Model Theory

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Contents

Chapter 1

Pure Model Theory

1.1 Basics

These first two sections follow Marker's book on Model Theory [1] with more emphasis on where things are happening, i.e. what signature we are working in, and some more general statements such as working with embeddings rather than subsets.

1.1.1 Signatures

Definition – First order language
We assume we have a tuple $\mathcal{L} = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathcal{V}, \{\neg, \vee, \forall, =, \top\})$ such that

- $|\mathcal{C}|, |\mathcal{F}|, |\mathcal{R}|$ each sufficiently large (say they have cardinality $\aleph_5$ or something).
- $|\mathcal{V}| = \aleph_0$. We index $\mathcal{V} = \{v_0, v_1, \dots\}$ using $\mathbb{N}$.
- $\mathcal{C}$,$\mathcal{F}$,$\mathcal{R}$, $\mathcal{V}$, $\{\neg, \vee, \forall, =\}$ do not overlap.

We call $\mathcal{L}$ the language and only really use it to get symbols to work with. Whenever we introduce new symbols to create larger signatures, we are pulling them out of this box.

Definition – Signature
In a language $\mathcal{L}$, a tuple $\Sigma = (C, F, n_\star, R, m_\star)$ is a signature[†] when

- $C \subseteq \mathcal{C}$. We call C the set of constant symbols.
- $F \subseteq \mathcal{F}$ and $n_\star : F \to \mathbb{N}$, which we call the function arity. We call F the set of function symbols.
- $R \subseteq \mathcal{R}$ and $m_\star : R \to \mathbb{N}$, which we call the relation arity. We call T the set of relation symbols.

Given a signature Σ, we may refer to its constant, function and relation symbol sets as $\Sigma_{\text{con}}, \Sigma_{\text{fun}}, \Sigma_{\text{rel}}$. We will always denote function arity using $n_\star$ and relation arity using $m_\star$.

[†]Many authors call Σ the language, but I have chosen this way of defining things instead.

Example. *The signature of rings will be used to define the theory of rings, the theory of integral domains, the theory of fields, and so on. The signature of binary relations will be used to define the theory of partial orders with the interpretation of the relation as $<$, to define the theory of equivalence relations with the interpretation of the relation as , and to define the theory* ZFC *with the relation interpreted as $\in$.*

Definition – Σ-terms

Given Σ a signature, its set of terms Σ_{ter} is inductively defined using three constructors:

| If $c \in \Sigma_{\text{con}}$ then c is a Σ-term.

| If $v_i \in \mathcal{V}$, v_i is a Σ-term.

| Given $f \in \Sigma_{\text{fun}}$ and a n_f-tuple of Σ-terms $t \in (\Sigma_{\text{ter}})^{n_f}$ then the symbol $f(t)$ is a Σ-term.

The terms v_i are called variables and will be referred to as elements of Σ_{var}.

Example. *In the signature of rings, terms will be multivariable polynomials over $\mathbb{Z}$ since they are sums and products of constant symbols $0, 1$ and variable symbols. In the signature of binary relations there are no constant or function symbols so the only terms are variables.*

Definition – Atomic formula, quantifier free formula

Given Σ a signature, its set of atomic Σ-formulas is defined as

| $\top$ is an atomic Σ-formula

| Given $t, s \in \Sigma_{\text{ter}}$, the string $(t = s)$ is an atomic Σ-formula.

| Given $r \in \Sigma_{\text{rel}}, t \in (\Sigma_{\text{ter}})^{m_r}$, the string $r(t)$ is an atomic Σ-formula.

This will be the Σ-formulas that are not built 'inductively'.

The set of quantifier free Σ-formulas is defined inductively:

| Given ϕ an atomic Σ-formula, ϕ is a quantifier free Σ-formula.

| Given ϕ a quantifier free Σ-formula, the string $\neg\phi$ is a quantifier free Σ-formula.

| Given ϕ, ψ both quantifier free Σ-formulas, the string $\phi \vee \psi$ is a quantifier free Σ-formula.

Notice quantifier free Σ-formulas are indeed Σ-formulas

Definition – Σ-formula, free variable

Given Σ a signature, its set of Σ-formulas Σ_{for} is inductively defined:

| Given ϕ a quantifier free Σ-formula, ϕ is a Σ-formula.

| Given $\phi \in \Sigma_{\text{for}}$ and $v_i \in \Sigma_{\text{var}}$, we take the replace all occurrences of v_i with an unused symbol such as z in the string $(\forall v_i, \phi)$ and call this a Σ-formula.

Shorthand for some Σ-formulas include

- $\bot := \neg\top$
- $\phi \wedge \psi := \neg((\neg\phi) \vee (\neg\psi))$
- $\phi \to \psi := (\neg\phi) \vee \psi$
- $\exists v, \phi := \neg(\forall v, \neg\phi)$

The symbol z is meant to be a 'bounded variable', and will not be considered when we want to evaluate variables in formulas. Not bound variables are called 'free variables'.

Remark. There are two different uses of the symbol '$=$' from now on, and context will allow us to tell them apart. Similarly, logical symbols might be used in our 'higher language' and will not be confused with symbols from formulas.

Formulas should be thought of as propositions with some bits loose, namely the free variables, since it does not make any sense to ask if $x = x \vee x = a$ without saying what x you are taking (where x is a variable as a is a constant symbol, say). When there are no free variables we get what intuitively looks like a proposition, and we will call these particular formulas sentences.

For the sake of learning the theory the following two lemmas should be skipped. They are essentially algorithms that tell us what the variables in terms and what the free variables in formulas are. We include them for formality.

Proposition – Terms have finitely many variables
For any term $t \in \Sigma_{\text{ter}}$ there exists a finite subset $S \subseteq \mathbb{N}$ indexing the variables v_i occuring in t.

Proof. If there exists a finite subset T of $\mathbb{N}$ such that if v_i occurs in t then $i \in T$ then we can take the intersection of all such sets and have the finite set S we're interested in.

We prove existence of such a T using the inductive definition of t:

- If $t = c$ a constant symbol, then $T = \varnothing$ satisfies the above.
- If $t = v_i$ a variable symbol, then $T = \{i\}$ satisfies the above.
- If $t = f(t_0, \ldots, t_{n_f})$, then by our induction hypothesis we have a T_i satisfying the condition for each t_i. Then $\cup_i T_i$ satisfies the condition for t.

□

Proposition – Formulas have finitely many *free* variables
Given $\phi \in \Sigma_{\text{for}}$, there exists a finite $S \subseteq \mathbb{N}$ indexing the *free* variables v_i occuring in ϕ.

Proof. Like in the terms case, we only need to show that there exists a T If v_i occurs freely in ϕ then $i \in T$. We induct on what ϕ is, noting that until the last case there are no quantifiers being considered so the variables in question are free:

- If ϕ is $\top$ then it has no variables.
- If ϕ is $t = s$, then we have S_t, S_s indexing the (free) variables of t and s by the previous proposition, and so we can pick $T = S_t \cup S_s$.
- If ϕ is $r(t_0, \ldots, t_{m_r})$, then for each t_i we have S_i indexing the variables of t_i. Hence we can pick $T = \cup_i S_i$.
- If ϕ is $\neg\,\psi$, then by the induction hypothesis we have T satisfying the above conditions for ψ. Pick this T for ϕ.
- If ϕ is $\psi \vee \chi$ then by the induction hypothesis we have T_ψ, T_χ satisfying the above conditions for ψ and χ. We take T to be the union of indexing sets for ψ and χ.
- If ϕ is $\forall v_i, \psi$ with v_i substituted for z, then by the induction hypothesis we have T_ψ satisfying the above conditions for ψ. Take $T = T_\psi \setminus \{v_i\}$. In fact taking T_ψ itself works as well.

□

Notation (Substituting Terms for Variables). *If a Σ-formula ϕ has a free variable v_i then to remind ourselves of the variable we can write $\phi = \phi(v_i)$ instead.*

If ϕ has S indexing its free variables and $t \in (\Sigma_{\text{ter}})^S$, then we write $\phi(t)$ to mean ϕ with t_i substituted for each v_i. We can show by induction on terms and formulas that this is still a Σ-formula.

Definition – Σ-structure, interpretation

Given a signature Σ, a set M and interpretation functions

- $\star^{\mathcal{M}}_{\Sigma_{\text{con}}} : \Sigma_{\text{con}} \to M$
- $\star^{\mathcal{M}}_{\Sigma_{\text{fun}}} : \Sigma_{\text{fun}} \to (M^{n_\star} \to M)$
- $\star^{\mathcal{M}}_{\Sigma_{\text{rel}}} : \Sigma_{\text{rel}} \to \mathcal{P}(M^{m_\star})$

we say that $\mathcal{M} := (M, \star^{\mathcal{M}}_{\Sigma})$ is a Σ-structure. The latter functions two are dependant types since the powers $n_\star, m_\star$ depend on the function and relation symbols given. The class (or set or whatever) of Σ-structures is denoted Σ_{str}. Given only the Σ-structure $\mathcal{M}$, we call its underlying carrier set M as $\mathcal{M}_{\text{car}}$.

We write $\star^{\mathcal{M}}_{\Sigma}$ to represent any of the three interpretation functions when the context is clear. Given $c \in \Sigma_{\text{con}}, f \in \Sigma_{\text{fun}}, r \in \Sigma_{\text{rel}}$, we might write the 'interpretations' of these symbols as any of the following

$$c^{\mathcal{M}}_{\Sigma_{\text{con}}} = c^{\mathcal{M}}_{\Sigma} = c^{\mathcal{M}} \qquad f^{\mathcal{M}}_{\Sigma_{\text{fun}}} = f^{\mathcal{M}}_{\Sigma} = f^{\mathcal{M}} \qquad r^{\mathcal{M}}_{\Sigma_{\text{rel}}} = r^{\mathcal{M}}_{\Sigma} = r^{\mathcal{M}}$$

The structures in a signature will become the models which we are interested in, in particular structures will be a models of theories. For example $\mathbb{Z}$ is a structure in the signature of rings, and models the theory of rings but not the theory of fields. In the signature of binary relations, $\mathbb{N}$ with the usual ordering $\leq$ is a structure that models of the theory of partial orders but not the theory of equivalence relations.

Definition – Interpretation of terms

Given a signature Σ, a Σ-structure $\mathcal{M}$ and a Σ-term t, let S be the unique set indexing the variables of t. Then there exists a unique induced map $t^{\mathcal{M}}_{T} : \mathcal{M}^{S}_{\text{car}} \to \mathcal{M}_{\text{car}}$, that commutes with the interpretation of constants and functions[†]. We then refer to this map as *the* interpretation of the term t. This in turn defines a dependant Π-type

$$\star^{\mathcal{M}}_{T} : \Sigma_{\text{ter}} \to (\mathcal{M}^{S_\star}_{\text{car}} \to \mathcal{M}_{\text{car}})$$

[†] See this more precisely stated in the proof

Proof. To define a map $t^{\mathcal{M}}_{T} : M^S \to M$ for each t we use the inductive definition of $t \in \Sigma_{\text{ter}}$. If M is empty we define $t^{\mathcal{M}}_{T}$ as the empty function. Otherwise let $a \in M^S$:

- If $t = c \in \Sigma_{\text{con}}$ then define $t^{\mathcal{M}}_{T} : a \mapsto c^{\mathcal{M}}_{C}$, the constant map. This type checks since $S = \varnothing$ therefore $t^{\mathcal{M}}_{T} : M^0 \to M$.
- If $t = v_i \in \Sigma_{\text{var}}$ then define $t^{\mathcal{M}}_{T} : a \mapsto a$, the identity. This type checks since $|S| = 1$.
- If $t = f(s)$ for some $f \in \Sigma_{\text{fun}}$ and $s \in (\Sigma_{\text{var}})^{n_f}$ then define $t^{\mathcal{M}}_{T} : a \mapsto f^{\mathcal{M}}_{F}(s^{\mathcal{M}}_{T}(a))$. This type checks since s has the same number of variables as t.

By definition, this map commutes with the interpretation of constants and functions, i.e.

$$\begin{tikzcd}
\Sigma_{\text{con}} \arrow[r, "\subseteq"] \arrow[dr, "\star^{\mathcal{M}}_{\Sigma_{\text{con}}}"'] & \Sigma_{\text{ter}} \arrow[d, "\star^{\mathcal{M}}_{T}"] \\
& (\mathcal{M}^{S_\star} \to \mathcal{M})
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod_{f \in \Sigma_{\text{fun}}} (\Sigma_{\text{ter}})^{n_f} \arrow[rrr, bend left] \arrow[drrr, bend right] & s \in (\Sigma_{\text{ter}})^{n_f} \arrow[r, mapsto] \arrow[d, mapsto] & f(s) & \Sigma_{\text{ter}} \arrow[d, "\star^{\mathcal{M}}_{T}"] \\
& f^{\mathcal{M}}_{\Sigma_{\text{fun}}}(s^{\mathcal{M}}_{T}(\star)) & & (\mathcal{M}^{S_\star} \to \mathcal{M})
\end{tikzcd}$$

The map is clearly unique. $\square$

Where there is no ambiguity, we write $t_T^{\mathcal{M}} = t^{\mathcal{M}}$. Furthermore, if we have a tuple $t \in (\Sigma_{\text{ter}})^k$, then we write $t_T^{\mathcal{M}} := (t_0^{\mathcal{M}}, \cdots, t_k^{\mathcal{M}})$

Definition – Sentences and satisfaction

Let Σ be a signature and ϕ a Σ-formula. Let $S \subseteq \mathbb{N}$ index the free variables of ϕ. We say $\phi \in \Sigma_{\text{for}}$ is a Σ-sentence when S is empty.

If Σ has no constant symbols then $\varnothing$ is a Σ-structure by interpreting functions and relations as $\varnothing$. Then given a Σ-sentence (not any Σ-formula) ϕ we want to define $\varnothing \vDash_\Sigma \phi$ using the inductive definition of ϕ:

- If ϕ is $\top$ then $\varnothing \vDash_\Sigma \phi$.
- ϕ cannot be $t = s$ since this contains a constant symbol or a variable.

- If ϕ is $\neg\psi$ for some $\psi \in \Sigma_{\text{for}}$, then $\varnothing \vDash_\Sigma \phi$ when $\varnothing \nvDash_\Sigma \psi$
- If ϕ is $(\psi \vee \chi)$, then $\varnothing \vDash_\Sigma \phi$ when $\varnothing \vDash_\Sigma \psi$ or $\varnothing \vDash_\Sigma \chi$.
- If ϕ is of the form $\forall v, \psi$, then $\varnothing \vDash_\Sigma \phi$.

Let $\mathcal{M}$ be a Σ-structure with non-empty carrier set. Then given $a \in (\mathcal{M}_{\text{car}})^S$, we want to define $\mathcal{M} \vDash_\Sigma \phi(a)$:

- If ϕ is $\top$ then $\mathcal{M} \vDash_\Sigma \phi$.[†]
- If ϕ is $t = s$ then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)$.
- If ϕ is $r(t)$, where $r \in \Sigma_{\text{rel}}$ and $t \in (\Sigma_{\text{ter}})^{m_r}$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $t^{\mathcal{M}}(a) \in r^{\mathcal{M}}$.

- If ϕ is $\neg\psi$ for some $\psi \in \Sigma_{\text{for}}$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $\mathcal{M} \nvDash_\Sigma \psi(a)$
- If ϕ is $(\psi \vee \chi)$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $\mathcal{M} \vDash_\Sigma \psi(a)$ or $\mathcal{M} \vDash_\Sigma \chi(a)$.
- If ϕ is $(\forall v, \psi(a)) \in \Sigma_{\text{for}}$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ if for any $b \in \mathcal{M}_{\text{car}}$, $\mathcal{M} \vDash_\Sigma \psi(a)(b)$.

We say $\mathcal{M}$ satisfies $\phi(a)$.

[†]We can omit the a when there are no free variables. Formally this a is the unique element in $\mathcal{M}^{\varnothing}$ given by the empty set.

Remark. Any Σ-structure $\mathcal{M}$ satisfies $\top$ and does not satisfy $\bot$. The empty set satisfies things of the form $\forall, \ldots$ but not $\exists, \ldots$, as we would expect. Note that for c a tuple of constant symbols $\mathcal{M} \vDash_\Sigma \phi(c)$ is the same thing as $\mathcal{M} \vDash_\Sigma \phi(c^{\mathcal{M}})$.

1.1.2 Theories and Models

Definition – Σ-theory

T is an Σ-theory when it is a subset of Σ_{for} such that all elements of T are Σ-sentences. We denote the set of Σ-theories as Σ_{the}.

Definition – Models

Given an Σ-structure $\mathcal{M}$ and Σ-theory T, we write $\mathcal{M} \vDash_\Sigma T$ and say $\mathcal{M}$ *is a Σ-model of T* when for all $\phi \in T$ we have $\mathcal{M} \vDash_\Sigma \phi$.

