDash_\Sigma & \text{if } \mathcal{M} \in Y \\ b_\mathcal{M} \in \mathcal{M} & \mathcal{M} \notin Y \end{cases}$$

  since each $\mathcal{M}$ is non-empty. By induction we have $\mathcal{N} \vDash_\Sigma \psi(\pi(a), \pi(b))$ and so $\mathcal{N} \vDash_\Sigma \exists v, \psi(\pi(a), v)$

$\square$

Corollary – The Compactness Theorem
A Σ-theory is consistent if and only if it is finitely consistent.

Proof. Suppose T is finitely consistent. For each finite subset $\Delta \subseteq T$ we let $\mathcal{M}_\Delta$ be the given non-empty model of Δ, which exists by finite consistency. We generate an ultrafilter $\mathcal{F}$ on $\mathfrak{M} := \{\mathcal{M}_\Delta \,|\, \Delta \in I\}$ and use Łos's Theorem to show that $\prod \mathfrak{M}/\mathcal{F}$ is a model of T. Let

$$I = \{\Delta \subseteq T \,|\, \Delta \text{ finite}\} \quad \text{and } [\star] : I \to \mathcal{P}(I) := \Delta \mapsto \{\Gamma \in I \,|\, \Delta \subseteq \Gamma\}$$

Writing $[I]$ for the image of I, we claim that $\mathcal{F} := \{U \in \mathcal{P}(I) \,|\, \exists V \in [I], V \subseteq U\}$ forms an ultrafilter on I (i.e. an ultrafilter on the Boolean algebra $\mathcal{P}(I)$). Indeed

- $\varnothing \in I$ thus $I = \{[\varnothing] \in [I] \subseteq \mathcal{F}\}$.
- Suppose $\varnothing \in \mathcal{F}$ then $\varnothing \in [I]$ and so there exists $\Delta \in I$ such that $[\Delta] = \varnothing$. This is a contradiction as $\Delta \in [\Delta]$.
- If $U, V \in \mathcal{F}$ then there exist $\Delta_U, \Delta_V \in I$ such that $[\Delta_U] \subseteq U$ and $[\Delta_V] \subseteq V$.

$$\begin{aligned}
[\Delta_U] \cap [\Delta_V] &= \{\Gamma \in I \,|\, \Delta_0 \subseteq \Gamma \text{ and } \Delta_1 \subseteq \Gamma\}\\
&= \{\Gamma \in I \,|\, \Delta_0 \cup \Delta_1 \subseteq \Gamma\}\\
&= [\Delta_0 \cup \Delta_1] \in [I] \subseteq \mathcal{F}
\end{aligned}$$

- Closure under superset is clear.

We identify each $\mathcal{M}_\Delta \in \mathfrak{M}$ with $\Delta \in I$ and generate the same filter (which we will still call $\mathcal{F}$) on $\mathfrak{M}$ (this is okay as the power sets are isomorphic as Boolean algebras.) By Łos's Theorem $\prod \mathfrak{M}/\mathcal{F}$ is a well-defined Σ-structure such that for any Σ-sentence ϕ

$$\prod \mathfrak{M}/\mathcal{F} \vDash_\Sigma \phi \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \phi\} \in \mathcal{F}$$

Let $\phi \in T$, then $\{\Delta \in I \,|\, \{\phi\} \subseteq \Delta\} \in \mathcal{F}$ and so

$$\{\Delta \in I \,|\, \{\phi\} \subseteq \Delta\} \subseteq \{\Delta \in I \,|\, \phi \in \Delta\} \subseteq \{\Delta \in I \,|\, \mathcal{M} \vDash_\Sigma \phi\} \in \mathcal{F}$$

The image of this under the isomorphism is $\{\mathcal{M}_\Delta \in X \,|\, \mathcal{M} \vDash_\Sigma \phi\}$ thus is in $\mathcal{F}$ and so $\prod \mathfrak{M}/\mathcal{F} \vDash_\Sigma \phi$. □

2.2.4 Stone Duality

This section of the appendix gives a more algebraic way of constructing Stone spaces.

Definition – A Boolean algebra on $F(\Sigma, n)$
Let T be a Σ-theory. We quotient out $F(\Sigma, n)$ by the equivalence relation

$$\phi \sim \psi \quad := \quad \phi \text{ and } \psi \text{ equivalent modulo } T := T \vDash_\Sigma \forall v, (\phi \leftrightarrow \psi)$$

Call the projection into the quotient π and the quotient $F(\Sigma, n)/T$. We make $F(\Sigma, n)/T$ into a Boolean algebra by interpreting 0 as $\pi(\bot)$, 1 as $\pi(\top)$, $\pi(\phi) \cap \pi(\psi)$ as $\pi(\phi \wedge \psi)$, $\pi(\phi) \cup \pi(\psi)$ as $\pi(\phi \vee \psi)$, $\smallsetminus \pi(\phi)$ as $\pi(\neg \phi)$ and $\pi(\phi) \leq \pi(\psi)$ as

$$\{(\pi(\phi), \pi(\psi)) \,|\, T \vDash_\Sigma \forall v, (\phi \rightarrow \psi)\}$$

One can verify that these are well-defined and satisfy the axioms of a Boolean algebra. Notice we need T (potentially chosen to be the empty set) to make $\rightarrow$ look like $\leq$ and that we had to quotient modulo T to make $\leq$ satisfy antisymmetry. Antisymmetry in this context looks very much like 'propositional extensionality'. Thus it makes sense to consider the Stone space of this Boolean algebra $S(F(\Sigma, n)/T)$.

Lemma
If $p \subseteq F(\Sigma, n)$ is a maximal subset ($\forall \phi \in F(\Sigma, n), \phi \in p$ or $\neg \phi \in p$) then $\pi(\phi) \in \pi(p)$ in the quotient implies $\phi \in p$.

Proof. If $\pi(\phi) \in \pi(p)$ then there exists $\psi \in p$ such that ψ is equivalent to ϕ modulo T. By consistency with T there exists a non-empty Σ-model $\mathcal{M}$ of T and $b \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma p(b)$, in particular $\mathcal{M} \vDash_\Sigma \psi(b)$. Equivalence modulo T then gives us that $\mathcal{M} \vDash_\Sigma \phi(b)$. By maximality of p, ϕ or $\neg \phi$ is in p but the latter would lead to $\mathcal{M} \nvDash_\Sigma \phi(b)$, a contradiction. □

Proposition – The Stone space is a set of ultrafilters
The Stone space of a Σ-theory T is homeomorphic to the set of ultrafilters from $S(F(\Sigma, n)/T)$ that have preimage consistent with T with the subspace topology. In other words, if π is the projection to the quotient then $S_n(T) \cong X$, where

$$X := \left\{\mathcal{F} \in S(F(\Sigma, n)/T) \,|\, \pi^{-1}(\mathcal{F}) \text{ consistent with } T\right\}$$

Proof. Warning: this proof uses π to be three different things, the quotient $F(\Sigma, n) \rightarrow F(\Sigma, n)/T$, the image map (of the quotient) $S_n(T) \rightarrow X$, and the map of clopen sets (the image map of the image map) $\mathcal{P}(S_n(T)) \rightarrow \mathcal{P}(X)$. The second will be a homeomorphism and the third will be a map between subsets of the topologies (in particular the clopen subsets).

We show that sending $p \in S_n(T)$ to its image under the projection to the quotient $\pi(p)$ is a homeomorphism. To show that it is well-defined it suffices to show that for any p, a maximal n-type over T, $\pi(p)$ is an ultrafilter of $F(\Sigma, n)/T$ with preimage consistent with T. Preimage being consistent with T follows from the definition of n-types over theories. To show that it is a proper filter:

- $\top \in p$ by consistency and maximality. Hence $\pi(\top) \in \pi(p)$.
- If $\pi(\bot) \in \pi(p)$ then $\bot \in p$ which is a contradiction with consistency.
- If $\pi(\phi), \pi(\psi) \in \pi(p)$ then $\phi, \psi \in p$ and so $\phi \wedge \psi \in p$ thus by definition of the Boolean algebra $F(\Sigma, n)/T$,

$$\pi(\phi) \cap \pi(\psi) = \pi(\phi \wedge \psi) \in \pi(p)$$

- If $\pi(\phi) \in \pi(p)$ and $\pi(\phi) \leq \pi(\psi)$ then $\phi \in p$ and by definition of $\leq$,

$$T \vDash_\Sigma \forall v, (\phi \to \psi)$$

  Since p is consistent with T there exists a non-empty Σ-structure $\mathcal{M}$ and $b \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma \phi(b) \to \psi(b)$ and $\mathcal{M} \vDash_\Sigma \phi(b)$. Hence $\mathcal{M} \vDash_\Sigma \psi(b)$ and by maximality of p we have $\psi \in p$ and $\pi(\psi) \in \pi(p)$.

The image $\pi(p)$ is an ultrafilter by the equivalent definition: if $\pi(\phi) \in F(\Sigma, n)/T$ then either $\phi \in p$ or $\smallsetminus\phi \in p$ by maximality of p, hence $\pi(\phi) \in \pi(p)$ or $\pi(\smallsetminus\phi) \in \pi(p)$. Thus we have π is a map into

$$\left\{\mathcal{F} \in S(F(\Sigma, n)/T) \,|\, \pi^{-1}(\mathcal{F}) \text{ consistent with } T\right\}$$

Injectivity: if $p, q \in S_n(T)$ and $\pi(p) = \pi(q)$ then if $\phi \in p$, we have $\pi(\phi) \in \pi(p) = \pi(q)$ and by our claim above $\phi \in q$. Surjectivity: let $\mathcal{F} \in S(F(\Sigma, n)/T)$ have its preimage consistent with T. Then its preimage is an n-type. If its preimage is a maximal n-type then we have surjectivity. Indeed since $\mathcal{F}$ is an ultrafilter if $\phi \in F(\Sigma, n)$ then $\pi(\phi) \in \mathcal{F}$ or $\pi(\neg\phi) = \smallsetminus\pi(\phi) \in \mathcal{F}$, hence $\phi \in \pi^{-1}(\mathcal{F})$ or $\neg\phi \in \pi^{-1}(\mathcal{F})$.

To show that the map is continuous in both directions, it suffices to show that images of clopen sets are clopen and preimages of clopen sets are clopen, as each topology is generated by their clopen sets. For $\phi \in F(\Sigma, n)$ since $\pi : S_n(T) \to X$ is a bijection we have that

$$\pi([\phi]_T) = \left\{\mathcal{F} \in X \,|\, \phi \in \pi^{-1}(\mathcal{F})\right\} = \{\mathcal{F} \in F(\Sigma, n)/T \,|\, \pi(\phi) \in \mathcal{F} \text{ and } \mathcal{F} \in X\} = [\pi(\phi)] \cap X$$

and similarly $\pi^{-1}([\pi(\phi)] \cap X) = [\phi]_T$. Hence there is a correspondence between clopen sets. $\square$

Lemma – Topological consistency
Let $\mathcal{F} \in S(F(\Sigma, n)/T)$ and T be a Σ-theory. $\pi^{-1}(\mathcal{F})$ is consistent with T if and only if

$$\pi^{-1}(\mathcal{F}) \in \bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T$$

if and only if

$$\bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T \text{ is non empty}$$

Proof. (1. $\Rightarrow$ 2. $\Rightarrow$ 3.) Suppose $\pi^{-1}(\mathcal{F})$ is consistent with T. Then $\pi^{-1}(\mathcal{F}) \in S_n(T)$ thus for any $\phi \in \pi^{-1}(\mathcal{F})$, $\pi^{-1}(\mathcal{F}) \in [\phi]_T$. Hence

$$\pi^{-1}(\mathcal{F}) \in \bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T$$

and it is non-empty.

(3. $\Rightarrow$ 1.) Suppose

$$p \in \bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T$$

then $\forall \phi \in \pi^{-1}(\mathcal{F}), \phi \in p$. As $\mathcal{F}$ is an ultrafilter, for any $\phi \in p$,

$$\phi \notin \pi^{-1}(\mathcal{F}) \Rightarrow \neg\phi \in \pi^{-1}(\mathcal{F}) \Rightarrow \neg\phi \in p \text{ a contradiction}$$

Hence $p = \pi^{-1}(\mathcal{F})$. Hence $\pi^{-1}(\mathcal{F}) \in S_n(T)$ and thus is consistent with T. □

Proposition – Topological compactness implies compactness for types
Let $\mathcal{F} \in S(F(\Sigma, n)/T)$ and T be a Σ-theory. Then $\pi^{-1}(\mathcal{F})$ is consistent with T if and only if $\pi^{-1}(\mathcal{F})$ if finitely consistent with T.

Proof. By definition $\pi^{-1}(\mathcal{F})$ is finitely consistent with T if and only if any finite subset of $\pi^{-1}(\mathcal{F})$ is consistent with T. Translating this to the topology, this is if and only if for any finite subset $\Delta \subseteq \pi^{-1}(\mathcal{F})$,

$$\bigcap_{\phi \in \Delta} [\phi]_T \text{ is non empty}$$

By topological compactness of $S_n(T)$ this is if and only if

$$\bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T \text{ is non empty}$$

Translating this back to model theory this is if and only if $\pi^{-1}(\mathcal{F})$ is consistent with T. □

2.3 Ordinals

[3]

Definition – Linear ordering, well-ordering, transitive
Let X be a set (or class or whatever) partially ordered by $\leq$. It is a linearly ordered if for each $x, y \in X$, $x \leq y$ or $y \leq x$. It is a well-ordered if each non-empty subset $S \subseteq X$ has a least element, i.e. there exists $s \in S$ such that for any $x \in S$, $s \leq x$.

A set X is transitive if for any member of X is a subset of X. An example of transitive sets are the naturals:

$$0 := \varnothing, 1 := \{0\}, 2 := \{0, 1\}, \ldots$$

Exercise (Transitive). *The reason for the use of the word transitive is due to its equivalent definition: x is transitive if for any $y \in x$ and any $z \in y$, $z \in x$.*

Exercise (Well-orderings are linear). *Take two elements and consider the set containing these two elements. This has a minimum.*

Definition – Ordinal, succesion, limit ordinal
A set in an ordinal if it is transitive and well-ordered by taking $<$ to be $\in$. The class of ordinals is denoted Ord.

If $\alpha \in \mathrm{Ord}$ then $\alpha + 1$ is defined to be $\alpha \cup \{\alpha\}$, $\alpha + 1$ is then called a successor ordinal. Check that the successor of an ordinal is an ordinal. If $\alpha \in \mathrm{Ord}$ is not a successor ordinal then it is called a limit ordinal.

We endow Ord with an ordering given by $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.

> **Lemma – Transitivity of $<$**
>
> Suppose $\leq$ is a partial order. The relation $<$ (defined by $\alpha < \beta$ if and only if $\alpha \leq \beta$ and $a \neq b$) is transitive.
>
> Hence $\alpha \leq \beta < \gamma$ implies $\alpha < \gamma$.

Proof. Suppose $a < b$ and $b < c$. Then $a \leq b$ and $b \leq c$ and by transitivity of $\leq$ we have $a \leq c$. It remains to show that $a \neq c$. Suppose $a = c$, then $a \leq b$ and $b \leq a$ and by antisymmetry $a = b$. This is false since $a < b$ and so $a \neq b$ by definition. $\square$

> **Proposition – Basic facts about ordinals**
>
> Most importantly, we show that Ord is well-ordered.
>
> 1. 0 is a limit ordinal.
> 2. Subsets of ordinals are well-ordered by $\in$.
> 3. The intersection of a non-empty subclass of is an ordinal.
> 4. Ordinals are closed under membership: if $\alpha \in \mathrm{Ord}$ and $\beta \in \alpha$ then $\beta \in \mathrm{Ord}$.
> 5. Let $\alpha \neq \beta$ be ordinals. If $\alpha \subseteq \beta$ then $\alpha \in \beta$.
> 6. Ord is well-ordered by $\subseteq$ (which is by the fifth part the same thing as $=$ or $\in$).

Proof.

1. 0 is trivially well-ordered and transitive. If $0 = \alpha \cup \{\alpha\}$ then $\alpha \in 0$, which is a contradiction.
2. Let $\alpha \in \mathrm{Ord}$ and $S \subseteq \alpha$. By moving elements into α we see that S inherits the linear ordering given by $\in$. Any subset of S is a subset of α, hence any subset has a minimal element. Thus S is well-ordered
3. Let S be a non-empty class of ordinals, containing α, say. $\bigcap S$ is a subset of α thus is well-ordered by the second part. It is transitive: any element in the intersection satisfies
$$\forall \beta \in S, x < \beta$$
hence for every $\beta \in S$, as β is transitive, $x \subseteq \beta$. Thus $x \subseteq \bigcap S$.
4. Let $\alpha \in \mathrm{Ord}$ and $\beta \in \alpha$. As α is transitive $\beta \subseteq \alpha$. By the second part we have that β is well-ordered by $<$.

   It remains to show that β is transitive. Let $\gamma \in \beta$ and suppose for a contradiction that $\gamma \nsubseteq \beta$. Then there exists $\delta \in \gamma$ such that $\delta \notin \beta$. By transitivity of $<$ (in this case $\in$), $\delta \in \gamma \in \beta \in \alpha$ implies $\delta \in \alpha$.

   As α is linearly-ordered we have that $\beta \leq \delta$ or $\delta \leq \beta$, equivalently three cases $\beta \in \delta$ or $\beta = \delta$ or $\delta \in \beta$. The last case is false by assumption. Then first case gives $\delta \in \gamma \in \beta \in \delta$, hence $\delta < \delta$ by transitivity of $<$. In the second case $\delta \in \gamma \in \beta = \delta$, hence by transitivity again $\delta < \delta$. In either case $\delta \neq \delta$, a contradiction.
5. Let γ be the minimum element of $\beta \setminus \alpha$, using well-ordering of β. Then we claim that $\alpha = \gamma$, which implies $\alpha \in \beta$. Indeed if $x \in \alpha$ then $x \in \beta$ by assumption. By linearity of β, $\gamma \leq x$ or $x < \gamma$. In the first case we have by transitivity of $<$ that $\gamma < \alpha$, which is a contradiction as $\gamma \in \beta \setminus \alpha$. Thus $x \in \gamma$ and so $\alpha \subseteq \gamma$.

   On the other hand suppose $x \in \gamma$ and $x \notin \alpha$. Then $x \in \beta \setminus \alpha$ and so by minimality $\gamma \leq x$. This is a contradiction as $\gamma \leq x < \gamma$ and by transitivity we have $\gamma < \gamma$.

   We can see α as the initial segment β given by γ, i.e. $\{x \in \beta \mid x < \gamma\}$.

6. Reflexivity, antisymmetry and transitivity are clear. It suffices to show that it is a well-ordering. Let S be a non-empty set of ordinals. $\bigcap_{\alpha \in S} \alpha$ is an ordinal by the third part. We want to show that there exists $\alpha \in S$ such that $\bigcap S = \alpha$. Suppose not, then by the fifth part we have for any $\alpha \in S$, $\bigcap S \in \alpha$. Hence $\bigcap S \in \bigcap S$, which is a contradiction.

□

Proposition – Transfinite induction

Let $C \subseteq \mathrm{Ord}$ be defined inductively:

- If α is a limit ordinal and $\forall \beta < \alpha, \beta \in C$ then $\alpha \in C$. (In particular $0 \in C$.)
- If $\alpha \in C$ then $\alpha + 1 \in C$.

Then $C = \mathrm{Ord}$. We have the first constructor as the limit case because the smallest ordinal 0 is a limit ordinal.

('Strong') Let $D \subseteq \mathrm{Ord}$ be defined with a single constructor:

- If $\alpha \in \mathrm{Ord}$ and $\forall \beta < \alpha, \beta \in C$ then $\alpha \in C$.

Then $D = \mathrm{Ord}$.

Proof. Suppose $C \neq \mathrm{Ord}$. Then as Ord is well-ordered, there exists $\beta \in \mathrm{Ord}$ that is the least ordinal such that $\beta \notin C$. If β is a successor $\alpha + 1$ then by minimality of β, $\alpha \in C$ and applying the first property gives $\beta \in C$, which is a contradiction. Otherwise, β is a limit ordinal. Thus by minimality $\forall \alpha < \beta, \alpha \in C$. Applying the second property of C gives $\beta \in C$.

Check that the $C \subseteq D$ which implies $D = \mathrm{Ord}$. □

Here is an example of transfinite induction in use:

Lemma – Less than and the successor

If $\alpha < \beta$ are ordinals then $\alpha + 1 \in \beta$ or $\alpha + 1 = \beta$.

Proof. We induct on β. Suppose β is a limit ordinal. Then $\alpha + 1 \notin \beta$ implies $\beta \leq \alpha + 1$ by Ord being well-ordered. This implies $\alpha < \alpha$ which is a contradiction. Hence in this case $\alpha + 1 \in \beta$.

Suppose $\beta = \gamma + 1$. then $\alpha < \gamma$ or $\alpha = \gamma$. In the second case $\alpha + 1 = \gamma + 1 = \beta$ and we are done. In the first case, by the induction hypothesis $\alpha + 1 \in \gamma$ or $\alpha + 1 = \gamma$. In either case $\alpha + 1 \in \beta$. □

To define a function on Ord, it then suffices to define a function on C, using the recursor (in Type theory) of C or of D. For our purposes it means that only need to say what the function does to successor and limit ordinals, or only what it does to ordinals given the images of all smaller ordinals.

Proposition – Transfinite recursion

We denote the class of all sets by V. Let $G : V \to V$. Then there exists a unique map $F : \mathrm{Ord} \to V$ such that for each ordinal α,

$$F(\alpha) = G(F\big|_{\alpha})$$

F can be thought of as a sequence indexed by Ord.[†]

[†] As we are in set theory it makes sense to consider $F\big|_{\alpha}$ as a set.

Proof. We show by transfinite induction that for each ordinal α there exists a unique $F_\alpha : \alpha \to V$ such that for each $\xi < \alpha$

$$F_\alpha(\xi) = G(F_\alpha\big|_{\xi})$$

- If α is a successor ordinal and there exists a unique $F_\beta : \beta \to V$ such that $\forall \xi < \beta$,
$$F_\beta(\xi) = G(F_\beta\big|_\xi)$$
Then take $F_\alpha : \alpha \to V$ such that it restricts to F_β on β and maps $\beta \mapsto G(F_\beta)$. F_α is the unique map that satisfies $\forall \xi < \alpha, F(\xi) = G(F\big|_\xi)$ since its restriction to β is the unique.

