Cubical Quotients

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Quotients in the category of Sets

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Surjections are in bijection with equivalence relations.

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Understand the higher-categorical version of this fact. This means:

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- Define **cubical objects** to generalize **equivalence relations**
- Describe the quotient of cubical objects, called **cubical quotients**.
- Regularity and exactness results for cubical quotients: classify cubical objects that are in bijection with their quotients - **cubical groupoids**.

Cartesian Cube category

The **(Cartesian) cube category $\square$**, is the free finite product category on

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & {[1]} \end{tikzcd}$$

such that [0] is the terminal object.

(Awodey 2023)

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Work in model category of **cubical sets** $\mathcal{S} = \mathrm{cSet} = \mathrm{Set}^{\boxslash^{\mathrm{op}}}$; think of fibrant cubical sets as “spaces” or “types”.

(Awodey 2023)

Cubical objects

Higher analogue of equivalence relations: cubical objects.

$$
\begin{tikzcd}
\vdots \arrow[d, shift left=3] \arrow[d, shift right=3] \arrow[d, phantom, "\cdots"] \\
G_1 \arrow[d, shift left=3, "d_1"] \arrow[d, shift right=3, "d_0"'] \\
G_0 \arrow[u]
\end{tikzcd}
$$

diagram in category $\mathcal{S}$

object $\mathbb{G}$ in category $\mathcal{S}^{\square^{\mathrm{op}}}$

Examples of cubical objects

$$
\begin{tikzcd}
\vdots \arrow[d, equal, shift right=4] \arrow[d, phantom, "\cdots"] \arrow[d, equal, shift left=4] \\
E \arrow[d, equal, shift right=4] \arrow[d, phantom, "\cdots"] \arrow[d, equal, shift left=4] \\
E \arrow[d, shift right=3, "\sigma"'] \arrow[d, shift left=3, "\tau"] \\
X \arrow[u]
\end{tikzcd}
$$

Examples of cubical objects

$$
\begin{tikzcd}
\vdots \arrow[d, shift right=4, equal] \arrow[d, phantom, "\cdots"] \arrow[d, shift left=4, equal] \\
E \arrow[d, shift right=4, equal] \arrow[d, phantom, "\cdots"] \arrow[d, shift left=4, equal] \\
E \arrow[d, shift right=3, "\sigma"'] \arrow[d, shift left=3, "\tau"] \\
X \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\vdots \arrow[d, shift right=4, equal] \arrow[d, phantom, "\cdots"] \arrow[d, shift left=4, equal] \\
\mathbb{Z} \times \mathbb{Z} \arrow[d, "+" description] \arrow[d, shift right=6, phantom, "\cdots"] \arrow[d, shift left=6, phantom, "\cdots"] \\
\mathbb{Z} \arrow[d, shift right=4, "!"'] \arrow[d, shift left=4, "!"] \\
1 \arrow[u, "0" description]
\end{tikzcd}
$$

$$\begin{tikzcd}[column sep=huge]
\vdots \arrow[d, equal, shift right=4] \arrow[d, equal, shift left=4] \arrow[d, phantom, "\cdots"] & \vdots \arrow[d, equal, shift right=4] \arrow[d, equal, shift left=4] \arrow[d, phantom, "\cdots"] & \vdots \arrow[d, shift left=4] \arrow[d, phantom, "\cdots"] \\
E \arrow[d, equal, shift right=4] \arrow[d, equal, shift left=4] \arrow[d, phantom, "\cdots"] & \mathbb{Z} \times \mathbb{Z} \arrow[d, "+" description] \arrow[d, phantom, "\cdots\qquad\cdots"] & X^4 \arrow[u, shift left=4] \arrow[d, shift left=4] \arrow[d, phantom, "\cdots"] \\
E \arrow[d, "\sigma"', shift right=4] \arrow[d, "\tau", shift left=4] & \mathbb{Z} \arrow[d, "!"', shift right=4] \arrow[d, "!", shift left=4] & X^2 \arrow[u, shift left=4] \arrow[d, "\pi_1"', shift right=4] \arrow[d, "\pi_2", shift left=4] \\
X \arrow[u] & 1 \arrow[u, "0" description] & X \arrow[u, "\delta" description]
\end{tikzcd}$$