EXAMPLE (THE EMPTY SIGNATURE AND THEORY). $\Sigma_\varnothing = (\varnothing, \varnothing, n_\star, \varnothing, m_\star)$ *is the empty signature. (We pick the empty functions for $n_\star, m_\star$.) The empty $\Sigma_\varnothing$-theory is given by $\varnothing$. Notice that any set is a $\Sigma_\varnothing$-structure and* *moreover a $\Sigma_\varnothing$-model of the empty $\Sigma_\varnothing$-theory.*

Example. *In the signature of rings, the rings axioms will be the theory of rings and structures satisfying the theory will be rings. The theory of ZFC consists of the ZFC axioms and a model of ZFC would be thought of as the 'class of all sets'.*

Definition – Consequence

Given a Σ-theory T and a Σ-sentence ϕ, we say ϕ is a consequence of T and say $T \vDash_\Sigma \phi$ when for all Σ-models $\mathcal{M}$ of T, we have $\mathcal{M} \vDash_\Sigma \phi$.

Remark. We have to be a bit careful when we go from something like $\mathcal{M} \vDash_\Sigma \phi(a)$ to deducing something about T. This is because there might not exist a Σ-constant c such that $c^\mathcal{M} = a$, it only makes sense to write $T \vDash_\Sigma \phi$ if ϕ is a *sentence*.

We have included the empty structure in this definition, often this will not make a difference since the signature could have constant symbols or the theory T could have an 'existential' sentence in it.

Exercise (Logical consequence). *Let T be a Σ-theory and ϕ and ψ be Σ-sentences. Show that the following are equivalent:*

- $T \vDash_\Sigma \phi \to \psi$
- $T \vDash_\Sigma \phi$ *implies* $T \vDash_\Sigma \psi$.

Definition – Consistent theory

A Σ-theory T is consistent if either of the following equivalent definitions hold:

- There does not exists a Σ-sentence ϕ such that $T \vDash_\Sigma \phi$ and $T \vDash_\Sigma \neg\phi$.
- There exists a Σ-model of T.

Thus the definition of consistent is intuitively 'T does not lead to a contradiction'.

A theory T is finitely consistent if all finite subsets of T are consistent. This will turn out to be another equivalent definition, given by the compactness theorem.

Proof. We show that the two definitions are equivalent. ($\Rightarrow$) Suppose no model exists. Take ϕ to be the Σ-sentence $\top$. Hence all Σ-models of T satisfy $\top$ and $\bot$ (there are none) so $T \vDash_\Sigma \top$ and $T \vDash_\Sigma \bot$. ($\Leftarrow$) Suppose T has a Σ-model $\mathcal{M}$ and $\mathcal{M} \vDash_\Sigma \phi$ and $\mathcal{M} \vDash_\Sigma \neg\phi$. This implies $\mathcal{M} \vDash_\Sigma \phi$ and $\mathcal{M} \nvDash_\Sigma \phi$, a contradiction. $\square$

1.1.3 The Compactness Theorem

Read ahead for the statement of the Compactness Theorem. The first two parts of the theorem are easy to prove. This chapter focuses on proving the final part.

Definition – Witness property

Given a signature Σ and a Σ-theory T, we say that Σ satisfies the witness property when for any Σ-formula ϕ with exactly one free variable v, there exists $c \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma (\exists v, \phi(v)) \to \phi(c)$.

This says that if for all Σ-model $\mathcal{M}$ of T, there exists an element $a \in \mathcal{M}$ such that $\mathcal{M} \vDash_\Sigma \phi(m)$ then there exists a constant symbol c of Σ such that $\phi(c^\mathcal{M})$ is true.

Definition – Maximal theory
A Σ theory T is Σ-maximal if for any Σ-formula ϕ, if ϕ is a Σ-sentences then $\phi \in T$ or $\neg\phi \in T$.

Proposition – Maximum property
Given a Σ-maximal and finitely consistent theory T and a Σ-sentence ϕ,

$$T \vDash_\Sigma \phi \text{ if and only if } \phi \in T \text{ if and only if } \neg\phi \notin T \text{ if and only if } \nvDash_\Sigma \neg\phi$$

Proof. First note that by maximality and finite consistency if $\phi, \neg\phi \in T$ then we have a finite subset $\{\phi, \neg\phi\} \subseteq T$, which is false. Hence

$$\phi \in T \Leftrightarrow \neg\phi \notin T$$

We prove the first if and only if and deduce the third by replacing ϕ with $\neg\phi$. $(\Rightarrow)$ Suppose $T \vDash_\Sigma \phi$. Since T is Σ-maximal, we have $\phi \in T$ or $\neg\phi \in T$. If $\neg\phi \in T$ then we have a finite subset $\{\phi, \neg\phi\} \subseteq T$. Hence T is not finitely consistent, thus the second case is false. $(\Leftarrow)$ Suppose $\phi \in T$. Case on $T \vDash_\Sigma \phi$ or $T \nvDash_\Sigma \phi$. If $T \nvDash_\Sigma \phi$ then there exists $\mathcal{N}$ a Σ-model of T such that $N \nvDash_\Sigma \phi$. But $\mathcal{N} \vDash_\Sigma \phi$ since $\phi \in T$. Thus the second case is false. $\square$

Notation (Ordering signatures). *We write $\Sigma \leq \Sigma(*)$ for two signatures if $\Sigma_{\text{con}} \subseteq \Sigma(*)_{\text{con}}$, $\Sigma_{\text{fun}} \subseteq \Sigma(*)_{\text{fun}}$ and $\Sigma_{\text{rel}} \subseteq \Sigma(*)_{\text{rel}}$.*

For the sake of formality we include the following two lemmas, neither of which are particularly inspiring or significant, but they do allow us to move freely between signatures.

Lemma – Moving models down signatures
Given two signatures such that $\Sigma \leq \Sigma(*)$ and $\mathcal{N}$ a $\Sigma(*)$-structure we can make $\mathcal{M}$ a Σ-structure such that

1. $\mathcal{M}_{\text{car}} = \mathcal{N}_{\text{car}}$
2. They have the same interpretation on Σ.
3. For any Σ-formula ϕ with free variables indexed by S and any $a \in \mathcal{M}^S$

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$

4. If T is a Σ-theory and $T(*)$ is a $\Sigma(*)$-theory such that $T \subseteq T(*)$ and $\mathcal{N}$ a $\Sigma(*)$-model of $T(*)$, then $\mathcal{M}$ is a Σ-model of T.

Technically the new structure is not $\mathcal{N}$, but for convenience we will write $\mathcal{N}$ to mean either of the two and let subscripts involving Σ and $\Sigma(*)$ describe which one we mean.

Proof. We let the carrier set be the same and define $\star_\Sigma^{\mathcal{M}}$ by restriction:

- $\star_{\Sigma_{\text{con}}}^{\mathcal{M}}$ is the restriction of $\star_{\Sigma(*)_{\text{con}}}^{\mathcal{N}}$ to Σ_{con}
- $\star_{\Sigma_{\text{fun}}}^{\mathcal{M}}$ is the restriction of $\star_{\Sigma(*)_{\text{fun}}}^{\mathcal{N}}$ to Σ_{fun}
- $\star_{\Sigma_{\text{rel}}}^{\mathcal{M}}$ is the restriction of $\star_{\Sigma(*)_{\text{rel}}}^{\mathcal{N}}$ to Σ_{rel}

We will need that for any Σ-term t with variables indexed by S, the interpretation of terms is equal: $t_\Sigma^{\mathcal{M}} = t_{\Sigma(*)}^{\mathcal{N}}$. Indeed:

- If t is a constant then $t_\Sigma^{\mathcal{M}} = c_\Sigma^{\mathcal{M}} = c_{\Sigma(*)}^{\mathcal{N}} = t_{\Sigma(*)}^{\mathcal{N}}$

- If t is a variable then $t_{\Sigma}^{\mathcal{M}} = \mathrm{id}_{\mathcal{M}} = \mathrm{id}_{\mathcal{N}} = t_{\Sigma(*)}^{\mathcal{N}}$
- If t is $f(s)$ then by induction $t_{\Sigma}^{\mathcal{M}} = f_{\Sigma}^{\mathcal{M}}(s_{\Sigma}^{\mathcal{M}}) = f_{\Sigma(*)}^{\mathcal{N}}(s_{\Sigma(*)}^{\mathcal{N}}) = t_{\Sigma(*)}^{\mathcal{N}}$

Let ϕ be a Σ-formula with variables indexed by $S \subseteq \mathbb{N}$. Let a be in $\mathcal{M}^S$. Case on ϕ to show that $\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$:

- If ϕ is $\top$ then both satisfy ϕ.
- If ϕ is $t = s$ then
$$\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow t_{\Sigma}^{\mathcal{M}} = s_{\Sigma}^{\mathcal{M}} \Leftrightarrow t_{\Sigma(*)}^{\mathcal{M}} = s_{\Sigma(*)}^{\mathcal{M}} \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$
Since the interpretation of terms are equal from above.
- If ϕ is $r(t)$ then by how we defined $r_{\Sigma}^{\mathcal{M}}$ and since interpretation of terms are equal
$$\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow t_{\Sigma}^{\mathcal{M}}(a) \in r_{\Sigma}^{\mathcal{M}} \Leftrightarrow t_{\Sigma(*)}^{\mathcal{N}}(a) \in r_{\Sigma(*)}^{\mathcal{N}} \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$
- If ϕ is $\neg\,\psi$ then using the induction hypothesis
$$\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow \mathcal{M} \nvDash_{\Sigma} \psi(a) \Leftrightarrow \mathcal{N} \nvDash_{\Sigma(*)} \psi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$
- If ϕ is $\psi \vee \chi$ then using the induction hypothesis
$$\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow \mathcal{M} \vDash_{\Sigma} \psi(a) \text{ or } \mathcal{M} \vDash_{\Sigma} \chi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \psi(a) \text{ or } \mathcal{N} \vDash_{\Sigma(*)} \chi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$
- If ϕ is $\forall v, \psi$ then
$$\begin{aligned} \mathcal{M} \vDash_{\Sigma} \phi(a) &\Leftrightarrow \forall b \in \mathcal{M}, \mathcal{M} \vDash_{\Sigma} \psi(a, b) \\ &\Leftrightarrow \forall b \in \mathcal{M}, \mathcal{N} \vDash_{\Sigma(*)} \psi(a, b) && \text{by the induction hypothesis} \\ &\Leftrightarrow \forall b \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma(*)} \psi(a, b) && \text{by the induction hypothesis} \\ &\Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a) \end{aligned}$$

Hence $\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$.

Suppose $T \subseteq T(*)$ are respectively Σ and $\Sigma(*)$-theories and $\mathcal{N} \vDash_{\Sigma(*)} T(*)$. If $\phi \in T \subseteq T(*)$ then by the previous part, $\mathcal{N} \vDash_{\Sigma(*)} \phi$ implies $\mathcal{M} \vDash_{\Sigma} \phi$. Hence $\mathcal{M} \vDash_{\Sigma} T$. $\square$

Lemma – Moving models and theories up signatures
Suppose $\Sigma \leq \Sigma(*)$.

1. Suppose $\mathcal{M}$ is a Σ-model of Σ-theory T. Then if there exists $\mathcal{M}(*)$ a $\Sigma(*)$-structure whose carrier set is the same as $\mathcal{M}$ and whose interpretation agrees with $\star_{\Sigma}^{\mathcal{M}}$ on constants, functions and relations of Σ, then $\mathcal{M}(*)$ is a $\Sigma(*)$-model of T.
2. Suppose T is a Σ-theory and ϕ is a Σ-sentence such that $T \vDash_{\Sigma} \phi$. Then $T \vDash_{\Sigma(*)} \phi$.

Again, if we have constructed such a $\mathcal{M}(*)$ from $\mathcal{M}$ we tend to just refer to it as $\mathcal{M}$ and let subscripts involving Σ and $\Sigma(*)$ describe which one we mean.

Proof.

1. Suppose $\mathcal{M} \vDash_{\Sigma} T$. Let $\phi \in T$. To show that $\mathcal{M}(*) \vDash_{\Sigma(*)} \phi$ we first we prove a useful claim: if $t \in \Sigma_{\mathrm{ter}}$ with no variables then $t_{\Sigma(*)}^{\mathcal{M}(*)} = t_{\Sigma}^{\mathcal{M}}$. Case on what t is:
   - If t is a constant symbol c in Σ_{con}, then since $\star_{\Sigma(*)}^{\mathcal{M}(*)} = \star_{\Sigma}^{\mathcal{M}}$ on Σ,
   $$t_{\Sigma(*)}^{\mathcal{M}(*)} = c_{\Sigma(*)}^{\mathcal{M}(*)} = c_{\Sigma}^{\mathcal{M}} = t_{\Sigma}^{\mathcal{M}}$$

- If t is a variable then it has one variable, thus false.
- If t is $f(s)$ then since $\star_{\Sigma(*)}^{\mathcal{M}(*)} = \star_{\Sigma}^{\mathcal{M}}$ on Σ,

$$t_{\Sigma(*)}^{\mathcal{M}(*)} = f(s)_{\Sigma(*)}^{\mathcal{M}(*)} = f_{\Sigma(*)}^{\mathcal{M}(*)}(s_{\Sigma(*)}^{\mathcal{M}(*)}) = f_{\Sigma}^{\mathcal{M}}(s_{\Sigma}^{\mathcal{M}}) = f(s)_{\Sigma}^{\mathcal{M}} = t_{\Sigma}^{\mathcal{M}}$$

Case on what ϕ is (ϕ has no variables):

- If ϕ is $\top$ then it is satisfied.
- If ϕ is $t = s$, then by the claim above,

$$t_{\Sigma(*)}^{\mathcal{M}(*)} = t_{\Sigma}^{\mathcal{M}} = s_{\Sigma}^{\mathcal{M}} = s_{\Sigma(*)}^{\mathcal{M}(*)}$$

- If ϕ is $r(t)$, then by the claim above and the fact that relations are interpreted the same way,

$$t_{\Sigma(*)}^{\mathcal{M}(*)} = t_{\Sigma}^{\mathcal{M}} \in r_{\Sigma}^{\mathcal{M}} = r_{\Sigma(*)}^{\mathcal{M}(*)}$$

- If ϕ is $\neg\psi$ then using the induction hypothesis on ψ,

$$\mathcal{M} \vDash_{\Sigma} \phi \;\Leftrightarrow\; \mathcal{M} \nvDash_{\Sigma} \psi \;\Leftrightarrow\; \mathcal{M}(*) \nvDash_{\Sigma(*)} \psi \;\Leftrightarrow\; \mathcal{M}(*) \vDash_{\Sigma(*)} \phi$$

- If ϕ is $\psi \vee \chi$ then using the induction hypothesis on ψ and χ,

$$\mathcal{M} \vDash_{\Sigma} \phi \Leftrightarrow \mathcal{M} \vDash_{\Sigma} \psi \text{ or } \mathcal{M} \vDash_{\Sigma} \chi \Leftrightarrow \mathcal{M}(*) \vDash_{\Sigma(*)} \psi \text{ or } \mathcal{M}(*) \vDash_{\Sigma(*)} \chi \Leftrightarrow \mathcal{M}(*) \vDash_{\Sigma(*)} \phi$$

- If ϕ is $\forall v, \psi(v)$ and $a \in \mathcal{M}(*) = \mathcal{M}$ then using the induction hypothesis on ψ, $\mathcal{M} \vDash_{\Sigma} \psi(a) \Rightarrow \mathcal{M}(*) \vDash_{\Sigma(*)} \psi(a)$. Hence $\mathcal{M}(*) \vDash_{\Sigma(*)} \phi$.

Thus $\mathcal{M}(*)$ is a $\Sigma(*)$-model of T.

2. Suppose $T \vDash_{\Sigma} \phi$. If $\mathcal{M}(*) \vDash_{\Sigma(*)} T$ then by moving $\mathcal{M}(*)$ down to Σ, we have a corresponding $\mathcal{M} \vDash_{\Sigma} T$ whose carrier set is the same as $\mathcal{M}(*)$. Hence $\mathcal{M} \vDash_{\Sigma} \phi$. Naturally the models agree on interpretation of constants, functions and relations of Σ and so $\mathcal{M}(*) \vDash_{\Sigma(*)} \phi$ by the previous part.

$\square$

The following lemma is the bulk of the proof of the compactness theorem.

> **Lemma – Henkin construction**
> Let Σ be a signature. Let $0 < \kappa$ be a cardinal such that $|\Sigma_{\text{con}}| \leq \kappa$. If a Σ-theory T
>
> - has the witness property
> - is Σ-maximal
> - is finitely consistent
>
> then it has a non-empty Σ-model $\mathcal{M}$ such that $|\mathcal{M}| \leq \kappa$.

