Model Theory

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Contents

Chapter 1

Pure Model Theory

1.1 Basics

This section follows Marker's book on Model Theory [12] but with more emphasis on where things are happening, i.e. what signature we are working in, and also working with many-sorted signatures. For a quick introduction to model theory we recommend reading up to where morphisms are introduced. Then it is worth following routes to particular results such as quantifier elimination in algebraically closed fields or Morley Rank being Krull dimension, which may help to motivate the rest of the theory.

1.1.1 Signatures

Definition – Signature

A signature [†] $\Sigma = (S, F, R)$ consists of

- S a set of *sort* symbols.
- F a set of *function* symbols. Each function symbol $f \in F$ has *arity* $(A_1, \ldots, A_n, B)$, a tuple of sort symbols in S. We write $f : A_1 \times \cdots \times A_n \to B$ to denote f with its arity.
- R a set of *relation* symbols. Each relation symbol $r \in R$ has *arity* $(A_1, \ldots, A_n)$, a tuple of sort symbols in S. We write $r \hookrightarrow A_1 \times \cdots \times A_n$ to denote r with its arity.

Given a signature Σ, we may refer to its sort, function and relation symbol sets as $\Sigma_{\text{sor}}, \Sigma_{\text{fun}}, \Sigma_{\text{rel}}$. We should think of function arity as the source and target of a function and the relation arity as the product that the relation is a subset of.

For convenience we single out 0-ary functions - $c \in F$ with arity (B) - and call them *constant* symbols. For each sort B we label the set of such constant symbols B_{con} and think of these as 'elements' of the sort B and write $c : B$. We denote the set of all constant symbols in a signature by

$$\Sigma_{\text{con}} := \bigcup_{A \in \Sigma_{\text{sor}}} A_{\text{con}}$$

[†] Many authors call Σ a *language* instead.

Our signatures are multi-sorted, meaning we can have elements and variables living in different spaces. For example the 'universe of all sets', groups, rings and partial orders are all 1-sorted with just one underlying set, whereas group actions and modules are 2-sorted. If we are working in the 1-sorted case we will not mention the sort at all, because there is only one.

EXAMPLE. *The signature of rings will be used to define the theory of rings, the theory of integral domains, the theory of fields, and so on. The signature with just a single binary relation will be used to define the theory of partial orders with the interpretation of the relation as $<$, to define the theory of equivalence relations with the interpretation of the relation as $\sim$, and to define the theory* ZFC *with the relation interpreted as $\in$. The signature of modules will be used to define the theory of modules, the theory of vector spaces, and so on.*

Definition – Σ-terms

Let Σ be a signature. For each sort $A \in \Sigma_{\text{sor}}$ we create A_{var}, a countable set of *variable symbols* of type A, (A_{var} and B_{var} are disjoint for distinct sorts).

We inductively define the set of terms Σ_{ter} with their types and their associated set of typed variables $\text{tv}_\star$. We write $t : A$ to mean t is a Σ-term of type A.

| If A is a sort and $v \in A_{\text{var}}$ then $v : A$ is a Σ-term. It has typed variables $\text{tv}_v = \{v : A\}$.

| If $f \in \Sigma_{\text{fun}}$ is a function symbol $f : A_1 \times \cdots \times A_n \to B$ and for each $1 \leq i \leq n$ we have $t_i : A_i$. We introduce notation $t = (t_1, \ldots, t_n) : A_1 \times \cdots \times A_n$. Then $f(t) : B$ is a Σ-term. It has typed variables $\text{tv}_{f(t)} = \{t_i : A_i\}_{i=1}^n$.

If there are repeated sorts we might write A^n instead of $A \times \cdots \times A$. Note that given no terms, constant symbols $c : B$ are terms of type B.

EXAMPLE. *In the signature of rings, terms will be multivariable polynomials over $\mathbb{Z}$ since they are sums and products of constant symbols $0, 1$ and variable symbols. In the signature of modules, terms will be finite linear combinations, where the coefficients are polynomials over $\mathbb{Z}$. In the signature of binary relations there are no constant or function symbols so the only terms are variables.*

Definition – Interpretation and structures

Let $\mathcal{C}$ be a category with finite products[†]. Given a signature Σ, a Σ-*structure* $\mathcal{M}$ - or an interpretation $\star_\Sigma^{\mathcal{M}}$ (which we just write as $\star^{\mathcal{M}}$ here) of the signature in $\mathcal{C}$ is the following

- Each sort symbol $A \in \Sigma_{\text{sor}}$ is interpreted as an object $A^{\mathcal{M}}$ of $\mathcal{C}$.
- Each function symbol $f : A_1 \times \cdots \times A_n \to B$ is interpreted as a morphism from the product in $\mathcal{C}$

$$f^{\mathcal{M}} : A_1^{\mathcal{M}} \times \cdots \times A_n^{\mathcal{M}} \to B^{\mathcal{M}}$$

- Each relation symbol $r \hookrightarrow A_1 \times \cdots \times A_n$ is interpreted as a sub-object of the product in **Set**

$$r^{\mathcal{M}} \hookrightarrow A_1^{\mathcal{M}} \times \cdots \times A_n^{\mathcal{M}}$$

[†] If you prefer you can just take $\mathcal{C}$ to be **Set** the category of sets.

The structures in a signature will become the models of theories. For example $\mathbb{Z}$ is a structure in the signature of rings, a model of the theory of rings but not a model of the theory of fields. In the signature of binary relations, $\mathbb{N}$ with the usual ordering $\leq$ is a structure that models of the theory of partial orders but not the theory of equivalence relations.

Definition – Σ-morphism, Σ-embedding

The collection of all Σ-structures over a category $\mathcal{C}$ forms a category denoted by $\mathbf{Str}(\Sigma, \mathcal{C})$, which has objects as Σ-structures and morphisms as Σ-morphisms:

Suppose $\mathcal{M}, \mathcal{N}$ are Σ-structures in $\mathcal{C}$ and for each sort symbol A we have $\iota_A : A^{\mathcal{M}} \to A^{\mathcal{N}}$. We say $\iota := (\iota_A)_{A \in \Sigma_{\text{sor}}}$ is a Σ-*morphism* from $\mathcal{M}$ to $\mathcal{N}$ when

- For all function symbols $f : A_1 \times \cdots \times A_n \to B$ this diagram commutes:

$$\begin{tikzcd} A_1^{\mathcal{M}} \times \cdots \times A_n^{\mathcal{M}} \arrow[r, "f^{\mathcal{M}}"] \arrow[d, "\iota_{A_1} \times \cdots \times \iota_{A_n}"'] & B^{\mathcal{M}} \arrow[d, "\iota_B"] \\ A_1^{\mathcal{N}} \times \cdots \times A_n^{\mathcal{N}} \arrow[r, "f^{\mathcal{N}}"] & B^{\mathcal{N}} \end{tikzcd}$$

- For all relation symbols $r \hookrightarrow A_1 \times \cdots \times A_n$ there exists a morphism $\iota_r : r^{\mathcal{M}} \to r^{\mathcal{N}}$ such that

$$\begin{tikzcd} r^{\mathcal{M}} \arrow[r, hook] \arrow[d, "\iota_r"'] & A_1^{\mathcal{M}} \times \cdots \times A_n^{\mathcal{M}} \arrow[d, "\iota_{A_1} \times \cdots \times \iota_{A_n}"] \\ r^{\mathcal{N}} \arrow[r, hook] & A_1^{\mathcal{N}} \times \cdots \times A_n^{\mathcal{N}} \end{tikzcd}$$

ι is called a Σ-embedding (and $\mathcal{N}$ a Σ-extension) when each ι_A is a monomorphism and $r^{\mathcal{M}}$ is the pullback of the last diagram. Restricting morphisms between objects to purely Σ-embeddings results in a subcategory of $\mathbf{Str}(\Sigma, C)$.

The notion of morphisms here will be the same as that of morphisms in the algebraic setting. For example in the signature of groups, preserving interpretation of function symbols says the identity is sent to the identity, multiplication is preserved, and inverses are sent to inverses.

Definition – Interpretation of terms

Given a Σ-structure $\mathcal{M}$ in category $\mathcal{C}$ and a Σ-term $t : B$ with $\text{tv}_t = \{v_1 : A_1, \ldots, v_n : A_n\}$. Then we can naturally interpret t in the Σ-structure $\mathcal{M}$ as a morphism in $\mathcal{C}$

$$t^{\mathcal{M}} : A_1^{\mathcal{M}} \times \cdots \times A_n^{\mathcal{M}} \to B^{\mathcal{M}}$$

that commutes with the interpretation of function symbols (see proof). We then refer to this map as *the* interpretation of the term t.[†]

The following notation will be used for the interpretation of a tuple of terms (which is the product of their interpretations)

$$s = (s_1, \ldots, s_n) \text{ gives us a map } s^{\mathcal{M}} := s_1^{\mathcal{M}} \times \cdots \times s_n^{\mathcal{M}}$$

[†]In type theory this is can be seen as the function type

$$\lambda v_1 : A_1, \ldots \lambda v_n : A_n, t^{\mathcal{M}} : A_1^{\mathcal{M}} \times \cdots \times A_n^{\mathcal{M}} \to B^{\mathcal{M}}$$

Proof. We use the inductive definition of t:

- If t is a variable $v : A$ then $\text{tv}_t = \{v : A\}$ and $t : A$. Thus it makes sense to define $t^{\mathcal{M}} : A^{\mathcal{M}} \to A^{\mathcal{M}}$ to be the identity.
- Suppose $t = f(s) : C$ for some function symbol $f : B_1 \times \cdots \times B_n \to C$ and terms $s : B_1 \times \cdots \times B_n$. Since $\text{tv}_t = \bigcup_{i=1}^n \text{tv}_{s_i}$ we must have $t^{\mathcal{M}} : \prod_{x:A \in \text{tv}_t} A^{\mathcal{M}} \to C^{\mathcal{M}}$. Induction gives each $s_i^{\mathcal{M}}$, and there is naturally the map
$$s^{\mathcal{M}} := s_1^{\mathcal{M}} \times \cdots \times s_n^{\mathcal{M}} : \prod_{x:A \in \text{tv}_t} A^{\mathcal{M}} \to B_1^{\mathcal{M}} \times \cdots \times B_n^{\mathcal{M}}$$
so we define the interpretation of t to be the composition $t^{\mathcal{M}} := f^{\mathcal{M}} \circ s^{\mathcal{M}}$.

□

Note that for constant symbols the interpretation has the empty product - a 'singleton' - as its domain hence is a constant map - an 'element' of the interpreted sort.

> Example. *Following the previous example, in the signature of rings terms will be multivariable integer polynomials and then terms interpreted in some structure - say a ring A - are then multivariable integer polynomial functions $A^n \to A$.*

The following is exactly what we expect - that terms are well behaved with respect to morphisms.

> **Proposition – Σ-morphisms commute with interpretation of terms**
> Given a Σ-morphism $\iota : \mathcal{M} \to \mathcal{N}$, we have that for any Σ-term t with $\mathrm{tv}_t = \{v_1 : A_1, \ldots, v_n : A_n\}$ this diagram commutes
>
> $$\begin{tikzcd} A_1^{\mathcal{M}} \times \cdots \times A_n^{\mathcal{M}} \arrow[r, "t^{\mathcal{M}}"] \arrow[d, "\iota_{A_1} \times \cdots \times \iota_{A_n}"'] & B^{\mathcal{M}} \arrow[d, "\iota_B"] \\ A_1^{\mathcal{N}} \times \cdots \times A_n^{\mathcal{N}} \arrow[r, "t^{\mathcal{N}}"] & B^{\mathcal{N}} \end{tikzcd}$$

Proof. We case on what t is:

- If t is a variable $v : A$ then $t^{\mathcal{M}} = \mathbb{1}_{A^{\mathcal{M}}}$ and $t^{\mathcal{N}} = \mathbb{1}_{A^{\mathcal{N}}}$ hence the following commutes

$$\begin{tikzcd} A^{\mathcal{M}} \arrow[r, "t^{\mathcal{M}}"] \arrow[d, "\iota_A"'] & A^{\mathcal{M}} \arrow[d, "\iota_A"] \\ A^{\mathcal{N}} \arrow[r, "t^{\mathcal{N}}"] & A^{\mathcal{N}} \end{tikzcd}$$

- Suppose $t = f(s) : C$ for some function symbol $f : B_1 \times \cdots \times B_n \to C$ and terms $s : B_1 \times \cdots \times B_n$. We consider the diagram

$$\begin{tikzcd} \prod_{x:A \in \mathrm{tv}_t} A^{\mathcal{M}} \arrow[rr, "t^{\mathcal{M}}"] \arrow[dr, "s_1^{\mathcal{M}} \times \cdots \times s_n^{\mathcal{M}}"] \arrow[dd, "\prod \iota_A"'] & & B^{\mathcal{M}} \arrow[dd, "\iota_B"] \\ & B_1^{\mathcal{M}} \times \cdots \times B_n^{\mathcal{M}} \arrow[ur, "f^{\mathcal{M}}"] \arrow[dd, "\iota_{B_1} \times \cdots \times \iota_{B_n}"] & \\ \prod_{x:A \in \mathrm{tv}_t} A^{\mathcal{N}} \arrow[rr, "t^{\mathcal{N}}"] \arrow[dr, "s_1^{\mathcal{N}} \times \cdots \times s_n^{\mathcal{N}}"] & & B^{\mathcal{N}} \\ & B_1^{\mathcal{N}} \times \cdots \times B_n^{\mathcal{N}} \arrow[ur, "f^{\mathcal{N}}"] & \end{tikzcd}$$

  Induction gives that the left face commutes and the definition of interpretation gives us that the top and bottom faces commute. By definition of interpretation of function symbols the right face commutes. From this we can deduce that the background of the following diagram commutes, as required.

□

> **Definition – Models in different signatures**
> Given two signatures $\Sigma \leq \Sigma^*$ and a Σ^*-structure $\mathcal{M}$ is naturally a Σ-structure $\mathcal{N}$ such that $\star_{\Sigma}^{\mathcal{M}} = \star_{\Sigma^*}^{\mathcal{N}}$. We will often just write $\mathcal{M}$ to mean either.

Proposition – Morphisms in different signatures
Suppose $\Sigma \le \Sigma^*$ and we have Σ^*-structures $\mathcal{M}$ and $\mathcal{N}$ (naturally Σ-structures). If $\iota : \mathcal{M} \to \mathcal{N}$ is a morphism in $\mathcal{C}$ then ι is a Σ-morphism if and only if it is a Σ^*-morphism, and an embedding in Σ if and only if in Σ^*.

Proof. If $f : \prod A \to B$ is a function symbol then $\prod A_{\Sigma}^{\mathcal{M}} = \prod A_{\Sigma^*}^{\mathcal{M}}$ and $B_{\Sigma}^{\mathcal{M}} = B_{\Sigma^*}^{\mathcal{M}}$ (similarly for $\mathcal{N}$). Hence the commuting diagram is the same diagram. Similarly the commuting diagram is the same for any relation symbol. Giving us the result. $\square$

1.1.2 Formulas in classical first order logic

From this point onwards we will work in classical first order logic.

Definition – Atomic formula, quantifier free formula
Given Σ a signature, its set of atomic Σ-formulas along with their associated set of typed free variables $\mathrm{tv}_\star$ is defined by

| $\top$ is an atomic Σ-formula with no typed free variables $\mathrm{tv}_\top = \varnothing$.

| Given $t, s \in \Sigma_{\mathrm{ter}}$ of the same type, the string $t = s$ is an atomic Σ-formula. The set of typed free variables is $\mathrm{tv}_{t=s} = \mathrm{tv}_t \cup \mathrm{tv}_s$.

| Given $r \in \Sigma_{\mathrm{rel}}$ such that $r \hookrightarrow A_1 \times \dots A_n$ and $t : A_1 \times \cdots \times A_n$, the string $r(t)$ is an atomic Σ-formula. The typed free variables are $\mathrm{tv}_{r(t)} = \bigcup_{i=1}^n \mathrm{tv}_{t_i}$.

We extend this to the set of quantifier free Σ-formulas inductively:

| Given ϕ an atomic Σ-formula, ϕ is a quantifier free Σ-formula.

| Given ϕ a quantifier free Σ-formula, the string $\neg\,\phi$ is a quantifier free Σ-formula with the same set of typed free variables as ϕ.

| Given ϕ, ψ both quantifier free Σ-formulas, the string $\phi \vee \psi$ is a quantifier free Σ-formula. The set of typed free variables is $\mathrm{tv}_{\phi\vee\psi} = \mathrm{tv}_\phi \cup \mathrm{tv}_\psi$.

The set of all Σ-formulas Σ_{for} is defined by extending the above:

| Given ϕ a quantifier free Σ-formula, ϕ is a Σ-formula.

| Given $\phi \in \Sigma_{\mathrm{for}}$, a sort A and a variable $v : A$, we take the string $\forall v : A, \phi$ and replace all occurrences of v with an unused symbol $z \in A_{\mathrm{var}}$, producing a new a Σ-formula $\psi = \forall z : A, \phi$. The set of typed free variables is $\mathrm{tv}_\psi = \mathrm{tv}_\phi \setminus \{v : A\}$.

Shorthand for some Σ-formulas include

- $\bot := \neg\,\top$
- $\phi \wedge \psi := \neg\,((\neg\,\phi) \vee (\neg\,\psi))$
- $\phi \to \psi := (\neg\,\phi) \vee \psi$
- $\exists v : A, \phi := \neg\,(\forall v : A, \neg\,\phi)$

The symbol z is thought of as a 'bounded variable' as oppose to a 'free variable', and will not be considered when we want to evaluate variables in formulas.

Often we will be in 1-sorted signatures, in which case we will just write

$$\forall v, \phi \text{ and } \exists v, \phi$$

since the type is obvious.

Remark. Formulas should be thought of as propositions with some bits loose, namely the free variables, since it does not make any sense to ask if $x = a$ without saying what x you are taking (where x is a variable as a is a constant symbol, say). When there are no free variables we get what intuitively looks like a proposition, and we will call these particular formulas sentences.

Definition – Substituting in terms

If a Σ-formula ϕ has a typed free variable $v : A$ then to remind ourselves of the variable we can write $\phi = \phi(v : A) = \phi(v)$ instead.

If we have term $t : A$ for some typed free variable $v : A$ in tv_ϕ, then we write $\phi(t)$ to mean $\phi(v : A)$ with t substituted for v. By induction on terms and formulas this is still a Σ-formula and has

$$\text{tv}_{\phi(t)} = \text{tv}_t \cup \left(\text{tv}_{\phi(v)} \setminus \{v : A\}\right)$$

1.1.3 Classical models, theories

From now on we will be interpreting only in the category $*/\mathbf{Set}$ of non-empty sets. We require non-emptiness because the classical proof of compactness, given by a Henkin construction relies on non-emptiness of our models.

Definition – Satisfaction

Let $\mathcal{M}$ be a Σ-structure (interpreted in $*/\mathbf{Set}$ and ϕ a Σ-formula. Let $a \in \prod_{x:A\in\text{tv}_\phi} A^{\mathcal{M}}$ be a tuple indexed by the typed free variables of ϕ. Then we define $\mathcal{M} \vDash_\Sigma \phi(a)$ by induction on formulas:

- If ϕ is $\top$ then $\mathcal{M} \vDash_\Sigma \phi$.[†]
- If ϕ is $t = s$ then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)$.
- If ϕ is $r(t)$, where $r \hookrightarrow A_1 \times \cdots \times A_n$ is a relation symbol and $t : A_1 \times \cdots \times A_n$ are terms, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $t^{\mathcal{M}}(a) \in r^{\mathcal{M}}$.[†]
- If ϕ is $\neg\psi$ for some $\psi \in \Sigma_{\text{for}}$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $\mathcal{M} \nvDash_\Sigma \psi(a)$
- If ϕ is $\psi \vee \chi$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $\mathcal{M} \vDash_\Sigma \psi(a)$ or $\mathcal{M} \vDash_\Sigma \chi(a)$.
- If ϕ is $\forall v : A, \psi$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ if for any $b \in A^{\mathcal{M}}$, $\mathcal{M} \vDash_\Sigma \psi(a, b)$.

[†]We can omit the a when there are no free variables. Formally a is the unique element in the empty product.
[†]$t^{\mathcal{M}}$ was the product of interpreted terms.

Remark. Any Σ-structure satisfies $\top$ and does not satisfy $\bot$. Note that for c a tuple of constant symbols $\mathcal{M} \vDash_\Sigma \phi(c)$ if and only if $\mathcal{M} \vDash_\Sigma \phi(c^{\mathcal{M}})$.

Definition – Sentences and theories

Let Σ be a signature and ϕ a Σ-formula. We say $\phi \in \Sigma_{\text{for}}$ is a Σ-sentence when it has no free variables, $\text{tv}_\phi = \varnothing$.

T is an Σ-theory when it is a subset of Σ_{for} such that all elements of T are Σ-sentences. We denote the set of Σ-theories as Σ_{the}.

Definition – Models

Given an Σ-structure $\mathcal{M}$ and Σ-theory T, we write $\mathcal{M} \vDash_\Sigma T$ and say $\mathcal{M}$ *is a* Σ*-model of* T when for all $\phi \in T$ we have $\mathcal{M} \vDash_\Sigma \phi$.

Example. *In the signature of rings, the rings axioms will be the theory of rings and each model of the theory will consist of a single sort - the ring. The theory of* ZFC *consists of the* ZFC *axioms and a model of* ZFC *would be a single sort thought of as the 'class of all sets'. In the signature of modules, the theory of modules will consist of the theory for rings, the theory for commutative groups, and the axioms for modules over a ring. A model of the theory of modules would consist of two sorts, one for the ring and one for the module.*

Definition – Consequence

Given a Σ-theory T and a Σ-sentence ϕ, we say ϕ is a consequence of T and say $T \vDash_\Sigma \phi$ when for all Σ-models $\mathcal{M}$ of T, we have $\mathcal{M} \vDash_\Sigma \phi$. We also write $T \vDash_\Sigma \Delta$ for Σ-theories T and Δ when for every $\phi \in \Delta$ we have $T \vDash_\Sigma \phi$.

Exercise (Logical consequence). *Let T be a Σ-theory and ϕ and ψ be Σ-sentences. Show that the following are equivalent:*

- $T \vDash_\Sigma \phi \to \psi$
- $T \vDash_\Sigma \phi$ *implies* $T \vDash_\Sigma \psi$.

Definition – Consistent theory

A Σ-theory T is consistent if either of the following equivalent definitions hold:

- There does not exists a Σ-sentence ϕ such that $T \vDash_\Sigma \phi$ and $T \vDash_\Sigma \neg\phi$.
- There exists a Σ-model of T.

Thus the definition of consistent is intuitively 'T does not lead to a contradiction'. A theory T is finitely consistent if all finite subsets of T are consistent. This will turn out to be another equivalent definition, given by the compactness theorem.

Proof. We show that the two definitions are equivalent. ($\Rightarrow$) Suppose no model exists. Take ϕ to be the Σ-sentence $\top$. Hence all Σ-models of T satisfy $\top$ and $\bot$ (there are none) so $T \vDash_\Sigma \top$ and $T \vDash_\Sigma \bot$. ($\Leftarrow$) Suppose T has a Σ-model $\mathcal{M}$ and $T \vDash_\Sigma \phi$ and $T \vDash_\Sigma \neg\phi$. This implies $\mathcal{M} \vDash_\Sigma \phi$ and $\mathcal{M} \nvDash_\Sigma \phi$, a contradiction. $\square$

Definition – Elementary equivalence

Let $\mathcal{M}$, $\mathcal{N}$ be Σ-structures. They are elementarily equivalent if for any Σ-sentence ϕ, $\mathcal{M} \vDash_\Sigma \phi$ if and only if $\mathcal{N} \vDash_\Sigma \phi$. We write $\mathcal{M} \equiv_\Sigma \mathcal{N}$.

Definition – Maximal and complete theories

A Σ-theory T is *maximal* if for any Σ-sentence ϕ, $\phi \in T$ or $\neg\phi \in T$.

T is *complete* when either of the following equivalent definitions hold:

- For any Σ-sentence ϕ, $T \vDash_\Sigma \phi$ or $T \vDash_\Sigma \neg\phi$.
- All models of T are elementarily equivalent.

Note that maximal theories are complete.

Proof. ($\Rightarrow$) Let $\mathcal{M}$ and $\mathcal{N}$ be models of T and ϕ be a Σ-sentence. If $T \vDash_\Sigma \phi$ then both satisfy ϕ. Otherwise $\neg\phi \in T$ and neither satisfy ϕ.

($\Leftarrow$) If ϕ is a Σ-sentence then suppose for a contradiction

$$T \nvDash_\Sigma \phi \text{ and } T \nvDash_\Sigma \neg\phi$$

Then there exist models of T such that $\mathcal{M} \nvDash_\Sigma \phi$ and $\mathcal{N} \nvDash_\Sigma \neg\phi$. By assumption they are elementarily equivalent and so $\mathcal{M} \vDash_\Sigma \neg\phi$ implies $\mathcal{N} \vDash_\Sigma \neg\phi$, a contradiction. □

Exercise (Not consistent, not complete). *Let T be a Σ-theory and ϕ is a Σ-sentence. Show that $T \nvDash_\Sigma \phi$ if and only if $T \cup \{\neg\phi\}$ is consistent. Furthermore, $T \nvDash_\Sigma \neg\phi$ if and only if $T \cup \{\phi\}$ is consistent.*

Note that by definition for Σ-structures and Σ-formulas we (classically) have that

$$\mathcal{M} \vDash_\Sigma \neg\phi(a) \Leftrightarrow \mathcal{M} \nvDash_\Sigma \phi(a)$$

Find examples of theories that do not satisfy

$$T \vDash_\Sigma \neg\phi \Leftrightarrow T \nvDash_\Sigma \phi$$

1.1.4 The Compactness Theorem

Read ahead for the statement of the Compactness Theorem. The first two parts of the theorem are easy to prove. This chapter focuses on proving the final part. Compactness is a useful tool in classical model theory, allowing us to reduce a proof of consistency to a proof about finite consistency, which is much easier to work with. In the finite case we can even just take conjunction of a set of sentences and make it just a question about the validity of one sentence. The term 'compactness' corresponds to the topological notion when looking at the Stone space of a theory, though it is easier to visualise once the notion of an n-type is introduced.

Notation. *Write the 'size of a model' to be*

$$|\mathcal{M}| := \left| \bigcup_{A \in \Sigma_{\mathrm{sor}}} A^{\mathcal{M}} \right|$$

Definition – Witness property

Given a signature Σ and a Σ-theory T, we say that T has the witness property when for any Σ-formula ϕ with exactly one typed free variable $v : A$, there exists a constant symbol $c : A$ such that if $\exists v : A, \phi(v) \in T$ then $\phi(c) \in T$.

We will try to extend our theory such that it is maximal, finitely consistent, and has the witness property in what's called a 'Henkin construction'. The following lemma says once we have such a theory then it is consistent, not just finitely consistent. The reason for wanting the witness property is that the Σ-structure we build in the lemma will interpret sorts as the sort sets themselves (modulo some equivalence), so we need the language to have enough constant symbols and for T to 'witness' all existence statements using constant symbols.

Lemma – From finitely consistent to consistent

Let Σ be a signature. If a Σ-theory T is maximal, finitely consistent and has the witness property then it is consistent.

> Furthermore, let $0 < \kappa$ be a cardinal such that $|\Sigma_{\text{con}}| \leq \kappa$ then we can construct a Σ-model $\mathcal{M}$ such that the 'size of $\mathcal{M}$' is bounded by $|\mathcal{M}| \leq \kappa$.

Proof. **The Σ-structure**: For each sort A in Σ we quotient the set A_{con} of constant symbols of type A by the equivalence relation $c \sim d$ if and only if $c = d \in T$. Let $\pi_A : A_{\text{con}} \to A_{\text{con}} / \sim$. This defines a Σ-structure $\mathcal{M}$ in the following way:

1. Interpret each sort A as $\pi_A(A)$. Clearly the size of $\mathcal{M}$ is bounded by κ.

2. To interpret functions we must use the witness property. Let $f : \prod A \to B$ be a function symbol and take an element of the interpretation of its domain $\prod A^{\mathcal{M}}$. Since each π_A is surjective this element is of the form $c^{\mathcal{M}} = (\pi_A(c_A))_A$ for some constant symbols $c = (c_A)_A : \prod A$. As T is maximal either $\exists v : B, f(c) = v$ or its negation is in T, but the latter would imply T is not finitely consistent: if there exists $\mathcal{M} \vDash_\Sigma \forall v : B, f(c) \neq v$ then the interpretation of $\mathcal{M} \vDash_\Sigma f(c) \neq f^{\mathcal{M}} c^{\mathcal{M}}$, a contradiction. Hence by the witness property there exists $b : B$ such that $f(c) = b \in T$. We define $f^{\mathcal{M}}$ to map $c^{\mathcal{M}} \mapsto b^{\mathcal{M}}$.

   To show that $f^{\mathcal{M}}$ is well-defined, suppose $c_0^{\mathcal{M}} = c_1^{\mathcal{M}}$ and $f(c_0) = b_0, f(c_1) = b_1 \in T$. We want that $b_0 = b_1 \in T$. By definition of $\mathcal{M}$, $c_0 = c_1 \in T$. If $b_0 = b_1 \notin T$ then its negation is, giving a finitely inconsistent subset
$$\{c_0 = c_1, f(c_0) = b_0, f(c_1) = b_1, b_0 \neq b_0\} \subseteq T$$
   which is a contradiction.

3. Let $r \hookrightarrow A_1 \times \cdots \times A_n$. We define
$$r^{\mathcal{M}} := \{t^{\mathcal{M}} \mid \text{variable free term } t : A_1^{\mathcal{M}} \times \cdots \times A_n^{\mathcal{M}} \text{ and } r(t) \in T\}$$

Now that we have a Σ-structure we want to show that it is a Σ-model of T.

Terms: We first need that terms are represented by constant symbols: given term $t : B$ with no typed variables $\text{tv}_t = \varnothing$ and constant symbol $d : B$ then $t = d \in T$ if and only if $\mathcal{M} \vDash_\Sigma t = d$. Note that since t has no typed variables it is not a variable itself. Hence $t = f(s)$ for some function symbol $f : \prod_i E_i \to B$ and term $s : \prod_i E_i$ with no typed variables. By induction we assume the statement to be true for each $s_i : E_i$.

($\Rightarrow$) If we can find constant symbols $e = (e_i)_i : \prod_i E_i$ such that each e_i satisfies $s_i = e_i \in T$ and $f(e) = d \in T$ then by definition of interpretation of functions $f^{\mathcal{M}}(e^{\mathcal{M}}) = d^{\mathcal{M}}$ and by induction $s_i^{\mathcal{M}} = e_i^{\mathcal{M}}$ so

$$t^{\mathcal{M}} = (f(s))^{\mathcal{M}} = f^{\mathcal{M}}(s^{\mathcal{M}}) = f^{\mathcal{M}}(e^{\mathcal{M}}) = (f(e))^{\mathcal{M}} = d^{\mathcal{M}}$$

Using the witness property for each i we can construct e inductively. Suppose we have $e_1 : E_1, \ldots, e_{i-1} : E_{i-1}$ such that
$$\Delta_i := \{e_1 = s_1, \ldots, e_{i-1} = s_{i-1}, \phi_1(e_1), \ldots, \phi_{i-1}(e_{i-1})\} \subseteq T$$
where each ϕ_j is the formula

$$\exists x_{j+1}, \ldots, \exists x_n, f(e_1, \ldots, e_{j-1}, v, x_{j+1}, \ldots, x_n) = d$$

with a single typed free variable $v : E_i$ and bounded variables $x_j : E_j$. To complete the induction we find e_i such that $e_i = s_i, \phi_i(e_i) \in T$. The witness property gives a constant symbol $e_i : E_i$ such that $\exists v : E_i, \phi_i(v) \in T$ implies $\phi_i(e_i) \in T$.

Suppose for a contradiction $\exists v : E_i, \phi_i(v) \notin T$ then we have a finite subset $\Delta_i \cup f(s) = d, \forall v : E_i, \neg\, \phi_i(v) \subseteq T$ which we claim is inconsistent. Suppose $\mathcal{N}$ is a model of the finite set. Then

$$\mathcal{N} \vDash_\Sigma \forall v, \forall x_{i+1}, \forall x_n, \neg\, \phi_i$$

So taking $v = s_i^{\mathcal{N}}$ and each $x_j = s_j^{\mathcal{N}}$

$$\mathcal{N} \vDash_\Sigma f(e_1, \ldots, e_{j-1}, s_i, s_{i+1}, \ldots, s_n) \neq d$$

Now for each $j < i$ we also have $\mathcal{N} \vDash_\Sigma e_j = s_j$ and so $\mathcal{N} \vDash_\Sigma f(s) \neq d$, a contradiction. With the induction done, we have the final ϕ_n giving

$$f(e_1, \ldots, e_n) = d \in T \text{ and all } e_i = s_i \in T$$

($\Leftarrow$) Note that each $s_i^{\mathcal{M}} = e_i^{\mathcal{M}}$ for some $e_i : E_i$ since each π_{E_i} is surjective. By the induction hypothesis for each i we have $s_i = e_i \in T$. Hence

$$f(e)^{\mathcal{M}} = f^{\mathcal{M}}(e^{\mathcal{M}}) = f^{\mathcal{M}}(s^{\mathcal{M}}) = t^{\mathcal{M}} = d^{\mathcal{M}}$$

Hence $f(e) = d \in T$ by induction. To conclude $f(s) = d \in T$ we note that

$$\{f(s) \neq d, s_1 = e_1, \ldots, s_n = e_n, f(e) = d\}$$

is a finite inconsistent subset of T. Thus $t = d \in T \Leftrightarrow t^{\mathcal{M}} = d^{\mathcal{M}}$.

Formulas: we now show $\mathcal{M} \vDash_\Sigma T$. It suffices to show a stronger statement which will be needed for the induction: for all Σ-sentences ϕ

$$\phi \in T \Leftrightarrow \mathcal{M} \vDash_\Sigma \phi$$

We case on what ϕ is:

- Case ϕ is $\top$: all Σ-structures satisfy $\top$ so $\mathcal{M} \vDash_\Sigma \top$. If $\bot \in T$ then T wouldn't be finitely consistent, hence $\top \in T$.
- Case ϕ is $t = s$ (note ϕ is a sentence so $\mathrm{tv}_t = \mathrm{tv}_s = \varnothing$): first note $\exists v, t = v \in T$, otherwise $\forall v, t \neq v \in T$ so T cannot be finitely consistent (any model interprets t as something). We apply the witness property to obtain constant symbol d such that $t = d \in T$. Since terms are represented by constant symbols we also deduce $t^{\mathcal{M}} = d^{\mathcal{M}}$.

  Note $t = s \in T$ if and only if $s = d \in T$: if $t = s \in T$ then $s \neq d$ cannot be in the theory by considering finite inconsistent subset $\{t = d, t = s, s \neq d\}$. Hence $s = d \in T$. The converse is similar. Since terms are represented by constant symbols,

  $$t = s \in T \quad \Leftrightarrow \quad d = s \in T \quad \Leftrightarrow \quad d^{\mathcal{M}} = s^{\mathcal{M}} \quad \Leftrightarrow \quad t^{\mathcal{M}} = s^{\mathcal{M}}$$

- Case ϕ is $r(t_1, \ldots, t_n)$ where $r \hookrightarrow A_1 \times \cdots \times A_n$ (again $\mathrm{tv}_{t_i} = \varnothing$): ($\Rightarrow$) Suppose $r(t) \in T$. By induction, apply the witness property to the formulas

  $$\phi_i := \exists x_{i+1} : A_{i+1}, \ldots, \exists x_n : A_n, r(\ldots, e_{i-1}, v, x_{i+1}, \ldots) \land v = t_i$$

  each time obtaining $e_i : A_i$ such that $\phi_i(e_i) \in T$. The result is $r(e) \in T$ and each $e_i = t_i \in T$. Using the claim for terms and how we interpreted relations in $\mathcal{M}$ this implies $t^{\mathcal{M}} = e^{\mathcal{M}} \in r^{\mathcal{M}}$, and so $\mathcal{M} \vDash_\Sigma r(t)$.

  ($\Leftarrow$) Suppose $\mathcal{M} \vDash_\Sigma r(t)$. Since π_{A_i} are all surjective, there exists $e \in \prod_i A_i$ such that $e^{\mathcal{M}} = t^{\mathcal{M}} \in r^{\mathcal{M}}$. Using the claim for terms again we obtain $t = e \in T$ and using how $\mathcal{M}$ interprets relations, $r(e) \in T$ so $r(t) \in T$.

