

Cubical Quotients

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Quotients in the category of Sets

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Surjections are in bijection with equivalence relations.

Understand the higher-categorical version of this fact. This means:

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- Define **cubical objects** to generalize **equivalence relations**
- Describe the quotient of cubical objects, called **cubical quotients**.
- Regularity and exactness results for cubical quotients: classify cubical objects that are in bijection with their quotients - **cubical groupoids**.

Cartesian Cube category

The **(Cartesian) cube category** \square , is the free finite product category on

$$[0] \begin{array}{c} \xrightarrow{0} \\ \xRightarrow{\quad} \\ \xrightarrow{1} \end{array} [1]$$

such that $[0]$ is the terminal object.

(Awodey 2023)

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Each $[n]$ is the n -cube, 0 and 1 are the endpoints of the interval.

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Work in model category of **cubical sets** $\mathcal{S} = \mathbf{cSet} = \mathbf{Set}^{\square^{\text{op}}}$; think of fibrant cubical sets as “spaces” or “types”.

(Awodey 2023)

Cubical objects

Higher analogue of equivalence relations: cubical objects.

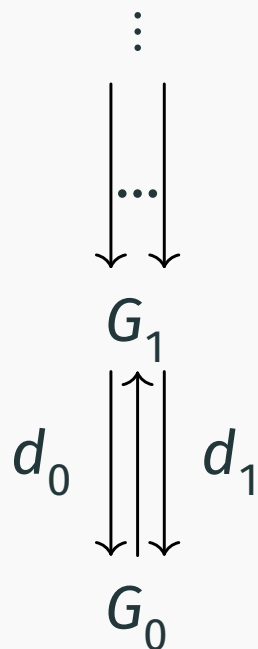


diagram in category \mathcal{S}

object \mathbb{G} in category $\mathcal{S}^{\square^{op}}$

Examples of cubical objects

$$\begin{array}{c} \vdots \\ \parallel \quad \parallel \\ \dots \\ E \\ \parallel \quad \parallel \\ \dots \\ E \\ \sigma \quad \uparrow \quad \tau \\ \downarrow \quad \downarrow \\ X \end{array}$$

Examples of cubical objects

$$\begin{array}{c}
 \begin{array}{|c|} \hline \vdots \\ \hline \end{array} \\
 \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\
 E \\
 \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\
 E \\
 \begin{array}{c} \sigma \quad \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \quad \tau \\
 \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \quad \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \\
 X
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{|c|} \hline \vdots \\ \hline \end{array} \\
 \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\
 \mathbb{Z} \times \mathbb{Z} \\
 \downarrow \\
 \begin{array}{|c|} \hline \dots + \dots \\ \hline \end{array} \\
 \downarrow \\
 \mathbb{Z} \\
 \begin{array}{c} ! \quad \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \quad ! \\
 \begin{array}{|c|} \hline 0 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \quad \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \\
 1
 \end{array}$$

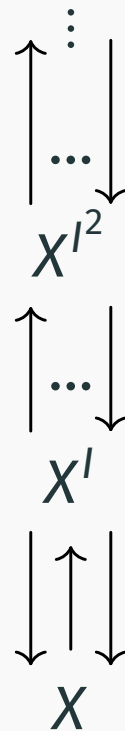
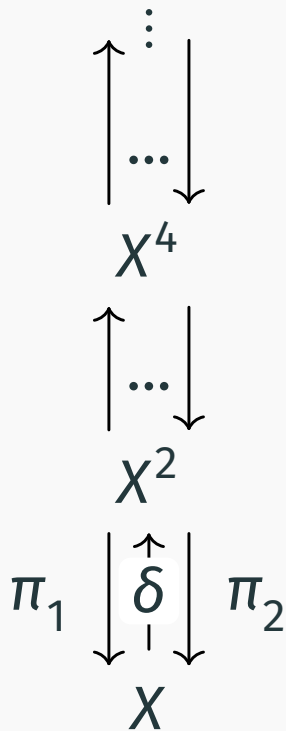
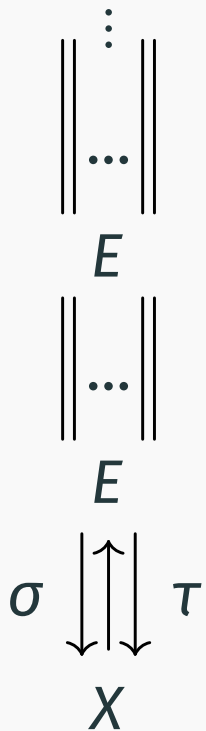
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 1
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{|c|} \hline \vdots \\ \hline \end{array} \\
 \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\
 X^4 \\
 \begin{array}{|c|} \hline \vdots \\ \hline \end{array} \\
 \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\
 X^2 \\
 \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \\
 \begin{array}{|c|} \hline \delta \\ \hline \end{array} \\
 \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \\
 X
 \end{array}$$