Proof. Without loss of generality Σ is non-empty, since we can add a constant symbol to Σ, see T as a theory in that signature, and then take the model we make of T back down to being a Σ-model of T. **The Σ-structure**: Consider quotienting Σ_{con} by the equivalence relation $c \sim d$ if and only if $T \vDash_{\Sigma} c = d$. Let $\pi : \Sigma_{\text{con}} \to \Sigma_{\text{con}}/\sim$. This defines a non-empty Σ-structure $\mathcal{M}$ in the following way:

1. We let the carrier set be the image of the quotient. We let the constant symbols be interpreted as their equivalence classes: $\star_{\Sigma_{\text{con}}}^{\mathcal{M}} = \pi$. We now have the desired cardinality for $\mathcal{M}$:[1] $|\mathcal{M}| \leq |\Sigma_{\text{con}}| \leq \kappa$ and Σ_{con} is non-empty so $\mathcal{M}$ is non-empty.

[1]From this point onwards when it is obvious what we mean we write $\mathcal{M}$ for $\mathcal{M}_{\text{car}}$.

2. To interpret functions we must use the witness property. Given $f \in \Sigma_{\text{fun}}$ and $\pi(c) \in \mathcal{M}^{n_f}$ (π is surjective), we obtain by the witness property some $d \in \Sigma_{\text{con}}$ such that
$$T \vDash_\Sigma (\exists v, f(c) = v) \to (f(c) = d)$$
We define $f^\mathcal{M}$ to map $\pi(c) \mapsto \pi(d)$.

To show that $f^\mathcal{M}$ is well-defined, let $c_0, c_1 \in (\Sigma_{\text{con}})^{n_f}$ such that $\pi(c_0) = \pi(c_1)$ and suppose $\pi(d_0), \pi(d_1)$ are their images. It suffices that $\pi(d_0) = \pi(d_1)$, i.e. $T \vDash_\Sigma d_0 = d_1$. Indeed, let Σ-structure $\mathcal{N}$ be a model of T. Then
$$\mathcal{N} \vDash_\Sigma (\exists v, f(c_0) = v) \Rightarrow \mathcal{N} \vDash_\Sigma f(c_0) = d_0$$
Similarly $f(c_1) = d_1$. Hence
$$d_0^\mathcal{N} = f^\mathcal{N}(c_0^\mathcal{N}) = f^\mathcal{N}(c_1^\mathcal{N}) = d_1^\mathcal{N}$$
and $N \vDash_\Sigma d_0 = d_1$. Hence $T \vDash_\Sigma d_0 = d_1$.

3. Let $r \in \Sigma_{\text{rel}}$. We define $r^\mathcal{M} := \{\pi(c) \,|\, c \in (\Sigma_{\text{con}})^{m_r} \wedge T \vDash_\Sigma r(c)\}$

Hence $\mathcal{M}$ is a Σ-structure. We want to show that $\mathcal{M}$ is a Σ-model of T.

Terms: to show that $\mathcal{M}$ satisfies formulas of T we first need that interpretation of terms is working correctly. Claim: if $t \in \Sigma_{\text{ter}}$ with variables indexed by S, $d \in \Sigma_{\text{con}}$ and c is in $(\Sigma_{\text{con}})^S$ then $T \vDash_\Sigma t(c) = d$ if and only if $t^\mathcal{M}(c^\mathcal{M}) = d^\mathcal{M}$. i.e. $\mathcal{M} \vDash_\Sigma t(c) = d$. Case on what t is:

- If t is a constant symbol, $T \vDash_\Sigma t(c) = d$ if and only if $T \vDash_\Sigma t = d$ if and only if $\pi(t) = \pi(d)$ if and only if $t^\mathcal{M} = t^\mathcal{M}(c^\mathcal{M}) = d^\mathcal{M}$.
- Suppose $t \in \Sigma_{\text{var}}$, then it suffices to show that $T \vDash_\Sigma c = d$ if and only if $c^\mathcal{M} = d^\mathcal{M}$, which we have already done above.
- With the induction hypothesis, suppose $t = f(s)$, where $f \in \Sigma_{\text{fun}}$ and $s \in (\Sigma_{\text{ter}})^{n_f}$. ($\Rightarrow$) If we can find $e = (e_1, \dots e_{n_f}) \in (\Sigma_{\text{con}})^{n_f}$ such that each $T \vDash_\Sigma s_i(c) = e_i$ and $f(e)^\mathcal{M} = d^\mathcal{M}$, then we have $s_i^\mathcal{M}(c^\mathcal{M}) = e_i^\mathcal{M}$ and so
$$t^\mathcal{M}(c^\mathcal{M}) = (f(s))^\mathcal{M}(c^\mathcal{M}) = f^\mathcal{M}(s^\mathcal{M}(c^\mathcal{M})) = f^\mathcal{M}(e^\mathcal{M}) = f(e)^\mathcal{M} = d^\mathcal{M}$$
Indeed, using the witness property n_f times we can construct e. Suppose we have by induction $e_1, \dots, e_{i-1} \in \Sigma_{\text{con}}$ such that for each $j < i$, they satisfy
$$T \vDash_\Sigma \exists x_{j+1}, \dots, \exists x_{n_f}, f(e_1, \dots, e_j, x_{j+1}, \dots, x_{n_f}) = d \wedge e_j = s_j(c)$$
For each i let ϕ_i be the formula
$$\exists x_{i+1}, \dots, \exists x_{n_f}, f(e_1, \dots, e_{i-1}, v, x_{i+1}, \dots, x_{n_f}) = d \wedge v = s_i(c)$$
with a single variable v. Then by the witness property, there exists an $e_i \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma \exists v, \phi_i(v) \to \phi_i(e_i)$. To complete the induction we show that $T \vDash_\Sigma \phi_i(e_i)$. Then we will be done since each $\phi_i(e_i)$ will give us $T \vDash_\Sigma s_i(c) = e_i$ and the last ϕ_{n_f} gives $f(e)^\mathcal{M} = d^\mathcal{M}$.

To this end, let $\mathcal{N}$ be a model of T. Then $\mathcal{N} \vDash_\Sigma \exists v, \phi_i(v) \to \phi_i(e_i)$. By assumption $T \vDash_\Sigma t(c) = d$ so $t^\mathcal{N}(c^\mathcal{N}) = d^\mathcal{N}$, hence
$$f^\mathcal{N}(s_1^\mathcal{N}(c), \dots, s_{n_f}^\mathcal{N}(c)) = d^\mathcal{N}$$
Taking v to be $s_i(c)$ and x_{i+k} to be $s_{i+k}(c)$ we have that $\mathcal{N} \vDash_\Sigma \exists v, \phi_i(v)$. Hence $\mathcal{N} \vDash_\Sigma \phi_i(e_i)$. Thus we have $T \vDash_\Sigma \phi_i(e_i)$.

($\Leftarrow$) Note that each $(s_i(c))^\mathcal{M} = e_i^\mathcal{M}$ for some $e_i \in \Sigma_{\text{con}}$ since π is surjective. By the induction hypothesis for each i we have $T \vDash_\Sigma s_i(c) = e_i$. Hence
$$f(e)^\mathcal{M} = f^\mathcal{M}(e^\mathcal{M}) = f^\mathcal{M}((s(c))^\mathcal{M}) = f^\mathcal{M}(s^\mathcal{M}(c^\mathcal{M})) = t^\mathcal{M}(c^\mathcal{M}) = d^\mathcal{M}$$
Hence $\pi(f(e)) = \pi(d)$ and $T \vDash_\Sigma f(e) = d$. It follows that $T \vDash_\Sigma t(c) = d$.

Thus $T \vDash_\Sigma t(c) = d \Leftrightarrow t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}}$.

Formulas: now we can show that $\mathcal{M} \vDash_\Sigma T$. Since for all $\phi \in T$ we have $T \vDash_\Sigma \phi$, it suffices to show that for all Σ-sentences $\phi, T \vDash_\Sigma \phi$ implies $\mathcal{M} \vDash_\Sigma \phi$. We prove a stronger statement which will be needed for the induction: for all Σ-formulas ϕ with variables indexed by S and $c \in (\Sigma_{\text{con}})^S$,

$$T \vDash_\Sigma \phi(c) \Leftrightarrow \mathcal{M} \vDash_\Sigma \phi(c^{\mathcal{M}})$$

We case on what ϕ is:

- Case ϕ is $\top$: all Σ-structures satisfy $\top$.
- Case ϕ is $t = s$: ($\Rightarrow$) Apply the witness property to $t(c) = v$, obtaining $d \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma (\exists v, t(c) = v) \to t(c) = d$. Since clearly $T \vDash_\Sigma \exists v, t(c) = v$ (take any model and it has an interpretation of $t(c)$) we have $T \vDash_\Sigma t(c) = d$. Also by assumption $T \vDash_\Sigma t(c) = s(c)$ so it follows that $T \vDash_\Sigma s(c) = d$. Using the claim from before for terms, we obtain $t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}} = s^{\mathcal{M}}(c^{\mathcal{M}})$. Hence $\mathcal{M} \vDash_\Sigma t(c^{\mathcal{M}}) = s(c^{\mathcal{M}})$.

  ($\Leftarrow$) If $\mathcal{M} \vDash_\Sigma t(c^{\mathcal{M}}) = s(c^{\mathcal{M}})$ then since π is surjective, there exists $d \in \Sigma_{\text{con}}$ such that

  $$t^{\mathcal{M}}(c^{\mathcal{M}}) = s^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}}$$

  Using the claim for terms we obtain $T \vDash_\Sigma t(c) = d$ and $T \vDash_\Sigma s(c) = d$. It follows that $T \vDash_\Sigma t(c) = s(c)$.
- Case ϕ is $r(t)$: ($\Rightarrow$) Suppose $T \vDash_\Sigma r(t(c))$. By induction, apply the witness property m_r times to the formulas

  $$\exists x_{i+1}, \dots, \exists x_{m_r}, r(\dots, e_{i-1}, v, x_{i+1}, \dots) \wedge v = t_i(c)$$

  each time obtaining $e_i \in \Sigma_{\text{con}}$ satisfying the formula. The result is $T \vDash_\Sigma r(e)$ and each $T \vDash_\Sigma t_i(c) = e_i$. Using the claim for terms and how we interpreted relations in $\mathcal{M}$ this implies $t^{\mathcal{M}}(c^{\mathcal{M}}) = e^{\mathcal{M}} \in r^{\mathcal{M}}$, and hence $\mathcal{M} \vDash_\Sigma r(t(c))$.

  ($\Leftarrow$) Suppose $\mathcal{M} \vDash_\Sigma r(t(c))$. Since π is surjective, there exists $e \in \Sigma_{\text{con}}$ such that $e^{\mathcal{M}} = t^{\mathcal{M}}(c^{\mathcal{M}}) \in r^{\mathcal{M}}$. Using the claim for terms again we obtain $T \vDash_\Sigma t(c) = e$ and using how $\mathcal{M}$ interprets relations, $T \vDash_\Sigma r(e)$. It follows that $T \vDash_\Sigma r(t(c))$.
- Case ϕ is $\neg \chi$: Using the maximal property of T for the first $\Leftrightarrow$ and the induction hypothesis for the second $\Leftrightarrow$ we have

  $$T \vDash_\Sigma \neg \chi(c) \;\Leftrightarrow\; T \nvDash_\Sigma \chi(c) \;\Leftrightarrow\; \mathcal{M} \nvDash_\Sigma \chi(c) \;\Leftrightarrow\; \mathcal{M} \vDash_\Sigma \neg \chi(c)$$
- Case ϕ is $\chi_0 \vee \chi_1$

  $$\begin{aligned} \mathcal{M} \vDash_\Sigma \chi_0(c^{\mathcal{M}}) \vee \chi_1(c^{\mathcal{M}}) &\Leftrightarrow \mathcal{M} \vDash_\Sigma \chi_0(c^{\mathcal{M}}) \text{ or } \mathcal{M} \vDash_\Sigma \chi_1(c^{\mathcal{M}}) \\ &\Leftrightarrow T \vDash_\Sigma \chi_0(c) \text{ or } T \vDash_\Sigma \chi_1(c) \quad \text{by the induction hypothesis} \end{aligned}$$

  Hence it suffices to show that

  $$T \vDash_\Sigma \chi_0(c) \text{ or } T \vDash_\Sigma \chi_1(c) \;\Leftrightarrow\; T \vDash_\Sigma \chi_0(c) \vee \chi_1(c)$$

  ($\Rightarrow$) Suppose $T \vDash_\Sigma \chi_0(c)$ or $T \vDash_\Sigma \chi_1(c)$. For $\mathcal{N}$ a Σ-model of T,

  $$\mathcal{N} \vDash_\Sigma \chi_0(c^{\mathcal{N}}) \text{ or } \mathcal{N} \vDash_\Sigma \chi_1(c^{\mathcal{N}}) \Rightarrow \mathcal{N} \vDash_\Sigma \chi_0(c^{\mathcal{N}}) \vee \chi_1(c^{\mathcal{N}})$$

  Thus $T \vDash_\Sigma \chi_0(c) \vee \chi_1(c)$.

  ($\Leftarrow$) Suppose $T \vDash_\Sigma \chi_0(c) \vee \chi_1(c)$. For $\mathcal{N}$ a Σ-model of T,

  $$\mathcal{N} \vDash_\Sigma \chi_0(c^{\mathcal{N}}) \vee \chi_1(c^{\mathcal{N}}) \Rightarrow \mathcal{N} \vDash_\Sigma \chi_0(c^{\mathcal{N}}) \text{ or } \mathcal{N} \vDash_\Sigma \chi_1(c^{\mathcal{N}})$$

  Thus $T \vDash_\Sigma \chi_0(c)$ or $T \vDash_\Sigma \chi_1(c)$.

- Case ϕ is $\forall v, \chi(v)$ ($\Rightarrow$) Let $d \in \mathcal{M}$, then since π surjective $\exists e \in \Sigma_{\text{con}}$ such that $\pi(e) = d$. Since $T \vDash_\Sigma \forall v, \chi(c, v)$ it follows that $T \vDash_\Sigma \chi(c, e)$ and by induction $\mathcal{M} \vDash_\Sigma \chi(c^{\mathcal{M}}, d)$. Hence $\mathcal{M} \vDash_\Sigma \forall v, \chi(c^{\mathcal{M}}, v)$.

  ($\Leftarrow$) We show the contrapositive. If $T \nvDash_\Sigma \forall v, \chi(c)(v)$, then by the maximal property of T, $T \vDash_\Sigma \exists v, \neg\chi(c, v)$. Applying the witness property to $\neg\chi(c, v)$, there exists $e \in \Sigma_{\text{con}}$ such that

  $$T \vDash_\Sigma (\exists v, \neg\chi(c)(v)) \to (\neg\chi(c)(v)) \quad \Rightarrow \quad T \vDash_\Sigma \neg\chi(c)(v)$$

  Thus $T \nvDash_\Sigma \chi(c)(v)$ by the maximal property of T and $\mathcal{M} \nvDash_\Sigma \chi(c^{\mathcal{M}})(v)$ by the induction hypothesis. Hence $\mathcal{M} \nvDash_\Sigma \forall v, \chi(c^{\mathcal{M}})(v)$.