Examples of cubical objects

$$
\begin{tikzcd}
\vdots \arrow[d, equal] \arrow[d, phantom, "\cdots"] \\
E \arrow[d, equal] \arrow[d, phantom, "\cdots"] \\
E \arrow[d, shift right=2, "\sigma"'] \arrow[d, shift left=2, "\tau"] \\
X \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \arrow[d, equal] \arrow[d, phantom, "\cdots"] \\
\mathbb{Z} \times \mathbb{Z} \arrow[d, "+"] \arrow[d, phantom, "\cdots\quad\cdots"] \\
\mathbb{Z} \arrow[d, shift right=2, "!"'] \arrow[d, shift left=2, "!"] \\
1 \arrow[u, "0"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \arrow[d, shift left] \arrow[d, phantom, "\cdots"] \\
X^4 \arrow[u, shift left] \arrow[d, shift left] \arrow[d, phantom, "\cdots"] \\
X^2 \arrow[u, shift left] \arrow[d, shift right=2, "\pi_1"'] \arrow[d, shift left=2, "\pi_2"] \\
X \arrow[u, "\delta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \arrow[d, shift left] \arrow[d, phantom, "\cdots"] \\
X^{I^2} \arrow[u, shift left] \arrow[d, shift left] \arrow[d, phantom, "\cdots"] \\
X^{I} \arrow[u, shift left] \arrow[d, shift right=2] \arrow[d, shift left=2] \\
X \arrow[u]
\end{tikzcd}
$$

I = interval = $y[1]$ $\qquad$ $y : \boxslash \to \mathcal{S}$

Path and quotient functors

$$\begin{tikzcd} \mathcal{S}^{\boxtimes^{\mathrm{op}}} \\ \mathcal{S} \arrow[u, "\mathbb{P}"'] \end{tikzcd}$$

$$\mathbb{P}X = X^{I^{(-)}}$$

Path and quotient functors

$$
\begin{tikzcd}
\mathcal{S}^{\boxslash^{\mathrm{op}}} \arrow[d, shift right=3ex, "B"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{S} \arrow[u, shift right=3ex, "\mathbb{P}"']
\end{tikzcd}
\qquad\qquad
\mathbb{P}X = X^{I^{(-)}}
$$

Path and quotient functors

$B\mathbb{G}$ = quotient of $\mathbb{G}$

$$\begin{tikzcd} \mathcal{S}^{\boxslash^{\mathrm{op}}} \arrow[d, shift right=3ex, "B"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{S} \arrow[u, shift right=3ex, "\mathbb{P}"'] \end{tikzcd}$$

$\mathbb{P}X = X^{I^{(-)}}$

Quotient universal property

$$\mathcal{S}(B\mathbb{G}, X) \cong \mathcal{S}^{\boxslash^{\mathrm{op}}}(\mathbb{G}, \mathbb{P}X)$$

$B\mathbb{G} =$ initial space with G_0 points, G_1 paths, G_2 homotopies, ...

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$$B\mathbb{G} = \text{ initial space with } G_0 \text{ points}, G_1 \text{ paths}, G_2 \text{ homotopies}, \ldots$$

Quotient is homotopy colimit of diagram $\mathbb{G}$. Well known in simplicial setting (Bousfield and Kan 1972, Ch XII, 4.3).

Examples of cubical quotients

$$\begin{tikzcd}
\vdots \arrow[d, equal, shift left=3] \arrow[d, equal, shift right=3] \arrow[d, phantom, "\cdots"] & \\
E \arrow[d, equal, shift left=3] \arrow[d, equal, shift right=3] \arrow[d, phantom, "\cdots"] & \\
E \arrow[d, shift right=3, "\sigma"'] \arrow[d, shift left=3, "\tau"] & \\
X \arrow[u] \arrow[r] & \mathbb{BG} = X/E
\end{tikzcd}$$

Examples of cubical quotients

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\begin{tikzcd}
\vdots \arrow[d, phantom, "\cdots"] & & \vdots \arrow[d, phantom, "\cdots"] \\
E \arrow[d, phantom, "\cdots"] & & \mathbb{Z} \times \mathbb{Z} \arrow[d, "+"] \\
E \arrow[d, shift right=2, "\sigma"'] \arrow[d, shift left=2, "\tau"] & & \mathbb{Z} \arrow[d, shift right=3, "!"'] \arrow[d, shift left=3, "!"] \\
X \arrow[u] \arrow[r] & B\mathbb{G} = X / E & 1 \arrow[u, "0"] \arrow[r] & B\mathbb{G} = S^1
\end{tikzcd}
$$