- Case ϕ is $\neg \chi$: Note that by finite consistency we cannot have the subset $\{\chi, \neg \chi\} \subseteq T$. Hence by maximality

  $$\neg \chi \in T \quad \Leftrightarrow \quad \chi \notin T$$

  Using the induction hypothesis for χ,

  $$\chi \notin T \quad \Leftrightarrow \quad \mathcal{M} \nvDash_\Sigma \chi \quad \Leftrightarrow \quad \mathcal{M} \vDash_\Sigma \neg \chi$$

- Case ϕ is $\chi_0 \lor \chi_1$: given the induction hypothesis

  $$\mathcal{M} \vDash_\Sigma \chi_0 \lor \chi_1 \quad \Leftrightarrow \quad \mathcal{M} \vDash_\Sigma \chi_0 \text{ or } \mathcal{M} \vDash_\Sigma \chi_1 \quad \Leftrightarrow \quad \chi_0 \in T \text{ or } \chi_1 \in T$$

Hence it suffices to show that

$$\chi_0 \in T \text{ or } \chi_1 \in T \quad \Leftrightarrow \quad \chi_0 \vee \chi_1 \in T$$

($\Rightarrow$) WLOG suppose χ_0 is in T. If $\chi_0 \vee \chi_1 \notin T$ then $\{\chi_0, \neg\chi_0 \wedge \neg\chi_1\}$ is a finite inconsistent subset of T, a contradiction.

($\Leftarrow$) Suppose $\chi_0 \vee \chi_1 \in T$. If neither χ_0 nor χ_1 is in T then $\{\neg\chi_0, \neg\chi_1, \chi_0 \vee \chi_1\}$ is a finite inconsistent subset of T.

- Case ϕ is $\forall v : A, \chi(v)$ ($\Rightarrow$) Take any element $c^{\mathcal{M}} \in A^{\mathcal{M}}$ (using surjectivity of π_A have symbol $c : A$). We see that $\phi(c) \in T$ since otherwise $\{\neg\phi(c), \forall v : A, \chi(v)\}$ is a finitely inconsistent subset of T. Hence by induction $\mathcal{M} \vDash_\Sigma \phi(c)$ and we conclude $\mathcal{M} \vDash_\Sigma \forall v : A, \chi(v)$.

  ($\Leftarrow$) We show the contrapositive. If $\forall v : A, \chi(v) \notin T$, then by maximality $\exists v : A, \neg\chi(v) \in T$. Applying the witness property we obtain constant symbol $c : A$ such that $\neg\chi(c) \in T$. By induction $\mathcal{M} \nvDash_\Sigma \chi(c)$ and so $\mathcal{M} \nvDash_\Sigma \forall v : A, \chi(v)$.

Thus $\phi \in T \Leftrightarrow \mathcal{M} \vDash_\Sigma \phi$ and we are done. □

> Notation. *We write $\Sigma \leq \Sigma^*$ for two signatures if $\Sigma_{\text{sor}} \subseteq {\Sigma^*}_{\text{sor}}$, $\Sigma_{\text{fun}} \subseteq {\Sigma^*}_{\text{fun}}$ and $\Sigma_{\text{rel}} \subseteq {\Sigma^*}_{\text{rel}}$.*

For the sake of formality we need the following two lemmas, neither of which are particularly exciting, but they do allow us to move freely between signatures without worry.

Lemma – Moving models down signatures

Given two signatures $\Sigma \leq \Sigma^*$ and a Σ^*-structure $\mathcal{N}$ we naturally have a Σ-structure $\mathcal{M}$ such that for each sort, function and relation symbol in Σ has the same interpration under $\mathcal{M}$ and $\mathcal{N}$. Then

- (Preserves satisfaction) For any Σ-formula ϕ with $\text{tv}_\phi = \{x_i : A_i\}_i$ and $a \in \prod_i A_i^{\mathcal{M}}$

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \phi(a)$$

- (Preserves theories) If T is a Σ-theory and T^* is a Σ^*-theory such that $T \subseteq T^*$ and $\mathcal{N}$ a Σ^*-model of T^*, then $\mathcal{M}$ is a Σ-model of T.

From now on we will write $\mathcal{N}$ to mean either of the two and let subscripts involving Σ and Σ^* describe which one we mean.

Proof. We will need that for any Σ-term t, the interpretation of terms is equal: $t_\Sigma^{\mathcal{M}} = t_{\Sigma^*}^{\mathcal{N}}$. Indeed:

- If t is a variable $x : A$ then $t_\Sigma^{\mathcal{M}} = \mathbb{1}_{A_{\Sigma^*}^{\mathcal{M}}} = \mathbb{1}_{A_{\Sigma^*}^{\mathcal{N}}} = t_{\Sigma^*}^{\mathcal{N}}$
- If t is $f(s)$ then $t_\Sigma^{\mathcal{M}} = f_\Sigma^{\mathcal{M}}(s_\Sigma^{\mathcal{M}}) = f_{\Sigma^*}^{\mathcal{M}}(s_\Sigma^{\mathcal{M}})$ which by induction is $f_{\Sigma^*}^{\mathcal{N}}(s_{\Sigma^*}^{\mathcal{N}}) = t_{\Sigma^*}^{\mathcal{N}}$

Let ϕ be a Σ-formula with $\text{tv}_\phi = \{x_i : A_i\}_i$ and $a \in \prod_i A_i^{\mathcal{M}}$. Case on ϕ to show that $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \phi(a)$:

- If ϕ is $\top$ then both satisfy ϕ.
- If ϕ is $t = s$ then since the interpretation of terms are equal

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow t_\Sigma^{\mathcal{M}} = s_\Sigma^{\mathcal{M}} \Leftrightarrow t_{\Sigma^*}^{\mathcal{M}} = s_{\Sigma^*}^{\mathcal{M}} \Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \phi(a)$$

- If ϕ is $r(t)$ then by definition of $r_\Sigma^{\mathcal{M}}$ and since interpretation of terms are equal

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow t_\Sigma^{\mathcal{M}}(a) \in r_\Sigma^{\mathcal{M}} \Leftrightarrow t_{\Sigma^*}^{\mathcal{N}}(a) \in r_{\Sigma^*}^{\mathcal{N}} \Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \phi(a)$$

- If ϕ is $\neg\psi$ then using the induction hypothesis

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \nvDash_\Sigma \psi(a) \Leftrightarrow \mathcal{N} \nvDash_{\Sigma^*} \psi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \phi(a)$$

- If ϕ is $\psi \vee \chi$ then using the induction hypothesis

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \vDash_\Sigma \psi(a) \text{ or } \mathcal{M} \vDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \psi(a) \text{ or } \mathcal{N} \vDash_{\Sigma^*} \chi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \phi(a)$$

- If ϕ is $\forall v : B, \psi$ then by induction

$$\begin{aligned} \mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow \forall b \in B^{\mathcal{M}}, \mathcal{M} \vDash_\Sigma \psi(a, b) \\ &\Leftrightarrow \forall b \in B^{\mathcal{M}}, \mathcal{N} \vDash_{\Sigma^*} \psi(a, b) \\ &\Leftrightarrow \forall b \in B^{\mathcal{N}}, \mathcal{N} \vDash_{\Sigma^*} \psi(a, b) \\ &\Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \phi(a) \end{aligned}$$

Hence $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \phi(a)$. It follows that if $T \subseteq T^*$ are respectively Σ and Σ^*-theories and $\mathcal{N} \vDash_{\Sigma^*} T^*$ then $\mathcal{M} \vDash_\Sigma T$. □

Lemma – Moving models and theories up signatures

Suppose $\Sigma \leq \Sigma^*$.

1. Suppose $\mathcal{M}$ is a Σ-structure. Then i whose interpretation agrees with $\mathcal{M}$ on symbols from Σ, then for any Σ-formula ϕ with $\mathrm{tv}_\phi = \{x_i : A_i\}_i$ and $a \in \prod_i A_i^{\mathcal{M}}$

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \phi(a)$$

   In particular if $\mathcal{M}$ is a model of Σ-theory T then $\mathcal{M}^*$ is a Σ^*-model of T viewed as a Σ^*-theory.

2. Suppose T is a Σ-theory and ϕ is a Σ-sentence such that $T \vDash_\Sigma \phi$. Then $T \vDash_{\Sigma^*} \phi$.

Again, if we have constructed such a $\mathcal{M}^*$ from $\mathcal{M}$ we tend to just refer to it as $\mathcal{M}$ and let subscripts involving Σ and Σ^* describe which one we mean.

Proof. The proof that satisfaction is preserved is exactly the same as for moving models down signatures. Suppose $T \vDash_\Sigma \phi$. If $\mathcal{M}^* \vDash_{\Sigma^*} T$ then by moving $\mathcal{M}^*$ down to Σ, we have a corresponding Σ-structure $\mathcal{M} \vDash_\Sigma T$. Hence $\mathcal{M} \vDash_\Sigma \phi$, and since both models agree on interpretation of constants, functions and relations of Σ we have $\mathcal{M}^* \vDash_{\Sigma^*} \phi$ by the previous part. □

Proposition – Henkin (witness)

Suppose $\Sigma(0)$-theory $T(0)$ is finitely consistent. Then there exists a signature Σ^* and Σ^*-theory T^* such that

1. $\Sigma(0)_{\mathrm{con}} \subseteq {\Sigma^*}_{\mathrm{con}}$ and otherwise share the same function and relation symbols.
2. $|{\Sigma^*}_{\mathrm{con}}| = |\Sigma(0)_{\mathrm{con}}| + \aleph_0$
3. $T(0) \subseteq T^*$
4. T^* is finitely consistent
5. Any maximal finitely consistent Σ^*-theory S such that $T^* \subseteq S$ has the witness property

Proof. We want to define $\Sigma(i), T(i)$, for each $i \in \mathbb{N}$. By induction, we assume we have signature $\Sigma(i)$ and Σ-theory $T(i)$ such that

1. $\Sigma(0)_{\mathrm{con}} \subseteq \Sigma(i)_{\mathrm{con}}$ and they share the same function and relation symbols.
2. $|\Sigma(i)_{\mathrm{con}}| = |\Sigma(0)_{\mathrm{con}}| + \aleph_0$
3. $T(0) \subseteq T(i)$
4. $T(i)$ is finitely consistent

Note that $\Sigma(0)$ satisfies this. Let
$$W(i) := \{\phi \in \Sigma(i)_{\text{for}} \mid |\text{tv}_\phi| = 1\}$$
We construct $\Sigma(i+1)$ by adding constant symbols $c_\phi : A$ for each $\phi \in W(i)$ with $\text{tv}_\phi = \{v : A\}$ and keeping all the other symbols from $\Sigma(i)$:
$$\Sigma(i+1)_{\text{fun}} := \Sigma(i)_{\text{fun}} \sqcup \{c_\phi : A \mid \phi \in W(i) \text{ and } \exists v : A, \phi(v) \in T(i)\}$$
Then add to $T(i)$ a witness formula for each $\phi \in W(i)$
$$T(i+1) := T(i) \cup \{(\exists v : A, \phi(v)) \to \phi(c_\phi) \mid \phi \in W(i)\}$$
Certainly $T(i+1)$ is a $\Sigma(i+1)$-theory such that $T(0) \subseteq T(i+1)$, $\Sigma(0)_{\text{con}} \subseteq \Sigma(i+1)_{\text{con}}$ with symbols otherwise unchanged. Since $W(i)$ is countibly infinite,
$$|\Sigma(i+1)_{\text{con}}| = |\Sigma(i)_{\text{con}}| + \aleph_0 = |\Sigma(0)_{\text{con}}| + \aleph_0$$
We need to check that $T(i+1)$ is finitely consistent. Take a finite subset of $T(i+1)$. It is a union of two finite sets
$$\Delta_T \subseteq T(i) \text{ and } \Delta \subseteq \{(\exists v : A, \phi(v)) \to \phi(c_\phi) \mid \dots\}$$
Since $T(i)$ is finitely consistent there exists a $\Sigma(i)$-model $\mathcal{M}$ of Δ_T.

We want $\mathcal{M}$ to be a $\Sigma(i+1)$-structure. For any sort, function and relation symbol from $\Sigma(i)$ we take the same interpretation as in $\mathcal{M}$. It remains to interpret the new constant symbols in $\Sigma(i+1)$, suppose $c_\phi : A$ and $\exists v : A, \phi(v) \in T(i)$ then if there exists some $a \in A^{\mathcal{M}}$ such that $\mathcal{M} \vDash_{\Sigma(i)} \phi(a)$ then we interpret c_ϕ as a. Otherwise (since interpretation of sorts are non-empty[1]) we take some (any) $a \in A^{\mathcal{M}}$ and interpret c_ϕ as a.

We check is is a $\Sigma(i+1)$-model of $\Delta_T \cup \Delta_w$. Since $\mathcal{M}$ is a $\Sigma(i)$-model of Δ_T it is a $\Sigma(i+1)$-model of Δ_T. Also for any $(\exists v : A, \phi(v)) \to \phi(c_\phi) \in \Delta$, if $\mathcal{M} \vDash_{\Sigma(i+1)} \exists v : A, \phi(v)$ then $\mathcal{M} \vDash_{\Sigma(i)} \exists v : A, \phi(v)$, hence $\mathcal{M}$ interprets $c_\phi : A$ as some element $a \in A^{\mathcal{M}(i)}$ such that $\mathcal{M} \vDash_{\Sigma(i)} \phi(a)$ so $\mathcal{M} \vDash_{\Sigma(i+1)} \phi(c_\phi)$. Hence $T(i+1)$ is finitely consistent.

Let Σ^* be the signature with its constant symbols $\Sigma^*{}_{\text{con}} = \bigcup_{i \in \mathbb{N}} \Sigma(i)_{\text{con}}$ and otherwise the same symbols as $\Sigma(0)$. Then
$$|\Sigma^*{}_{\text{con}}| = |\bigcup_{i \in \mathbb{N}} \Sigma(i)_{\text{con}}| = \aleph_0 \times (\aleph_0 + \Sigma(0)_{\text{con}}) = \aleph_0 + \Sigma(0)_{\text{con}}$$

Let $T^* = \bigcup_{i \in \mathbb{N}} T(i)$. Any finite subset of T^* is a subset of some $T(i)$, hence has a $\Sigma(i)$-model $\mathcal{M}$. By interpreting the added constant symbols as any element in the corresponding interpreted sort, we make $\mathcal{M}^*$ a Σ^*-model of the finite set. Hence T^* is finitely consistent.

Suppose S is a maximal finitely consistent Σ^*-theory such that $T^* \subseteq S$, and ϕ is a Σ^*-formula of exactly one typed variable. There exists an $i \in \mathbb{N}$ such that $\phi \in \Sigma(i)_{\text{for}}$. Hence $(\exists v : A, \phi(v)) \to \phi(c_\phi)$ is in $T(i+1) \subseteq S$. We deduce $\phi(c_\phi) \in S$ since S is maximal and finitely consistent, by considering finite inconsistent set
$$\{\exists v : A, \phi(v), (\exists v : A, \phi(v)) \to \phi(c_\phi), \neg\, \phi(c_\phi)\}$$
□

> **Lemma – Henkin (adding formulas to finitely consistent theories)**
> If T is a finitely consistent Σ-theory and ϕ is a Σ-sentence then at one of $T \cup \{\phi\}$ or $T \cup \{\neg\, \phi\}$ is finitely consistent.

Proof. We show that for any finite $\Delta \subseteq T \cup \{\phi\}$ and finite $\Delta_\neg \subseteq T \cup \{\neg\, \phi\}$, one of Δ or $\Delta_\neg$ is consistent. By finite consistency of T the finite subset
$$(\Delta \setminus \{\phi\}) \cup (\Delta_\neg \setminus \{\neg\, \phi\}) \subseteq T$$
is consistent. Let $\mathcal{M}$ be the model of $(\Delta \setminus \{\phi\}) \cup (\Delta_\neg \setminus \{\neg\, \phi\})$. If $\mathcal{M} \vDash_\Sigma \phi$ then $\mathcal{M} \vDash_\Sigma \Delta$, otherwise $\mathcal{M} \vDash_\Sigma \neg\, \phi$ so $\mathcal{M} \vDash_\Sigma \Delta_\neg$. Hence $T \cup \{\phi\}$ or $T \cup \{\neg\, \phi\}$ is finitely consistent. □

[1] This is the reason we need to interpret into $*/\mathbf{Set}$.

Exercise. *Find a signature Σ, a consistent Σ-theory T and Σ-sentence ϕ such that $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are both consistent.*

Proposition – Extending a finitely consistent theory to a maximal theory (Zorn)

Given a finitely consistent Σ-theory $T(0)$ there exists a Σ-theory T^* such that

1. $T(0) \subseteq T^*$
2. T^* is finitely consistent.
3. T^* is a maximal Σ^*-theory.

Proof. We use Zorn's Lemma. Consider

$$\{T \in \Sigma_{\text{the}} \mid T \text{ finitely consistent and } T(0) \subseteq T\}$$

be ordered by inclusion. Let $(T_i)_{i\in I}$ be a chain. Then $\bigcup_{i\in I} T_i$ is a Σ-theory such that any finite subset is a subset of some T_i, hence is consistent by finite consistency of T_i. Zorn's lemma implies there exists a T^* in the set that is maximal (in the order theory sense).

To show that it is maximal as a Σ-theory, take a Σ-sentence ϕ, then either $T^* \cup \{\phi\}$ or $T^* \cup \{\neg\phi\}$ is finitely consistent. Hence $T^* \cup \{\phi\} = T^*$ or $T^* \cup \{\neg\phi\} = T^*$ by (order theoretic) maximality, so $\phi \in T^*$ or $\neg\phi \in T^*$. □

Notation (Cardinalities of signatures and structures). *Given a signature Σ, we write $|\Sigma| := |\Sigma_{\text{con}}| + |\Sigma_{\text{fun}}| + |\Sigma_{\text{rel}}|$ and call this the size of the signature Σ.*

Proposition – The compactness theorem

If T is a Σ-theory, then the following are equivalent:

1. T is finitely consistent.
2. T is consistent
3. For any infinite cardinal κ such that $|\Sigma| \leq \kappa$, there exists a Σ-model of T with size $\leq \kappa$.

Proof. (3. $\Rightarrow$ 2.) and (2. $\Rightarrow$ 1.) are both obvious. For (1. $\Rightarrow$ 3.) suppose $\Sigma(0)$-theory $T(0)$ is finitely consistent. Let κ be an infinite cardinal such that $|\Sigma(0)| \leq \kappa$. Then $|\Sigma(0)_{\text{con}}| \leq |\Sigma(0)| \leq \kappa$. We have shown that there exists a signature $\Sigma(1)$ and $\Sigma(1)$-theory $T(1)$ such that

1. $\Sigma(0)_{\text{con}} \subseteq \Sigma(1)_{\text{con}}$ with other symbols unchanged.
2. $|\Sigma(1)_{\text{con}}| = |\Sigma(0)_{\text{con}}| + \aleph_0$
3. $T(0) \subseteq T(1)$ is finitely consistent.
4. Any maximal $\Sigma(1)$-theory T such that $T(1) \subseteq T$ has the witness property.

As $T(1)$ is finitely consistent, there exists a $\Sigma(1)$-theory $T(2)$ such that

5. $T(1) \subseteq T(2)$
6. $T(2)$ is finitely consistent.
7. $T(2)$ is a maximal $\Sigma(1)$-theory.

Furthermore, $T(2)$ has the witness property by design of $T(1)$. Finally is a maximal, finitely consistent $\Sigma(1)$-theory with the witness property, hence has a $\Sigma(1)$-model $\mathcal{M}$ such that $|\mathcal{M}| \leq \kappa$. $\mathcal{M} \vDash_{\Sigma(1)} T(0)$ since $T(0) \subseteq T(1) \subseteq T(2)$. Moving $\mathcal{M}$ down we have a $\Sigma(0)$-model of T of the required size. □

1.1.5 The Category of Structures in Set

Definition – Partial Σ-morphism, Σ-embedding
We define partial Σ-morphisms and embeddings. Suppose $\mathcal{M}, \mathcal{N}$ are Σ-structures, and for each sort symbol A we have a subset $S_A \subseteq A^{\mathcal{M}}$. and $\iota_A : S_A \to A^{\mathcal{N}}$. We say $\iota := (\iota_A)_{A \in \Sigma_{\text{sor}}}$ a *partial Σ-morphism* from $\mathcal{M}$ to $\mathcal{N}$ when

- For all function symbols $f : A_1 \times \cdots \times A_n \to B$ and $a = (a_i) \in \prod S_{A_i}$ such that $f^{\mathcal{M}}(a) \in S_B$,

$$\iota_B \circ f^{\mathcal{M}}(a) = f^{\mathcal{N}}(\iota_{A_i}(a_i))$$

- For all relation symbols $r \hookrightarrow A_1 \times \cdots \times A_n$ and $a = (a_i) \in \prod S_{A_i}$,

$$a \in r^{\mathcal{M}} \Rightarrow (\iota_{A_i}(a_i)) \in r^{\mathcal{N}}$$

Furthermore if ι is injective and we have the pullback condition translated to **Set**:

$$a \in r^{\mathcal{M}} \Leftarrow (\iota_{A_i}(a_i)) \in r^{\mathcal{N}}$$

then ι is called a *partial Σ-embedding* (or extension). In the case that all $S_A = A^{\mathcal{M}}$ we can reproduce the original definitions of Σ-morphisms $\iota : \mathcal{M} \to \mathcal{N}$ as well as a Σ-embeddings.

Example. *Given the category of structures $\mathbf{Str}(\Sigma_{\text{RNG}}, */\mathbf{Set})$ for the signature of rings, we can take the subcategory whose objects are models of the theory of rings (namely rings), obtaining the category of rings. Similarly taking the subcategory whose objects are models of the theory of fields (namely fields) produces the category of fields.*

Definition – Elementary embedding
A partial Σ-embedding $\iota : \mathcal{M} \to \mathcal{N}$ *preserves ϕ* if for any $a \in \prod_{x:A \in \text{tv}_\phi} A^{\mathcal{M}}$,

$$\mathcal{M} \vDash_\Sigma \phi(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

where we write $\iota(a)$ to mean $(\iota_A(a_A))_{x:A \in \text{tv}_\phi}$.

We say ι *preservers ϕ upwards* when only $\Rightarrow$ holds and *downwards* when only $\Leftarrow$ holds. The embedding is elementary if it preserves all Σ-formulas.

Proposition – Embeddings preserve quantifier free formulas
Let $\iota : \mathcal{M} \to \mathcal{N}$ be a Σ-embedding, then

1. ι preserves any atomic Σ-formula $\top$, $t = s$ or $r(s)$.
2. If ι preserves Σ-formula χ then it is preserves $\neg\chi$.
3. If ι preserves Σ-formulas χ_0 and χ_1 then preserves $\chi_0 \vee \chi_1$.

Thus from the above we can deduce by induction that if ι is elementary with respect to all quantifier free Σ-formulas.

Proof.

- Trivial.

- If ϕ is $t = s$ then

$$\begin{aligned}
\mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow t^\mathcal{M}(a) = s^\mathcal{M}(a) \\
&\Leftrightarrow \iota(t^\mathcal{M}(a)) = \iota(s^\mathcal{M}(a)) && \text{by injectivity} \\
&\Leftrightarrow t^\mathcal{N}(\iota(a)) = s^\mathcal{N}(\iota(a)) && \text{morphisms commute with} \\
&\Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a)) && \text{interpretation of terms}
\end{aligned}$$

- If ϕ is $r(s)$ then

$$\begin{aligned}
\mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow a \in r^\mathcal{M} \\
&\Leftrightarrow \iota(a) \in r^\mathcal{N} && \text{pullback} \\
&\Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))
\end{aligned}$$

- If ϕ is $\neg\chi$ and $\mathcal{M} \vDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi(\iota(a))$ then

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \nvDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \nvDash_\Sigma \chi(\iota(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

- If ϕ is $\chi_0 \vee \chi_1$ and $\mathcal{M} \vDash_\Sigma \chi_i(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_i(\iota(a))$

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \vDash_\Sigma \chi_0(a) \text{ or } \mathcal{M} \vDash_\Sigma \chi_1(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_0(\iota(a)) \text{ or } \mathcal{N} \vDash_\Sigma \chi_1(\iota(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

□

Definition – Universal Formula, Universal Sentence

A Σ-formula is universal if it can be built inductively by the following two constructors:

| If ϕ is a quantifier free Σ-formula then it is a universal Σ-formula.

| If ϕ is a universal Σ-formula then $\forall v, \phi(v)$ is a universal Σ-formula.

In other words universal Σ-formulas are formulas that start with a bunch of 'for alls' followed by a quantifier free formula.

Proposition – Embeddings preserve universal formulas downwards

Given $\iota : \mathcal{M} \to \mathcal{N}$ a Σ-embedding and ψ a Σ-formula such that ι preserves ψ then ι preserves $\forall v : B, \psi$ *downwards*.

Dually we can show that embeddings preserve existential Σ-formulas *upwards*.

Proof. By assumption, for any $a \in \prod_{x:A \in \mathrm{tv}_\phi} A^\mathcal{M}$ and $b \in B^\mathcal{M}$,

$$\mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b)) \quad \Rightarrow \quad \mathcal{M} \vDash_\Sigma \psi(a, b)$$

Then for any $a \in \prod_{x:A \in \mathrm{tv}_\phi} A^\mathcal{M}$

$$\begin{aligned}
&\mathcal{N} \vDash_\Sigma \phi(\iota(a)) \\
\Rightarrow &\forall b \in \mathcal{M}, \mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b)) \\
\Rightarrow &\forall b \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \psi(\iota(a), \iota(b)) \\
\Rightarrow &\mathcal{M} \vDash_\Sigma \phi(a)
\end{aligned}$$

□

Proposition – Isomorphisms are Elementary

If two Σ-structures $\mathcal{M}$ and $\mathcal{N}$ are Σ-isomorphic then the isomorphism is elementary.

Proof. Both the isomorphism and its inverse are Σ-embeddings hence the result follows from induction on formulas using the fact that embeddings preserve quantifier free formulas, embeddings preserving universal formulas downwards. There is a subtlety of having to induct here, using the individual induction steps proven before, due to the complexity of formulas such as $\neg(\forall v : A, v = v)$. □

1.1.6 Vaught's Completeness Test

Read ahead to the statement of Vaught's Completeness Test.

Proposition – Infinitely modelled theories have arbitrary large models

Given Σ a signature, A a sort symbol in the signature, T a Σ-theory with a model $\mathcal{M}$ such that $A^{\mathcal{M}}$ is infinite, and a cardinal κ such that $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$, then there exists $\mathcal{N}$ a Σ-model of T such that $\kappa = |A^{\mathcal{N}}|$.

One can extend this to any sort and obtain the same result replacing $|A^{\mathcal{N}}|$ with $|\mathcal{N}|$.

Proof. Enrich only the signature's constant symbols of type A to create Σ^* a signature such that ${\Sigma^*}_{\text{con}} = \Sigma_{\text{con}} \cup \{c_\alpha : A \mid \alpha \in \kappa\}$. Let $T^* = T \cup \{c_\alpha \neq c_\beta \mid \alpha, \beta \in \kappa \wedge \alpha \neq \beta\}$ be a Σ^*-theory.

Using the compactness theorem, we show that T^* is finitely consistent. Take a finite subset of T^*. This is the union of a finite subset $\Delta_T \subseteq T$, and a finite subset of $\Delta_\kappa \subseteq \{c_\alpha \neq c_\beta \mid \alpha, \beta \in \kappa \wedge \alpha \neq \beta\}$. We want to make the given model $\mathcal{M}$ a Σ^*-model of $\Delta_T \cup \Delta_\kappa$ by interpreting the new symbols of $\{c_\alpha : A \mid \alpha \in \kappa\}$ in a sensible way.

The set $I \subset \kappa$ indexing the constant symbols appearing in Δ_κ is finite and $A^{\mathcal{M}}$ is infinite, so we can find distinct elements of $A^{\mathcal{M}}$ to interpret the elements of $\{c_\alpha : A \mid \alpha \in I\}$. Interpret the rest of the new constant symbols as the same element of non-empty $A^{\mathcal{M}}$ (these don't matter), then $\mathcal{M} \vDash_{\Sigma^*} \Delta_T \cup \Delta_\kappa$. Thus T^* is finitely consistent hence consistent.

Using the third equivalent version of consistency of T^*, there exists $\mathcal{N}$ a Σ^*-model of T^* with $|A^{\mathcal{N}}| \leq |\mathcal{N}| \leq \kappa$. If $|A^{\mathcal{N}}| < \kappa$ then there would be c_α, c_β that are interpreted as equal, hence $\mathcal{N} \vDash_{\Sigma^*} c_\alpha = c_\beta$ and $\mathcal{N} \nvDash_{\Sigma^*} c_\alpha = c_\beta$, a contradiction. Thus $|A^{\mathcal{N}}| = \kappa$. We can move $\mathcal{N}$ down a signature to make it a Σ-model of T. □

Definition – Categoricity

Given a single-sorted signature Σ and a cardinal κ, a Σ-theory T is called κ-categorical if any two models of T of size κ are isomorphic.

Notation. *In single-sorted model theory we often use a Σ-structure $\mathcal{M}$ to also denote the interpretation of the single sort in Σ.*

Proposition – Vaught's Completeness Test

Let signature Σ be single-sorted. Suppose that Σ-theory T is consistent with no finite model, and κ-categorical for some cardinal satisfying $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$. Then T is complete.

Proof. Suppose not: if T is not complete then there exists Σ-formula ϕ such that $T \nvDash_\Sigma \neg\phi$ and $T \nvDash_\Sigma \phi$. These imply $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ are both consistent. Let $\mathcal{M}_\neg$ and $\mathcal{M}$ be models of $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ respectively. Then each are models of T so they are infinite and so $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ and hence are infinite.

Since we have κ such that $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$, there exists $\mathcal{N}_\neg, \mathcal{N}$ respectively Σ-models of $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ such that $\kappa = |\mathcal{N}_\neg| = |\mathcal{N}|$. Since T is κ-categorical $\mathcal{N}$ and $\mathcal{N}_\neg$ are isomorphic. As isomorphisms are elementary Σ-embeddings we have a contradiction: $\mathcal{N} \vDash_\Sigma \phi$ and $\mathcal{N} \vDash_\Sigma \neg\phi$. □

1.1.7 Elementary embeddings and diagrams of models

Proposition – Tarski-Vaught Elementary Embedding Test
Let $\iota : \mathcal{M} \to \mathcal{N}$ be a Σ-embedding, then the following are equivalent:

1. ι is elementary
2. For any $\phi \in \Sigma_{\text{for}}$ preserved by ι, any $x : B \in \text{tv}_\phi$ and any $a \in \prod_{B \neq A \in \text{tv}_\phi} A^\mathcal{M}$,
$$\forall b \in B^\mathcal{M}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b)) \quad \Rightarrow \quad \mathcal{N} \vDash_\Sigma \forall x : B, \phi(\iota(a), x),$$
which we call the Tarski-Vaught condition.
3. For any $\phi \in \Sigma_{\text{for}}$ preserved by ι, any $x : B \in \text{tv}_\phi$ and any $a \in \prod_{B \neq A \in \text{tv}_\phi} A^\mathcal{M}$,
$$\mathcal{N} \vDash_\Sigma \exists x : B, \phi(\iota(a), x) \quad \Rightarrow \quad \exists b \in B^\mathcal{M}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b))$$
This is dual to the previous.

Proof. We only show the first two statements are equivalent. ($\Rightarrow$) Since ι is elementary it suffices to show
$$\mathcal{M} \vDash_\Sigma \forall v : B, \phi(a, v)$$
Let $b \in B^\mathcal{M}$, then by assumption $\mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b))$, which is implies $\mathcal{M} \vDash_\Sigma \phi(a, b)$ as ι is elementary. Thus we indeed have $\mathcal{M} \vDash_\Sigma \forall v : B, \phi(a, v)$.

($\Leftarrow$) We induct on ϕ, though most of the work was already done before.

- If ϕ does not start with a $\forall$ then each case of ϕ follows from embeddings preserving quantifier free formulas.
- Embeddings preserve universal formulas downwards so we only need to check that if ι preserves ψ then ι preserves $\forall x : B, \psi$ upwards.

$$\begin{aligned} \mathcal{M} \vDash_\Sigma \forall v : B, \psi(a, v) &\Rightarrow \forall b : B^\mathcal{M}, \mathcal{M} \vDash_\Sigma \psi(a, b) \\ &\Rightarrow \forall b : B^\mathcal{M}, \mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b)) && \text{induction hypothesis} \\ &\Rightarrow \mathcal{N} \vDash_\Sigma \forall x : B, \psi(\iota(a), x) && \text{Tarski-Vaught condition} \end{aligned}$$

□

Lemma – Moving elementary embeddings down signatures
Suppose $\Sigma \leq \Sigma(*)$. If $\iota : \mathcal{M} \to \mathcal{N}$ is an elementary Σ^*-embedding then ι is an elementary Σ-embedding.

Proof. Naturally ι is a Σ-embedding, we use Tarski-Vaught to show it elementary: let $\phi \in \Sigma_{\text{for}}$ be preserved by ι, let $x : B \in \text{tv}_\phi$ and let $a \in \prod_{B \neq A \in \text{tv}_\phi} A^\mathcal{M}$, Then for any $b \in \mathcal{N}$
$$\mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b)) \Leftrightarrow \mathcal{N} \vDash_{\Sigma^*} \phi(\iota(a), \iota(b))$$

and $B^{\mathcal{M}}_{\Sigma} = B^{\mathcal{M}}_{\Sigma^*}$. Hence

$$\begin{aligned}
&\forall b \in B^{\mathcal{M}}_{\Sigma}, \mathcal{N} \vDash_{\Sigma} \phi(\iota(a), \iota(b)) \\
\Rightarrow &\forall b \in B^{\mathcal{M}}_{\Sigma^*}, \mathcal{N} \vDash_{\Sigma^*} \phi(\iota(a), \iota(b)) \\
\Rightarrow &\mathcal{N} \vDash_{\Sigma^*} \forall x : B, \phi(\iota(a), x) && \iota \text{ is elementary in } \Sigma^* \\
\Rightarrow &\mathcal{N} \vDash_{\Sigma} \forall x : B, \phi(\iota(a), x)
\end{aligned}$$

□

NOTATION. *Let $\mathcal{M}$ be a Σ-structure. Let $X \subseteq A^{\mathcal{M}}$ be a subset of the interpreted sort. Enriching only the constant symbols of Σ we can create a signature $\Sigma(X)$ such that*

$$\Sigma(X)_{\text{con}} := \Sigma_{\text{con}} \cup \{c_a : A \,|\, a \in X\}$$

In particular we write $\Sigma(\mathcal{M})$ to mean the above process applied to every sort interpreted by $\mathcal{M}$.

$$\Sigma(\mathcal{M})_{\text{con}} := \Sigma_{\text{con}} \cup \{c_a : A \,|\, A \in \Sigma_{\text{sor}}, a \in A^{\mathcal{M}}\}$$

Definition – Atomic and elementary diagrams of a structure

Let $\mathcal{M}$ be a Σ-structure, we move $\mathcal{M}$ up to the signature $\Sigma(\mathcal{M})$ by interpreting each new constant symbol c_a as a. We define the atomic diagram of $\mathcal{M}$ over Σ:

| If ϕ is an atomic $\Sigma(\mathcal{M})$-sentence such that $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi$, then $\phi \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$.

| If $\phi \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$ then $\neg\phi \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$.

We define the elementary diagram of $\mathcal{M}$ over Σ as

$$\mathrm{ElDiag}(\Sigma, \mathcal{M}) := \left\{\phi \in \Sigma(\mathcal{M})\text{-sentences } \,|\, \mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi\right\}$$

The elementary diagram of $\mathcal{M}$ is a maximal $\Sigma(\mathcal{M})$-theory with $\mathcal{M}$ as a model of it. Notice that is *not* the same as the set of all Σ-sentences satisfied by $\mathcal{M}$, which is known as the theory of $\mathcal{M}$ in Σ.

Proposition – Models of the elementary diagram are elementary extensions

Let $\mathcal{M}$ be a Σ-structure. $\mathcal{N}$ a $\Sigma(\mathcal{M})$-model of $\mathrm{AtDiag}(\Sigma, \mathcal{M})$ is naturally a Σ-extension of $\mathcal{M}$. Furthermore if $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$ then the embedding is elementary.

Conversely, given a Σ-embedding from $\mathcal{M}$ into a Σ-structure $\mathcal{N}$, $\mathcal{N}$ is naturally a $\Sigma(\mathcal{M})$-model $\mathrm{AtDiag}(\Sigma, \mathcal{M})$. If the embedding is elementary then it is a $\Sigma(\mathcal{M})$-model of $\mathrm{ElDiag}(\Sigma, \mathcal{M})$.

Proof. ($\Rightarrow$) Suppose $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{AtDiag}(\Sigma, \mathcal{M})$. Firstly we work in $\Sigma(\mathcal{M})$ to define the embedding: move $\mathcal{M}$ up a signature by taking the same interpretation as used in the definition of $\Sigma(\mathcal{M})$:

$$\star^{\mathcal{M}}_{\Sigma(\mathcal{M})} : c_a \mapsto a$$

and preserving the same interpretation for symbols of Σ. We can then write any elements of an interpreted sort $A^{\mathcal{M}}$ as $c^{\mathcal{M}}_{\Sigma(\mathcal{M})}$, for some constant symbol $c : A$.