Thus $T \vDash_\Sigma \phi \Leftrightarrow \mathcal{M} \vDash_\Sigma \phi$ and we are done. ☐

Proposition – Giving Theories the Witness Property

Suppose $\Sigma(0)$-theory $T(0)$ is finitely consistent. Then there exists a signature $\Sigma(*)$ and $\Sigma(*)$-theory $T(*)$ such that

1. $\Sigma(0)_{\text{con}} \subseteq \Sigma(*)_{\text{con}}$ and they share the same function and relation symbols.
2. $|\Sigma(*)_{\text{con}}| = |\Sigma(0)_{\text{con}}| + \aleph_0$
3. $T(0) \subseteq T(*)$
4. $T(*)$ is finitely consistent
5. Any $\Sigma(*)$-theory T' such that $T(*) \subseteq T'$ has the witness property

Proof. Again without loss of generality $\Sigma(0)$ is non-empty. We want to define $\Sigma(i), T(i)$, for each $i \in \mathbb{N}$. By induction, we assume we have $\Sigma(i)$ non-empty a signature and $T(i)$ a Σ-theory such that

1. $\Sigma(0)_{\text{con}} \subseteq \Sigma(i)_{\text{con}}$ and they share the same function and relation symbols.
2. $|\Sigma(i)_{\text{con}}| = |\Sigma(0)_{\text{con}}| + \aleph_0$
3. $T(0) \subseteq T(i)$
4. $T(i)$ is finitely consistent

Let

$$W(i) := \{\phi \in \Sigma(i)_{\text{for}} \mid \phi \text{ has exactly one free variable}\}$$

We construct $\Sigma(i+1)$ by adding constant symbols c_ϕ for each $\phi \in W(i)$ and keeping the same function and relation symbols of $\Sigma(i)$:

$$\Sigma(i+1)_{\text{con}} := \Sigma(i)_{\text{con}} \sqcup \{c_\phi \mid \phi \in W(i)\}$$

We create a witness formula $w(\phi)$ for each formula $\phi \in W$:

$$\begin{aligned} w : W(i) &\to \Sigma(i+1)_{\text{for}} \\ \phi &\mapsto ((\exists v, \phi(v)) \to \phi(c_\phi)) \end{aligned}$$

Then let

$$T(i+1) := T(i) \cup w(W(i))$$

Certainly $T(i+1)$ is a $\Sigma(i+1)$-theory such that $T(0) \subseteq T(i+1)$, $\Sigma(0)_{\text{con}} \subseteq \Sigma(i+1)_{\text{con}}$ where the function and relation symbols are unchanged. Since $W(i)$ is countibly infinite, $|\Sigma(i+1)_{\text{con}}| = |\Sigma(i)_{\text{con}}| + \aleph_0 = |\Sigma(0)_{\text{con}}| + \aleph_0$. We need to check that $T(i+1)$ is finitely consistent. Take a finite subset of $T(i+1)$. It is a union of two finite sets $\Delta_T \subseteq T(i)$ and $\Delta_w \subseteq w(W(i))$. Since $T(i)$ is finitely consistent there exists a $\Sigma(i)$-model $\mathcal{M}(i)$ of Δ_T. Let $\mathcal{M}(i+1)$ be defined to have carrier set $\mathcal{M}(i)$. There exists some $b \in \mathcal{M}(i) = \mathcal{M}(i+1)$ since $\Sigma(i)$ is non-empty. To define interpretation for $\mathcal{M}(i+1)$, let $c \in \Sigma(i+1)_{\text{con}}$:

$$c^{\mathcal{M}(i+1)}_{\Sigma(i+1)} := \begin{cases} c^{\mathcal{M}(i)}_{\Sigma(i)} & \text{when } c \in \Sigma(i)_{\text{con}} \\ a & \text{when } c = c_\phi \text{ and } \exists a \in \mathcal{M}(i), \mathcal{M}(i) \vDash_{\Sigma(i)} \phi(a) \\ b & \text{when } c = c_\phi \text{ and } \forall a \in \mathcal{M}(i), \mathcal{M}(i) \nvDash_{\Sigma(i)} \phi(a) \end{cases}$$

Then $\mathcal{M}(i+1)$ is a well defined $\Sigma(i+1)$-structure. We check is is a $\Sigma(i+1)$-model of $\Delta_T \cup \Delta_w$. Since $\star_{\Sigma(i+1)}^{\mathcal{M}(i+1)}$ agrees with $\star_{\Sigma(i)}^{\mathcal{M}(i)}$ for constants, functions and relations from $\mathcal{M}(i)$ - a $\Sigma(i)$-model of Δ_T - it is a $\Sigma(*)$-model of Δ_T. If $\psi \in \Delta_w$ then it is $\exists v, \phi(v) \to \phi(c_\phi)$ for some $\phi \in W(i)$. Supposing that $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \exists v, \phi(v)$ it suffices to show $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \phi(c_\phi)$. Then there exists $a \in \mathcal{M}(i+1) = \mathcal{M}(i)$ such that $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \phi(a)$. Hence $\mathcal{M}(i) \vDash_{\Sigma(i)} \phi(a)$ and so c_ϕ is interpreted as a in $\mathcal{M}(i+1)$. Hence $\mathcal{M}(i+1) \vDash_{\Sigma(i)} \phi(c_\phi)$. Thus the induction is complete.

Let $\Sigma(*)$ be the signature such that its function and relations are the same as $\Sigma(0)$ and $\Sigma(*)_{\text{con}} = \bigcup_{i\in\mathbb{N}} \Sigma(i)_{\text{con}}$. Then

$$|\Sigma(*)_{\text{con}}| = |\bigcup_{i\in\mathbb{N}} \Sigma(i)_{\text{con}}| = \aleph_0 \times (\aleph_0 + \Sigma(0)_{\text{con}}) = \aleph_0 + \Sigma(0)_{\text{con}}$$

Let $T(*) = \bigcup_{i\in\mathbb{N}} T(i)$. Any finite subset of $T(*)$ is a subset of some $T(i)$, hence has a non-empty $\Sigma(i)$-model $\mathcal{M}$. Checking the relevant conditions for moving models up signatures, we have $\mathcal{M}(*)$ a $\Sigma(*)$-model of the finite subset (by interpreting the new constant symbols as the element of the non-empty carrier set.). Hence $T(*)$ is finitely consistent.

If T' is a $\Sigma(*)$-theory such that $T(*) \subseteq T'$, and ϕ is a $\Sigma(*)$-formula of exactly one variable. There exists an $i \in \mathbb{N}$ such that $\phi \in \Sigma(i)_{\text{for}}$. Since $c_\phi \in \Sigma(i+1)$ satisfies $T(i) \vDash_{\Sigma(i+1)} (\exists v, \phi(v)) \to \phi(c_\phi)$, by moving the logical consequence up to $\Sigma(*)$, we have $T(i) \vDash_{\Sigma(*)} (\exists v, \phi(v)) \to \phi(c_\phi)$. If $\mathcal{N}$ is a $\Sigma(*)$-model of T' then it is a $\Sigma(*)$-model of $T(i)$, then $\mathcal{N} \vDash_{\Sigma(*)} (\exists v, \phi(v)) \to \phi(c_\phi)$. Hence $T' \vDash_{\Sigma(*)} (\exists v, \phi(v)) \to \phi(c_\phi)$, satisfying the witness property. □

Lemma – Adding Formulas to Consistent Theories

If T is a finitely consistent Σ-theory and ϕ is a Σ-sentence then at least one of $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely consistent.

Proof. We show that for any finite $\Delta \subseteq T \cup \{\phi\}$ and for any finite $\Delta_\neg \subseteq T \cup \{\neg\phi\}$, one of Δ or $\Delta_\neg$ is consistent. The finite subset

$$(\Delta \setminus \{\phi\}) \cup (\Delta_\neg \setminus \{\neg\phi\}) \subseteq T$$

is consistent by finite consistency of T. Let $\mathcal{M}$ be the model of $(\Delta \setminus \{\phi\}) \cup (\Delta_\neg \setminus \{\neg\phi\})$. Case on whether $\mathcal{M} \vDash_\Sigma \phi$ or not. In the first case $\mathcal{M} \vDash_\Sigma \Delta$ and in the second $\mathcal{M} \vDash_\Sigma \Delta_\neg$. Hence $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely consistent. □

Exercise. *Find a signature Σ, a consistent Σ-theory T and Σ-sentence ϕ such that $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are both consistent.*

Proposition – Extending a finitely consistent theory to a maximal theory (Zorn)

Given a finitely consistent Σ-theory $T(0)$ there exists a Σ-theory $T(*)$ such that

1. $T(0) \subseteq T(*)$
2. $T(*)$ is finitely consistent.
3. $T(*)$ is Σ-maximal.

Proof. We use Zorn's Lemma. Let

$$Z := \{T \in \Sigma_{\text{the}} \,|\, T \text{ finitely consistent and } T(0) \subseteq T\}$$

be ordered by inclusion. Let $T(0) \subseteq T(1) \subseteq \cdots$ be a chain. Then $\bigcup_{i\in\mathbb{N}} T(i)$ is a Σ-theory such that any finite subset is a subset of some $T(i)$, hence is consistent by finite consistency of $T(i)$. Zorn's lemma implies there exists a $T(*) \in Z$ that is maximal (in the order theory sense). Since $T(*)$ is in Z, we have that it is a finitely consistent Σ-theory containing $T(0)$.

To show that it is Σ-maximal we take a Σ-sentence ϕ. By the previous result, $T(*) \cup \{\phi\}$ or $T(*) \cup \{\neg\phi\}$ is finitely consistent. Hence $T(*) \cup \{\phi\} = T(*)$ or $T(*) \cup \{\neg\phi\} = T(*)$ by (order theoretic) maximality, so $\phi \in T(*)$ or $\neg\phi \in T(*)$. □

> Notation (Cardinalities of signatures and structures). *Given a signature Σ, we write $|\Sigma| := |\Sigma_{\text{con}}| + |\Sigma_{\text{fun}}| + |\Sigma_{\text{rel}}|$ and call this the cardinality of the signature Σ.*

> **Proposition – The compactness theorem**
>
> If T is a Σ-theory, then the following are equivalent:
>
> 1. T is finitely consistent.
> 2. T is consistent
> 3. For any infinite cardinal κ such that $|\Sigma| \leq \kappa$, there exists a non-empty Σ-model of T with cardinality $\leq \kappa$.

Proof. 3. implies 2. and 2. implies 1. are both obvious. A proof of 1. implies 3. follows.

Suppose an $\Sigma(0)$-theory $T(0)$ is finitely consistent. Let κ be an infinite cardinal such that $|\Sigma(0)| \leq \kappa$. Then $|\Sigma(0)_{\text{con}}| \leq |\Sigma(0)| \leq \kappa$. We wish to find a Σ-model of T with cardinality $\leq \kappa$. We have shown that there exists a signature $\Sigma(1)$ and $\Sigma(1)$-theory $T(1)$ such that

1. $\Sigma(0)_{\text{con}} \subseteq \Sigma(1)_{\text{con}}$ and they share the same function and relation symbols.
2. $|\Sigma(1)_{\text{con}}| = |\Sigma(0)_{\text{con}}| + \aleph_0$
3. $T(0) \subseteq T(1)$
4. $T(1)$ is finitely consistent.
5. Any $\Sigma(1)$-theory T such that $T(1) \subseteq T$ has the witness property.

$T(1)$ is finitely consistent so there exists a $\Sigma(1)$-theory $T(2)$ such that

6. $T(1) \subseteq T(2)$
7. $T(2)$ is finitely consistent.
8. $T(2)$ is $\Sigma(1)$-maximal.

Furthermore, $T(2)$ has the witness property due to point 5. Since $T(2)$ has the witness property, is $\Sigma(1)$-maximal and finitely consistent, $T(2)$ has a non-empty $\Sigma(1)$-model $\mathcal{M}$ such that $|\mathcal{M}| \leq \kappa$ by Henkin Construction. $\mathcal{M} \vDash_{\Sigma(1)} T(0)$ since $T(0) \subseteq T(1) \subseteq T(2)$. We can move $\mathcal{M}$ down to $\Sigma(0)$, obtaining $\mathcal{M} \vDash_{\Sigma(0)} T(0)$. □

1.1.4 The Category of Structures

> **Definition – Σ-morphism, Σ-embedding, Σ-isomorphism**
>
> Given Σ a signature, $\mathcal{M}, \mathcal{N}$ both Σ-structures, $A \subseteq \mathcal{M}_{\text{car}}$ and $\iota : A \to \mathcal{N}_{\text{car}}$, we call ι a partial Σ-morphism from $\mathcal{M}$ to $\mathcal{N}$ when
>
> - For all $c \in C$ (such that $c^{\mathcal{M}} \in A$),
> $$\iota(c^{\mathcal{M}}) = c^{\mathcal{N}}$$
> - For all $f \in F$ and all $a \in M^{n_f}$ (such that $f^{\mathcal{M}}(a) \in A$),
> $$\iota \circ f^{\mathcal{M}}(a) = f^{\mathcal{N}} \circ \iota(a)$$

- For all $r \in R$, for all $a \in M^{m_r} \cap A^{m_r}$,

$$a \in r^{\mathcal{M}} \Rightarrow \iota(a) \in r^{\mathcal{N}}$$

If in addition for relations we have

$$a \in r^{\mathcal{M}} \Leftarrow \iota(a) \in r^{\mathcal{N}} \quad \text{and} \quad \iota \text{ is injective,}$$

then ι is called a partial Σ-embedding (the word extension is often used interchangably with embedding).

In the case that $A = \mathcal{M}_{\text{car}}$ we write $\iota : \mathcal{M} \to \mathcal{N}$ and call ι a Σ-morphism.

The notion of morphisms here will be the same as morphisms in the algebraic setting. For example in the signature of monoids (groups), preserving interpretation of constant symbols says the identity is sent to the identity and preserving interpretation of function symbols says the multiplication is preserved.

Definition – Elementary Embedding
A partial Σ-embedding $\iota : A \to \mathcal{N}$ (for $A \subseteq \mathcal{M}$) is elementary if for any Σ-formula ϕ with variables indexed by S and $a \in A^S$,

$$\mathcal{M} \vDash_\Sigma \phi(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

The following is exactly what we expect - that terms are well behaved with respect to morphisms.

Lemma – Σ-morphisms commute with interpretation of terms
Given a Σ-morphism $\iota : \mathcal{M} \to \mathcal{N}$, we have that for any Σ-term t with variables indexed by S and $a \in \mathcal{M}^S$,

$$\iota(t^{\mathcal{M}}(a)) = t^{\mathcal{N}}(\iota(a))$$

Proof. We case on what t is:

- If $t = c \in \Sigma_{\text{con}}$ then
$$\iota(t^{\mathcal{M}}(a)) = \iota(c^{\mathcal{M}}) = c^{\mathcal{N}} = t^{\mathcal{N}}(\iota(a))$$
- If $t = v \in \Sigma_{\text{var}}$ then
$$\iota(t^{\mathcal{M}}(a)) = \iota(v^{\mathcal{M}}(a)) = \iota(a) = v^{\mathcal{N}}(\iota(a)) = t^{\mathcal{N}}(\iota(a))$$
- If $t = f(s)$ then
$$\iota(t^{\mathcal{M}}(a)) = \iota \circ f^{\mathcal{M}}(s^{\mathcal{M}}(a)) = f^{\mathcal{N}} \circ \iota(s^{\mathcal{M}}(a)) = f^{\mathcal{N}} \circ s^{\mathcal{N}}(\iota(a)) = t^{\mathcal{N}}(\iota(a))$$
Using the induciton hypothesis in the penultimate step.

□

It is worth knowing that the set of Σ-structures form a category:

Definition – The category of Σ-structures
Given a signature Σ, we have $\mathbb{M}\mathbf{od}(\Sigma)$ - a category where objects are Σ-structures and morphisms are Σ-morphisms.

Clearly for any $\mathcal{M}$, the identity exists and is a Σ-morphism. We show that composition of morphisms are morphisms. Furthermore, composition of embeddings are embeddings and composition of elementary embeddings are elementary. Thus we could also define morphisms between objects to be embeddings or elementary embeddings and obtain a subcategory.

Hence we inherit a notion of isomorphism of Σ-structures from category theory.

Proof. Let $\iota_1 : \mathcal{M}_0 \to \mathcal{M}_1$ and $\iota_2 : \mathcal{M}_1 \to \mathcal{M}_2$ be Σ-morphisms. We show that the composition is a Σ-morphism:

- If $c \in \Sigma_{\text{con}}$ then
$$\iota_1 \circ \iota_0(c^{\mathcal{M}_0}) = \iota_1(c^{\mathcal{M}_1}) = c^{\mathcal{M}_2}$$
- If $f \in \Sigma_{\text{fun}}$ and $a \in {\mathcal{M}_0}^{n_f}$ then
$$\iota_1 \circ \iota_0 \circ f^{\mathcal{M}_0}(a) = \iota_1 \circ f^{\mathcal{M}_1} \circ \iota_0(a) = f^{\mathcal{M}_2} \circ \iota_1 \circ \iota_0(a)$$
- If $r \in \Sigma_{\text{rel}}$ and $a \in {\mathcal{M}_0}^{m_f}$ then
$$a \in r^{\mathcal{M}_0} \Rightarrow \iota_1(a) \in r^{\mathcal{M}_1} \Rightarrow \iota_2 \circ \iota_1(a) \in r^{\mathcal{M}_2}$$

To show that embeddings compose to be embeddings we note that the composition of injective functions is injective and if $r \in \Sigma_{\text{rel}}$ and $a \in {\mathcal{M}_0}^{m_f}$ then

$$a \in r^{\mathcal{M}_0} \Leftrightarrow \iota_1(a) \in r^{\mathcal{M}_1} \Leftrightarrow \iota_2 \circ \iota_1(a) \in r^{\mathcal{M}_2}$$

To show that composition of elementary embeddings are elementary, let $\phi \in \Sigma_{\text{for}}$ and a in $\mathcal{M}_0$ be chosen suitably. Then

$$\mathcal{M}_0 \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M}_1 \vDash_\Sigma \phi(\iota_1(a)) \Leftrightarrow \mathcal{M}_2 \vDash_\Sigma \phi(\iota_2 \circ \iota_1(a))$$

□

Example. *Given the category of structures from the signature of rings, we can take the subcategory whose objects are models of the theory of rings (namely rings), hence producing the category of rings. Similarly taking the subcategory whose objects are models of the theory of fields (namely fields) produces the category of fields.*

Proposition – Embeddings Preserve Satisfaction of Quantifier Free Formulas

Given $\iota : \mathcal{M} \to \mathcal{N}$ a Σ-embedding and ϕ a Σ-formula with variables indexed by S, and $a \in \mathcal{M}^S$,

1. If ϕ is $\top$ then it is satisfied by both.
2. If ϕ is $t = s$ then $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$.
3. If ϕ is $r(s)$ then $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$.
4. If ϕ is $\neg\chi$ and $\mathcal{M} \vDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi(\iota(a))$ then
$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$
5. If ϕ is $\chi_0 \vee \chi_1$ and $\mathcal{M} \vDash_\Sigma \chi_i(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_i(\iota(a))$ then
$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

Thus from the above we can immediately conclude by induction that if ϕ is a quantifier free Σ-formula,

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

Note that our original result is stronger than this since we didn't assume the formula to be quantifier free.

