

Number Theory Notes

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Date

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Chapter 1

Finite Fields

1.1 Generalities

1.1.1 Finite fields

Definition – Characteristic of a field

If K is a field then the map $\mathbb{Z} \rightarrow K$ induced by $1 \mapsto 1$ is a ring morphism. The image of this morphism is an integral domain since K is a field, hence the kernel is a prime ideal. Since \mathbb{Z} is a PID, we can define the characteristic of K , denoted $\text{Char}(K)$ to be the positive generator of the kernel.

Proposition – Frobenius map

If K is a field and $\text{Char}(K)$ is prime then

$$\sigma_p : K \rightarrow K \quad := \quad x \mapsto x^p$$

is an injection.

Proof. Easy to show $\sigma_p(0) = 0, \sigma_p(1) = 1$. Also

$$\sigma_p(ab) = (ab)^p = a^p b^p = \sigma_p(a) \sigma_p(b)$$

$$\sigma_p(a + b) = (a + b)^p = a^p + b^p = \sigma_p(a) + \sigma_p(b)$$

by expanding the binomial and noting that when $1 \leq k \leq p$, $p \mid \binom{p}{k} k!(p-k)!$ and is coprime to the latter two, thus $p \mid \binom{p}{k}$. Since σ_p is a morphism of fields it is injective. \square

Proposition – Classification of finite fields

Let K be a finite field and suppose $\Omega \models \text{ACF}_p$ where p is prime and q is a non-trivial power of p . Then

1. $\text{Char}(K) \neq 0$ and $|K| = p^{[K:\mathbb{F}_p]}$

2. $\mathbb{F}_q := \{x \in \Omega \mid x^q = x\}$ is the unique subfield of Ω with q elements.
3. If $|K| = q$ then $K \cong \mathbb{F}_q$.

Proof.

1. If $\text{Char}(K) = 0$ then \mathbb{Z} injects into K thus $\aleph_0 \leq |\mathbb{Z}| \leq |K|$ which is false. Since $[K : \mathbb{F}_p]$ is the cardinality of any basis B of K as a vector space over \mathbb{F}_p and $K \cong \mathbb{F}_p^B$, $|K| = |\mathbb{F}_p^B| = p^{|B|} = p^{[K:\mathbb{F}_p]}$.
2. Easy to show elementarily that \mathbb{F}_q is a subfield. As polynomials over a field are separable if and only the gcd of the derivative and the polynomial is 1,

$$D(X^q - X) = qX^{q-1} - 1 = -1$$

Hence it has q distinct roots in the algebraic closure of Ω , namely Ω itself. Hence $|\mathbb{F}_q| = q$. Uniqueness: if $L \subseteq \Omega$ and $|L| = q$ then for any unit $x \in L \setminus \{0\}$, $x^{q-1} = 1$ by Lagrange and so $x \in \mathbb{F}_q$. Thus $L \subseteq \mathbb{F}_q$ and they have equal finite cardinality, so $L = \mathbb{F}_q$.

3. If L is a field such that $|L| = q$ then the image of \mathbb{Z} in L has cardinality dividing q by Lagrange. Hence $\text{Char}(L) = p$ and the image of \mathbb{Z} is \mathbb{F}_p . Finitely generate L over \mathbb{F}_p and for each generator a the minimal polynomial of a over \mathbb{F}_p splits in Ω since it is algebraically closed. By ‘embedding finite extensions via conjugates’ in Galois Theory, there is a map $L \rightarrow F_q$ which is injective. It is an isomorphism since they have the same finite cardinality.

□

1.1.2 Multiplicative group of a finite field

Definition – Euler’s Totient Function

If $1 \leq a \leq d$ in \mathbb{Z} then a is coprime to d if and only if $\bar{a} \in \mathbb{Z}/d\mathbb{Z}$ is a generator since

$$\begin{aligned} (a, d) &= 1 \\ \Leftrightarrow \exists \lambda, \mu \in \mathbb{Z}, \lambda a + \mu d &= 1 \\ \Leftrightarrow \exists \lambda \in \mathbb{Z}, \overline{\lambda a} &= 1 \\ \Leftrightarrow \langle \bar{a} \rangle &= \mathbb{Z}/d\mathbb{Z} \end{aligned}$$

We define Euler’s totient function

$$\phi(d) := |\{a \in \mathbb{Z}/d\mathbb{Z} \mid \langle a \rangle = \mathbb{Z}/d\mathbb{Z}\}| = |\{a \in \mathbb{Z} \mid 1 \leq a \leq d \wedge (a, d) = 1\}|$$

NOTATION. For any cyclic group G , let $\Phi(G) = \{g \in G \mid \langle g \rangle = G\}$ be the set of generators.

Proposition – Partitioning cyclic groups

If $n \in \mathbb{Z}_{>0}$ then $n = \sum_{d|n} \phi(d)$.

Proof. Let $n \in \mathbb{Z}_{>0}$ and let d divide n . Then by some cyclic group theory there exists a unique cyclic subgroup $C_d \leq \mathbb{Z}/n\mathbb{Z}$ with cardinality d . We want to show that $\mathbb{Z}/n\mathbb{Z} = \bigsqcup_{d|n} \Phi(C_d)$. Indeed if $x \in \mathbb{Z}/n\mathbb{Z}$ then $\langle x \rangle$ has some order d dividing n by Lagrange. Hence $x \in \Phi(\langle x \rangle) = \Phi(C_d)$. Thus $\mathbb{Z}/n\mathbb{Z} \subseteq \cup_{d|n} \Phi(C_d)$.

To show it is disjoint notice that if x is in $\Phi(C_d) \cap \Phi(C_e)$ then d and e are both the order of x . \square

Proposition – Sufficient condition for cyclic

Let G be a group such that for any $d \mid |G|$,

$$|\{x \in G \mid x^d = e\}| \leq d$$

then G is cyclic.

Proof. We show that for all divisors of $|G|$ there is an element of G of that order. Then in particular $|G| \mid |G|$ and so there is a generator of G .