This allows us to define a $\Sigma(\mathcal{M})$-morphism $\iota : \mathcal{M} \to \mathcal{N}$ such that for each constant symbol $c : A$,

$$\iota_A : c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \to c^{\mathcal{N}}_{\Sigma(\mathcal{M})}$$

To check that ι is well defined, take $c, d : A$ such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} = d^{\mathcal{M}}_{\Sigma(\mathcal{M})}$.

$$\begin{aligned}
&\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} c = d \\
&\Rightarrow c = d \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\
&\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} c = d \\
&\Rightarrow c^{\mathcal{N}}_{\Sigma(\mathcal{M})} = d^{\mathcal{N}}_{\Sigma(\mathcal{M})}
\end{aligned}$$

Thus ι is well defined. In fact adding 'not' gives us injectivity in the same way: Take $c, d \in \Sigma(\mathcal{M})_{\text{con}}$ such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \neq d^{\mathcal{M}}_{\Sigma(\mathcal{M})}$.

$$\begin{aligned}
&\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} c \neq d \\
&\Rightarrow c \neq d \in \operatorname{AtDiag}(\Sigma, \mathcal{M}) \\
&\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} c \neq d \\
&\Rightarrow c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \neq d^{\mathcal{N}}_{\Sigma(\mathcal{M})}
\end{aligned}$$

Thus ι is injective. To check that ι is a $\Sigma(\mathcal{M})$-morphism, we check interpretation of (non-constant) functions and relations. The non-constant function and relation symbols are the same for Σ and $\Sigma(\mathcal{M})$. Let $f : \prod A \to B$ be a non-constant function symbol in Σ and $c : \prod A$. By design we can find $d : B$ such that $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} f(c) = d$. Hence $f(c) = d \in \operatorname{AtDiag}(\Sigma, \mathcal{M})$. Hence $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} f(c) = d$ and

$$\iota \circ f^{\mathcal{M}}_{\Sigma(\mathcal{M})}(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = \iota(d^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = d^{\mathcal{N}}_{\Sigma(\mathcal{M})} = f^{\mathcal{N}}_{\Sigma(\mathcal{M})}(c^{\mathcal{N}}_{\Sigma(\mathcal{M})}) = f^{\mathcal{N}}_{\Sigma(\mathcal{M})} \circ \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})})$$

Let $r \hookrightarrow \prod A$ be a relation symbol in Σ and let $c : \prod A$.

$$\begin{aligned}
c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \in r^{\mathcal{M}}_{\Sigma(\mathcal{M})} &\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} r(c) \\
&\Rightarrow r(c) \in \operatorname{AtDiag}(\Sigma, \mathcal{M}) \\
&\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} r(c) \\
&\Rightarrow \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \in r^{\mathcal{N}}_{\Sigma(\mathcal{N})}
\end{aligned}$$

To show that ι is an embedding it remains to show pullback, which we contrapositive.

$$\begin{aligned}
c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \notin r^{\mathcal{M}}_{\Sigma(\mathcal{M})} &\Rightarrow \mathcal{M} \nvDash_{\Sigma(\mathcal{M})} r(c) \\
&\Rightarrow \neg\, r(c) \in \operatorname{AtDiag}(\Sigma, \mathcal{M}) \\
&\Rightarrow \mathcal{N} \nvDash_{\Sigma(\mathcal{M})} r(c) \\
&\Rightarrow \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \notin r^{\mathcal{N}}_{\Sigma(\mathcal{N})}
\end{aligned}$$

Assume furthermore that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \operatorname{ElDiag}(\Sigma, \mathcal{M})$. We show that the embedding is elementary. Let ϕ be a $\Sigma(\mathcal{M})$-formula and $c^{\mathcal{M}} \in \prod_{x : A \in \mathrm{tv}_\phi} A^{\mathcal{M}}$.

$$\begin{aligned}
\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi(c^{\mathcal{M}}) &\Rightarrow \phi(c) \in \operatorname{ElDiag}(\Sigma, \mathcal{M}) \\
&\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi(c) \\
&\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi(\iota(c^{\mathcal{M}}))
\end{aligned}$$

and again to show the reverse we contrapositive

$$\begin{aligned}
\mathcal{M} \nvDash_{\Sigma(\mathcal{M})} \phi(c^{\mathcal{M}}) &\Rightarrow \neg\, \phi(c) \in \operatorname{ElDiag}(\Sigma, \mathcal{M}) \\
&\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \neg\, \phi(c) \\
&\Rightarrow \mathcal{N} \nvDash_{\Sigma(\mathcal{M})} \phi(\iota(c^{\mathcal{M}}))
\end{aligned}$$

Hence ι is an (elementary) $\Sigma(\mathcal{M})$-embedding hence an (elementary) Σ-embedding.

($\Leftarrow$) Sketch: Suppose $\iota : \mathcal{M} \to \mathcal{N}$ is an (elementary) embedding. Make $\mathcal{M}$ and $\mathcal{N}$ into $\Sigma(\mathcal{M})$-structures by $\star^{\mathcal{M}}_{\Sigma(\mathcal{M})} : c_a \to a$ and $\star^{\mathcal{N}}_{\Sigma(\mathcal{M})} : c_a \to \iota(a)$, for any $a \in A^{\mathcal{M}}$. It follows that ι becomes an (elementary) $\Sigma(\mathcal{M})$-embedding. $\Sigma(\mathcal{M})$-embeddings preserve quantifier free formulas so $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \operatorname{AtDiag}(\Sigma, \mathcal{M})$. When the embedding is elementary, any $\Sigma(\mathcal{M})$-sentence satisfied by $\mathcal{M}$ will be satisfied by $\mathcal{N}$ and so $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \operatorname{ElDiag}(\Sigma, \mathcal{M})$. $\square$

1.1.8 Universal axiomatization

Definition – Axiomatization, universal theory, universal axiomatization

A Σ-theory A is an axiomatization of a Σ-theory T if for all Σ-structures $\mathcal{M}$,

$$\mathcal{M} \vDash_\Sigma T \Leftrightarrow \mathcal{M} \vDash_\Sigma A$$

If A is a set of universal Σ-sentences is called a universal Σ-theory. We are interested in universal axiomatizations of theories.

Lemma – Lemma on constants

Suppose $\Sigma_{\text{con}} \subseteq {\Sigma^*}_{\text{con}}$, T is a Σ-theory and ϕ a Σ-formula with $\text{tv}_\phi = \{x_1 : A_1, \ldots, x_n : A_n\}$. If there exists $\mathcal{M}$ a Σ-model of T such that

$$\mathcal{M} \vDash_\Sigma \exists x : \prod_{i=1}^{n} A_i, \phi$$

then $\mathcal{M}$ can be extended to a Σ^*-model of T such that for any tuple of new constant symbols $c = (c_i)$, where $c_i : A_i \in {\Sigma^*}_{\text{con}} \setminus \Sigma_{\text{con}}$, we have

$$\mathcal{M} \vDash_{\Sigma^*} \phi(c)$$

Packaged differently using the contrapositive we have: if there exist $c_i : A_i \in {\Sigma^*}_{\text{con}} \setminus \Sigma_{\text{con}}$ such that $T \vDash_{\Sigma^*} \phi(c)$ then

$$T \vDash_\Sigma \forall x : \prod_{i=1}^{n} A_i, \phi$$

Proof. Suppose there exists $\mathcal{M}$ a Σ-model of T and $a \in \prod A_i^{\mathcal{M}}$ such that $\mathcal{M} \vDash_\Sigma \phi(a)$.

We extend $\mathcal{M}$ to being a Σ^*-model of T by interpreting to the new constant symbols $c : B \in {\Sigma^*}_{\text{con}} \setminus \Sigma_{\text{con}}$ as

$$c^{\mathcal{M}}_{\Sigma(\mathcal{M})} := \begin{cases} a_i & \text{, if } B = A_i \\ b \in B^{\mathcal{M}} & \text{, otherwise, since each } B^{\mathcal{M}} \text{ is non-empty} \end{cases}$$

Then $\mathcal{M}$ is a Σ^*-model of T such that $\mathcal{M} \vDash_{\Sigma^*} \phi(a)$, which by design gives us that for any $c_i : A_i \in {\Sigma^*}_{\text{con}} \setminus \Sigma_{\text{con}}$

$$\mathcal{M} \nvDash_{\Sigma^*} \phi(c)$$

□

NOTATION. *Let T be a Σ-theory, then*

$$T_\forall := \{\phi \text{ universal } \Sigma\text{-sentences} \mid T \vDash_\Sigma \phi\}$$

is called the set of universal consequences of T.

Proposition – Universal axiomatizations make substructures models

T a Σ-theory has a universal axiomatization if and only if for any Σ-model $\mathcal{N}$ of T and any Σ-embedding from some Σ-structure $\mathcal{M} \to \mathcal{N}$, $\mathcal{M}$ is a Σ-model of T.

Proof. ($\Rightarrow$) Suppose A is a universal axiomatization of T, $\mathcal{N}$ is a Σ-model of T and $\mathcal{M} \to \mathcal{N}$ is a Σ-embedding. Let $\phi \in T$. Then $\mathcal{N} \vDash_\Sigma T$ implies $\mathcal{N} \vDash_\Sigma A$ by definition of A. $\mathcal{N} \vDash_\Sigma A$ implies $\mathcal{M} \vDash_\Sigma A$ since embeddings preserve the satisfaction of quantifier free formulas downwards. Finally $\mathcal{M} \vDash_\Sigma A$ implies $\mathcal{M} \vDash_\Sigma T$ by definition of A.

($\Leftarrow$) We show that $T_\forall$ is a universal axiomatization of T. One direction is obvious: any Σ-model of T is a Σ-model of $T_\forall$.

Suppose $\mathcal{M} \vDash_\Sigma T_\forall$. We first show that $T \cup \mathrm{AtDiag}(\Sigma, \mathcal{M})$ is consistent. By the compactness theorem it suffices to show that for any subset Δ of $\mathrm{AtDiag}(\Sigma, \mathcal{M})$, $T \cup \Delta$ is consistent. Write $\Delta = \{\psi_1, \ldots, \psi_n\}$. Let $\psi = \bigwedge_{1 \le i \le n} \psi_i$. We can find S the set of constant symbols $c_i : A_i \in \Sigma(\mathcal{M})_{\mathrm{con}} \setminus \Sigma_{\mathrm{con}}$ that appear in ψ and create $\phi \in \Sigma_{\mathrm{for}}$ such that $\mathrm{tv}_\phi = \{x_i : A_i\}_{c_i : A_i \in S}$ and $\phi(c) = \psi$, where $c = (c_i)$. Since $\Delta \subseteq \mathrm{AtDiag}(\Sigma, \mathcal{M})$ we have $\forall i, \mathcal{M} \vDash_{\Sigma(\mathcal{M})} \psi_i$. Hence $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi(c)$. Then

$$\mathcal{M} \vDash_\Sigma \exists x : \prod A_i, \phi \text{ and so } \mathcal{M} \nvDash_\Sigma \forall x : \prod A_i, \neg\, \phi(v)$$

Since each ψ_i is from the the atomic diagram of $\mathcal{M}$ they are all quantifier free. Thus ϕ is a quantifier free Σ-formula and $\forall v, \neg\, \phi(v)$ is universal. Hence $T \nvDash_\Sigma \forall v, \neg\, \phi(v)$ by the definition of $T_\forall$. By the lemma on constants this implies that $T \nvDash_{\Sigma(\mathcal{M})} \neg\, \phi(c)$. Hence there exists a $\Sigma(\mathcal{M})$-model of $T \cup \phi(c)$. Then it follows that this is also a $\Sigma(\mathcal{M})$-model of $T \cup \Delta$. Thus $T \cup \Delta$ is consistent so $T \cup \mathrm{AtDiag}(\Sigma, \mathcal{M})$ is consistent.

Thus there exists $\mathcal{N}$ a Σ-model of $T \cup \mathrm{AtDiag}(\Sigma, \mathcal{M})$. This is a model of $\mathrm{AtDiag}(\Sigma, \mathcal{M})$ so there is a $\Sigma(\mathcal{M})$-embedding $\mathcal{M} \to \mathcal{N}$. Seeing this as a Σ-embedding the theorem's hypothesis tells us $\mathcal{M}$ is a Σ-model of T. $\square$

The following appears as an exercise in the second chapter of Marker's book [12] and is a consequence of the lemma on constants.

Corollary – Amalgamation

Suppose Σ is single sorted. Let $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures, and suppose we have partial elementary Σ-embeddings $\iota_{\mathcal{M}} : \mathcal{A} \to \mathcal{M}$ and $\iota_{\mathcal{N}} : \mathcal{A} \to \mathcal{N}$, such that $\varnothing \neq A \subseteq \mathcal{A}$ is the domain for $\iota_{\mathcal{M}}$ and $\iota_{\mathcal{N}}$. Then there exists a common elementary extension $\mathcal{P}$ of $\mathcal{M}$ and $\mathcal{N}$ such that the following commutes:

$$\begin{tikzcd} \mathcal{M} \arrow[r] & \mathcal{P} \\ A \arrow[u, "\iota_{\mathcal{M}}"] \arrow[r, "\iota_{\mathcal{N}}"'] & \mathcal{N} \arrow[u] \end{tikzcd}$$

$\mathcal{P}$ is the 'amalgamation' of $\mathcal{M}$ and $\mathcal{N}$.

Proof. We will denote the only sort symbol by S. We show that the theory $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \mathrm{ElDiag}(\Sigma, \mathcal{N})$ is a consistent $\Sigma(\mathcal{M}, \mathcal{N})$-theory, where $\Sigma(\mathcal{M}, \mathcal{N})_{\mathrm{con}}$ is defined to be

$$\{c_a \mid a \in A\} \cup \{c_a \mid a \in \mathcal{M} \setminus \iota_{\mathcal{M}}(A)\} \cup \{c_a \mid a \in \mathcal{N} \setminus \iota_{\mathcal{N}}(A)\}$$

and other symbols are the same as Σ. For the rest of the proof we identify

$$\Sigma(\mathcal{M})_{\mathrm{con}} \cong \Sigma(A)_{\mathrm{con}} \cup \{c_a \mid a \in \mathcal{M} \setminus \iota_{\mathcal{M}}(A)\}$$

by taking $c_{\iota_{\mathcal{M}}(a)} \mapsto c_a$ (similarly with $\mathcal{N}$). Which is why we don't bother writing $\Sigma(A, \mathcal{M}, \mathcal{N})_{\mathrm{con}}$.

By the compactness theorem it suffices to show that for any finite subset $\Delta \subseteq \mathrm{ElDiag}(\Sigma, \mathcal{N})$, $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \Delta$ is consistent. Let $\phi(v)$ be the $\Sigma(A)$-formula and $a \in \mathcal{N}^n$ such that [1]

$$\phi(a) = \bigwedge_{\psi \in \Delta} \psi$$

Then $\phi(a)$ is naturally a $\Sigma(\mathcal{N})$-sentence.

Suppose for a contradiction $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \Delta$ is inconsistent. Then any $\Sigma(\mathcal{M}, \mathcal{N})$-model of $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ is not a model of Δ, which implies it satisfies $\neg\, \phi(a)$ so

$$\mathrm{ElDiag}(\Sigma, \mathcal{M}) \vDash_{\Sigma(\mathcal{M}, \mathcal{N})} \neg\, \phi(a)$$

[1] Take out all the finitely many constants appearing from $\mathcal{N} \setminus \iota_{\mathcal{N}}(A)$ in Δ and make them into a tuple a, replacing them with free variables. What remains is a finite set of $\Sigma(A)$-formulas. We take the conjunction of all of them to be $\phi(v)$.

By the lemma on constants applied to $\Sigma(\mathcal{M}) \leq \Sigma(\mathcal{M},\mathcal{N})$, $\mathrm{ElDiag}(\Sigma,\mathcal{M})$ and $a \in \Sigma(\mathcal{M},\mathcal{N})_{\mathrm{con}} \setminus \Sigma(\mathcal{M})_{\mathrm{con}}$ we have

$$\mathrm{ElDiag}(\Sigma,\mathcal{M}) \vDash_{\Sigma(\mathcal{M})} \forall v : S^n, \neg\, \phi(v)$$

$\mathcal{M}$ is a $\Sigma(\mathcal{M})$-model of its elementary diagram.

$$\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \forall v, \neg\, \phi(v) \Rightarrow \mathcal{M} \vDash_{\Sigma(A)} \forall v, \neg\, \phi(v)$$

Since $\mathcal{A} \to \mathcal{M}$ and $\mathcal{A} \to \mathcal{N}$ are partial elementary Σ-embeddings (hence partial elementary $\Sigma(A)$-embeddings) and $\forall v, \phi(v,w)$ is a $\Sigma(A)$-formula we have that

$$\mathcal{M} \Rightarrow \mathcal{A} \Rightarrow \mathcal{N} \vDash_{\Sigma(A)} \forall v, \neg\, \phi(v) \Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{N})} \forall v, \neg\, \phi(v)$$

However $\phi(a)$ was a conjunction of formulas in $\mathrm{ElDiag}(\Sigma,\mathcal{N})$ so $\mathcal{N} \vDash_{\Sigma(\mathcal{N})} \phi(a)$, which gives us our contradiction.

Hence $\mathrm{ElDiag}(\Sigma,\mathcal{M}) \cup \mathrm{ElDiag}(\Sigma,\mathcal{N})$ is consistent as a $\Sigma(\mathcal{M},\mathcal{N})$-theory. Let $\mathcal{P}$ be a $\Sigma(\mathcal{M},\mathcal{N})$-model of this (and naturally $\Sigma(\mathcal{M})$ and $\Sigma(\mathcal{N})$-models of the respective elementary diagrams). Then there exist elementary $\Sigma(\mathcal{M})$ and $\Sigma(\mathcal{N})$-extensions $\lambda_{\mathcal{M}} : \mathcal{M} \to \mathcal{P}$ and $\lambda_{\mathcal{N}} : \mathcal{N} \to \mathcal{P}$.

Naturally, we can move everything down to $\Sigma(A)$. Thus for any $a \in A$ let c be the constant symbol for a in $\Sigma(A)$:

$$\lambda_{\mathcal{M}} \circ \iota_{\mathcal{M}}(a) = \lambda_{\mathcal{M}}(c^{\mathcal{M}}_{\Sigma(A)}) = \lambda_{\mathcal{M}}(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c^{\mathcal{P}}_{\Sigma(\mathcal{M})} = c^{\mathcal{P}}_{\Sigma(A)}$$

By symmetry we have

$$\lambda_{\mathcal{M}} \circ \iota_{\mathcal{M}}(a) = c^{\mathcal{P}}_{\Sigma(A)} = \lambda_{\mathcal{N}} \circ \iota_{\mathcal{N}}(a)$$

□

1.1.9 The Löwenheim-Skolem Theorems

Proposition – Upward Löwenheim-Skolem Theorem

Let Σ be a signature and A a sort symbol in the signature. If $\mathcal{M}$ is a Σ-structure with $A^{\mathcal{M}}$ infinite, and κ a cardinal such that $|A^{\mathcal{M}}| + |\Sigma_{\mathrm{con}}| \leq \kappa$, there exists a Σ-structure $\mathcal{N}$ with $|A^{\mathcal{N}}| = \kappa$ as well as an elementary Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$.

One can extend this to any sort and obtain the same result replacing $|A^{\mathcal{N}}|$ with $|\mathcal{N}|$.

Proof. $\mathcal{M}$ is a $\Sigma(\mathcal{M})$-model, of $\mathrm{ElDiag}(\Sigma,\mathcal{M})$. Thus $\mathrm{ElDiag}(\Sigma,\mathcal{M})$ is a $\Sigma(\mathcal{M})$-theory with a model $\mathcal{M}$ such that $A^{\mathcal{M}}$ is infinite, hence it has a $\Sigma(\mathcal{M})$-model $\mathcal{N}$ with $\kappa = |A^{\mathcal{N}}|$. Making $\mathcal{N}$ a Σ-structure, we obtain a Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$. □

Definition – Skolem Functions

We say that a Σ-theory T *has built in Skolem functions* when for any Σ-formula ϕ, with $\mathrm{tv}_\phi = \{v : B\} \cup W$ there exists a function symbol $f : \prod_{x:A\in W} A \to B$ such that

$$\forall w : \prod_{x:A\in W} A, ([\exists v : B, \phi(v,w)] \to \phi(f(w),w))$$

is a sentence in T. In particular if W is empty then we have the special case where f is a constant symbol

$$T \vDash_\Sigma \exists v : B, \phi(v) \to \phi(f)$$

which looks like the witness property.

> **Proposition – Skolemization**
> Let $T(0)$ be a $\Sigma(0)$-theory, then there exists T a Σ theory such that
>
> 1. $|\Sigma_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$
> 2. $\Sigma(0)_{\text{fun}} \subseteq \Sigma_{\text{fun}}$, and they share the same relation symbols
> 3. $T(0) \subseteq T$
> 4. All $\Sigma(0)$-models of $T(0)$ are naturally Σ-models of T.
> 5. T has built in Skolem functions
>
> We call T the Skolemization of $T(0)$.

Proof. Similarly to the Witness Property proof, we define $\Sigma(i), T(i)$ for each $i \in \mathbb{N}$. Suppose by induction that we have $T(i) \in \Sigma_{\text{the}}$, such that

1. $|\Sigma(i)_{\text{fun}}| \leq |\Sigma(0)_{\text{fun}}| + \aleph_0$
2. $\Sigma(0)_{\text{fun}} \subseteq \Sigma(i)_{\text{fun}}$ and they share the same relation symbols
3. $T(0) \subseteq T(i)$
4. All $\Sigma(0)$-models of $T(0)$ are naturally $\Sigma(i)$-models of $T(i)$.

Then define $\Sigma(i+1)$ such that only the function symbols are enriched:

$$\Sigma(i+1)_{\text{fun}} := \Sigma(i)_{\text{fun}} \cup \left\{ f_\phi : \prod_{x:A\in W} A \to B \,|\, \phi \in \Sigma(i)_{\text{for}} \text{ and } \text{tv}_\phi = \{v : B\} \cup W \right\}$$

There are countably infinite $\Sigma(i)$-formulas, thus $|\Sigma(i)_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$ as required.

For each new function symbol f_ϕ define

$$\Psi(f_\phi) := \forall w : \prod_{x:A\in W} A, ([\exists v : B, \phi(v, w)] \to \phi(f_\phi(w), w))$$

We then form $\Sigma(i+1)$-theory $T(i+1)$ by adding each $\Psi(\phi)$ to $T(i)$, noting that $T(0) \subseteq T(i) \subseteq T(i+1)$.

Let $\mathcal{M}$ be a $\Sigma(0)$-model of $T(0)$, then it is naturally a $\Sigma(i)$-model of $T(i)$. To extend interpretation to $\Sigma(i+1)$, for each new function symbol $f_\phi : \prod_{x:A\in W} A \to B$ define

$$f_\phi^{\mathcal{M}} : \prod_{x:A\in W} A^{\mathcal{M}} \to B^{\mathcal{M}}$$
$$a \mapsto \begin{cases} b & \text{, if there is some } b \in B^{\mathcal{M}} \text{ such that } \mathcal{M} \vDash_{\Sigma(i)} \phi(b, a) \\ \text{anything} & \text{, otherwise, since each } B^{\mathcal{M}} \text{ is non-empty} \end{cases}$$

Then by construction for each f_ϕ, $\mathcal{M} \vDash_{\Sigma(i+1)} \Psi(f_\phi)$. By induction we had $\mathcal{M} \vDash_{\Sigma(i+1)} T(i)$, which together imply $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} T(i+1)$.

Let Σ^* be the signature with relation symbols from $\Sigma(0)$ and ${\Sigma^*}_{\text{fun}} = \bigcup_{i\in\mathbb{N}} \Sigma(i)_{\text{fun}}$. Then

$$|{\Sigma^*}_{\text{fun}}| = |\bigcup_{i\in\mathbb{N}} \Sigma(i)_{\text{fun}}| = \aleph_0 \times (\aleph_0 + \Sigma(0)_{\text{fun}}) = \aleph_0 + \Sigma(0)_{\text{fun}}$$

Let $T^* = \bigcup_{i\in\mathbb{N}} T(i)$. We show that T^* has built in Skolem functions. Let ϕ be a Σ^*-formula with a free variable $v : B$. Then $\phi \in \Sigma(i)_{\text{for}}$ for some $i \in \mathbb{N}$. Thus $\Psi(f_\phi) \in T(i+1) \subseteq T^*$ as required.

If $\mathcal{M} \vDash_\Sigma T$ then extend the interpretation to Σ^* such that for all $i \in \mathbb{N}$, and $f \in T(i)$, $f^{\mathcal{M}}_{\Sigma^*} = f^{\mathcal{M}}_{\Sigma(i)}$. Since all interpretations agree upon intersection this is well-defined. Since for each i we have $\mathcal{M} \vDash_{\Sigma(i)} T(i)$ we have $\mathcal{M} \vDash_{\Sigma^*} T(i)$ and so $\mathcal{M}$ is a Σ^*-model of T^*. □

Notation. *For $\mathcal{M}$, a Σ-structure we write $X \subseteq \mathcal{M}$ to mean: for each sort A there exists $X_A \subseteq A^{\mathcal{M}}$.*

We define the theory of a Σ-structure $\mathcal{M}$ to be

$$\mathrm{Th}_{\mathcal{M}}(\Sigma) := \{\phi \in \Sigma_{\text{for}} \mid \phi \text{ is a } \Sigma\text{-sentence and } \mathcal{M} \vDash_\Sigma \phi\}$$

If the signature is obvious we just write $\mathrm{Th}_{\mathcal{M}}$. Note that $\mathrm{Th}_{\mathcal{M}}(\Sigma)$ is a consistent and complete Σ-theory as it is modelled by $\mathcal{M}$ and any formula is either satisfied by $\mathcal{M}$ or not.

Let $\mathcal{M}$ be a Σ-structure and let $X \subseteq \mathcal{M}$. $\mathcal{M}$ is naturally a $\Sigma(X)$-structure. The theory of $\mathcal{M}$ over X is defined by

$$\mathrm{Th}_{\mathcal{M}}(X) := \mathrm{Th}_{\mathcal{M}}(\Sigma(X))$$

Note that $\mathrm{Th}_{\mathcal{M}}(\mathcal{M})$ is the elementary diagram $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ (where $\mathcal{M}$ is seen as the collection of all the $A^{\mathcal{M}}$ for each sort A).

Proposition – Downward Löwenheim-Skolem Theorem

Let $\mathcal{N}$ be an infinite $\Sigma(0)$-structure and $M \subseteq \mathcal{N}$ (noting $M \subseteq \mathcal{N}$ is notation) for each sort symbol A. Then there exists a $\Sigma(0)$-structure $\mathcal{M}$ such that

- $M \subseteq \mathcal{M} \hookrightarrow \mathcal{N}$ and the 'inclusion' $\mathcal{M} \hookrightarrow \mathcal{N}$ is an elementary $\Sigma(0)$-embedding.
- $|\mathcal{M}| \leq |M| + |\Sigma(0)_{\text{fun}}| + \aleph_0$, where $M := \bigcup_{A\text{sort symbol}} M_A$

Proof. We first take the Skolemization of $\mathrm{Th}_{\mathcal{N}}$ and call the new signature and theory Σ and T. By assumtion $\mathcal{N} \vDash_{\Sigma(0)} \mathrm{Th}_{\mathcal{N}}$, and by construction of T, $\mathcal{N}$ becomes a Σ-model T. Also note that by construction $|\Sigma| \leq |\Sigma(0)| + \aleph_0$.

We want to make the smaller model $\mathcal{M}$. It must be closed under taking functions, so we define it inductively. Suppose by induction we have for each sort symbol A a set $M_A \subseteq M_A(i) \subseteq A^{\mathcal{N}}$ such that

$$|M(i)| \leq |M| + |\Sigma_{\text{fun}}| + \aleph_0$$

where $M(i) = \bigcup_{A\text{sort symbol}} M_A(i)$. We inductively define each $M_B(i+1)$:

$$M_B(i+1) := M_B(i) \cup \left\{ f^{\mathcal{N}}(a) \mid f : \prod A \to B \text{ is a function symbol and } a \in \prod M_A(i) \right\}$$

Then

$$\begin{aligned}
|M(i+1)| &\leq |M(i)| + |\Sigma_{\text{fun}}| \times |\prod M_A(i)| \\
&\leq |M(i)| + |\Sigma_{\text{fun}}| \times (|M_A(i)| \times \aleph_0) \\
&\leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0 + |\Sigma_{\text{fun}}| \times (|M(0)| + |\Sigma_{\text{fun}}| + \aleph_0) \\
&\leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0
\end{aligned}$$

Then $A^{\mathcal{M}} := \bigcup_i M_A(i)$ and

$$|\mathcal{M}| \leq (|M(0)| + |\Sigma_{\text{fun}}| + \aleph_0) \times \aleph_0 \leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0 \leq |M(0)| + |\Sigma(0)_{\text{fun}}| + \aleph_0$$

To interpret function symbols, let $f : \prod A \to B$ be a function symbol and let $a \in \prod A^{\mathcal{M}}$. Then there exists an i such that $a \in \prod M_A(i)$ so we let $f^{\mathcal{M}}$ take a to $f^{\mathcal{N}}(a) \in M_B(i+1)$. We define the interpretation of relations $r \hookrightarrow \prod A$ as the product of intersections (pullbacks) $r^{\mathcal{M}} := \prod A^{\mathcal{M}} \cap r^{\mathcal{N}}$.

By construction the inclusion $\subseteq$ is a Σ-embedding. We check that it is elementary using the third equivalent condition in the Tarski-Vaught Test: let $\phi \in \Sigma_{\text{for}}$ be preserved by $\subseteq$, $x : B \in \text{tv}_\phi$ and $a \in \prod_{B \neq A \in \text{tv}_\phi} A^{\mathcal{M}}$. Suppose

$$\mathcal{N} \vDash_\Sigma \exists x : B, \phi(a, x)$$

T has built in Skolem functions and $\mathcal{N} \vDash_\Sigma T$ so there exists $f \in \Sigma_{\text{fun}}$ such that

$$\mathcal{N} \vDash_\Sigma (\exists x : B, \phi(a, v)) \to \phi(a, f(a))$$

Hence $\mathcal{N} \vDash_\Sigma \phi(a, f(a))$. Hence the embedding $\mathcal{M} \to \mathcal{N}$ is elementary. Then $\subseteq$: $\mathcal{M} \to \mathcal{N}$ is an elementary $\Sigma(0)$-embedding. $\square$

1.2 Types

This section mainly follows material from Tent and Ziegler's book [11]. There is an unfortunate terminology clash with types in type theory. Model theoretic n-types have nothing to do with type theoretic types and n-types. However, the model theoreic notions of 'sorts' and 'terms of type A a sort' *does* line up with the notion of a term of of type A (usually A would be called a 'type' in type theory).

For this chapter we fix a signature Σ and $V_n = v_1 : A_1, \ldots, v_n : A_n$ a finite set of typed variables in Σ.

1.2.1 Types

Definition – $F(\Sigma, V_n)$ and n-dimensional theories

Let $F(\Sigma, V_n)$ be the set of Σ-formulas whose typed free variables are amongst those n variables. We call $p \subseteq F(\Sigma, V_n)$ an n-dimensional theory, noting that if $n = 0$ we have the original notion of a Σ-theory. We will define n-types later as n-dimensional theories that have some notion of consistency. We say p is a maximal if for any $\phi \in F(\Sigma, V_n)$, $\phi \in p$ or $\neg\phi \in p$, generalising the notion of maximality of a theory.

For any tuple of constant symbols $c : \prod_{i=1}^n A_i$ and n-dimensional theory $p \subseteq F(\Sigma, V_n)$ we write

$$p(c) = \{\phi(c) \mid \phi \in p\}$$

and if $\mathcal{M}$ is a Σ-structure with $a \in \prod_{i=1}^n A_i^{\mathcal{M}}$ we write

$$\mathcal{M} \vDash_\Sigma p(a)$$

to mean for every $\phi \in p$, $\mathcal{M} \vDash_\Sigma \phi(a)$.

Definition – Realisation

Let T be a Σ-theory. Let $p \subseteq F(\Sigma, V_n)$ be an n-dimensional theory. Let $\mathcal{M}$ be a Σ-structure.

- p is *realised* in $\mathcal{M}$ by $a \in \prod_{i=1}^n A_i^{\mathcal{M}}$ over A if

  $$\mathcal{M} \vDash_\Sigma p(a)$$

  We also just say p is realised in $\mathcal{M}$. If p is not realised in $\mathcal{M}$ then we say $\mathcal{M}$ *omits* p.

- p is *finitely realised* in $\mathcal{M}$ over A if for each finite subset $\Delta \subseteq p$ there exists $a \in \prod_{i=1}^n A_i^{\mathcal{M}}$ such that Δ is realised in $\mathcal{M}$ by a.

EXAMPLE. *We have a notion of compactness for n-dimensional theories, but there is no such notion for realisation. For example, let Σ be the signature with just a binary relation $<$. For the Σ-theory of dense linear orders we have a Σ-model $\mathbb{Q}$ which finitely realises the 1-dimensional theory*

$$p = x < q \,|\, q \in \mathbb{Q}$$

since any finite set has an infemum. However there is no element of $\mathbb{Q}$ realising p. We will later see that finite realisation of p implies existence of an elementary extension of $\mathbb{Q}$ realising p. An example of this would be $\mathbb{Q} \cup -\infty$ with the obvious ordering.

Definition – n-types and n-dimensional compactness

Let T be a Σ-theory and $p \subseteq F(\Sigma, V_n)$ be an n-dimensional Σ-theory. Let $\Sigma(c)$ be the signature with added constant symbols $c_1 : A_1, \ldots, c_n : A_n$. The following are equivalent (see proof):

1. $T \cup p(c)$ is a consistent $\Sigma(c)$-theory.
2. (Consistent with T) There exists $\mathcal{M} \vDash_\Sigma T$ such that p is realised in $\mathcal{M}$.
3. (Finitely consistent with T) For any finite subset $\Delta \subseteq p$, there exists $\mathcal{M} \vDash_\Sigma T$ such that p is realised in $\mathcal{M}$.

If any of the above is true then we say p an *n-type consistent with T* (also called a *partial* n-type on T). The second and third definitions being equivalent is the n-dimensional compactness theorem (also called *compactness for types*).

Furthermore if $\mathcal{M}$ is a Σ-structure with subset $X \subseteq \mathcal{M}$ (noting $X \subseteq \mathcal{M}$ is notation), we say that p is an *n-type on $\mathcal{M}$ over A* if p is an n-type consistent with $\mathrm{Th}_{\mathcal{M}}(A)$.

Proof. (1. $\Leftrightarrow$ 2.) Suppose we have a $\Sigma(c)$-structure $\mathcal{M} \vDash_{\Sigma(c)} T \cup p(c)$. Then by taking the images of the interpretation of each c_i in $A_i^{\mathcal{M}}$ we obtain $a = c^{\mathcal{M}} \in \prod A_i^{\mathcal{M}}$ such that $\mathcal{M} \vDash_{\Sigma(c)} p(a)$. Moving this down to Σ we have

$$\mathcal{M} \vDash_\Sigma T \cup p(a)$$

Conversely suppose we have $\mathcal{M} \vDash_\Sigma T$ and $a \in \prod A_i^{\mathcal{M}}$ such that $\mathcal{M} \vDash_\Sigma p(a)$. We can make $\mathcal{M}$ a $\Sigma(c)$-structure such that everything from Σ is interpreted in the same way and each constant symbol c_i is interpreted as a_i. Thus for any $\phi(c) \in p(c)$,

$$\mathcal{M} \vDash_\Sigma \phi(a) \Rightarrow \mathcal{M} \vDash_{\Sigma(c)} \phi(a) \Rightarrow \mathcal{M} \vDash_{\Sigma(c)} \phi(c)$$

Hence $\mathcal{M} \vDash_{\Sigma(c)} T \cup p(c)$ and $T \cup p(c)$ is consistent in $\Sigma(c)$.

(2. $\Leftrightarrow$ 3.)

$$\begin{aligned}
&p \text{ consistent with } T\\
\Leftrightarrow\ & T \cup p(c) \text{ consistent in } \Sigma(c) \quad \text{by (1. } \Leftrightarrow \text{ 2.)}\\
\Leftrightarrow\ & \text{for any finite } \Delta(c) \subseteq p(c), T \cup \Delta(c) \text{ consistent in } \Sigma(c) \quad \text{by compactness}\\
\Leftrightarrow\ & \text{for any finite } \Delta \subseteq p, T \cup \Delta(c) \text{ consistent in } \Sigma(c)\\
\Leftrightarrow\ & \text{for any finite } \Delta \subseteq p, \Delta \text{ consistent with } T \quad \text{by (1. } \Leftrightarrow \text{ 2.)}
\end{aligned}$$

□

Definition – Type of an element

Let $\mathcal{M}$ be a Σ-structure and $a \in \prod_{i=1}^n A_i^{\mathcal{M}}$ Then

$$\mathrm{tp}_{\Sigma,n}^{\mathcal{M}}(a) := \{\phi \in F(\Sigma, V_n) \,|\, \mathcal{M} \vDash_\Sigma \phi(a)\}$$

is the type of a in $\mathcal{M}$ over A. One can verify that if $\mathcal{M}$ is any Σ-structure then $\mathrm{tp}(a)$ (we get rid of superscripts and subscripts when clear) is a maximal n-type consistent with any Σ-theory modelled by $\mathcal{M}$. If $X \subseteq \mathcal{M}$ and we want to do the same but with signature $\Sigma(X)$ we write $\operatorname{tp}^{\mathcal{M}}_{X,n}(a)$.