Examples of cubical objects



```
l = interval = y[1]
```

$$y : \Box \rightarrow \mathcal{S}$$

Path and quotient functors

 $\mathcal{S}^{\square \text{op}}$  \mathbb{P} \mathcal{S}

$$\mathbb{P}X = X^{I^{(-)}}$$

Path and quotient functors

$$\begin{array}{ccc} & \mathcal{S}^{\square \text{op}} & \\ B \downarrow & \dashv & \uparrow \mathbb{P} \\ & \mathcal{S} & \end{array}$$

$$\mathbb{P}X = X^{I^{(-)}}$$

Path and quotient functors

$B\mathbb{G} = \text{quotient of } \mathbb{G}$

$$\begin{array}{ccc} & \mathcal{S}^{\square \text{op}} & \\ \downarrow & & \uparrow \\ B & \dashv & \mathbb{P} \\ \uparrow & & \downarrow \\ & \mathcal{S} & \end{array}$$

$$\mathbb{P}X = X^{I^{(-)}}$$

Quotient universal property

$$\mathcal{S}(B\mathbb{G}, X) \cong \mathcal{S}^{\square^{\text{op}}}(\mathbb{G}, \mathbb{P}X)$$

$B\mathbb{G}$ = initial space with G_0 points, G_1 paths, G_2 homotopies, ...

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Quotient is homotopy colimit of diagram \mathbb{G} . Well known in simplicial setting (Bousfield and Kan 1972, Ch XII, 4.3).

Examples of cubical quotients

$$\begin{array}{c}
 \begin{array}{|c|} \hline \vdots \\ \hline \end{array} \\
 \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\
 E \\
 \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\
 E \\
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 X \rightarrow BG = X/E
 \end{array}$$

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$$\begin{array}{ccc}
 \begin{array}{c} \vdots \\ \parallel \quad \dots \quad \parallel \\ E \\ \parallel \quad \dots \quad \parallel \\ E \\ \sigma \downarrow \uparrow \downarrow \tau \\ X \end{array} & \longrightarrow & BG = X/E \\
 \\
 \begin{array}{c} \vdots \\ \parallel \quad \dots \quad \parallel \\ \mathbb{Z} \times \mathbb{Z} \\ \downarrow \\ \dots \quad + \quad \dots \\ \downarrow \\ \mathbb{Z} \\ \downarrow \uparrow \downarrow \\ ! \quad 0 \quad ! \\ \downarrow \uparrow \downarrow \\ 1 \end{array} & \longrightarrow & BG = S^1
 \end{array}$$

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 \vdots \\
 \parallel \quad \dots \quad \parallel \\
 E \\
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$$X \rightarrow BG = X/E$$

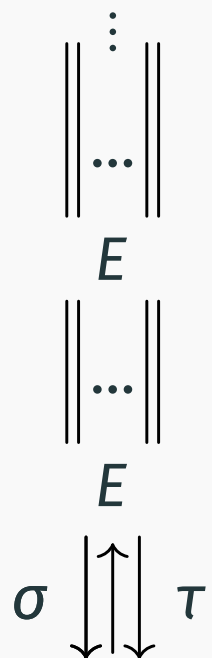
$$\begin{array}{c}
 \vdots \\
 \parallel \quad \dots \quad \parallel \\
 \mathbb{Z} \times \mathbb{Z} \\
 \downarrow \\
 \dots + \dots \\
 \downarrow \\
 \mathbb{Z} \\
 \downarrow \uparrow \\
 ! \quad 0 \quad ! \\
 \downarrow \uparrow
 \end{array}$$