Let $d \mid |G|$. Consider $\{x \in G \mid x \text{ has order } d\}$. If it is non-empty, then take such an x :

$$\langle x \rangle \subseteq \{g \in G \mid g^d = e\}$$

and so $d \leq |\langle x \rangle| \leq |\{g \in G \mid g^d = e\}| \leq d$. Then $\langle x \rangle = \{g \in G \mid g^d = e\}$. Hence for $g \in G$,

$$\begin{aligned} g \text{ has order } d &\Leftrightarrow g \text{ has order } d \wedge g^d = e \\ &\Leftrightarrow g \text{ has order } d \wedge g \in \langle x \rangle \\ &\Leftrightarrow \langle g \rangle = \langle x \rangle \end{aligned}$$

Hence $|\{x \in G \mid x \text{ has order } d\}| = \phi(d)$. In either case, (empty or not), $|\{x \in G \mid x \text{ has order } d\}| \leq \phi(d)$

Assume for a contradiction that there exists a d such that $\{x \in G \mid x \text{ has order } d\}$ is empty. Then partitioning

$$G = \bigsqcup_{d \mid |G|} \{x \in G \mid x \text{ has order } d\}$$

we have that

$$|G| = \sum_{d \mid |G|} |\{x \in G \mid x \text{ has order } d\}| < \sum_{d \mid |G|} \phi(d) = |G|$$

a contradiction. \square

Proposition – \mathbb{F}_q^* is cyclic

Suppose $d \mid |\mathbb{F}_q^*|$. Then since $\mathbb{F}_q[X]$ has division algorithm,

$$|\{x \in \mathbb{F}_q^* \mid x^d = 1\}| \leq d$$

Hence \mathbb{F}_q^* is cyclic.

1.2 Equations over a finite field

Proposition

Power sums lemma Let $u \in \mathbb{N}$ and K be field with $|K| = q$ a power of a non-trivial prime. Then

$$\sum_{x \in K} x^u = \begin{cases} -1 & , 1 \leq u \wedge q-1 \mid u \\ 0 & , \text{otherwise} \end{cases}$$

Proof. Case $u = 0$ then $\sum_{x \in K^n} x^u = \sum_{x \in K^n} 0 = 0$.

Case $1 \leq u \wedge q-1 \mid u$ then for some d ,

$$\sum_{x \in K} x^u = \sum_{x \in K} (x^{q-1})^d = \sum_{x \in K^*} 1^d = (q-1)1 = -1$$

Case $1 \leq u \wedge q-1 \nmid u$ then there exist $d, r \in \mathbb{N}$ such that $u = (q-1)d + r$ and $0 < r < q-1$. Let y be a generator of K^* (K^* is cyclic). Then suppose for a contradiction that $y^u = 1$, then $q-1 \mid u$ since $q-1$ is the order of y , a contradiction. Multiplying by y is a bijection on the group, hence

$$\sum_{x \in K^n} x^u = \sum_{x \in K^n} (yx)^u = y^u \sum_{x \in K^n} x^u$$

Thus $(1 - y^u) \sum_{x \in K^n} x^u = 0$ and so $\sum_{x \in K^n} x^u = 0$, as $y^u \neq 1$. □

Definition – Vanishing

Suppose for all $I \subseteq K[x_1, \dots, x_n]$ We define the vanishing of I , $V(I) := \{x \in K^n \mid \forall f \in I, f(x) = 0\}$

Proposition – Chevalley

Suppose for all $f \in I \subseteq K[x_1, \dots, x_n]$ (finite),

$$\sum_{f \in I} \deg f < n$$

Then

$$|V(I)| \stackrel{p}{=} 0$$

Proof. Consider $P := \prod_{f \in I} (1 - f^{q-1})$. This is well defined as I is finite. We show that $V(I) = P^{-1}(1)$. Let $x \in K^n$.

$$x \in V(I) \Rightarrow \forall f \in I, f(x) = 0 \Rightarrow f(x)^{q-1} = 0 \Rightarrow P(x) = 1$$

$$x \notin V \Rightarrow \exists f \in I, f \neq 0 \Rightarrow f(x)^{q-1} = 1 \Rightarrow P(x) = 0$$

Let $S : K[x_1, \dots, x_n] \rightarrow K := f \rightarrow \sum_{x \in K^n} f(x)$ Then $S(P) = \sum_{x \in V(I)} 1 \stackrel{p}{=} |V(I)|$. Thus we need show that $S(P) = 0$.

$$\deg P = \sum_{f \in I} (q-1) \deg f = (q-1) \sum_{f \in I} \deg f < n \Rightarrow < (q-1)n$$

by assumption. Hence there exists a finite set T and $\lambda_i \in K$ such that

$$P = \sum_{i \in T} \lambda_i \prod_{j=1}^n x_j^{u_{ij}}$$

and for all $i \in T$, $\sum_{j=1}^n u_{ij} < (q-1)n$. Then

$$S(P) = \sum_{x \in K^n} P(x) \tag{1.1}$$

$$= \sum_{x \in K^n} \sum_{i \in T} \lambda_i \prod_{j=1}^n x_j^{u_{ij}} \tag{1.2}$$

$$= \sum_{i \in T} \lambda_i \sum_{x \in K^n} \prod_{j=1}^n x_j^{u_{ij}} \tag{1.3}$$

$$\tag{1.4}$$

Let $i \in T$ then there exists a k such that $u_{ik} < q-1$ so

$$\sum_{x \in K^n} \prod_{j=1}^n x_j^{u_{ij}} \tag{1.5}$$

$$= \sum_{x_1 \in K} \cdots \sum_{x_n \in K} \prod_{j=1}^n x_j^{u_{ij}} \tag{1.6}$$

$$= \sum_{x_1 \in K} \cdots \sum_{x_k \in K} \cdots \sum_{x_n \in K} \prod_{j \neq k} x_j^{u_{ij}} \sum_{x_k \in K} x_k^{u_{ik}} \tag{1.7}$$

$$= \sum_{x_1 \in K} \cdots \sum_{x_k \in K} \cdots \sum_{x_n \in K} \prod_{j \neq k} x_j^{u_{ij}} 0 \tag{1.8}$$

The last part using the [power sum lemma](#). Hence $|V(I)| \stackrel{p}{=} S(P) = 0$ □

Corollary – Non-trivial vanishing

Suppose for all $f \in I \subseteq K[x_1, \dots, x_n]$ (finite),

$$\sum_{f \in I} \deg f < n$$

and $0 \in V(I)$ then $\exists x \in V(I) \setminus \{0\}$.