Proof.

1. Trivial.
2. If ϕ is $t = s$ then

$$\begin{aligned} \mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a) \\ &\Leftrightarrow \iota(t^{\mathcal{M}}(a)) = \iota(s^{\mathcal{M}}(a)) && \text{by injectivity} \\ &\Leftrightarrow t^{\mathcal{N}}(\iota(a)) = s^{\mathcal{N}}(\iota(a)) && \text{morphisms commute with} \\ &\Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a)) && \text{interpretation of terms} \end{aligned}$$

3. If ϕ is $r(s)$ then

$$\begin{aligned} \mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow a \in r^{\mathcal{M}} \\ &\Leftrightarrow \iota(a) \in r^{\mathcal{N}} && \text{embeddings property} \\ &\Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a)) \end{aligned}$$

4. If ϕ is $\neg\chi$ and $\mathcal{M} \vDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi(\iota(a))$ then

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \nvDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \nvDash_\Sigma \chi(\iota(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

5. If ϕ is $\chi_0 \vee \chi_1$ and $\mathcal{M} \vDash_\Sigma \chi_i(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_i(\iota(a))$

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \vDash_\Sigma \chi_0(a) \text{ or } \mathcal{M} \vDash_\Sigma \chi_1(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_0(\iota(a)) \text{ or } \mathcal{N} \vDash_\Sigma \chi_1(\iota(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

□

Definition – Universal Formula, Universal Sentence
A Σ-formula is universal if it can be built inductively by the following two constructors:

| If ϕ is a quantifier free Σ-formula then it is a universal Σ-formula.

| If ϕ is a universal Σ-formula then $\forall v, \phi(v)$ is a universal Σ-formula.

In other words universal Σ-formulas are formulas that start with a bunch of 'for alls' followed by a quantifier free formula.

Proposition – Embeddings preserve satisfaction of universal formulas downwards
Given $\iota : \mathcal{M} \to \mathcal{N}$ a Σ-embedding and ϕ a universal Σ-formula with variables indexed by S (χ is quantifier free). For any $a \in \mathcal{M}^S$

$$\mathcal{N} \vDash_\Sigma \phi(\iota(a)) \quad \Rightarrow \quad \mathcal{M} \vDash_\Sigma \phi(a)$$

By taking the contrapositive we can show that embeddings preserve satisfaction of 'existential' Σ-formulas upwards.

Proof. We induct on ϕ:

- If ϕ is a quantifier free then since embeddings preserve satisfaction of quantifier free formulas, $\mathcal{N} \vDash_\Sigma \phi(\iota(a)) \Rightarrow \mathcal{M} \vDash_\Sigma \phi(a)$.
- If ϕ is $\forall v_i, \psi$ with S indexing the variables of ψ. Let $T := S \setminus \{i\}$ Assuming the inductive hypothesis: for any $a \in \mathcal{M}^T$ and $b \in \mathcal{M}$,

$$\mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b)) \quad \Rightarrow \quad \mathcal{M} \vDash_\Sigma \psi(a, b)$$

Then for any $a \in \mathcal{M}^T$

$$\begin{aligned} & \mathcal{N} \vDash_\Sigma \phi(\iota(a)) \\ \Rightarrow\ & \forall b \in \mathcal{M}, \mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b)) \\ \Rightarrow\ & \forall b \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \psi(\iota(a), \iota(b)) && \text{by the induction} \\ \Rightarrow\ & \mathcal{M} \vDash_\Sigma \phi(a) \end{aligned}$$

□

Proposition – Isomorphisms are Elementary

If two Σ-structures $\mathcal{M}$ and $\mathcal{N}$ are Σ-isomorphic then the isomorphism is elementary.

Proof. Let $\iota : \mathcal{M} \to \mathcal{N}$ be a Σ-isomorphism. We case on what ϕ is:

- If ϕ is quantifier free, then each case is follows from applying embeddings preserve satisfaction of quantifier free formulas.
- If ϕ is $\forall v, \chi(v)$ then $(\Rightarrow)$ Let $b \in \mathcal{N}_{\text{con}}$ then $\iota^{-1}(b) \in \mathcal{M}_{\text{con}}$ is well defined by surjectivity. Hence $\mathcal{M} \vDash_\Sigma \chi(\iota^{-1}(b), a)$ and so $\mathcal{N} \vDash_\Sigma \chi(b, \iota(a))$ by the induction hypothesis. Hence $\mathcal{N} \vDash_\Sigma \phi(\iota(a))$. $(\Leftarrow)$ The same.

□

1.1.5 Vaught's Completeness Test

Read ahead to the statement of Vaught's Completeness Test. We aim to prove it in this section.

Definition – Finitely modelled, infinitely modelled

A Σ-theory T is finitely modelled when there exists a Σ-model of T with finite carrier set.

A Σ-theory T is finitely modelled when there exists a Σ-model of T with infinite carrier set.

Finitely modelled is *not* the same as finitely consistent.

Proposition – Infinite Modelled Theories Have Arbitrary Large Models

Given Σ a signature, T a Σ-theory that is infinitely modelled, and a cardinal κ such that $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$, there exists $\mathcal{M}$ a Σ-model of T such that $\kappa = |\mathcal{M}|$.

Proof. Enrich only the signature's constant symbols to create $\Sigma(*)$ a signature such that $\Sigma(*)_{\text{con}} = \Sigma_{\text{con}} \cup \{c_\alpha \mid \alpha \in \kappa\}$. Let $T(*) = T \cup \{c_\alpha \neq c_\beta \mid \alpha, \beta \in \kappa \wedge \alpha \neq \beta\}$ be a $\Sigma(*)$-theory.

Using the compactness theorem, it suffices to show that $T(*)$ is finitely consistent. Take a finite subset of $T(*)$. This is the union of a finite subset $\Delta_T \subseteq T$, and a finite subset of $\Delta_\kappa \subseteq \{c_\alpha \neq c_\beta \mid \alpha, \beta \in \kappa \wedge \alpha \neq \beta\}$. Let $\mathcal{M}$ be the Σ-model of T with infinite cardinality. We want to make $\mathcal{M}$ a $\Sigma(*)$-model of $\Delta_T \cup \Delta_\kappa$ by interpreting the new symbols of $\{c_\alpha \mid \alpha \in \kappa\}$ in a sensible way.

Since Δ_κ is finite, we can find a finite subset $I \subset \kappa$ that indexes the constant symbols appearing in Δ_κ. Since $\mathcal{M}$ is infinite and I is finite, we can find distinct elements of $\mathcal{M}$ to interpret the elements of $\{c_\alpha \mid \alpha \in I\}$. Interpret the rest of the new constant symbols however, for example let them all be sent to the same element, then $\mathcal{M} \vDash_{\Sigma*} \Delta_T \cup \Delta_\kappa$. Hence $T*$ is consistent.

Using the third equivalence of $T(*)$ being consistent, there exists $\mathcal{M}$ a $\Sigma(*)$-model of $T(*)$ with $|\mathcal{M}| \leq \kappa$. If $|\mathcal{M}| < \kappa$ then there would be c_α, c_β that are interpreted as equal, hence $\mathcal{M} \vDash_{\Sigma(*)} c_\alpha = c_\beta$ and $\mathcal{M} \nvDash_{\Sigma(*)} c_\alpha = c_\beta$, a contradiction. Thus $|\mathcal{M}| = \kappa$. Move $\mathcal{M}$ down a signature to make it a Σ-model of T. This doesn't change the cardinality of $\mathcal{M}$, so we have a Σ-model of T with cardinality κ. □

Definition – Elementary equivalence

Let $\mathcal{M}$, $\mathcal{N}$ be Σ-structures. They are elementarily equivalent if for any Σ-sentence ϕ, $\mathcal{M} \vDash_\Sigma \phi$ if and only if $\mathcal{N} \vDash_\Sigma \phi$. We write $\mathcal{M} \equiv_\Sigma \mathcal{N}$.

Definition – Complete

A Σ-theory T is complete when either of the following equivalent definitions hold:

- For any Σ-sentence ϕ, $T \vDash_\Sigma \phi$ or $T \vDash_\Sigma \neg\phi$.
- All models of T are elementarily equivalent.

Proof. ($\Rightarrow$) Let $\mathcal{M}$ and $\mathcal{N}$ be models of T and ϕ be a Σ-sentence. If $\phi \in T$ then both satisfy ϕ. Otherwise $\neg\phi \in T$ and neither satisfy ϕ.

($\Leftarrow$) If ϕ is a Σ-sentence then suppose for a contradiction

$$T \nvDash_\Sigma \phi \text{ and } T \nvDash_\Sigma \neg\phi$$

Then there exist models of T such that $\mathcal{M} \nvDash_\Sigma \phi$ and $\mathcal{N} \nvDash_\Sigma \neg\phi$. By assumption they are elementarily equivalent and so $\mathcal{M} \vDash_\Sigma \neg\phi$ implies $\mathcal{N} \vDash_\Sigma \neg\phi$, a contradiction. □

Note that if we have completeness and compactness, then for any Σ-sentence ϕ, *either* $T \vDash_\Sigma \phi$ or $T \vDash_\Sigma \neg\phi$.

Proposition – Not a consequence is consistent

Let T be a Σ-theory and ϕ is a Σ-sentence then $T \nvDash_\Sigma \phi$ if and only if $T \cup \{\neg\phi\}$ is consistent. Furthermore, $T \nvDash_\Sigma \neg\phi$ if and only if $T \cup \{\phi\}$ is consistent.

Proof. For the first statement: ($\Rightarrow$) Unfolding $T \nvDash_\Sigma \phi$, we have that there exists a Σ-model $\mathcal{M}$ of T such that $\mathcal{M} \nvDash_\Sigma \phi$. Hence $\mathcal{M} \vDash_\Sigma \neg\phi$ and we are done. The backward proof is straightward.

For the second statement, apply the first to $\neg\phi$ and obtain $T \nvDash_\Sigma \neg\phi$ if and only if $T \cup \{\neg\neg\phi\}$ is consistent. Note that for any Σ-structure $\mathcal{M}$, $\mathcal{M} \vDash_\Sigma \neg\neg\phi$ if and only if $\mathcal{M} \vDash_\Sigma \phi$. This completes the proof. □

Proposition – Vaught's Completeness Test

Suppose that Σ-theory T is consistent, not finitely modelled, and κ-categorical for some cardinal satisfying $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$. Then T is complete.

Proof. Suppose not: If T is not complete then there exists Σ-formula ϕ such that $T \nvDash_\Sigma \phi$ and $T \nvDash_\Sigma \neg\phi$. These imply $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ are respectively consistent by the previous proposition. Let $\mathcal{M}_\neg$ and $\mathcal{M}$ be models of $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ respectively. Then each are models of T so they are infinite and so $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ are infinitely modelled.

Since we have κ such that $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$, there exists $\mathcal{N}_\neg, \mathcal{N}$ respectively Σ-models of $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ such that $\kappa = |\mathcal{N}_\neg| = |\mathcal{N}|$. Since T is κ-categorical $\mathcal{N}$ and $\mathcal{N}_\neg$ are isomorphic by an elementary Σ-embedding. As ϕ has no free variables this implies that $\mathcal{N} \vDash_\Sigma \phi$ and $\mathcal{N} \vDash_\Sigma \neg\phi$, a contradiction. □

1.1.6 Elementary embeddings and diagrams of models

Proposition – Tarski-Vaught Elementary Embedding Test

Let $\iota : \mathcal{M} \to \mathcal{N}$ be a Σ-embedding, then the following are equivalent:

1. ι is elementary

2. For any $\phi \in \Sigma_{\text{for}}$ with free variables indexed by S, any $i \in S$ and any $a \in (\mathcal{M})^{S \setminus \{i\}}$,

$$\forall b \in \mathcal{M}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b)) \quad \Rightarrow \quad \forall c \in \mathcal{N}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), c),$$

which we call the Tarski-Vaught condition. Note that if $i \in S$ then a 'fills in' all but one of the free variables of ϕ, leaving only one free variable for substitution. (S can be empty, but then this doesn't say very much.)

3. For any $\phi \in \Sigma_{\text{for}}$ with free variables indexed by S, any $i \in S$ and any $a \in (\mathcal{M})^{S \setminus \{i\}}$,

$$\exists c \in \mathcal{N}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), c) \quad \Rightarrow \quad \exists b \in \mathcal{M}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b))$$

This is essentially the contrapositive of the previous statement, and is included because it is more commonly version of the statement.

Proof. We only show the first two statements are equivalent and leave the third as an exercise. ($\Rightarrow$) First show that $\mathcal{M} \vDash_\Sigma \forall v, \phi(a, v)$. Let $b \in \mathcal{M}$, then by assumption $\mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b))$, which is implies $\mathcal{M} \vDash_\Sigma \phi(a, b)$ as ι is an elementary embedding. Thus we indeed have $\mathcal{M} \vDash_\Sigma \forall v, \phi(a, v)$ which in turn implies $\mathcal{N} \vDash_\Sigma \forall v, \phi(\iota(a), v)$ and we are done.

($\Leftarrow$) We case on what ϕ is, though most of the work was already done before.

- If ϕ is quantifier free, then each case follows from applying embeddings preserve satisfaction of quantifier free formulas.
- The backwards implication follows from applying embeddings preserve satisfaction of universal formulas downwards.

  For the forwards implication we use the Tarski-Vaught condition (so far ι just needed to be a Σ-embedding)

$$\begin{aligned} \mathcal{M} \vDash_\Sigma \forall v, \psi(a, v) &\Rightarrow \forall b \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \psi(a, b) & \\ &\Rightarrow \forall b \in \mathcal{M}, \mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b)) & \text{by the induction hypothesis} \\ &\Rightarrow \forall c \in \mathcal{N}, \mathcal{N} \vDash_\Sigma \psi(\iota(a), c) & \text{by the Tarski-Vaught condition} \\ &\Rightarrow \mathcal{N} \vDash_\Sigma \phi & \end{aligned}$$

□

Proposition – Moving Morphisms Down Signatures

Suppose $\Sigma \leq \Sigma(*)$. If $\iota : \mathcal{M} \to \mathcal{N}$ is a $\Sigma(*)$-morphism then

1. ι can be made into a Σ-morphism.
2. If ι is an embedding then it remains an embedding.
3. If ι is an elementary embedding then it remains elementary.

Proof. 1. Move $\mathcal{M}$ and $\mathcal{N}$ down to being Σ structures (by picking $T(*) = T = \varnothing$). We show that the same set morphism $\iota : \mathcal{M} \to \mathcal{N}$ is a Σ-morphism.

- If $c \in \Sigma_{\text{con}}$ then since moving structures down signatures preserves interpretation on the lower signature, and since ι is a $\Sigma(*)$ embedding,

$$\iota(c_\Sigma^\mathcal{M}) = \iota(c_{\Sigma(*)}^\mathcal{M}) = c_{\Sigma(*)}^\mathcal{N} = c_\Sigma^\mathcal{N}$$

- If $f \in \Sigma_{\text{fun}}$ and $a \in (\mathcal{M})^{n_f}$ then similarly

$$\iota \circ f_\Sigma^\mathcal{M}(a) = \iota \circ f_{\Sigma(*)}^\mathcal{M}(a) = f_{\Sigma(*)}^\mathcal{N}(\iota(a)) = f_\Sigma^\mathcal{N}(\iota(a))$$

- If $r \in \Sigma_{\text{rel}}$ and $a \in (\mathcal{M})^{m_r}$ then

$$a \in r_\Sigma^{\mathcal{M}}(a) = r_{\Sigma(*)}^{\mathcal{M}} \Rightarrow \iota(a) \in r_{\Sigma(*)}^{\mathcal{N}} = r_\Sigma^{\mathcal{N}}$$

2. If we also have that it is an embedding in $\Sigma(*)$, then injectivity is preserved as it is a property of set morphisms. Given $r \in \Sigma_{\text{rel}}$ and $a \in (\mathcal{M})^{m_r}$,

$$\iota(a) \in r_\Sigma^{\mathcal{N}} = r_{\Sigma(*)}^{\mathcal{N}} \Rightarrow a \in r_{\Sigma(*)}^{\mathcal{M}}(a) = r_\Sigma^{\mathcal{M}}$$

3. If we also have that ι is elementary in $\Sigma(*)$ then we use the Tarski-Vaught Test: let $\phi \in \Sigma_{\text{for}}$ have free variables indexed by S, let $i \in S$ and let $a \in (\mathcal{M})^{S \setminus \{i\}}$. Then due to the construction in moving $\mathcal{M}$ and $\mathcal{N}$ down a signature we have that for any $b \in \mathcal{N}$,

$$\mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b)) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(\iota(a), \iota(b))$$

and similarly for $\mathcal{M}$. Hence

$$\begin{aligned} &\forall b \in \mathcal{N}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b)) \\ \Rightarrow &\forall b \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma(*)} \phi(\iota(a), \iota(b)) \\ \Rightarrow &\forall c \in \mathcal{M}, \mathcal{M} \vDash_{\Sigma(*)} \phi(a, c) && \iota \text{ is elementary in } \Sigma(*) \\ \Rightarrow &\forall c \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \phi(a, c) \end{aligned}$$

Hence ι is elementary in Σ.