Definition – Equivalence modulo a theory
We say two Σ-formulas ϕ and ψ in $F(\Sigma, V_n)$ are equivalent modulo a Σ-theory T if

$$T \vDash_\Sigma \forall v : \prod_{i=1}^{n} A_i, (\phi \Leftrightarrow \psi)$$

Definition – Stone space of a theory
Let T be a Σ-theory. Let the stone space of T be the set $S_n(T)$ of all maximal n-types on T. (The signature of the n-types of the on T is implicit, given by the signature of T.) We give a topology on $S_n(T)$ by specifying an open basis, which consists of the subsets

$$[\phi]_T := \{p \in S_n(T) \mid \phi \in p\}$$

for each $\phi \in F(\Sigma, V_n)$.

Lemma – Extending to maximal n-types
Any n-type can be extended to a maximal n-type by taking the type of an element. In particular p is a maximal n-type consistent with theory T if and only if $p = \operatorname{tp}^{\mathcal{M}}_n(a)$ for a model $\mathcal{M}$ of T and some $a \in \prod_{i=1}^n A_i^{\mathcal{M}}$. Hence elements of the Stone space of a theory are types of elements.

Proof. Let T be a Σ-theory and p be a n-type. Then by definition p is realised by some $a \in \prod_{i=1}^n A_i^{\mathcal{M}}$, where $\mathcal{M}$ is a Σ-model of T. Taking the type of element a we have $p \subseteq \operatorname{tp}^{\mathcal{M}}_n(a)$ is a maximal n-type extending p. □

Proposition – Elementary properties of the Stone space
Let T be a Σ-theory, $p \in S_n(T)$, and $\phi, \psi \in F(\Sigma, V_n)$.

- $(\neg\, \phi) \in p$ if and only if $p \notin [\phi]_T$.
- $[\phi]_T = [\psi]_T$ if and only if ϕ and ψ are equivalent modulo T; $[\phi]_T \subseteq [\psi]_T$ if and only if ϕ and ψ

  $$T \vDash_\Sigma \forall v : \prod_{i=1}^{n} A, \phi \to \psi$$

  The basis elements are closed under Boolean operations
- $[\neg\, \phi]_T = S_n(T) \setminus [\phi]_T$
- $[\phi \vee \psi]_T = [\phi]_T \cup [\psi]_T$
- $[\phi \wedge \psi]_T = [\phi]_T \cap [\psi]_T$
- $[\top]_T = S_n(T)$ and $[\bot]_T = \varnothing$ and in particular if $[\phi]_T = S_n(T) = [\top]_T$ then ϕ and $\top$ are equivalent modulo T hence $T \vDash_\Sigma \forall v, \phi$.

Proof. We will just prove a couple of these.

- Suppose $(\neg\, \phi) \in p$. If $p \in [\phi]_T$ then $phi, \neg\, \phi$ are both in p, but p is consistent with T so there exists a model $\mathcal{M}$ and a from $\mathcal{M}$ such that $\mathcal{M} \vDash_\Sigma \phi(a)$ and $\mathcal{M} \nvDash_\Sigma \phi(a)$, a contradiction. For the other direction, $p \notin [\psi]_T$ and so $\psi \notin p$ and by maximality $\neg\, \phi \in p$.

- We show the $\leftrightarrow$ statement. ($\Rightarrow$) Let $\mathcal{M} \vDash_\Sigma T$ and $a \in \prod_{i=1}^n A_i^\mathcal{M}$. We want that $\mathcal{M} \vDash_\Sigma \phi(a) \leftrightarrow \psi(a)$. By symmetry it suffices to assume $\mathcal{M} \vDash_\Sigma \phi(a)$ and show $\mathcal{M} \vDash_\Sigma \psi(a)$. Taking the type of element a gives us maximal n-type $\mathrm{tp}_n^\mathcal{M}(a) \in [\phi]_T \subseteq [\psi]_T$. Hence $\mathcal{M} \vDash_\Sigma \psi(a)$. ($\Leftarrow$) Suppose $T \vDash_\Sigma \forall v, (\phi \leftrightarrow \psi)$. Let $p \in [\phi]_T$; by symmetry it suffices to show that $p \in [\psi]_T$. Since p is consistent with T there exists a Σ-structure $\mathcal{M} \vDash_\Sigma T$ and $a \in \prod_{i=1}^n A_i^\mathcal{M}$ such that $\mathcal{M} \vDash_\Sigma p(a)$. By assumption $\mathcal{M} \vDash_\Sigma (\phi \to \psi)(a)$ and $p \in [\phi]_T$ so $\mathcal{M} \vDash_\Sigma \psi(a)$. By maximality of p we just need to note that the case $\neg \psi \in p$ implies $\mathcal{M} \nvDash_\Sigma \psi(a)$ which is false.

□

Proposition – Topological properties of the Stone space
Let T be a theory.

- Elements of the basis of $S_n(T)$ are clopen.
- $S_n(T)$ is Hausdorff.
- $S_n(T)$ is compact.

Proof.

- By maximality of each p the complement of U is also in the open basis:
$$\{p \in S_n(T) \mid \phi \notin p\} = \{p \in S_n(T) \mid (\neg \phi) \in p\}$$
Hence each element of the basis is clopen.

- Let $p, q \in S_n(T)$ and suppose they share the same neighbourhoods. then for any formula $\phi \in F(\Sigma, V_n)$ we have
$$p \in [\phi]_T \Leftrightarrow q \in [\phi]_T$$
Hence $\phi \in p$ if and only if $\phi \in q$, so $p = q$.

- We use the compactness theorem for types. Let C be a collection of closed sets with the finite intersection property. Each closed $U \in C$ can be written as a possibly infinite intersection of a finite union of basis elements. A finite union of closed sets is still a basis element since $[\phi] \cup [\psi]_T = [\phi \vee \psi]_T$, so for each U there exists $\Gamma_U \subseteq F(\Sigma, V_n)$ such that
$$U = \bigcap_{\phi_U \in \Gamma_U} [\phi_U]_T$$
Sets of closed sets will correspond (roughly) to sets of formulas, and non-emptiness of the intersection of the closed sets corresponds to consistency of that set of formulas with T: an n-type p in the intersection of all the closed sets gives consistency of the set of formulas with respect to T. The right set of formulas (corresponding to C) to take is
$$\Gamma := \{\phi_U \in \Gamma_U \mid U \in C\} \quad \text{which is the same data as} \quad [\Gamma]_T := \{[\phi_U]_T \mid \phi_U \in \Gamma\}$$
The intersection any finite subset $[\Delta]_T \subseteq [\Gamma]_T$ (due to finite $\Delta \subseteq \Gamma$) is non-empty as it contains a finite intersection of elements in C.
$$\bigcap_{\phi_U \in \Delta} [\phi_U]_T \supseteq \bigcap_{\phi_U \in \Delta} U$$
Thus any finite subset $\Delta \subseteq \Gamma$ is consistent with T. Hence Γ is finitely consistent with T and so Γ is consistent with T. Extend Γ to a maximal n-type p. Then for any $U \in C$ we have
$$p \in \bigcap_{\phi_U \in \Gamma_U} [\phi_U] = U$$

□

EXAMPLE (A VISUALISATION OF THE STONE SPACE). *Suppose Σ is the signature of rings. Let K be an algebraically closed field, viewed as a Σ-structure. Let $T = \mathrm{ElDiag}(\Sigma, K)$, and $V_n = x_1 : A, \ldots, x_n : A$ (A is the only sort in the signature of rings). Then there is a continuous bijection from the Stone space $S_n(T)$ to $\mathrm{Spec}(K[x_1, \ldots, x_n])$ that is not an isomorphism. In other words the Stone space is a finer topology on the affine space $\mathrm{Spec}(K[x_1, \ldots, x_n]) \cong \mathbb{A}^n$. Notice that there cannot be a isomorphism since the Stone space is Hausdorff but this particular Spec cannot be Hausdorff.*

In this example of the Stone space, an example of a clopen set in the basis would be given by ($[\star]_T$ of) the formula

$$f(x_1, \ldots, x_n) = 0 \wedge g(x_1, \ldots, x_n) \neq 0$$

where f and g are polynomials in $K(x_1, \ldots, x_n)$ (noting that we are actually working in $\Sigma(K)$). In Spec, this looks like the intersection of a closed set $\{f = 0\}$ and an open set $\{g \neq 0\}$.

EXAMPLE. *This is a solution to exercise 4.5.2 in Marker's book [12]. Let Σ_s be a signature with just a single sort A and just a single function symbol $s : A \to A$, the 'successor function'. Consider the Σ_s-structure $\mathbb{Z}$ with $s^{\mathbb{Z}} : x \mapsto x + 1$. Consider $S_n(\mathrm{Th}_{\mathbb{Z}}(\Sigma_s))$, the Stone spaces of the theory of $\mathbb{Z}$ for each list of variables. Let $p \in S_n(\mathrm{Th}_{\mathbb{Z}})$*

If $n = 0$ then $p = \mathrm{Th}_{\mathbb{Z}}$ and $S_n(\mathrm{Th}_{\mathbb{Z}})$ is just a single point. This is true in general for any maximal theory T: the list of variables is empty, so p is a maximal Σ_s-theory consistent with T. Hence $T \subseteq p$, but T is maximal so they are equal.

Suppose p is 1-dimensional. Elements of the Stone space are types of elements in elementary extensions of $\mathbb{Z}$, so there exists $\mathbb{Z} \to \mathcal{N}$ and elementary extension and $a \in A^{\mathcal{N}}$ such that $p = \mathrm{tp}^{\mathcal{N}}(a)$. If a is in the image of $\mathbb{Z}$ then $p = \mathrm{tp}^{\mathcal{M}}(a)$; we claim this implies $p = \mathrm{tp}^{\mathcal{M}}(b)$ for any $b \in \mathbb{Z}$.

Proof. WLOG $a \in \mathbb{Z}$. We show that $+b - a : \mathbb{Z} \to \mathbb{Z}$ is a Σ_s-automorphism. Indeed it commutes with interpretation of $s^{\mathcal{M}}$ (it *is* $s^{\mathcal{M}}$ applied $b - a$ times) and has two sided inverse $-b + a$ which is a Σ_s-morphism for the same reason. Isomorphisms are elementary embeddings and a is sent to b, so $\phi \in \mathrm{tp}^{\mathcal{M}}(a)$ if and only if $\phi \in \mathrm{tp}^{\mathcal{M}}(b)$. □

If a is not in the image of $\mathbb{Z}$ then we need to determine what the rest of $\mathcal{N}$ could look like. Noting that $s^{\mathbb{Z}}$ is injective - which can be expressed as a Σ_s-sentence, we have that $s^{\mathcal{N}}$ is also injective since the embedding is elementary. Intuitively the only elementary extensions of $\mathbb{Z}$ can be disjoint unions of $\mathbb{Z}$ with other injective looking successor interpretations, for example $Z \sqcup \mathbb{N}$ with the successor on $\mathbb{N}$ interpreted as usual; moreover $\mathbb{Z}$ and $\mathbb{N}$ are the only (basic) Σ_s-structures on which we have an injective looking successor function.

We confirm and formalise this intuition by showing that the only other cases for p are when a behaves like some $n \in \mathbb{N}$, characterised by the fact that it has exactly $n - 1$ predecessors. Let $n \in \mathbb{N}$ and suppose

$$p \in [\forall y : A, s^{n+1}(y) \neq x] \cap [\exists y : A, s^n(y) = x]$$

Then we claim that p is isolated by the conjunction of these two formulas, i.e.

$$\{p\} = [\forall y : A, s^{n+1}(y) \neq x] \cap [\exists y : A, s^n(y) = x]$$

Proof. Suppose $q \in S_1(\mathrm{Th})$ is another point in the intersection, (by embedding into the disjoint union of $\mathcal{N}$ and the model realising q) WLOG $q = \mathrm{tp}^{\mathcal{N}}(b)$ for some $b \in A^{\mathcal{N}}$. Define the orbit of an element $x \in A^{\mathcal{N}}$ by

$$\mathrm{Orb}(x) = \left\{y \in A^{\mathcal{N}} \mid \exists k, l \in \mathbb{N}, \mathcal{N} \vDash_{\Sigma_s} s^k(x) = s^l(y)\right\}$$

First we show that the substructures $\mathrm{Orb}(a)$ and $\mathrm{Orb}(b)$ are isomorphic substructures of $\mathcal{N}$; in fact they are both isomorphic to $\mathbb{N}$ with $a \mapsto n \mapsto b$. By symmetry we just construct $\mathbb{N} \to \mathrm{Orb}(a)$. Let a_0 be the element of $\mathrm{Orb}(a)$ such that $\mathcal{N} \vDash_{\Sigma_s} s^n(a_0) = a$. Then it follows that

$$\mathrm{Orb}(a) = \mathrm{Orb}(a_0) = \left\{ s^k(a_0) \mid k \in \mathbb{N} \right\}$$

and we can define the map $\mathbb{N} \to \mathrm{Orb}(a)$ by $k \mapsto s^k(a_0)$, which is a Σ_s-isomorphism.

These isomorphisms are (possibly distinct) elementary embeddings $\mathbb{N} \to \mathcal{N}$. Hence $\mathrm{tp}^{\mathcal{N}}(a) = n^{\mathbb{N}} = \mathrm{tp}^{\mathcal{N}}(b)$. $\square$

Now it remains to consider the case when p is not in any of the above singleton sets. This says that for each $n \in \mathbb{N}$

$$p \in [\exists y : A, s^{n+1}(y) = x] \cup [\forall y : A, s^n(y) \neq x]$$

By induction, noting that $\forall y : A, s^0(y) \neq x \notin p$ for the base case (take y to be a), p is always in the left side of the union (and never in the right side). Hence

$$p \in \bigcap_{n \in \mathbb{N}} [\exists y : A, s^{n+1}(y) = x]$$

This brings us back to the first case, $p = \mathrm{tp}^{\mathcal{N}}(a) = \mathrm{tp}^{\mathbb{Z}}(0)$ (this does not imply a is in the image of $\mathbb{Z}$; we could have two disjoint copies of $\mathbb{Z}$ for example): mapping $\mathbb{Z}$ to $\mathrm{Orb}(a)$ by taking $n \mapsto s^n(a)$ (we can deal with the negatives using injectivity) we have an isomorphism, hence an elementary embedding (maybe not the original elementary embedding into $\mathcal{N}$) indentifying $\mathrm{tp}^{\mathcal{N}}(a)$ with $\mathrm{tp}^{\mathbb{Z}}(0)$.

1.2.2 Types on Structures and Omitting Types

Lemma – Finite realisation and embeddings [7]
Let $\mathcal{M}$ be a Σ-structure, $A \subseteq \mathcal{M}$ and $p \subseteq F(\Sigma(A), V_n)$ an n-dimensional theory. Then the following are equivalent

- p is consistent with $\mathrm{ElDiag}(\Sigma, \mathcal{M})$.
- There exists an elementary embedding $\mathcal{M} \to \mathcal{N}$ such that p is realised in $\mathcal{N}$.
- There exists an elementary embedding $\mathcal{M} \to \mathcal{N}$ such that p is finitely realised in $\mathcal{N}$.
- p is finitely realised in $\mathcal{M}$.

The elementary embeddings can be seen as both Σ-embeddings or $\Sigma(A)$-embeddings for any subset $A \subseteq \mathcal{M}$.

Proof. (1. $\Rightarrow$ 2.) There exists $\mathcal{N}$ and a $b \in \prod_{i=1}^n A_i^{\mathcal{N}}$ such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$ and $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} p(b)$. Then since models of the elementary diagram correspond to elementary extensions, there exists an elementary $\Sigma(\mathcal{M})$-embedding $\mathcal{M} \to \mathcal{N}$ (which can be moved down to being a $\Sigma(A)$-embedding for any subset $A \subseteq \mathcal{M}$.)

(2. $\Rightarrow$ 3.) Clear.

(3. $\Rightarrow$ 4.) Let $\Delta \subseteq p$ be finite. Then by assumption there exists an elementary Σ-embedding $\iota : \mathcal{M} \to \mathcal{N}$ and $b \in \prod_{i=1}^n A_i^{\mathcal{N}}$ such that $\mathcal{N} \vDash_\Sigma \Delta(b)$. Hence

$$\mathcal{N} \vDash_\Sigma \exists v : \prod_{i=1}^n A_i, \bigwedge_{\psi \in \Delta} \psi(v) \quad \Rightarrow \quad \mathcal{M} \vDash_\Sigma \exists v : \prod_{i=1}^n A_i, \bigwedge_{\psi \in \Delta} \psi(v)$$

by the embedding being elementary. Hence there exists $a \in \prod_{i=1}^n A_i^{\mathcal{M}}$ realising Δ.

(4. $\Rightarrow$ 1.) By compactness for types, it suffices to show that p is finitely consistent with the elementary

diagram. Let $\Delta \subseteq p$ be finite. Then by assumption there is $a \in \prod_{i=1}^n A_i^{\mathcal{M}}$ such that $\mathcal{M} \vDash_\Sigma \Delta(a)$ and so $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \Delta(a)$. Clearly $\mathcal{M}$ is a model of its elementary diagram. □

We note again that finitely realised does not imply realised in general.

Definition – Isolated types

Let $p \subseteq F(\Sigma, V_n)$ be an n-dimensional theory and $\phi \in F(\Sigma, V_n)$. Then ϕ isolates p with respect to Σ-theory T if

- ϕ is consistent relative to T and
- For each $\psi \in p$

$$T \vDash_\Sigma \forall v : \prod_{i=1}^n A, \phi \to \psi$$

We say p is isolated if there exists such a ϕ (some also say p is principal).

Proposition – Isolation topologically

If $p \in S_n(T)$ then p is isolated by ϕ if and only if $\{p\} = [\phi]_T$ - it is an 'isolated point'.

Proof. $(\Rightarrow)$ By maximality $\phi \in p$ and $p \in [\phi]_T$. Let $q \in [\phi]_T$. For any element $\psi \in p$ we have

$$T \vDash_\Sigma \forall v, \phi \to \psi$$

hence $q \in [\phi]_T \subseteq [\psi]_T$ and $\psi \in q$. Hence $p \subseteq q$ and by maximality $p = q$.

$(\Leftarrow)$ Let $\psi \in p$

$$[\phi \to \psi] = [\neg\, \phi] \cup [\psi] = (S_n(T) \setminus \{p\}) \cup [\psi] = S_n(T)$$

Hence $T \vDash_\Sigma \forall v, \phi \to \psi$. □

Proposition – Isolated types are realised in every model of a complete theory

Any finite n-type on a *complete* theory is realised in every model of the theory. Hence any n-type is finitely realised in every model of the theory. Hence an n-type isolated with respect to a *complete* theory is realised in every model of the theory.

Proof. By taking the conjunction of formulas it suffices to show that a single formula ϕ consistent with complete Σ-theory T is realised in any model $\mathcal{M}$ of T, or equivalently

$$T \vDash_\Sigma \exists v, \phi$$

By completeness of T either the above holds or

$$T \vDash_\Sigma \forall v, \neg\, \phi$$

By consistency of ϕ with T there exists a model which shows that the latter cannot hold. □

Remark. To demonstrate that completeness of T is required we take the example of the incomplete theory of rings as T, the 1-type to be $\{x^2 + 1 = 0\}$ and the model in question to be $\mathbb{Z}$. Clearly the 1-type is realised in $\mathbb{Z}[i]$ but not in $\mathbb{Z}$.

Definition – Substructure generated by a subset

Let $\mathcal{M}$ be a Σ-structure. Let $X \subseteq \mathcal{M}$. Then the following are equivalent definitions for a substructure $\langle A \rangle$ of $\mathcal{M}$ generated by X:

- The Σ-structure $\langle X \rangle$ defined inductively such that $X \subseteq \langle X \rangle$ and if $f : \prod A \to B \in \Sigma_{\text{fun}}$ and $a \in \prod A^{\langle X \rangle}$ then $f^{\mathcal{M}}(a) \in B^{\langle X \rangle}$.

> - $\bigcap\{\mathcal{N} \text{ substructure of } \mathcal{M} \mid X \subseteq \mathcal{N}\}$, where by the intersection we mean intersecting in each interpreted sort.
>
> We say a substructure is finitely generated if there exists a finite set X such that it is equal to $\langle X \rangle$.

Proof. We fix the first definition as $\langle X \rangle$ and call the second $\cap$ with each sort of the intersection as $\cap_A$. We show $\langle X \rangle$ forms a substructure of $\mathcal{M}$ containing X: By definition $f^{\langle X \rangle} := f^{\mathcal{M}}$ is well defined. Each relation $r \hookrightarrow \prod A$ is naturally interpreted as the pullback

$$r^{\mathcal{M}} \cap \prod A^{\langle X \rangle}$$

Hence $\langle X \rangle$ is a substructure of $\mathcal{M}$ containing X. Furthermore the second definition is a contained in $\langle X \rangle$.

For the other direction note that if $b \in B^{\langle X \rangle}$ was built via the first definition then $b \in X_B$ or $b = f^{\mathcal{M}}(a)$ for some tuple $a \in \prod A^{\langle X \rangle}$. In the first case it is trivially in any substructure containing X. In the second case any substructure $\mathcal{M}$ containing X is closed under $f^{\mathcal{N}} = f^{\mathcal{M}}$ and by induction $a \in \prod A^{\mathcal{N}}$ for any substructure $\mathcal{N}$. Hence $b = f^{\mathcal{M}}(a) \in \mathcal{N}$ for any substructure. Thus $\langle A \rangle \subseteq \bigcap \mathcal{N}$ and we are done. □

> **Lemma – Proofs are finite**
>
> Suppose T is a Σ-theory and ϕ a Σ-sentence such that $T \vDash_\Sigma \phi$. Then there exists a finite subset Δ of T such that $\Delta \vDash_\Sigma \phi$.

Proof. We show the contrapositive. Suppose for all finite subsets Δ of T, $\Delta \nvDash_\Sigma \phi$, then each $\Delta \cup \{\neg \phi\}$ is consistent and by compactness $T \cup \{\neg \phi\}$ is consistent. Hence $T \nvDash_\Sigma \phi$. □

The omitting types theorem is a kind-of converse to 'an n-type isolated with respect to a *complete* theory is realised in every model of the theory.'

> **Proposition – Omitting types**
>
> [8] Let Σ be a countable signature, T a countable Σ-theory and p a non-isolated n-type on T. Then there exists a countable Σ-model of T that omits p.

Proof. Write $V_n = \{x_1 : A_1, \ldots, x_n : A_n\}$. Let Σ_C be the signature with symbols from Σ plus a countably infinite set of constant symbols C_A for each sort A. We build a Σ_C-theory T_C and a countable model of T_C that omits p. Our T_C will satisfy the following three conditions

- $T \subseteq T_C$ is maximal and consistent
- (Witness) For any formula ϕ with only one free variable $v : A$ there exists $c \in C_A$ such that if the sentence $\exists v : A, \phi(v)$ is in the theory T_C then $\phi(c) \in T_C$.
- (Omitting) For any tuple $c \in C_{A_1} \times \cdots \times C_{A_n}$ there exists $\phi \in p$ such that $\neg \phi(c) \in T_C$.

Given such a T_C we build our omitting model $\mathcal{N}$. By consistency of T_C we have a model $\mathcal{M}$, from which we can take the collection X of interpreted constant symbols: for each A let

$$X_A := \{c^{\mathcal{M}} \in A^{\mathcal{M}} \mid c \in C_A\}$$

We show that $\mathcal{N} := \langle X \rangle$ the substructure it generates has sorts interpreted as $A^{\mathcal{N}} = X_A$. Suppose $f : \prod A \to B$ is a function symbol from Σ and $c \in \prod A$ represents an element of $\prod X_A$ then since T_C is maximal either $\exists v : B, f(c) = v$ or its negation is in T_C, but consistency of T_C implies the latter is false. Hence $\exists v : B, f(c) = v$ is in T_C and so by the witness property there exists $d \in C_B$ such that $f(c) = d$ is in T_C. In particular $f^{\mathcal{M}}(c^{\mathcal{M}}) \in \mathcal{N}$. We interpret relation symbols as the pullback and we have a Σ_C-embedding $\mathcal{N} \to \mathcal{M}$.

To show $\mathcal{N}$ is a Σ_C-model of T_C it suffices that the embedding is elementary, which we show by Tarski-Vaught: suppose ϕ is a Σ_C-sentence preserved by the inclusion. If $\exists v : A, \phi$ is a Σ_C-sentence satisfied by $\mathcal{M}$

then by maximality of T_C we have $\exists v : A, \phi \in T_C$ and so there exists $c \in C$ such that $\phi(c) \in T_C$. Since ϕ is preserved by the inclusion, $\mathcal{M} \vDash_{\Sigma_C} \phi(c)$ if and only if $\mathcal{N} \vDash_{\Sigma_C} \phi(c)$, concluding Tarski-Vaught. Furthermore, any tuple of elements in $\prod A_i^{\mathcal{N}}$ are represented by some $c \in \prod C_n$, hence by the last property of T_C we have some $\phi \in p$ such that $\mathcal{N} \nvDash_{\Sigma_C} \phi(c)$. So $\mathcal{N}$ is a countable (as Σ and each C_A is countable) Σ-model of T that omits p.

It remains to construct such a theory. Let $T_0 := T$, and we build Σ_C-theory T_{i+1} from Σ_C-theory T_i, guaranteeing that at each step *only finitely many formulas are added*. Enumerate the (countable) set of Σ_C-sentences by $\{\phi_i\}_{i\in\mathbb{N}}$; enumerate the (countable) set of tuples $C_1 \times \cdots \times C_n$ by $\{c_i\}_{i\in\mathbb{N}}$; enumerate the (countable) set of Σ-formulas ψ in a single variable $v : A$ such that $\exists v : A, \psi \in T$ by $\{\psi_i\}_{i\in\mathbb{N}}$.

Maximality: At least one of $T_i \cup \{\phi_i\}$ and $T_i \cup \{\neg\, \phi_i\}$ is a consistent Σ_C-theory (since T_i is consistent) so we take the consistent one to be T^1. If the added formula is equivalent to $\exists v : A, \psi(v)$ where ψ has exactly one free variable $v : A$ then we take $T^2 := T' \cup \{\psi(c)\}$ for some $c \in C_A$ that has not appeared yet in T^1, otherwise leaving $T^2 := T^1$.

Witness: We want the witness property for ψ_i. We pick a constant symbol $c \in C_A$ that has not appeared in T^2 and let $T^3 = T^2 \cup \{\psi_i(c)\}$. This is still consistent since T^2 has a model, which is also a model of T, hence c can be interpreted in a sensible way.

Omitting: We wish find $\delta \in p$ such that $T^3 \cup \{\neg\, \delta(c_i)\}$ is consistent, which (by chucking $\delta(c_i)$ in at each step) allows us to omit p once the induction is complete. Since T^3 is an extension of T by finitely many formulas we can take their conjunction to find Σ_C-sentence $\chi(d) \in \Sigma_C$ where χ is a Σ-formula and d represents all the constant symbols from C appearing in the conjunction. Assume for a contradiction for every $\delta \in p$ we have $T^3 \cup \{\neg\, \delta(c_i)\}$ is inconsistent. Then for each δ,

$$T^3 \vDash_{\Sigma_C} \delta(c_i) \quad \Rightarrow \quad T \vDash_{\Sigma_C} \chi(d) \to \delta(c_i)$$

By the lemma on constants this implies for every $\delta \in p$

$$T \vDash_{\Sigma} \forall x : \prod A_i, \forall v, \chi(x, v) \to \delta(x) \quad \Rightarrow \quad T \vDash_{\Sigma} \forall x : \prod A_i, (\exists v, \chi(x, v)) \to \delta(x)$$

where x are variables corresponding to c_i and v makes up to the rest appearing in d. This implies that p is isolated by $\exists v, \chi(x, v)$, which is a contradiction. Hence we can extend $T_{i+1} := T^3 \cup \{\neg\, \delta(c_i)\}$ for some $\delta \in p$. Again, if the added formula is equivalent to $\exists v : A, \psi(v)$ where ψ has exactly one free variable $v : A$ then we add in a formula to compensate for this.

Now we take $T_C := \bigcup_{i\in\mathbb{N}} T_i$. It is maximal and it is finitely consistent (each finite subset is in some T_i, which is consistent) hence consistent. The construction guaranteed T_C having the witness property and omitting p. □

1.3 Quantifier elimination and model completeness

Written whilst following section on algebraically closed fields.

1.3.1 Quantifier elimination

Definition – Quantifier elimination

Let T be a Σ-theory and ϕ a Σ-formula. We say the quantifiers of ϕ can be eliminated if there exists a quantifier free Σ-formula ψ that is equivalent to ϕ modulo T. We say ϕ is reduced to ψ.

We say T has quantifier elimination if the quantifiers of any Σ-formula can be eliminated.

Lemma – Deduction

Let T be a Σ-theory, Δ a finite Σ-theory and ψ a Σ-sentence. Then $T \cup \Delta \vDash_\Sigma \psi$ if and only if

$$T \vDash_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$$

Proof. We first case on if Δ is empty or not. If it is empty then $T \cup \Delta \vDash_\Sigma \psi$ if and only if $T \vDash_\Sigma \psi$ if and only if $T \vDash_\Sigma \top \to \psi$ if and only if

$$T \vDash_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$$

($\Rightarrow$) Suppose $\mathcal{M} \vDash_\Sigma T$ then we need to show $\mathcal{M} \vDash_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$. Indeed, suppose $\mathcal{M} \vDash_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right)$ then by induction $\mathcal{M} \vDash_\Sigma T \cup \Delta$ and so by assumption that $T \cup \Delta \vDash_\Sigma \psi$ we have $\mathcal{M} \vDash_\Sigma \psi$. Hence $\mathcal{M} \vDash_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$.

($\Leftarrow$) Suppose $\mathcal{M} \vDash_\Sigma T \cup \Delta$ then $\mathcal{M} \vDash_\Sigma T$ thus by assumption that $T \vDash_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$ we have $\mathcal{M} \vDash_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$. By induction $\mathcal{M} \vDash_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right)$ thus we have $\mathcal{M} \vDash_\Sigma \psi$. □

Lemma – Proofs are finite

Suppose T is a Σ-theory and ϕ a Σ-sentence such that $T \vDash_\Sigma \phi$. Then there exists a finite subset Δ of T such that $\Delta \vDash_\Sigma \phi$.

Proof. We show the contrapositive. Suppose for all finite subsets Δ of T, $\Delta \nvDash_\Sigma \phi$, then $\Delta \cup \{\phi\}$ is consistent and by compactness $T \cup \{\phi\}$ is consistent. Hence $T \nvDash_\Sigma \phi$. □

Proposition – Eliminating quantifiers of a formula

Let Σ be a signature such that $\Sigma_{\text{con}} \neq \varnothing$. Suppose T is a Σ-theory and ϕ is a Σ-formula with free-variables $v = (v_1, \dots, v_n)$. Then the quantifiers of ϕ can be eliminated if and only if the following holds: for any two Σ-models $\mathcal{M}, \mathcal{N}$ of T and any Σ-structure $\mathcal{A}$ that with Σ-embeddings into both $\mathcal{M}$ and $\mathcal{N}$ $(\iota_\mathcal{M}, \iota_\mathcal{N})$, if $a \in \mathcal{A}^n$ then

$$\mathcal{M} \vDash_\Sigma \phi(\iota_\mathcal{M}(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota_\mathcal{N}(a))$$

Proof. ($\Rightarrow$) Let $a \in \mathcal{A}^n$. By assumption there exists $\psi \in \Sigma_{\text{for}}$ such that $T \vDash_\Sigma \forall v, (\phi(v) \leftrightarrow \psi(v))$ Then $\mathcal{M} \vDash_\Sigma \phi(\iota_\mathcal{M}(a))$ if and only if $\mathcal{M} \vDash_\Sigma \psi(\iota_\mathcal{M}(a))$ if and only if $\mathcal{A} \vDash_\Sigma \psi(a)$, since embeddings preserve the satisfaction of quantifier free formulas. Similarly, this is if and only if $\mathcal{N} \vDash_\Sigma \phi(\iota_\mathcal{N}(a))$.

($\Leftarrow$) Let

$$\Gamma := \{\psi \text{ quantifier free } \Sigma_{\text{for}} \,|\, T \vDash_\Sigma \forall v, (\phi \to \psi)\}$$

and let $\Sigma(*)$ be such that $\Sigma(*)_{\text{con}} = \Sigma_{\text{con}} \cup \{d_1, \dots, d_n\}$ for some new constant symbols d_i (indexed according to the free-variables of ϕ). We claim that $T \cup \{\psi(d) \,|\, \psi \in \Gamma\} \vDash_{\Sigma(*)} \phi(d)$. We first look at how this would complete the proof. If it is true then as proofs are finite we have a finite subsets $\Delta \subseteq \Gamma$ such that $T \cup \{\psi(d) \,|\, \psi \in \Delta\} \vDash_{\Sigma(*)} \phi(d)$. By deduction we have

$$T \vDash_{\Sigma(*)} \left(\bigwedge_{\psi \in \Delta} \psi(d) \right) \to \phi(d)$$

and by the lemma on constants

$$T \vDash_\Sigma \forall v, \left(\bigwedge_{\psi \in \Delta} \psi(v)\right) \to \phi(v)$$

where $\left(\bigwedge_{\psi \in \Delta} \psi(v)\right)$ is quantifier free. By the definition of Δ we have the other implication as well:

$$T \vDash_\Sigma \forall v, \left(\bigwedge_{\psi \in \Delta} \psi(v)\right) \leftrightarrow \phi(v)$$

hence the result.

Suppose for a contradiction $T \cup \{\psi(d) \mid \psi \in \Gamma\} \nvDash_{\Sigma(*)} \phi(d)$. Then there exists a model $\mathcal{M}$ of $T \cup \{\psi(d) \mid \psi \in \Gamma\}$ such that $\mathcal{M} \nvDash_\Sigma \phi(d)$.

Suppose for a second contradiction that the $\Sigma(*)(\mathcal{M})$-theory $T \cup \mathrm{AtDiag}(\Sigma(*), \mathcal{M}) \cup \{\phi(d)\}$ is inconsistent. Then by compactness some subset $T \cup \Delta \cup \{\phi(d)\}$ is inconsistent, where $\Delta \subseteq \mathrm{AtDiag}(\Sigma(*), \mathcal{M})$ is finite. This implies $T \cup \Delta \vDash_{\Sigma(*)(\mathcal{M})} \neg\phi(d)$. Hence by deduction we have

$$T \vDash_{\Sigma(*)(\mathcal{M})} \left(\bigwedge_{\psi(d) \in \Delta} \psi(d)\right) \to \neg\phi(d)$$

By the lemma on constants applied to $\Sigma_{\mathrm{con}} \subseteq \Sigma(*)(\mathcal{M})_{\mathrm{con}}$

$$T \vDash_\Sigma \forall v, \left[\left(\bigwedge_{\psi(d) \in \Delta} \psi(v)\right) \to \neg\phi(v)\right]$$

Taking the contrapositive,

$$T \vDash_\Sigma \forall v, \left[\phi(v) \to \left(\bigvee_{\psi(d) \in \Delta} \neg\psi(v)\right)\right]$$

Hence $\bigvee_{\psi(d) \in \Delta} \neg\psi(v) \in \Gamma$ and so $\mathcal{M} \vDash_{\Sigma(*)} \bigvee_{\psi(d) \in \Delta} \neg\psi(v)$ by definition of $\mathcal{M}$. However each $\Delta \subseteq \mathrm{AtDiag}(\Sigma(*), \mathcal{M})$ and so $\mathcal{M} \vDash_{\Sigma(*)} \bigwedge_{\psi(d) \in \Delta} \psi(v)$, a contradiction. Thus there exists a model

$$\mathcal{N} \vDash_{\Sigma(*)(\mathcal{M})} T \cup \mathrm{AtDiag}(\Sigma(*), \mathcal{M}) \cup \{\phi(d)\}$$

Since $\mathcal{N} \vDash_{\Sigma(*)(\mathcal{M})} \mathrm{AtDiag}(\Sigma(*), \mathcal{M})$ there exists a $\Sigma(*)(\mathcal{M})$ morphism $\iota : \mathcal{M} \to \mathcal{N}$. Move this morphism down to Σ, then by assumption with $\mathcal{A} := \mathcal{M}$, for any sentence χ

$$\mathcal{M} \vDash_\Sigma \chi \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi$$

Since $\mathcal{N} \vDash_{\Sigma(*)(\mathcal{M})} \phi(d)$ by the lemma on constants $\mathcal{N} \vDash_\Sigma \forall v, \phi(v)$ and so $\mathcal{M} \vDash_\Sigma \forall v, \phi(v)$. Which is a contradiction because $\mathcal{M} \vDash_{\Sigma(*)} \neg\phi(d)$ and so by the lemma on constants $\mathcal{M} \vDash_\Sigma \forall v, \neg\phi(v)$. (We have $\mathcal{M} \vDash_\Sigma \phi(c^{\mathcal{M}}, \dots c^{\mathcal{M}})$ and $\mathcal{M} \nvDash_\Sigma \phi(c^{\mathcal{M}}, \dots c^{\mathcal{M}})$.) □

> **Lemma – Sufficient condition for quantifier elimination**
> Let T be a Σ-theory and suppose for any quantifier free Σ-formula ψ with at least one free variable w, the quantifier of $\forall w, \psi(w)$ can be eliminated. Then T has quantifier elimination.