- If α is a limit ordinal and for any $\beta < \alpha$ there exists a unique $F_\beta : \beta \to V$ satisfying $\forall \xi < \beta$
$$F_\beta(\xi) = G(F_\beta\big|_\xi)$$
Define $F_\alpha : \alpha \to V$ as the union of all the F_β. F_α is well-defined since all of the F_β agree upon restriction by uniqueness. F_α satisfies $\forall \xi < \alpha, F_\alpha(\xi) = G(F\big|_\xi)$ by construction. F_α is the unique map satisfying this: if H satisfies the same then for any $\beta \in \alpha$, $F_\alpha\big|_\beta$ (similarly $H\big|_\beta$) is the unique map f satisfying
$$\forall \xi < \beta, f(\xi) = G(f\big|_\xi)$$
Thus for any $\beta \in \alpha$, $F_\alpha\big|_\beta = H\big|_\beta$
$$F_\alpha(\beta) = G(F_\alpha\big|_\beta) = G(H\big|_\beta) = H(\beta)$$
and so $F_\alpha = H$.

We define F to map any ordinal α to $G(F_\alpha)$. By strong induction on α we show that $F_\alpha = F\big|_\alpha$: suppose for any $\beta < \alpha$, $F_\beta = F\big|_\beta$, then for any $\beta < \alpha$
$$F_\alpha(\beta) = G(F_\alpha\big|_\beta) = G(F_\beta) = F(\beta) = F\big|_\alpha(\beta)$$
and so $F_\alpha = F\big|_\alpha$ and for any α,
$$F(\alpha) = G(F_\alpha) = G(F\big|_\alpha)$$
Lastly F is unique: if H also satisfies the conditions then suppose $F(\beta) = H(\beta)$ for all $\beta < \alpha$. This implies $F\big|_\alpha = H\big|_\alpha$ and
$$F(\alpha) = G(F\big|_\alpha) = G(H\big|_\alpha) = H(\alpha)$$
thus again by strong induction on α we have $F = H$. □

2.3.1 Pregeometries

This subsection walks through the basics of pregeometries, leading to the theorem on bases having the same cardinality, allowing us to define dimension. It follows the lecture notes by Prof. C. Ward Henson [7].

Definition – Pregeometry

Let X be a set and cl be a map from the power set of X to itself, called the 'closure map'. (X, cl) is a pregeometry the following hold

1. cl is a morphism of ordered sets: for any $U \subseteq V \subseteq X$, $\mathrm{cl}(U) \subseteq \mathrm{cl}(V)$.
2. Idempotence: for any $U \subseteq X$, $\mathrm{cl}(\mathrm{cl}(U)) = \mathrm{cl}(U)$.
3. Finite character: if $U \subseteq X$ and $a \in \mathrm{cl}(U)$ then there exists a finite subset $F \subseteq U$ such that $a \in \mathrm{cl}(F)$.
4. Exchange: if $a \in \mathrm{cl}(U \cup \{b\})$ then $a \in \mathrm{cl}(U)$ or $b \in \mathrm{cl}(U \cup \{a\})$.

Definition – Span, independence, basis

Let (X, cl) be a pregeometry. Let $U, V \subseteq X$.

- U spans V if $U \subseteq V$ and $\mathrm{cl}(U) = \mathrm{cl}(V)$.
- U is independent if for any $a \in U$, $a \notin \mathrm{cl}(U \setminus \{a\})$.

- U is a basis of V if U spans V and is independent.

Proposition – Independence and span
Let (X, cl) be a pregeometry, $a \in X$ and $U, V \subseteq X$.

1. If U is independent then any subset of U is independent.
2. If U is independent and $a \notin \mathrm{cl}(U)$ then then $U \cup \{a\}$ is independent.
3. U is independent if and only if every finite subset of U is independent.
4. U spans V if and only if there exists a subset of U that is a basis of V.
5. U is a basis of V if and only if U is a maximally independent subset of V if and only if U is a minimally spanning subset of V, i.e. for any independent $W \subseteq V$, $U \not\subset W$; for any spanning $W \subseteq V$, $W \not\subset U$.
6. Any independent subset U is contained in a basis.

Proof.

1. Let $a \in A \subseteq U$. Suppose for a contradiction $a \in \mathrm{cl}(A \setminus \{a\})$ then as cl is a morphism of ordered sets $a \in \mathrm{cl}(U \setminus \{a\})$, which is false as U is independent. Hence A is independent.
2. The forward direction follows from the part 1.

   $(\Leftarrow)$ Let $a \in U$ and suppose for a contradiction $a \in \mathrm{cl}(U \setminus \{a\})$. As cl has 'finite character' there exists a finite subset $F \subseteq U \setminus \{a\}$ such that $a \in \mathrm{cl}(F)$. This is a contradiction as every subset of U is independent so F is independent.
3. Let $x \in U \cup \{a\}$. Suppose for a contradiction that $x \in \mathrm{cl}(U \cup \{a\} \setminus \{x\})$. We will show that $a \in \mathrm{cl}(U)$, which is false. If $x = a$ then $a \in \mathrm{cl}(U \cup \{a\} \setminus \{a\}) = \mathrm{cl}(U)$ and we are done. Otherwise $x \in U$ then by the exchange property we have $x \in \mathrm{cl}(U \setminus \{x\})$ or $a \in \mathrm{cl}(U)$. The first case is false since U is independent, hence we are done.
4. The backward direction follows from the fact that cl is a morphism of ordered sets.

   $(\Rightarrow)$(Zorn) Suppose U spans V and consider
   $$\{W \subseteq U \,|\, W \text{ is independent}\}$$
   This set is non-empty since the empty set is independent. Let C be a chain in the set. Then $\bigcup C$ is a subset of U. It is independent because every finite subset of it is independent by part 3.
5. We show that U is basis of V if and only if it is a maximally independent subset of V.

   $(\Rightarrow)$ Suppose for a contradiction that $U \subset W$ and W is independent. Let $a \in W \setminus U$. Since U is a basis $a \in \mathrm{cl}(U) \subseteq \mathrm{cl}(W \setminus \{a\})$, contradicting independence of W.

   $(\Leftarrow)$ Suppose U is maximally independent. Then to show U is also spanning we show that $V \subseteq \mathrm{cl}(U)$. Let $a \in V$. If $a \in U$ then we are done. If $a \notin U$ then $U \cup \{a\}$ is dependent by maximality and so there exists $b \in U \cup \{a\}$ such that $b \in \mathrm{cl}(U \cup \{a\} \setminus \{b\})$. In the case that $b \in U$ by independence of U and the exchange property
   $$b \in \mathrm{cl}(U \setminus b \cup \{a\}) \wedge b \notin \mathrm{cl}(U \setminus \{b\}) \Rightarrow a \in \mathrm{cl}(U)$$
   In the case that $b = a$ then
   $$a \in \mathrm{cl}(U \cup \{a\} \setminus \{a\}) = \mathrm{cl}(U)$$

   Now we show that U is a basis of V if and only if it is a minimally spanning subset of V.

($\Rightarrow$) Suppose for a contradiction we have $W \subset U$ such that W spans V. Then let $a \in U \setminus W$. Since W is spanning we have

$$a \in \mathrm{cl}(W) \subseteq \mathrm{cl}(U \setminus \{a\})$$

contradicting independence of U.

($\Leftarrow$) Let U be a minimally spanning subset. To show it is independent, let $a \in U$ and suppose for a contradiction $a \in \mathrm{cl}(U \setminus \{a\})$. Then $U \subseteq \mathrm{cl}(U \setminus \{a\})$ and so

$$V \subseteq \mathrm{cl}(U) \subseteq \mathrm{cl}(U \setminus \{a\})$$

and $U \setminus \{a\}$ spans V which contradicts the minimality of U.

6. (Zorn) Consider $\{V \subseteq X \mid U \subseteq V \text{ and } V \text{ independent}\}$. This is non-empty as it contains U. The union of any chain in it independent if and only if every finite subset of it is independent if and only if each set in it is independent by part 3.

□

Lemma – Support

Let (X, cl) be a pregeometry and let U be a basis of $V \subseteq X$. Then for any $a \in \mathrm{cl}(V)$ there exists a minimal finite subset $F \subseteq U$ such that $a \in \mathrm{cl}(F)$, i.e. if $a \in \mathrm{cl}(G)$ for some $G \subseteq V$ then $F \subseteq G$. We call F the support of a in the basis U.

Proof. Since cl has finite character we have that there exists a finite subset of U containing a in its closure, and by well ordering of the naturals (and choice) we can take one such subset F with minimal cardinality. Let G be another subset containing a in its closure. Suppose for a contradiction there exists $b \in F \setminus G$. Then $a \in \mathrm{cl}(G) \subseteq \mathrm{cl}(U \setminus \{b\})$. Hence

$$U \setminus \{b\} \cup \{a\} \subseteq \mathrm{cl}(U \setminus \{b\}) \Rightarrow \mathrm{cl}(U \setminus \{b\} \cup \{a\}) = \mathrm{cl}(U \setminus \{b\})$$

We show that this contradicts the independence of U. Since $b \in F$ and F is minimal, $a \notin \mathrm{cl}(F \setminus \{b\})$ and by exchange

$$b \in \mathrm{cl}(F \setminus \{b\} \cup \{a\}) \subseteq \mathrm{cl}(U \setminus \{b\} \cup \{a\}) = \mathrm{cl}(U \setminus \{b\})$$

contradicting the independence of U. □

Proposition – Existence of dimension

Let (X, cl) be a pregeometry. Then for any U there exists a basis of U.

Proof. We apply Zorn's lemma. The set $\{W \subseteq U \mid W \text{ is independent}\}$ contains the empty set so is non-empty. Let C be a chain in there. Then $\bigcup C$ is a subset of U. $\bigcup C$ is independent if and only if every finite subset of it is independent, which is true since any part of the chain is independent. Hence there exists a maximally independent subset of U, i.e. a basis of U. □

Proposition – Uniqueness of dimension

Let (X, cl) be a pregeometry. Then if U and V are bases of $Y \subseteq X$ then $|U| = |V|$.

Proof. Without loss of generality $|U| \leq |V|$. It suffices to show that $|V| \leq |U|$. We case on whether V is finite or not.

If V is infinite then we reconstruct V in the following way: for each $a \in U$ there exists a finite subset F_a of V such that $a \in \mathrm{cl}(F_a)$. By choice we can construct $(F_a)_{a \in U}$ and take the union $\bigcup_{a \in U} F_a$. For any $a \in U$, the union satisfies $a \in \mathrm{cl}(F_a) \subseteq \mathrm{cl}(\bigcup F_a)$ thus $U \subseteq \mathrm{cl}(\bigcup F_a)$ and

$$Y \subseteq \mathrm{cl}(U) \subseteq \mathrm{cl}(\bigcup F_a)$$

Since $\bigcup F_a$ is a subset of V that spans Y, and bases are minimally spanning sets, we have $V = \bigcup F_a$.

Suppose for a contradiction U is finite. Then $V = \bigcup_{a\in U} F_a$ would be a finite union of finite sets which is finite, contradicting our assumption about V. Hence

$$|V| = \left|\bigcup_{a\in U} F_a\right| \leq |U|$$

Thus we are done.

If V is finite then we use induction on $n \in \mathbb{N}$ to show that if V_0 is a independent set of cardinality n then there exists $W \subseteq U$ such that V_0 and W are disjoint, $V_0 \cup W$ is a basis of Y and $|V_0 \cup W| = |U|$. This will imply the existance of such a W for V such that

$$|V| \leq |V \cup W| = |U|,$$

completing the proof.

For the base case we take $W = U$. For the induction step we suppose V_0 is non-empty (containing a) and obtain via the induction hypothesis $W \subseteq U$ such that $V_0 \setminus \{a\}$ and W are disjoint, $V_0 \setminus \{a\} \cup W$ is a basis of Y and $|V_0 \setminus \{a\} \cup W| = |U|$. If $a \in W$ then we can take our new W to be $W \setminus \{a\}$ and see that V_0 and $W \setminus \{a\}$ are disjoint, $V_0 \cup W \setminus \{a\}$ is a basis of Y and $|V_0 \cup W \setminus \{a\}| = |U|$.

Otherwise, take F the support of a in the basis $V_0 \setminus \{a\} \cup W$. Supposing for a contradiction that $F \cap W$ were empty, we get $F \subseteq V_0 \setminus \{a\}$, which implies $a \in \mathrm{cl}(F) \subseteq \mathrm{cl}(V_0 \setminus \{a\})$, contradicting independence of V_0. Hence there exists $b \in F \cap W$.

We claim $W \setminus \{b\}$ is our new W. They are disjoint: since $V_0 \setminus \{a\} \cap W$ is empty and we are in the case where $a \notin W$, we have $V_0 \cap W$ is empty and so $V_0 \cap W \setminus \{b\}$ is empty.

To show that their union is independent: we first note that as subsets of independent sets are independent

$$(V_0 \setminus \{a\}) \cup (W \setminus \{b\}) = (V_0 \cup W \setminus \{b\}) \setminus \{a\}$$

is independent. It suffices to show that

$$a \notin \mathrm{cl}((V_0 \setminus \{a\}) \cup (W \setminus \{b\}))$$

Suppose for a contradiction $a \in \mathrm{cl}((V_0 \setminus \{a\}) \cup (W \setminus \{b\}))$. Then by minimality of F we have

$$b \in F \subseteq (V_0 \setminus \{a\}) \cup (W \setminus \{b\}) \Rightarrow b \in V_0 \setminus \{a\} \Rightarrow V_0 \setminus \{a\} \cap W \neq \varnothing$$

The last of which is a contradiction.

To show that their union is spanning it suffices to show that $b \in \mathrm{cl}(V_0 \cup W \setminus \{b\})$ since this implies

$$V_0 \setminus \{a\} \cup W \subseteq V_0 \cup W \subseteq \mathrm{cl}(V_0 \cup W \setminus \{b\})$$

and so $Y \subseteq \mathrm{cl}(V_0 \setminus \{a\} \cup W) \subseteq \mathrm{cl}(V_0 \cup W \setminus \{b\})$. Suppose $b \in V_0$, then $b \in \mathrm{cl}(V_0 \cup W \setminus \{b\})$ and we are done. Otherwise

$$(V_0 \cup W \setminus \{b\}) \setminus \{a\} = (V_0 \setminus \{a\}) \cup (W \setminus \{b\}) = (V_0 \setminus \{a\} \cup W) \setminus \{b\}$$

Since $V_0 \cup W \setminus \{b\}$ is independent $a \notin \mathrm{cl}(V_0 \setminus \{a\} \cup W \setminus \{b\})$. Thus by the exchange principle

$$a \in \mathrm{cl}((V_0 \setminus \{a\}) \cup (W \setminus \{b\}) \cup \{b\}) \Rightarrow b \in \mathrm{cl}(V_0 \cup W \setminus \{b\})$$

and so the union is spanning.

Finally we note that

$$|V_0 \cup W \setminus \{b\}| = |V_0| + |W| - 1 = |V_0 \setminus \{a\} \cup W| = |U|$$

and the proof is complete. □

Definition – Dimension
Let U be a subset of (X, cl) a pregeometry. Then the dimension of U is defined to be the cardinality of any basis of U. The existence and uniqueness of dimension are given by the previous two theorems.

Chapter 3

Model Theory of Fields

3.1 Ax-Grothendieck

This section studies the theories of fields in the language of rings, with particular focus on algebraically closed fields.

3.1.1 Language of Rings

We introduce rings and fields and construct the field of fractions of integral domains to see the models in action.

Definition – Signature of rings, theory of rings
We define $\Sigma_{\mathrm{RNG}} := (\{0,1\}, \{+,-,\cdot\}, n_\star, \varnothing, m_\star)$ to be the signature of rings, where $n_+ = n_\cdot = 2$, $n_- = 1$ and $m_\star$ is the empty function.

Using the obvious abbreviations $x + (-y) = x - y, x \cdot y = xy$ and so on, we define the theory of rings RNG as the set containing:

| Assosiativity of addition: $\forall x \forall y \forall z, (x+y)+z = x+(y+z)$

| Identity for addition: $\forall x, x + 0 = x$

| $\forall x, x - x = 0$

| Commutativity of addition: $\forall x \forall y, x + y = y + x$

| Assosiativity of multiplication: $\forall x \forall y \forall z, (x \cdot y) \cdot z = x \cdot (y \cdot z)$

| Identity for multiplication: $\forall x, x \cdot 0 = x$

| Commutativity of multiplication: $\forall x \forall y, x \cdot y = y \cdot x$

| Distributivity: $\forall x \forall y \forall z, x \cdot (y + z) = x \cdot y + x \cdot z$

Note that we don't have axioms for closure of functions and existence or uniqueness of inverses as it is encoded by interpretation of $+, -, \cdot$ being well-defined. Note that the theory of rings is universal.

Definition – Theory of integral domains and fields
We define the Σ_{RNG}-theory of integral domains

$$\mathrm{ID} := \mathrm{RNG} \cup \{0 \neq 1, \forall x \forall y, xy = 0 \to (x = 0 \vee y = 0)\}$$

and the Σ_{RNG}-theory of fields

$$\mathrm{FLD} := \mathrm{RNG} \cup \{\forall x, x = 0 \vee \exists y, xy = 1\}$$

Note that the theory of integral domains is universal but the theory of fields is not.

Proposition – Field of fractions

Suppose $\mathcal{A} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$. Then there exists an Σ_{RNG}-embedding $\iota : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{B} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$. We call $\mathcal{B}$ the field of fractions.

Proof. We construct $X = \left\{(x, y) \in \mathcal{A}^2 \,|\, y \neq 0\right\}$ and and equivalence relation $(x, y) \sim (v, w) \Leftrightarrow xw = yv$. (Use $\mathcal{A} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ to show that this is an equivalence relation.) Let $\mathcal{B} = X/\sim$ with $\pi : X \to \mathcal{B}$ as the quotient map. Denote $\pi(x, y) := \frac{x}{y}$, interpret $0^{\mathcal{B}} = \frac{0^{\mathcal{A}}}{1^{\mathcal{A}}}$ and $1^{\mathcal{B}} = \frac{1^{\mathcal{A}}}{1^{\mathcal{A}}}$. Interpret $+$ and $\cdot$ as standard fraction addition and multiplication and use $\mathcal{A} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ to check that these are well defined.

Check that $\mathcal{B}$ is an Σ_{RNG} structure and that $\mathcal{B} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$. Define $\iota : \mathcal{A} \to \mathcal{B} := a \mapsto \frac{a}{1}$ and show that this well defined and injective. Check that ι is a Σ_{RNG}-morphism and note that since there are no relation symobls in Σ_{RNG} it is also an embedding. □

Proposition – Universal property of field of fractions

Suppose $A \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ and K its field of fractions. Then if $L \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$ and there exists a Σ_{RNG}-embedding $\iota_L : A \to L$, then there exists a unique Σ_{RNG}-embedding $K \to L$ that commutes with the other embeddings:

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & K \arrow[d, dashed] \\
& L
\end{tikzcd}$$

Proof. Define the map $\iota : K \to L$ sending $\frac{a}{b} \mapsto \frac{\iota_L(a)}{\iota_L(b)}$. Check that this is well-defined and a Σ_{RNG}-morphism. It is injective because ι_L is injective:

$$\frac{\iota_L(a)}{\iota_L(b)} = 0 \Rightarrow \iota_L(a) = 0 \Rightarrow a = 0$$

Thus it is an embedding.

It is unique: suppose $\phi : K \to L$ is a Σ_{RNG}-embedding that commutes with the diagram. Then for any $a \in K$, $\phi(\frac{a}{1}) = \iota_L(a) = \iota(\frac{a}{1})$. Since both ϕ, ι are embeddings they commute with taking the inverse for $a \neq 0$: $\phi(\frac{1}{a}) = \iota(\frac{1}{a})$. Since any element of K can be written as $\frac{a}{b}$, we have shown that $\phi = \iota$. □

3.1.2 Algebraically closed fields

Definition – Theory of algebraically closed fields

We define the Σ_{RNG} theory of algebraically closed fields

$$\mathrm{ACF} := \mathrm{FLD} \cup \left\{ \mathop{\forall}\limits_{i=0}^{n-1} a \exists x,\ x^n + \sum_{i=0}^{n-1} a_i x^i = 0 \,|\, n \in \mathbb{N}_{>0}, a \in {\Sigma_{\mathrm{RNG}}}_{\mathrm{var}}^{n-1} \right\}$$

Unlike the theories $\mathrm{RNG}, \mathrm{ID}, \mathrm{FLD}$ this theory is countably infinite.

Proposition
ACF is not complete.

Proof. Take the Σ_{RNG}-formula $\forall x, x + x = 0$. This is satisfied by the algebraic closure of $\mathbb{F}_2$ but not by that of $\mathbb{F}_3$, since field embeddings preserve characteristic. □

Definition – Algebraically closed fields of characteristic p
For $p \in \mathbb{Z}_{>0}$ prime define

$$\phi_p := \forall x, \sum_{i=1}^{p} x = 0$$

and let $\mathrm{ACF}_p := \mathrm{ACF} \cup \{\phi_p\}$. Furthermore, let

$$\mathrm{ACF}[0] := \mathrm{ACF} \cup \{\neg\, \phi_p \mid p \in \mathbb{Z}_{>0} \text{ prime}\}$$

An important fact about algebraically closed fields of characteristic p:

Proposition – Transcendence degree and characteristic determine algebraically closed fields of characteristic p up to isomorphism
If K_0, K_1 are fields with same characteristic and transcendence degree over their minimal subfield ($\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$) then they are (non-canonically) isomorphic.

Proof. See appendix. □

NOTATION. *If $K \vDash_{\Sigma_{\text{RNG}}} ACF[p]$, write $\mathrm{t.deg}(K)$ to mean the transcendence degree over its minimal subfield ($\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$).*

Lemma – Cardinality of algebraically closed fields
If L is an algebraically closed field then it has cardinality $\aleph_0 + \mathrm{t.deg}(L)$.