$\square$

NOTATION. *Let A be a set and Σ be a signature, enriching only the constant symbols of Σ we can create a signature $\Sigma(A)$ such that*

$$\Sigma(A)_{\text{con}} := \Sigma_{\text{con}} \cup \{c_a \mid a \in A\}$$

In the case that $A = \mathcal{M}$ for some model $\mathcal{M}$ then we write $\Sigma(\mathcal{M}) := \Sigma(A)$.

Definition – $\Sigma(\mathcal{M})$, Diagram and the Elementary Diagram of a Structure
Let $\mathcal{M}$ be a Σ-structure, we move $\mathcal{M}$ up to the signature $\Sigma(\mathcal{M})$ by interpreting each new constant symbol c_a as a. ($\mathcal{M}$ satisfies the conditions of our lemma for moving models up signatures by choosing $T = \varnothing$). Thus we may treat $\mathcal{M}$ as a $\Sigma(\mathcal{M})$ structure. We define the atomic diagram of $\mathcal{M}$ over Σ:

| If ϕ is an atomic $\Sigma(\mathcal{M})$-sentence such that $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi$, then $\phi \in \operatorname{AtDiag}(\Sigma, \mathcal{M})$.

| If $\phi \in \operatorname{AtDiag}(\Sigma, \mathcal{M})$ then $\neg\phi \in \operatorname{AtDiag}(\Sigma, \mathcal{M})$.

We define the elementary diagram of $\mathcal{M}$ over Σ:

| If $\phi \in \Sigma(\mathcal{M})_{\text{for}}$ and ϕ is a $\Sigma(\mathcal{M})$-sentence and $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi$, then $\phi \in \operatorname{ElDiag}(\Sigma, \mathcal{M})$.

Proposition – Models of the elementary diagram correspond to elementary extensions
Given $\mathcal{M}$ a Σ-structure and $\mathcal{N}$ a $\Sigma(\mathcal{M})$-structure such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \operatorname{AtDiag}(\Sigma, \mathcal{M})$, we can make $\mathcal{N}$ into a Σ-structure and find a Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$. Furthermore $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \operatorname{ElDiag}(\Sigma, \mathcal{M})$ then the embedding is elementary.

Conversely, given an elementary Σ-embedding from $\mathcal{M}$ into a Σ-structure $\mathcal{N}$, we can move $\mathcal{N}$ up to being a $\Sigma(\mathcal{M})$ structure such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \operatorname{ElDiag}(\Sigma, \mathcal{M})$. Note that we don't show the converse for the atomic case.

Proof. ($\Rightarrow$) Suppose $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{AtDiag}(\Sigma, \mathcal{M})$. Firstly we work in $\Sigma(\mathcal{M})$ to define the embedding: move $\mathcal{M}$ up a signature by taking the same interpretation as used in the definition of $\Sigma(\mathcal{M})$:

$$\star^{\mathcal{M}}_{\Sigma(\mathcal{M})} : c_a \mapsto a$$

and preserving the same interpretation for symbols of Σ. This makes $\star^{\mathcal{M}}_{\Sigma(\mathcal{M})_{\mathrm{con}}}$ surjective. Thus we write elements of $\mathcal{M}$ as $c^{\mathcal{M}}_{\Sigma(\mathcal{M})}$, for some $c \in \Sigma(\mathcal{M})_{\mathrm{con}}$

Next we define the $\Sigma(\mathcal{M})$-morphism $\iota : \mathcal{M} \to \mathcal{N}$ such that $\iota : c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \to c^{\mathcal{N}}_{\Sigma(\mathcal{M})}$. To check that ι is well defined, take $c, d \in \Sigma(\mathcal{M})_{\mathrm{con}}$ such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} = d^{\mathcal{M}}_{\Sigma(\mathcal{M})}$.

$$\begin{aligned} &\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} c = d \\ &\Rightarrow c = d \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} c = d \\ &\Rightarrow c^{\mathcal{N}}_{\Sigma(\mathcal{M})} = d^{\mathcal{N}}_{\Sigma(\mathcal{M})} \end{aligned}$$

Thus ι is well defined. In fact doing 'not' gives us injectivity in the same way: Take $c, d \in \Sigma(\mathcal{M})_{\mathrm{con}}$ such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \neq d^{\mathcal{M}}_{\Sigma(\mathcal{M})}$.

$$\begin{aligned} &\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} c \neq d \\ &\Rightarrow c \neq d \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} c \neq d \\ &\Rightarrow c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \neq d^{\mathcal{N}}_{\Sigma(\mathcal{M})} \end{aligned}$$

Thus ι is injective. To check that ι is a $\Sigma(\mathcal{M})$-morphism, we check interpretation of functions and relations. Let $f \in \Sigma(\mathcal{M})_{\mathrm{fun}} = \Sigma_{\mathrm{fun}}$ and $c \in (\Sigma(\mathcal{M})_{\mathrm{con}})^{n_f}$. $\star^{\mathcal{M}}_{\Sigma(\mathcal{M})_{\mathrm{con}}}$ is surjective thus we can find $d \in \Sigma\mathcal{M}_{\mathrm{con}}$ such that $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} f(c) = d$. Hence $f(c) = d \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$. Hence $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} f(c) = d$.

$$\begin{aligned} \iota \circ f^{\mathcal{M}}_{\Sigma(\mathcal{M})}(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) &= \iota(d^{\mathcal{M}}_{\Sigma(\mathcal{M})}) \\ &= d^{\mathcal{N}}_{\Sigma(\mathcal{M})} \\ &= f^{\mathcal{N}}_{\Sigma(\mathcal{M})}(c^{\mathcal{N}}_{\Sigma(\mathcal{M})}) \\ &= f^{\mathcal{N}}_{\Sigma(\mathcal{M})} \circ \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) \end{aligned}$$

Let $r \in \Sigma(\mathcal{M})_{\mathrm{rel}} = \Sigma_{\mathrm{rel}}$ and $c \in (\Sigma(\mathcal{M})_{\mathrm{con}})^{m_r}$.

$$\begin{aligned} c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \in r^{\mathcal{M}}_{\Sigma(\mathcal{M})} &\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow r(c) \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \in r^{\mathcal{N}}_{\Sigma(\mathcal{N})} \end{aligned}$$

To show that ι is an embedding it remains to show the backward implication for relations. Let $r \in \Sigma(\mathcal{M})_{\mathrm{rel}} = \Sigma_{\mathrm{rel}}$ and $c \in (\Sigma(\mathcal{M})_{\mathrm{con}})^{m_r}$.

$$\begin{aligned} c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \notin r^{\mathcal{M}}_{\Sigma(\mathcal{M})} &\Rightarrow \mathcal{M} \nvDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow \neg\, r(c) \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \nvDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \notin r^{\mathcal{N}}_{\Sigma(\mathcal{N})} \end{aligned}$$

Assume furthermore that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$. We show that this embedding is elementary. Let ϕ be a $\Sigma(\mathcal{M})$-formula with variables indexed by S and $a \in (\mathcal{M})^S$. Let $c \in (\Sigma_{\mathrm{con}})^S$ be such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} = a$.

$$\begin{aligned} \mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi(a) &\Rightarrow \phi(c) \in \mathrm{ElDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi(c) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi(\iota(a)) \end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{M} \nvDash_{\Sigma(\mathcal{M})} \phi(a) &\Rightarrow \phi(c) \notin \mathrm{ElDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \nvDash_{\Sigma(\mathcal{M})} \phi(c) \\ &\Rightarrow \mathcal{N} \nvDash_{\Sigma(\mathcal{M})} \phi(\iota(a))\end{aligned}$$

Hence ι is an elementary embedding. Moving $\iota : \mathcal{M} \to \mathcal{N}$ down to being a Σ-morphism of Σ-structures completes the proof. ($\Leftarrow$) Sketch: Suppose $\iota : \mathcal{M} \to \mathcal{N}$ is an elementary embedding. Make both $\mathcal{M}$ and $\mathcal{N}$ $\Sigma(\mathcal{M})$-structures by $\star^{\mathcal{M}}_{\Sigma(\mathcal{M})} : c_a \to a$ and $\star^{\mathcal{N}}_{\Sigma(\mathcal{M})} : c_a \to \iota(a)$, where $a \in \mathcal{M}$. Show that ι is still an elementary embedding when moved up to $\Sigma(\mathcal{M})$. Then for any $\phi \in \mathrm{ElDiag}(\Sigma, \mathcal{M})$, $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi$ and so by the embedding being elementary $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi$. Hence $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$. □

1.1.7 Universal axiomatization

Definition – Axiomatization

A Σ-theory A is an axiomatization of a Σ-theory T if for all Σ-structures $\mathcal{M}$,

$$\mathcal{M} \vDash_\Sigma T \Leftrightarrow \mathcal{M} \vDash_\Sigma A$$

Definition – Universal Theory, Universal Axiomatization

If A is a subset of Σ_{uni} then it is called a universal theory.

A Σ-theory A is a universal axiomatization of a Σ-theory T if it is a universal theory that is an axiomatization of T.

Lemma – Lemma on constants

Suppose $\Sigma_{\mathrm{con}} \subseteq \Sigma(*)_{\mathrm{con}}$, $T \in \Sigma_{\mathrm{the}}$, $\phi \in \Sigma_{\mathrm{for}}$ with variables indexed by S. Suppose there exists a list of constant symbols not from Σ, i.e. $c \in (\Sigma(*)_{\mathrm{con}} \setminus \Sigma_{\mathrm{con}})^S$ such that $T \vDash_{\Sigma(*)} \phi(c)$. Then

$$T \vDash_\Sigma \forall v, \phi(v)$$

Proof. We prove the contrapositive. Suppose $T \nvDash_\Sigma \forall v, \phi(v)$ then there exists $\mathcal{M}$ a Σ-model of T such that $\mathcal{M} \nvDash_\Sigma \forall v, \phi(v)$. Thus there exists $a \in \mathcal{M}^S$ such that $\mathcal{M} \nvDash_\Sigma \phi(a)$.

Let b be the element of $\mathcal{M}$. We move $\mathcal{M}$ up a signature by extending the interpretation to the new constant symbols. Suppose $d \in \Sigma(*)_{\mathrm{con}} \setminus \Sigma_{\mathrm{con}}$,

$$d^{\mathcal{M}}_{\Sigma(\mathcal{M})} = \begin{cases} a_i & \text{, if } \exists i \in S, d = c_i, \\ b & \text{, otherwise} \end{cases}$$

Then $\mathcal{M}$ is a $\Sigma(*)$-model of T such that $\mathcal{M} \nvDash_{\Sigma(*)} \phi(a)$, which by construction is equivalent to $\mathcal{M} \nvDash_{\Sigma(*)} \phi(c)$. □

Notation. *Universal consequences of T Let T be a Σ-theory, then*

$$T_\forall := \{\phi \in \Sigma_{\mathrm{uni}} \,|\, T \vDash_\Sigma \phi\}$$

is called the set of universal consequences of T.

Proposition – Universal axiomatizations make substructures models

T a Σ-theory has a universal axiomatization if and only if for any Σ-model $\mathcal{N}$ of T and any Σ-embedding $\mathcal{M} \to \mathcal{N}$, $\mathcal{M}$ is a Σ-model of T.

Proof. ($\Rightarrow$) Suppose A is a universal axiomatization of T, $\mathcal{N}$ is a Σ-model of T and $\mathcal{M} \to \mathcal{N}$ is a Σ-embedding. $\mathcal{N} \vDash_\Sigma T$ if and only if $\mathcal{N} \vDash_\Sigma A$ by definition of A. $\mathcal{N} \vDash_\Sigma A$ if and only if $\mathcal{M} \vDash_\Sigma A$ since embeddings preserve the satisfaction of quantifier free formulas Finally $\mathcal{M} \vDash_{Si} A$ if and only if $\mathcal{M} \vDash_\Sigma T$ by definition of A.

($\Leftarrow$) We show that $T_\forall$ is a universal axiomatization of T. Let $\mathcal{M} \vDash_\Sigma T$ and let $\phi \in T_\forall$. Then by definition of $T_\forall$, $T \vDash_\Sigma \phi$. Hence $\mathcal{M} \vDash_\Sigma \phi$ and any Σ-model of T is a Σ-model of $T_\forall$.

Suppose $\mathcal{M} \vDash_\Sigma T_\forall$. We first show that $T \cup \mathrm{AtDiag}(\Sigma, \mathcal{M})$ is consistent. By the compactness theorem it suffices to show that for any subset Δ of $\mathrm{AtDiag}(\Sigma, \mathcal{M})$, $T \cup \Delta$ is consistent. Write $\Delta = \{\psi_1, \ldots, \psi_n\}$. Let $\psi = \bigwedge_{1 \leq i \leq n} \psi_i$. We can find S that indexes the constant symbols in $\Sigma(\mathcal{M})_{\mathrm{con}} \setminus \Sigma_{\mathrm{con}}$ that appear in ψ (in the same way as we made indexing sets of the variables). Then we can create $\phi \in \Sigma_{\mathrm{for}}$ with variables indexed by S such that $\phi(c) = \psi$, where c is a list of constant symbols in $\Sigma(\mathcal{M})_{\mathrm{con}} \setminus \Sigma_{\mathrm{con}}$ indexed by S. Since $\Delta \subseteq \mathrm{AtDiag}(\Sigma, \mathcal{M})$ we have $\forall i, \mathcal{M} \vDash_\Sigma \psi_i$. Hence $\mathcal{M} \vDash_\Sigma \phi(c)$. We can then show that $\mathcal{M} \vDash_\Sigma \exists v, \phi(v)$ and so $\mathcal{M} \nvDash_\Sigma \forall v, \neg \phi(v)$.

Since each ψ_i is from the the atomic diagram of $\mathcal{M}$ they are all quantifier free. Thus ϕ is a quantifier free Σ-formula and $\forall v, \neg \phi(v) \in \Sigma_{\mathrm{uni}}$. Hence $T \nvDash_\Sigma \forall v, \neg \phi(v)$ by the definition of $T_\forall$. By the lemma on constants this implies that $T \nvDash_{\Sigma(\mathcal{M})} \neg \phi(a)$. Hence there exists a $\Sigma(\mathcal{M})$-model of $T \cup \phi(a)$. Then it follows that this is also a $\Sigma(\mathcal{M})$-model of $T \cup \Delta$. Thus $T \cup \Delta$ is consistent so $T \cup \mathrm{AtDiag}(\Sigma, \mathcal{M})$ is consistent.

Thus there exists a Σ-model $\mathcal{N}$ of $T \cup \mathrm{AtDiag}(\Sigma, \mathcal{M})$. This is a model of $\mathrm{AtDiag}(\Sigma, \mathcal{M})$ thus by there is a $\Sigma(\mathcal{M})$-embedding $\mathcal{M} \to \mathcal{N}$ there is a $\Sigma(\mathcal{M})$-embedding $\mathcal{M} \to \mathcal{N}$. We make this a Σ-embedding, hence using the theorem's hypothesis $\mathcal{M}$ is a Σ-model of T. □

The following result has doesn't come up at all until much later, but is included here as another demonstration of the lemma on constants in use. It appears as an exercise in the second chapter of Marker's book [1].

Corollary – Amalgamation

Let $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures, and suppose we have partial elementary Σ-embeddings $\iota_\mathcal{M} : A \to \mathcal{M}$ and $\iota_\mathcal{N} : A \to \mathcal{N}$, for $\varnothing \neq A \subseteq \mathcal{A}$. Then there exists a common elementary extension $\mathcal{L}$ of $\mathcal{M}$ and $\mathcal{N}$ such that the following commutes:

$$\begin{tikzcd} \mathcal{M} \arrow[r] & \mathcal{L} \\ A \arrow[u, "\iota_\mathcal{M}"] \arrow[r, "\iota_\mathcal{N}"] & \mathcal{N} \arrow[u] \end{tikzcd}$$

$\mathcal{L}$ is the 'amalgamation' of $\mathcal{M}$ and $\mathcal{N}$.