Proof. Induct on what ϕ is.

- If ϕ is $\top$, an equality or a relation then it is already quantifier free.

- If ϕ is $\neg\chi$ and there exists a quantifier free Σ-formula ψ such that $T \vDash_\Sigma \forall v, \chi \leftrightarrow \psi$. Then $T \vDash_\Sigma \forall v, \neg\chi \leftrightarrow \neg\psi$. Hence ϕ can be reduced to $\neg\psi$ which is quantifier free.
- If ϕ is $\chi_0 \vee \chi_1$ and there exist respective reductions of these ψ_0 and ψ_1 then ϕ reduces to $\psi_0 \vee \psi_1$ which is quantifier free.
- If ϕ is $\forall w, \chi(w)$ and there exists quantifier free ψ such that

$$T \vDash_\Sigma \forall w, \bigforall_{v \in S} v, (\chi \leftrightarrow \psi)$$

where S indexes the rest of the free variables in χ and ψ. Then we can show that

$$T \vDash_\Sigma \bigforall_{v \in S} (\phi \leftrightarrow (\forall w, \psi))$$

By assumption there exists ω a quantifier free Σ-formula such that

$$T \vDash_\Sigma \bigforall_{v \in S} (\omega \leftrightarrow (\forall w, \psi))$$

Hence ϕ can be reduced to ω.

$\square$

Corollary – Improvement: Sufficient condition for quantifier elimination
If T be a Σ-theory if for any quanfier free Σ-formula ϕ with at least one free variable w (index the rest by S), for any $\mathcal{M}, \mathcal{N}$ Σ-models of T, for any Σ-structure $\mathcal{A}$ that embeds into $\mathcal{M}$ and $\mathcal{N}$ (via $\iota_\mathcal{M}, \iota_\mathcal{N}$) and any $a \in (\mathcal{A})^S$,

$$\mathcal{M} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{M}(a)) \Rightarrow \mathcal{N} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{N}(a))$$

then T has quantifier elimination.

Equivalently we can use the statement

$$\mathcal{M} \vDash_\Sigma \exists w, \phi(\iota_\mathcal{M}(a)) \Rightarrow \mathcal{N} \vDash_\Sigma \exists w, \phi(\iota_\mathcal{N}(a))$$

by negating ϕ.

Proof. To show that T has quantifier elimination it suffices to show that for any quanfier free Σ-formula ϕ with at least one free variable w (index the rest by S), the quantifiers of $\forall w, \phi$ can be eliminated. This is true if and only if for any $\mathcal{M}, \mathcal{N}$ Σ-models of T, for any Σ-structure $\mathcal{A}$ that injects into $\mathcal{M}$ and $\mathcal{N}$ (via $\iota_\mathcal{M}, \iota_\mathcal{N}$) and any $a \in (\mathcal{A})^S$,

$$\mathcal{M} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{M}(a)) \Rightarrow \mathcal{N} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{N}(a))$$

By symmetry of $\mathcal{M}$ and $\mathcal{N}$ we only require one implication. Hence the proposition. $\square$

Remark. For quantifier elimination it also suffices to show that for any Σ-models $\mathcal{M}$ of T, for any Σ-structure $\mathcal{A}$ that embeds into $\mathcal{M}$ (via $\iota_\mathcal{M}$) and any $a \in (\mathcal{A})^S$,

$$\mathcal{M} \vDash_\Sigma \forall w, \phi(\iota(a)) \Rightarrow \mathcal{A} \vDash_\Sigma \forall w, \phi(a)$$

since embeddings preserve satisfaction of universal formulas downwards. Equivalently we can use

$$\mathcal{A} \vDash_\Sigma \exists w, \phi(a) \Rightarrow \mathcal{M} \vDash_\Sigma \exists w, \phi(\iota(a))$$

1.3.2 Back and Forth

'Back and forth' is a technique used to determine elementary equivalence of models, quantifier elimination of theories and completeness of theories. This section draws together work from Poizat [14], OLP [2], and Pillay [13]. It is motivated by the example at the end, which should be looked at first.

Proposition – Image of generators are generators of the image

The image of a substructure generated by a subset is a substructure generated by the image of a set. In particular, a finitely generated substructure has finitely generated image under a Σ-morphism given by the image of the generators.

Proof. Let $\iota : \langle A \rangle \to \mathcal{N}$ be a Σ-morphism. We show that $\langle \iota(A) \rangle = \iota(\langle A \rangle)$. If $b \in \langle \iota(A) \rangle$ then $b = c^{\mathcal{N}}$ or $b = f^{\mathcal{N}}(\iota(\alpha))$ for $al \in \langle A \rangle$. Hence $b = c^{\mathcal{N}} = \iota(c^{\mathcal{M}}) \in \iota(\langle A \rangle)$ or

$$b = f^{\mathcal{N}}(\iota(\alpha)) = \iota(f^{\mathcal{M}}(\alpha)) \in \iota(\langle A \rangle)$$

Thus $\langle \iota(A) \rangle \subseteq \iota(\langle A \rangle)$. The other direction is similar. □

Definition – Partial isomorphisms

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. A partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$ is a Σ-isomorphism p with finitely generated domain in of $\mathcal{M}$ and codomain in $\mathcal{N}$.

Proposition – Equivalent definition of partial isomorphism

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. Let $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$. The following are equivalent:

- There exists a partial isomorphism $p : \langle a \rangle \to \langle b \rangle$ such that $p(a) = b$.
- $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$

Proof. ($\Rightarrow$) We induct on terms to show that $t^{\mathcal{M}}(a) = t^{\langle a \rangle}(a)$ for each term t:

- If t is a constant symbol or a variable then by definition of the the substructure interpretation they are equal.
- If t is $f(s)$ and we have the inductive hypothesis $s^{\mathcal{M}}(a) = s^{\langle a \rangle}(a)$ then by definition of the substructure interpretation
$$t^{\mathcal{M}}(a) = f^{\mathcal{M}}(s^{\mathcal{M}}(a)) = f^{\mathcal{M}}(s^{\langle a \rangle}(a)) = f^{\langle a \rangle}(s^{\langle a \rangle}(a)) = t^{\langle a \rangle}(a)$$

Let ϕ be a quantifier free Σ-formula with up to n variables. We show by induction on ϕ that

$$\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow \langle a \rangle \vDash_{\Sigma} \phi(a)$$

- If ϕ is $\top$ it is trivial.
- If ϕ is $t = s$ then it is clear that
$$t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a) \Leftrightarrow t^{\langle a \rangle}(a) = s^{\langle a \rangle}(a)$$
by what we showed for terms.
- If ϕ is $r(t)$ then
$$(a_{i_1}, \ldots, a_{i_m}) \in r^{\mathcal{M}} \Leftrightarrow (a_{i_1}, \ldots, a_{i_m}) \in r^{\mathcal{M}} \cap \langle a \rangle = r^{\langle a \rangle}$$
- If ϕ is $\neg \psi$ or $\psi \vee \chi$ then it is clear by induction.

As p is an Σ-isomorphism, for any quantifier free Σ-formula with up to n variables,

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \langle a\rangle \vDash_\Sigma \phi(a) \Leftrightarrow \langle b\rangle \vDash_\Sigma \phi(b)\mathcal{N} \vDash_\Sigma \phi(b)$$

($\Leftarrow$) Suppose $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$. We define $p : \langle a\rangle \to \mathcal{N}$ by the following: if $\alpha \in \langle a\rangle$ then one can write α as a term t evaluated at a: $\alpha = t^{\mathcal{M}}(a)$; p maps a to $t^{\mathcal{N}}(b)$. To show that p is well-defined, note that if two terms t and s are such that $t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)$ then $t = s$ is a formula in $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$ and so $t^{\mathcal{N}}(b) = s^{\mathcal{N}}(b)$. it is injective because if two terms t and s are such that $t^{\mathcal{N}}(b) = s^{\mathcal{N}}(b)$ then $t = s$ is a formula in $\mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b) = \mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a)$ and so $t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)$.

By definition p commutes with the interpretation of constant symbols, function symbols, and relations. Furthermore, for each i, $p(a_i) = b_i$ by taking the term to be a variable and evaluating at a_i. The image of p is $\langle b\rangle$ as the image of a is b. Hence it is a partial isomorphism $\langle a\rangle \to \langle b\rangle$ such that $p(a) = b$. ☐

Proposition – Basic facts about partial isomorphisms

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures.

- The inverse of a partial isomorphism is a partial isomorphism.
- The restriction of a partial isomorphism is a partial isomorphism.
- The composition of partial isomorphisms is a partial isomorphism.

Definition – Partially isomorphic structures

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. A partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$ is said to have the back and forth property if

- (Forth) For each $a \in \mathcal{M}$ there exists a partial isomorphism q such that q extends p and $a \in \operatorname{dom} p$.
- (Back) For each $p \in I$ there exists a partial isomorphism q such that q extends p and $b \in \operatorname{codom} q$.

We say $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent when all partial isomorphisms from $\mathcal{M}$ to $\mathcal{N}$ have the back and forth property.

Proposition – Equivalent definition of back and forth property

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. Let $p : \langle a\rangle \to \langle b\rangle$ for $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ be a partial isomorphism such that $p(a) = b$. It has the back and forth property if and only if the two conditions hold

- (Forth) For any $\alpha \in \mathcal{M}$, there exists $\beta \in \mathcal{N}$ such that $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a,\alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b,\beta)$
- (Back) For any $\beta \in \mathcal{N}$, there exists $\alpha \in \mathcal{M}$ such that $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a,\alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b,\beta)$

Proof. ($\Rightarrow$) Suppose p has the back and forth property. We only show 'forth' as the 'back' case is similar. Let $\alpha \in \mathcal{M}$. By 'forth' there exists q a partial isomorphism extending p such that $\alpha \in \mathrm{dom}(q)$. By restriction and the fact that the image of generators generates the image, there exists $\beta \in \mathcal{N}$ such that

$$q\big|_{\langle a,\alpha\rangle \to \langle b,\beta\rangle}$$

is a local isomorphism. Using the the equivalent definition we obtain $\mathrm{qftp}(a,\alpha) = \mathrm{qftp}(b,\beta)$.

($\Leftarrow$) We show that p has the 'forth' property. Let $\alpha \in \mathcal{M}$. By assumption there exists $\beta \in \mathcal{M}$ such that

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a,\alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b,\beta)$$

Thus there exists $q : \langle a,\alpha\rangle \to \langle b,\beta\rangle$ such that $q(a) = b$ and $q(\alpha) = \beta$. Hence p is extended by q with α in its domain. ☐

Proposition – Quantifier elimination for types

Let T be a Σ-theory. T has quantifier elimination if and only if for any $n \in \mathbb{N}$, any two Σ-models of T and any $a \in \mathcal{M}^n, b \in \mathcal{N}^n$, if

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

then

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

Proof. ($\Rightarrow$) Let $\phi \in \mathrm{tp}(a)$. By quantifier elimination there exists quantifier free ψ such that they are equivalent modulo T. Then $\mathcal{M} \vDash_\Sigma \psi(a)$ and $\psi \in \mathrm{qftp}(a) = \mathrm{qftp}(b)$. Thus $\mathcal{N} \vDash_\Sigma \psi(b)$ and by equivalence modulo T $\mathcal{N} \vDash_\Sigma \phi(b)$. Hence $\phi \in \mathrm{tp}(b)$. The other inclusion is similar.

($\Leftarrow$) Let $n \in \mathbb{N}$. Define a map $f : S_n(T) \to S^{\mathrm{qf}}_n(T)$ that takes a maximal n-type p to $p \cap QFF(\Sigma, n)$. It is well-defined as the image is indeed a maximal n-type. It is a surjection as any quantifier free maximal n type is an n-type and therefore can be extended to a maximal n-type. To show injectivity we note that any two elements of $S_n(T)$ can be written as types of elements $\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a)$ and $\mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$. If their images are equal then

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

thus by assumption they are equal.

To show that f is continuous we show that elements of the clopen basis have clopen preimage. Let $[\phi]^{\mathrm{qf}}_T$ be in the clopen basis of $S^{\mathrm{qf}}_n(T)$. Then $p \in [\phi]_T$ if and only if $\phi \in p$ if and only if $\phi \in f(p)$ if and only if $f(p) \in [\phi]^{\mathrm{qf}}_T$. Hence the preimage is $[\phi]_T$ which is clopen.

A continuous bijection between Hausdorff compact spaces is a homeomorphism. Hence for any $\phi \in F(\Sigma, n)$ the image of the clopen set generated by ϕ is clopen: there exists $\psi \in QFF(\Sigma, n)$ such that $[\phi]_T = f^{-1}[\psi]^{\mathrm{qf}}_T = [\psi]_T$. $[\phi]_T = [\psi]_T$ if and only if they are equivalent modulo T. Thus we can eliminate quantifiers for any $\phi \in F(\Sigma, n)$ for any n. Thus T has quantifier elimination. □

Lemma – Back and forth equivalence implies quantifier elimination for types

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. If $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent and $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ are such that

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

then

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

Proof. Let $\phi \in F(\Sigma, n)$. If ϕ is quantifier free then $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(b)$. By induction on formulas it suffices to show that if ϕ is the formula $\forall v, \psi$ and for any $\alpha \in \mathcal{M}$ there exists $\beta \in \mathcal{N}$ such that $\mathcal{M} \vDash_\Sigma \psi(a, \alpha) \Leftrightarrow \mathcal{N} \vDash_\Sigma \psi(b, \beta)$, then we have $\mathcal{M} \vDash_\Sigma \forall v, \psi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \forall v, \psi(b)$.

By the equivalent definition of partial isomorphisms, there exists $p : \langle a \rangle \to \langle b \rangle$ a partial isomorphism in p such that $p(a) = b$. Suppose $\mathcal{M} \vDash_\Sigma \forall v, \psi(a)$ and let $\beta \in \mathcal{N}$, then $\mathcal{M} \vDash_\Sigma \forall v, \psi(a, \alpha)$. By 'back' in the equivalent definition of the back and forth property there exists $\alpha \in \mathcal{M}$ such that $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a, \alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b, \beta)$ Hence $\mathcal{N} \vDash_\Sigma \forall v, \psi(a, \alpha)$. The other direction is similar. □

Corollary – Back and forth implies elementary equivalence

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. If $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent then they are elementarily equivalent.

Proof. Let ϕ be a quantifier free Σ-formula with 0 variables, i.e. a quantifier free sentence. As the empty set is a partial isomorphism. Thus by the equivalent definition of a partial isomorphism,

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing,0}(\varnothing) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing,0}(\varnothing)$$

By the fact that back and forth equivalence implies quantifier elimination for types,

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,0}(\varnothing) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,0}(\varnothing)$$

Thus for any Σ-sentence ϕ, $\mathcal{M} \vDash_\Sigma \phi$ if and only if $\phi \in \mathrm{tp}^{\mathcal{M}}_{\varnothing,0}(\varnothing) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,0}(\varnothing)$ if and only if $\mathcal{N} \vDash_\Sigma \phi$. □

Definition – ω-saturation
Let $\mathcal{M}$ be a Σ-structure. $\mathcal{M}$ is ω-saturated if for every finite subset $A \subseteq \mathcal{M}$, every $n \in \mathbb{N}$ and every $p \in S_n(\mathrm{Th}_{\mathcal{M}}(A))$, p is realised in $\mathcal{M}$.

See the general version κ-saturated here.

Proposition – ∞-equivalence
Let $\mathcal{M}$ and $\mathcal{N}$ be ω-saturated Σ-structures. If $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ satisfy

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,n}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,n}(b)$$

then

- (Forth) For any $\alpha \in \mathcal{M}$ there exists $\beta \in \mathcal{N}$ such that

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,n+1}(a,\alpha) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,n+1}(b,\beta)$$

- (Back) For any $\beta \in \mathcal{N}$ there exists $\alpha \in \mathcal{M}$ such that

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,n+1}(a,\alpha) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,n+1}(b,\beta)$$

If this property holds for any pair a, b related by a partial isomorphism we say $\mathcal{M}$ and $\mathcal{N}$ are ∞-equivalent.

Proof. Let $\alpha \in \mathcal{M}$ and consider

$$p(a,v) := \mathrm{tp}^{\mathcal{M}}_{a,1}(\alpha) \in S_1(\mathrm{Th}_{\mathcal{M}}(a))$$

Any formula in $p(a,v)$ can be written as a Σ-formula $\phi(w,v)$ with variables w replaced with elements of a (v represents a single variable to be replaced by α). Let

$$p(w,v) := \{\phi(w,v) \mid \phi(a,v) \in p(a,v)\}$$

We claim that

$$p(b,v) := \{\phi(b,v) \mid \phi \in p(w,v)\} \in S_1(\mathrm{Th}_{\mathcal{N}}(b))$$

To this end, we note that it is indeed a maximal subset of $F(\Sigma(b),1)$ since for any $\phi(b) \in F(\Sigma(b),1)$

$$\phi(a) \in p(a,v) \text{ or } \neg\phi(a) \in p(a) \Rightarrow \phi(b) \in p(b,v) \text{ or } \neg\phi(b) \in p(b)$$

We just need to show that it is consistent with $\mathrm{Th}_{\mathcal{N}}(b)$.

By compactness for types and noting that $\mathcal{N}$ is a $\Sigma(b)$-model of $\mathrm{Th}_{\mathcal{N}}(b)$, it suffices to show that for any finite subset $\Delta(w,v) \subseteq p(w,v)$ there exists $\beta \in \mathcal{N}^m$ such that $\mathcal{N} \vDash_{\Sigma(b)} \Delta(b,\beta)$.

$$\begin{aligned}
&\mathcal{M} \vDash_{\Sigma(a)} \bigwedge_{\phi\in\Delta} \phi(a,\alpha)\\
\Rightarrow&\mathcal{M} \vDash_{\Sigma} \exists v, \bigwedge_{\phi\in\Delta} \phi(a,v)\\
\Rightarrow&\left(\exists v, \bigwedge_{\phi\in\Delta} \phi(a,v)\right) \in \mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)\\
\Rightarrow&\mathcal{N} \vDash_{\Sigma} \exists v, \bigwedge_{\phi\in\Delta} \phi(b,v)\\
\Rightarrow&\exists\beta \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma} \bigwedge_{\phi\in\Delta} \phi(b,\beta)\\
\Rightarrow&\exists\beta \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma(b)} \Delta(b,\beta)
\end{aligned}$$

Thus $p(b,v) \in S_1(\mathrm{Th}_{\mathcal{N}}(b))$ and since $\mathcal{N}$ is ω-saturated $p(b,v)$ is realised in $\mathcal{N}$ by some β. Thus by maximality, $p(b,v) = \mathrm{tp}^{\mathcal{N}}_{b,1}(\beta)$.

Finally, for $\phi(v,w) \in F(\Sigma, n+1)$

$$\begin{aligned}
&\phi(v,w) \in \mathrm{tp}^{\mathcal{M}}_{\varnothing}(a,\alpha) \Leftrightarrow && \mathcal{M} \vDash_{\Sigma} \phi(a,\alpha) \Leftrightarrow \mathcal{M} \vDash_{\Sigma(a)} \phi(a,\alpha)\\
\Leftrightarrow&\phi(a,v) \in \mathrm{tp}^{\mathcal{M}}_{a,1}(\alpha) = p(a,v)\\
\Leftrightarrow&\phi(b,v) \in p(b,v) = \mathrm{tp}^{\mathcal{M}}_{b,1}(\beta)\\
\Leftrightarrow&\mathcal{N} \vDash_{\Sigma(b)} \phi(b,\beta) \Leftrightarrow \mathcal{N} \vDash_{\Sigma} \phi(b,\beta)\\
\Leftrightarrow&\phi(w,v) \in \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b,\beta)
\end{aligned}$$

□

Proposition – Back and forth method for showing quantifier elimination

Let T be a Σ-theory. If T has quantifier elimination then for any two ω-saturated Σ-models of T are back and forth equivalent.

If any two Σ-models of T are back and forth equivalent then T has quantifier elimination. [†]

[†] We could also phrase this as T has quantifier elimination if and only if any two ω-saturated Σ-models of T are back and forth equivalent, but the saturation requirement becomes redundant in one direction.

Proof. ($\Rightarrow$) Let p be a partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$. By the equivalent definition of partial isomorphisms there exists $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ such that $p(a) = b$ and

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

By quantifier elimination for types

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

The models are ω-saturated, hence by ∞-equivalence for any $\alpha \in \mathcal{M}$ there exists $\beta \in \mathcal{N}$ such that

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a,\alpha) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b,\beta)$$

Taking only the quantifier free elements, we obtain

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a,\alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b,\beta)$$

and by the equivalent definition of the back and forth property we have that p has the back and forth property.

($\Leftarrow$) Let $n \in \mathbb{N}$, $\mathcal{M}$ and $\mathcal{N}$ be models of T, $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$. By quantifier elimination for types it suffices to show that if

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

then

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

This is satisfied as any two models of T are back and forth equivalent. $\square$

Corollary – Back and forth condition for completeness
Let T be a Σ-theory. If any two models of T are back and forth equivalent then T is complete.

Proof. If any two models are back and forth equivalent then any two non-empty models are elementarily equivalent (the non-empty is redundant information). Hence T is complete. $\square$

We end this section with a nice example of all of this in action.

Example (Infinite infinite equivalence classes).

$$\Sigma_E := (\varnothing, \varnothing, n_f, \{E\}, m_r)$$

where $m_E = 2$ and n_f is the empty function, defines the signature of binary relations. We write for variables x and y, we write $x \sim y$ as notation for $E(x, y)$ The theory of equivalence relations ER *is set set containing the following formulas:*

Reflexivity - $\forall x, x \sim x$
Symmetry - $\forall x \forall y, x \sim y \to y \sim x$
Transitivity - $\forall x \forall y \forall z, (x \sim y \wedge y \sim z) \to x \sim z$

For $n \in \mathbb{N}_{>1}$ define

$$\phi_n := \exists_{i=1}^{n} x_i, \bigwedge_{i<j} x_i \nsim x_j$$
$$\psi_n := \forall x, \exists_{i=1}^{n} x_i, \bigwedge_{i=1}^{n} (x \sim x_i) \wedge \bigwedge_{i<j} (x_i \neq x_j)$$

Show that the theory $T = \mathrm{ER} \cup \phi_n, \psi_{n1<i}$ has quantifier elimination and is complete. (You may wonder if it is indeed a theory and what nasty induction must be done to show that its formulas can be constructed.)

Proof. We first define the projection into the quotient: if $\mathcal{M} \vDash_{\Sigma_E} T$ and $a \in \mathcal{M}$ then

$$\pi_{\mathcal{M}}(a) := \{b \in \mathcal{M} \,|\, \mathcal{M} \vDash_{\Sigma_E} a \sim b\}$$

If $A \subseteq \mathcal{M}$ we write $\pi_{\mathcal{M}}(A)$ to be the image

$$\{\pi_{\mathcal{M}}(a) \,|\, \exists a \in A\}$$

Note that the quotient is $\pi_{\mathcal{M}}(\mathcal{M})$.

Let $\mathcal{M}, \mathcal{N}$ be Σ_E-models of T and let p be a partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$. By the back and forth condition for quantifier elimination and the back and forth condition for completeness it suffices to show that p has the back and forth property.

We only show 'forth'. Let $\alpha \in \mathcal{M}$. Suppose $\pi_{\mathcal{M}}(\alpha) \cap \operatorname{dom} p$ is empty. We can show that $\pi_{\mathcal{N}}(\mathcal{N})$ is infinite whilst $\pi_{\mathcal{N}}(\operatorname{codom} p)$ is finite, hence there exists $\beta \in \mathcal{N}$ such that $\pi(\beta) \in \pi_{\mathcal{N}}(\mathcal{N}) \setminus \pi_{\mathcal{N}}(\operatorname{codom} p)$ is non-empty. Then define $q : \operatorname{dom} p \cup \{\alpha\} \to \operatorname{codom} p \cup \{\beta\}$ to agree with p on its domain and send α to β. Note that the domain and codomain of q are substructures as the language only contains a relation symbol (thus all subsets are substructures). We show that q is an isomorphism. It is clearly bijective, and to be an embedding it just needs to preserve interpretation of the relation. Let $a, b \in \operatorname{dom} q$, if both are in $\operatorname{dom} p$ then as p is a partial isomorphism

$$a \sim^{\mathcal{M}} b \Leftrightarrow p(a) \sim^{\mathcal{N}} p(b) \Leftrightarrow q(a) \sim^{\mathcal{N}} q(b)$$

Otherwise WLOG $a = \alpha$. If $b = \alpha$ then it is clear. If $b \in \operatorname{dom} p$ then by assumption $b \notin \pi_{\mathcal{M}}(\alpha) = \pi(a)$ hence $\neg\, a \sim^{\mathcal{M}} b$. By construction

$$q(a) = q(\alpha) = \beta \Rightarrow \pi_{\mathcal{N}}(q(a)) \notin \pi_{\mathcal{N}}(\operatorname{codom} p) \quad \text{and} \quad q(b) = p(b) \in \operatorname{codom} p$$

hence $\neg\, q(a) \sim^{\mathcal{N}} q(b)$. Thus q is a local isomorphism extending p.

Suppose $\pi_{\mathcal{M}}(\alpha) \cap \operatorname{dom} p$ is non-empty, i.e. there exists $a \in \operatorname{dom} p$ such that $\alpha \sim^{\mathcal{M}} a$ We can show that $\pi_{\mathcal{N}}(p(a))$ is infinite and $\operatorname{codom} p$ is finite hence there exists $\beta \in \pi_{\mathcal{N}}(p(a)) \setminus \operatorname{codom} p$. Then define $q : \operatorname{dom} p \cup \{\alpha\} \to \operatorname{codom} p \cup \{\beta\}$ to agree with p on its domain and send α to β. Again p is clearly a bijection on substructures, and we show that the relation is preserved. Let $b, c \in \operatorname{dom} q$. If $b, c \in \operatorname{dom} p$ then it is clear as p is an isomorphism, it is also clear if $b, c = \alpha$. Otherwise WLOG $c = \alpha$ and $b \in \operatorname{dom} p$. Then $c = \alpha \sim^{\mathcal{M}} a$ and by construction of β

$$q(c) = q(\alpha) = \beta \sim^{\mathcal{N}} p(a)$$

Noting $a \sim^{\mathcal{M}} b$ if and only if $p(a) \sim^{\mathcal{N}} p(b)$ as p is a partial isomorphism thus $c \sim^{\mathcal{M}} a \sim^{\mathcal{M}} b$ if and only if $q(c) \sim^{\mathcal{N}} p(a) \sim^{\mathcal{N}} p(b) = q(b)$. Hence q is a local isomorphism extending p. Thus p has the 'forth' property (and similarly the 'back' property). $\square$

> **Proposition – Countable back and forth equivalent structures are isomorphic**
> Let $\mathcal{M}$ and $\mathcal{N}$ be countably infinite Σ-structures. If $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Proof. Write $\mathcal{M} = \{a_i\}_{i \in \mathbb{N}}$ and $\mathcal{N} = \{b_i\}_{i \in \mathbb{N}}$. Inductively define partial isomorphisms p_n for $n \in \mathbb{N}$:

- Take p_0 to be the empty function.
- If $n+1$ is odd then ensure $a_{n/2}$ is in the domain: by the 'forth' property of p there exists p_{n+1} extending p_n such that $a_{n/2} \in \operatorname{dom}(p_{n+1})$.
- If $n+1$ is even then ensure $b_{(n+1)/2}$ is in the codomain: by the 'back' property of p there exists p_{n+1} extending p_n such that $b_{(n-1)/2} \in \operatorname{codom}(p_{n+1})$.

We claim that p, the union of the partial isomorphisms p_n for each $n \in \mathbb{N}$, is an isomorphism. Note that it is well-defined and has image $\mathcal{N}$ as the p_i are nested and for any $a_i \in \mathcal{M}$ and $b_i \in \mathcal{N}$, $a_i \in \operatorname{dom}(p_{2i+1})$ and $b_i \in \operatorname{dom}(p_{2i+2})$. It is injective: if $a_i, a_j \in \mathcal{M}$ and $p(a_i) = p(a_j)$ then $p_{2i+2}(a_i) = p_{2i+2}(a_j)$ and so $a_i = a_j$ as p_{2i+2} is a partial isomorphism. One can show that it is an Σ-embedding. $\square$

1.3.3 Model completeness

> **Definition – Model Completeness**
> We say a Σ-theory T is model complete when given two Σ-models of T and a Σ-embedding $\iota : \mathcal{M} \to \mathcal{N}$, the embedding is elementary.

Remark. Any Σ-theory T with quantifier elimination is model complete. If ϕ is a Σ-formula and $a \in (\mathcal{M})^S$. Then given two Σ-models of T and a Σ-embedding $\iota : \mathcal{M} \to \mathcal{N}$ we can take ψ a quantifier free formula such that $T \vDash_\Sigma \forall v, \phi \leftrightarrow \psi$. Since embeddings preserve satisfaction of quantifier free formulas

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \vDash_\Sigma \psi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \psi(\iota(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

Thus the extension is elementary.

1.3.4 Quantifier free types

Quantifier free types become important when we start making back and forth constructions. All the results carry through, thus there isn't much to say until they are relevant. Thus this section is worth skipping for now.

Definition – $QFF(\Sigma, n)$ and formulas consistent with a theory
We define $QFF(\Sigma, n)$ to be the subset of $F(\Sigma, n)$ of quantifier free formulas.

For subsets $p \subseteq QFF(\Sigma, n) \subseteq F(\Sigma, n)$ the definitions of consistency carry through and we can apply compactness for types on these subsets too. The definition of maximality is the same but restricted to $QFF(\Sigma, n)$.

Let T be a Σ-theory. Any subset $p \subseteq QFF(\Sigma, n)$ that is consistent with T is called a quantifier free n-type on T. Note that any quantifier free n-type is an n-type.

Definition – Quantifier free Stone space of a theory
Let T be a Σ-theory. Let the stone space of T, $S_n^{\mathrm{qf}}(T)$ be the set of all maximal quantifier free n-types on T. We give the same topology: $U \subseteq S_n(T)$ is an element of the basis when there exists $\phi \in QFF(\Sigma, n)$ such that

$$U = [\phi]_T^{\mathrm{qf}} := \left\{p \in S_n^{\mathrm{qf}}(T) \mid \phi \in p\right\}$$

Proposition – Extending to maximal quantifier free n-types
Any quantifier free n-type on a theory can be extended to a maximal quantifier free n-type.

Proof. Any quantifier free n-type is an n-type, hence can be extended to a maximal n-type. The intersection of a maximal n-type with QFF is a maximal quantifier free n-type. This intersection extends the quantifier free n-type and we are done. □

Proposition – Properties of the Stone space
Let T be a Σ-theory, $\phi, \psi \in QFF(\Sigma, n)$. Elementary properties:

- $(\neg\, \phi) \in p$ if and only if $p \notin [\psi]_T^{\mathrm{qf}}$.
- $[\phi]_T = [\psi]_T^{\mathrm{qf}}$ if and only if ϕ and ψ are equivalent modulo T.
- $[\top]_T^{\mathrm{qf}} = S_n^{\mathrm{qf}}(T)$
- $[\neg\, \phi]_T^{\mathrm{qf}} = S_n(T) \setminus [\phi]_T^{\mathrm{qf}}$
- $[\phi \vee \psi]_T^{\mathrm{qf}} = [\phi]_T^{\mathrm{qf}} \cup [\psi]_T^{\mathrm{qf}}$
- $[\bot]_T^{\mathrm{qf}} = \varnothing$
- $[\phi \wedge \psi]_T^{\mathrm{qf}} = [\phi]_T^{\mathrm{qf}} \cap [\psi]_T^{\mathrm{qf}}$

Topological properties:

- Elements of the basis of $S_n^{\mathrm{qf}}(T)$ are clopen.
- $S_n^{\mathrm{qf}}(T)$ is Hausdorff.
- $S_n^{\mathrm{qf}}(T)$ is compact.

Proof. The proofs of these are exactly the same as before, some of them can even be bypassed by using the previous results. □

Definition – Quantifier free type of an element
Let $\mathcal{M}$ be a Σ-structure, $A \subseteq \mathcal{M}$ and $a \in \mathcal{M}^n$ Then

$$\mathrm{qftp}_{A,n}^{\mathcal{M}}(a) := \left\{\phi \in QFF(\Sigma(A), n) \,|\, \mathcal{M} \vDash_{\Sigma(A)} \phi(a)\right\}$$

is the quantifier free type of a in $\mathcal{M}$ over A. One can verify that this is a maximal quantifier free n-type on T if $\mathcal{M}$ is a model of T. We will often drop parts of the subscripts and superscripts when it is clear. In fact the n can be deduced by the length of a and serves only to explicitely spell things out.

1.4 Examples

Here is a compilation of some examples of signatures and theories.

Definition – Signature and theory of groups
Let the following be the signature of groups:

- There is only one sort symbol G which stands for the group.
- Function symbols: the constant symbol for the identity $0 : G$, the symbol for addition $+ : G^2 \to G$ and the symbol for inverses $- : G \to G$.
- There are no relation symbols.

In the signature of groups we define the theory of (commutative) groups (we write $a + b$ to mean $+(a, b)$ and so on):

| Assosiativity: $\forall x, y, z : G, (x + y) + z = x + (y + z)$

| Identity: $\forall x : G, x + 0 = x$

| Inverse: $\forall x : G, x - x = 0$

| (For commutative only): $\forall x, y : G, x + y = y + x$

Note that we don't have axioms for 'closure of functions' and 'existence or uniqueness of inverses' as it is encoded by interpretation of $+, -, \times$ being well-defined.

Removing the function symbol $-$ and the sentence 'inverse for addition' we obtain the language and theory of (commutative) monoids.

Definition – Signature of monoid actions
Let the following be the signature of monoid actions:

- The sort symbols are: A the sort for the monoid and X the sort for the A-set.
- The functions symbols are $1 : A$ and $\times : A^2 \to A$ purely for the monoid together with $\rho : A \times X \to X$ for the action of the monoid on the set.

- There are no relation symbols.

In this signature we define the theory of monoid actions (we write ax to mean $\rho(a, x)$ and so on):

- The theory of monoids with 1 as the identity and $\times$ as symbol for addition.
- Identity: $\forall x : X, 1x = x$
- Compatibility: $\forall a, b : A, \forall x : X, (ab)x = a(bx)$

Definition – Signature and theory of modules
Let the following be the signature of modules over a ring:

- The sort symbols are: A the sort for the ring and M the sort for the module.
- The function symbols for the ring are constants $0_A, 1_A : A$, addition and multiplication $+_A, \times_A : A^2 \to A$ and inverse $-_A : A \to A$. For the module we have constant $0_M : M$, addition $+_M : M^2 \to M$ and inverse $-_M : M \to M$. Lastly we have $\rho : A \times M \to M$ for the action of the ring on the module.
- There are no relation symbols.

In the signature of modules we define the theory of modules over a ring as the set containing (we write am to mean $\rho(a, m)$ and so on):

- The theory of rings with the obvious substitutions.
- The theory of commutative groups where G is replaced with M, and the function symbols for the group are the function symbols for the module.
- The theory of monoid actions with the monoid as A under multiplication and the A-set as M.
- Distributivity of ρ over $+_A$: $\forall a, b : A, \forall x : M, (a + b)x = ax + bx$
- Distributivity of ρ over $+_M$: $\forall a : A, \forall x, y : M, a(x + y) = ax + ay$

Definition – Singature of a single binary relation
The signature $\Sigma_<$ of a single binary relation is given by

- A sort symbol P for the set.
- No function symbols.
- A relation symbol $< \hookrightarrow P^2$

For variables x and y of type P, we write $x < y$ as notation for $<(x, y)$.

Definition – Order theories
The $\Sigma_<$-theory of partial orders is given by the two formulas

- Non-reflexivity - $\forall x : P, \neg(x < x)$
- Transitivity - $\forall x, y, z : P, x < y \wedge y < z \to x < z$

The $\Sigma_<$-theory of linear orders is the theory of partial orders plus

- Linearity - $\forall x, y : P, x < y \vee x = y \vee y < x$

The $\Sigma_<$-theory of dense linear orders is the theory of linear orders plus

- Density - $\forall x, y : P, x < y \to \exists z : P, x < z \wedge z < y$.