$$1 \rightarrow BG = S^1$$

$$\begin{array}{c}
 \vdots \\
 \uparrow \quad \dots \quad \downarrow \\
 X^4 \\
 \uparrow \quad \dots \quad \downarrow \\
 X^2 \\
 \downarrow \uparrow \delta \downarrow \\
 \pi_1 \quad \downarrow \uparrow \quad \pi_2
 \end{array}$$

$$X \rightarrow BG = \|X\|_{-1}$$

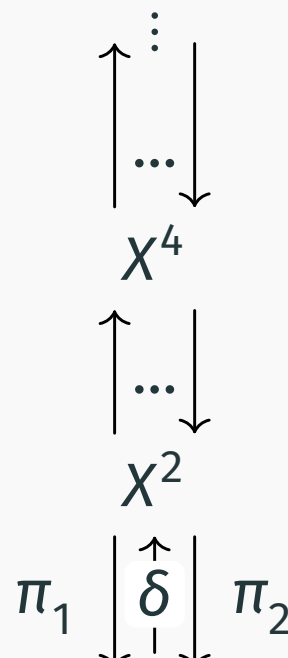
Examples of cubical quotients



$$X \rightarrow BG = X/E$$



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$$X \rightarrow BG = \|X\|_{-1}$$



$$X \rightarrow BG = X$$

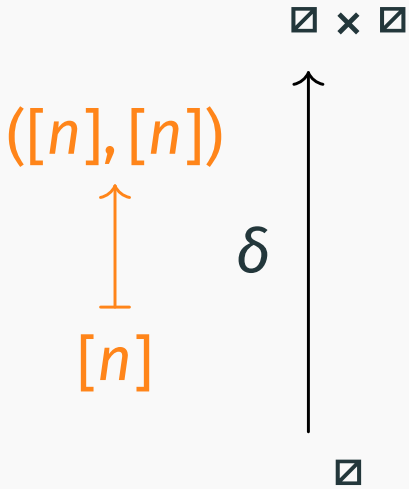
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$$\begin{array}{c} \square \times \square \\ \uparrow \delta \\ \square \end{array}$$

$([n], [n])$

\uparrow

$[n]$

$$\begin{array}{c} \mathcal{S}^{\square^{\text{op}}} \cong \text{Set}^{(\square \times \square)^{\text{op}}} \\ \downarrow B \cong \delta^* \\ \mathcal{S} \end{array}$$

Why Cartesian cubes?

δ has a right adjoint $\Rightarrow \mathbb{P}$ is also a restriction functor.

$$\begin{array}{c}
 \begin{array}{c} ([n], [n]) \\ \uparrow \\ [n] \end{array} \quad \begin{array}{c} \delta \\ \uparrow \\ \mathbb{Q} \times \mathbb{Q} \end{array} \quad \begin{array}{c} \dashv \\ \downarrow \end{array} \quad \begin{array}{c} \times \\ \downarrow \\ [n + m] \end{array} \quad \begin{array}{c} ([n], [m]) \\ \downarrow \\ [n + m] \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{S}^{\mathbb{Q}^{op}} \cong \text{Set}^{(\mathbb{Q} \times \mathbb{Q})^{op}} & & \\
 \downarrow & \dashv & \uparrow \\
 B \cong \delta^* & & \mathbb{P} \cong \times^* \\
 & & \downarrow \\
 & & \mathcal{S}
 \end{array}$$

More nice adjoint computation

Recall from e.g. (Joyal and Tierney 2007) skeletal adjunction

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$$\begin{array}{ccc} & \square \times \square & \\ \begin{array}{c} ([n], [m]) \\ \downarrow \\ [m] \end{array} & \begin{array}{c} \downarrow \pi_2 \\ \dashv \\ \uparrow i_2 \\ \square \end{array} & \end{array}$$