Proof. If $|V| = 1$ then $p \nmid |V|$ which is a contradiction. Thus the vanishing is non-trivial. □

Definition – Homogeneous

$f \in K[x_1, \dots, x_n]$ is homogeneous with degree m if all monomials are of degree m .

Corollary – Conics over a finite field

If $3 \leq n$ then if $f \in K[x_1, \dots, x_n]$ is homogeneous with degree 2 then it has a non-trivial zero.

1.3 Quadratic reciprocity

Proposition – Exact sequence

If K is a finite field,

- If $\text{Char}(K) = 2$ then all elements are square.
- If $\text{Char}(K) \neq 2$ then the non-zero squares form a subgroup of index 2, and is the kernel of the group morphism $x \rightarrow x^{\frac{q-1}{2}}$ into $\langle -1 \rangle$.

I can't be bothered to make the exact sequence.

Proof.

- If $\text{Char}(K) = 2$ then the [Frobenius map](#) $\sigma_2 : x \mapsto x^2$ is an automorphism of K . Hence the preimage of any element squares to that element.
- If $\text{Char}(K) \neq 2$ then generate $K^* = \langle g \rangle$ since it is cyclic. The map $x \rightarrow x^{\frac{q-1}{2}}$ has kernel $\{x \in K \mid x \text{ square}\}$ since (writing any element as a multiple of g)

$$g^n \in \ker \Leftrightarrow g^{\frac{n(q-1)}{2}} = 1 \Leftrightarrow q-1 \mid \frac{n(q-1)}{2} \Leftrightarrow n \text{ even} \Leftrightarrow x \text{ square}$$

We check where the generator g is sent. If $g^{\frac{q-1}{2}} = 1$ then the order of g is less than $q-1$ which is a contradiction hence the image is non-trivial. Any element of the image of the map squares to 1 hence solves $x^2 - 1 = 0$, which only has two solutions in K . Thus the image is $\langle -1 \rangle$ and the index of the kernel is 2.

□

Definition – Legendre symbol

If p is prime that is not 2 and $x \in \mathbb{F}_p$ then

$$\left(\frac{x}{p} \right) := \begin{cases} x^{\frac{p-1}{2}} & , x \text{ unit} \\ 0 & , x = 0 \end{cases}$$

Check that for each p this is a homomorphism $\mathbb{F}_p \rightarrow \langle -1 \rangle$.

Definition – $\varepsilon(n)$

If $n \in \mathbb{Z}$ is odd

$$\varepsilon(n) := \frac{n-1}{2} \pmod{2}$$

Proposition – Computations

$$\begin{aligned} \left(\frac{1}{p}\right) &= 1 \\ \left(\frac{-1}{p}\right) &= (-1)^{\varepsilon(p)} \end{aligned}$$

Proposition – Quadratic reciprocity

Let $l \neq p$ be primes that aren't 2. Then

$$\left(\frac{l}{p}\right) \left(\frac{p}{l}\right) = (-1)^{\varepsilon(l)\varepsilon(p)}$$

Proof. Let w be order l element of Ω , the algebraic closure of \mathbb{F}_p . For $x \in \mathbb{F}_l$ write w^x to be w^r for any $r \in \mathbb{Z}$ such that $x = \bar{r} \in \mathbb{F}_l$ (independent of choice of r by $w^l = 1$). Let

$$y = \sum_{x \in \mathbb{F}_l} \left(\frac{x}{l}\right) w^x \in \Omega$$

We first show that $y^2 = (-1)^{\varepsilon(l)} \bar{l}$, where $\bar{l} \in \mathbb{F}_p$.

$$\begin{aligned} y^2 &= \left(\sum_{x \in \mathbb{F}_l} \left(\frac{x}{l}\right) w^x \right) \left(\sum_{y \in \mathbb{F}_l} \left(\frac{y}{l}\right) w^y \right) \\ &= \sum_{x \in \mathbb{F}_l} \sum_{y \in \mathbb{F}_l} \left(\frac{x}{l}\right) w^x \left(\frac{y}{l}\right) w^y \\ &= \sum_{x \in \mathbb{F}_l} \sum_{y \in \mathbb{F}_l} \left(\frac{xy}{l}\right) w^{x+y} \\ &= \sum_{u \in \mathbb{F}_l} \sum_{x \in \mathbb{F}_l} \left(\frac{x(u-x)}{l}\right) w^u \end{aligned}$$

Case on what x is:

$$\begin{aligned}
 x \neq 0 &\Rightarrow \left(\frac{x(u-x)}{l} \right) = \left(\frac{xu - x^2}{l} \right) \\
 &= \left(\frac{x^2}{l} \right) \left(\frac{-1}{l} \right) \left(\frac{1 - \frac{u}{x}}{l} \right) \\
 &= x^{p-1} \left(\frac{-1}{l} \right) \left(\frac{1 - \frac{u}{x}}{l} \right) \\
 &= (-1)^{\varepsilon(l)} \left(\frac{1 - \frac{u}{x}}{l} \right)
 \end{aligned}$$

If $x = 0$ then clearly $\left(\frac{x(u-x)}{l} \right) = 0$. Hence

$$y^2 = \sum_{u \in \mathbb{F}_l} \sum_{x \in \mathbb{F}_l^*} (-1)^{\varepsilon(l)} \left(\frac{1 - \frac{u}{x}}{l} \right) = (-1)^{\varepsilon(l)} \sum_{u \in \mathbb{F}_l} \sum_{x \in \mathbb{F}_l^*} \left(\frac{1 - \frac{u}{x}}{l} \right)$$