Examples of cubical quotients

$$
\begin{tikzcd}
\vdots \arrow[d, equal, shift right=2] \arrow[d, equal, shift left=2] \arrow[d, phantom, "\cdots"] \\
E \arrow[d, equal, shift right=2] \arrow[d, equal, shift left=2] \arrow[d, phantom, "\cdots"] \\
E \arrow[d, shift right=3, "\sigma"'] \arrow[d, shift left=3, "\tau"] \\
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\mathbb{Z} \times \mathbb{Z} \arrow[d, "+" description] \arrow[d, phantom, shift right=4, "\cdots"] \arrow[d, phantom, shift left=4, "\cdots"] \\
\mathbb{Z} \arrow[d, shift right=3, "!"'] \arrow[d, shift left=3, "!"] \\
1 \arrow[u, "0" description] \arrow[r] & B\mathbb{G} = S^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \arrow[d, shift left=3] \arrow[d, phantom, "\cdots"] \\
X^4 \arrow[u, shift left=3] \arrow[d, shift left=3] \arrow[d, phantom, "\cdots"] \\
X^2 \arrow[u, shift left=3] \arrow[d, shift right=3, "\pi_1"'] \arrow[d, shift left=3, "\pi_2"] \\
X \arrow[u, "\delta" description] \arrow[r] & B\mathbb{G} = \|X\|_{-1}
\end{tikzcd}
$$

Examples of cubical quotients

$$\begin{tikzcd}
\vdots \arrow[d, equal, shift right=3] \arrow[d, equal, shift left=3] \arrow[d, phantom, "\cdots"] \\
E \arrow[d, equal, shift right=3] \arrow[d, equal, shift left=3] \arrow[d, phantom, "\cdots"] \\
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\mathbb{Z} \arrow[d, shift right=5, "!"'] \arrow[d, shift left=5, "!"] \\
1 \arrow[u, "0" description] \arrow[r] & B\mathbb{G} = S^1
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\vdots \arrow[d, shift left=3] \arrow[d, phantom, "\cdots"] \\
X^4 \arrow[u, shift left=3] \arrow[d, shift left=3] \arrow[d, phantom, "\cdots"] \\
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Why Cartesian cubes?

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δ has a right adjoint $\Rightarrow$ $\mathbb{P}$ is also a restriction functor.

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$$
\begin{tikzcd}
([n],[n]) & \boxslash \times \boxslash \\
{[n]} \arrow[u, mapsto] & \boxslash \arrow[u, "\delta"]
\end{tikzcd}
$$

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$$\begin{tikzcd} & & \square \times \square \\ ([n],[n]) & & \\ {[n]} \arrow[u, mapsto] & & \square \arrow[uu, "\delta"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}^{\square^{\mathrm{op}}} \cong \mathrm{Set}^{(\square \times \square)^{\mathrm{op}}} \arrow[d, "B \cong \delta^*"'] \\ \mathcal{S} \end{tikzcd}$$

Why Cartesian cubes?

δ has a right adjoint $\Rightarrow$ $\mathbb{P}$ is also a restriction functor.

$$
\begin{array}{ccc}
([n],[n]) & & ([n],[m]) \\
\uparrow & & \downarrow \\
[n] & & [n+m]
\end{array}
\qquad
\begin{tikzcd}
\boxslash \times \boxslash \arrow[d, shift left=2, "\times"] \\
\boxslash \arrow[u, shift left=2, "\delta"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{S}^{\boxslash^{\mathrm{op}}} \cong \mathrm{Set}^{(\boxslash\times\boxslash)^{\mathrm{op}}} \arrow[d, shift right=2, "B \cong \delta^*"'] \\
\mathcal{S} \arrow[u, shift right=2, "\mathbb{P} \cong \times^*"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

More nice adjoint computation

Recall from e.g. (Joyal and Tierney 2007) skeleta adjunction

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$$([n],[m]) \mapsto [m]$$

$$\begin{tikzcd} \boxslash \times \boxslash \arrow[d, shift right=3, "\pi_2"'] \arrow[d, phantom, "\dashv"] \\ \boxslash \arrow[u, shift right=3, "i_2"'] \end{tikzcd}$$