More nice adjoint computation

Recall from e.g. (Joyal and Tierney 2007) skeleta adjunction

$$\begin{array}{c}
 \begin{array}{ccc}
 ([n], [m]) & & ([0], [m]) \\
 \downarrow & & \uparrow \\
 \pi_2 & \dashv & i_2 \\
 \downarrow & & \uparrow \\
 \square & & [m]
 \end{array}
 \end{array}$$

$$\begin{array}{ccccc}
 \mathcal{S}^{\square^{\text{op}}} & \cong & \text{Set}^{(\square \times \square)^{\text{op}}} \\
 \uparrow & & \downarrow & & \uparrow \\
 \text{sk}_0 & & \text{tr}_0 \cong i_2^* & & \text{cosk}_0 \\
 & & \downarrow & & \\
 & & \mathcal{S} & &
 \end{array}$$

More nice adjoint computation

$$\begin{array}{ccc} \square \times \square & & \\ \uparrow & \dashv & \downarrow \\ \delta & & \times \\ \square & & \end{array}$$

$$\begin{array}{ccc} \square \times \square & & \\ \downarrow & \dashv & \uparrow \\ \pi_2 & & i_2 \\ \square & & \end{array}$$

More nice adjoint computation

$$\begin{array}{ccc}
 \square \times \square & & \square \times \square \\
 \uparrow & & \downarrow \\
 \delta & \dashv & \pi_2 \\
 \downarrow & & \uparrow \\
 \square & & \square
 \end{array} \times \implies \begin{array}{ccc}
 \square \times \square & & \square \times \square \\
 \downarrow & & \uparrow \\
 \pi_2 & \dashv & i_2 \\
 \uparrow & & \downarrow \\
 \square & & \square
 \end{array}$$

More nice adjoint computation

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \square \times \square \\ \uparrow \quad \downarrow \\ \delta \quad \dashv \quad \times \\ \downarrow \quad \uparrow \\ \square \end{array} \quad \begin{array}{c} \Rightarrow \\ \Rightarrow \end{array} \quad \begin{array}{c} \square \times \square \\ \downarrow \quad \uparrow \\ \pi_2 \quad \dashv \quad i_2 \\ \downarrow \quad \uparrow \\ \square \end{array} \quad \begin{array}{c} \Leftarrow \\ \Leftarrow \end{array}$$

More nice adjoint computation

$$\begin{array}{c}
 \square \times \square \\
 \uparrow \quad \downarrow \\
 \delta \quad \dashv \quad \times \\
 \downarrow \quad \uparrow \\
 \square
 \end{array}
 \xRightarrow{\quad}
 \begin{array}{c}
 \square \times \square \\
 \downarrow \quad \uparrow \\
 \pi_2 \quad \dashv \quad i_2 \\
 \downarrow \quad \uparrow \\
 \square
 \end{array}$$

$$\begin{array}{c}
 \mathcal{S}^{\square^{\text{op}}} \\
 \uparrow \mathbb{P} \\
 \mathcal{S}
 \end{array}
 \begin{array}{c}
 \xRightarrow{\quad} \\
 \xRightarrow{\quad}
 \end{array}
 \begin{array}{c}
 \text{sk}_0 \\
 \text{cosk}_0
 \end{array}$$

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$$\text{sk}_0 \quad \begin{array}{c} \curvearrowright \\ \text{---} \Rightarrow \text{---} \Rightarrow \text{---} \\ \text{---} \curvearrowleft \end{array} \quad \begin{array}{c} \mathcal{S}^{\square^{\text{op}}} \\ \uparrow \text{---} \downarrow \\ \mathbb{P} \\ \downarrow \text{---} \uparrow \\ \mathcal{S} \end{array} \quad \text{cosk}_0$$

$$\text{tr}_0 \quad \begin{array}{c} \curvearrowright \\ \text{---} \Rightarrow \text{---} \Rightarrow \text{---} \\ \text{---} \curvearrowleft \end{array} \quad \begin{array}{c} \mathcal{S}^{\square^{\text{op}}} \\ \downarrow \text{---} \uparrow \\ B \\ \downarrow \text{---} \uparrow \\ \mathcal{S} \end{array}$$

Quotient types?