Given $x \neq 0$, case on what u is:

$$\begin{aligned}
 u = 0 &\Rightarrow \sum_{x \in \mathbb{F}_l^*} \left(\frac{1 - \frac{u}{x}}{l} \right) \\
 &= \sum_{x \in \mathbb{F}_l^*} \left(\frac{1}{l} \right) \\
 &= \sum_{x \in \mathbb{F}_l^*} 1 \\
 &= \bar{l} - 1
 \end{aligned}$$

$$\begin{aligned}
 u \neq 0 &\Rightarrow \sum_{x \in \mathbb{F}_l^*} \left(\frac{1 - \frac{u}{x}}{l} \right) \\
 &= \sum_{x \in \mathbb{F}_l^*} \left(\frac{1 - \frac{1}{x}}{l} \right) \\
 &= \sum_{s \in \mathbb{F}_l^*} \left(\frac{1 - s}{l} \right) \\
 &= \sum_{s \in \mathbb{F}_l \setminus \{1\}} \left(\frac{s}{l} \right) \\
 &= \sum_{s \in \mathbb{F}_l} \left(\frac{s}{l} \right) - \left(\frac{1}{l} \right) \\
 &= -1
 \end{aligned}$$

Since the index of the kernel of $\left(\frac{\star}{l}\right)$ is 2, and the cosets have equal cardinality. Hence

$$\begin{aligned} y^2(-1)^{\varepsilon(l)} &= \sum_{u \in \mathbb{F}_l} \sum_{x \in \mathbb{F}_l^*} \left(\frac{1 - \frac{u}{x}}{l} \right) \\ &= \bar{l} - 1 - \sum_{u \in \mathbb{F}_l^*} w^u \\ &= \bar{l} - (1 + w + w^2 + \cdots + w^l) \end{aligned}$$

since l is prime. Note that $0 = w^l - 1 = (w+1)(1+w+\cdots+w^l)$. Hence $1+w+\cdots+w^l = 0$ and $y^2 = (-1)^{\varepsilon(l)}\bar{l}$.

Next we show that $y^{p-1} = \left(\frac{p-1}{l}\right)$.

$$\begin{aligned} y^p &= \sum_{x \in \mathbb{F}_l} \left(\frac{x}{l}\right)^p w^x p && \text{'Freshman's dream'} \\ &= \sum_{x \in \mathbb{F}_l} \left(\frac{x}{l}\right) w^x p && \left(\frac{x}{l}\right) = \pm 1 \text{ and } p \text{ is odd} \\ &= \sum_{z \in \mathbb{F}_l} \left(\frac{zp^{-1}}{l}\right) w^z \\ &= \left(\frac{p^{-1}}{l}\right) \left(\sum_{z \in \mathbb{F}_l} \left(\frac{z}{l}\right) w^z\right) \\ &= \left(\frac{p^{-1}}{l}\right) y \end{aligned}$$

Hence

$$y^{p-1} = \left(\frac{p^{-1}}{l}\right) = \left(\frac{pl}{l}\right)^{-1}$$

thus

$$\begin{aligned} \left(\frac{l}{p}\right) \left(\frac{p}{l}\right) &= \left(\frac{l}{p}\right) y^{1-p} \\ &= \left(\frac{l}{p}\right) (y^2)^{\frac{1-p}{2}} \\ &= \left(\frac{l}{p}\right) ((-1)^{\varepsilon(l)}\bar{l})^{\frac{1-p}{2}} \\ &= \left(\frac{l}{p}\right) \left(\frac{(-1)^{\varepsilon(l)}l}{p}\right)^{-1} \\ &= \left(\frac{(-1)^{\varepsilon(l)}}{p}\right)^{-1} \\ &= (-1)^{\varepsilon(l)\varepsilon(p)} \end{aligned}$$

□

Chapter 2

p-adic Fields

2.1 p-adic Integers and Rationals

Definition – Projective system A

Define a contravariant functor $A : (\mathbb{N}, \leq) \rightarrow \mathbf{Ring}$ such that for each n

$$A_n := \mathbb{Z}/p^n\mathbb{Z} \quad \text{and} \quad \pi_n : \mathbb{Z} \rightarrow A_n \text{ is the projection}$$

and for any n such that $1 \leq n$, there exists a surjective ring morphism $\phi_n : A_n \rightarrow A_{n-1}$ such that $\phi_n \circ \pi_n = \pi_{n-1}$ and $\ker(\phi_n) = p^{n-1}A_n$.

EXERCISE. Check that such a ϕ_n exists.

Definition – p-adic integers

Let

$$\mathbb{Z}_p = \{x \in \prod_{n \in \mathbb{N}} A_n \mid (\forall n \in \mathbb{N}, x_n \in A_n) \wedge (\forall n > 0, \phi_n(x_n) = x_{n-1})\}$$

Define addition and multiplication pointwise. Verify that this \mathbb{Z}_p is a ring with $0 = (0)_{n \in \mathbb{N}}$ and $1 = (1)_{n \in \mathbb{N}}$.

For each $n \in \mathbb{N}$ let $\varepsilon_n : \mathbb{Z}_p \rightarrow A_n$ be the ring morphisms mapping $x \mapsto x_n$. Note that by definition $\phi_n \circ \varepsilon_n = \varepsilon_{n-1}$.

In addition, provide each A_n with the discrete topology, giving $\prod_{n \in \mathbb{N}} A_n$ the product topology and \mathbb{Z}_n the subset topology.

Proposition – \mathbb{Z}_p is compact

Since each A_n is finite, each A_n is compact. Hence by Tychonoff's theorem the product is compact. Since closed in compact is compact we just need to show that \mathbb{Z}_p is closed.