Proof. Let S be a transcendence basis and call the minimal subfield K. Since L is algebraically closed it splits the seperable polynomials $x^n - 1$ for each n. Hence L is infinite. Also $S \subseteq L$ and so $\aleph_0 + \mathrm{t.deg}(L) \le |L|$. For the other direction note that

$$M = \bigcup_{f \in I} \{a \in M \mid f = \min(a, K)\}$$

where $I \subseteq K(S)[x]$ is the set of monic and irreducible polynomials over $K(S)$. Thus

$$\begin{aligned}
|M| &\le |I| \times \aleph_0 \le |K(S)[x]| \times \aleph_0 \\
&\le |K(S)| \times \aleph_0 \le |K[S]| \times |K[S]| \times \aleph_0 \\
&= |K[S]| \times \aleph_0 \le \left| \bigcup_{n \in \mathbb{N}} (K \cup S)^n \right| \times \aleph_0 \\
&= \left| \bigcup_{n \in \mathbb{N}} K \cup S \right| \times \aleph_0 = |K \cup S| \times \aleph_0 \\
&= |S| \times \aleph_0
\end{aligned}$$

Noting that $K = \mathbb{Q}$ or $\mathbb{F}_p$ and so is at most countable. By Schröder–Bernstein we have $\aleph_0 + \mathrm{t.deg}(L) = |L|$. □

Proposition
ACF_p is κ-categorical for uncountable κ, consistent and complete.

Proof. Suppose $K, L \models_{\Sigma_{\mathrm{RNG}}} \mathrm{ACF}_p$ and $|K| = |L| = \kappa$. Then $\mathrm{t.deg}(K) + \aleph_0 = |K| = \kappa$ and so $\mathrm{t.deg}(K) = \kappa$ (as κ is uncountable). Similarly $\mathrm{t.deg}(L) = \kappa$ and so $\mathrm{t.deg}(K) = \mathrm{t.deg}(L)$. Thus K and L are isomorphic.

ACF_p is consistent due to the existence of the algebraic closures for any characteristic, it is not finitely modelled and is $\aleph_1$-categorical with $\Sigma_{\mathrm{RNG\,con}} + \aleph_0 \leq \aleph_1$, hence it is complete by Vaught's test. □

3.1.3 Ax-Grothendieck

Proposition – Lefschetz principle
Let ϕ be a Σ_{RNG}-sentence. Then the following are equivalent:

1. There exists a Σ_{RNG}-model of ACF_0 that satisfies ϕ. (If you like $\mathbb{C} \models_{\Sigma_{\mathrm{RNG}}} \phi$.)
2. $\mathrm{ACF}_0 \models_{\Sigma_{\mathrm{RNG}}} \phi$
3. There exists $n \in \mathbb{N}$ such that for any prime p greater than n, $\mathrm{ACF}_p \models_{\Sigma_{\mathrm{RNG}}} \phi$
4. There exists $n \in \mathbb{N}$ such that for any prime p greater than n there exists a non-empty Σ_{RNG}-model of ACF_p that satisfies ϕ.

Proof.

1. $\Rightarrow$ 2. If $\mathbb{C} \models_{\Sigma_{\mathrm{RNG}}} \phi$ then since ACF_0 is complete $\mathrm{ACF}_0 \models_{\Sigma_{\mathrm{RNG}}} \phi$ or $\mathrm{ACF}_0 \models_{\Sigma_{\mathrm{RNG}}} \neg\phi$. In the latter case we obtain a contradiction.

2. $\Rightarrow$ 3. Suppose $\mathrm{ACF}[0] \models_{\Sigma_{\mathrm{RNG}}} \phi$ then since 'proofs are finite' there exists a finite subset Δ of $\mathrm{ACF}[0]$ such that $\Delta \models_{\Sigma_{\mathrm{RNG}}} \phi$. Let n be maximum of all $q \in \mathbb{N}$ such that $\neg\phi_q \in \Delta$. By uniqueness of characteristic, if p is prime and greater than n and q is prime such that $\neg\phi_q \in \Delta$ then $\mathrm{ACF}_p \models_{\Sigma_{\mathrm{RNG}}} \neg\phi_q$. Thus if $\mathcal{M}$ is a Σ_{RNG}-model of ACF_p then $\mathcal{M} \models_{\Sigma_{\mathrm{RNG}}} \Delta$ and so $\mathcal{M} \models_{\Sigma_{\mathrm{RNG}}} \phi$. Hence for all primes p greater than n, $\mathrm{ACF}_p \models_{\Sigma_{\mathrm{RNG}}} \phi$.

3. $\Rightarrow$ 4. ACF_p is consistent thus there exists a non-empty Σ_{RNG}-model of ACF_p. Our hypothesis implies it satisfies ϕ.

4. $\Rightarrow$ 1. Let $n \in \mathbb{N}$ such that for any prime p greater than n there exists a non-empty Σ_{RNG}-model of ACF_p that satisfies ϕ. Then because ACF_p is complete $\mathrm{ACF}_p \models_{\Sigma_{\mathrm{RNG}}} \phi$. Suppose for a contradiction $\mathrm{ACF}_0 \not\models_{\Sigma_{\mathrm{RNG}}} \phi$. Then by completeness $\mathrm{ACF}_0 \models_{\Sigma_{\mathrm{RNG}}} \neg\phi$. Hence by the above we obtain there exists m such that for all p greater than m, $\mathrm{ACF}_p \models_{\Sigma_{\mathrm{RNG}}} \neg\phi$. Then since there are infinitely many primes, take p greater than both m and n, then ACF_p is inconsistent, a contradiction. Hence $\mathrm{ACF}_0 \models_{\Sigma_{\mathrm{RNG}}} \phi$ and in particular $\mathbb{C} \models_{\Sigma_{\mathrm{RNG}}} \phi$.

□

Lemma – Ax-Grothendieck for algebraic closures of finite fields
If Ω is an algebraic closure of a finite field then any injective polynomial map over Ω is surjective.

Proof. See appendix. □

Lemma – Construction of Ax-Grothendieck formula
There exists a Σ_{RNG}-sentence $\Phi_{n,d}$ such that for any field K, $K \models_{\Sigma} \Phi_{n,d}$ if and only if for all $d, n \in \mathbb{N}$ any injective polynomial map $f : K^n \to K^n$ of degree less than or equal to d is surjective.

Proof. We first need to be able to express polynomials in n varibles of degree less than or equal to d in an elementary way. We first note that for any $n, d \in \mathbb{N}$ there exists a finite set S and powers $r_{s,j} \in \mathbb{N}$ (for each $(s, j) \in S \times \{1, \ldots, n\}$). such that any polynomial $f \in K[x_1, \ldots, x_n]$ can be written as

$$\sum_{s \in S} \lambda_s \prod_{j=1}^{n} x_j^{r_{s,j}}$$

for some $\lambda_s \in K$. Now we have a way of quantifying over all such polynomials, which is by quantifying over all the coefficients. We define $\Phi_{n,d}$:

$$\Phi_{n,d} := \forall_{i=1}^{n} \forall_{s \in S} \lambda_{s,i}, \left[\forall_{j=1}^{n} x_j \forall_{j=1}^{n} y_j, \bigwedge_{i=1}^{n} \left(\sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} x_j^{r_{s,j}} = \sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} y_j^{r_{s,j}} \right) \longrightarrow \bigwedge_{i=1}^{n} x_i = y_i \right]$$
$$\longrightarrow \forall_{j=1}^{n} x_j, \exists_{i=1}^{n} z_i, \bigwedge_{i=1}^{n} \left(z_i = \sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} x_j^{r_{s,j}} \right)$$

At first it quantifies over all of the coefficients of all the f_i. The following part says that if the polynomial map is injective then it is surjective. Thus $K \vDash_{\Sigma} \Phi_{n,d}$ if and only if for all $d, n \in \mathbb{N}$ any injective polynomial map $f : K^n \to K^n$ of degree less than or equal to d is surjective. □

Proposition – Ax-Grothendieck

If K is an algebraically closed field of characterstic 0 then any injective polynomial map over K is surjective. In particular injective polynomial maps over $\mathbb{C}$ are surjective.

Proof. We show an equivalent statement: for any $n, d \in \mathbb{N}$, any injective polynomial map $f : K^n \to K^n$ of degree less than or equal to d is surjective. This is true if and only if $K \vDash_{\Sigma_{\mathrm{RNG}}} \Phi_{n,d}$ (by construction of the A-G formula) which is true if and only if for all p prime greater than some natural number there exists an algebraically closed field of characteristic p that satisfies $\Phi_{n,d}$, by the Lefschetz principle. Indeed, take the natural 0 and let p be a prime greater than 0. Take Ω an algebraic closure of $\mathbb{F}_p$, which indeed models ACF_p. $\Omega \vDash_{\Sigma_{\mathrm{RNG}}} \Phi_{n,d}$ if and only if for any $n, d \in \mathbb{N}$, any injective polynomial map $f : \Omega^n \to \Omega^n$ of degree less than or equal to d is surjective (by construction of the A-G formula). The final statement is true due to A-G for algebraic closures of finite fields. □

3.1.4 Quantifier elimination in algebraically closed fields and Nullstellensatz

Lemma

Let X be a subset of a ring A. Then $\Sigma_{\mathrm{RNG}}(X)$-terms are interpreted as polynomials with coefficients from $\langle X \rangle$, the smallest subring of A generated by X.

Proof. Let t be a $\Sigma_{\mathrm{RNG}}(X)$-term.

- If t is a constant c then it's interpretation is $c^{\mathcal{M}}$, a constant polynomial. Since c is 0, 1 or something in X, this is in any polynomial ring over $\langle X \rangle$.
- If t is a variable v_i then it is interpreted as the single variable polynomial $v_s \in \langle X \rangle[v_s]$.
- If t is $f(s_1, \ldots, s_n)$ and each s_i is interpreted as a polynomial q_i, take the polynomial ring $\langle X \rangle[v_1, \ldots, v_k]$ containing all the q_i (everything is finite). Then $f(q_1, \ldots, q_n)$ is still a polynomial in $\langle X \rangle[v_1, \ldots, v_k]$ as f is either $+, -$ or $\cdot$.

□

Lemma – Disjunctive normal form for rings

Let A be a ring or the empty set. Let ϕ be a quantifier free $\Sigma_{\mathrm{RNG}}(A)$-formula with variables indexed by S. Then there exist $\Sigma_{\mathrm{RNG}}(A)$-terms (i.e. polynomials) p_{ij}, q_{ij} such that for any $\Sigma_{\mathrm{RNG}}(A)$-structure $\mathcal{M}$

$$\mathcal{M} \vDash_{\Sigma} \underset{s\in S}{\overset{v_s}{\forall}}, \left[\phi(v) \leftrightarrow \bigvee_{i\in I} \left(\bigwedge_{j\in J_{i0}} p_{ij}(v) = 0 \wedge \bigwedge_{j\in J_{i1}} q_{ij}(v) \neq 0 \right)\right]$$

Proof. Applying the general disjunctive normal form we have that any formula of the form $s = t$ or $r(t)$ is just a polynomial equation, since there are no relation symbols. Moving everything to one side we have that they are of the form $p_{ij}(v) = 0$ and $q_{ij}(v) \neq 0$. □

Remark – Substructures of algebraically closed fields. Since $\mathrm{GRP}, \mathrm{RNG}, \mathrm{ID}$ are all universal axomatizations of themselves, by 'universal axiomatizations make substructures models' we have that substructures of groups are groups, substructures of rings are rings, and substructures of integral domains are integral domains. Furthermore, substructures of algebraically closed fields are substructures of integral domains hence are integral domains. This becomes relevant when proving the equivalent condition on quantifier elimination.

Notation. *If A is a ring and $I \subseteq A[x_1, \dots, x_n]$ is a set of polynomials then the vanishing of I over A is*

$$\mathbb{V}_A(I) := \{c \in A^n \mid \forall f \in I, f(c) = 0\}$$

Proposition

ACF has quantifier elimination.

Proof. Let ϕ be a quantifier free Σ_{RNG}-formula with a free variable w and index the rest by S. Let K, L be algebraically closed field with A embedding into each via ι_K, ι_L. Let $a \in A^S$. It suffices to show that

$$K \vDash_{\Sigma_{\mathrm{RNG}}} \exists w, \phi(\iota_K(a), w) \Rightarrow L \vDash_{\Sigma_{\mathrm{RNG}}} \exists w, \phi(\iota_L(a), w)$$

Crucially by the remark before this proposition A is an integral domain thus we can consider $\iota : A \to \Omega$ the embedding into an algebraic closure of the field of fractions of A. As K and L are fields extending the fraction field of A, by the properties of field of fractions and algebraic closures there are field extensions $\kappa : \Omega \to K$ and $\lambda : \Omega \to L$ such that $\iota_K = \kappa \circ \iota$ and $\iota_L = \lambda \circ \iota$. Let we write $a' := \iota(a)$, and have $\iota_K(a) = \kappa(a')$ and $\iota_L(a) = \lambda(a')$.

$$\begin{tikzcd}
 & & K \\
A \arrow[r] \arrow[urr] \arrow[drr] & F \arrow[r] \arrow[ur] \arrow[dr] & \Omega \arrow[u] \arrow[d] \\
 & & L
\end{tikzcd}$$

We can find the 'disjunctive normal form' of ϕ, i.e. Σ_{RNG}-terms p_{ij}, q_{ij} such that any ring $\mathcal{M}$ satisfies

$$\mathcal{M} \vDash_{\Sigma} \underset{s\in S}{\overset{v_s}{\forall}}, \forall w, \left[\phi(v, w) \leftrightarrow \bigvee_{i\in I} \left(\bigwedge_{j\in J_{i0}} p_{ij}(v, w) = 0 \wedge \bigwedge_{j\in J_{i1}} q_{ij}(v, w) \neq 0 \right)\right]$$

Hence assuming for some $b \in K$ that

$$K \vDash_{\Sigma_{\mathrm{RNG}}} \bigvee_{i \in I} \left(\bigwedge_{j \in J_{i0}} p_{ij}(\kappa(a'), b) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(\kappa(a'), b) \neq 0 \right)$$

it suffices to show that there exists a $c \in L$ such that

$$L \vDash_{\Sigma_{\mathrm{RNG}}} \bigvee_{i \in I} \left(\bigwedge_{j \in J_{i0}} p_{ij}(\lambda(a'), c) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(\lambda(a'), c) \neq 0 \right)$$

Assume for a contradiction that I is empty then by convension we have the disjunctive normal form is $\perp$ and so $K \vDash_{\Sigma_{\mathrm{RNG}}} \perp$ which implies K is empty, which is a contradiction as the constant symbol 0 has an interpretation in K.

Thus there exists $i \in I$ (we keep this i for the rest of the proof) such that

$$K \vDash_{\Sigma_{\mathrm{RNG}}} \bigwedge_{j \in J_{i0}} p_{ij}(\kappa(a'), b) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(\kappa(a'), b) \neq 0$$

Now case on whether or not there exists a $j \in J_{i0}$ such that p^K_{ij} is not the zero polynomial. If it exists then $p^K_{ij}(\kappa(a'), b) = 0$. As p_{ij} is a Σ_{RNG}-term and so $\kappa p^{\Omega}_{ij} = p^K_{ij}$. (Intuitively it is a polynomial over $\langle 0, 1 \rangle$ as shown in 'disjunctive normal form'.) Thus b is algebraic over Ω, which is algebraically closed. Hence there is a $c \in \Omega$ such that $\kappa(c) = b$:

$$\begin{aligned}
& \bigwedge_{j \in J_{i0}} p^K_{ij}(\kappa(a'), \kappa(c)) = 0 \wedge \bigwedge_{j \in J_{i1}} q^K_{ij}(\kappa(a'), \kappa(c)) \neq 0 \\
\Rightarrow & \bigwedge_{j \in J_{i0}} \kappa\left(p^{\Omega}_{ij}(a', c)\right) = 0 \wedge \bigwedge_{j \in J_{i1}} \kappa\left(q^{\Omega}_{ij}(a', c)\right) \neq 0 \\
\Rightarrow & \bigwedge_{j \in J_{i0}} \left(p^{\Omega}_{ij}(a', c)\right) = 0 \wedge \bigwedge_{j \in J_{i1}} \left(q^{\Omega}_{ij}(a', c)\right) \neq 0 \qquad \kappa \text{ is injective} \\
\Rightarrow & \bigwedge_{j \in J_{i0}} \lambda\left(p^{\Omega}_{ij}(a', c)\right) = 0 \wedge \bigwedge_{j \in J_{i1}} \lambda\left(q^{\Omega}_{ij}(a', c)\right) \neq 0 \\
\Rightarrow & \bigwedge_{j \in J_{i0}} \left(p^L_{ij}(\lambda(a'), \lambda(c))\right) = 0 \wedge \bigwedge_{j \in J_{i1}} \left(q^L_{ij}(\lambda(a'), \lambda(c))\right) \neq 0 \\
\Rightarrow & L \vDash_{\Sigma_{\mathrm{RNG}}} \bigvee_{i \in I} \left(\bigwedge_{j \in J_{i0}} p_{ij}(\lambda(a'), \lambda(c))) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(\lambda(a'), \lambda(c))) \neq 0 \right)
\end{aligned}$$

Thus we are done with this case.

In the other case we turn our attention to J_{i1}. If $j \in J_{i1}$, we see $q^{\Omega}_{ij}(a', w)$ as a polynomial in $\Omega[w]$. $q^K_{ij}(\kappa(a'), b) \neq 0$ and so $q^{\Omega}_{ij}(a', w)$ is not the zero polynomial hence $q^L_{ij}(\lambda(a'), w)$ has finitely many zeros in L by the division algorithm in $\Omega[w]$. Let

$$\mathbb{V}_L(q_{ij})_j = \left\{ c \in L \mid \exists j \in J_{i1} \mid q^L_{ij}(\lambda(a'), c) = 0 \right\}$$

be the vanishing of the q_{ij} for every $j \in J_{j1}$. Then $\mathbb{V}$ is finite and L is infinite as it is algebraically closed hence there exists a $c \in L$ $L \vDash_{\Sigma_{\mathrm{RNG}}} \bigwedge_{j \in J_{i1}} q_{ij}(\lambda(a'), c) \neq 0$. Since each f_{ij} are the zero polynomial we also have $L \vDash_{\Sigma_{\mathrm{RNG}}} \bigwedge_{j \in J_{i0}} p_{ij}(\lambda(a'), \lambda(c))) = 0$ Hence we are done. $\square$

Proposition – Weak Nullstellensatz
If K is an algebraically closed field and $\mathfrak{p}$ is a prime ideal of $K[x_1, \dots, x_n]$, then $\mathbb{V}_K(\mathfrak{p}) \neq \varnothing$.

Proof. By Hilbert's basis theorem $K[x_1, \dots, x_n]$ is Noetherian so we can find a finite set $\{f_1, \dots, f_m\}$ generating $\mathfrak{p}$. It suffices to find a common zero of $\{f_1, \dots, f_m\}$. We can write each f_i as $\sum_{j \in S_i} a_{ij} \prod x_j^{n_{ijk}}$ Write a to represent $(a_{ij})_{1 \le i \le m, j \in S_i}$, all the coefficients of all the f_i. We can construct some Σ-formula $\phi(v, x)$ (a polynomial in variables x with coefficients v_{ij} corresponding to the a_{ij}) such that for any $b \in K^n$,

$$K \vDash_{\Sigma_{\mathrm{RNG}}} \phi(a, b) \Leftrightarrow \bigwedge_{j=1}^{m} f_i(b) = 0$$

and for any extension $\iota : K \to L$ and $b \in L^n$,

$$L \vDash_{\Sigma_{\mathrm{RNG}}} \phi(\iota(a), b) \Leftrightarrow \bigwedge_{j=1}^{m} \iota(f_i, b) = 0$$

We can quotient $K[x_1, \dots, x_n]/\mathfrak{p}$ and take an algebraic closure L (quotienting by prime ideals gives an integral domain).

$$\begin{tikzcd}
K[x_1, \dots, x_n] \arrow[r, "\alpha"] & L \\
K \arrow[u, "\kappa"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

We can then take $b := (x_1, \dots, x_n) \in (K[x_1, \dots, x_n])^n$ and send it through to $\alpha(b) \in L^n$.

$$\begin{aligned}
&\bigwedge_{j=1}^{m} \kappa(f_i, b) = 0 \\
&\Rightarrow \bigwedge_{j=1}^{m} \iota(f_i, \alpha(b)) = 0 \\
&\Rightarrow L \vDash_{\Sigma_{\mathrm{RNG}}} \phi(\iota(a), \alpha(b)) \\
&\Rightarrow L \vDash_{\Sigma_{\mathrm{RNG}}} \mathop{\exists}\limits_{i=1}^{n} x_i \phi(\iota(a), x)
\end{aligned}$$

since K and L are both algebraically closed and ACF has quantifier elimination so it is model complete, which implies that the embedding is elementary. Hence we have

$$\begin{aligned}
&\Rightarrow K \vDash_{\Sigma_{\mathrm{RNG}}} \mathop{\exists}\limits_{i=1}^{n} x_i \phi(a, x) \\
&\Rightarrow \exists c \in K^n, K \vDash_{\Sigma_{\mathrm{RNG}}} \phi(a, c) \\
&\Rightarrow \exists c \in K^n, \bigwedge_{j=1}^{m} f_i(b) = 0
\end{aligned}$$

$\square$

3.1.5 The Classical Zariski Topology, Chevalley, Vanishings

Definition – Classical Zariski Topology

Let K be an algebraically closed field and let

$$\{\mathbb{V}_K(E) \mid E \subseteq K[x_1, \ldots, x_n]\}$$

be a closed basis for a topology on K^n. We call these Zariski closed sets.

The closed sets in this setting correspond to closed sets in $\mathrm{Spec}(K[x_1, \ldots, x_n])$, though the spaces are not homeomorphic.

Proposition – Closed sets are finitely generated vanishings

If K is an algebraically closed field and V is a closed set of K^n under the Zariski topology then $V = \mathbb{V}_K(S)$ for some finite subset of the polynomial ring $S \subseteq K[x_1, \ldots, x_n]$.

Proof. By definition of Zariski closed sets any element of the closed basis is $\mathbb{V}_K(E)$ for some $E \subseteq K[x_1, \ldots, x_n]$ We consider the ideal generated by E. Important point: This is finitely generated by the Hilbert basis theorem, and so we can just require E to be finite without loss of generality.

Moreover, arbitrary intersection of such sets is also finitely generated: Let $E \in I$ be a collection of subsets of $K[x_1, \ldots, x_n]$. Then $a \in \bigcap_{E \in I} \mathbb{V}_K(E)$ if and only if for every $E \in I$ and every $f \in E$, $f(a) = 0$. This is true if and only if $a \in \mathbb{V}_K(\bigcup_{E \in I} E)$. Thus $\bigcap_{E \in I} \mathbb{V}_K(E) = \mathbb{V}_K(\bigcup_{E \in I} E)$ which is finitely generated by the first point.