Recall from e.g. (Joyal and Tierney 2007) skeleta adjunction

$$
\begin{tikzcd}
([n],[m]) \arrow[d, mapsto] & \boxslash \times \boxslash \arrow[d, shift right=4, "\pi_2"'] \arrow[d, phantom, "\dashv"] & ([0],[m]) \\
{[m]} & \boxslash \arrow[u, shift right=4, "i_2"'] & {[m]} \arrow[u, mapsto]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{S}^{\boxslash^{\mathrm{op}}} \cong \mathrm{Set}^{(\boxslash\times\boxslash)^{\mathrm{op}}} \arrow[d, "\mathrm{tr}_0 \cong i_2^*" description] \\
\mathcal{S} \arrow[u, bend left=60, "\mathrm{sk}_0"] \arrow[u, bend right=60, "\mathrm{cosk}_0"']
\end{tikzcd}
$$

More nice adjoint computation

$$
\begin{tikzcd}
\boxslash \times \boxslash \arrow[d, shift left=3ex, "\times"] \\
\boxslash \arrow[u, shift left=3ex, "\delta"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\boxslash \times \boxslash \arrow[d, shift right=3ex, "\pi_2"'] \\
\boxslash \arrow[u, shift right=3ex, "i_2"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxslash \times \boxslash \arrow[d, shift left=4, "\times"] \arrow[d, phantom, "\dashv"] \\
\boxslash \arrow[u, shift left=4, "\delta"]
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\boxslash \times \boxslash \arrow[d, shift right=4, "\pi_2"'] \arrow[d, phantom, "\dashv"] \\
\boxslash \arrow[u, shift right=4, "i_2"']
\end{tikzcd}
$$

More nice adjoint computation

$$
= \qquad
\begin{tikzcd}
\boxslash \times \boxslash \arrow[d, shift left=2, "\times"] \\
\boxslash \arrow[u, shift left=2, "\delta"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\qquad \Longrightarrow \qquad
\begin{tikzcd}
\boxslash \times \boxslash \arrow[d, shift right=2, "\pi_2"'] \\
\boxslash \arrow[u, shift right=2, "i_2"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\qquad \Longleftarrow
$$

More nice adjoint computation

$$\begin{tikzcd}[column sep=large] & \boxslash\times\boxslash \arrow[dd, shift left=3, "\times"] & & \boxslash\times\boxslash \arrow[dd, shift right=3, "\pi_2"'] & \\ {} \arrow[r, Rightarrow, no head, color=orange] & {} \arrow[r, phantom, "\dashv"] & {} \arrow[r, Rightarrow, color=blue] & {} \arrow[r, phantom, "\dashv"] & {} & {} \arrow[l, Rightarrow, color=orange] \\ & \boxslash \arrow[uu, shift left=3, "\delta"] & & \boxslash \arrow[uu, shift right=3, "i_2"'] & \end{tikzcd}$$

$$\begin{tikzcd} & \mathcal{S}^{\boxslash^{\mathrm{op}}} & \\ {} \arrow[r, Rightarrow, color=blue] & \mathbb{P} \arrow[r, Rightarrow, color=orange] & {} \\ & \mathcal{S} \arrow[uu, bend left=80, "\mathrm{sk}_0"] \arrow[uu, bend right=80, "\mathrm{cosk}_0"'] \arrow[uu] & \end{tikzcd}$$

More nice adjoint computation

$$
\color{orange}{=}\quad
\begin{tikzcd}
\boxslash \times \boxslash \arrow[d, shift left=3ex, "\times"] \\
\boxslash \arrow[u, shift left=3ex, "\delta"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\quad\color{blue}{\Longrightarrow}\quad
\begin{tikzcd}
\boxslash \times \boxslash \arrow[d, shift right=3ex, "\pi_2"'] \\
\boxslash \arrow[u, shift right=3ex, "i_2"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\quad\color{orange}{\Longleftarrow}
$$

$$
\begin{tikzcd}
\mathcal{S}^{\boxslash^{\mathrm{op}}} \\
\mathcal{S} \arrow[u, "\mathbb{P}" description] \arrow[u, bend left=90, "\mathrm{sk}_0"] \arrow[u, bend right=90, "\mathrm{cosk}_0"']
\end{tikzcd}
\qquad \mathrm{sk}_0 \color{blue}{\Longrightarrow} \mathbb{P} \color{orange}{\Longrightarrow} \color{black}{\mathrm{cosk}_0}
\qquad\qquad
\begin{tikzcd}
\mathcal{S}^{\boxslash^{\mathrm{op}}} \arrow[d, "B"] \arrow[d, bend right=90, "\mathrm{tr}_0"'] \\
\mathcal{S}
\end{tikzcd}
\qquad \mathrm{tr}_0 \overset{\color{orange}{q}}{\color{orange}{\Longrightarrow}} B
$$

Quotient types?