1.

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2.

-
-

3.

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1. Quotient $B\mathbb{G}$ not necessarily fibrant.

-
-

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-
-

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3. 1-Groupoid quotients could be implemented in HoTTLean's groupoid semantics of HoTT (Veltri and Weide 2021)

Regularity

Let $f : X \rightarrow Y$ be a function between sets. Compute kernel K_f by

$$\begin{array}{ccc} K_f & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \delta \\ X^2 & \xrightarrow{f^2} & Y^2 \end{array}$$

$$\begin{array}{ccc} K_f & \longrightarrow & Y \\ \downarrow \downarrow & & \parallel \parallel \\ X & \xrightarrow{f} & Y \end{array}$$

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Forgetful functor $\mathbf{ER} \rightarrow \mathbf{Set}$ is fibered, and above diagram is the cartesian lift of f .

Regularity

Obtain (epi, mono) factorization of f by taking the quotient

$$\begin{array}{ccccc}
 X & \dashrightarrow & K_f & \longrightarrow & Y \\
 \delta \downarrow & & \downarrow & \lrcorner & \downarrow \delta \\
 X^2 & \xrightarrow{1^2} & X^2 & \xrightarrow{f^2} & Y^2
 \end{array}$$

$$\begin{array}{ccccc}
 X & \dashrightarrow & K_f & \longrightarrow & Y \\
 \parallel \parallel & & \downarrow \downarrow & & \parallel \parallel \\
 X & \xlongequal{\quad} & X & \xrightarrow{f} & Y
 \end{array}$$

$$\begin{array}{ccccc}
 B\delta_X = X & \twoheadrightarrow & BKer_X = \text{Im } f & \twoheadrightarrow & B\delta_Y = Y \\
 & \searrow & & \nearrow & \\
 & & f & &
 \end{array}$$

Regularity

$\text{tr}_0 : \mathcal{S}^{\square^{\text{op}}} \rightarrow \mathcal{S}$ taking $\mathbb{G} \mapsto G_0$ is fibered. For $f : X \rightarrow Y$ in \mathcal{S} , define **nerve** Nf as Cartesian lift of f , into path cubical object on Y .

$$\bar{f} : Nf \rightarrow \mathbb{P}Y$$

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \uparrow \downarrow & & \uparrow \downarrow \\
 \dots & & \dots \\
 (Nf)_2 & \longrightarrow & Y^{I^2} \\
 \uparrow \downarrow & & \uparrow \downarrow \\
 \dots & & \dots \\
 (Nf)_1 & \longrightarrow & Y^I \\
 \uparrow \downarrow & & \uparrow \downarrow \\
 X & \xrightarrow{f} & Y
 \end{array}$$

Regularity

Obtain a (strong-epi, mono) factorization of f (w.r.t model structure on \mathcal{S}).

$$\begin{array}{ccccc}
 \mathbb{P}X & \overset{\cdot\cdot\cdot\cdot}{\longrightarrow} & Nf & \xrightarrow{\quad \overline{f} \quad} & \mathbb{P}Y \\
 & \searrow & & \nearrow & \\
 & & \mathbb{P}f & & \\
 \\
 B(\mathbb{P}X) = X & \longrightarrow\!\!\!\twoheadrightarrow & B(Nf) = \operatorname{Im} f & \xrightarrow{\quad B\overline{f} \quad} & B(\mathbb{P}Y) = Y \\
 & \searrow & & \nearrow & \\
 & & B(\mathbb{P}f) = f & &
 \end{array}$$

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 & \searrow & & \nearrow & \\
 & & B(\mathbb{P}f) = f & &
 \end{array}$$

Related lower dimensional analogue in (Bourke and Garner 2013).

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Exactness

All equivalence relations are kernels of their quotients. Not all cubical objects are nerves of their quotients

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Task ahead is to classify those \mathbb{G} .

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Task ahead is to classify those \mathbb{G} .

Should be the cubical objects admitting certain “ ∞ -groupoid structure”. See “Segal space” in (Joyal and Tierney 2007) for simplicial analogue.

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