Furthermore, any finite union of such sets is also finitely generated. We prove this by induction. For the empty case:

$$\bigcup_{E \in \varnothing} \mathbb{V}_K(E) = \varnothing = \mathbb{V}_K(K[x_1, \ldots, x_n])$$

which is finitely generated by the first point. It suffices to show that the union of two such sets is also finitely generated.

$$a \in \mathbb{V}(F) \cup \mathbb{V}(G) \Leftrightarrow \left(\bigwedge_{f \in F} f(a) = 0 \right) \vee \left(\bigwedge_{g \in G} g(a) = 0 \right)$$

$$\Leftrightarrow \bigwedge_{f \in F} \bigwedge_{g \in G} f(a) = 0 \vee g(a) = 0 \Leftrightarrow \bigwedge_{f \in F} \bigwedge_{g \in G} (fg)(a) = 0$$

The last step is due to $K[x_1, \ldots, x_n]$ being an integral domain. Hence we have a finite intersection $\mathbb{V}(F) \cup \mathbb{V}(G) = \mathbb{V}(\bigcap (fg)(a) = 0)$ which is finitely generated by the first point. □

Proposition – Constructable

Let K be an algebraically closed field with the Zariski topology on K^n. Define the set C inductively:

| If $X \subseteq K^n$ is closed then it is in C.

| If $X \subseteq K^n$ is in C then $K^n \setminus X$ is in C.

| If $X, Y \subseteq K^n$ are in C then $X \cup Y$ is in C.

Then C is the set of constructable and equivalently the set of of definable sets in K^n (by quantifier elimination).

C consists of 'finite boolean combinations' of closed sets, and corresponds to the original definition 'constructable'.

Proof. ($\Rightarrow$) First we show by induction on the set C that if X is in C then X is constructble. If X is closed then $X = \mathbb{V}(S)$ for some finite S. Therefore X is defined by $\phi(v, b) := \bigwedge_{f \in S} f(a) = 0$, where each f is some Σ_{RNG} formula evaluated at $b \in K^m$. This is a finite and which is the negation of a finite or which is (by induction) constructable. The rest of the induction follows immediately.

($\Leftarrow$) If X is constructable then we show by induction that it is in C. If X is defined by an atomic formula then it is either $\top$ or $t = s$. If ϕ is $\top$ then $X = K^n$ which is closed hence in C. If ϕ is $t = s$ then $t^K(b)$ and $s^K(b)$ are polynomials in $K[x_1, \dots, x_n]$. Writing $f = t^K(b) - s^K(b)$ we have $X = \{a \in K^n \mid f(a) = 0\}$, which is closed hence in C. The rest of the induction follows immediately. □

Proposition – Chevalley

Over an algebraically closed field, the image of a constructable set under a polynomial map is constructable.

Proof. Let $\rho : K^n \to K^m$ be a polynomial map defined by $(f_i)_{i=1}^m$. Suppose $X \subseteq K^n$ is constructable. Then as constructable is equivalent to definable over K there exists Σ-formula ϕ and $b \in K^l$ such that

$$X = \{a \in K^n \mid K \vDash_{\Sigma_{\text{RNG}}} \phi(a, b)\}$$

Then

$$\begin{aligned}
\rho(X) &= \{c \in K^m \mid \exists a \in K^m, K \vDash_{\Sigma_{\text{RNG}}} \wedge \rho(a) = c\} \\
&= \left\{c \in K^m \mid \exists a \in K^m, K \vDash_{\Sigma_{\text{RNG}}} \wedge \bigwedge_{i=1}^m f_i(a) = c_i\right\} \\
&= \left\{c \in K^m \mid K \vDash_{\Sigma_{\text{RNG}}} \exists_{j=1}^n x_j, \phi(x, b) \wedge \bigwedge_{i=1}^m \phi_i(x, d) = c_i\right\}
\end{aligned}$$

The d appearing in $\phi_i(x, d)$ is due to the fact that the polynomials f_i may have coefficients not from the language. Thus the image is constructable. □

Notation (Radical). *We write $r(\mathfrak{a})$ to be the radical of $\mathfrak{a}$.*

Definition – Ideal generated by subsets of K^n

If K is a field, for a subset $X \subseteq K^n$, we write $I(X)$ to mean the ideal of X in $K[x_1, \dots, x_n]$ to mean

$$\{f \in K[x_1, \dots, x_n] \mid \forall a \in X, f(a) = 0\}$$

Exercise (Taking ideals and vanishings are order reversing). *Show that if $X \subseteq Y \subseteq K^n$ then $I(Y) \subseteq I(X)$. Show that if $E \subseteq F \subseteq K[x_1, \dots, x_n]$ then $\mathbb{V}_K(F) \subseteq \mathbb{V}_K(E)$.*

Proposition – Strong Nullstellensatz

Let K be an algebraically closed field and suppose $\mathfrak{a}$ is an ideal of $K[x_1, \dots, x_n]$. Then $r(\mathfrak{a}) = I(\mathbb{V}(\mathfrak{a}))$.

Proof. See appendix. □

Proposition – The Zariski closed sets are Artinian
Any descending chain of Zariski closed sets in K^n for K a field stabilises.

Proof. Let $\ldots \subseteq \mathbb{V}(\mathfrak{a}_1) \subseteq \mathbb{V}(\mathfrak{a}_0)$ be a chain of Zariski closed sets. Then taking the ideals generated each of them we have an ascending chain

$$I(\mathbb{V}(\mathfrak{a}_0)) \subseteq I(\mathbb{V}(\mathfrak{a}_1)) \subseteq \ldots$$

This stabilises because $K[x_1, \ldots, x_n]$ is Noetherian. Hence by strong Nullstellensatz we have that

$$r(\mathfrak{a}_0) \subseteq r(\mathfrak{a}_1) \subseteq \ldots$$

stabilises and taking the vanishing gives back the descending chain $\ldots \subseteq \mathbb{V}(\mathfrak{a}_1) \subseteq \mathbb{V}(\mathfrak{a}_0)$ which stabilises. □

Definition – Irreducible, Variety
If X is a topological space then the following are equivalent:

1. Any non-empty open set is dense in X.
2. Any pair of non-empty open subsets intersect non-trivially.
3. Any two closed proper subsets do not form a cover of X.

If any of the above hold then X is said to be irreducible, and a subset of X is irreducible if it is irreducible under the subspace topology.

A variety is a Zariski closed set that is irreducible.

Corollary
Zariski closed sets are finite unions of varieties.

Proof. For a contradiction suppose $\mathbb{V}_0$ is a Zariski closed set that is not a finite union of varieties. Then $\mathbb{V}_0$ is not irreducible and so there exists two closed proper subsets $\mathbb{V}_1, \mathbb{V}_1'$ that cover $\mathbb{V}_0$. If both $\mathbb{V}_1$ and $\mathbb{V}_1'$ are finite unions of varieties then we have a contradiction, hence without loss of generality $\mathbb{V}_1$ is not a finite union of varieties. By induction we obtain

$$\cdots \subset \mathbb{V}_1 \subset \mathbb{V}_0$$

which stabilises, a contradiction. □

3.1.6 The Stone space and Spec

Proposition – Any polynomial is a formula
Let K be an algebraically closed field and $v = (v_1, \ldots, v_n)$ be variables. Then there exists a map

$$\text{eqzero} : K[x_1, \ldots, x_n] \to F(\Sigma_{\text{RNG}}(K), v)$$

such that for any $a \in K^n$ and any $f \in K[x_1, \ldots, x_n]$,

$$K \vDash_{\Sigma_{\text{RNG}}(K)} \text{eqzero}_f(a) \Leftrightarrow f(a) = 0$$

where eqzero_f is the image of an f.

Proof. This would be some nasty induction. I guess show that monomials can be made into formulas and then sums of monomials can be made into formulas. Everything is finite so it should be okay. □

Proposition – Bijection between the Stone space and spec
Let K be an algebraically closed field. Then define the map

$$\begin{aligned} I : S_n(\mathrm{ElDiag}(\Sigma, K)) &\to \mathrm{Spec}(K[x_1, \dots, x_n]) \\ p &\mapsto \left\{f \in K[x_1, \dots, x_n] \mid \mathrm{eqzero}_f \in p\right\} \end{aligned}$$

We show that this is well-defined, continuous and a bijection. Hence $\mathrm{Spec}(K[x_1, \dots, x_n])$ is compact. However, the spaces are *not* homeomorphic.

Proof. Let $p \in S_n(\mathrm{ElDiag}(\Sigma, K))$. First we check that $I_p := \left\{f \in K[x_1, \dots, x_n] \mid \mathrm{eqzero}_f \in p\right\}$ is a prime ideal. We will repeatedly use the following fact: since p is consistent with $\mathrm{ElDiag}(\Sigma, K)$ we have that it is finitely realised in K.

Let $f, g \in I_p$, then $\mathrm{eqzero}_f, \mathrm{eqzero}_g \in p$. Suppose for a contradiction $f + g \notin I_p$, then by maximality of p, $\neg\, \mathrm{eqzero}_{f+g} \in p$. Taking the finite subset of p to be $\left\{\mathrm{eqzero}_f, \phi_g, \neg\, \mathrm{eqzero}_{f+g}\right\}$, by the above fact we obtain $a \in K^n$ such that

$$K \vDash_{\Sigma_{\mathrm{RNG}(K)}} \left\{\mathrm{eqzero}_f, \mathrm{eqzero}_g, \neg\, \mathrm{eqzero}_{f+g}\right\}(a)$$

By definition of eqzero this implies

$$f(a) = 0, g(a) = 0, (f + g)(a) \neq 0$$

which is a contradiction. Similarly we can let $f \in I_p$, $\lambda \in K$ and suppose $\lambda f \notin I_p$ and so $\neg\, \mathrm{eqzero}_{\lambda f} \in p$. We take the finite subset $\{f, \lambda f\} \subseteq p$ and get a contradiction.

Let the product of two polynomials fg be in I_p. Suppose for a contradiction $f, g \notin I_p$. Then take the finite subset $\left\{\neg\, \mathrm{eqzero}_f, \neg\, \mathrm{eqzero}_g, \mathrm{eqzero}_{f+g}\right\} \subseteq p$ and obtain an $a \in K^n$ such that $f(a) \neq 0, g(a) \neq 0$ but $f(a)g(a) = 0$, a contradiction.

To show that the map is continuous let $V(E) \subseteq \mathrm{Spec}(K[x_1, \dots, x_n])$ be closed, where E is a subset of $K[x_1, \dots, x_n]$ (V denotes the spec vanishing - see appendix). Then $p \in I^{-1}(U)$ if and only if $I_p \in U$ if and only if $E \subseteq I_p$ if and only if

$$\forall f \in E, \mathrm{eqzero}_f \in p$$

Hence the preimage is closed:

$$I^{-1}(U) = \left\{p \in S_n(T) \mid \forall f \in E, \mathrm{eqzero}_f \in p\right\} = \bigcap_{f \in E} [\mathrm{eqzero}_f]$$

as the basis elements $[\phi]$ are clopen. Thus I is continuous.

To show that it is injective suppose $I_p = I_q$ and let $\phi \in p$. By quantifier-elimination in ACF we have ϕ is equivalent to quantifier free ψ modulo T. Then $\psi \in p$ as p is consistent with T and $\phi \in q$ if and only if $\psi \in q$ as q is consistent with T. Take the disjunctive normal form of ψ

$$\bigvee_{i \in I} \left(\bigwedge_{j \in J_{i0}} f_{ij}(v) = 0 \wedge \bigwedge_{j \in J_{i1}} g_{ij}(v) \neq 0 \right)$$

Then (by maximality of p) there exists some i such that for all j, $f_{ij} = 0$ and $g_{ij} \neq 0$ are in p. Thus $f_{ij} = 0$ and $g_{ij} \neq 0$ are in q as they can be made into polynomials in $I_p = I_q$. Hence by maximality of q we have $\psi \in q$ and so $\phi \in q$. By symmetry $p = q$.

For surjectivity, let $\mathfrak{p}$ be a prime ideal of the polynomial ring. Consider the set

$$\mathrm{eqzero}(\mathfrak{p}) \cup \{\neg\, \phi \mid \phi \in \mathrm{eqzero}(K[x_1, \dots, x_n] \setminus \mathfrak{p})\}$$

Assuming for now that this set is consistent with the elementary diagram, it can be extended to p a maximal n-type of elementary diagram. Then

$$f \in \mathfrak{p} \Rightarrow \mathrm{eqzero}_f \in p \Rightarrow f \in I_p$$
$$f \notin \mathfrak{p} \Rightarrow (\neg\ \mathrm{eqzero}_f) \in p \Rightarrow f \notin I_p$$

and so $I_p = \mathfrak{p}$ and we have found a preimage p.

To show that the set is consistent it suffices to show that it is finitely realised in K. Let $\Delta \subseteq \mathfrak{p}$ and $\Gamma \subseteq K[x_1, \ldots, x_n] \setminus \mathfrak{p}$ be finite subsets. It suffices to show that

$$\exists a \in K^n, \forall f \in \Delta, \forall g \in \Gamma, f(a) = 0 \wedge g(a) \neq 0$$

which is equivalent to

$$\mathbb{V}_K(\Delta) \cap K^n \setminus \mathbb{V}_k(\Gamma) \neq \varnothing$$

Suppose for a contradiction it is empty then $\mathbb{V}_k(\Delta) \subseteq \mathbb{V}_k(\Gamma)$ and taking ideals gives us

$$\Gamma \subseteq r(\Gamma) = I(\mathbb{V}_K(\Gamma)) \subseteq I(\mathbb{V}_K(\Delta)) = r(\Delta) \subseteq \mathfrak{p}$$

Here we used that taking ideals is order reversing, strong Nullstellensatz, and the fact that the radical is a subset of any prime ideal containing its generators. Thus $\Gamma \subseteq \mathfrak{p}$ which is a contradiction. Hence we have surjectivity.

Suppose for a contradiction that the two spaces were homeomorphic. Then as the Stone space is Hausdorff, $\mathrm{Spec}(K[x_1, \ldots, x_n])$ would also be Hausdorff, which is a contradiction. $\square$

3.2 Morley Rank in Algebraically Closed Fields and Dimension

We work towards the result that Krull dimension for algebraically closed fields is the same thing as Morley rank for varieties. The idea is that Morley rank of a variety corresponds to the Morley rank of types from elements of the variety, corresponds to model theoretic dimension, corresponds to transcendence degree, which is the same thing as Krull dimension. Strong minimality will play an important role in defining model theoretic algebraic closure, which is then used to define dimension.

3.2.1 Dimension

In this section we will set up the important definitions needed to talk about dimension. A prerequisite for this section is knowledge of the basic results for pregeometries, covered in the appendix.

Definition – Algebraic, algebraic closure

Let $\mathcal{M}$ be a Σ-structure and let D be a subset of $\mathcal{M}$. Let A be a subset of D, $a \in \mathcal{M}$ is algebraic over A if a belongs to a finite $\Sigma(A)$-definable set . Define the algebraic closure of A over D to be

$$\mathrm{acl}_{\Sigma,D}(A) := \{a \in D \,|\, a \text{ is algebraic over } A\}$$

We drop the subscripts Σ and D when it is sufficiently obvious.

Definition – Minimal, strongly minimal [5]

Let $\mathcal{M}$ be a Σ-structure. Let D be an infinite $\Sigma(\mathcal{M})$-definable subset of $\mathcal{M}^n$. D is minimal in $\mathcal{M}$ if any $\Sigma(\mathcal{M})$-definable subset of D if finite or cofinite. D is strongly minimal if it is minimal in $\mathcal{N}$ for any elementary extension $\mathcal{N}$ of $\mathcal{M}$. A Σ-theory T is strongly minimal if any Σ-model of T is strongly minimal (note that any Σ-structure is definable by the formula $v = v$).

We have an equivalent definition of strong minimality, which if you like you can skip.

Proposition – Strong minimality in terms of Morley rank and degree

Let X be a definable subset of $\mathcal{M}$, a Σ-structure. Then X is strongly minimal if and only if $\mathrm{MR}(X) = \mathrm{MD}\, X = 1$.

Proof. $(\Rightarrow)$ Suppose X is strongly minimal. Then X is infinite hence $1 \leq \mathrm{MR}(X)$. Let $\mathbb{M}$ be an ω-saturated extension of $\mathcal{M}$. If $2 \leq \mathrm{MR}(X)$ then there would be infinitely many disjoint $\Sigma(\mathbb{M})$ definable subsets of X of Morley rank 1 (hence they are infinite). By strong minimality the only $\Sigma(\mathbb{M})$-definable subsets of X are finite or cofinite. Thus these subsets must all be cofinite and so any two will intersect, a contradiction. Thus $1 = \mathrm{MR}(X)$.

Since $\mathrm{MR}(X) = 1 \in \mathrm{Ord}$ we have that $\mathrm{MD}\, X \in \mathbb{N}_{>0}$. Again if we have two disjoint $\Sigma(\mathbb{M})$-definable subsets of X of Morley rank 1 we have a contradiction, hence $\mathrm{MD}\, X \leq 1$ and so $\mathrm{MD}\, X = 1$.

$(\Leftarrow)$ Suppose X has Morley rank and degree 1. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. Let $A \subseteq X$ be a $\Sigma(\mathcal{N})$-definable subset of X; its complement is also $\Sigma(\mathcal{N})$-definable. Suppose both are infinite then they both have Morley rank greater than or equal to 1 and are disjoint, thus $2 \leq \mathrm{MD}\, X = 1$, which is a contradiction. Hence one is finite and the other cofinite. □

Lemma – Some definable sets

Let $\mathcal{M}$ be a Σ-structure and let $B, C \subseteq \mathcal{M}$ be Σ-definable set (i.e. $\Sigma(\varnothing)$-definable). Let $\phi(x)$ be a Σ-formula with n free variables. For $b \in B^n$, let $\psi(x, b)$ be a $\Sigma(B)$-formula with m free variables ($\psi(x, y)$ is a Σ-formula with $n + m$ free variables).

Then the following sets are definable by a Σ-formula:

- The intersection of B and C, the union of B and C and the complement of B.
- The set of $b \in B^n$ that satisfy $\phi(x)$:
$$\{b \in B^n \mid \mathcal{M} \vDash_\Sigma \phi(b)\}$$
- The elements $b \in \mathcal{M}^n$ such that $\psi(x, b)$ defines a set of at most k elements.
- The elements $b \in \mathcal{M}^n$ such that $\psi(x, b)$ defines a set of at least k elements.
- The elements $b \in \mathcal{M}^n$ such that $\psi(x, b)$ defines a set of cardinality k.
$$\{b \in \mathcal{M}^n \mid |\psi(\mathcal{M}, b)| = k\}$$
and even $\{b \in B^n \mid |\psi(\mathcal{M}, b)| = k\}$ by taking the intersection of two definable sets.

We will become lazier when dealing with definable sets as we gain an idea of what should and should not be definable.

Proof.
- This is clear.
- Since B is Σ-definable we can take $\chi(x)$ as the Σ-formula defining B and consider the Σ-formula
$$\phi(x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{n} \chi(x_i)$$
Clearly this defines $\{b \in B^n \mid \mathcal{M} \vDash_\Sigma \phi(b)\}$.
- To make $\{b \in \mathcal{M}^n \mid |\psi(\mathcal{M}, b)| \leq k\}$ we take the Σ-formula $\chi(x)$:
$$\chi(x) = \forall_{i=1}^{k+1} x_i, \bigwedge_{i=1}^{k+1} \psi(x_i, y) \to \bigvee_{i \neq j}^{x_i = x_j}$$
where potentially x_i represents m variables, which we can quantify over as it is finite.

- To make $\{b \in \mathcal{M}^n \mid k \leq |\psi(\mathcal{M}, b)|\}$ we take the Σ-formula $\chi(x)$:

$$\chi(x) = \exists_{i=1}^{k} x_i, x_i \neq x_j$$

$\square$

Proposition – Algebraic closure is a pregeometry
Let $\mathcal{M}$ be a Σ-structure. Let D be a minimal subset of $\mathcal{M}$. Then $(D, \mathrm{acl}_{\Sigma,D})$ is a pregeometry.

Proof. [6] The signature we work in will always be Σ and the strongly minimal subset will always be D so we drop the subscript here. Preserves order: *if* $A \subseteq B \subseteq D$ *then* $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$. Let $a \in \mathrm{acl}(A)$. Then there exists a finite $\Sigma(A)$-definable set containing a. Any $\Sigma(A)$-formula is naturally a $\Sigma(B)$-formula thus $a \in \mathrm{acl}(B)$.

Idempotence: *for any* $A \subseteq D$, $\mathrm{acl}(A) = \mathrm{acl}(\mathrm{acl}(A))$. ($\Rightarrow$) We first show that for any subset $A \subseteq D$, $A \subseteq \mathrm{acl}(A)$. Let $a \in A$ then $a = x$ is a $\Sigma(A)$-formula that is defines a finite set. Thus $a \in \mathrm{acl}(A)$. Directly we have the corollary $\mathrm{acl}(A) \subseteq \mathrm{acl}(\mathrm{acl}(A))$.

($\Leftarrow$) We show that $\mathrm{acl}(\mathrm{acl}(A)) \subseteq \mathrm{acl}(A)$. Let $a \in \mathrm{acl}(\mathrm{acl}(A))$. Then there exists $\phi(x, v) = \phi(x, v_0, \ldots, v_n)$ a Σ-formula and $b_0, \ldots, b_n \in \mathrm{acl}(A)$ such that $\phi(x, b)$ defines a finite subset of $\mathcal{M}$ containing a. Let k be the finite cardinality of $\phi(\mathcal{M}, b)$. There exists a Σ-formula $\psi(v)$ that defines the set $\{b \in B \mid |\phi(\mathcal{M}, b)| \leq n\}$

$$\phi'(x, v) := \phi(x, v) \wedge \psi(v)$$

We have that $a \in \phi(\mathcal{M}, b) = \phi'(\mathcal{M}, b)$ and for any $c \in \mathcal{M}^n$, $\phi'(\mathcal{M}, c)$ is finite.