1.
   -
   -
2.
   -
   -
3.

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1. Quotient $B\mathbb{G}$ not necessarily fibrant.
   -
   -
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   - Analyse bicubical sets with fibrant diagonal (i.e. those that $B\mathbb{G}$ is automatically fibrant. (Moerdijk 1989)
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3. 1-Groupoid quotients could be implemented in HoTTLean's groupoid semantics of HoTT (Veltri and Weide 2021)

Regularity

Let $f : X \to Y$ be a function between sets. Compute kernel K_f by

$$
\begin{tikzcd}
K_f \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "\delta"] \\
X^2 \arrow[r, "f^2"'] & Y^2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
K_f \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & Y \arrow[d, equal, shift right] \arrow[d, equal, shift left] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Regularity

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$$\begin{tikzcd} K_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\delta"] \\ X^2 \arrow[r, "f^2"'] & Y^2 \end{tikzcd} \qquad \begin{tikzcd} K_f \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & Y \arrow[d, equal, shift left] \arrow[d, equal, shift right] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Forgetful functor $\mathrm{ER} \to \mathrm{Set}$ is fibered, and above diagram is the cartesian lift of f.

Regularity

Obtain (epi, mono) factorization of f by taking the quotient

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, "\delta"'] & K_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\delta"] \\
X^2 \arrow[r, "1^2"'] & X^2 \arrow[r, "f^2"'] & Y^2
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, shift right=2, equal] \arrow[d, shift left=2, equal] & K_f \arrow[r] \arrow[d, shift right=2] \arrow[d, shift left=2] & Y \arrow[d, shift right=2, equal] \arrow[d, shift left=2, equal] \\
X \arrow[r, equal] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
B\delta_X = X \arrow[r, two heads] \arrow[rr, bend right, "f"'] & B\mathrm{Ker}_X = \mathrm{Im}\, f \arrow[r, tail] & B\delta_Y = Y
\end{tikzcd}$$

Regularity

$\mathrm{tr}_0 : \mathcal{S}^{\square^{\mathrm{op}}} \to \mathcal{S}$ taking $\mathbb{G} \mapsto G_0$ is fibered. For $f : X \to Y$ in $\mathcal{S}$, define **nerve** Nf as Cartesian lift of f, into path cubical object on Y.

$$\overline{f} : Nf \to \mathbb{P}Y$$

$$\begin{tikzcd}
\vdots \arrow[d, shift left=2] \arrow[d, phantom, "\cdots"] & \vdots \arrow[d, shift left=2] \arrow[d, phantom, "\cdots"] \\
(Nf)_2 \arrow[u, shift left=2] \arrow[r] \arrow[d, shift left=2] \arrow[d, phantom, "\cdots"] & Y^{I^2} \arrow[u, shift left=2] \arrow[d, shift left=2] \arrow[d, phantom, "\cdots"] \\
(Nf)_1 \arrow[u, shift left=2] \arrow[r] \arrow[d, shift left=2] \arrow[d, shift right=2] & Y^{I} \arrow[u, shift left=2] \arrow[d, shift left=2] \arrow[d, shift right=2] \\
X \arrow[u] \arrow[r, "f"'] & Y \arrow[u]
\end{tikzcd}$$

Regularity

Obtain a (strong-epi, mono) factorization of f (w.r.t model structure on $\mathcal{S}$).

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\mathbb{P}X \arrow[r, dashed] \arrow[rr, bend right, "\mathbb{P}f"'] & Nf \arrow[r, "\overline{f}"] & \mathbb{P}Y
\end{tikzcd}$$

$$\begin{tikzcd}
B(\mathbb{P}X) = X \arrow[r, two heads] \arrow[rr, bend right, "B(\mathbb{P}f) = f"'] & B(Nf) = \operatorname{Im} f \arrow[r, tail, "B\overline{f}"] & B(\mathbb{P}Y) = Y
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Related lower dimensional analogue in (Bourke and Garner 2013).

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Should be the cubical objects admitting certain "∞-groupoid structure". See "Segal space" in (Joyal and Tierney 2007) for simplicial analogue.

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