We want to write \mathbb{Z}_p as the intersection of closed sets

$$D_k := \left\{ x \in \prod_{n \in \mathbb{N}} A_n \mid \phi_k(x_k) = x_{k-1} \right\}$$

for $k \in \mathbb{N}$. Clearly

$$\bigcap_{k \in \mathbb{N}} D_k = \mathbb{Z}_p$$

and

$$D_k = \bigcup_{x_{k-1} \in A_{k-1}} \left(\varepsilon_{k-1}^{-1}(x_{k-1}) \cap \bigcup \{ \varepsilon_k^{-1}(x_k) \mid x_k \in A_k \wedge \phi_k(x_k) = x_{k-1} \} \right)$$

Since each $\{x_k\}$ is closed in A_k , each preimage $\varepsilon_k^{-1}(x_k)$ is closed. Thus the finite union of the preimages

$$\bigcup \{ \varepsilon_k^{-1}(x_k) \mid x_k \in A_k \wedge \phi_k(x_k) = x_{k-1} \}$$

is closed. Since each $\{x_{k-1}\}$ is closed in A_{k-1} , each preimage $\varepsilon_{k-1}^{-1}(x_{k-1})$ is closed. Thus intersection

$$\left(\varepsilon_{k-1}^{-1}(x_{k-1}) \cap \bigcup \{ \varepsilon_k^{-1}(x_k) \mid x_k \in A_k \wedge \phi_k(x_k) = x_{k-1} \} \right)$$

is closed. Hence the finite union is closed and D_k is closed. Arbitrary intersection of closed is closed so \mathbb{Z}_p is closed and thus compact.

Proposition – Universal property of \mathbb{Z}_p

Suppose R is a ring with ring morphisms $\rho_n : R \rightarrow A_n$ for each $n \in \mathbb{N}$ such that for each $n > 0$, $\phi_n \circ \rho_n = \rho_{n-1}$. Then there exists a unique ring morphism $f : R \rightarrow \mathbb{Z}_p$ such that for each n , $\varepsilon_n \circ f = \rho_n$.

Proof. If there exists such a map then it is unique: suppose f, g both satisfy the given properties. Then for any n and any $a \in R$ $\varepsilon_n \circ f(a) = \rho_n(a) = \varepsilon_n \circ g(a)$. Thus $f(a) = g(a)$, by the property of products (if they agree on all the projections they are equal).

For existence we let $a \in R$ and consider the set

$$\bigcap_{n \in \mathbb{N}} \varepsilon_n^{-1} \circ \rho_n(a)$$

show that it has cardinality 1, and let f map a to this unique element. If $x, y \in \bigcap_{n \in \mathbb{N}} \varepsilon_n^{-1} \circ \rho_n(a)$ then for any $n \in \mathbb{N}$, $\varepsilon_n(x) = \rho_n(a) = \varepsilon_n(y)$. Thus $x = y$ by the property of products. Hence the cardinality is ≤ 1 .

To show that the set is non-empty, take $x = (\rho_n(a))_{n \in \mathbb{N}}$. This is in \mathbb{Z}_p since for each $n > 0$, $\phi_n \circ \rho_n(a) = \rho_{n-1}(a)$. Also it is in the intersection since for each n , $\varepsilon_n(x) = \rho_n(a)$. Hence the cardinality is 1. Hence f is well-defined and for all $n \in \mathbb{N}$, $\varepsilon_n \circ f = \rho_n$.

For any n ,

$$\varepsilon_n \circ f(a + b) = \rho_n(a + b) = \rho_n(a) + \rho_n(b) = \varepsilon_n \circ f(a) + \varepsilon_n \circ f(b) = \varepsilon_n(f(a) + f(b))$$

Hence by property of products $f(a + b) = f(a) + f(b)$ and similarly for multiplication. Note that for any n , $\varepsilon_n \circ f(1) = \rho_n(1) = 1$. Hence $f(1) = 1$. Thus f is a ring morphism. \square

Corollary – \mathbb{Z} injects into \mathbb{Z}_p

Then there exists a unique injective ring morphism $\iota : \mathbb{Z} \rightarrow \mathbb{Z}_p$ such that for each n , $\varepsilon_n \circ \iota = \pi_n$.

Proof. By the previous theorem the morphism exists and is unique. It must send $1 \mapsto 1$ hence $\iota(x) = 0$ would imply $\pi_n(x) = \varepsilon_n \circ \iota(x) = 0$ for all $n \in \mathbb{N}$. Hence for any $n \in \mathbb{N}$, $p^n \mid x$. Thus $x = 0$. \square

Proposition – Multiplying by p^n is injective and $x_n = 0$ implies $x \in p^n \mathbb{Z}_p$

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{p^n} \mathbb{Z}_p \xrightarrow{\varepsilon_n} A_n \longrightarrow 0$$

is a short exact sequence of abelian groups.

Proof. To check that the morphism $\mathbb{Z}_p \rightarrow \mathbb{Z}_p$ multiplying by p^n is injective it suffices to show that multiplying by p is injective. Suppose x is in the kernel of this map, then $px = 0$ thus for any n , $px_{n+1} = \varepsilon_{n+1}(px) = 0$. We show that for any n , $x_n = 0$. There exists $a \in \mathbb{Z}$ such that $\pi_{n+1}(a) = x_{n+1}$. Since $\pi_{n+1}(pa) = px_{n+1} = 0$, $pa = p^{n+1}b$ for some $b \in \mathbb{Z}$. Hence $a = p^n b$ since \mathbb{Z} is an integral domain. Thus $\pi_n(a) = x_n = 0$. Thus $x = 0$.

To check that the $p^n \mathbb{Z}_p = \ker(\varepsilon_n)$ we note that for any $x \in \mathbb{Z}_p$, $\varepsilon_n(p^n x) = p^n x_n = 0 \in A_n$. Hence $p^n \mathbb{Z}_p \subseteq \ker(\varepsilon_n)$. For the other direction suppose $\varepsilon_n(x) = 0$. Suppose $n \leq m \in \mathbb{Z}$. Then there exists a unique $a_m \in \mathbb{Z}$ such that $0 \leq a < p^m$ and $\pi_m(a_m) = \varepsilon_m(x)$. Then

$$\pi_n(a_m) = \phi_m \circ \cdots \circ \phi_{n+1} \pi_m(a_m) = \phi_m \circ \cdots \circ \phi_{n+1} \varepsilon_m(x) = \varepsilon_n(x) = 0$$

Thus there exists a unique $b_m \in \mathbb{Z}$ such that $a_m = p^n b_m$.