For each b_i appearing in b there exists a $\Sigma(A)$-formula $\psi_i(v_i)$ such that $b_i \in \psi_i(\mathcal{M})$ and this definable set is finite. Define the $\Sigma(A)$-formula

$$\phi''(x) := \exists_{i=1}^{n} v_i, \phi'(x, v_0, \ldots, v_n) \wedge \bigwedge_{i=1}^{n} \psi_i(v_i)$$

Then $a \in \phi''(\mathcal{M})$ by taking the v_i to be b_i and

$$\begin{aligned} d \in \phi''(\mathcal{M}) &\Rightarrow \exists c \in \mathcal{M}^n, \mathcal{M} \vDash_\Sigma \phi'(d, c) \text{ and for each } i, \mathcal{M} \vDash_\Sigma \psi_i(c_i) \\ &\Rightarrow \text{there exist for each } i \ c_i \in \psi_i(\mathcal{M}), \mathcal{M} \vDash_\Sigma \phi'(d, c) \\ &\Rightarrow d \in \bigcup_{i=0}^{n} \bigcup_{c_i \in \psi_i(\mathcal{M})} \phi'(\mathcal{M}, c) \end{aligned}$$

The last expression is a finite union of finite sets which is finite. Hence $\phi''(\mathcal{M})$ is finite and $a \in \mathrm{acl}(A)$

Finite character: *if* $A \subseteq D$ *and* $a \in \mathrm{acl}(A)$ *then there exists a finite subset* $F \subseteq A$ *such that* $a \in \mathrm{acl}(F)$. Take the $\Sigma(A)$-formula defining the finite set containing a. Pick out the (finitely many) constant symbols from A, forming a finite subset $F \subseteq A$. Then $a \in \mathrm{acl}(F)$.

Exchange: *if* $A \subseteq D$ *and* $a, b \in D$ *such that* $a \in \mathrm{acl}(A, b)$ (*shorthand for* $A, \{a\}$) *then* $a \in \mathrm{acl}(A)$ *or* $b \in \mathrm{acl}(A, a)$. Since $a \in \mathrm{acl}(A, b)$ there exists a $\Sigma(A)$-formula $\phi(v, w)$ such that $a \in \phi(\mathcal{M}, b)$ and $\phi(\mathcal{M}, b)$ is finite - say it has cardinality n (if b does not appear in the formula then we immediately have $a \in \mathrm{acl}(A)$). There exists a $\Sigma(A)$-formula $\psi(w)$ defining the set

$$\psi(\mathcal{M}) = \{b' \in D \mid n = |\phi(\mathcal{M}, b')|\}$$

As $\psi(\mathcal{M}) \subseteq D$ and D is minimal, $\psi(\mathcal{M})$ is finite or cofinite. If it is finite then $b \in \psi(\mathcal{M})$ and so $b \in \mathrm{acl}(A) \subseteq \mathrm{acl}(A, a)$.

If it is $\psi(\mathcal{M})$ then consider the $\Sigma(A)$-formula $\phi(v,w) \wedge \psi(w)$. For each $a' \in D$ let $X(a')$ be the subset of D defined by $\phi(a',w) \wedge \psi(w)$. Consider $b \in X(a)$, and case on whether it is finite or cofinite. If it is finite then $b \in \mathrm{acl}(A,a)$ as $\phi(a,w) \wedge \psi(w)$ is a $\Sigma(A)$-formula defining a finite set.

If $X(a)$ is cofinite then let $m = |D \setminus X(a)| \in \mathbb{N}$. There exists a $\Sigma(A)$-formula $\chi(v)$ defining the set

$$\chi(\mathcal{M}) = \{a' \in D \,|\, m = |D \setminus X(a')|\}$$

If $\chi(\mathcal{M})$ is finite then $a \in \chi(\mathcal{M})$ and so $a \in \mathrm{acl}(A)$. If $\chi(\mathcal{M})$ is confinite then there exist $n+1$ distinct elements $a_i \in \chi(\mathcal{M})$ since D is infinite by definition. Take the (finite) intersection of the cofinite $X(a_i)$, producing a non-empty (infinite) set. Take

$$b' \in \bigcap_{i=1}^{n+1} X(a_i) = \bigcap_{i=1}^{n+1} \phi(a_i, \mathcal{M}) \cap \psi(\mathcal{M})$$

Then for each i, $\mathcal{M} \vDash_\Sigma \phi(a_i, b')$, hence $n+1 \leq |\phi(\mathcal{M}, b')|$. However $\mathcal{M} \vDash_\Sigma \psi(b')$ implies $n = |\phi(\mathcal{M}, b')|$, a contradiction. □

The definition of dimension for pregeometries thus carries through for subsets of D.

Definition

Let $\mathcal{M}$ be a Σ-structure and let $X \subseteq D \subseteq \mathbb{M}$, where D is minimal. We write $\dim_{\Sigma,D}(X)$ to mean the dimension of X in the pregeometry $(D, \mathrm{acl}_{\Sigma,D})$. We call this the Σ-dimension of X in D.

Lemma – acl preserves dimension

Let $\mathcal{M}$ be a Σ-structure and let $X \subseteq D \subseteq \mathbb{M}$, where D is minimal. Then $\dim_{\Sigma,D}(X) = \dim_{\Sigma,D}(\mathrm{acl}_{\Sigma,D}(X))$.

Proof. Let $S \subseteq X$ be a basis of X. Then it is an independent subset of $X \subseteq \mathrm{acl}(X)$ such that

$$\mathrm{acl}(X) \subseteq \mathrm{acl}(S) \quad \text{and by idempotence} \quad \mathrm{acl}(\mathrm{acl}(X)) \subseteq \mathrm{acl}(X) \subseteq \mathrm{acl}(S)$$

Hence S is a basis for $\mathrm{acl}(X)$. □

3.2.2 The theory of algebrically closed fields is strongly minimal

Definition – Strongly minimal theory

A Σ-theory T is (strongly) minimal if any Σ-model of T is (strongly) minimal.

Lemma – Disjunctive normal form of definable sets

Let K be an algebraically closed field. Any definable set in K^n can be written in the form

$$\bigcup_{i \in S} (V_i \cap U_i)$$

where V_i is a variety and U_i is the complement of a variety in K^n.

Proof. Let X be a definable set:

$$X = \{a \in K^m \,|\, K \vDash_{\Sigma_{\mathrm{RNG}}} \phi(a,b)\}$$

where $b \in K^n$ and ϕ is some Σ_{RNG}-formula with $n+m$ free variables. Then by quantifier elimination in ACF we have a quantifier free Σ_{RNG}-formula ψ such that

$$X = \{a \in K^m \,|\, K \vDash_{\Sigma_{\mathrm{RNG}}} \psi(a,b)\}$$

We can find the 'disjunctive normal form' of ψ as it is quantifier free. Hence for $a \in K^m$

$$\begin{aligned} & a \in X \\ \Leftrightarrow & \bigvee_{i \in I} \left(\bigwedge_{j \in J_{i0}} p_{ij}(a,b) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(a,b) \neq 0 \right) \\ \Leftrightarrow & \, a \in \bigcup_{i \in I} \left(\bigcap_{j \in J_{i0}} \mathbb{V}_K(p_{ij}(x,b)) \cap \bigcap_{j \in J_{i1}} K^m \setminus \mathbb{V}_K(q_{ij}(x,b)) \right) \end{aligned}$$

□

Lemma – Non-trivial vanishings are finite
If K is a field and $S \subseteq K[x]$ then $\mathbb{V}_K(S)$ is finite or $S = \{0\}$. In particular $\mathbb{V}_K(S)$ is either finite or cofinite in K.

Proof. If $\mathbb{V}(f)$ is finite we are done. If $\mathbb{V}(f)$ is infinite then each $f \in S$ has infinitely many distinct roots so $f = 0$ by the division algorithm.

In particular, if $S = \{0\}$ then $\mathbb{V}(S)$ is K and it is cofinite. □

Proposition
ACF is strongly minimal.

Proof. Let K be an algebraically closed field.

Let $D \subseteq K$ be definable. Any elementary extension of K is also algebraically closed so without loss of generality we only need to show minimality rather than strong minimality. Then there exist $p_{ij}(x,b), q_{ij}(x,b) \in K[x]$ (note that the polynomials are in only one variable) such that

$$D = \bigcup_{i \in I} \left(\bigcap_{j \in J_{i0}} \mathbb{V}_K(p_{ij}(x,b)) \cap \bigcap_{j \in J_{i1}} K \setminus \mathbb{V}_K(q_{ij}(x,b)) \right)$$

Which is a finite union and intersection of finite and cofinite sets, which is finite or cofinite. Hence D is finite or cofinite and ACF is strongly minimal. □

3.2.3 Dimension and transcendence degree

We show that dimension and transcendence degree are the same thing.

Lemma – Formulas defining finite sets and polynomials
Suppose K is an algebraically closed field, S is a subset of an extension field. ϕ is a $\Sigma_{\mathrm{RNG}}(K,S)$-formula with exactly 1 free variable. If ϕ defines a finite set containing $a \in K(S)$ then there exists a non-zero polynomial $p \in K(S)[x]$ such that $p(a) = 0$.

Proof. First take the disjunctive normal form of ϕ

$$K(S) \vDash_{\Sigma(K,S)} \forall v, \phi \leftrightarrow \bigvee_{i \in I} \left(\bigwedge_{j \in J_{i0}} p_{ij}(v) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(v) \neq 0 \right)$$

where $\Sigma(K,S)$-terms p_{ij}, q_{ij} are naturally polynomials in $K(S)[x]$. We see that there is some $i \in I$ such that

$$K(S) \vDash_{\Sigma_{\mathrm{RNG}}(K,S)} \bigwedge_{j \in J_{i0}} p_{ij}(a) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(a) \neq 0$$

The set defined by $\bigwedge_{j \in J_{i0}} p_{ij}(v) = 0$ in $K(S)$ is $\mathbb{V}_{K(S)}(\{p_{ij} \mid j \in J_{i0}\})$, hence is either finite (there exists a non-zero polynomial) or all of $K(S)$ (all polynomials are zero). In the first case we obtain a non-zero polynomial $p \in K(S)[x]$ such that $p(a) = 0$ and we are done.

Assume for a contradiction the second case holds. For each $j \in J_{i1}$ consider the set defined by $q_{ij}(v) \neq 0$ in $K(S)$, which is cofinite or empty. If it is empty then it is not satisfied by a which is a contradiction. Hence

$$\bigwedge_{j \in J_{i0}} p_{ij}(v) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(v) \neq 0$$

defines a cofinite subset of $\phi(K(S))$, and so $\phi(K(S))$ is infinite, a contradiction. □

A result that uses the previous lemma is that the usual algebraic closure for fields is indeed a special case of our definition of algebraic closure. This is included only because it is noteworthy and can be skipped.

Proposition – Algebraic closure in ACF is an field theoretic algebraic closure
Let $K \to M$ be a field extension with M algebraically closed, then as ACF is strongly minimal we have that M is strongly minimal. Consider a subset $A \subseteq M$. Then $\mathrm{acl}_{\Sigma(K),M}(A)$ is an algebraic closure of $K(A)$.

Proof. We write acl instead of $\mathrm{acl}_{\Sigma(K),M}$. We must show that $\mathrm{acl}(A)$ is a field that contains $K(A)$, is algebraically closed and is an algebraic extension of $K(A)$.

To show that it is a field we only show closure for addition and leave the rest as an exercise. Let $a, b \in \mathrm{acl}(A)$. Then there exist $\Sigma(K,A)$-formulas ϕ_a, ϕ_b that define finite subsets of M containing a and b respectively. Then their sum is in the finite set defined by the formula with one free variable z:

$$\exists x, \exists y, \phi_a(x) \wedge \phi_b(y) \wedge z = x + y$$

It contains $K(A)$ if and only if it contains A and the image of K. It contians K since for any element $k \in K$ we can take the $\Sigma(K,A)$-formula $x = k$. Similarly for A.

To show that it is algebraically closed let $p \in \mathrm{acl}(A)[x]$ be a polynomial. Then we write p out in terms of its coefficients $a_i \in \mathrm{acl}(A)$

$$p = \sum_{i=1}^{m} a_i x^i$$

Each coefficient is in a finite subset of M defined by a $\Sigma(K,A)$-formula ϕ_i. The formula with free variable x

$$\exists_{i=1}^{m} v_i, \bigwedge_{i=1}^{m} \phi_i(v_i) \wedge \sum_{i=1}^{m} v_i x^i$$

defines the set of roots of p in M. Since M is algebraically closed this is all the roots and since p only has finitely many roots it is finite. Hence all the roots of p are in $\mathrm{acl}(A)$.

To show that it is an algebraic extension of $K(A)$ we take $a \in \mathrm{acl}(A)$ and obtain a formula defining a finite subset of M containing a. Thus there exists a polynomial $p \in K(A)[x]$ with a as a root, and so a is algebraic over $K(A)$. □

Proposition – Transcendence degree is dimension

Let $K \to L$ be a field extension. Suppose $S \subseteq L$ and M is an algebraically closed field extension of L. We consider the pregeometry $(M, \mathrm{acl}_{\Sigma_{\mathrm{RNG}}(K),M})$.

1. S is algebraically independent over K if and only if S is independent in the pregeometry.
2. $K(S) \to L$ is an algebraic extension if and only if S spans L in the pregeometry.
3. S is a transcendence basis for the extension $K \to L$ if and only if S is a basis for L in the pregeometry.
4. Transcendence degree is the same thing as dimension:

$$\mathrm{t.deg}(K \to K(S)) = \dim_{\Sigma(K),M}(L)$$

Proof. We will write acl to mean $\mathrm{acl}_{\Sigma_{\mathrm{RNG}}(K),M}$.

1. ($\Rightarrow$) Let $a \in S$ and suppose for a contradiction that $a \in \mathrm{acl}(S \setminus \{a\})$. Then there exists a $\Sigma_{\mathrm{RNG}}(K, S \setminus \{a\})$-formula ϕ defining a finite subset of $\mathbb{M}$ containing a. Then there exists a non-zero polynomial $p \in K(S \setminus \{a\})[x]$ such that $p(a) = 0$. This contradicts algebraic independence of S.

   ($\Leftarrow$) Suppose $p \in K[x_0, \ldots, x_n]$ with distinct $s_0, \ldots, s_n \in S$ such that $p(s_0, \ldots, s_n) = 0$. Then we have a polynomial in one variable
   $$p(x_0, s_1, \ldots, s_n) \in K(S \setminus \{s_0\})[x_0]$$
   with s_0 as a root. Elements of $K(S \setminus \{s_0\})$ can be written as polynomials over K in variables from $S \setminus \{s_0\}$, which are $\Sigma_{\mathrm{RNG}}(K, S \setminus \{s_0\})$-terms; hence $p(x_0, s_1, \ldots, s_n)$ is naturally a $\Sigma_{\mathrm{RNG}}(K, S \setminus \{s_0\})$-term as well. Call $\phi(x_0)$ the $\Sigma_{\mathrm{RNG}}(K, S \setminus \{s_0\})$-formula '$p(x_0, s_1, \ldots, s_n) = 0$'. Since non-zero polynomials have finitely many roots $\phi(M)$ is finite or $p = 0$; since s_0 is a root,
   $$M \vDash_{\Sigma_{\mathrm{RNG}}(K, S \setminus \{s_0\})} \phi(s_0)$$
   and so if it is finite then $s_0 \in \mathrm{acl}(S \setminus \{s_0\})$, contradicting pregeometrical independence. Hence $p = 0$ and so S is algebraically independent.

2. ($\Rightarrow$) By idempotence of acl, for S to be spanning in the pregeometry it suffices to show that $L \subseteq \mathrm{acl}(S)$ (take acl of both sides). Let $a \in L$. Since $K(S) \to L$ is an algebraic extension there exists a non-zero polynomial $p \in K(S)[x]$ such that $p(a) = 0$. We write p as a polynomial over K in variables from S, which is a $\Sigma_{\mathrm{RNG}}(K, S)$-term, and so the $\Sigma_{\mathrm{RNG}}(K, S)$-formula '$p = 0$' defines a finite set containing a.

   ($\Leftarrow$) Let $a \in L$. We want to show that a is algebraic over $K(S)$ Since S spans L
   $$a \in L \subseteq \mathrm{acl}(L) \subseteq \mathrm{acl}(S)$$
   and so there exists a $\Sigma(K, S)$-formula defining a finite set containing a. Hence there exists a non-zero polynomial $p \in K(S)[x]$ with a as a root. Hence the extension is algebraic.

3. S is a transcendence basis of the extension $K \to L$ if and only if $K(S) \to L$ is algebraic and S is algebraically independent over K. By the last two parts this is if and only if S is independent and spanning L in the pregeometry, which is if and only if S is pregeometrical basis for L.

4. This is clear.

□

3.2.4 Morley rank of types and dimension

This section is purely model theoretic. It allows us to connect our definition dimension together with Morley rank for types of tuples.

Proposition – Projection and Morley rank

Suppose $\mathbb{M}$ is an ω-saturated Σ-structure. Let ϕ be a $\Sigma(\mathbb{M})$-formula and let v be a variable symbol. Then

$$\mathrm{MR}(\exists v, \phi) \leq \mathrm{MR}(\phi)$$

Proof. First we remove the case where v is not a free variable of ϕ by noting that if this is the case then $\exists v, \phi$ and ϕ have the same number of free variables and so $a \in (\exists v, \phi)(\mathbb{M})$ if and only if $\mathbb{M} \vDash_\Sigma \exists v, \phi(a)$ if and only if $\mathbb{M} \vDash_\Sigma \phi(a)$ if and only if $a \in \phi(\mathbb{M})$. Hence they have the same Morley rank.

Then we induct on $\alpha \in \mathrm{Ord}$ to show that $\alpha \leq \mathrm{MR}(\exists v, \phi)$ implies $\alpha \leq \mathrm{MR}(\phi)$. We will not bother with the cases $-\infty$ and ∞ and the former is trivial and the latter follows from the induction result. If $0 \leq \mathrm{MR}(\exists v, \phi)$ then there exists $a \in \mathbb{M}^n$ such that $\mathbb{M} \vDash_\Sigma \exists v, \phi(a, v)$. Hence there is $b \in \mathbb{M}$ such that $\mathbb{M} \vDash_\Sigma \phi(a, b)$ and so $0 \leq \mathrm{MR}(\phi)$.

The non-zero limit ordinal case is trivial. Suppose $\alpha + 1 \leq \mathrm{MR}(\exists v, \phi)$. Then there exist $\Sigma(\mathbb{M})$-formulas ψ_i defining disjoint subsets of $(\exists v, \phi)(\mathbb{M})$ such that $\alpha \leq \mathrm{MR}(\psi_i)$. Then take $\Sigma(\mathbb{M})$-formulas χ_i to be $\psi_i \wedge \phi$.[1] If $a \in (\exists v, \phi)(\mathbb{M})$ then $a \in (\exists v, \psi \wedge \phi)(\mathbb{M})$ and so

$$\psi_i(\mathbb{M}) \subseteq (\exists v, \phi)(\mathbb{M}) \subseteq (\exists v, \psi \wedge \phi)(\mathbb{M}) \quad \Rightarrow \quad \alpha \leq \mathrm{MR}(\psi_i) \leq \mathrm{MR}(\exists v, \chi_i)$$

by the induction hypothesis we have $\alpha \leq \chi_i$ and the χ_i define disjoint subsets of $\phi(\mathbb{M})$, hence $\alpha + 1 \leq \mathrm{MR}(\phi)$. ☐

Lemma – Morley rank of extended types [5]

Suppose $\mathbb{M}$ is a κ-saturated, strongly minimal Σ-structure. Let $A \subseteq \mathbb{M}$ be such that $|A| < \kappa$. Let $a \in \mathbb{M}^n$ and $b \in \mathbb{M}$. Then

$$\mathrm{MR}(\mathrm{tp}_A^{\mathbb{M}}(a)) \leq \mathrm{MR}(\mathrm{tp}_A^{\mathbb{M}}(a, b))$$

Furthermore 'if b is dependent then removing it preserves Morley rank'

$$b \in \mathrm{acl}_{\Sigma(A),\mathbb{M}}(\{a_1, \ldots, a_n\}) \Rightarrow \mathrm{MR}(\mathrm{tp}_A^{\mathbb{M}}(a)) = \mathrm{MR}(\mathrm{tp}_A^{\mathbb{M}}(a, b))$$

This says we can reduce finite sets to subsets that are independent in the pregeometry whilst preserving Morley rank of the type.

Proof. We write $\mathrm{MR}(\star)$ to mean $\mathrm{MR}(\mathrm{tp}_A^{\mathbb{M}}(\star))$. We use x or $x_1, \ldots, x_n$ to denote the variables corresponding to a and use v to denote the variable corresponding to b.

Let ϕ be a rank representative for $\mathrm{tp}(a, b)$. Then $a \in (\exists v, \phi)(\mathbb{M})$ and so $(\exists v, \phi) \in \mathrm{tp}(a)$. Hence

$$\mathrm{MR}(a) \leq \mathrm{MR}(\exists v, \phi) \leq \mathrm{MR}(\phi) \leq \mathrm{MR}(a, b)$$

For the other inequality it suffices to show by induction on $\alpha \in \mathrm{Ord}$ that If $a \in \mathbb{M}^n$ and $b \in \mathbb{M}$. Then 'if b is dependent' then

$$\alpha \leq \mathrm{MR}(a, b) \Rightarrow \alpha \leq \mathrm{MR}(a)$$

For the base case we note that $a \in \phi(\mathbb{M})$ for any rank representative ϕ of $\mathrm{tp}(a)$ and so $\phi(\mathbb{M})$ is non-empty and

$$0 \leq \mathrm{MR}(\phi) = \mathrm{MR}(a)$$

The non-zero limit ordinal case is trivial.

[1] It is important here that we notice $\psi_i \wedge \phi$ and ϕ have the same number of free variables, and so we can say that $(\psi_i \wedge \phi)(\mathbb{M}) \subseteq \phi(\mathbb{M})$, which is not the case for if we just took ψ_i, since it is one free variable short. (In general $\mathbb{M}^n \cap \mathbb{M}^{n+1} = \varnothing$!)

Suppose for the successor case $\alpha + 1 \leq \mathrm{MR}(a, b)$. Then $\alpha \leq \mathrm{MR}(a, b)$ and so by the induction hypothesis $\alpha \leq \mathrm{MR}(a)$. There exists a 'smallest' rank representative $\phi(x)$ of $\mathrm{tp}(a)$ such that there is no $\Sigma(A)$-formula ψ such that

$$\mathrm{MR}(\phi \wedge \psi) = \mathrm{MR}(\phi \wedge \neg \psi) = \alpha$$

Since $b \in \mathrm{acl}(a)$ we have a $\Sigma(A)$-formula $\psi(x, v)$ such that $b \in \psi(a, \mathbb{M})$ and $\psi(a, \mathbb{M})$ is finite with cardinality n. Let $|\psi(x, \mathbb{M})| = n$ denote the formula defining the set of $a' \in \mathbb{M}^n$ such that $|\psi(a', \mathbb{M})| = n$. Then define the $\Sigma(A)$-formula

$$\Phi(x, v) := \phi(x) \wedge \psi(x, v) \wedge |\psi(x, \mathbb{M})| = n$$

Note that $\Phi \in \mathrm{tp}(a, b)$ and so $\alpha + 1 \leq \mathrm{MR}(a, b) \leq \mathrm{MR}(\Phi)$. Thus for $i \in \mathbb{N}$ there exist formulas θ_i with rank at least α defining disjoint subsets of $\Phi(\mathbb{M})$.

