

Limits in the category of categories

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1 Introduction

In this project, I show that $\text{Cat}.u\ u$, the category of small categories is complete, where “small” in `lean` is taken to mean the universe levels (sizes) for objects and morphisms is the same. My work can easily be adapted to show that relaxing the “small” condition, $\text{Cat}.v\ u$ has all finite limits. Since the category of categories is a bicategory, and it is quite rare in practice to work with strict equality of functors, the “morally correct” limits to consider should really be 2-limits, where the functors are considered up to natural isomorphism. However, these are not considered here, and existence of 1-limits holds anyway, which should deduce the 2-limit case.

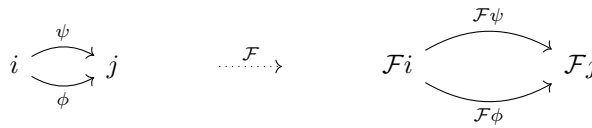
An obvious next step is to show results for colimits, which I have not yet done.

2 Mathlib Results

The category theory library in `mathlib` already contains definitions for limits, using cones over diagrams.

Definition – Diagrams, cones and limits

A diagram in a category \mathcal{C} is simply a functor from some indexing category $F : \mathcal{I} \rightarrow \mathcal{C}$. Diagrams are not explicitly defined in `lean`, as this would lead to extra unnecessary definitional equalities.

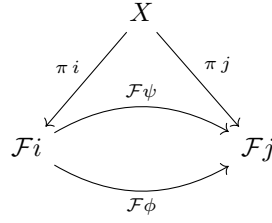


A diagram for equalizers

A cone over a diagram $F : \mathcal{J} \rightarrow \mathcal{C}$ consists of an apex X an object in \mathcal{C} and a natural transformation from (the constant functor at) X to F . This natural transformation captures the data of legs from X into the diagram that commute with the maps in the diagram.

```
structure cone (F : J => C) :=  
  (X : C) (pi : (const J).obj X -> F)
```

Here `lean` recognizes that F is a functor, so it infers that the categorical map into F should be in the functor category, i.e. a natural transformation.



A cone over the equalizer diagram, or a “fork”

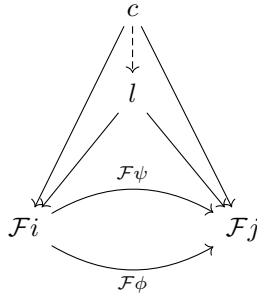
Finally, the data of a cone l being the limit of a diagram consists of

- (lift) For any cone c , a map $\text{lift} : c \rightarrow l$ in the category \mathcal{C} .
- (fac) A proof that lift commutes with everything else in the diagram.
- (uniq) Any other map $c \rightarrow l$ commuting with everything is equal to lift .

Put another way, a limit of a diagram is the terminal object when considering the cones as a category.

```
structure is_limit (t : cone F) :=
  (lift :  $\prod$  (s : cone F), s.X  $\rightarrow$  t.X)
  (fac' :  $\forall$  (s : cone F) (j : J), lift s  $\gg$  t. $\pi$ .app j = s. $\pi$ .app j . obviously)
  (uniq' :  $\forall$  (s : cone F) (m : s.X  $\rightarrow$  t.X) (w :  $\forall$  j : J, m  $\gg$  t. $\pi$ .app j = s. $\pi$ .app j),
    m = lift s . obviously)
```

Note that lift is data, whereas the others are propositional. Also note that fac' and uniq' are appended with `obviously`. This means that when we construct a limit, we could only supply the `lift`, and let `lean` figure the rest out. Many of the structures in the category theory library use this so that making instances of categories, functors, natural transformations, and limits less work.



l is the equalizer of $\mathcal{F}\phi$ and $\mathcal{F}\psi$