Let $b = (\pi_m(b_m))_{m \in \mathbb{N}} \in \mathbb{Z}_p$. Note that multiplying by p^n commutes with all the map as they are ring homomorphisms. Then for any $m \in \mathbb{N}$,

$$\begin{aligned} \phi_{m+1} \varepsilon_{m+1}(b) &= \phi_{m+1} \circ \pi_{m+1}(b_{m+1}) &= \phi_{m+1} \circ \pi_{m+1}(p^n a_{m+1}) \\ &= p^n \phi_{m+1} \circ \pi_{m+1}(a_{m+1}) &= p^n \pi_m(a_m) \\ &= \pi_m(b_m) &= \varepsilon_m(b) \end{aligned}$$

Hence $b \in \mathbb{Z}_p$. Furthermore, let $m \in \mathbb{N}$ then

$$\varepsilon_m(p^n b) = p^n \pi_m(b_m) = \pi_m(p^n b_m) = \pi_m(a_m) = \varepsilon_m(x)$$

Hence $p^n b = x$. Thus $x \in p^n \mathbb{Z}_p$. \square

Proposition – \mathbb{Z}_p is a local ring, decomposition of non-zero elements

If $x \in \mathbb{Z}_p$ then

1. $x_n \in A_n$ is a unit if and only if $x_n \notin pA_n$.
2. $x \in \mathbb{Z}_p$ is a unit if and only if $x \notin p\mathbb{Z}_p$.
3. For any $x \in \mathbb{Z}_p \setminus \{0\}$ there exist unique $n \in \mathbb{N}$ and $u \in \mathbb{Z}_p$ such that u is a unit and $p^n u = x$.

Proof.

1. If x_n is a unit and $x_n \in pA_n$ then write $x_n = py_n$ for $y_n \in A_n$. We see that p is a unit since $x_n^{-1}py_n = 1$. However p is nilpotent since $p^n = 0$ a contradiction. Hence $x_n \notin pA_n$. Conversely if $x_n \notin pA_n$ then supposing $x_1 = 0$ deduces $x \in p\mathbb{Z}_p$ by the [previous proposition](#). Hence $x_n \in pA_n$ a contradiction. Thus $x_1 \neq 0 \in A_1$, a field, so x_1 is a unit in A_1 . Hence there exist $x_{\mathbb{Z}}, y_{\mathbb{Z}}, z_{\mathbb{Z}} \in \mathbb{Z}$ such that $\iota(x_{\mathbb{Z}}) = x$ and

$$\begin{aligned} x_{\mathbb{Z}}y_{\mathbb{Z}} + pz_{\mathbb{Z}} &= 1 \\ \Rightarrow \pi_n(x_{\mathbb{Z}}y_{\mathbb{Z}} + pz_{\mathbb{Z}}) &= 1 \\ \Rightarrow x_ny_n + pz_n &= 1 \\ \Rightarrow x_ny_n(1 + \dots + (pz_n)^{n-1}) &= 1 - (pn)^z = 1 \in A_n \\ \Rightarrow x_n &\text{ is a unit} \end{aligned}$$

Hence x_n is a unit if and only if $x_n \notin pA_n$.

2. If x is a unit of \mathbb{Z}_p then in particular x_1 is a unit. Suppose $x \in p\mathbb{Z}_p$ then $x_1 = 0$ by the [previous proposition](#). Hence x_1 is not a unit, a contradiction. Thus $x \notin p\mathbb{Z}_p$.

For the converse suppose $x \notin p\mathbb{Z}_p$ then by the [previous proposition](#) $x_1 \neq 0$. For any $n \in \mathbb{N}$, if $x_n \in A_n$ then $x_1 = \phi_n \circ \dots \circ \phi_2 x_n = 0$ which is false. Hence for any $n \in \mathbb{N}$, $x_n \notin pA_n$ which by the first part implies there exists a unique $y_n \in A_n$, $x_n y_n = 1$. We show that $y := (y_n)_{n \in \mathbb{N}}$ is the inverse of x in \mathbb{Z}_p . To show that $y \in \mathbb{Z}_p$ let $n \in \mathbb{N}$.

$$x_n \phi_{n+1}(y_{n+1}) = \phi_{n+1}(x_{n+1}) \phi_{n+1}(y_{n+1}) \phi_{n+1}(x_{n+1} y_{n+1}) = \phi(1) = 1$$

Hence $\phi_{n+1}(y_{n+1}) = y_n$ by uniqueness of inverses in A_n . To show that $xy = 1$ note that for any $n \in \mathbb{N}$, $\varepsilon_n(xy) = x_n y_n = 1$. Hence $xy = 1$.

3. Let $x \in \mathbb{Z}_p$ be non-zero and consider the set

$$\{n \in \mathbb{N} \mid \varepsilon_n(x) = 0\}$$

This is non-empty since $\varepsilon_0(x) = 0$. By induction there exists a maximum of this set, call this n . Since $\varepsilon_n(x) = 0$ by the [previous proposition](#) $x = p^n y$ for some $y \in \mathbb{Z}_p$. Suppose $y \in p\mathbb{Z}_p$ then $\varepsilon_{n+1}(x) = 0$ which is a contradiction with maximality. Hence by the previous part of this proposition y is a unit.

Suppose we have another decomposition $x = p^m z$ with z a unit. Then by maximality of n , $m \leq n$. By the [previous proposition](#) we have that multiplication by p^m is injective. Hence $p^n y = p^m z$ implies $p^{n-m} y = z$. Since z is a unit, $n - m = 0$. Hence $n = m$ and $y = p^{n-m} y = z$.