We show that for each $m \in \mathbb{N}_{>0}$

$$\alpha \leq \mathrm{MR}(\bigwedge_{i=1}^{m} \exists y, \theta_i)$$

We first show for each i that $\alpha \leq \mathrm{MR}(\exists v, \theta_i)$, covering the base case $m = 1$. There exist representative types, element $c \in \mathbb{M}^n$ and $d \in \mathbb{M}$: $\theta_i \in \mathrm{tp}(c, d)$ and $\mathrm{MR}(\theta_i) = \mathrm{MR}(c, d)$. If $d \in \mathrm{acl}(c)$ then by induction we have

$$\alpha \leq \mathrm{MR}(c, d) \Rightarrow \alpha \leq \mathrm{MR}(c) \leq \mathrm{MR}(\exists v, \theta_i)$$

Indeed $d \in \mathrm{acl}(c)$ since $\theta_i(\mathbb{M}) \subseteq \Phi(\mathbb{M})$ which says

$$d \in \psi(c, \mathbb{M}) \text{ and } |\psi(c, \mathbb{M})| = n$$

Suppose it is true for m. Write $\chi(x) := \bigwedge_{i=1}^{m} \exists v, \theta_i$. Then for each i, $(\exists v, \theta_i)(\mathbb{M}) \subseteq \phi(\mathbb{M})$ implies $\chi(\mathbb{M}) \subseteq \phi(\mathbb{M})$ and

$$\alpha \leq \mathrm{MR}(\chi) \leq \mathrm{MR}(\phi) = \alpha \Rightarrow \alpha = \mathrm{MR}(\chi) = \mathrm{MR}(\phi \wedge \chi)$$

We showed above that $\alpha \leq \exists y, \theta_m$. Partition $(\exists y, \theta_m)(\mathbb{M})$ into its intersection with $\chi(\mathbb{M})$ and $\neg \chi(\mathbb{M})$. Supposing for a contradiction

$$\alpha \not\leq \mathrm{MR}(\chi \wedge \exists v, \theta_m)$$

we see that $\alpha \leq \mathrm{MR}(\exists y, \theta_m)$ is the maximum of the two parts which must be $\mathrm{MR}(\neg \chi \wedge \exists v, \theta_m)$. Then

$$\alpha \leq \mathrm{MR}(\exists v, \theta_m \wedge \neg \chi) \leq \mathrm{MR}(\phi \wedge \neg \chi) \leq \mathrm{MR}(\phi) = \alpha \Rightarrow \alpha = \mathrm{MR}(\phi \wedge \neg \chi)$$

This contradicts the property of ϕ being the 'smallest' rank representative.

In particular we have shown, applying compactness, that $\{\exists v, \theta_i\}$ is consistent and thus can be extended to an element of $S_n(\mathrm{Th}_{\mathbb{M}}(A))$, which is realised by some $c \in \mathcal{M}^n$ as it is κ-saturated. Thus we have for each i that $\mathbb{M} \vDash_{\Sigma(A)} \exists v, \theta_i(c, v)$, giving us $d_i \in \mathbb{M}$ such that $\mathbb{M} \vDash_{\Sigma(A)} \theta_i(c, d_i)$. However the $\theta_i(c, \mathbb{M})$ are disjoint subsets of $\Phi(c, \mathbb{M}) \subseteq \psi(c, \mathbb{M})$ and so the d_i must be distinct elements of $\psi(c, \mathbb{M})$ and so $\psi(c, \mathbb{M})$ is infinite. However, $(c, d_1) \in \Phi(\mathbb{M})$ so $|\psi(c, \mathbb{M})| = n$, a contradiction. □

Lemma – Independent tuples have the same type [5]

Let $\mathcal{M}$ be a Σ-structure with $A \subseteq D \subseteq \mathcal{M}$, where D is minimal and $\Sigma(A)$-definable. Let $a \in D^k$ and $b \in D^k$. If $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are both independent in the pregeometry $(D, \mathrm{acl}_{\Sigma(A), D})$ then

$$\mathrm{tp}_A^{\mathcal{M}}(a) = \mathrm{tp}_A^{\mathcal{M}}(b)$$

Proof. We induct on $k \in \mathbb{N}$. We write tp for $\mathrm{tp}_A^{\mathcal{M}}$.

For the base case $k = 0$ we note that empty tuples define the same type

$$\mathrm{tp}(a) = \mathrm{tp}(\varnothing) = \mathrm{tp}(b)$$

More concretely this is equal to $\mathrm{Th}_{\mathcal{M}}(A)$.

Suppose it is true for k. Let $\{a_1, \ldots, a_{k+1}\}$ and $\{b_1, \ldots, b_{k+1}\}$ be independent in the pregeometry. Then by induction $\mathrm{tp}(a_1, \ldots, a_k) = \mathrm{tp}(b_1, \ldots, b_k)$. Let $\psi \in \mathrm{tp}(a_1, \ldots, a_{k+1})$. It suffices to show that $\psi \in \mathrm{tp}(b_1, \ldots, b_{k+1})$. Note that $a_k \in \psi(a_1, \ldots, a_k, \mathcal{M}) \cap D$, which is a $\Sigma(A, a_1, \ldots, a_k)$-definable subset of D. Since by assumption $\{a_1, \ldots, a_{k+1}\}$ is independent this definable set must be infinite, and by minimality of D its complement $\neg\,\psi(a_1, \ldots, a_k, \mathcal{M}) \cap D$ is finite with cardinality n, say.

There exists a $\Sigma(A)$-formula χ in free-variables $x_1, \ldots, x_k$ defining the set of $c \in D^k$ such that

$$|\neg\,\psi(c, \mathcal{M}) \cap D| = n$$

Since $\chi \in \mathrm{tp}(a_1, \ldots, a_k) = \mathrm{tp}(b_1, \ldots, b_k)$ implies $|\neg\,\psi(b_1, \ldots, b_k, \mathcal{M}) \cap D| = n$,

$$b_{k+1} \notin \psi(b_1, \ldots, b_k, \mathcal{M}) \Rightarrow b_{k+1} \in \neg\,\psi(b_1, \ldots, b_k, \mathcal{M}) \text{ finite} \Rightarrow b_{k+1} \text{ is dependent, contradiction}$$

We have $b_{k+1} \in \psi(b_1, \ldots, b_k, \mathcal{M})$ and so $\psi \in \mathrm{tp}(b_1, \ldots, b_k)$. $\square$

> **Proposition – Morley rank of types and dimension**
> Suppose $\mathbb{M}$ is strongly minimal and κ-saturated. Let $A \subseteq \mathbb{M}$ such that $|A| < \kappa$ and $\dim_{\Sigma(A),\mathbb{M}}(\mathbb{M})$ is infinite. Let $a \in \mathbb{M}^k$. Then
> $$\mathrm{MR}(\mathrm{tp}^{\mathbb{M}}_A(a)) = \dim_{\Sigma(A),\mathbb{M}}(\{a_1, \ldots, a_k\})$$

Proof. We work in the pregeometry $(\mathbb{M}, \mathrm{acl}_{\Sigma(A),\mathbb{M}})$ and write $\dim$ for $\dim_{\Sigma(A),\mathbb{M}}$ and acl for $\mathrm{acl}_{\Sigma(A),\mathbb{M}}$. We also write $\mathrm{MR}(a)$ for $\mathrm{MR}(\mathrm{tp}^{\mathbb{M}}_A(a))$ and tp for $\mathrm{tp}^{\mathbb{M}}_A$.

Let us first show that without loss of generality $\{a_1, \ldots, a_k\}$ is independent in the pregeometry. Concretely, we prove by induction on k that there exists an independent subset $s \subseteq \{a_1, \ldots, a_k\}$ such that

$$\mathrm{MR}(s) = \mathrm{MR}(a) \quad \text{and} \quad \dim(s) = \dim(a)$$

If $k = 0$ then it is okay as the empty set is trivially independent. If $0 < k$ then we case on if $\{a_1, \ldots, a_k\}$ is independent or not. If it is then we are done. Otherwise remove a dependent element a_i. By the theorem on Morley rank of extended types we have that

$$\mathrm{MR}(a \setminus \{a_i\}) = \mathrm{MR}(a)$$

Since acl preserves dimension and a_i is dependent

$$\dim(a \setminus \{a_i\}) = \dim(\mathrm{acl}(a \setminus \{a_i\})) = \dim(\mathrm{acl}(a)) = \dim(a)$$

Hence by induction there is a subset $s \subseteq a \setminus \{a_i\} \subseteq a$ such that

$$\mathrm{MR}(s) = \mathrm{MR}(a \setminus \{a_i\}) = \mathrm{MR}(a) \text{ and } \dim(s) = \dim(a \setminus \{a_i\}) = \dim(a)$$

Hence a can be replaced by an independent subset.

Now we show that for independent $a_1, \ldots, a_k$, $\mathrm{MR}(a) = k$. This will complete the proof since independent sets are bases for themselves, which implies $\dim(a) = k = \mathrm{MR}(a)$. We prove this by induction on k. If $k = 0$ then $\mathrm{tp}^{\mathbb{M}}_A(a)$ consists of $\Sigma(A)$-sentences satisfied by $\mathbb{M}$, which by convension each define the set $\{\varnothing\}$. Hence everything in $\mathrm{tp}(a)$ has Morley rank 0 and so $\mathrm{MR}(a) = 0$.

For the induction step suppose $a_1, \ldots, a_{k+1}$ form an independent set in the pregeometry. We first show that $k + 1 \leq \mathrm{MR}(a)$. Let $\phi(x_1, \ldots, x_{k+1})$ be a rank representative of a. Since the dimension of $\mathbb{M}$ is infinite we can take a countably infinite subset of a basis $\{b_i\}_{i\in\mathbb{N}}$. Let the formulas

$$\psi_i := \phi \wedge (x_1 = b_i)$$

define disjoint subsets of $\phi(\mathbb{M})$. We need to show that these formulas have Morley rank k. Since independent sets of the same length have the same type, for each $i \in \mathbb{N}$

$$\mathrm{tp}(a_1, \dots, a_{k+1}) = \mathrm{tp}(b_i, \dots, b_{i+k})$$

In particular $\phi \in \mathrm{tp}(b_i, \dots, b_{i+k})$ and so $\psi_i \in \mathrm{tp}(b_i, \dots, b_{i+k})$. Hence and by induction

$$k \leq \mathrm{MR}(b_i, \dots, b_{i+k-1}) \leq \mathrm{MR}(b_i, \dots, b_{i+k}) \leq \mathrm{MR}(\psi_i)$$

Hence we have $k + 1 \leq \mathrm{MR}(\phi) = \mathrm{MR}(a)$.

Suppose for a contradiction that $k + 2 \leq \mathrm{MR}(\phi)$. Then we have for each $i \in \mathbb{N}$, $\Sigma(\mathbb{M})$-formulas χ_i of Morley rank $k + 1$ defining disjoint subset of $\phi(\mathbb{M})$. As they are disjoint there exists an i such that $a \notin \chi_i(\mathbb{M})$. We take $\mathrm{tp}(c) \in S_n(\Theta_{\mathbb{M}}(A))$, a type representing this χ_i realised by some $c \in \mathbb{M}^{k+1}$ by κ-saturation:

$$\mathrm{MR}(c) = \mathrm{MR}(\chi_i) = k + 1 \text{ and } \chi_i \in \mathrm{tp}(c)$$

Suppose for a contradiction that $c_1, \dots, c_{k+1}$ are independent in the pregeometry. Then as independent sets of the same length have the same type we have $\mathrm{tp}(a) = \mathrm{tp}(c)$, which implies $\chi_i \in \mathrm{tp}(a)$ so $a \in \chi_i(\mathbb{M})$, a contradiction. There is a strictly smaller subset $B \subsetneq \{c_1, \dots, ck + 1\}$ that is a pregeometrical basis for $\{c_1, \dots, ck + 1\}$. This will satisfy $\mathrm{MR}(B) = \mathrm{MR}(c)$ (by induction using the lemma on extended types). By the induction hypothesis, noting that B is independent, we have

$$\mathrm{MR}(\chi_i) = \mathrm{MR}(c) = \mathrm{MR}(B) = |B| < k + 1$$

which contradicts $\mathrm{MR}(\chi_i) = k + 1$. Hence $k + 1 = \mathrm{MR}(\phi) = \mathrm{MR}(a)$. □

3.2.5 Morley rank is Krull dimension

We are now ready to show that Morley rank is the same thing as Krull dimension. A prerequisite is the result in algebraic geometry that transcendence degree is the same thing as Krull dimension.

Remark – Defining formula for Zariski closed sets. Let K be an alebraically closed field and $V \subseteq K^n$ a Zariski closed set. Then it is a finitely generated vanishing so there exists a finite $S_V \subseteq K[x_1, \dots, x_n]$ such that V is defined by the $\Sigma(K)$-formula

$$\bigwedge_{p \in S_V} p = 0$$

where each polynomial is naturally a $\Sigma(K)$-term.

> **Proposition – Morley rank is Krull dimension for algebraically closed fields [5]**
> Let K be an algebraically closed field and $V \subseteq K^n$ a variety. Then the Morley rank of V is equal to its Krull dimension.

Proof. We write tp for $\mathrm{tp}_K^{\mathbb{M}}$. We show by induction that for each $n \in \mathbb{N}$ if $\mathrm{k.dim}(V) = n$ then $\mathrm{MR}(V) = n$. Since Krull dimension is always finite this is sufficient. If $n = 0$ then V is a singleton hence $\mathrm{MR}(V) = 0$.

Suppose $n > 0$ and $\mathrm{k.dim}(V) = n$. Let $\mathbb{M}$ be a κ-saturated elementary extension of K for some cardinal κ strictly greater than $|K|$. We replace K and V with their images in $\mathbb{M}$ to make things simpler. The variety V is Zariski closed by definition and so it is defined by a $\Sigma(K)$-formula

$$\phi := \bigwedge_{p \in S_V} p = 0$$

for some finite $S_V \subseteq K[x_1, \dots, x_n]$.

Note that since formulas are represented by types and κ-saturation we have

$$\mathrm{MR}(V) = \mathrm{MR}(\phi) = \max \{\mathrm{MR}(q) \mid \phi \in q \in S_n(\mathrm{Th}_{\mathbb{M}}(K))\} = \max \{\mathrm{MR}(\mathrm{tp}(a)) \mid a \in \phi(\mathbb{M})\}$$

Thus there exists $a \in \phi(\mathbb{M})$ such that $\mathrm{MR}(\mathrm{tp}(a)) = \mathrm{MR}(V)$. It suffices to show that $\mathrm{MR}(\mathrm{tp}(a)) = n$.

Let $I(a)$ denote the ideal in $K[x_1, \ldots, x_n]$ of polyomials vanishing at a (extending our previous definition of I beyond K^n). Then $V_a := \mathbb{V}_K(I(a))$ is Zariski closed and so it is defined by a $\Sigma(K)$-formula

$$\psi_a := \bigwedge_{p \in S_{V_a}} p = 0$$

for some finite $S_{V_a} \subseteq K[x_1, \ldots, x_n]$. We first show that $V_a = V$. To show that $V_a \subseteq V$, we take $b \in V_a$ and note that for each $p \in S$, $p(b) = 0$ by definition and so $b \in \phi(K) = V$. Suppose for a contradiction $V_a \subset V$. Then the $\mathrm{k.dim}(V_a) < \mathrm{k.dim}(V) = n$. Hence by the induction hypothesis $\mathrm{MR}(V_a) < n$ and since $a \in \psi_a(\mathbb{M})$

$$\begin{aligned}
\mathrm{MR}(V) &= \mathrm{MR}(\mathrm{tp}(a)) \\
&\leq \max \{\mathrm{MR}(\mathrm{tp}(b) \mid b \in \psi_a(\mathbb{M}))\} \\
&= \max \{\mathrm{MR}(q) \mid \psi_a \in q \in S_n(\mathrm{Th}_{\mathbb{M}}(K))\} \\
&= \mathrm{MR}(\psi) \\
&= \mathrm{MR}(V_a) < n
\end{aligned}$$

Hence by the induction hypothesis V has Krull dimension strictly less than n, a contradiction. Thus $V_a = V$.

Now we have that $K(a)$ is isomorphic to the function field $K(V)$:

$$K(a_1, \ldots, a_n) \cong K[x_1, \ldots, x_n]/I(V_a) = K[x_1, \ldots, x_n]/I(V) = K(V)$$

As Krull dimension is transcendence degree, $n = \mathrm{k.dim}(V) = \mathrm{t.deg}(K \to K(V)) = \mathrm{t.deg}(K \to K(a))$. This in turn is the same as $\dim_{\Sigma_{\mathrm{RNG}}(K),\mathbb{M}}(\{a_1, \ldots, a_n\})$ since transcendence degree is dimension. Now ACF is strongly minimal and $\dim_{\Sigma_{\mathrm{RNG}}(K),\mathbb{M}}(\mathbb{M})$ is infinite thus dimension corresponds to Morley rank of the type

$$k = \dim_{\Sigma_{\mathrm{RNG}}(K),\mathbb{M}}(\{a_1, \ldots, a_n\}) = \mathrm{MR}(\mathrm{tp}(a)) = \mathrm{MR}(V)$$

$\square$

Chapter 4

Appendix

4.1 Algebraic closure

Definition

Let K be a field. L is an algebraic closure of K if there exists an algebraic field extension $\iota : K \to L$ and L is algebraically closed.

Proposition – Existence of algebraic closure of Fields [1]

Let K be a field. Then there exists an algebraic closure of K.

Proof. We build L by induction, at each step making an algebraic extension to a greater field containing roots of irreducible polynomials in the previous field. Without loss of generality we just do this for K.

Let F be the set of irreducible polynomials $f \in K[x]$. Consider $K[x_f]_{f\in F}$, where each x_f is a variable and the ideal $\mathfrak{a} = \sum_{f\in F} K f(x_f)$. We want to quotient by this ideal, but we also want this to result in a field. Thus we try to find a maximal ideal containing it. By an application of Zorn, all proper ideals are contained in a maximal ideal, hence it suffices to show it is proper. Clearly it is non-trivial (it contains $x_f - 1$ for $f = x - 1$). Assume for a contradiction that it is the whole ring, then there is a finite set S such that

$$1 = \sum_{f\in S} c_f f(x_f)$$

where S is finite and each $c_f \in K$. By the existance of splitting fields over base field K, we take the splitting field K' of $\prod_{f\in S} f$. Then from $K'[x_f]_{f\in S}$ we can evaluate at the roots r_f for each f:

$$1 = \sum_{f\in S} c_f f(r_f) = \sum_{f\in S} 0 = 0$$

a contradiction.

Let $\mathfrak{m}$ be the maximal ideal containing $\mathfrak{a}$ and let $K(1) = K[x_f]_{f\in F}/\mathfrak{m}$. There is an obvious ring morphism $K \to K(1)$. If $f \in K[x]$ is irreducible then $f(x_f) \in \mathfrak{m}$ so $0 = \overline{f(x_f)} = f(\overline{x_f})$ in $K(1)$. Hence f has a root x_f in $K(1)$.

By the above construction, we have a chain $K : \mathbb{N} \to \mathbb{M}\mathbf{od}(\Sigma_{\mathrm{RNG}})$ and so we can take the direct limit; call this L. We obtain for free that L is a field and there is a ring morphism $K \to L$ that commutes with all the morphisms in the chain. If $f \in L[x]$ then there exists a $\beta \in \mathbb{N}$ and $b \in K(\beta)[x]$ such that $b \mapsto f$ in the map $K(\beta)[x] \to L[x]$. Then b has finite degree n, thus completely splits in $K(\beta + n)$. Hence b splits in L, f splits in L so L is algebraically closed.

To show that L is an algebraic extension let $a \in L$. Then there exists a $\beta \in \mathbb{N}$ and $b \in K(b)$ such that $b \mapsto a$ in the map $K(\beta) \to L$. Since b is algebraic over K (each extension is finite hence algebraic) a is algebraic over K. $\square$

Proposition – Basic facts about algebraic closures
Suppose K has an algebraic closure Ω. Then

1. If M is an algebraically closed field extending K then there exists a field extension $\Omega \to M$ such that the diagram commutes

$$\begin{tikzcd} K \arrow[r] \arrow[dr] & \Omega \arrow[d, dashed] \\ & M \end{tikzcd}$$

2. If $K \to L$ is an algebraic field extension then there exists a field extension $L \to \Omega$ such that the diagram commutes

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \Omega \\ L \arrow[ur, dashed] & \end{tikzcd}$$

Proof.

1. The map $K \to \Omega$ is algebraic and so we can consider Ω as $K(S)$ for some minimal generating set $S \subseteq \Omega$. M is algebraically closed thus it splits all minimal polynomials of $a \in S$ over K, thus by the embedding theorem (Galois theory), we have an embedding $\Omega \to M$ that commutes with the diagram.
2. If $K \to L$ is algebraic then we can find a generating set $S \subseteq L$ such that $K(S) = L$. Ω is algebraically closed thus it splits all minimal polynomials of $a \in S$ over K, thus by the embedding theorem (Galois theory), we have an embedding $L \to \Omega$ that commutes with the diagram.

$\square$

Proposition – Algebraic closures of isomorphic fields are isomorphic
Suppose $K_0 \cong K_1$. Then any algebraic closures of these fields are (non-canonically) isomorphic.

Proof. Call the algebraic closures Ω_0 and Ω_1:

$$\begin{tikzcd} K_0 \arrow[r, "\iota_0"] \arrow[d, "\sigma", "\sim"'] & \Omega_0 \\ K_1 \arrow[r, "\iota_1"] & \Omega_1 \end{tikzcd}$$

By the properties of algebraic closures there exist $f : \Omega_0 \to \Omega_1$ and $g : \Omega_1 \to \Omega_0$ that individually commute with the above diagram. We claim that the composition $g \circ f$ is a surjection.