Chapter 2

Appendix

2.1 Boolean Algebras, Ultrafilters and the Stone Space

2.1.1 Boolean Algebras

There is a very detailed wikipedia page [5] on Boolean algebras, which can be used as references for elementary proofs.

Definition – Partially ordered set

The signature of partially ordred sets Σ_{PO} consists of $(\varnothing, \varnothing, n_f, \{\leq\}, m_f)$, where $n_\leq = 2$. The theory of partially ordered sets PO consists of

| Reflexivity: $\forall x, x \leq x$ (this is just notation for $\leq(x, x)$)

| Antisymmetry: $\forall x \forall y, (x \leq y \wedge y \leq x) \to x = y$

| Transitivity: $\forall x \forall y \forall z, (x \leq y \wedge y \leq z) \to x \leq z$

Definition – Boolean algebra

The signature of Boolean algebras Σ_{BLN} consists of $(\{1, 0\}, \{\leq, \sqcap, \sqcup, -\}, n_f, \varnothing, m_f)$, where $n_\leq = 2$, $n_\sqcap = n_\sqcup = 2$ and $n_- = 1$. The theory of Boolean algebras BLN consists of the theory of partially ordered sets[†] PO together with the formulas

| Assosiativity of conjunction: $\forall x \forall y \forall z, (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$

| Assosiativity of disjunction: $\forall x \forall y \forall z, (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$

| Identity for conjunction: $\forall x, x \sqcap 1 = x$

| Identity for disjunction: $\forall x, x \sqcup 0 = x$

| Commutativity of conjunction: $\forall x \forall y, x \sqcap y = y \sqcap x$

| Commutativity of disjunction: $\forall x \forall y, x \sqcup y = y \sqcup x$

| Distributivity of conjunction: $\forall x \forall y \forall z, x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$

| Distributivity of disjunction: $\forall x \forall y \forall z, x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$

| Negation on conjunction: $\forall x, x \sqcap -x = 0$

| Negation on disjunction: $\forall x, x \sqcup -x = 1$

| Order on conjunction: $\forall x \forall y, (x \sqcap y) \leq x$

| Maximal property of conjunction: $\forall x \forall y \forall z, (x \leq y) \wedge (x \leq z) \rightarrow (x \leq y \sqcap z)$

| Order on disjunction: $\forall x \forall y, x \leq (x \sqcup y)$

| Minimal property of disjunction: $\forall x \forall y \forall z, (x \leq z) \wedge (y \leq z) \rightarrow (x \sqcup y \leq z)$

Often 'absorption' is also included, but it can be deduced from the other axioms. I have not used the usual logical symbols due to obvious clashes with our notation, and we will be using this in the context of sets anyway. If B is a Σ_{BLN}-model of BLN we call it a Boolean algebra.

[†]Often the ordering is defined afterwards as $a \leq b$ if and only if $a = a \sqcap b$.

Definition – Filters and ultrafilters on Boolean algebras
Let P be a partial order. A subset $\mathcal{F}$ of P is an filter on P if

- $\mathcal{F}$ is non-empty.
- Closure under 'superset': if $a \in \mathcal{F}$ then any $b \in P$ such that $a \leq b$ is also a member of $\mathcal{F}$.
- Closure under 'finite intersection': for any two members $a, b \in P$ there is a common smaller $c \in P$: $c \leq a$ and $c \leq b$. For Boolean algebras this is equivalent to the conjunction $a \sqcap b$ being in $\mathcal{F}$.

We say a filter on P is proper if it is not equal to P - for Boolean algebras this is equivalent to not containing 0. A proper filter $\mathcal{F}$ on P is an ultrafilter (maximal filter) when for any filter $\mathcal{G}$ on P such that $\mathcal{F} \subseteq \mathcal{G}$ we have $\mathcal{G} = \mathcal{F}$ or $\mathcal{G} = P$.

Lemma – Facts about Boolean algebras
Let B be a Boolean algebra, let $\mathcal{F}$ be an ultrafilter on B, let $a, b \in B$ and let $f : B \to C$ be a morphism.

- $a \sqcap a = a$ and $a \sqcup a = a$.
- $a \sqcup 1 = 1$ and $a \sqcap 0 = 0$.
- If $a \sqcap b = 0$ and $a \sqcup b = 1$ then $a = -b$ (negations are unique.)
- $-(a \sqcup b) = (-a) \sqcap (-b)$ and its dual. (De Morgan)
- ($a \in \mathcal{F}$ or $b \in \mathcal{F}$) if and only if $a \sqcup b \in \mathcal{F}$.
- Morphisms are order preserving.
- Morphisms commute with negation.
- $a \sqcap b = 1$ if and only if $a = 1$ and $b = 1$. Similarly for $\sqcup$ with 0.
- If $a \sqcap b = 0$ then $b \leq -a$.

Proposition – Equivalent definition of ultrafilter for Boolean algebras
Let B be a boolean algebra. Let $\mathcal{F}$ be a proper filter on B. The following are equivalent:

1. $\mathcal{F}$ is an ultrafilter over B.
2. If $a \sqcup b \in \mathcal{F}$ then $a \in \mathcal{F}$ or $b \in \mathcal{F}$. '$\mathcal{F}$ is prime'.
3. For any a of B, $a \in \mathcal{F}$ or $(-a) \in \mathcal{F}$.

For 2. the or is in fact 'exclusive or' since if both a and its negation were in $\mathcal{F}$ then $\mathcal{F}$ would not be proper.

Proof. (1. $\Rightarrow$ 2.) Suppose $a_0 \sqcup a_1 \in \mathcal{F}$.

$$\mathcal{G}_{a_0} = \{b \in B \mid \exists c \in \mathcal{F}, c \sqcap a_0 \leq b\}$$

Then clearly $\mathcal{G}_{a_0}$ is a filter containing $\mathcal{F}$ and a_0. Similarly have $\mathcal{G}_{a_1}$. Since $\mathcal{F}$ is an ultrafilter, we can case on whether $\mathcal{G}_{a_i}$ is $\mathcal{F}$ or B. If either is $\mathcal{F}$ then $a_i \in \mathcal{F}$ and we are done. If $\mathcal{G}_{a_0} = \mathcal{G}_{a_1} = B$ then both contain 0 and so there exist $c_i \in \mathcal{F}$ such that $c_i \sqcap a_i = 0$. Hence $c_i \leq -a_i$, and so $(-a_i) \in \mathcal{F}$. Thus $-(a_0 \sqcup a_1) = -a_0 \sqcap -a_1 \in \mathcal{F}$, and so $0 \in \mathcal{F}$, a contradiction.

($2. \Rightarrow 3.$) Let $a \in \mathcal{F}$. Then we have that $a \sqcup -a = 1$, which is by definition a member of $\mathcal{F}$. By assumption this implies that $a \in \mathcal{F}$ or $-a \in \mathcal{F}$.

($3. \Rightarrow 1.$) Let $\mathcal{G}$ be a proper filter such that $\mathcal{F} \subseteq \mathcal{G}$. It suffices to show that $\mathcal{G} = \mathcal{F}$. Then let $a \in \mathcal{G}$. Suppose $a \notin \mathcal{F}$. Then $-a \in \mathcal{F}$ and so $-a \in \mathcal{G}$. Thus $0 = (a \sqcap -a) \in \mathcal{G}$ thus $\mathcal{G}$ is not proper, a contradiction. Hence $\mathcal{G} = \mathcal{F}$. ☐

Proposition – Extending filters to ultrafilters on Boolean algebras

Any Boolean algebra can be extended to an ultrafilter.

Proof. Let $\mathcal{F}$ be a proper filter on B a Boolean algebra. The set of proper filters on B that contain $\mathcal{F}$ is a non-empty set ordered by inclusion. Any chain of proper filters is a proper filter so by Zorn we have a maximal element. ☐

Definition

The category of Boolean algebras consists of Boolean algebras as the objects and for any two Boolean algebras B, C morphisms $f : B \to C$ such that for any $a, b \in B$

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f(a \sqcup b) &= f(a) \sqcup f(b) \\ f(a \sqcap b) &= f(a) \sqcap f(b) \end{aligned}$$

Definition – Stone space

A topological space is 0-dimensional if it has a clopen basis.

The category of Stone spaces has 0-dimensional, compact and Hausdorff topological spaces as objects and continuous maps as morphisms.

Proposition – Contravariant functor from Boolean algebras to Stone spaces [11]

Given B a Boolean algebra, the following topological spaces are isomorphic:

- $S(B)$, the set of ultrafilters on B with a clopen basis given by collections of ultrafilters containing some element of B.
- $\mathrm{Mor}(B, 2)$, the set of Boolean algebra morphisms from $B \to 2$ with the subspace topology from 2^B, where 2 is given the discrete topology and $2^B = \prod_{b \in B} 2$ inherits a product topology.

This is a Stone space and there is a contravariant functor from the category of Boolean algebras to the category of Stone spaces taking each B to $S(B)$ and each Boolean algebra morphism $f : A \to B$ to a continuous map of Stone spaces:

$$S(f) := f^{-1}(\star) : S(B) \to S(A)$$

Note: In the second definition the functor would take f to the precomposition map $\star \circ f : \mathrm{Mor}(B, 2) \to \mathrm{Mor}(A, 2)$.

Proof. We first show that the spaces are isomorphic. In fact their isomorphism is the restriction of a larger isomorphism between 2^B and the power set of B. We take the bijection

$$2^B \to \mathcal{P}(B) := f \mapsto f^{-1}(1)$$

inducing a topology on the power set of B. We must show that $S(B)$ is the image of $\mathrm{Mor}(B, 2)$ under this isomorphism and that the subspace topology on $S(B)$ from $\mathcal{P}(B)$ is the topology defined by the basis given.

$S(B)$ **is the image of** $\mathrm{Mor}(B, 2)$: Let $f : B \to 2$ be a Boolean algebra morphism. We must show that $\mathcal{F} := f^{-1}(1)$ is an ultrafilter. Since $f(1) = 1$, $1 \in \mathcal{F}$. If $a, b \in \mathcal{F}$ then $f(a) = f(b) = 1$ so $f(a \sqcap b) = f(a) \sqcap f(b) = 1$ thus $\mathcal{F}$ is closed under intersection. If $a \leq b$ and $a \in \mathcal{F}$ then as f is order preserving $f(a) \leq f(b)$. It is a proper filter as $f(0) = 0 \neq 1$. To show it is an ultrafilter we use the equivalent definition: let $a \in \mathcal{P}(B)$. If $f(a) = 1$ then we are done, otherwise

$$f(-a) = -f(a) = -0 = 1 \Rightarrow -a \in \mathcal{F}$$

and we are done. To show that this is a surjection we use the inverse:

$$\mathcal{F} \mapsto \left(a \to \begin{cases} 1, a \in \mathcal{F} \\ 0, a \notin \mathcal{F} \end{cases} \right)$$

To show that this is a morphism we note that $\mathcal{F}$ is proper and contains 1 thus $f(0) = 0$ and $f(1) = 1$. Also

$$\begin{aligned} & f(a \sqcap b) = 1 \Leftrightarrow a \sqcap b \in \mathcal{F} \\ & \Leftrightarrow a \in \mathcal{F} \text{ and } b \in \mathcal{F} \quad \text{by closure under finite intersection and superset} \\ & \Leftrightarrow f(a) = 1 \text{ and } f(b) = 1 \\ & \Leftrightarrow f(a) \sqcap f(b) = 1 \quad \text{as proven before} \end{aligned}$$

Hence $f(a \sqcap b) = f(a) \sqcap f(b)$. Lastly since

$$-f(a) = 1 \Leftrightarrow f(a) = 0 \Leftrightarrow a \notin \mathcal{F} \Leftrightarrow -a \in \mathcal{F} \Leftrightarrow f(-a) = 1$$

thus by De Morgan and uniqueness of negations we have

$$f(a \sqcup b) = f(-(-a \sqcap -b)) = -[-f(a) \sqcap -f(b)] = (- - f(a)) \sqcup - - f(b) = f(a) \sqcup f(b)$$

Thus the inverse map gives back a Boolean algebra morphism and so the image of $\mathrm{Mor}(B, 2)$ is $S(B)$ under the isomorphism.

The topologies agree: It suffices that any open set in the basis of each can be written as a open set in the other. For each b we let π_b be the continuous projection/evaluation maps from $2^B \to 2$ mapping $f \mapsto f(b)$. $(\Rightarrow)$ Let $[b]$ be an element of the basis for $S(B)$ under the Stone topology. Then this is the image of the open subset $\pi_b^{-1}\{1\}$ under the isomorphism:

$$\pi_b^{-1}\{1\} = \left\{f \in 2^B \,|\, f(b) = 1\right\} \leftrightarrow \left\{f^{-1}(1) \,|\, f(b) = 1\right\} = \{\mathcal{F} \text{ ultrafilter} \,|\, b \in \mathcal{F}\}$$

$(\Leftarrow)$ Conversely, any open in $\mathrm{Mor}(B, 2)$ is of the form $\pi_b^{-1}X$ where $b \in B$ and $X \subseteq 2$. When X is empty or 2 then $\pi_b^{-1}X$ is empty or the whole space so it is open in $S(B)$. Otherwise if $X = \{1\}$ we have the case above again. The case $X = \{0\}$ is also covered by taking the clopen complement. Thus the topologies are the same under this isomorphism.

We now show that the equivalent topologies are Stone spaces. $S(B)$ is 0-dimensional: an element of the open basis is given by

$$[b] := \{\mathcal{F} \subseteq B \,|\, b \in \mathcal{F}\}$$

for some $b \in B$. The complement of each open in the basis $S(B) \setminus [b] = [-b]$ is again in the open basis. $S(B)$ is Hausdorff: if $\mathcal{F}, \mathcal{G} \in S(B)$ are not equal then without loss of generality there exists $b \in \mathcal{F}$ such that $b \notin \mathcal{G}$. Then the points $\mathcal{F}$ and $\mathcal{G}$ are separated $\mathcal{F} \in [b]$ and $\mathcal{G} \in [-b]$.

$\mathrm{Mor}(B, 2)$ is compact: note that $2 = \{0, 1\}$ with the discrete topology is compact and so $2^B = \prod_{a \in B} 2$ with the product topology is also compact by Tychonoff. Closed in compact is compact, so it remains to show that $\mathrm{Mor}(B, 2)$ is closed in 2^B:

$$\mathrm{Mor}(B, 2) = \{f \mid f(0) = 0\} \cap \{f \mid f(1) = 1\} \cap \{f \mid f \text{ commutes with } \sqcap\} \cap \{f \mid f \text{ commutes with } \sqcup\}$$

It suffices that these four sets are closed. The first is $\pi_0^{-1}(0)$ and the second $\pi_1^{-1}(1)$, which are both closed (and open). For the third

$$\{f \mid f \text{ commutes with } \sqcap\} = \bigcap_{a,b \in B} \{f \in 2^B \mid f(a \sqcap b) = f(a) \sqcap f(b)\}$$

So it suffices that each set in the intersection is closed:

$$\begin{aligned} &2^B \setminus \{f \mid f(a \sqcap b) = f(a) \sqcap f(b)\} \\ &\qquad = \{f \mid f(a \sqcap b) = 0 \text{ and } f(a) \sqcap f(b) = 1\} \cup \{f \mid f(a \sqcap b) = 1 \text{ and } f(a) \sqcap f(b) = 0\} \\ &\qquad = \left(\pi_{a \sqcap b}^{-1} \{0\} \cap \pi_a^{-1} \{1\} \cap \pi_b^{-1} \{1\}\right) \cup \left(\pi_{a \sqcap b}^{-1} \cap (\pi_a^{-1} \{0\}) \cup \pi_b^{-1} \{0\}\right) \end{aligned}$$

Each projection map's preimage is open so this whole complement is open. Hence $\{f \mid f \text{ commutes with } \sqcap\}$ is closed. Similarly for $\{f \mid f \text{ commutes with } \sqcup\}$ so we have shown that $S(B) \cong \mathrm{Mor}(B, 2)$ is compact.

To show that $S(\star)$ is a contravariant functor we need to check that the morphism map

$$S(f) := f^{-}1(\star) : S(B) \to S(A)$$

is a well-defined, respects the identity and composition. We show that $S(f)$ is continuous: it suffices that preimages of clopen elements are clopen. Let $[b] \subseteq S(A)$ be clopen.

$$\begin{aligned} &S(f)^{-1}[b] \\ =&\left\{\mathcal{F} \in S(B) \mid f^{-1}(\mathcal{F}) \in [b]\right\} \\ =&\{\mathcal{F} \in S(B) \mid f(b) \in \mathcal{F}\} \\ =&[f(b)] \end{aligned}$$

which is clopen. □

Proposition – Stone Duality

There is an equivalence between the category of Stone spaces and the category of Boolean algebras. Given by the functor $\mathcal{B}_\star$ sending any Stone space X to the set of its clopen subsets (this is a basis of X as it is 0-dimensional):

$$\text{Boolean algebras} \underset{\mathcal{B}_\star}{\overset{S(\star)}{\rightleftarrows}} \text{Stone spaces}$$

and its inverse $S(\star)$.

Proof. Let X be a 0-dimensional compact Hausdorff topological space. There is an obvious Boolean algebra to take on

$$\mathcal{B}_X := \{a \subseteq X \mid a \text{ is clopen}\}$$

which is interpreting 0 to be $\varnothing$, 1 to be X, $\leq$ as $\subseteq$, conjunction as intersection, disjunction as union and negation to be taking the complement in X. One can check that this is a Boolean algebra.

We make this a contravariant functor by taking any continuous map $f : X \to Y$ to an induced map $f^\diamond := f^{-1}(\star) : \mathcal{B}_Y \to \mathcal{B}_X$. One can check that this is a well-defined functor.

It remains to show the equivalence of categories by giving natural transformations $S(\mathcal{B}_\star) \to \mathbb{1}_\star$ in the category of topological spaces and $\mathbb{1}_\star \to \mathcal{B}_{S(\star)}$ in the category of Stone spaces. This is omitted. □

2.1.2 Isolated points of the Stone space

Definition – Isolated point
Let X be a topological space and $x \in X$. We say x is isolated if $\{x\}$ is open.

Definition – Derived set
Let X be a topological space. The derived set of X is defined as

$$X' := X \setminus \{x \in X \mid x \text{ isolated}\}$$

Exercise (Equivalent definition of derived set). *Let U be a subspace of X a topological space. $x \in U$ is not isolated (in the subspace topology) if and only if for any open set O_x containing x, $O_x \cap U \setminus \{x\}$ is non-empty. (it is a limit point of U.)*

Definition – Atom
Let B be a Boolean algebra. We say $a \in B$ is an atom if it is non-zero and for any $b \in B$ if $b \leq a$ then $b = a$ or $b = 0$.

Definition – Principle filter
Let B be a Boolean algebra and $a \in B$. Suppose a is non-zero. Then principle filter of a is defined as

$$a^{\uparrow} := \{b \in B \mid a \leq b\}$$

One should check that this is a proper filter.

Proposition
Let $a \in B$ a Boolean algebra. Then $a^{\uparrow}$ is an ultrafilter if and only if a is atomic.

Proof. ($\Rightarrow$) Suppose $b \leq a$. As $a^{\uparrow}$ is ultrafilter either $b \in a^{\uparrow}$ or $-b \in a^{\uparrow}$. In the first case $a \leq b$ hence $a = b$. If $-b \in a^{\uparrow}$ then $b \leq a \leq -b$ and so

$$1 = b \sqcup -b \leq -b \sqcup -b = -b$$

and so $1 = -b$ and $b = 0$.

($\Leftarrow$) Suppose a is an atom. We show that for any $b \in B$, it is in $a^{\uparrow}$ or its negation is in $a^{\uparrow}$ Since $1 \in a^{\uparrow}$, $b \sqcup -b \in a^{\uparrow}$ and so $a \leq b \sqcup -b$. Thus

$$a = a \sqcap (b \sqcup -b) = (a \sqcap b) \sqcup (a \sqcap -b)$$

Since a is non-zero, either $a \sqcap b$ or $a \sqcap -b$ is non-zero. If $a \sqcap b \neq 0$ then together with the fact that $a \sqcap b \leq a$ we conclude that $a \sqcap b = a$ as a is atomic. Hence $a \leq b$ and $b \in a^{\uparrow}$. Similarly the other case results in $-b \in a^{\uparrow}$. □

Proposition – Correspondence between atoms and isolated points
Let $a \in B$ a Boolean algebra. If a is an atom if and only if $[a]$ is a singleton in $S(B)$. Hence $a^{\uparrow}$ is an isolated point in $S(B)$.

Proof. ($\Rightarrow$) If a is an atom then $[a] = \{a^{\uparrow}\}$ is the only ultrafilter containing a is the principle filter of a: Let $\mathcal{F}$ be an ultrafilter containing a. Then for any $b \in \mathcal{F}$, $a \sqcap b \in \mathcal{F}$ and non-zero. Therefore $a \sqcap b = a$ as a is an atom and $a \leq b$. Hence $b \in a^{\uparrow}$. By maximality $\mathcal{F} = a^{\uparrow}$. Hence $[a]$ is a singleton.

($\Leftarrow$) Suppose a is not atomic. Then there exists $b \leq a$ such that $b \neq 0$ and $b \neq a$.

$$\begin{aligned} & b \sqcup (a \sqcap -b) = (a \sqcap b) \sqcup (a \sqcap -b) \\ =& a \sqcup (b \sqcap -b) = a \sqcup 0 = a \end{aligned}$$

There exist ultrafilters extending the principle filters of b and $a \sqcap -b$. These are not equal since b and $-b$ cannot be in the same proper filter. Both filters contain a. Hence $[a]$ is not a singleton. □

2.1.3 Ultraproducts and Łos's Theorem

This section introduces ultrafilters and ultraproducts and uses Łos's Theorem to prove the compactness theorem. Łos's theorem appears as an exercise in Tent and Ziegler's book [11].

Definition – Filters on sets

Let X be a set. The power set of X is a Boolean algebra with 0 interpreted as $\varnothing$, 1 interpreted as X and $\leq$ interpreted as $\subseteq$. We say $\mathcal{F}$ is a filter on X if it is a filter on the power set. The definition translates to:

- $X \in \mathcal{F}$.
- For any two members of $\mathcal{F}$ their intersection is in $\mathcal{F}$.
- If $a \in \mathcal{F}$ then any b in the power set of X such that $a \subseteq b$ is also a member of $\mathcal{F}$.

Translating definitions over we have that a filter on X is proper if and only if it does not contain the empty set, if and only if the filter is not equal to the power set. Furthermore a proper filter $\mathcal{F}$ is an ultrafilter if and only if for any filter $\mathcal{G}$, if $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{F} = \mathcal{G}$ or $\mathcal{G}$ is the power set of X.

Definition – Ultraproduct

Let $\mathcal{F}$ be an ultrafilter on X. We define a relation on $\prod_{x \in X} x$ by

$$(a_x)_{x \in X} \sim (b_x)_{x \in X} := \{x \in X \mid a_x = b_x\} \in \mathcal{F}$$

This is an equivalence relation as

- $(a_x)_{x \in X} \sim (a_x)_{x \in X} \Leftrightarrow \{x \in X \mid a_x = a_x\} = X \in \mathcal{F}$
- Symmetry is due to symmetry of $=$.
- If $\{x \in X \mid a_x = b_x\} \in \mathcal{F}$ and $\{x \in X \mid b_x = c_x\} \in \mathcal{F}$ then $\{x \in X \mid a_x = b_x = c_x\}$ is their intersection and so is in $\mathcal{F}$. Thus its superset $\{x \in X \mid a_x = c_x\}$ is in $\mathcal{F}$.

We define the ultraproduct of X over $\mathcal{F}$:

$$\prod X/\mathcal{F} := \prod_{x \in X} x/\sim$$

Proposition – Equivalent definition of ultrafilter (translated to the power set)

Let X be a set. Let $\mathcal{F}$ be a proper filter on X. $\mathcal{F}$ is an ultrafilter over X if and only if for every subset $U \subseteq X$, *either* $U \in \mathcal{F}$ or $X \setminus U \in \mathcal{F}$.

Proof. Follows immediately from the equivalent definition of an ultrafilter. □

Proposition – Łos's Theorem

Let $\mathfrak{M} \subseteq \mathbf{Str}(\Sigma)$ where Σ is a signature such that each carrier set is non-empty. Suppose $\mathcal{F}$ is an ultrafilter on $\mathfrak{M}$ (i.e. an ultrafilter on the Boolean algebra $P(\mathfrak{M})$). Then we want to make $\mathcal{N} := \prod \mathfrak{M}/\mathcal{F}$ into a Σ-structure. Let π be the natural surjection $\prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M} \to \prod \mathfrak{M}/\mathcal{F}$. If $a = (a_1, \dots, a_n) \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ then write $a_{\mathcal{M}} := ((a_1)_{\mathcal{M}}, \dots, (a_n)_{\mathcal{M}})$.

- Constant symbols $c \in \Sigma_{\text{con}}$ are interpreted as
$$c^{\mathcal{N}} := \pi(c^{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}}$$
- Any function symbol $f \in \Sigma_{\text{fun}}$ is interpreted as the function
$$f^{\mathcal{N}} : \left(\prod \mathfrak{M}/\mathcal{F}\right)^n \to \prod \mathfrak{M}/\mathcal{F} := \pi\left((a_{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}}\right) \mapsto \pi(f^{\mathcal{M}}(a_{\mathcal{M}}))_{\mathcal{M}\in\mathfrak{M}}$$
where $\pi\left((a_{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}}\right) = (\pi((a_i)_{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}})_{i=1}^n$.
- Any relation symbols $r \in \Sigma_{\text{rel}}$ is interpreted as the subset such that
$$\pi(a) \in r^{\mathcal{N}} \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, a_{\mathcal{M}} \in r^{\mathcal{M}}\} \in \mathcal{F}$$
where $a = a_1, \dots, a_m$ and $\pi(a) = (\pi(a_i))_{i=1}^m$.

Then for any Σ-formula ϕ with free variables indexed by S, If $a = (a_1, \dots, a_n) \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ then
$$\mathcal{N} \vDash_\Sigma \phi(\pi(a)) \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \phi(a_{\mathcal{M}})\} \in \mathcal{F}$$

Proof. We show that the interpretation of functions is well defined. Let $a, b \in \left(\prod_{\mathcal{M}\in\mathfrak{M}}\right)^{n_f}$. Suppose for each $i \in \{1, \dots, n_f\}$, $a_i \sim b_i \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$. Then

$$\begin{aligned}
&\text{for each } i, \{\mathcal{M} \in \mathfrak{M} \,|\, (a_i)_{\mathcal{M}} = (b_i)_{\mathcal{M}}\} \in \mathcal{F} \\
&\Rightarrow \left\{\mathcal{M} \in \mathfrak{M} \,|\, \bigwedge_{i=1}^{n} (a_i)_{\mathcal{M}} = (b_i)_{\mathcal{M}}\right\} \in \mathcal{F} \text{ by closure under finite conjunction} \\
&\Rightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, a_{\mathcal{M}} = b_{\mathcal{M}}\} \in \mathcal{F} \\
&\Rightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, f^{\mathcal{M}}(a_{\mathcal{M}}) = f^{\mathcal{M}}(b_{\mathcal{M}})\} \in \mathcal{F} \text{ by closure under superset} \\
&\Rightarrow \pi(f^{\mathcal{M}}(a_{\mathcal{M}})) = \pi(f^{\mathcal{M}}(b_{\mathcal{M}})) \text{ by definition of the quotient}
\end{aligned}$$

We use the following claim: If t is a term with variables S and $a \in (\prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M})^S$ then there exists $b \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ such that
$$t^{\mathcal{N}} \circ \pi(a) = \pi(b) \text{ and } \forall \mathcal{M} \in \mathfrak{M}, t^{\mathcal{M}}(a_{\mathcal{M}}) = b_{\mathcal{M}}$$
We prove this by induction on t:

- If t is a constant symbol c then pick $b := (c^{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}}$.
- If t is a variable then let $a \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ (only one varible), pick $b := a$.
- If t is a $f(s)$ then let $a \in (\prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M})^S$ by the inducition hypothesis there exists $c \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ such that
$$s^{\mathcal{N}} \circ \pi(a) = \pi(c) \text{ and } \forall \mathcal{M} \in \mathfrak{M}, s^{\mathcal{M}}(a_{\mathcal{M}}) = c_{\mathcal{M}}$$
Then we can take $b = (f^{\mathcal{M}}(c_{\mathcal{M}}))_{\mathcal{M}\in\mathfrak{M}}$. Thus
$$t^{\mathcal{N}} \circ \pi(a) = f^{\mathcal{N}}(s^{\mathcal{N}} \circ (\pi(a))) = f^{\mathcal{N}}(\pi(c)) = \pi(f^{\mathcal{M}}(c_{\mathcal{M}}))_{\mathcal{M}\in\mathfrak{M}} = \pi(b)$$
and for any $\mathcal{M} \in \mathfrak{M}$,
$$t^{\mathcal{M}}(a_{\mathcal{M}}) = f^{\mathcal{M}} \circ s^{\mathcal{M}}(a_{\mathcal{M}}) = f^{\mathcal{M}}(c_{\mathcal{M}}) = b_{\mathcal{M}}$$

We now induct on ϕ to show that for any appropriate a,

$$\mathcal{N} \vDash_\Sigma \phi(\pi(a)) \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \phi(a_\mathcal{M})\} \in \mathcal{F}$$

- The case where ϕ is $\top$ is trivial (noting that anything models $\top$ and $\mathfrak{M} \in \mathcal{F}$).
- If ϕ is $s = t$ then it suffices to show that

  $$s^\mathcal{N} \circ \pi(a) = t^\mathcal{N} \circ \pi(a) \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma t^\mathcal{M}(a_\mathcal{M}) = s^\mathcal{M}(a_\mathcal{M})\} \in \mathcal{F}$$

  ($\Rightarrow$) If for two terms s, t we have $s^\mathcal{N} \circ \pi(a) = t^\mathcal{N} \circ \pi(a)$ then by the claim there exists $b \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ such that

  $$\mathfrak{M} = \{\mathcal{M} \in \mathfrak{M} \,|\, s^\mathcal{M}(a_\mathcal{M}) = b_\mathcal{M} = t^\mathcal{M}(a_\mathcal{M})\} = \{\mathcal{M} \in \mathfrak{M} \,|\, t^\mathcal{M}(a_\mathcal{M}) = s^\mathcal{M}(a_\mathcal{M})\}$$

  which is therefore in the filter $\mathcal{F}$. ($\Leftarrow$) If for two terms s, t we have $s^\mathcal{N} \circ \pi(a) \neq t^\mathcal{N} \circ \pi(a)$ then by the claim there exist $b \neq c \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ such that

  $$\{\mathcal{M} \in \mathfrak{M} \,|\, t^\mathcal{M}(a_\mathcal{M}) = s^\mathcal{M}(a_\mathcal{M})\} = \{\mathcal{M} \in \mathfrak{M} \,|\, b_\mathcal{M} = c_\mathcal{M}\} = \varnothing$$

  which is not in the filter $\mathcal{F}$ as it is proper.
- If ϕ is $r(t)$ then by the claim we have $b \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ with the desired properties. It suffices to show that

  $$\pi(b) \in r^\mathcal{N} \Leftrightarrow \{\mathcal{M} \in \mathcal{M} \,|\, b_\mathcal{M} \in r^\mathcal{M}\} \in \mathcal{F}$$

  This follows from our definition of interpretation of relation symbols.
- If ϕ is $\neg\psi$ then $\mathcal{N} \vDash_\Sigma \phi(\pi(a))$ if and only if $\{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M})\} \notin \mathcal{F}$ by induction. This holds if and only if its complement is in the filter: $\{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \nvDash_\Sigma \psi(a_\mathcal{M})\} \in \mathcal{F}$ which is if and only if $\{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \phi(a_\mathcal{M})\} \notin \mathcal{F}$
- Without loss of generality we can use $\wedge$ instead of $\vee$ to make things simpler (replacing this comes down to dealing with a couple of $\neg$ statements). If ϕ is $\psi \wedge \chi$ then one direction follows filters being closed under intersection:

  $$\begin{aligned} &\mathcal{N} \vDash_\Sigma \phi(\pi(a)) \\ &\Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M})\} \in \mathcal{F} \text{ and } \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \chi(a_\mathcal{M})\} \in \mathcal{F} \\ &\Rightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M})\} \cap \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \chi(a_\mathcal{M})\} \in \mathcal{F} \\ &\Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M}) \wedge \chi(a_\mathcal{M})\} \in \mathcal{F} \end{aligned}$$

  To make second implication a double implication we note that each of the two sets

  $$\{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M})\} \in \mathcal{F} \text{ and } \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \chi(a_\mathcal{M})\} \in \mathcal{F}$$

  are supersets of the intersection which is in $\mathcal{F}$.
- Without loss of generality we can use $\exists$ instead of $\forall$ to make things simpler. ($\Rightarrow$) Suppose $\mathcal{N} \vDash_\Sigma \exists v, \psi(\pi(a), v)$. Then there exists $b \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ such that $\mathcal{N} \vDash_\Sigma \psi(\pi(a), \pi(b))$. Then by induction

  $$\{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M}, b_\mathcal{M})\} \in \mathcal{F}$$

  This is a subset of

  $$\{\mathcal{M} \in \mathfrak{M} \,|\, \exists c \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \psi(a_\mathcal{M}, c)\} = \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \exists v, \psi(a_\mathcal{M}, v)\}$$

  ($\Leftarrow$) Suppose $Y := \{\mathcal{M} \in \mathfrak{M} \,|\, \mathcal{M} \vDash_\Sigma \exists v, \psi(a_\mathcal{M}, v)\} \in \mathcal{F}$. Then by the axiom of choice we have for each $\mathcal{M}$

  $$\begin{cases} b_\mathcal{M} \in \mathcal{M}, \mathcal{M} \vDash_\Sigma & \text{if } \mathcal{M} \in Y \\ b_\mathcal{M} \in \mathcal{M} & \mathcal{M} \notin Y \end{cases}$$

  since each $\mathcal{M}$ is non-empty. By induction we have $\mathcal{N} \vDash_\Sigma \psi(\pi(a), \pi(b))$ and so $\mathcal{N} \vDash_\Sigma \exists v, \psi(\pi(a), v)$

$\square$

Corollary – The Compactness Theorem
A Σ-theory is consistent if and only if it is finitely consistent.

Proof. Suppose T is finitely consistent. For each finite subset $\Delta \subseteq T$ we let $\mathcal{M}_\Delta$ be the given non-empty model of Δ, which exists by finite consistency. We generate an ultrafilter $\mathcal{F}$ on $\mathfrak{M} := \{\mathcal{M}_\Delta \mid \Delta \in I\}$ and use Łos's Theorem to show that $\prod \mathfrak{M}/\mathcal{F}$ is a model of T. Let

$$I = \{\Delta \subseteq T \mid \Delta \text{ finite}\} \qquad \text{and } [\star] : I \to \mathcal{P}(I) := \Delta \mapsto \{\Gamma \in I \mid \Delta \subseteq \Gamma\}$$

Writing $[I]$ for the image of I, we claim that $\mathcal{F} := \{U \in \mathcal{P}(I) \mid \exists V \in [I], V \subseteq U\}$ forms an ultrafilter on I (i.e. an ultrafilter on the Boolean algebra $\mathcal{P}(I)$). Indeed

- $\varnothing \in I$ thus $I = \{[\varnothing] \in [I] \subseteq \mathcal{F}\}$.
- Suppose $\varnothing \in \mathcal{F}$ then $\varnothing \in [I]$ and so there exists $\Delta \in I$ such that $[\Delta] = \varnothing$. This is a contradiction as $\Delta \in [\Delta]$.
- If $U, V \in \mathcal{F}$ then there exist $\Delta_U, \Delta_V \in I$ such that $[\Delta_U] \subseteq U$ and $[\Delta_V] \subseteq V$.

$$\begin{aligned} [\Delta_U] \cap [\Delta_V] &= \{\Gamma \in I \mid \Delta_0 \subseteq \Gamma \text{ and } \Delta_1 \subseteq \Gamma\} \\ &= \{\Gamma \in I \mid \Delta_0 \cup \Delta_1 \subseteq \Gamma\} \\ &= [\Delta_0 \cup \Delta_1] \in [I] \subseteq \mathcal{F} \end{aligned}$$

- Closure under superset is clear.

We identify each $\mathcal{M}_\Delta \in \mathfrak{M}$ with $\Delta \in I$ and generate the same filter (which we will still call $\mathcal{F}$) on $\mathfrak{M}$ (this is okay as the power sets are isomorphic as Boolean algebras.) By Łos's Theorem $\prod \mathfrak{M}/\mathcal{F}$ is a well-defined Σ-structure such that for any Σ-sentence ϕ

$$\prod \mathfrak{M}/\mathcal{F} \vDash_\Sigma \phi \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_\Sigma \phi\} \in \mathcal{F}$$

Let $\phi \in T$, then $\{\Delta \in I \mid \{\phi\} \subseteq \Delta\} \in \mathcal{F}$ and so

$$\{\Delta \in I \mid \{\phi\} \subseteq \Delta\} \subseteq \{\Delta \in I \mid \phi \in \Delta\} \subseteq \{\Delta \in I \mid \mathcal{M} \vDash_\Sigma \phi\} \in \mathcal{F}$$

The image of this under the isomorphism is $\{\mathcal{M}_\Delta \in X \mid \mathcal{M} \vDash_\Sigma \phi\}$ thus is in $\mathcal{F}$ and so $\prod \mathfrak{M}/\mathcal{F} \vDash_\Sigma \phi$. □

2.1.4 Stone Duality

This section of the appendix gives a more algebraic way of constructing Stone spaces.