□

Definition – \mathbb{N}_∞

On the set $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ define commutative addition such that if $n, m \in \mathbb{N}$ then it is the usual addition and for any $x \in \mathbb{N}_\infty$, $x + \infty = \infty$. We order the set using \leq , where it is the usual $m \leq n$ for $m, n \in \mathbb{N}$ and for any $x \in \mathbb{N}_\infty$, $x \leq \infty$ and if $\infty \leq x$ then $x = \infty$. This is a total order hence we have a well defined infimum for any non-empty set.

Definition – p -adic valuation

Given p a prime, define $v_p : \mathbb{Z}_p \rightarrow \mathbb{N}_\infty$ sending any non-zero x to n , where $n \in \mathbb{N}$ and $u \in \mathbb{Z}_p$ is a unit such that $x = p^n u$. In the other case we define $v_p(0) := \infty$.

Proposition

For any p prime and $x, y \in \mathbb{Z}_p$

$$v_p(xy) = v_p(x) + v_p(y), \quad \inf \{v_p(x), v_p(y)\} \leq v_p(x + y)$$

Proof. Case on what x, y are. □

Corollary

\mathbb{Z}_p is an integral domain.

Proof. Let $x, y \in \mathbb{Z}_p$ be such that $xy = 0$. Suppose for a contradiction both x, y are non-zero. Then $v_p(x), v_p(y) \in \mathbb{N}$ hence $\infty = v_p(xy) = v_p(x) + v_p(y) \in \mathbb{N}$, a contradiction. □

Definition – Metric on \mathbb{Z}_p

Define a norm on \mathbb{Z}_p by

$$|\star| : \mathbb{Z}_p \rightarrow \mathbb{R}_{\geq 0} := x \mapsto \begin{cases} 0 & , x = 0 \\ p^{-v_p(x)} & , x \neq 0 \end{cases}$$

This satisfies

1. $|x| = 0 \Leftrightarrow x = 0$
2. $|x + y| \leq \max(|x|, |y|) \leq |x| + |y|$
3. $|xy| \leq |x| |y|$
4. $|1| = 1$

This induces a metric on \mathbb{Z}_p .

Proof. Straight forward. □

Proposition – Cosets are clopen balls

For any n and $a \in \mathbb{Z}$ the coset $a + p^n \mathbb{Z}_p$ is a clopen ball $B_\delta(a)$ for some $\delta \in \mathbb{R} - > 0$.

Proof. $b \in a + p^n \mathbb{Z}_p$ if and only if $n \leq v_p(b-a)$ if and only if $|b-a| \leq p^{-n}$ if and only if $|b-a| < \frac{p^{-n} + p^{1-n}}{2} =: \delta$, as the image of the norm is discrete. Hence $a + p^n \mathbb{Z}_p = \overline{B_{p^{-n}}(a)} = B_\delta(a)$ and is clopen. \square

Proposition – Induced topologies are equivalent

The metric topology \mathcal{T}_0 is the same as the subspace topology \mathcal{T}_1 from $\prod_{n \in \mathbb{N}} A_n$.

Proof. We first show that the neighbourhoods of points are the same. Call the neighbourhood filter for a point a in the metric topology $N_0(a)$ and the other $N_1(a)$. We use $\langle \star | \dots \rangle$ to mean the neighbourhood filter generated by $\{\star | \dots\}$.

$$\begin{aligned} N_1(a) &= \langle U \cap \mathbb{Z}_p \mid a \in U \in \text{product topology on } \prod A_n \rangle \\ &= \langle \varepsilon_n^{-1}(U) \cap \mathbb{Z}_p \mid \exists n \in \mathbb{N}, a_n \in U \subseteq A_n \rangle \\ &= \langle U \subseteq \mathbb{Z}_p \mid \exists n \in \mathbb{N}, a + \ker(\varepsilon_n) \subseteq U \rangle \\ &= \langle U \subseteq \mathbb{Z}_p \mid \exists n \in \mathbb{N}, a + p^n \mathbb{Z}_p \subseteq U \rangle \\ &= \langle U \subseteq \mathbb{Z}_p \mid \exists \delta > 0, B_\delta(a) \subseteq U \rangle \\ &= N_0(a) \end{aligned}$$

The penultimate equality is due to [cosets being clopen balls](#) for one inclusion and the other inclusion follows from finding $n \in \mathbb{N}$ such that $p^{-(n+1)} < \delta < p^{-n}$.

Since a subset U is open in a topology if and only if for all points $a \in U$, $U \in N(p)$ we see that $U \in \mathcal{T}_0$ if and only if $\forall p \in U, U \in N_0(p)$ if and only if $\forall p \in U, U \in N_1(p)$ if and only if $U \in \mathcal{T}_1$. \square

Proposition – Topological properties of \mathbb{Z}_p

\mathbb{Z}_p is complete in the topological sense and the image of \mathbb{Z} is dense in \mathbb{Z}_p .

Proof. Any Cauchy sequence in \mathbb{Z}_p has a subsequence converging to $x \in \mathbb{Z}_p$ as \mathbb{Z}_p is a [compact](#) metric space. This is also the unique limit of the original sequence as it is Cauchy. Hence \mathbb{Z}_p is complete.

Clearly $\overline{\iota(\mathbb{Z})} \subseteq \mathbb{Z}_p$. Let $x \in \mathbb{Z}_p$. We want to show that there exists a sequence in $\iota(\mathbb{Z})$ converging to x , hence showing that $x \in \overline{\iota(\mathbb{Z})}$. For any $n \in \mathbb{N}$ there exists an element $b \in \mathbb{Z}$ such that $\pi_n(b) = \varepsilon_n(x)$. Define the sequence $y : \mathbb{N} \rightarrow \mathbb{Z}_p := n \rightarrow \iota(b)$. Then we claim that $\lim_{n \in \mathbb{N}} y(n) = x$. Let $\delta \in \mathbb{R}_{>0}$. There exists $N \in \mathbb{N}$ such that $p^{-N} < \delta$. Let $n \in \mathbb{N}$ be such that $N \leq n$. Then $\varepsilon_n(x - y(n)) = 0$ [implies](#) $x - y(n) \in p^n A_n$ and so

$$|x - y(n)| = p^{-v_p(x-y(n))} \leq p^{-n} \leq p^{-N} < \delta$$

Thus the limit exists and is x . Hence $\overline{\iota(\mathbb{Z})} = \mathbb{Z}_p$. \square

Definition – \mathbb{Q}_p

Since \mathbb{Z}_p is an [integral domain](#), we can construct its field of fractions. We call this \mathbb{Q}_p .