Proof. We show first that the theory $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \mathrm{ElDiag}(\Sigma, \mathcal{N})$ is consistent as a $\Sigma(\mathcal{M}, \mathcal{N})$-theory, where $\Sigma(\mathcal{M}, \mathcal{N})_{\mathrm{con}}$ is defined to be

$$\{c_a \mid a \in A\} \cup \{c_a \mid a \in \mathcal{M} \setminus \iota_\mathcal{M}(A)\} \cup \{c_a \mid a \in \mathcal{N} \setminus \iota_\mathcal{N}(A)\}$$

where terms and formulas from $\Sigma(A), \Sigma(\mathcal{M}), \Sigma(\mathcal{N})$ are interpreted in the natural way: the constants $c_{\iota_\mathcal{M}(a)} \mapsto c_a$ (similarly for $\mathcal{N}$). For the rest of the proof we identify $\Sigma(\mathcal{M})_{\mathrm{con}}$ with $\Sigma(A)_{\mathrm{con}} \cup \{c_a \mid a \in \mathcal{M} \setminus \iota_\mathcal{M}(A)\}$ (similarly with $\mathcal{N}$).

By the compactness theorem it suffices to show that for any finite subset $\Delta \subseteq \mathrm{ElDiag}(\Sigma, \mathcal{N})$, $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \Delta$ is consistent. Let ϕ be the $\Sigma(\mathcal{M})$-formula and $a \in \mathcal{N}^\star$ be such that [1]

$$\phi(a) = \bigwedge_{\psi \in \Delta} \psi$$

$\phi(a)$ is naturally a $\Sigma(\mathcal{M}, \mathcal{N})$-sentence such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M},\mathcal{N})} \phi(a)$.

Suppose for a contradiction $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \Delta$ is inconsistent. Then any $\Sigma(\mathcal{M}, \mathcal{N})$-model of $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ is not a model of Δ, which implies it does not satisfy $\phi(a)$. Hence

$$\mathrm{ElDiag}(\Sigma, \mathcal{M}) \vDash_{\Sigma(\mathcal{M},\mathcal{N})} \neg \phi(a)$$

By the lemma on constants applied to $\Sigma(\mathcal{M}) \leq \Sigma(\mathcal{M}, \mathcal{N})$, $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ and $a \in \Sigma(\mathcal{M}, \mathcal{N})_{\mathrm{con}} \setminus \Sigma(\mathcal{M})_{\mathrm{con}}$ we have

$$\mathrm{ElDiag}(\Sigma, \mathcal{M}) \vDash_{\Sigma(\mathcal{M})} \forall v, \neg \phi(v)$$

Noting that $\mathcal{M}$ is a Σ-model of its elementary diagram, and moving $\mathcal{M}$ down a signature we have that

$$\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \forall v, \neg \phi(v) \Rightarrow \mathcal{M} \vDash_{\Sigma(A)} \forall v, \neg \phi(v) \Rightarrow$$

Since $A \to \mathcal{M}$ and $A \to \mathcal{N}$ are partial elementary Σ-embeddings (and thus naturally $\Sigma(A)$-embeddings) we have that $\mathcal{A} \vDash_{\Sigma(A)} \forall v, \neg \phi(v)$ and so $\mathcal{N} \vDash_{\Sigma(A)} \forall v, \neg \phi(v)$. Move this up to $\Sigma(\mathcal{M}, \mathcal{N})$ we have a contradiction, by choosing v to be a: $\mathcal{N} \vDash_{\Sigma(\mathcal{M},\mathcal{N})} \neg \phi(a)$, but we remarked before that $\mathcal{N} \vDash_{\Sigma(\mathcal{M},\mathcal{N})} \phi(a)$.

Hence $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \mathrm{ElDiag}(\Sigma, \mathcal{N})$ is consistent as a $\Sigma(\mathcal{M}, \mathcal{N})$-theory. Let $\mathcal{L}$ be a $\Sigma(\mathcal{M}, \mathcal{N})$-model of this (and naturally a $\Sigma(\mathcal{M})$ or a $\Sigma(\mathcal{N})$ structure). Then there exist elementary $\Sigma(\mathcal{M})$ and $\Sigma(\mathcal{N})$-extensions $\lambda_{\mathcal{M}} : \mathcal{M} \to \mathcal{L}$ and $\lambda_{\mathcal{N}} : \mathcal{N} \to \mathcal{L}$ such that $\lambda_{\mathcal{M}}(c_{m_{\Sigma(\mathcal{M})}}^{\mathcal{M}}) = c_{m_{\Sigma(\mathcal{M})}}^{\mathcal{L}}$ for each constant symbol c_m for $m \in \mathcal{M} \setminus \iota_{\mathcal{M}}(A)$ and $\lambda_{\mathcal{M}}(c_{a_{\Sigma(\mathcal{M})}}^{\mathcal{M}}) = c_{a_{\Sigma(\mathcal{M})}}^{\mathcal{L}}$ for each constant symbol c_a for $a \in A$ (similarly with $\mathcal{N}$).

Naturally, we can move everything down to $\Sigma(A)$. Thus for any $a \in A$

$$\lambda_{\mathcal{M}} \circ \iota_{\mathcal{M}}(a) = \lambda_{\mathcal{M}}(c_{a_{\Sigma(A)}}^{\mathcal{M}}) = \lambda_{\mathcal{M}}(c_{a_{\Sigma(\mathcal{M})}}^{\mathcal{M}}) = c_{a_{\Sigma(\mathcal{M})}}^{\mathcal{L}} = c_{a_{\Sigma(A)}}^{\mathcal{L}}$$

By symmetry we have $\lambda_{\mathcal{M}} \circ \iota_{\mathcal{M}}(a) = c_{a_{\Sigma(A)}}^{\mathcal{L}} = \lambda_{\mathcal{N}} \circ \iota_{\mathcal{N}}(a)$. □

1.1.8 The Löwenheim-Skolem Theorems

The results in this subsection aren't used until much later.It is worth skipping for now, but the material can be covered with the foundations made so far.

> **Proposition – Upward Löwenheim-Skolem Theorem**
>
> If $\mathcal{M}$ is an infinite Σ-structure and κ a cardinal such that $|\mathcal{M}| + |\Sigma_{\mathrm{con}}| \leq \kappa$, there exists a Σ-structure $\mathcal{N}$ with cardinality κ as well as an elementary Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$.

Proof. Make $\mathcal{M}$ a $\Sigma(\mathcal{M})$ structure as in the definition of $\Sigma(\mathcal{M})$, then clearly $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$. Thus $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ is a $\Sigma(\mathcal{M})$-theory with an infinite model $\mathcal{M}$, hence there exists a $\Sigma(\mathcal{M})$-structure $\mathcal{N}$ of cardinality κ such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$. Hence we can make $\mathcal{N}$ a Σ-structure and find a Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$. □

> **Definition – Skolem Functions**
>
> We say that a Σ-theory T *has built in Skolem functions* when for any Σ-formula ϕ that is not a sentence,

[1] Take out all the finitely many constants appearing from $\mathcal{N} \setminus \iota_{\mathcal{N}}(A)$ in Δ and make them into a tuple a, replacing them with free variables. What remains is a finite set of $\Sigma(A)$-formulas, which are naturally also $\Sigma(\mathcal{M})$-formulas. We take the 'and' of all of them to be ϕ.

with free variables indexed by S, there exists $f \in \Sigma_{\text{fun}}$ such that $n_f = k$ and

$$T \vDash_\Sigma \bigforall_{i \in S} w_i, (\exists v, \phi(v, w) \to \phi(f(w), w)),$$

Note that w can be length 0, in which case f has arity 0 and so would be interpreted as a constant map. We would have

$$T \vDash_\Sigma \exists v, \phi(v) \to \phi(f)$$

Proposition – Skolemization

Let $T(0)$ be a $\Sigma(0)$-theory, then there exists T a Σ theory such that

1. $|\Sigma_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$
2. $\Sigma(0)_{\text{fun}} \subseteq \Sigma_{\text{fun}}$, and they share the same constant and relation symbols
3. $T(0) \subseteq T$
4. T has built in Skolem functions
5. All models of $T(0)$ can be moved up to being models of T with interpretations agreeing on Σ.

We call T the Skolemization of $T(0)$.

Proof. Similarly to the Witness Property proof, we define $\Sigma(i), T(i)$ for each $i \in \mathbb{N}$. Suppose by induction that we have $T(i) \in \Sigma_{\text{the}}$, such that

1. $|\Sigma(i)_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$
2. $\Sigma(0)_{\text{fun}} \subseteq \Sigma(i)_{\text{fun}}$ and they share the same constant and relation symbols
3. $T(0) \subseteq T(i)$
4. All models of $T(0)$ can be moved up to being models of $T(i)$ with interpretations agreeing on Σ

Then define $\Sigma(i+1)$ such that only the function symbols are enriched:

$$\Sigma(i+1)_{\text{fun}} := \Sigma(i)_{\text{fun}} \cup \{f_\phi \mid \phi \in \Sigma(i)_{\text{for}} \text{ and } \phi \text{ is not a sentence}\}$$

extending the arity $n_\star$ to by having $n_{f_\phi} = |S| - 1$, where S is indexes the free variables of ϕ. There are countably infinite $\Sigma(i)$-formulas, thus $|\Sigma(i)_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$.

Define $\Psi : \Sigma(i)_{\text{for}} \to \Sigma(i+1)_{\text{for}}$ mapping

$$\phi \mapsto \forall w, (\exists v, \phi(v, w)) \to (\phi(f_\phi(w), w)),$$

where w is a list of variables of the suitable length. We then define

$$T(i+1) := T(i) \cup \Psi(\Sigma(i)_{\text{for}})$$

which is a $\Sigma(i+1)$-theory because the image of Ψ has only Σ-sentences. Note that $T(0) \subseteq T(i) \subseteq T(i+1)$.

Let $\mathcal{M}(0)$ be a $\Sigma(0)$-model of $T(0)$, then we have $\mathcal{M}(i)$ a $\Sigma(i)$-model of $T(i)$ with the same carrier set and same interpretation on $\Sigma(0)$ as $\mathcal{M}(0)$. Let $\mathcal{M}(i+1)$ have the same carrier set as $\mathcal{M}(0)$. To extend interpretation to $\Sigma(i+1)$, we first deal with the case where $\mathcal{M}(i)$ is empty by simply interpreting all new function symbols as the empty function. Otherwise we have a $c \in \mathcal{M}(0)$. For $f_\phi \in \Psi(\Sigma(i)_{\text{for}})$ define

$$f_\phi^{\mathcal{M}(i+1)} : \mathcal{M}(i+1)^{n_{f_\phi}} \to \mathcal{M}(i+1)$$
$$a \mapsto \begin{cases} b & \text{, if } \exists b \in \mathcal{M}, \mathcal{M}(i) \vDash_{\Sigma(i)} \phi(b, a) \\ c & \text{, otherwise} \end{cases}$$

Then by construction, $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \Psi(\Sigma(i)_{\text{for}})$. By checking the conditions on moving $\mathcal{M}(i)$ up to $\mathcal{M}(i+1)$, we can also conclude $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} T(i)$. Hence $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} T(i+1)$.

Let $\Sigma(*)$ be the signature such that its constants and relations agree with $\Sigma(0)$ and $\Sigma(*)_{\text{fun}} = \bigcup_{i\in\mathbb{N}} \Sigma(i)_{\text{fun}}$. Then

$$|\Sigma(*)_{\text{fun}}| = |\bigcup_{i\in\mathbb{N}} \Sigma(i)_{\text{fun}}| = \aleph_0 \times (\aleph_0 + \Sigma(0)_{\text{fun}}) = \aleph_0 + \Sigma(0)_{\text{fun}}$$

Let $T(*) = \bigcup_{i\in\mathbb{N}} T(i)$. We show that $T(*)$ has built in Skolem functions. Let ϕ be a $\Sigma(*)$-formula that is not a Σ-sentence. Then $\phi \in \Sigma(i)_{\text{for}}$ for some $i \in \mathbb{N}$. Thus $\Psi(\phi) \in T(i+1) \subseteq T(*)$, hence

$$T(*) \vDash_{\Sigma(*)} \forall w, (\exists v, \phi(v,w)) \to (\phi(f_\phi(w), w))$$

Thus $T(*)$ has built in Skolem functions.

If $\mathcal{M} \vDash_\Sigma T$ then let $\mathcal{M}(*) = \mathcal{M}$ and define the interpretation such that for all $i \in \mathbb{N}$, and $f \in T(i)$, $f^{\mathcal{M}(*)}_{\Sigma(*)} = f^{\mathcal{M}(i)}_{\Sigma(i)}$. Since all interpretations agree upon intersection this is well-defined. To show that $\mathcal{M}(*)$ is a $\Sigma(*)$-model of $T(*)$, let ϕ be in $T(*)$; there is some $i \in \mathbb{N}$ such that $\phi \in T(i)$. Using our lifted $\mathcal{M}(i)$ from before we have $\mathcal{M}(i) \vDash_{\Sigma(i)} \phi$. By checking the conditions on moving $\mathcal{M}(i)$ up to $\mathcal{M}(*)$, we can also conclude $\mathcal{M}(*) \vDash_{\Sigma(*)} \phi$ (by taking the $\Sigma(*)$ theory $\{\phi\}$). Hence $\mathcal{M}(*) \vDash_{\Sigma(*)} T(*)$. □

Definition – Theory of a Structure
We define the theory of a Σ-structure $\mathcal{M}$ to be

$$\mathrm{Th}_{\mathcal{M}} := \{\phi \in \Sigma_{\text{for}} \,|\, \phi \text{ is a } \Sigma\text{-sentence and } \mathcal{M} \vDash_\Sigma \phi\}$$

Proposition – Downward Löwenheim-Skolem Theorem
Let $\mathcal{N}$ be a $\Sigma(0)$-structure and $M(0) \subseteq \mathcal{N}$. Then there exists a $\Sigma(0)$-structure $\mathcal{M}$ such that

- $M(0) \subseteq \mathcal{M} \subseteq \mathcal{N}$
- $|\mathcal{M}| \leq |M(0)| + |\Sigma(0)_{\text{fun}}| + \aleph_0$
- The inclusion $\subseteq : \mathcal{M} \to \mathcal{N}$ is an elementary embedding.

Proof. We first take the Skolemization of $\mathrm{Th}_{\mathcal{N}}$ and call the new signature and theories Σ and T. Since $\mathcal{N} \vDash_{\Sigma(0)} \mathrm{Th}_{\mathcal{N}}$, we can move it up to being a Σ-structure so that $\mathcal{N} \vDash_\Sigma T$.

We want to create the carrier set of $\mathcal{M}$, it has to be big enough so that interpreted functions are closed on $\mathcal{M}$. Given $M(i)$ such that $|M(i)| \leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0$, we inductively define $M(i+1)$:

$$M(i+1) := M(i) \cup \{f^{\mathcal{N}}_\Sigma(a) \,|\, f \in \Sigma_{\text{fun}} \wedge a \in M(i)^{n_f}\}$$

Then

$$\begin{aligned}
|M(i+1)| &\leq |M(i)| + |\Sigma_{\text{fun}}| \times |M(i)^{n_f}| \\
&\leq |M(i)| + |\Sigma_{\text{fun}}| \times (|M(i)| + \aleph_0) \\
&\leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0 + |\Sigma_{\text{fun}}| \times (|M(0)| + |\Sigma_{\text{fun}}| + \aleph_0) \\
&\leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0
\end{aligned}$$

Then $\mathcal{M} := \bigcup_i M(i)$ and $|\mathcal{M}| \leq |M(i)| \times \aleph_0 = |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0 \leq |M(0)| + |\Sigma(0)_{\text{fun}}| + \aleph_0$.

We first interpret function symbols, which will give us a way to interpret constant symbols. For $f \in \Sigma_{\text{fun}}$ and $a \in (\mathcal{M})^{n_f}$, define $f^{\mathcal{M}}_\Sigma(a) = f^{\mathcal{N}}_\Sigma(a)$. This is well-defined as there exists $i \in \mathbb{N}$ such that $a \in (M(i))^{n_f}$,

$$f^{\mathcal{M}}_\Sigma(a) \in M(i+1) \subseteq \mathcal{M}$$

Then to interpret constant symbols, we consider for each $c \in \Sigma_{\text{con}}$ the formula $v = c$. Since T has built in Skolem functions and $\mathcal{N} \vDash_\Sigma T$, there exists f with arity $n_f = 0$ such $\mathcal{N} \vDash_\Sigma (\exists v, v = c) \to f = c$. Since $\mathcal{N} \vDash_\Sigma \exists v, v = c$, we have $f_\Sigma^{\mathcal{N}} = c_\Sigma^{\mathcal{N}}$. Since $f_\Sigma^{\mathcal{N}} = f_\Sigma^{\mathcal{M}} : (\mathcal{M})^0 \to \mathcal{M}$ we can define $c_\Sigma^{\mathcal{M}} = c_\Sigma^{\mathcal{N}} = f_\Sigma^{\mathcal{N}} = f_\Sigma^{\mathcal{M}} \in \mathcal{M}$.