Definition – A Boolean algebra on $F(\Sigma, n)$
Let T be a Σ-theory. We quotient out $F(\Sigma, n)$ by the equivalence relation

$$\phi \sim \psi \quad := \quad \phi \text{ and } \psi \text{ equivalent modulo } T := T \vDash_\Sigma \forall v, (\phi \leftrightarrow \psi)$$

Call the projection into the quotient π and the quotient $F(\Sigma, n)/T$. We make $F(\Sigma, n)/T$ into a Boolean algebra by interpreting 0 as $\pi(\bot)$, 1 as $\pi(\top)$, $\pi(\phi) \sqcap \pi(\psi)$ as $\pi(\phi \wedge \psi)$, $\pi(\phi) \sqcup \pi(\psi)$ as $\pi(\phi \vee \psi)$, $-\pi(\phi)$ as $\pi(\neg \phi)$ and $\pi(\phi) \leq \pi(\psi)$ as

$$\{(\pi(\phi), \pi(\psi)) \mid T \vDash_\Sigma \forall v, (\phi \to \psi)\}$$

One can verify that these are well-defined and satisfy the axioms of a Boolean algebra. Notice we need T (potentially chosen to be the empty set) to make $\to$ look like $\leq$ and that we had to quotient modulo T to make $\leq$ satisfy antisymmetry. Antisymmetry in this context looks very much like 'propositional extensionality'. Thus it makes sense to consider the Stone space of this Boolean algebra $S(F(\Sigma, n)/T)$.

Lemma

If $p \subseteq F(\Sigma, n)$ is a maximal subset ($\forall \phi \in F(\Sigma, n), \phi \in p$ or $\neg\phi \in p$) then $\pi(\phi) \in \pi(p)$ in the quotient implies $\phi \in p$.

Proof. If $\pi(\phi) \in \pi(p)$ then there exists $\psi \in p$ such that ψ is equivalent to ϕ modulo T. By consistency with T there exists a non-empty Σ-model $\mathcal{M}$ of T and $b \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma p(b)$, in particular $\mathcal{M} \vDash_\Sigma \psi(b)$. Equivalence modulo T then gives us that $\mathcal{M} \vDash_\Sigma \phi(b)$. By maximality of p, ϕ or $\neg\phi$ is in p but the latter would lead to $\mathcal{M} \nvDash_\Sigma \phi(b)$, a contradiction. $\square$

Proposition – The Stone space is a set of ultrafilters

The Stone space of a Σ-theory T is homeomorphic to the set of ultrafilters from $S(F(\Sigma, n)/T)$ that have preimage consistent with T with the subspace topology. In other words, if π is the projection to the quotient then $S_n(T) \cong X$, where

$$X := \left\{\mathcal{F} \in S(F(\Sigma, n)/T) \,|\, \pi^{-1}(\mathcal{F}) \text{ consistent with } T\right\}$$

Proof. Warning: this proof uses π to be three different things, the quotient $F(\Sigma, n) \to F(\Sigma, n)/T$, the image map (of the quotient) $S_n(T) \to X$, and the map of clopen sets (the image map of the image map) $\mathcal{P}(S_n(T)) \to \mathcal{P}(X)$. The second will be a homeomorphism and the third will be a map between subsets of the topologies (in particular the clopen subsets).

We show that sending $p \in S_n(T)$ to its image under the projection to the quotient $\pi(p)$ is a homeomorphism. To show that it is well-defined it suffices to show that for any p, a maximal n-type over T, $\pi(p)$ is an ultrafilter of $F(\Sigma, n)/T$ with preimage consistent with T. Preimage being consistent with T follows from the definition of n-types over theories. To show that it is a proper filter:

- $\top \in p$ by consistency and maximality. Hence $\pi(\top) \in \pi(p)$.
- If $\pi(\bot) \in \pi(p)$ then $\bot \in p$ which is a contradiction with consistency.
- If $\pi(\phi), \pi(\psi) \in \pi(p)$ then $\phi, \psi \in p$ and so $\phi \wedge \psi \in p$ thus by definition of the Boolean algebra $F(\Sigma, n)/T$,
$$\pi(\phi) \sqcap \pi(\psi) = \pi(\phi \wedge \psi) \in \pi(p)$$
- If $\pi(\phi) \in \pi(p)$ and $\pi(\phi) \leq \pi(\psi)$ then $\phi \in p$ and by definition of $\leq$,
$$T \vDash_\Sigma \forall v, (\phi \to \psi)$$
Since p is consistent with T there exists a non-empty Σ-structure $\mathcal{M}$ and $b \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma \phi(b) \to \psi(b)$ and $\mathcal{M} \vDash_\Sigma \phi(b)$. Hence $\mathcal{M} \vDash_\Sigma \psi(b)$ and by maximality of p we have $\psi \in p$ and $\pi(\psi) \in \pi(p)$.

The image $\pi(p)$ is an ultrafilter by the equivalent definition: if $\pi(\phi) \in F(\Sigma, n)/T$ then either $\phi \in p$ or $\neg\phi \in p$ by maximality of p, hence $\pi(\phi) \in \pi(p)$ or $\pi(\neg\phi) \in \pi(p)$. Thus we have π is a map into

$$\left\{\mathcal{F} \in S(F(\Sigma, n)/T) \,|\, \pi^{-1}(\mathcal{F}) \text{ consistent with } T\right\}$$

Injectivity: if $p, q \in S_n(T)$ and $\pi(p) = \pi(q)$ then if $\phi \in p$, we have $\pi(\phi) \in \pi(p) = \pi(q)$ and by our claim above $\phi \in q$. Surjectivity: let $\mathcal{F} \in S(F(\Sigma, n)/T)$ have its preimage consistent with T. Then its preimage is an n-type. If its preimage is a maximal n-type then we have surjectivity. Indeed since $\mathcal{F}$ is an ultrafilter if $\phi \in F(\Sigma, n)$ then $\pi(\phi) \in \mathcal{F}$ or $\pi(\neg\phi) = -\pi(\phi) \in \mathcal{F}$, hence $\phi \in \pi^{-1}(\mathcal{F})$ or $\neg\phi \in \pi^{-1}(\mathcal{F})$.

To show that the map is continuous in both directions, it suffices to show that images of clopen sets are clopen and preimages of clopen sets are clopen, as each topology is generated by their clopen sets. For $\phi \in F(\Sigma, n)$ since $\pi : S_n(T) \to X$ is a bijection we have that

$$\pi([\phi]_T) = \left\{\mathcal{F} \in X \,|\, \phi \in \pi^{-1}(\mathcal{F})\right\} = \{\mathcal{F} \in F(\Sigma, n)/T \,|\, \pi(\phi) \in \mathcal{F} \text{ and } \mathcal{F} \in X\} = [\pi(\phi)] \sqcap X$$

and similarly $\pi^{-1}([\pi(\phi)] \sqcap X) = [\phi]_T$. Hence there is a correspondence between clopen sets. $\square$

Lemma – Topological consistency
Let $\mathcal{F} \in S(F(\Sigma, n)/T)$ and T be a Σ-theory. $\pi^{-1}(\mathcal{F})$ is consistent with T if and only if

$$\pi^{-1}(\mathcal{F}) \in \bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T$$

if and only if

$$\bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T \text{ is non empty}$$

Proof. (1. $\Rightarrow$ 2. $\Rightarrow$ 3.) Suppose $\pi^{-1}(\mathcal{F})$ is consistent with T. Then $\pi^{-1}(\mathcal{F}) \in S_n(T)$ thus for any $\phi \in \pi^{-1}(\mathcal{F})$, $\pi^{-1}(\mathcal{F}) \in [\phi]_T$. Hence

$$\pi^{-1}(\mathcal{F}) \in \bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T$$

and it is non-empty.

(3. $\Rightarrow$ 1.) Suppose

$$p \in \bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T$$

then $\forall \phi \in \pi^{-1}(\mathcal{F}), \phi \in p$. As $\mathcal{F}$ is an ultrafilter, for any $\phi \in p$,

$$\phi \notin \pi^{-1}(\mathcal{F}) \Rightarrow \neg\phi \in \pi^{-1}(\mathcal{F}) \Rightarrow \neg\phi \in p \text{ a contradiction}$$

Hence $p = \pi^{-1}(\mathcal{F})$. Hence $\pi^{-1}(\mathcal{F}) \in S_n(T)$ and thus is consistent with T. □

Proposition – Topological compactness implies compactness for types
Let $\mathcal{F} \in S(F(\Sigma, n)/T)$ and T be a Σ-theory. Then $\pi^{-1}(\mathcal{F})$ is consistent with T if and only if $\pi^{-1}(\mathcal{F})$ if finitely consistent with T.

Proof. By definition $\pi^{-1}(\mathcal{F})$ is finitely consistent with T if and only if any finite subset of $\pi^{-1}(\mathcal{F})$ is consistent with T. Translating this to the topology, this is if and only if for any finite subset $\Delta \subseteq \pi^{-1}(\mathcal{F})$,

$$\bigcap_{\phi \in \Delta} [\phi]_T \text{ is non empty}$$

By topological compactness of $S_n(T)$ this is if and only if

$$\bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T \text{ is non empty}$$

Translating this back to model theory this is if and only if $\pi^{-1}(\mathcal{F})$ is consistent with T. □

Chapter 3

Model Theory of Fields

3.1 Ax-Grothendieck

This section studies the theories of fields in the language of rings, with particular focus on algebraically closed fields.

3.1.1 Language of Rings

We introduce rings and fields and construct the field of fractions of integral domains.

> **Definition – Signature and theory of rings**
> Let the following be the signature of rings:
>
> - There is only one sort symbol A which stands for the ring.
> - The function symbols are the constant symbols $0, 1 : A$, the symbols for addition and multiplication $+, \times : A^2 \to A$ and taking for inverse $- : A \to A$.
> - There are no relation symbols.
>
> We define the theory of rings RNG as the set containing (we write $a \times b$ or ab to mean $\times(a, b)$ and so on):
>
> | The theory of commutative groups with function symbols $0, +, -$.
>
> | The theory of commutative monoids with function symbols $1, \times$ (assosiativity, identity and commutativity for $\times$).
>
> | Distributivity: $\forall x, y, z : A, x \times (y + z) = x \times y + x \times z$
>
> Note that the theory of groups and rings are universal. We often omit the sort A since there is only one sort. We adopt the usual convention for short hand, e.g. $2 := 1 + 1$ is a term of type A and $xy := x \times y$ is a term of type A when x and y are variables.

> **Definition – Theory of integral domains and fields**
> We define the Σ_{RNG}-theory of integral domains
>
> $$\mathrm{ID} := \mathrm{RNG} \cup \{0 \neq 1, \forall x \forall y, xy = 0 \to (x = 0 \vee y = 0)\}$$

and the Σ_{RNG}-theory of fields

$$\mathrm{FLD} := \mathrm{RNG} \cup \{\forall x, x = 0 \vee \exists y, xy = 1, 0 \neq 1\}$$

Note that the theory of integral domains is universal but the theory of fields is not.

Going through the details of the construction of a field of fractions written from the model theoretic perspective is a good exercise.

Definition – Field of fractions

Suppose $A \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ and with a Σ-embedding into a field $A \to K \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$. If for every $L \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$ and Σ_{RNG}-embedding $\iota_L : A \to L$, there exists a unique Σ_{RNG}-embedding $K \to L$ that commutes with the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & K \arrow[d, dashed] \\ & L \end{tikzcd}$$

then K has the *universal property* of the field of fractions of A. Note that this definition is entirely without mention of interpretation in '$\mathbf{Set}^*$', not mentioning a set theoretic construction of the field K.

We verify the standard construction of the field of fractions has the universal property.

Proof. We construct $X = \{(x,y) \in A^2 \mid y \neq 0\}$ and an equivalence relation $(x,y) \sim (v,w) \Leftrightarrow xw = yv$. (Use $A \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ to show that this is an equivalence relation.) Let $K = X/\sim$ with $\pi : X \to K$ as the quotient map. Denote $\pi(x,y) := \frac{x}{y}$, interpret $0^K = \frac{0^A}{1^A}$ and $1^K = \frac{1^A}{1^A}$. Interpret $+$ and $\cdot$ as standard fraction addition and multiplication and use $A \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ to check that these are well defined.

Check that K is an Σ_{RNG} structure and that $K \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$. Define $\iota : A \to K := a \mapsto \frac{a}{1}$ and show that this well defined and injective. Check that ι is a Σ_{RNG}-morphism and note that since there are no relation symobls in Σ_{RNG} it is also an embedding.

Let L be another field with an embedding from A. Define the map $\iota : K \to L$ sending $\frac{a}{b} \mapsto \frac{\iota_L(a)}{\iota_L(b)}$. Check that this is well-defined and a Σ_{RNG}-morphism. It is injective because ι_L is injective:

$$\frac{\iota_L(a)}{\iota_L(b)} = 0 \Rightarrow \iota_L(a) = 0 \Rightarrow a = 0$$

Thus it is an embedding. It is unique: suppose $\phi : K \to L$ is a Σ_{RNG}-embedding that commutes with the diagram. Then for any $a \in K$, $\phi(\frac{a}{1}) = \iota_L(a) = \iota(\frac{a}{1})$. Since both ϕ, ι are embeddings they commute with taking the inverse for $a \neq 0$: $\phi(\frac{1}{a}) = \iota(\frac{1}{a})$. Since any element of K can be written as $\frac{a}{b}$, we have shown that $\phi = \iota$. $\square$

3.1.2 Algebraically closed fields

Definition – Theory of algebraically closed fields

We define the Σ_{RNG} theory of algebraically closed fields. Fixing variables $\{a_i : A\}_{i \in \mathbb{N}}$ and $x : A$ we define

$$\mathrm{ACF} := \mathrm{FLD} \cup \left\{ \forall a_0, \ldots, \forall a_{n-1}, \exists x : A,\ x^n + \sum_{i=0}^{n-1} a_i x^i = 0 \,|\, n \in \mathbb{N}_{>0} \right\}$$

Unlike the theories RNG, ID, FLD this theory is countably infinite.

Proposition

ACF is not complete.

Proof. Take the Σ_{RNG}-formula $2 = 0$. This is satisfied by the algebraic closure of $\mathbb{F}_2$ but not by that of $\mathbb{F}_3$, since field embeddings preserve characteristic. □

Definition – Algebraically closed fields of characteristic p

For $p \in \mathbb{Z}_{>0}$ prime let $\mathrm{ACF}_p := \mathrm{ACF} \cup \{p = 0\}$ and let

$$\mathrm{ACF}_0 := \mathrm{ACF} \cup \{p \neq 0 \,|\, p \in \mathbb{Z}_{>0} \text{ prime}\}$$

An important fact about algebraically closed fields of characteristic p:

Proposition – Transcendence degree and characteristic determine algebraically closed fields of characteristic p up to isomorphism

If K_0, K_1 are fields with same characteristic and transcendence degree over their minimal subfield ($\mathbb{F}_p$ or $\mathbb{Q}$) then they are (non-canonically) isomorphic.

Proof. See appendix. □

Notation. *If $K \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ACF}_p$, write $\mathrm{t.deg}(K)$ to mean the transcendence degree over its minimal subfield ($\mathbb{F}_p$ or $\mathbb{Q}$).*

Proposition

ACF_p is κ-categorical for all uncountable κ, consistent and complete.

Proof. Suppose $K, L \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ACF}_p$ and $|K| = |L| = \kappa$. Then $\mathrm{t.deg}(K) + \aleph_0 = |K| = \kappa$ and so $\mathrm{t.deg}(K) = \kappa$ (as κ is uncountable). Similarly $\mathrm{t.deg}(L) = \kappa$ and so $\mathrm{t.deg}(K) = \mathrm{t.deg}(L)$. Thus K and L are isomorphic.

ACF_p is consistent due to the existence of the algebraic closures for any characteristic, it is not finitely modelled and is $\aleph_1$-categorical with $\Sigma_{\mathrm{RNG}\,\mathrm{con}} + \aleph_0 \leq \aleph_1$, hence it is complete by Vaught's test. □

3.1.3 Ax-Grothendieck

In this section we provide a proof of Ax-Grothendieck, which says injective polynomial maps are surjective over certain algebraically closed fields.

Definition – Polynomial map

Let K be a field and n a natural. Let $f : K^n \to K^n$ such that for each $a \in K^n$,

$$f(a) = (f_1(a), \ldots, f_n(a))$$

for $f_1, \dots, f_n \in K[x_1, \dots, x_n]$. Then we call f a polynomial map over K.

Proposition – Lefschetz principle

Let ϕ be a Σ_{RNG}-sentence. Then the following are equivalent:

1. There exists a Σ_{RNG}-model of ACF_0 that satisfies ϕ. (If you like $\mathbb{C} \vDash_{\Sigma_{\mathrm{RNG}}} \phi$.)
2. $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \phi$
3. There exists $n \in \mathbb{N}$ such that for any prime p greater than n, $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \phi$
4. There exists $n \in \mathbb{N}$ such that for any prime p greater than n there exists a Σ_{RNG}-model of ACF_p that satisfies ϕ.

Proof. 1. $\Leftrightarrow$ 2. ACF_0 is complete and consistent (with $\mathbb{C}$).

2. $\Leftrightarrow$ 3. Suppose $\mathrm{ACF}[0] \vDash_{\Sigma_{\mathrm{RNG}}} \phi$ then since 'proofs are finite' there exists a finite subset Δ of $\mathrm{ACF}[0]$ such that $\Delta \vDash_{\Sigma_{\mathrm{RNG}}} \phi$. Let n be maximum of all primes $p \in \mathbb{N}$ such that $p \neq 0 \in \Delta$ so we have

$$\Delta \subseteq \mathrm{ACF} \cup \{p \neq 0 \,|\, p \leq n \text{ prime }\}$$

By uniqueness of characteristic, if q is prime and greater than n then for each prime $p \leq n$ we have $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} p \neq 0$ so $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \Delta$. It follows that for all primes p greater than n, $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \phi$.

Conversely, suppose for any prime p greater than n $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \phi$. By completeness of ACF_0 it suffices to assume $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \neg\phi$. Using the forward direction we have that there exists m such that for all p greater than m $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \neg\phi$. Since there are infinitely many primes we find some prime p such that ACF_p is inconsistent - a contradiction.

3. $\Leftrightarrow$ 4. ACF_p is consistent and complete. $\square$

Definition – Locally finite fields [4]

Let K be a field of characteristic p a prime. Then the following are equivalent:

1. The minimal subfield generated by any finite subset of K is finite.
2. $\mathbb{F}_p \to K$ is algebraic.
3. K embeds into an algebraic closure of $\mathbb{F}_p$.

Proof. 1. $\Rightarrow$ 2. Let $a \in K$. Then $\mathbb{F}_p(a)$ is the minimal subfield generated by a, and is finite by assumption. Finite field extensions are algebraic hence $F_p(a)$ (in particular a) is algebraic over $\mathbb{F}_p$.

2. $\Rightarrow$ 1. We show by induction that K is locally finite. Let S be a finite subset of K. If S is empty then $\mathbb{F}_p(S) = \mathbb{F}_p$ and so it is finite. If $S = T \cup s$ and $\mathbb{F}_p(T)$ is finite, then $s \in K$ is algebraic so by some basic field theory we can take the quotient by the minimal polynomial of s giving

$$\mathbb{F}_p(T)[x]/\min(s, \mathbb{F}_p(T)) \cong \mathbb{F}_p(S)$$

The left hand side is finite because it is a finite dimensional vector space over a finite field. Hence K is locally finite.

2. $\Leftrightarrow$ 3. These are the properties of algebraic closures. $\square$

Proposition – Ax-Grothendieck for locally finite fields

Let L be a locally finite field. Then any injective polynomial map from $L^n \to L^n$ is surjective.

Proof. Let $b = (b_1, \dots, b_n) \in L^n$. We find a subfield K such that $b \in K^n = f(K^n)$ Writing $f = (f_1, \dots, f_n)$ for $f_i \in \Omega[x_1, \dots, x_n]$ we can find $A \subseteq L$, the set of all the coefficients of all of the f_i. $A \cup \{b_1, \dots, b_n\}$ is finite, so the subfield K generated by it is also finite.

The restriction $f\big|_{K^n}$ is injective and has image inside K^n since each polynomial has coefficients in K and is evaluated at an element of K^n. Hence $f\big|_{K^n}$ is an injective endomorphism of a finite set thus is surjective. We conclude that $b \in K^n = f(K^n)$. ☐

Corollary – Ax-Grothendieck for algebraic closure of finite fields

If Ω is an algebraic closure of a finite field then any injective polynomial map over Ω is surjective.

Proof. It suffices that Ω is locally finite. Any finite field is an algebraic extension over $\mathbb{F}_p$ where p is its prime characteristic. Hence its algebraic closure Ω is an algebraic extension of $\mathbb{F}_p$ and so it is locally finite. ☐

Definition – The Ax-Grothendieck formula

We define a Σ_{RNG}-sentence $\Phi_{n,d}$ expressing the following (for any field K): for all $d, n \in \mathbb{N}$, any injective polynomial map $f : K^n \to K^n$ with componenets of degree at most d is surjective.

We first need to be able to express polynomials in n varibles of degree at most d in an elementary way. Note that for any $n, d \in \mathbb{N}$ there exists a finite set $S_{n,d}$ and powers $r_{s,j} \in \mathbb{N}$ (for each $(s, j) \in S \times \{1, \dots, n\}$). such that any polynomial $f \in K[x_1, \dots, x_n]$ of degree at most d can be written as

$$\sum_{s \in S} \lambda_s \prod_{j=1}^{n} x_j^{r_{s,j}}$$

for some $\lambda_s \in K$. Now we have a way of quantifying over all such polynomials, which is by quantifying over all the coefficients. We define $\Phi_{n,d}$:

$$\Phi_{n,d} := \bigvee_{i=1}^{n} \bigvee_{s \in S} \lambda_{s,i}, \left[\bigvee_{j=1}^{n} x_j \bigvee_{j=1}^{n} y_j, \bigwedge_{i=1}^{n} \left(\sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} x_j^{r_{s,j}} = \sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} y_j^{r_{s,j}} \right) \longrightarrow \bigwedge_{i=1}^{n} x_i = y_i \right]$$
$$\longrightarrow \bigvee_{j=1}^{n} x_j, \bigexists_{i=1}^{n} z_i, \bigwedge_{i=1}^{n} \left(z_i = \sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} x_j^{r_{s,j}} \right)$$

At first it quantifies over all of the coefficients of all the f_i. The following part says that if the polynomial map is injective then it is surjective. Thus $K \vDash_{\Sigma} \Phi_{n,d}$ if and only if for all $d, n \in \mathbb{N}$ any injective polynomial map $f : K^n \to K^n$ of degree less than or equal to d is surjective.

Proposition – Ax-Grothendieck

If K is an algebraically closed field of characterstic 0 then any injective polynomial map over K is surjective. In particular injective polynomial maps over $\mathbb{C}$ are surjective.

Proof. We show an equivalent statement: for any $n, d \in \mathbb{N}$, any injective polynomial map $f : K^n \to K^n$ of degree at most d is surjective. This is true if and only if $K \vDash_{\Sigma_{\text{RNG}}} \Phi_{n,d}$ (by construction) which is true if and only if for all p prime greater than some natural number there exists an algebraically closed field of characteristic p that satisfies $\Phi_{n,d}$, by the Lefschetz principle. Indeed, take the natural 0 and let p be a prime greater than 0. Take Ω an algebraic closure of $\mathbb{F}_p$, which indeed models ACF_p. Ω satisfies $\Phi_{n,d}$ if and only if for any $n, d \in \mathbb{N}$, any injective polynomial map $f : \Omega^n \to \Omega^n$ of degree less than or equal to d is surjective (by construction). The final statement is true due to A-G for algebraic closures of finite fields. ☐

3.1.4 Quantifier elimination in algebraically closed fields and Nullstellensatz

Lemma – Terms in the language of rings are polynomials

Let X be a subset of a ring A. Then $\Sigma_{\mathrm{RNG}}(X)$-terms are interpreted as polynomials with coefficients from $\langle X \rangle$, the smallest subring of A generated by X.

Proof. Let t be a $\Sigma_{\mathrm{RNG}}(X)$-term.

- If t is a constant c then it's interpretation is $c^{\mathcal{M}}$, a constant polynomial. Since c is 0, 1 or something in X, this is in any polynomial ring over $\langle X \rangle$.
- If t is a variable v_i then it is interpreted as the single variable polynomial $v_s \in \langle X \rangle[v_s]$.
- If t is $f(s_1, \ldots, s_n)$ and each s_i is interpreted as a polynomial q_i, take the polynomial ring $\langle X \rangle[v_1, \ldots, v_k]$ containing all the q_i (everything is finite). Then $f(q_1, \ldots, q_n)$ is still a polynomial in $\langle X \rangle[v_1, \ldots, v_k]$ as f is either $+, -$ or $\cdot$.

□

Lemma – Disjunctive normal form for rings

Let A be a ring or the empty set. Let ϕ be a quantifier free $\Sigma_{\mathrm{RNG}}(A)$-formula with variables indexed by S. Then there exist $\Sigma_{\mathrm{RNG}}(A)$-terms (i.e. polynomials) p_{ij}, q_{ij} such that for any $\Sigma_{\mathrm{RNG}}(A)$-structure $\mathcal{M}$

$$\mathcal{M} \vDash_{\Sigma} \overset{v_s}{\underset{s \in S}{\forall}}, \left[\phi(v) \leftrightarrow \bigvee_{i \in I} \left(\bigwedge_{j \in J_{i0}} p_{ij}(v) = 0 \wedge \bigwedge_{j \in J_{i1}} q_{ij}(v) \neq 0 \right) \right]$$

Proof. Applying the general disjunctive normal form we have that any formula of the form $s = t$ or $r(t)$ is just a polynomial equation, since there are no relation symbols. Moving everything to one side we have that they are of the form $p_{ij}(v) = 0$ and $q_{ij}(v) \neq 0$. □

Remark – Substructures of algebraically closed fields. Since $\mathrm{GRP}, \mathrm{RNG}, \mathrm{ID}$ are all universal axomatizations of themselves, by 'universal axiomatizations make substructures models' we have that substructures of groups are groups, substructures of rings are rings, and substructures of integral domains are integral domains. Furthermore, substructures of algebraically closed fields are substructures of integral domains hence are integral domains. This becomes relevant when proving the equivalent condition on quantifier elimination.

NOTATION. *If A is a ring and $I \subseteq A[x_1, \ldots, x_n]$ is a set of polynomials then the vanishing of I over A is*

$$\mathbb{V}_A(I) := \{c \in A^n \mid \forall f \in I, f(c) = 0\}$$

Proposition

ACF has quantifier elimination.

Proof. Let ϕ be a quantifier free Σ_{RNG}-formula with a free variable w and index the rest by S. Let K, L be algebraically closed field with A embedding into each via ι_K, ι_L. Let $a \in A^S$. It suffices to show that

$$K \vDash_{\Sigma_{\mathrm{RNG}}} \exists w, \phi(\iota_K(a), w) \Rightarrow L \vDash_{\Sigma_{\mathrm{RNG}}} \exists w, \phi(\iota_L(a), w)$$

Crucially by the remark before this proposition A is an integral domain thus we can consider $\iota : A \to \Omega$ the embedding into an algebraic closure of the field of fractions of A. As K and L are fields extending the fraction field of A, by the properties of field of fractions and algebraic closures there are field extensions $\kappa : \Omega \to K$ and $\lambda : \Omega \to L$ such that $\iota_K = \kappa \circ \iota$ and $\iota_L = \lambda \circ \iota$. Let we write $a' := \iota(a)$, and have $\iota_K(a) = \kappa(a')$ and $\iota_L(a) = \lambda(a')$.

$$\begin{tikzcd} & & K \\ A \arrow[r] \arrow[urr] \arrow[drr] & F \arrow[r] \arrow[ur] \arrow[dr] & \Omega \arrow[u] \arrow[d] \\ & & L \end{tikzcd}$$

We can find the 'disjunctive normal form' of ϕ, i.e. Σ_{RNG}-terms p_{ij}, q_{ij} such that any ring $\mathcal{M}$ satisfies

$$\mathcal{M} \vDash_\Sigma \overset{v_s}{\underset{s\in S}{\forall}}, \forall w, \left[\phi(v,w) \leftrightarrow \bigvee_{i\in I}\left(\bigwedge_{j\in J_{i0}} p_{ij}(v,w) = 0 \wedge \bigwedge_{j\in J_{i1}} q_{ij}(v,w) \neq 0\right)\right]$$

Hence assuming for some $b \in K$ that

$$K \vDash_{\Sigma_{\mathrm{RNG}}} \bigvee_{i\in I}\left(\bigwedge_{j\in J_{i0}} p_{ij}(\kappa(a'),b) = 0 \wedge \bigwedge_{j\in J_{i1}} q_{ij}(\kappa(a'),b) \neq 0\right)$$

it suffices to show that there exists a $c \in L$ such that

$$L \vDash_{\Sigma_{\mathrm{RNG}}} \bigvee_{i\in I}\left(\bigwedge_{j\in J_{i0}} p_{ij}(\lambda(a'),c) = 0 \wedge \bigwedge_{j\in J_{i1}} q_{ij}(\lambda(a'),c) \neq 0\right)$$

Assume for a contradiction that I is empty then by convension we have the disjunctive normal form is $\perp$ and so $K \vDash_{\Sigma_{\mathrm{RNG}}} \perp$ which implies K is empty, which is a contradiction as the constant symbol 0 has an interpretation in K.

Thus there exists $i \in I$ (we keep this i for the rest of the proof) such that

$$K \vDash_{\Sigma_{\mathrm{RNG}}} \bigwedge_{j\in J_{i0}} p_{ij}(\kappa(a'),b) = 0 \wedge \bigwedge_{j\in J_{i1}} q_{ij}(\kappa(a'),b) \neq 0$$

Now case on whether or not there exists a $j \in J_{i0}$ such that p_{ij}^K is not the zero polynomial. If it exists then $p_{ij}^K(\kappa(a'),b) = 0$. As p_{ij} is a Σ_{RNG}-term and so $\kappa p_{ij}^\Omega = p_{ij}^K$. (Intuitively it is a polynomial over $\langle 0,1\rangle$ as shown in 'disjunctive normal form'.) Thus b is algebraic over Ω, which is algebraically closed. Hence there is a $c \in \Omega$ such that $\kappa(c) = b$:

$$\begin{aligned}
&\bigwedge_{j\in J_{i0}} p_{ij}^K(\kappa(a'),\kappa(c)) = 0 \wedge \bigwedge_{j\in J_{i1}} q_{ij}^K(\kappa(a'),\kappa(c)) \neq 0 \\
\Rightarrow &\bigwedge_{j\in J_{i0}} \kappa\left(p_{ij}^\Omega(a',c)\right) = 0 \wedge \bigwedge_{j\in J_{i1}} \kappa\left(q_{ij}^\Omega(a',c)\right) \neq 0 \\
\Rightarrow &\bigwedge_{j\in J_{i0}} \left(p_{ij}^\Omega(a',c)\right) = 0 \wedge \bigwedge_{j\in J_{i1}} \left(q_{ij}^\Omega(a',c)\right) \neq 0 \qquad \kappa \text{ is injective} \\
\Rightarrow &\bigwedge_{j\in J_{i0}} \lambda\left(p_{ij}^\Omega(a',c)\right) = 0 \wedge \bigwedge_{j\in J_{i1}} \lambda\left(q_{ij}^\Omega(a',c)\right) \neq 0 \\
\Rightarrow &\bigwedge_{j\in J_{i0}} \left(p_{ij}^L(\lambda(a'),\lambda(c))\right) = 0 \wedge \bigwedge_{j\in J_{i1}} \left(q_{ij}^L(\lambda(a'),\lambda(c))\right) \neq 0 \\
\Rightarrow &L \vDash_{\Sigma_{\mathrm{RNG}}} \bigvee_{i\in I}\left(\bigwedge_{j\in J_{i0}} p_{ij}(\lambda(a'),\lambda(c))) = 0 \wedge \bigwedge_{j\in J_{i1}} q_{ij}(\lambda(a'),\lambda(c))) \neq 0\right)
\end{aligned}$$

Thus we are done with this case.

In the other case we turn our attention to J_{i1}. If $j \in J_{i1}$, we see $q_{ij}^{\Omega}(a', w)$ as a polynomial in $\Omega[w]$. $q_{ij}^{K}(\kappa(a'), b) \neq 0$ and so $q_{ij}^{\Omega}(a', w)$ is not the zero polynomial hence $q_{ij}^{L}(\lambda(a'), w)$ has finitely many zeros in L by the division algorithm in $\Omega[w]$. Let

$$\mathbb{V}_L(q_{ij})_j = \left\{c \in L \mid \exists j \in J_{i1} \mid q_{ij}^{L}(\lambda(a'), c) = 0\right\}$$

be the vanishing of the q_{ij} for every $j \in J_{j1}$. Then $\mathbb{V}$ is finite and L is infinite as it is algebraically closed hence there exists a $c \in L$ $L \vDash_{\Sigma_{\mathrm{RNG}}} \bigwedge_{j \in J_{i1}} q_{ij}(\lambda(a'), c) \neq 0$. Since each f_{ij} are the zero polynomial we also have $L \vDash_{\Sigma_{\mathrm{RNG}}} \bigwedge_{j \in J_{i0}} p_{ij}(\lambda(a'), \lambda(c))) = 0$ Hence we are done. □

> **Proposition – Weak Nullstellensatz**
> If K is an algebraically closed field and $\mathfrak{p}$ is a prime ideal of $K[x_1, \ldots, x_n]$, then $\mathbb{V}_K(\mathfrak{p}) \neq \varnothing$.

Proof. By Hilbert's basis theorem $K[x_1, \ldots, x_n]$ is Noetherian so we can find a finite set $\{f_1, \ldots, f_m\}$ generating $\mathfrak{p}$. It suffices to find a common zero of $\{f_1, \ldots, f_m\}$. We can write out each f_i and use a to represent all the coefficients of all the f_i as a tuple. We can construct some Σ-formula $\phi(v, x_1, \ldots, x_n)$ (a polynomial in variables x_i with coefficients v corresponding to the tuple a) such that for any $b \in K^n$,

$$K \vDash_{\Sigma_{\mathrm{RNG}}} \phi(a, b) \Leftrightarrow \bigwedge_{j=1}^{m} f_i(b) = 0$$

and for any extension $\iota : K \to L$ and $b \in L^n$,

$$L \vDash_{\Sigma_{\mathrm{RNG}}} \phi(\iota(a), b) \Leftrightarrow \bigwedge_{j=1}^{m} \iota(f_i)(b) = 0$$

We can quotient $K[x_1, \ldots, x_n]/\mathfrak{p}$ and take an algebraic closure L (quotienting by prime ideals gives an integral domain).

$$\begin{tikzcd}
K[x_1, \ldots, x_n] \arrow[r] & K[x_1, \ldots, x_n]/\mathfrak{p} \arrow[d, "\alpha"] \\
K \arrow[u] \arrow[ur, "\kappa"] & L
\end{tikzcd}$$

We can then take $b := (x_1, \ldots, x_n) \in (K[x_1, \ldots, x_n])^n$ and send it through to $\alpha(b) \in L^n$.

$$\begin{aligned}
&\bigwedge_{j=1}^{m} \kappa(f_i)(b) = 0 \\
&\Rightarrow \bigwedge_{j=1}^{m} \kappa \circ \alpha(f_i)(\alpha(b)) = 0 \\
&\Rightarrow \bigwedge_{j=1}^{m} \iota(f_i)(\alpha(b)) = 0 \\
&\Rightarrow L \vDash_{\Sigma_{\mathrm{RNG}}} \phi(\iota(a), \alpha(b)) \\
&\Rightarrow L \vDash_{\Sigma_{\mathrm{RNG}}} \coprod_{i=1}^{n} x_i, \phi(\iota(a), x)
\end{aligned}$$

since K and L are both algebraically closed and ACF has quantifier elimination so it is model complete, which implies that the embedding is elementary. Hence we have

$$\begin{aligned}
&\Rightarrow K \vDash_{\Sigma_{\mathrm{RNG}}} \exists_{i=1}^{n} x_i, \phi(a,x) \\
&\Rightarrow \exists c \in K^n, K \vDash_{\Sigma_{\mathrm{RNG}}} \phi(a,c) \\
&\Rightarrow \exists c \in K^n, \bigwedge_{j=1}^{m} f_i(b) = 0
\end{aligned}$$

□

3.1.5 The Classical Zariski Topology, Chevalley, Vanishings

Definition – Classical Zariski Topology

Let K be an algebraically closed field and let

$$\{\mathbb{V}_K(E) \mid E \subseteq K[x_1, \ldots, x_n]\}$$

be a closed basis for a topology on K^n. We call these Zariski closed sets.