The composition fixes K_0 because $\iota_0 = g \circ \iota_1 \circ \sigma = g \circ f \circ \iota_0$. Fix $a \in \Omega_0$. Let Λ be the set of roots of the minimal polynomial $\min(a, K_0)$. $g \circ f$ is a bijection on Λ: let $\lambda \in \Lambda$, then $g \circ f(\min(a, K_0))(\lambda) = \min(a, K_0)(g \circ f(\lambda))$ because $g \circ f$ fixes K_0. Thus $g \circ f(\Lambda) \subseteq \Lambda$. It is a therefore a bijection on Λ as Λ is finite and $g \circ f$ is injective (it is a field embedding). Thus there exists an element that maps to a under the composition. Hence

$$\Omega_1 = g \circ f(\Omega_1) \subseteq f(\Omega_0)$$

Thus f is surjective and injective (as it is a field embedding). Hence f is an isomorphism. □

4.2 Transcendence degree and characteristic determine ACF_p up to isomorphism

This part of the appendix mainly follows Hungerford's book [2].

Definition
Suppose $\iota : K \to L$ is a field embedding and $S \subseteq L$. Then S is algebraically independent over K when for any $n \in \mathbb{N}$, any $f \in K[x_1, \dots, x_n]$ and any distinct elements $s_i \in S$,

$$f(s) = 0 \Rightarrow f = 0$$

Remark. 1. If S is algebraically independent then any subset is algebraically independent over K.

2. In particular $\varnothing$ is algebraically over K.
3. For the embedding $K \to K$, any non-empty subset if algebraically dependent.
4. The concept of algbebraic independence extends that of linear independence.

Definition
Suppose $K \to L$ is a field embedding. Then $B \subseteq L$ is a transcendence basis of the extension if B is algebraically independent over K and is maximal with respect to $\subseteq$.

Remark. This is the analogue of a vector space basis.

Proposition – Existence of transcendence basis
Suppose $K \to L$ is a field embedding. There there exists a transcendence basis.

Proof. Application of Zorn's lemma on the set of algebraically independent subsets of L (non-empty due to $\varnothing$ being algebraically independent) with respect to inclusion. □

Proposition – Algebraic elements over $K(S)$
Let $K \to L$ be a field embedding and let $S \subseteq L$ be algebraically independent over K. Let $u \in L \setminus S$. $S \cup \{u\}$ is algebraically dependent if and only if u is algebraic over $K(S)$.

Proof. ($\Rightarrow$) Suppose $S \cup \{u\}$ is algebraically dependent. Then there exists non-zero $f \in K[x_0, \dots, x_n]$ and distinct $s_i \in S$ such that $f(u, s_1, \dots, s_n) = 0$. Write

$$f = \sum_{i=0}^{m} h_i(x_0)^i$$

for $h_i \in K[x_1, \dots, x_n]$. Then let

$$g(x_n) := \sum_{i=0}^{m} h_i(s_1, \dots, s_n)(x_0)^i \in K(S)[x_0]$$

which has the root u. Assuming for a contradiction that u is not algebraic over $K(S)$. Since $g(u) = 0$ we must have $g(x_0) = 0$ and so for each i, $h_i(s_1, \dots, s_n) = 0$. By algebraic independence of S over K we have that each $h_i = 0$. Thus $f = 0$, a contradiction.

($\Leftarrow$) Suppose u is algebraic over $K(S)$. Then there exists non-zero $f(x) \in K(S)[x]$ such that $f(u) = 0$. We can write $f(x) = \sum_{i=1}^{n} \frac{h_i}{g_i}(s_1, \ldots, s_m)x^i$ for $s_i \in K(S)$. Then we can factor all the g_i out, leaving

$$f(x) = \frac{1}{\prod_{j=1}^{n} g_j} \sum_{i=1}^{n} f_i(s_1, \ldots, s_m) \prod_{j \neq i} g_j(s_1, \ldots, s_m)x^i$$

Let

$$h = \sum_{i=1}^{n} f_i(x_1, \ldots, x_m) \prod_{j \neq i} g_j(x_1, \ldots, x_m){x_{m+1}}^i \in K[x_1, \ldots, x_{m+1}]$$

We see that $h(s_1, \ldots, s_m, u) = 0$ as $g(s_1, \ldots, s_m) \neq 0$ (S is algebraically independent). Suppose for a contradiction that $S \cup \{u\}$ is algebraically independent, then $h = 0$ and so for each i,

$$f_i(x_1, \ldots, x_m) \prod_{j \neq i} g_j(x_1, \ldots, x_m) = 0$$

but g_i were on the bottom of the fractions so they are non-zero, thus $f_i(x_1, \ldots, x_m) = 0$ as the polynomial ring is an integral domain. This implies that $f = 0$, a contradiction. □

Lemma – Composition of algebraic extensions

We use extensively the fact that the composition of algebraic extensions is algebraic.

Proposition – (Key) Identifying transcendence bases

Let $K \to L$ be a field embedding. Suppose $S \subseteq L$ is algebraically independent over K. Then L is algebraic over $K(S)$ if and only if S is a transcendence basis.

Proof. $\forall a \in L$, a is algebraic over $K(S)$ if and only if $\forall a \in L$, $S \cup \{a\}$ is algebraically independent if and only if S is maximally algebraically independent if and only if S is a transcendence basis. □

Corollary – Subsets containing transcendence bases

Let $K \to L$ be a field embedding and let $X \subseteq L$. If $K(X) \to L$ is algebraic then X contains a transcendence basis of the extension $K \to L$.

Proof. Using Zorn we have a maximally algebraically independent subset of X; call this S. This is a transcendence basis of the extension $K \to K(X)$. Thus by the previous proposition $K(S) \to K(X)$ is algebraic. The composition of algebraic extensions is algebraic, thus $K(S) \to L$ is algebraic. Hence S is a transcendence basis. □

Proposition – Uniqueness of finite transcendence degree

Let $K \to L$ be a field embedding and let S be a finite transcendence basis of the extension. Then any other transcendence basis has the same cardinality as S.

Proof. If S is empty then it is the unique transcendence basis.

Otherwise, let $S = \{s_1, \ldots, s_n\}$ and T be another transcendence basis. We show that there exists a $t \in T$ such that $\{t, s_2, \ldots, s_n\}$ is a transcendence basis.

To find such a t, assume for a contradiction that for any $t \in T$, $\{t, s_2, \ldots, s_n\}$ is algebraically dependent. Then each $t \in T$ is algebraic over $\{s_2, \ldots, s_n\}$ and so $K(\{s_2, \ldots, s_n\})(T)$ is algebraic over $K(\{s_2, \ldots, s_n\})$. Furthermore L is algebraic over $K(\{s_2, \ldots, s_n\})(T)$ hence L is algebraic over $K(s_2, \ldots, s_n)$. Hence s_1 is algebraic over $K(s_2, \ldots, s_n)$, a contradiction.

Thus there exists such a $t \in T$. It suffices to show that L is algebraic over $K(t, s_2, \dots, s_n)$. This is true if and only if $K(S)$ is algebraic over $K(t, s_2, \dots, s_n)$, if and only if s_1 is algebraic over $K(t, s_2, \dots, s_n)$, if and only if $\{t, s_1, s_2, \dots, s_n\}$ is algebraically dependent over K, which it is since it is contains S as a proper subset. Thus we have that $\{t, s_1, s_2, \dots, s_n\}$ is a transcendence basis of the same cardinality as S.

By induction, replacing s_i at each step, we obtain a subset of T that is a transcendence basis of the same cardinality as S. By maximality of transcendence bases this subset is T, thus $|S| = |T|$. □

Notation (Minimal polynomial). *If $K \to L$ is a field extension, let the minimal polynomial of $a \in L$ over K be denoted as* $\min(a, K)$

Lemma – Infinite transcendence bases inject into each other

Let field embedding $K \to L$ have infinite transcendence bases S and T. Then $|T| \leq |S|$.

Proof. S is infinite and hence non-empty; let $s \in S$ and consider $\min(s, K(T)) \in K(T)[x]$. Because polynomials are finite, there exists T_s, a finite subset of T such that $\min(s, K(T)) \in K(T_s)[x]$. Hence s is algebraic over $K(T_s)$.

We claim that $\bigcup_{s\in S} T_s$ is a transcendence basis of $K \to L$. Indeed it is a subset of T hence it is algebraically independent; by construction $K(S)$ is algebraic over $K(\bigcup_{s\in S} T_s)$ and L is algebraic over K and so L is algebraic over $K(\bigcup_{s\in S} T_s)$. Thus $\bigcup_{s\in S} T_s$ is a transcendence basis of $K \to L$. By maximality of transcendence bases $T = \bigcup_{s\in S} T_s$.

We inject T into S by writing it as a disjoint union of subsets of T_s. By the well-ordering principle (choice) we well-order S and define

$$X_s := T_s \setminus \bigcup_{i<s} T_i$$

Since $X_s \subseteq T_s$ we have $\bigcup_{s\in S} X_s \subseteq \bigcup_{s\in S} T_s$. Conversely, if there exist $s \in S$ and $x \in T_s$ then the set of $i \in S$ such that $x \in T_i$ is non-empty and thus has a minimum element m (by well-ordering). Thus $x \in X_m$ and so $\bigcup_{s\in S} X_s = \bigcup_{s\in S} T_s = T$.

Define a map from $\bigcup_{s\in S} X_s \to S \times \mathbb{N}$ by the following: let $s \in S$. Then since $X_s \subseteq T_s$ and T_s is finite we can write $X_s = \{x_0, \dots, x_{n_s}\}$. we send x_i to the element $(s, i) \in S \times \mathbb{N}$. This is well-defined because the X_i are disjoint and is clearly injective. Since S is infinite,

$$|T| = \left|\bigcup_{s\in S} X_s\right| \leq |S \times \mathbb{N}| = |S|$$

□

Proposition – Uniqueness of infinite transcendence degree

Let field embedding $K \to L$ have a transcendence basis. Then any other transcendence basis has the same cardinality.

Proof. Let S and T be two transcendence bases. If one of them were finite then by uniqueness of finite transcendence degree S and T have the same cardinality. Otherwise both are infinite and by the previous lemma $|T| \leq |S|$ and $|S| \leq |T|$. By Schröder–Bernstein $|S| = |T|$. □

Definition – Transcendence degree

If $\iota : K \to L$ is a field embedding then the transcendence degree is defined as the cardinality of a transcendence basis. It is well-defined as we showed that a basis exists and any two bases have the same cardinality. We use $\operatorname{t.deg}(\iota)$ to denote the degree.

NOTATION. *$K[x_1, \ldots, x_n]$ is the polynomial ring. $K(x_1, \ldots, x_n)$ is the field of fractions of the polynomial ring.*

Lemma – Isomorphism with field of polynomial fractions

Suppose $\iota : K \to L$ is a field embedding and $S \subseteq L$ is a finite set algebraically independent over K. Then there exists a (non-canonical) field isomorphism

$$K(S) \cong K(x_s)_{s \in S}$$

Proof. From Galois theory we have a (non-canonical) surjective ring morphism $K[x_s]_{s\in S} \to K[S]$ given by $x_s \mapsto s$. It is injective due to S being algebraically independence. By the universal property of field of fractions there is a unique isomorphism $K(S) \cong K(x_s)_{s\in S}$ that commutes with the other isomorphism.

$$\begin{tikzcd}
K[S] \arrow[r, "\subseteq"] \arrow[d, "\cong"] & K(S) \arrow[d, dashed] \\
K[x_s]_{s\in S} \arrow[r, "\subseteq"] & K(x_s)_{s\in S}
\end{tikzcd}$$

□

Proposition – Embedding algebraically independent sets

Suppose we have the field embeddings

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[d, "\sigma"] & F_0 \\
K_1 \arrow[r] & F_1
\end{tikzcd}$$

and let $S \subseteq F_0$ be an algebraically independent over K_0. Suppose we have an injection $\phi : S \to F_1$ such that the image is algebraically independent over K_1. Then there exists a unique field embedding $\overline{\sigma} : K_0(S) \to F_1$ such that $\overline{\sigma}\big|_S = \phi$ and the following commutes

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[d, "\sigma"] & K_0(S) \arrow[d, "\overline{\sigma}"] \\
K_1 \arrow[r] & F_1
\end{tikzcd}$$

Furthermore, if σ is an isomorphism then $\overline{\sigma}$ is an isomorphism $K_0(S) \to K_1(\phi(S))$.

Proof. We define $\overline{\sigma} : K_0(S) \to F_1$ by

$$\frac{f(s_1, \ldots, s_n)}{g(s_1, \ldots, s_n)} \mapsto \frac{\sigma(f)(\phi s_1, \ldots, \phi s_n)}{\sigma(g)(\phi s_1, \ldots, \phi s_n)}$$

where σ takes a polynomial over K_0 as an argument (the induced map on the polynomial rings). To check that $\overline{\sigma}$ is well-defined we just need to check uniqueness of the image. Suppose $\frac{f}{g}(s_1, \ldots, s_n) \in K_0(S)$. Due to the universal property of field of fractions, there is a unique field embedding $\widehat{\sigma} : K_0(x_i)_{i\leq n} \to K_1(x_i)_{i\leq n}$ that commutes with the injective polynomial ring morphism $\sigma : K_0[x_i]_{i\leq n} \to K_1[x_i]_{i\leq n}$ (which was induced by σ). By the previous lemma we have an isomorphisms $K_0(s_i)_{i\leq n} \cong K_0(x_i)_{i\leq n}$ and $K_1(\phi(s_i))_{i\leq n} \cong K_0(x_i)_{i\leq n}$ induced by the (not unique but suitably chosen) isomorphisms $K_0[s_i]_{i\leq n} \cong K_0[x_i]_{i\leq n}$ and $K_1[\phi(s_i)]_{i\leq n} \cong K_1[x_i]_{i\leq n}$. Hence we have the diagram:

$$\begin{tikzcd}
 & K_0[s_i]_i \arrow[r] \arrow[d, "\cong"] & K_0(s_i)_i \arrow[d, "\cong"] \\
K_0 \arrow[ur] \arrow[r] \arrow[d, "\sigma"] & K_0[x_i]_i \arrow[r] \arrow[d, "\sigma"] & K_0(x_i)_i \arrow[d, dashed, "\widehat{\sigma}"] \\
K_1 \arrow[r] \arrow[dr] & K_1[x_i]_i \arrow[r] \arrow[d, "\cong"] & K_1(x_i)_i \arrow[d, "\cong"] \\
 & K_1[\phi(s_i)]_i \arrow[r] & K_1(\phi(s_i))_i
\end{tikzcd}$$

The composition of the three maps on the right hand side is $\overline{\sigma}$ restricted to $K_0(s_i)_{i\leq n}$. Note that the composition is a well-defined injective ring morphism that commutes with everything else. Thus $\frac{f}{g}(s)$ is sent to a unique element of F_1. If $q \in K_0(S)$ maps to the same image under $\overline{\sigma}$ then it lies in the image of the composition so it is $\frac{f}{g}(s)$. The composition commutes with everything and so for anything from K_0, going to F_1 via σ is the same as going via $\widehat{\sigma}$.

Thus it is well-defined, injective and commutes. It is clearly a field embedding. It is unique because the map from $K_0[s_i] \to K_1[\phi(s_i)]$ was unique (though the intermediate isomorphisms were not unique). By definition, $\overline{\sigma}\big|_S = \phi$.

The above construction shows that if σ is an isomorphism then $\widehat{\sigma}$ is an isomorphism. Hence $\overline{\sigma}$ restricted to the finite subset is an isomorphism. Since this is for any subset, $\overline{\sigma}$ is an isomorphism $K_0(S) \to K_1(\phi(S))$. □

Proposition – Algebraically closed extensions of same transcendence degree are isomorphic

Suppose we have fields $K_0 \cong K_1$ and field extensions $K_0 \to L_0$ and $K_1 \to L_1$ of equal transcendence degree such that L_0, L_1 are algebraically closed, then L_0 and L_1 are (non-canonically) isomorphic.

Proof. Let σ be the isomorphism $K_0 \to K_1$ Let S_0, S_1 be transcendence bases of $K_0 \to L_0$ and $K_1 \to L_1$. They have the same cardinality thus we can biject S_0, S_1 and produce an isomorphism $\overline{\sigma} : K_0(S_0) \to K_1(S_1)$. The extensions $K_0(S_0) \to L_0$ and $K_1(S_1) \to L_1$ are algebraic and L_0, L_1 are algebraically closed. Hence they are algebraic closures of isomorphic fields, which implies they they are (non-canonically) isomorphic.

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[d, "\sigma"', "\sim"] & K_0(S_0) \arrow[r] \arrow[d, "\overline{\sigma}"', "\sim"] & L_0 \arrow[d, dashed] \\
K_1 \arrow[r] & K_1(S_1) \arrow[r] & L_1
\end{tikzcd}$$

□

Corollary – Transcendence degree and characteristic determine algebraically closed fields of characteristic p up to isomorphism

If K_0, K_1 are fields of the same characteristic and have the same transcendence degree over their minimal subfield ($\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$). Then they are (non-canonically) isomorphic.

Proof. K_0 and K_1 have the same characteristic p so they are extensions of isomorphic subfields (their minimal subfields). Thye are algebraically closed. They have the same transcendence degree thus by the previous proposition they are (non-canonically) isomorphic. □

Proposition – Tower law of transcendence degree

Suppose $K \xrightarrow{\iota_L} L \xrightarrow{\iota_M} M$ are field embeddings. Then

$$\mathrm{t.deg}(\iota_L) + \mathrm{t.deg}(\iota_M) = \mathrm{t.deg}(\iota_M \circ \iota_L)$$

Proof. Let B_L and B_M be transcendence bases for the extensions ι_L, ι_M respectively. We show that $B_L \cup B_M$ is a transcendence basis for the composition.

Since B_L is a basis, $K(B_L) \to L$ is algebraic and hence we can show that $K(B_L \cup B_M) \to L(B_M)$ is algebraic. $L(B_M) \to M$ is algebraic as B_M is a basis, thus the composition $K(B_L \cup B_M) \to M$ is algebraic.

To show that it is algebraically independent, we first note that B_L, B_M are disjoint, otherwise there exists b in the intersection, which is both in B_M and in L causing B_M to be algebraically dependent over L. Let $f \in K[x_1, \ldots, x_n]$ and let $l_1, \ldots, l_r, m_{r+1} \in B_L$ and $m_{r+1}, \ldots, m_n \in B_M$ be distinct elements such that

$$f(l_1, \ldots, l_r, m_{r+1}, \ldots, m_n) = 0$$

We can find some finite set I, $h_i \in K[x_1, \ldots, x_r]$ and $k_i \in K[x_{r+1}, \ldots, x_n]$ such that

$$f(x_1, \ldots, x_n) = \sum_{i \in I} h_i(x_1, \ldots, x_r) k_i(x_{r+1}, \ldots, x_n)$$

and each k_i are linearly independent.

$$g := \sum_{i \in I} h_i(l_1, \ldots, l_r) k_i(x_{r+1}, \ldots, x_n) \in L[x_{r+1}, \ldots, x_n] \quad \wedge \quad g(m_{r+1}, \ldots, m_n) = 0$$

Since B_M is algebraically independent $g = 0$. Thus (by linear independence) of k_i each $h_i(l_1, \ldots, l_r) = 0$ and hence each $h_i(x_1, \ldots, x_r) = 0$ as B_L is algebraically independent. Thus $f = 0$ and the union forms a transcendental basis and

$$\mathrm{t.deg}(\iota_M \circ \iota_L) = |B_L \cup B_M| = |B_L| + |B_M| - |B_L \cap B_M| = \mathrm{t.deg}(B_L) + \mathrm{t.deg}(B_M)$$

□

Lemma – Isomorphic extensions have same transcendence degree

Suppose $K \to L, K \to M$ are field extensions and $L \to M$ is a an isomorphism that preserves K, then

$$\mathrm{t.deg}(K \to L) = \mathrm{t.deg}(K \to M)$$

Proof. Let S be a transcendence basis for $K \to L$. We claim the image of S under the isomorphism $\sigma : L \to M$ is a transcendence basis for $K \to M$.

Algebraic independence: Let $n \in \mathbb{N}$ and let $p \in K[x_1, \ldots, x_n]$. Let $a \in \sigma(S)^n$ and suppose $p(a) = 0$. Then we apply σ^{-1} to both sides, noting that p has coefficients from K and therefore commutes with σ^{-1}:

$$0 = \sigma^{-1}(p(a)) = p(\sigma^{-1}(a))$$

But $\sigma^{-1}(a)$ is an element of S^n and by algebraic independence of S we have $p = 0$.

Maximality: let $a \in M \setminus \sigma(S)$. We show that $a \cup \sigma(S)$ is algebraically dependent. Consider $\sigma^{-}1(a) \in L$. Since S is a basis $\sigma^{-1}(a)$ is algebraic over $K(S)$. This we have $p \in K(S)[x_0]$ such that $p(\sigma^{-1}(a)) = 0$. We can identify p with a polynomial q in $K[x_0, \dots, x_n]$ with the remaining n coefficients from S, such that

$$p(x_0) = q(x_0, s_1, \dots, s_n)$$

Since σ preserves K we have that

$$0 = \sigma(q(\sigma^{-1}(a), s_1, \dots, s_n)) = q(a, \sigma(s_1), \dots, \sigma(s_n))$$

so $a \cup \sigma(S)$ is algebraically dependent.

Since σ is a field embedding it is injective and so S and $\sigma(S)$ have the same cardinality. □

4.3 Locally finite fields and polynomial maps

Definition – Locally finite
We say that a field is locally finite if for any finite subset, the minimal subfield '$K(S)$' containing the subset is finite.

Definition – Polynomial map
Let K be a field and n a natural. Let $f : K^n \to K^n$ such that for each $a \in K^n$,

$$f(a) = (f_1(a), \dots, f_n(a))$$

for $f_1, \dots, f_n \in K[x_1, \dots, x_n]$. Then we call f a polynomial map over K.

Lemma – Equivalences for locally finite over prime characteristic [12]
Let K be a field of characteristic p a prime. Then the following are equivalent:

1. K is locally finite.
2. $\mathbb{F}_p \to K$ is algebraic.
3. K embeds into an algebraic closure of $\mathbb{F}_p$.

In particular, the algebraic closure of a finite field is locally finite.

Proof.

1. $\Rightarrow$ 2. We show the contrapositive. Suppose there exists $a \in K$ such that a is not algebraic over $\mathbb{F}_p$. Then $\mathbb{F}_p(a)$ is a isomorphic to the field of fractions of a polynomial ring over K and so is infinite. Hence K is not locally finite.

2. $\Rightarrow$ 1. We show by induction that K is locally finite. Let S be a finite subset of K. If S is empty then $\mathbb{F}_p(S) = \mathbb{F}_p$ and so it is finite. If $S = S' \cup s$ and $\mathbb{F}_p(S')$ is finite, then $s \in K$ is algebraic so by some Galois theory we can write

$$\mathbb{F}_p(S')[x]/(\min(s, \mathbb{F}_p(S'))) \cong \mathbb{F}_p(S)$$

Where the left hand side is finite because it is a finite dimensional vector space over a finite field. Hence K is locally finite.