Lastly define the interpretation of relations as $r_\Sigma^{\mathcal{M}} = (\mathcal{M})^{m_r} \cap r_\Sigma^{\mathcal{N}}$.

By construction the inclusion $\subseteq$ is a Σ-embedding. We check that it is elementary using the third equivalent condition in the Tarski Vaught Test: let $\phi \in \Sigma_{\text{for}}$ with free variables indexed by S, $i \in S$ and $a \in (\mathcal{M})^{S \setminus \{i\}}$. Suppose $\exists c \in \mathcal{N}, \mathcal{N} \vDash_\Sigma \phi(a, c)$. T has built in Skolem functions, and $\mathcal{N} \vDash_\Sigma T$. Hence there exists $f \in \Sigma_{\text{fun}}$ such that

$$\mathcal{N} \vDash_\Sigma (\exists v, \phi(a, v)) \to \phi(a, f(a))$$

We can deduce $\mathcal{N} \vDash_\Sigma \phi(a, f(a))$. Noting that $f_\Sigma^{\mathcal{M}}(a) = f_\Sigma^{\mathcal{N}}(a)$ completes the Tarski Vaught Test. Hence $\subseteq$ is an elementary Σ-embedding.

We move $\subseteq : \mathcal{M} \to \mathcal{N}$ down a signature since by Skolemization we have $\Sigma(0) \leq \Sigma$. Then $\subseteq : \mathcal{M} \to \mathcal{N}$ is an elementary Σ-embedding. $\square$

Chapter 2

Model Theory of Fields

2.1 Ax-Grothendieck

This section studies the theories of fields in the language of rings, with particular focus on algebraically closed fields.

2.1.1 Language of Rings

We introduce rings and fields and construct the field of fractions of integral domains to see the models in action.

> **Definition – Signature of rings, theory of rings**
> We define $\Sigma_{\mathrm{RNG}} := (\{0,1\}, \{+,-,\cdot\}, n_\star, \varnothing, m_\star)$ to be the signature of rings, where $n_+ = n_\cdot = 2$, $n_- = 1$ and $m_\star$ is the empty function.
>
> Using the obvious abbreviations $x + (-y) = x - y, x \cdot y = xy$ and so on, we define the theory of rings RNG as the set containing:
>
> | Assosiativity of addition: $\forall x \forall y \forall z, (x+y)+z = x+(y+z)$
>
> | Identity for addition: $\forall x, x + 0 = x$
>
> | $\forall x, x - x = 0$
>
> | Commutativity of addition: $\forall x \forall y, x + y = y + x$
>
> | Assosiativity of multiplication: $\forall x \forall y \forall z, (x \cdot y) \cdot z = x \cdot (y \cdot z)$
>
> | Identity for multiplication: $\forall x, x \cdot 0 = x$
>
> | Commutativity of multiplication: $\forall x \forall y, x \cdot y = y \cdot x$
>
> | Distributivity: $\forall x \forall y \forall z, x \cdot (y + z) = x \cdot y + x \cdot z$
>
> Note that we don't have axioms for closure of functions and existence or uniqueness of inverses as it is encoded by interpretation of $+, -, \cdot$ being well-defined. Note that the theory of rings is universal.

> **Definition – Theory of integral domains and fields**
> We define the Σ_{RNG}-theory of integral domains
>
> $$\mathrm{ID} := \mathrm{RNG} \cup \{0 \neq 1, \forall x \forall y, xy = 0 \to (x = 0 \lor y = 0)\}$$

and the Σ_{RNG}-theory of fields

$$\mathrm{FLD} := \mathrm{RNG} \cup \{\forall x, x = 0 \vee \exists y, xy = 1\}$$

Note that the theory of integral domains is universal but the theory of fields is not.

Proposition – Field of fractions

Suppose $\mathcal{A} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$. Then there exists an Σ_{RNG}-embedding $\iota : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{B} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$. We call $\mathcal{B}$ the field of fractions.

Proof. We construct $X = \left\{(x, y) \in \mathcal{A}^2 \,|\, y \neq 0\right\}$ and and equivalence relation $(x, y) \sim (v, w) \Leftrightarrow xw = yv$. (Use $\mathcal{A} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ to show that this is an equivalence relation.) Let $\mathcal{B} = X/\sim$ with $\pi : X \to \mathcal{B}$ as the quotient map. Denote $\pi(x, y) := \frac{x}{y}$, interpret $0^{\mathcal{B}} = \frac{0^{\mathcal{A}}}{1^{\mathcal{A}}}$ and $1^{\mathcal{B}} = \frac{1^{\mathcal{A}}}{1^{\mathcal{A}}}$. Interpret $+$ and $\cdot$ as standard fraction addition and multiplication and use $\mathcal{A} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ to check that these are well defined.

Check that $\mathcal{B}$ is an Σ_{RNG} structure and that $\mathcal{B} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$. Define $\iota : \mathcal{A} \to \mathcal{B} := a \mapsto \frac{a}{1}$ and show that this well defined and injective. Check that ι is a Σ_{RNG}-morphism and note that since there are no relation symobls in Σ_{RNG} it is also an embedding. □

Proposition – Universal property of field of fractions

Suppose $A \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ and K its field of fractions. Then if $L \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$ and there exists a Σ_{RNG}-embedding $\iota_L : A \to L$, then there exists a unique Σ_{RNG}-embedding $K \to L$ that commutes with the other embeddings:

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & K \arrow[d, dashed] \\ & L \end{tikzcd}$$

Proof. Define the map $\iota : K \to L$ sending $\frac{a}{b} \mapsto \frac{\iota_L(a)}{\iota_L(b)}$. Check that this is well-defined and a Σ_{RNG}-morphism. It is injective because ι_L is injective:

$$\frac{\iota_L(a)}{\iota_L(b)} = 0 \Rightarrow \iota_L(a) = 0 \Rightarrow a = 0$$

Thus it is an embedding.

It is unique: suppose $\phi : K \to L$ is a Σ_{RNG}-embedding that commutes with the diagram. Then for any $a \in K$, $\phi(\frac{a}{1}) = \iota_L(a) = \iota(\frac{a}{1})$. Since both ϕ, ι are embeddings they commute with taking the inverse for $a \neq 0$: $\phi(\frac{1}{a}) = \iota(\frac{1}{a})$. Since any element of K can be written as $\frac{a}{b}$, we have shown that $\phi = \iota$. □

2.1.2 Algebraically closed fields

Definition – Theory of algebraically closed fields

We define the Σ_{RNG} theory of algebraically closed fields

$$\mathrm{ACF} := \mathrm{FLD} \cup \left\{ \bigforall_{i=0}^{n-1} a \exists x,\ x^n + \sum_{i=0}^{n-1} a_i x^i = 0 \,|\, n \in \mathbb{N}_{>0}, a \in {\Sigma_{\mathrm{RNG}}}_{\mathrm{var}}^{n-1} \right\}$$

Unlike the theories $\mathrm{RNG}, \mathrm{ID}, \mathrm{FLD}$ this theory is countably infinite.

Proposition
ACF is not complete.

Proof. Take the Σ_{RNG}-formula $\forall x, x+x=0$. This is satisfied by the algebraic closure of $\mathbb{F}_2$ but not by that of $\mathbb{F}_3$, since field embeddings preserve characteristic. □

Definition – Algebraically closed fields of characteristic p
For $p \in \mathbb{Z}_{>0}$ prime define

$$\phi_p := \forall x, \sum_{i=1}^{p} x = 0$$

and let $\text{ACF}_p := \text{ACF} \cup \{\phi_p\}$. Furthermore, let

$$\text{ACF}[0] := \text{ACF} \cup \{\neg\, \phi_p \,|\, p \in \mathbb{Z}_{>0} \text{ prime}\}$$

An important fact about algebraically closed fields of characteristic p:

Proposition – Transcendence degree and characteristic determine algebraically closed fields of characteristic p up to isomorphism
If K_0, K_1 are fields with same characteristic and transcendence degree over their minimal subfield ($\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$) then they are (non-canonically) isomorphic.

Proof. See appendix. □

NOTATION. *If $K \vDash_{\Sigma_{\text{RNG}}} ACF[p]$, write $\text{t.deg}(K)$ to mean the transcendence degree over its minimal subfield ($\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$).*

Lemma – Cardinality of algebraically closed fields
If L is an algebraically closed field then it has cardinality $\aleph_0 + \text{t.deg}(L)$.

Proof. Let S be a transcendence basis and call the minimal subfield K. Since L is algebraically closed it splits the seperable polynomials $x^n - 1$ for each n. Hence L is infinite. Also $S \subseteq L$ and so $\aleph_0 + \text{t.deg}(L) \leq |L|$. For the other direction note that

$$M = \bigcup_{f \in I} \{a \in M \,|\, f = \min(a, K)\}$$

where $I \subseteq K(S)[x]$ is the set of monic and irreducible polynomials over $K(S)$. Thus

$$\begin{aligned}
|M| &\leq |I| \times \aleph_0 \leq |K(S)[x]| \times \aleph_0 \\
&\leq |K(S)| \times \aleph_0 \leq |K[S]| \times |K[S]| \times \aleph_0 \\
&= |K[S]| \times \aleph_0 \leq \left| \bigcup_{n \in \mathbb{N}} (K \cup S)^n \right| \times \aleph_0 \\
&= \left| \bigcup_{n \in \mathbb{N}} K \cup S \right| \times \aleph_0 = |K \cup S| \times \aleph_0 \\
&= |S| \times \aleph_0
\end{aligned}$$

Noting that $K = \mathbb{Q}$ or $\mathbb{F}_p$ and so is at most countable. By Schröder–Bernstein we have $\aleph_0 + \text{t.deg}(L) = |L|$. □

Proposition

ACF_p is κ-categorical for uncountable κ, consistent and complete.

Proof. Suppose $K, L \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ACF}_p$ and $|K| = |L| = \kappa$. Then $\mathrm{t.deg}(K) + \aleph_0 = |K| = \kappa$ and so $\mathrm{t.deg}(K) = \kappa$ (as κ is uncountable). Similarly $\mathrm{t.deg}(L) = \kappa$ and so $\mathrm{t.deg}(K) = \mathrm{t.deg}(L)$. Thus K and L are isomorphic.

ACF_p is consistent due to the existence of the algebraic closures for any characteristic, it is not finitely modelled and is $\aleph_1$-categorical with $\Sigma_{\mathrm{RNG}\,\mathrm{con}} + \aleph_0 \leq \aleph_1$, hence it is complete by Vaught's test. □

2.1.3 Ax-Grothendieck

Proposition – Lefschetz principle

Let ϕ be a Σ_{RNG}-sentence. Then the following are equivalent:

1. There exists a Σ_{RNG}-model of ACF_0 that satisfies ϕ. (If you like $\mathbb{C} \vDash_{\Sigma_{\mathrm{RNG}}} \phi$.)
2. $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \phi$
3. There exists $n \in \mathbb{N}$ such that for any prime p greater than n, $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \phi$
4. There exists $n \in \mathbb{N}$ such that for any prime p greater than n there exists a non-empty Σ_{RNG}-model of ACF_p that satisfies ϕ.

Proof.

1. $\Rightarrow$ 2. If $\mathbb{C} \vDash_{\Sigma_{\mathrm{RNG}}} \phi$ then since ACF_0 is complete $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \phi$ or $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \neg\phi$. In the latter case we obtain a contradiction.

2. $\Rightarrow$ 3. Suppose $\mathrm{ACF}[0] \vDash_{\Sigma_{\mathrm{RNG}}} \phi$ then since 'proofs are finite' there exists a finite subset Δ of $\mathrm{ACF}[0]$ such that $\Delta \vDash_{\Sigma_{\mathrm{RNG}}} \phi$. Let n be maximum of all $q \in \mathbb{N}$ such that $\neg\phi_q \in \Delta$. By uniqueness of characteristic, if p is prime and greater than n and q is prime such that $\neg\phi_q \in \Delta$ then $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \neg\phi_q$. Thus if $\mathcal{M}$ is a Σ_{RNG}-model of ACF_p then $\mathcal{M} \vDash_{\Sigma_{\mathrm{RNG}}} \Delta$ and so $\mathcal{M} \vDash_{\Sigma_{\mathrm{RNG}}} \phi$. Hence for all primes p greater than n, $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \phi$.

3. $\Rightarrow$ 4. ACF_p is consistent thus there exists a non-empty Σ_{RNG}-model of ACF_p. Our hypothesis implies it satisfies ϕ.

4. $\Rightarrow$ 1. Let $n \in \mathbb{N}$ such that for any prime p greater than n there exists a non-empty Σ_{RNG}-model of ACF_p that satisfies ϕ. Then because ACF_p is complete $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \phi$. Suppose for a contradiction $\mathrm{ACF}_0 \nvDash_{\Sigma_{\mathrm{RNG}}} \phi$. Then by completeness $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \neg\phi$. Hence by the above we obtain there exists m such that for all p greater than m, $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \neg\phi$. Then since there are infinitely many primes, take p greater than both m and n, then ACF_p is inconsistent, a contradiction. Hence $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \phi$ and in particular $\mathbb{C} \vDash_{\Sigma_{\mathrm{RNG}}} \phi$.

□

Lemma – Ax-Grothendieck for algebraic closures of finite fields

If Ω is an algebraic closure of a finite field then any injective polynomial map over Ω is surjective.

Proof. See appendix. □

Lemma – Construction of Ax-Grothendieck formula

There exists a Σ_{RNG}-sentence $\Phi_{n,d}$ such that for any field K, $K \vDash_{\Sigma} \Phi_{n,d}$ if and only if for all $d, n \in \mathbb{N}$ any injective polynomial map $f : K^n \to K^n$ of degree less than or equal to d is surjective.

Proof. We first need to be able to express polynomials in n varibles of degree less than or equal to d in an elementary way. We first note that for any $n, d \in \mathbb{N}$ there exists a finite set S and powers $r_{s,j} \in \mathbb{N}$ (for each $(s, j) \in S \times \{1, \ldots, n\}$). such that any polynomial $f \in K[x_1, \ldots, x_n]$ can be written as

$$\sum_{s \in S} \lambda_s \prod_{j=1}^{n} x_j^{r_{s,j}}$$

for some $\lambda_s \in K$. Now we have a way of quantifying over all such polynomials, which is by quantifying over all the coefficients. We define $\Phi_{n,d}$:

$$\Phi_{n,d} := \forall_{i=1}^{n} \forall_{s \in S} \lambda_{s,i}, \left[\forall_{j=1}^{n} x_j \forall_{j=1}^{n} y_j, \bigwedge_{i=1}^{n} \left(\sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} x_j^{r_{s,j}} = \sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} y_j^{r_{s,j}} \right) \longrightarrow \bigwedge_{i=1}^{n} x_i = y_i \right]$$
$$\longrightarrow \forall_{j=1}^{n} x_j, \exists_{i=1}^{n} z_i, \bigwedge_{i=1}^{n} \left(z_i = \sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} x_j^{r_{s,j}} \right)$$

At first it quantifies over all of the coefficients of all the f_i. The following part says that if the polynomial map is injective then it is surjective. Thus $K \models_{\Sigma} \Phi_{n,d}$ if and only if for all $d, n \in \mathbb{N}$ any injective polynomial map $f : K^n \to K^n$ of degree less than or equal to d is surjective. $\square$

> **Proposition – Ax-Grothendieck**
> If K is an algebraically closed field of characterstic 0 then any injective polynomial map over K is surjective. In particular injective polynomial maps over $\mathbb{C}$ are surjective.

Proof. We show an equivalent statement: for any $n, d \in \mathbb{N}$, any injective polynomial map $f : K^n \to K^n$ of degree less than or equal to d is surjective. This is true if and only if $K \models_{\Sigma_{\text{RNG}}} \Phi_{n,d}$ (by construction of the A-G formula) which is true if and only if for all p prime greater than some natural number there exists an algebraically closed field of characteristic p that satisfies $\Phi_{n,d}$, by the Lefschetz principle. Indeed, take the natural 0 and let p be a prime greater than 0. Take Ω an algebraic closure of $\mathbb{F}_p$, which indeed models ACF_p. $\Omega \models_{\Sigma_{\text{RNG}}} \Phi_{n,d}$ if and only if for any $n, d \in \mathbb{N}$, any injective polynomial map $f : \Omega^n \to \Omega^n$ of degree less than or equal to d is surjective (by construction of the A-G formula). The final statement is true due to A-G for algebraic closures of finite fields. $\square$

Bibliography

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