Proposition – Inclusions into \mathbb{Q}_p

There is a unique injective ring morphism $\mathbb{Z}_p \rightarrow \mathbb{Q}_p$ which (without confusion) we treat as \subseteq and there is a unique injective extension of the ring morphism $\iota : \mathbb{Z} \rightarrow \mathbb{Z}_p$ to $\mathbb{Q} \rightarrow \mathbb{Q}_p$.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\subseteq} & \mathbb{Q} \\ \iota \downarrow & & \downarrow \\ \mathbb{Z}_p & \xrightarrow{\subseteq} & \mathbb{Q}_p \end{array}$$

Proof. The inclusion $\mathbb{Z}_p \rightarrow \mathbb{Q}_p$ is a result of the construction of the field of fractions. We extend ι by mapping $\frac{a}{b} \in \mathbb{Q}$ to $\frac{\iota(a)}{\iota(b)} \in \mathbb{Q}_p$. Check that it is well-defined and injective, a ring morphism and that the diagram above commutes. \square

Proposition

$\mathbb{Q}_p \cong \mathbb{Z}_p[\frac{1}{p}]$ canonically and any unit of \mathbb{Q}_p can be uniquely written in the form $p^n u$ for $n \in \mathbb{Z}$ and u a unit in the image of \mathbb{Z}_p under the isomorphism.

Proof. Let $f : \mathbb{Z}_p[\frac{1}{p}] \rightarrow \mathbb{Q}_p$ such that $\sum_{i=0}^n x_i (\frac{1}{p})^i \mapsto \sum_{i=0}^n \frac{x_i}{p^i}$. Clearly f is well defined and injective. To show that it is surjective note that for any element $\frac{a}{b} \in \mathbb{Q}_p$ with $a, b \in \mathbb{Z}_p, b \neq 0$ we can write $b = p^n u$ for unique $n \in \mathbb{N}$ and u a unit. Hence $\frac{a}{b} = \frac{a}{p^n u} = \frac{au^{-1}}{p^n}$ which is due to an element of $\mathbb{Z}_p[\frac{1}{p}]$ via f .

The same trick gives us the decomposition of units in \mathbb{Q}_p . \square

Definition – p -adic valuation for \mathbb{Q}_p

Extend the definition of v_p to \mathbb{Q}_p by taking $x \neq 0$ to n such that $p^n u = x$.

Note that $0 \leq v_p(x)$ if and only if x is a p -adic integer.

Definition – Addition is a homeomorphism on \mathbb{Q}_p

Let $a \in \mathbb{Q}_p$. Then the map $\mathbb{Q}_p \rightarrow \mathbb{Q}_p$ sending $b \mapsto a + b$ is a homeomorphism.

Proof. Let $b \in \mathbb{Q}_p$ and let $\delta \in \mathbb{R}_{>0}$. It suffices that $a + B_\delta(b) \subseteq B_\delta(a + b)$. Indeed if $c \in B_\delta(b)$ then $|a + c - (a + b)| = |c - b| < \delta$.

This map has inverse $-a$ which is continuous for the same reasons. Hence $a + \star$ is a homeomorphism. \square

Proposition

For any n $p^n\mathbb{Z}_p$ is clopen in \mathbb{Q}_p , in particular \mathbb{Z}_p is open in \mathbb{Q}_p . Hence \mathbb{Q}_p is locally compact and $\iota(\mathbb{Q})$ is dense in \mathbb{Q}_p .

Proof. Since \mathbb{Z}_p and \mathbb{Q}_p share the same metric Each $p^n\mathbb{Z}_p$ is clopen in \mathbb{Q}_p . We first note that \mathbb{Q}_p is locally compact at 0 since \mathbb{Z}_p is an open compact neighbourhood of 0. Furthermore, for any $a \in \mathbb{Q}_p$, $a + \star$ is a homeomorphism so the coset $a + \mathbb{Z}_p$ is the image of an open compact set which is open and compact. Clearly $a \in a + \mathbb{Z}_p$. Hence \mathbb{Q}_p is locally compact.

Clearly $\overline{\iota(\mathbb{Q})} \subseteq \mathbb{Q}_p$. Let $x \in \mathbb{Q}_p$, then $x = p^n u$ for $n \in \mathbb{N}$ and $u \in \mathbb{Z}_p$ a unit. Then $u \in \overline{\iota(\mathbb{Z})} \subseteq \overline{\iota(\mathbb{Q})}$ and so $x \in p^n \iota(\mathbb{Q}) \subseteq \iota(\mathbb{Q})$. Thus \mathbb{Q} is dense in \mathbb{Q}_p . \square

Proposition – Series converge iff terms converge

Let $x : \mathbb{N} \rightarrow \mathbb{Q}_p$ be a sequence. Then x converges if and only if $\lim_{n \in \mathbb{N}} (x(n+1) - x(n)) = 0$.

Proof. Since \mathbb{Q}_p is complete it suffices to show that x is Cauchy if and only if $\lim_{n \in \mathbb{N}} (x(n+1) - x(n)) = 0$. The forward implication is straightforward. For the other direction take $\delta \in \mathbb{R}_{>0}$. By assumption

$$\exists N \in \mathbb{N}, \forall n \in \mathbb{N}_{>N}, |x(n+1) - x(n)| < \frac{\delta}{2}$$

Let $n, m \in \mathbb{N}$ be such that $N \leq n \leq m$. By induction we can show that $|x(m) - x(n)| \leq \frac{\delta}{2} < \delta$, using $|x + y| \leq \max(|x|, |y|)$ for the induction. \square