2. $\Rightarrow$ 3. If $\mathbb{F}_p \to K$ is algebraic then it can be written as $\mathbb{F}_p(S)$ for some set S algebraic over $\mathbb{F}_p$. Any algebraic closure of $\mathbb{F}_p$ splits each $a \in S$ hence by the embedding theorem (Galois Theory)K embeds into the algebraic closure.

3. $\Rightarrow$ 2. If $a \in K$ then a can be embedded into the algebraic closure of $\mathbb{F}_p$ by assumption. Then it is algebraic over $\mathbb{F}_p$ thus a is algebraic over $\mathbb{F}_p$.

Any finite field is an algebraic extension over $\mathbb{F}_p$ where p is its prime characteristic. Hence its algebraic closure is an algebraic extension over $\mathbb{F}_p$ and so it is locally finite. $\square$

Corollary – Ax-Grothendieck for algebraic closure of finite fields

If Ω is an algebraic closure of a finite field then any injective polynomial map over Ω is surjective.

Proof. Suppose for a contradiction $f : \Omega^n \to \Omega^n$ is an injective polynomial map that is not surjective. Then there exists $b \in \Omega^n$ such that $b \notin f(\Omega^n)$. Writing $f = (f_1, \ldots, f_n)$ for $f_i \in \Omega[x_1, \ldots, x_n]$ we can find $A \subseteq \Omega^n$, the set of all the coefficients of all of the f_i. We want to find a contradiction by showing that f is surjective on the subfield K generated by $A \cup \{b\}$, which contains b. $A \cup \{b\}$ is finite and Ω is locally finite, thus K is finite. The restriction $f\big|_{K^n}$ is injective and has image inside K^n since each polynomial has coefficients in K and is evaluated at an element of K^n. Hence $f\big|_{K^n}$ is an injective endomorphism of a finite set thus is surjective, a contradiction. $\square$

4.4 Noetherian Modules

All of the following material is from Atiyah and McDonald's book [1].

Definition – Ascending chain

If $(\Sigma, \leq)$ is a partially ordered set then an ascending chain is a functor $F : (\mathbb{N}, \leq) \to (\Sigma, \leq)$.

A chain is stationary if there exists a $m \in \mathbb{N}$ such that for any $n \in \mathbb{N}_{\geq m}$, $F(m) = F(n)$.

Proposition – Noetherian chains

Let Σ be a partially ordered set. Then every non-empty subset has a maximal element if and only if every ascending chain is stationary.

Proof. ($\Rightarrow$) If F is a chain then $I = \{F(i) \mid i \in \mathbb{N}\}$ has a maximal element $F(m)$. Hence for any n greater than or equal to m, $F(m) = F(n)$. ($\Leftarrow$) Let $\varnothing \neq I \subseteq \Sigma$ and suppose for a contradiction that for each $m \in I$ there exists $a \in I$ such that $m < a$. Then create a chain by recursively taking such an a, starting with the non-empty element: $a_0 < a_1 < \ldots$. This chain is not stationary. $\square$

Definition – Noetherian Module

Order the submodules of an A-module M by $\subseteq$. M is Noetherian if it satisfies on one of the following equivalent definitions:

- Every non-empty set of submodules has a maximal element.
- Every ascending chain of submodules is stationary.
- Any submodule of M is finitely generated.

Proof. 1. $\Leftrightarrow$ 2. follows from Noetherian chains. 1. $\Rightarrow$ 3.: Let $N \leq M$ and take the set

$$I = \{S \leq N \mid S \text{ finitely generated}\}$$

Then $0 \in I$ hence it is non-empty hence has a maximal element, which we call N_0. Suppose for a contradiction that $N_0 < N$. Then there exists $x \in N \setminus N_0$ such that $N_0 < N_0 + Ax \leq N$. This is finitely generated hence belongs to I, contradicting with the maximality of N_0. 3. $\Rightarrow$ 2.: Let $N_0 \leq N_1 \leq \dots$ be a chain of submodules. Their union N is a submodule of M hence is finitely generated. There exists N_m such that all the generators are in N_m. At m the chain becomes stationary. □

Proposition – Exactness and Noetherian modules
If

$$0 \to N \to M \to P \to 0$$

is exact then M is Noetherian if and only if both N and P are Noetherian.

Proof. ($\Rightarrow$) If M is Noetherian then it is finitely generated, hence the image of N is finitely generated and so N is finitely generated. Furthermore any element in P has a preimage in M which can be written as a linear combination of the generators. Thus the image of the generators finitely generate P. ($\Leftarrow$) Let $(M_i)_{i\in\mathbb{N}}$ be an ascending chain of submodules in M. Its preimage (N_i) in N and its image (P_i) in P become stationary at m_n, m_p. We claim that (N_i) becomes stationary at the maximum of the two. We first show that

$$N_i \to M_i \to P_i$$

is exact. Indeed

$$\begin{aligned}&\ker(M_i \to P_i)\\ &= M_i \cap \ker(M \to P)\\ &= M_i \cap \Im(N \to M)\\ &= \Im(N_i \to M_i)\end{aligned}$$

We apply the snake lemma to

$$\begin{array}{ccccc} N_i & \longrightarrow & M_i & \longrightarrow & P_i \\ \downarrow & & \downarrow & & \downarrow \\ N_{i+1} & \longrightarrow & M_{i+1} & \longrightarrow & P_{i+1} \end{array}$$

Obtaining the exact sequences

$$0 = \ker(N_i \to N_{i+1}) \to \ker(M_i \to M_{i+1}) \to \ker(P_i \to P_{i+1}) = 0$$

$$0 = \mathrm{Coker}(N_i \to N_{i+1}) \to \mathrm{Coker}(M_i \to M_{i+1}) \to \mathrm{Coker}(P_i \to P_{i+1}) = 0$$

Noting that by stability the end points are 0. Hence by exactness $M_i \to M_{i+1}$ is an isomorphism (via $\subseteq$) and $M_i = M_{i+1}$. □

Proposition – Direct sum of Noetherian is Noetherian
If M_i are Noetherian A-modules then the direct sum $\bigoplus_{i=1}^n M_i$ is Noetherian.

Proof. Induction. Consider the exact sequence

$$0 \to M_n \to \bigoplus_{i=1}^{n} M_i \to \bigoplus_{i=1}^{n-1} M_i \to 0$$

Apply the proposition on exact sequences. □

Definition – Noetherian ring
A ring is Noetherian if it is a Noetherian module over itself.

Proposition
Finitely generated modules of Noetherian rings If A is a Noetherian ring then any finitely generated A-module is Noetherian.

Proof. M is the image of a projection of A^n for some n. It suffices that A^n is Noetherian by considering the exact sequence

$$0 \to \ker \to A^n \to M \to 0$$

A^n is a direct sum of Noetherian modules hence is Noetherian. □

Proposition – Hilbert basis theorem
If A is Noetherian then the polynomial ring $A[x_1, \dots, x_n]$ is Noetherian. In particular for any field K, $K[x_1, \dots, x_n]$ is Noetherian.

Proof. By induction it suffices to show that $A[x]$ is Noetherian. Let $\mathfrak{a}$ be an ideal of $A[x]$. We show that $\mathfrak{a}$ is finitely generated. Let I be the ideal of A formed by the leading coefficients of polynomials in $\mathfrak{a}$. I finitely generated as A is Noetherian. Hence $I = \sum i \in SAa_i$ for some leading coefficients a_i of polynomials f_i. We consider the ideal $\sum_{i \in S} A[x]f_i$.

We show that $\mathfrak{a} \leq \sum_{i \in S} A[x]f_i + \sum_{i=0}^{d-1} A[x]x^i$ which is finitely generated, and hence $\mathfrak{a}$ is finitely generated.

Let $p \in \mathfrak{a}$ then if $d := \max_{i \in S} \deg f_i \leq \deg p$ we can write the leading coefficient of p as a sum $\sum_{i \in S} \lambda_i a_i$ (for $\lambda_i \in A$). So $p - x^{\deg p - d} \sum_{i \in S} \lambda_i f_i$ had degree strictly less than before. By induction (stopping when the degree of p is less than d) we obtain $p \in \sum_{i \in S} A[x]f_i + \sum_{i=0}^{d-1} A[x]x^i$. Hence $A[x_1, \dots, x_n]$ is Noetherian.

For the 'in particular' we note that fields only have two ideals, hence trivially satisfy the ascending chain condition. □

4.5 Strong Nullstellensatz and Prime Spectra

We use the 'Rabinowitsch trick' to prove strong Nullstellensatz from the weak version. This is also from Atiyah and McDonald's book [1].

Proposition – Strong Nullstellensatz
Let K be an algebraically closed field and suppose $\mathfrak{a}$ is an ideal of $K[x_1, \dots, x_n]$. Then $r(\mathfrak{a}) = I(\mathbb{V}(\mathfrak{a}))$.

Proof. ($\Rightarrow$) We show $r(\mathfrak{a}) \subseteq I(\mathbb{V}(\mathfrak{a}))$. Clearly $I(\mathbb{V}(\mathfrak{a})) \subseteq r(I(\mathbb{V}(\mathfrak{a})))$ and if f is in the radical then $f^n(a) = 0$ and by induction (using that $K[x_1, \dots, x_n]$ is an integral domain) $f(a) = 0$. Hence $I(\mathbb{V}(\mathfrak{a})) = r(I(\mathbb{V}(\mathfrak{a})))$. By opening up the definition of $I(\mathbb{V}(\mathfrak{a}))$ we can show that $\mathfrak{a} \subseteq I(\mathbb{V}(\mathfrak{a}))$. Thus

$$r(\mathfrak{a}) \subseteq r(I(\mathbb{V}(\mathfrak{a}))) = I(\mathbb{V}(\mathfrak{a}))$$

($\Leftarrow$) Let $g \in I(\mathbb{V}(\mathfrak{a}))$. Consider the injective ring morphism $\iota : K[x_1, \dots, x_n] \to K[x_1, \dots, x_n][y]$ and the polynomial $1 - y\iota(g)$ of the codomain. Evaluation of this polynomial (in y) at any element of $\mathbb{V}_K(\iota(\mathfrak{a}))$ gives us 1. If $\iota(\mathfrak{a}) + \langle 1 - y\iota(g)\rangle$ were a proper ideal it would be contained in a maximal ideal [1] which is prime, but that maximal ideal would have an empty vanishing as everything evaluates to 1. By the weak Nullstellensatz this is a contradiction. Hence $\iota(\mathfrak{a}) + \langle 1 - y\iota(g)\rangle = \langle 1 \rangle$ and there exists a finite sum resulting in 1:

$$1 = \sum_{f \in S} \iota(f)h_f(y) + (1 - y\iota(g))h(y)$$

If $\iota(g) = 0$ then since ι is injective $g = 0$ and so is in $r(\mathfrak{a})$. Otherwise, we can make $A[\frac{1}{\iota(g)}]$ and evaluate the polynomial at $\frac{1}{\iota(g)}$:

$$1 = \sum_{f \in S} \iota(f) h_f(\frac{1}{\iota(g)})$$

Hence there exist $H_f \in \iota(\mathfrak{a})$ and $m \in \mathbb{N}$ such that

$$1 = \sum_{f \in S} \iota(f) \frac{H_f}{(\iota(g))^m}$$

Hence $\iota(g) \in \iota(\mathfrak{a})$ and so $g \in \mathfrak{a}$ by injectivity of ι. □

Corollary – Galois correspondence between ideals and vanishings

Let K be an algebraically closed field. If $X \subseteq K^n$ is Zariski closed then $\mathbb{V}(I(X)) = X$. Thus we have order reversing bijections:

$$\{\mathbb{V} \subseteq K^n \,|\, \text{vanishing}\} \underset{\mathbb{V}(\star)}{\overset{I(\star)}{\rightleftarrows}} \{\mathfrak{a} \leq K[x_1, \ldots, x_n] \,|\, r(\mathfrak{a}) = \mathfrak{a}\}$$

Proof. There exists a prime ideal $\mathfrak{a} \leq K[x_1, \ldots, x_n]$ such that $X = \mathbb{V}(\mathfrak{a})$ and $\mathfrak{a}$ is the radical of itself. By strong Nullstellensatz we have that $I(\mathbb{V}(\mathfrak{a})) = \mathfrak{a}$ thus $\mathbb{V}(I(\mathbb{V}(\mathfrak{a}))) = \mathbb{V}(\mathfrak{a})$. Hence $\mathbb{V}(I(X)) = X$. □

Proposition – Irreducible

If X is a non-empty topological space then the following are equivalent:

1. Any non-empty open set is dense in X.
2. Any pair of non-empty open subsets intersect non-trivially.
3. Any two closed proper subsets do not form a cover of X.

Proof. (1. $\Rightarrow$ 2.) Let $U, V \subseteq Y$ be open. $\overline{V} = X$ by assumption. Hence $\varnothing \neq U \subseteq \overline{V}$ and so their intersection is non-trivial. Hence $U \cap V$ is non-trivial.

(2. $\Rightarrow$ 1.) Let U be open and non-empty. Then if $\overline{U} \neq X$ then its complement is non-empty and so by assumption $U \cap X \setminus \overline{U}$ is non-empty, a contradiction.

(2. $\Leftrightarrow$ 3.) is clear. □

Definition – Regular map

Let $X \subseteq K^n$ and $Y \subseteq K^m$ be Zariski closed sets over field K. Then $\rho : X \to Y$ is regular when there exist polynomials $\{f_i\}_{i=1}^m \subseteq K[x_1, \ldots, x_n]$ such that ρ is the restriction to X of the map $K^n \to K^m$

$$a \mapsto (f_i(a))_{i=1}^m$$

Definition – Prime spectrum (Zariski topology)

In commutative algebra, for any ring A there is a topology $\mathrm{Spec}(A)$ (the spectrum of A), the set of all prime ideals in A. This is generated by the closed sets, namely for any $E \subseteq A$ the set

$$V(E) := \{\mathfrak{p} \in \mathrm{Spec}(A) \,|\, E \subseteq \mathfrak{p}\}$$

the 'vanishing' is closed. (These will generalise the vanishings in the classical Zariski topolgy.) It can be shown that under finite union and arbitrary intersection of these sets are still closed thus it defines a topology on $\mathrm{Spec}(A)$. Furthermore we have that for any $E \subseteq A$, $V(E) = V(\langle E \rangle)$ where the latter is the ideal generated by E.

We wish to relate all this to the classical setting. We take our ring A to be $K[x_1, \dots, x_n]$. We will show that the set of vanishings in $\mathrm{Spec}(K[x_1, \dots, x_n])$ bijects with the set of Zariski closed sets in K^n. Then we will take the varieties as the closed sets in $K[x_1, \dots, x_n]$.

Notation (Ideal generated by varieties). *For a subset $X \subseteq K^n$, we write $I(X)$ to mean the ideal of X in $K[x_1, \dots, x_n]$ to mean*

$$\{f \in K[x_1, \dots, x_n] \mid \forall a \in X, f(a) = 0\}$$

Proposition – Correspondence between Zariski topology and prime spectrum

Given K a field and a vanishing in the spectrum of $K[x_1, \dots, x_n]$, there exists a unique variety of some finite subset of $K[x_1, \dots, x_n]$. and vice versa.

Proof. Let $V(E) \in B$ then $\langle E \rangle$ is finitely generated by the Hilbert basis theorem so there exists some finite subset $S \subseteq K[x_1, \dots, x_n]$ such that

$$V(E) = V(\langle E \rangle) = V(\langle S \rangle)$$

We send $V(E)$ to $\mathbb{V}_K(S)$. This is well defined: suppose $V(\langle S \rangle) = V(\langle T \rangle)$ then for any prime ideal $\mathfrak{p}$, $\langle S \rangle \subseteq \mathfrak{p} \Leftrightarrow \langle T \rangle \subseteq \mathfrak{p}$. Hence

$$r(S) = \bigcap_{\langle S \rangle \subseteq \mathfrak{p} \text{ prime}} \mathfrak{p} = \bigcap_{\langle T \rangle \subseteq \mathfrak{p} \text{ prime}} \mathfrak{p} = r(T)$$

Let $a \in \mathbb{V}_K(S)$, then for any $f \in < S >$, $f(a) = 0$. To show that $a \in \mathbb{V}_K(T)$, let $f \in T$, then there exists some n such that $f^n \in < S >$ as $r(S) = r(T)$. Hence $f^n(a) = 0$ and since $K[x_1, \dots, x_n]$ is an integral domain, by induction $f(a) = 0$. Thus $a \in \mathbb{V}_K(T)$.

This map is clearly surjective and by strong Nullstellensatz it is injective:

$$\mathbb{V}_K(S) = \mathbb{V}_K(T) \Rightarrow I(\mathbb{V}_K(S)) = I(\mathbb{V}_K(T)) \Rightarrow r(S) = I(\mathbb{V}_K(S)) = I(\mathbb{V}_K(T)) = r(T)$$

hence

$$\langle S \rangle \subseteq \mathfrak{p} \Leftrightarrow r(S) \subseteq \mathfrak{p} \Leftrightarrow r(T) \subseteq \mathfrak{p} \Leftrightarrow \langle T \rangle \subseteq \mathfrak{p}$$

Thus $V(S) = V(T)$. $\square$

Proposition

If K is an algebraically closed field then $\mathrm{Spec}(K[x_1, \dots, x_n])$ is not Hausdorff.

Proof. Let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct prime ideals. We show that any opens containing each of them have non-trivial intersection.

Let $U_{\mathfrak{p}}, U_{\mathfrak{q}}$ be open sets containing the respective ideals. Closed sets are of the form $V(\mathfrak{a})$ for $\mathfrak{a}$ an ideal of $K[x_1, \dots, x_n]$ which by the Hilbert basis theorem is finitely generated. Hence there exist finite sets $S_{\mathfrak{p}}, S_{\mathfrak{q}} \subseteq K[x_1, \dots, x_n]$ such that

$$U_{\mathfrak{p}} = \{y \in \mathrm{Spec} \mid r(S_{\mathfrak{p}}) \nsubseteq y\}$$

and similarly for $U_{\mathfrak{q}}$. If $S_{\mathfrak{p}}$ contains only 0 then $U_{\mathfrak{p}}$ would be empty. Thus there exist $f \in S_{\mathfrak{p}} \setminus \{0\}, g \in S_{\mathfrak{p}} \setminus \{0\}$. It suffices to show that there is a prime ideal that contains neither f nor g.

As algebraically closed fields are infinite and so we can inject $\iota : \mathbb{N} \to K$. Define for each $m \in \mathbb{N}$ a prime ideal $\mathfrak{a}_m := \langle x_1 + \iota(m) \rangle$. If for all m, $f \in \mathfrak{a}_m$ then by the division algorithm f must be the 0 polynomial, a contradiction. (Similarly for g.) Hence there is some ideal $\mathfrak{a}_m$ containing neither f nor g. $\square$

4.6 Krull Dimension

Lemma – Varieties and prime ideals

Let K be an algebraically closed field and $X \subseteq K^n$ a Zariski closed set. Then X is a variety (i.e. X is irreducible) if and only if $I(X)$ is prime.

Proof. ($\Rightarrow$) Suppose X is irreducible. If $f, g \notin I(X)$ then there exist $a, b \in X$ such that $f(a) \neq 0$ and $g(b) \neq 0$. Then $X \cap \mathbb{V}(f)$ and $X \cap \mathbb{V}(g)$ are proper subsets of X and are Zariski closed. Hence

$$X \cap (\mathbb{V}(fg)) = X \cap (\mathbb{V}(f) \cup \mathbb{V}(g)) \subset X$$

This is a proper subset since X is irreducible. Thus there exists $c \in X$ such that $fg(a) \neq 0$ and so $fg \notin I(X)$.

($\Leftarrow$) Suppose X is reducible, i.e. there are $U, V \subset X$ Zariski closed proper subsets such that $U \cup V = X$. By the Galois correspondence between ideals and vanishings we have $I(X) \subset I(U), I(V)$. Take $f \in I(U) \setminus I(X), g \in I(V) \setminus I(X)$ and note that for any $a \in X$, $a \in U$ or $a \in V$ so $f(a) = 0$ or $g(a) = 0$, hence $fg(a) = 0$ and $fg \in I(X)$. Hence $I(X)$ is not prime. □

Definition – Krull dimension

Let K be an algebraically closed field and $V \subseteq K^n$ be a variety. Let $S \subseteq \mathbb{N}$ be the set of naturals m such that there exists a chain of closed irreducible subspaces ending with V with length m:

$$V_0 \subset V_1 \subset \cdots \subset V_m = V$$

By the Galois correspondence between ideals and vanishings this is the same as existance of a chain of prime ideals starting with the prime ideal $I(V)$

$$I(V) = \mathfrak{p}_m \subset \cdots \subset \mathfrak{p}_1 \subset \mathfrak{p}_0 \subset K[x_1, \ldots, x_n]$$

The set S is bounded since $K[x_1, \ldots, x_n]$ is Noetherian. Thus we can define the Krull dimension[†] of V to be the maximum element of S. We denote this as $\mathrm{k.dim}(V)$.

[†]We could define Krull dimension simply for any topological space by not requiring the last closed irreducible subspace to be equial to the whole space. However, this may not just be a natural number.

Remark. V has Krull dimension 0 if and only if $I(V)$ is a maximal ideal if and only if V is a singleton: the first equivalence is clear. If V is a singleton then trivially it has no irreducible subsets (as irreducible requires non-empty) hence has dimension 0.

If V has dimension 0 firstly note that V is non-empty by definition of irreducible. Then assume for a contradiction there are two distinct points a, b in V. The singleton $\{a\}$ is closed as it is the vanishing of the polynomials $x_1 - a_1 \ldots, x_n - a_n$. Then $\{a\}$ is a closed and irreducible subset of V, so the Krull dimension of V is greater than or equal to 1, a contradiction.

Definition – Function field

Let K be an algebraicaly closed field and $V \subseteq K^n$ a variety. Then we write $K(V)$ to mean the field of fractions of

$$K[x_1, \ldots, x_n]/I(V)$$

noting that this is well-defined since $I(V)$ is a prime ideal and so the quotient is an integral domain. We call $K(V)$ the function field of V.

Proposition – Equivalent definition of Krull dimension
Let K be an algebraically closed field and $V \subseteq K^n$ be a variety. Then the Krull dimension of V is equal to the transcendence degree of the function field $K(V)$ over K.

Proof. See [1]. □

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