The closed sets in this setting correspond to closed sets in $\mathrm{Spec}(K[x_1, \ldots, x_n])$, though the spaces are not homeomorphic.

Proposition – Closed sets are finitely generated vanishings

If K is an algebraically closed field and V is a closed set of K^n under the Zariski topology then $V = \mathbb{V}_K(S)$ for some finite subset of the polynomial ring $S \subseteq K[x_1, \ldots, x_n]$.

Proof. By definition of Zariski closed sets any element of the closed basis is $\mathbb{V}_K(E)$ for some $E \subseteq K[x_1, \ldots, x_n]$ We consider the ideal generated by E. Important point: This is finitely generated by the Hilbert basis theorem, and so we can just require E to be finite without loss of generality.

Moreover, arbitrary intersection of such sets is also finitely generated: Let $E \in I$ be a collection of subsets of $K[x_1, \ldots, x_n]$. Then $a \in \bigcap_{E \in I} \mathbb{V}_K(E)$ if and only if for every $E \in I$ and every $f \in E$, $f(a) = 0$. This is true if and only if $a \in \mathbb{V}_K(\bigcup_{E \in I} E)$. Thus $\bigcap_{E \in I} \mathbb{V}_K(E) = \mathbb{V}_K(\bigcup_{E \in I} E)$ which is finitely generated by the first point.

Furthermore, any finite union of such sets is also finitely generated. We prove this by induction. For the empty case:

$$\bigcup_{E \in \varnothing} \mathbb{V}_K(E) = \varnothing = \mathbb{V}_K(K[x_1, \ldots, x_n])$$

which is finitely generated by the first point. It suffices to show that the union of two such sets is also finitely generated.

$$\begin{aligned}
a \in \mathbb{V}(F) \cup \mathbb{V}(G) &\Leftrightarrow \left(\bigwedge_{f \in F} f(a) = 0 \right) \vee \left(\bigwedge_{g \in G} g(a) = 0 \right) \\
&\Leftrightarrow \bigwedge_{f \in F} \bigwedge_{g \in G} f(a) = 0 \vee g(a) = 0 \Leftrightarrow \bigwedge_{f \in F} \bigwedge_{g \in G} (fg)(a) = 0
\end{aligned}$$

The last step is due to $K[x_1, \ldots, x_n]$ being an integral domain. Hence we have a finite intersection $\mathbb{V}(F) \cup \mathbb{V}(G) = \mathbb{V}(\bigcap (fg)(a) = 0)$ which is finitely generated by the first point. □

Proposition – Constructable
Let K be an algebraically closed field with the Zariski topology on K^n. Define the set C inductively:

| If $X \subseteq K^n$ is closed then it is in C.

| If $X \subseteq K^n$ is in C then $K^n \setminus X$ is in C.

| If $X, Y \subseteq K^n$ are in C then $X \cup Y$ is in C.

Then C is the set of constructable and equivalently the set of of definable sets in K^n (by quantifier elimination).

C consists of 'finite boolean combinations' of closed sets, and corresponds to the original definition 'constructable'.

Proof. ($\Rightarrow$) First we show by induction on the set C that if X is in C then X is constructble. If X is closed then $X = \mathbb{V}(S)$ for some finite S. Therefore X is defined by $\phi(v, b) := \bigwedge_{f \in S} f(a) = 0$, where each f is some Σ_{RNG} formula evaluated at $b \in K^m$. This is a finite and which is the negation of a finite or which is (by induction) constructable. The rest of the induction follows immediately.

($\Leftarrow$) If X is constructable then we show by induction that it is in C. If X is defined by an atomic formula then it is either $\top$ or $t = s$. If ϕ is $\top$ then $X = K^n$ which is closed hence in C. If ϕ is $t = s$ then $t^K(b)$ and $s^K(b)$ are polynomials in $K[x_1, \ldots, x_n]$. Writing $f = t^K(b) - s^K(b)$ we have $X = \{a \in K^n \mid f(a) = 0\}$, which is closed hence in C. The rest of the induction follows immediately. □

Proposition – Chevalley
Over an algebraically closed field, the image of a constructable set under a polynomial map is constructable.

Proof. Let $\rho : K^n \to K^m$ be a polynomial map defined by $(f_i)_{i=1}^m$. Suppose $X \subseteq K^n$ is constructable. Then as constructable is equivalent to definable over K there exists Σ-formula ϕ and $b \in K^l$ such that

$$X = \{a \in K^n \mid K \vDash_{\Sigma_{\text{RNG}}} \phi(a, b)\}$$

Then

$$\begin{aligned}
\rho(X) &= \{c \in K^m \mid \exists a \in K^m, K \vDash_{\Sigma_{\text{RNG}}} \wedge \rho(a) = c\} \\
&= \left\{ c \in K^m \mid \exists a \in K^m, K \vDash_{\Sigma_{\text{RNG}}} \wedge \bigwedge_{i=1}^m f_i(a) = c_i \right\} \\
&= \left\{ c \in K^m \mid K \vDash_{\Sigma_{\text{RNG}}} \coprod_{j=1}^n x_j, \phi(x, b) \wedge \bigwedge_{i=1}^m \phi_i(x, d) = c_i \right\}
\end{aligned}$$

The d appearing in $\phi_i(x, d)$ is due to the fact that the polynomials f_i may have coefficients not from the language. Thus the image is constructable. □

NOTATION (RADICAL). *We write $r(\mathfrak{a})$ to be the radical of $\mathfrak{a}$.*

Definition – Ideal generated by subsets of K^n
If K is a field, for a subset $X \subseteq K^n$, we write $I(X)$ to mean the ideal of X in $K[x_1, \ldots, x_n]$ to mean

$$\{f \in K[x_1, \ldots, x_n] \mid \forall a \in X, f(a) = 0\}$$

EXERCISE (TAKING IDEALS AND VANISHINGS ARE ORDER REVERSING). *Show that if* $X \subseteq Y \subseteq K^n$ *then* $I(Y) \subseteq I(X)$. *Show that if* $E \subseteq F \subseteq K[x_1, \ldots, x_n]$ *then* $\mathbb{V}_K(F) \subseteq \mathbb{V}_K(E)$.

Proposition – Strong Nullstellensatz
Let K be an algebraically closed field and suppose $\mathfrak{a}$ is an ideal of $K[x_1, \ldots, x_n]$. Then $r(\mathfrak{a}) = I(\mathbb{V}(\mathfrak{a}))$.

Proof. See appendix. □

Proposition – The Zariski closed sets are Artinian
Any descending chain of Zariski closed sets in K^n for K a field stabilises.

Proof. Let $\ldots \subseteq \mathbb{V}(\mathfrak{a}_1) \subseteq \mathbb{V}(\mathfrak{a}_0)$ be a chain of Zariski closed sets. Then taking the ideals generated each of them we have an ascending chain

$$I(\mathbb{V}(\mathfrak{a}_0)) \subseteq I(\mathbb{V}(\mathfrak{a}_1)) \subseteq \ldots$$

This stabilises because $K[x_1, \ldots, x_n]$ is Noetherian. Hence by strong Nullstellensatz we have that

$$r(\mathfrak{a}_0) \subseteq r(\mathfrak{a}_1) \subseteq \ldots$$

stabilises and taking the vanishing gives back the descending chain $\ldots \subseteq \mathbb{V}(\mathfrak{a}_1) \subseteq \mathbb{V}(\mathfrak{a}_0)$ which stabilises. □

Definition – Irreducible, Variety
If X is a topological space then the following are equivalent:

1. Any non-empty open set is dense in X.
2. Any pair of non-empty open subsets intersect non-trivially.
3. Any two closed proper subsets do not form a cover of X.

If any of the above hold then X is said to be irreducible, and a subset of X is irreducible if it is irreducible under the subspace topology.

A variety is a Zariski closed set that is irreducible.

Corollary
Zariski closed sets are finite unions of varieties.

Proof. For a contradiction suppose $\mathbb{V}_0$ is a Zariski closed set that is not a finite union of varieties. Then $\mathbb{V}_0$ is not irreducible and so there exists two closed proper subsets $\mathbb{V}_1, \mathbb{V}_1'$ that cover $\mathbb{V}_0$. If both $\mathbb{V}_1$ and $\mathbb{V}_1'$ are finite unions of varieties then we have a contradiction, hence without loss of generality $\mathbb{V}_1$ is not a finite union of varieties. By induction we obtain

$$\cdots \subset \mathbb{V}_1 \subset \mathbb{V}_0$$

which stabilises, a contradiction. □

3.1.6 The Stone space and Spec

Proposition – Any polynomial is a formula
Let K be an algebraically closed field and $v = (v_1, \ldots, v_n)$ be variables. Then there exists a map

$$\text{eqzero} : K[x_1, \ldots, x_n] \to F(\Sigma_{\text{RNG}}(K), v)$$

such that for any $a \in K^n$ and any $f \in K[x_1, \dots, x_n]$,

$$K \vDash_{\Sigma_{\mathrm{RNG}(K)}} \mathrm{eqzero}_f(a) \Leftrightarrow f(a) = 0$$

where eqzero_f is the image of an f.

Proof. This would be some nasty induction. I guess show that monomials can be made into formulas and then sums of monomials can be made into formulas. Everything is finite so it should be okay. □

Proposition – Continuous bijection between the Stone space and spec
Let K be an algebraically closed field. Then define the map

$$\begin{aligned} I : S_n(\mathrm{ElDiag}(\Sigma, K)) &\to \mathrm{Spec}(K[x_1, \dots, x_n]) \\ p &\mapsto \left\{ f \in K[x_1, \dots, x_n] \mid \mathrm{eqzero}_f \in p \right\} \end{aligned}$$

We show that this is well-defined, continuous and a bijection. Hence $\mathrm{Spec}(K[x_1, \dots, x_n])$ is compact. However, the spaces are *not* homeomorphic.

Proof. Let $p \in S_n(\mathrm{ElDiag}(\Sigma, K))$. First we check that $I_p := \left\{ f \in K[x_1, \dots, x_n] \mid \mathrm{eqzero}_f \in p \right\}$ is a prime ideal. We will repeatedly use the following fact: since p is consistent with $\mathrm{ElDiag}(\Sigma, K)$ we have that it is finitely realised in K.

Let $f, g \in I_p$, then $\mathrm{eqzero}_f, \mathrm{eqzero}_g \in p$. Suppose for a contradiction $f + g \notin I_p$, then by maximality of p, $\neg\, \mathrm{eqzero}_{f+g} \in p$. Taking the finite subset of p to be $\left\{ \mathrm{eqzero}_f, \phi_g, \neg\, \mathrm{eqzero}_{f+g} \right\}$, by the above fact we obtain $a \in K^n$ such that

$$K \vDash_{\Sigma_{\mathrm{RNG}(K)}} \left\{ \mathrm{eqzero}_f, \mathrm{eqzero}_g, \neg\, \mathrm{eqzero}_{f+g} \right\}(a)$$

By definition of eqzero this implies

$$f(a) = 0, g(a) = 0, (f + g)(a) \neq 0$$

which is a contradiction. Similarly we can let $f \in I_p$, $\lambda \in K$ and suppose $\lambda f \notin I_p$ and so $\neg\, \mathrm{eqzero}_{\lambda f} \in p$. We take the finite subset $\{f, \lambda f\} \subseteq p$ and get a contradiction.

Let the product of two polynomials fg be in I_p. Suppose for a contradiction $f, g \notin I_p$. Then take the finite subset $\left\{ \neg\, \mathrm{eqzero}_f, \neg\, \mathrm{eqzero}_g, \mathrm{eqzero}_{f+g} \right\} \subseteq p$ and obtain an $a \in K^n$ such that $f(a) \neq 0, g(a) \neq 0$ but $f(a)g(a) = 0$, a contradiction.

To show that the map is continuous let $V(E) \subseteq \mathrm{Spec}(K[x_1, \dots, x_n])$ be closed, where E is a subset of $K[x_1, \dots, x_n]$ (V denotes the spec vanishing - see appendix). Then $p \in I^{-1}(U)$ if and only if $I_p \in U$ if and only if $E \subseteq I_p$ if and only if

$$\forall f \in E, \mathrm{eqzero}_f \in p$$

Hence the preimage is closed:

$$I^{-1}(U) = \left\{ p \in S_n(T) \mid \forall f \in E, \mathrm{eqzero}_f \in p \right\} = \bigcap_{f \in E} [\mathrm{eqzero}_f]$$

as the basis elements $[\phi]$ are clopen. Thus I is continuous.

To show that it is injective suppose $I_p = I_q$ and let $\phi \in p$. By quantifier-elimination in ACF we have ϕ is equivalent to quantifier free ψ modulo T. Then $\psi \in p$ as p is consistent with T and $\phi \in q$ if and only if $\psi \in q$ as q is consistent with T. Take the disjunctive normal form of ψ

$$\bigvee_{i \in I} \left(\bigwedge_{j \in J_{i0}} f_{ij}(v) = 0 \wedge \bigwedge_{j \in J_{i1}} g_{ij}(v) \neq 0 \right)$$

Then (by maximality of p) there exists some i such that for all j, $f_{ij} = 0$ and $g_{ij} \neq 0$ are in p. Thus $f_{ij} = 0$ and $g_{ij} \neq 0$ are in q as they can be made into polynomials in $I_p = I_q$. Hence by maximality of q we have $\psi \in q$ and so $\phi \in q$. By symmetry $p = q$.

For surjectivity, let $\mathfrak{p}$ be a prime ideal of the polynomial ring. Consider the set

$$\mathrm{eqzero}(\mathfrak{p}) \cup \{\neg\,\phi \,|\, \phi \in \mathrm{eqzero}(K[x_1, \ldots, x_n] \setminus \mathfrak{p})\}$$

Assuming for now that this set is consistent with the elementary diagram, it can be extended to p a maximal n-type of elementary diagram. Then

$$f \in \mathfrak{p} \Rightarrow \mathrm{eqzero}_f \in p \Rightarrow f \in I_p$$
$$f \notin \mathfrak{p} \Rightarrow (\neg\ \mathrm{eqzero}_f) \in p \Rightarrow f \notin I_p$$

and so $I_p = \mathfrak{p}$ and we have found a preimage p.

To show that the set is consistent it suffices to show that it is finitely realised in K. Let $\Delta \subseteq \mathfrak{p}$ and $\Gamma \subseteq K[x_1, \ldots, x_n] \setminus \mathfrak{p}$ be finite subsets. It suffices to show that

$$\exists a \in K^n, \forall f \in \Delta, \forall g \in \Gamma, f(a) = 0 \wedge g(a) \neq 0$$

which is equivalent to

$$\mathbb{V}_K(\Delta) \cap K^n \setminus \mathbb{V}_k(\Gamma) \neq \varnothing$$

Suppose for a contradiction it is empty then $\mathbb{V}_k(\Delta) \subseteq \mathbb{V}_k(\Gamma)$ and taking ideals gives us

$$\Gamma \subseteq r(\Gamma) = I(\mathbb{V}_K(\Gamma)) \subseteq I(\mathbb{V}_K(\Delta)) = r(\Delta) \subseteq \mathfrak{p}$$

Here we used that taking ideals is order reversing, strong Nullstellensatz, and the fact that the radical is a subset of any prime ideal containing its generators. Thus $\Gamma \subseteq \mathfrak{p}$ which is a contradiction. Hence we have surjectivity.

Suppose for a contradiction that the two spaces were homeomorphic. Then as the Stone space is Hausdorff, $\mathrm{Spec}(K[x_1, \ldots, x_n])$ would also be Hausdorff, which is a contradiction. $\square$

Chapter 4

Appendix

4.1 Transcendence degree and characteristic determine ACF_p up to isomorphism

This part of the appendix mainly follows Hungerford's book [9].

Definition
Suppose $\iota : K \to L$ is a field embedding and $S \subseteq L$. Then S is algebraically independent over K when for any $n \in \mathbb{N}$, any $f \in K[x_1, \ldots, x_n]$ and any distinct elements $s_i \in S$,

$$f(s) = 0 \Rightarrow f = 0$$

Remark. 1. If S is algebraically independent then any subset is algebraically independent over K.

2. $\varnothing$ is algebraically over K.
3. For the embedding $K \to K$, any non-empty subset is algebraically dependent.
4. The concept of algbebraic independence extends that of linear independence.

Definition
Suppose $K \to L$ is a field embedding. Then $B \subseteq L$ is a transcendence basis of the extension if B is algebraically independent over K and is maximal with respect to $\subseteq$.

Remark. This is the analogue of a vector space basis.

Proposition – Existence of transcendence basis
Suppose $K \to L$ is a field embedding. There there exists a transcendence basis.

Proof. Application of Zorn's lemma on the set of algebraically independent subsets of L (non-empty due to $\varnothing$ being algebraically independent) with respect to inclusion. □

Proposition – Algebraic elements over $K(S)$
Let $K \to L$ be a field embedding and let $S \subseteq L$ be algebraically independent over K. Let $u \in L \setminus S$. $S \cup \{u\}$ is algebraically dependent if and only if u is algebraic over $K(S)$.

Proof. ($\Rightarrow$) Suppose $S \cup \{u\}$ is algebraically dependent. Then there exists non-zero $f \in K[x_0, \dots, x_n]$ and distinct $s_i \in S$ such that $f(u, s_1, \dots, s_n) = 0$. Write

$$f = \sum_{i=0}^{m} h_i(x_0)^i$$

for $h_i \in K[x_1, \dots, x_n]$. Then let

$$g(x_n) := \sum_{i=0}^{m} h_i(s_1, \dots, s_n)(x_0)^i \in K(S)[x_0]$$

which has the root u. Assuming for a contradiction that u is not algebraic over $K(S)$. Since $g(u) = 0$ we must have $g(x_0) = 0$ and so for each i, $h_i(s_1, \dots, s_n) = 0$. By algebraic independence of S over K we have that each $h_i = 0$. Thus $f = 0$, a contradiction.

($\Leftarrow$) Suppose u is algebraic over $K(S)$. Then there exists non-zero $f(x) \in K(S)[x]$ such that $f(u) = 0$. We can write $f(x) = \sum_{i=1}^{n} \frac{h_i}{g_i}(s_1, \dots, s_m)x^i$ for $s_i \in K(S)$. Then we can factor all the g_i out, leaving

$$f(x) = \frac{1}{\prod_{j=1}^{n} g_j} \sum_{i=1}^{n} f_i(s_1, \dots, s_m) \prod_{j \neq i} g_j(s_1, \dots, s_m)x^i$$

Let

$$h = \sum_{i=1}^{n} f_i(x_1, \dots, x_m) \prod_{j \neq i} g_j(x_1, \dots, x_m){x_{m+1}}^i \in K[x_1, \dots, x_{m+1}]$$

We see that $h(s_1, \dots, s_m, u) = 0$ as $g(s_1, \dots, s_m) \neq 0$ (S is algebraically independent). Suppose for a contradiction that $S \cup \{u\}$ is algebraically independent, then $h = 0$ and so for each i,

$$f_i(x_1, \dots, x_m) \prod_{j \neq i} g_j(x_1, \dots, x_m) = 0$$

but g_i were on the bottom of the fractions so they are non-zero, thus $f_i(x_1, \dots, x_m) = 0$ as the polynomial ring is an integral domain. This implies that $f = 0$, a contradiction. □

Lemma – Composition of algebraic extensions

We use extensively the fact that the composition of algebraic extensions is algebraic.

Proposition – (Key) Identifying transcendence bases

Let $K \to L$ be a field embedding. Suppose $S \subseteq L$ is algebraically independent over K. Then L is algebraic over $K(S)$ if and only if S is a transcendence basis.

Proof. Every $a \in L$ is algebraic over $K(S)$ if and only if Every $a \in L$ makes $S \cup \{a\}$ algebraically dependent if and only if S is maximally algebraically independent if and only if S is a transcendence basis. □

Corollary – Subsets containing transcendence bases

Let $K \to L$ be a field embedding and let $X \subseteq L$. If $K(X) \to L$ is algebraic then X contains a transcendence basis of the extension $K \to L$.

Proof. Using Zorn we have a maximally algebraically independent subset of X; call this S. This is a transcendence basis of the extension $K \to K(X)$. Thus by the previous proposition $K(S) \to K(X)$ is algebraic. The composition of algebraic extensions is algebraic, thus $K(S) \to L$ is algebraic. Hence S is a transcendence basis. □

> **Proposition – Uniqueness of finite transcendence degree**
> Let $K \to L$ be a field embedding and let S be a finite transcendence basis of the extension. Then any other transcendence basis has the same cardinality as S.

Proof. If S is empty then it is the unique transcendence basis.

Otherwise, let $S = \{s_1, \ldots, s_n\}$ and T be another transcendence basis. We show that there exists a $t \in T$ such that $\{t, s_2, \ldots, s_n\}$ is a transcendence basis.

To find such a t, assume for a contradiction that for any $t \in T$, $\{t, s_2, \ldots, s_n\}$ is algebraically dependent. Then each $t \in T$ is algebraic over $\{s_2, \ldots, s_n\}$ and so $K(\{s_2, \ldots, s_n\})(T)$ is algebraic over $K(\{s_2, \ldots, s_n\})$. Furthermore L is algebraic over $K(\{s_2, \ldots, s_n\})(T)$ hence L is algebraic over $K(s_2, \ldots, s_n)$. Hence s_1 is algebraic over $K(s_2, \ldots, s_n)$, a contradiction.

Thus there exists such a $t \in T$. It suffices to show that L is algebraic over $K(t, s_2, \ldots, s_n)$. This is true if and only if $K(S)$ is algebraic over $K(t, s_2, \ldots, s_n)$, if and only if s_1 is algebraic over $K(t, s_2, \ldots, s_n)$, if and only if $\{t, s_1, s_2, \ldots, s_n\}$ is algebraically dependent over K, which it is since it is contains S as a proper subset. Thus we have that $\{t, s_1, s_2, \ldots, s_n\}$ is a transcendence basis of the same cardinality as S.

By induction, replacing s_i at each step, we obtain a subset of T that is a transcendence basis of the same cardinality as S. By maximality of transcendence bases this subset is T, thus $|S| = |T|$. □

> Notation (Minimal polynomial). *If $K \to L$ is a field extension, let the minimal polynomial of $a \in L$ over K be denoted as* $\min(a, K)$

> **Lemma – Infinite transcendence bases inject into each other**
> Let field embedding $K \to L$ have infinite transcendence bases S and T. Then $|T| \leq |S|$.

Proof. S is infinite and hence non-empty; let $s \in S$ and consider $\min(s, K(T)) \in K(T)[x]$. Because polynomials are finite, there exists T_s, a finite subset of T such that $\min(s, K(T)) \in K(T_s)[x]$. Hence s is algebraic over $K(T_s)$.

We claim that $\bigcup_{s \in S} T_s$ is a transcendence basis of $K \to L$. Indeed it is a subset of T hence it is algebraically independent; by construction $K(S)$ is algebraic over $K(\bigcup_{s \in S} T_s)$ and L is algebraic over K and so L is algebraic over $K(\bigcup_{s \in S} T_s)$. Thus $\bigcup_{s \in S} T_s$ is a transcendence basis of $K \to L$. By maximality of transcendence bases $T = \bigcup_{s \in S} T_s$.

We inject T into S by writing it as a disjoint union of subsets of T_s. By the well-ordering principle (choice) we well-order S and define

$$X_s := T_s \setminus \bigcup_{i < s} T_i$$

Since $X_s \subseteq T_s$ we have $\bigcup_{s \in S} X_s \subseteq \bigcup_{s \in S} T_s$. Conversely, if there exist $s \in S$ and $x \in T_s$ then the set of $i \in S$ such that $x \in T_i$ is non-empty and thus has a minimum element m (by well-ordering). Thus $x \in X_m$ and so $\bigcup_{s \in S} X_s = \bigcup_{s \in S} T_s = T$.

Define a map from $\bigcup_{s \in S} X_s \to S \times \mathbb{N}$ by the following: let $s \in S$. Then since $X_s \subseteq T_s$ and T_s is finite we can write $X_s = \{x_0, \ldots, x_{n_s}\}$. we send x_i to the element $(s, i) \in S \times \mathbb{N}$. This is well-defined because the X_i are disjoint and is clearly injective. Since S is infinite,

$$|T| = \left| \bigcup_{s \in S} X_s \right| \leq |S \times \mathbb{N}| = |S|$$

□

Proposition – Uniqueness of infinite transcendence degree
Let field embedding $K \to L$ have a transcendence basis. Then any other transcendence basis has the same cardinality.

Proof. Let S and T be two transcendence bases. If one of them were finite then by uniqueness of finite transcendence degree S and T have the same cardinality. Otherwise both are infinite and by the previous lemma $|T| \leq |S|$ and $|S| \leq |T|$. By Schröder–Bernstein $|S| = |T|$. ☐

Definition – Transcendence degree
If $\iota : K \to L$ is a field embedding then the transcendence degree is defined as the cardinality of a transcendence basis. It is well-defined as we showed that a basis exists and any two bases have the same cardinality. We use $\mathrm{t.deg}(\iota)$ to denote the degree.

Notation. *$K[x_1, \ldots, x_n]$ is the polynomial ring. $K(x_1, \ldots, x_n)$ is the field of fractions of the polynomial ring.*

Lemma – Isomorphism with field of polynomial fractions
Suppose $\iota : K \to L$ is a field embedding and $S \subseteq L$ is a finite set algebraically independent over K. Then there exists a (non-canonical) field isomorphism

$$K(S) \cong K(x_s)_{s \in S}$$

Proof. From Galois theory we have a (non-canonical) surjective ring morphism $K[x_s]_{s\in S} \to K[S]$ given by $x_s \mapsto s$. It is injective due to S being algebraically independence. By the universal property of field of fractions there is a unique isomorphism $K(S) \cong K(x_s)_{s\in S}$ that commutes with the other isomorphism.

$$\begin{tikzcd}
K[S] \arrow[r, "\subseteq"] \arrow[d, "\cong"] & K(S) \arrow[d, dashed] \\
K[x_s]_{s\in S} \arrow[r, "\subseteq"] & K(x_s)_{s\in S}
\end{tikzcd}$$

☐

Proposition – Embedding algebraically independent sets
Suppose we have the field embeddings

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[d, "\sigma"] & F_0 \\
K_1 \arrow[r] & F_1
\end{tikzcd}$$

and let $S \subseteq F_0$ be an algebraically independent over K_0. Suppose we have an injection $\phi : S \to F_1$ such that the image is algebraically independent over K_1. Then there exists a unique field embedding $\overline{\sigma} : K_0(S) \to F_1$ such that $\overline{\sigma}\big|_S = \phi$ and the following commutes

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[d, "\sigma"] & K_0(S) \arrow[d, "\overline{\sigma}"] \\
K_1 \arrow[r] & F_1
\end{tikzcd}$$

Furthermore, if σ is an isomorphism then $\overline{\sigma}$ is an isomorphism $K_0(S) \to K_1(\phi(S))$.

Proof. We define $\overline{\sigma} : K_0(S) \to F_1$ by

$$\frac{f(s_1, \ldots, s_n)}{g(s_1, \ldots, s_n)} \mapsto \frac{\sigma(f)(\phi s_1, \ldots, \phi s_n)}{\sigma(g)(\phi s_1, \ldots, \phi s_n)}$$

where σ takes a polynomial over K_0 as an argument (the induced map on the polynomial rings). To check that $\overline{\sigma}$ is well-defined we just need to check uniqueness of the image. Suppose $\frac{f}{g}(s_1, \ldots, s_n) \in K_0(S)$. Due to the universal property of field of fractions, there is a unique field embedding $\hat{\sigma} : K_0(x_i)_{i \leq n} \to K_1(x_i)_{i \leq n}$ that commutes with the injective polynomial ring morphism $\sigma : K_0[x_i]_{i \leq n} \to K_1[x_i]_{i \leq n}$ (which was induced by σ). By the previous lemma we have an isomorphisms $K_0(s_i)_{i \leq n} \cong K_0(x_i)_{i \leq n}$ and $K_1(\phi(s_i))_{i \leq n} \cong K_0(x_i)_{i \leq n}$ induced by the (not unique but suitably chosen) isomorphisms $K_0[s_i]_{i \leq n} \cong K_0[x_i]_{i \leq n}$ and $K_1[\phi(s_i)]_{i \leq n} \cong K_1[x_i]_{i \leq n}$. Hence we have the diagram:

$$\begin{tikzcd}
 & K_0[s_i]_i \arrow[r] \arrow[d, "\cong"] & K_0(s_i)_i \arrow[d, "\cong"] \\
K_0 \arrow[ur] \arrow[r] \arrow[d, "\sigma"] & K_0[x_i]_i \arrow[r] \arrow[d, "\sigma"] & K_0(x_i)_i \arrow[d, dashed, "\hat{\sigma}"] \\
K_1 \arrow[r] \arrow[dr] & K_1[x_i]_i \arrow[r] \arrow[d, "\cong"] & K_1(x_i)_i \arrow[d, "\cong"] \\
 & K_1[\phi(s_i)]_i \arrow[r] & K_1(\phi(s_i))_i
\end{tikzcd}$$

The composition of the three maps on the right hand side is $\overline{\sigma}$ restricted to $K_0(s_i)_{i \leq n}$. Note that the composition is a well-defined injective ring morphism that commutes with everything else. Thus $\frac{f}{g}(s)$ is sent to a unique element of F_1. If $q \in K_0(S)$ maps to the same image under $\overline{\sigma}$ then it lies in the image of the composition so it is $\frac{f}{g}(s)$. The composition commutes with everything and so for anything from K_0, going to F_1 via σ is the same as going via $\hat{\sigma}$.

Thus it is well-defined, injective and commutes. It is clearly a field embedding. It is unique because the map from $K_0[s_i] \to K_1[\phi(s_i)]$ was unique (though the intermediate isomorphisms were not unique). By definition, $\overline{\sigma}\big|_S = \phi$.

The above construction shows that if σ is an isomorphism then $\hat{\sigma}$ is an isomorphism. Hence $\overline{\sigma}$ restricted to the finite subset is an isomorphism. Since this is for any subset, $\overline{\sigma}$ is an isomorphism $K_0(S) \to K_1(\phi(S))$. $\square$

Proposition – Algebraically closed extensions of same transcendence degree are isomorphic

Suppose we have fields $K_0 \cong K_1$ and field extensions $K_0 \to L_0$ and $K_1 \to L_1$ of equal transcendence degree such that L_0, L_1 are algebraically closed, then L_0 and L_1 are (non-canonically) isomorphic.

Proof. Let σ be the isomorphism $K_0 \to K_1$ Let S_0, S_1 be transcendence bases of $K_0 \to L_0$ and $K_1 \to L_1$. They have the same cardinality thus we can biject S_0, S_1 and produce a (non-canonical) isomorphism $\overline{\sigma} : K_0(S_0) \to K_1(S_1)$. The extensions $K_0(S_0) \to L_0$ and $K_1(S_1) \to L_1$ are algebraic and L_0, L_1 are algebraically closed. Hence they are algebraic closures of isomorphic fields, which implies they they are (non-canonically) isomorphic.

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[d, "\sigma"', "\sim"] & K_0(S_0) \arrow[r] \arrow[d, "\overline{\sigma}"', "\sim"] & L_0 \arrow[d, dashed] \\
K_1 \arrow[r] & K_1(S_1) \arrow[r] & L_1
\end{tikzcd}$$

□

Corollary – Transcendence degree and characteristic determine algebraically closed fields of characteristic p up to isomorphism

If K_0, K_1 are fields of the same characteristic and have the same transcendence degree over their minimal subfield ($\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$). Then they are (non-canonically) isomorphic.

Proof. K_0 and K_1 have the same characteristic p so they are extensions of isomorphic subfields (their minimal subfields). Thye are algebraically closed. They have the same transcendence degree thus by the previous proposition they are (non-canonically) isomorphic. □

Proposition – Tower law of transcendence degree

Suppose $K \xrightarrow{\iota_L} L \xrightarrow{\iota_M} M$ are field embeddings. Then

$$\mathrm{t.deg}(\iota_L) + \mathrm{t.deg}(\iota_M) = \mathrm{t.deg}(\iota_M \circ \iota_L)$$

Proof. Let B_L and B_M be transcendence bases for the extensions ι_L, ι_M respectively. We show that $B_L \cup B_M$ is a transcendence basis for the composition.

Since B_L is a basis, $K(B_L) \to L$ is algebraic and hence we can show that $K(B_L \cup B_M) \to L(B_M)$ is algebraic. $L(B_M) \to M$ is algebraic as B_M is a basis, thus the composition $K(B_L \cup B_M) \to M$ is algebraic.

To show that it is algebraically independent, we first note that B_L, B_M are disjoint, otherwise there exists b in the intersection, which is both in B_M and in L causing B_M to be algebraically dependent over L. Let $f \in K[x_1, \ldots, x_n]$ and let $l_1, \ldots, l_r, m_{r+1} \in B_L$ and $m_{r+1}, \ldots, m_n \in B_M$ be distinct elements such that

$$f(l_1, \ldots, l_r, m_{r+1}, \ldots, m_n) = 0$$

We can find some finite set I, $h_i \in K[x_1, \ldots, x_r]$ and $k_i \in K[x_{r+1}, \ldots, x_n]$ such that

$$f(x_1, \ldots, x_n) = \sum_{i \in I} h_i(x_1, \ldots, x_r) k_i(x_{r+1}, \ldots, x_n)$$

and each k_i are linearly independent.

$$g := \sum_{i \in I} h_i(l_1, \ldots, l_r) k_i(x_{r+1}, \ldots, x_n) \in L[x_{r+1}, \ldots, x_n] \quad \wedge \quad g(m_{r+1}, \ldots, m_n) = 0$$

Since B_M is algebraically independent $g = 0$. Thus (by linear independence) of k_i each $h_i(l_1, \ldots, l_r) = 0$ and hence each $h_i(x_1, \ldots, x_r) = 0$ as B_L is algebraically independent. Thus $f = 0$ and the union forms a transcendental basis and

$$\mathrm{t.deg}(\iota_M \circ \iota_L) = |B_L \cup B_M| = |B_L| + |B_M| - |B_L \cap B_M| = \mathrm{t.deg}(B_L) + \mathrm{t.deg}(B_M)$$

□

Lemma – Isomorphic extensions have same transcendence degree

Suppose $K \to L, K \to M$ are field extensions and $L \to M$ is a an isomorphism that preserves K, then

$$\mathrm{t.deg}(K \to L) = \mathrm{t.deg}(K \to M)$$

Proof. Let S be a transcendence basis for $K \to L$. We claim the image of S under the isomorphism $\sigma : L \to M$ is a transcendence basis for $K \to M$.

Algebraic independence: Let $n \in \mathbb{N}$ and let $p \in K[x_1, \ldots, x_n]$. Let $a \in \sigma(S)^n$ and suppose $p(a) = 0$. Then we apply σ^{-1} to both sides, noting that p has coefficients from K and therefore commutes with σ^{-1}:

$$0 = \sigma^{-1}(p(a)) = p(\sigma^{-1}(a))$$

But $\sigma^{-1}(a)$ is an element of S^n and by algebraic independence of S we have $p = 0$.

Maximality: let $a \in M \setminus \sigma(S)$. We show that $a \cup \sigma(S)$ is algebraically dependent. Consider $\sigma^-1(a) \in L$. Since S is a basis $\sigma^{-1}(a)$ is algebraic over $K(S)$. This we have $p \in K(S)[x_0]$ such that $p(\sigma^{-1}(a)) = 0$. We can identify p with a polynomial q in $K[x_0, \ldots, x_n]$ with the remaining n coefficients from S, such that

$$p(x_0) = q(x_0, s_1, \ldots, s_n)$$

Since σ preserves K we have that

$$0 = \sigma(q(\sigma^{-1}(a), s_1, \ldots, s_n)) = q(a, \sigma(s_1), \ldots, \sigma(s_n))$$

so $a \cup \sigma(S)$ is algebraically dependent.

Since σ is a field embedding it is injective and so S and $\sigma(S)$ have the same cardinality. □

Lemma – Cardinality of polynomial rings
If A is a ring and S is a non-empty set of variables (algebraically independent over A) then

$$|A[S]| = |A| + |S| + |\aleph_0|$$

Proof. Since S is non-empty, the inequality

$$|A| + |S| + |\aleph_0| \leq |A[S]|$$

can easily be constructed. We induct (transfinitely) on S to show the other inequality, which suffices by Schröder–Bernstein. The cases we have are when S is a singleton, when $S = T \sqcup x$ and when S bijects with a limit ordinal.

If S is a singleton $\{x\}$ then we can partition $A[S] = A[x]$ by degree:

$$A[x] = \bigcup_{d \in \mathbb{N}} \{f \in A[x] \mid \deg f = d\}$$

For each d the set $\{f \in A[x] \mid \deg f = d\}$ bijects with A^d. When A is finite, A^d is also finite and so

$$|A[x]| \leq \aleph_0 \times \aleph_0 = \aleph_0 \leq |A| + |S| + |\aleph_0|$$

When A is infinite we have that $\left|A^d\right| = |A|$, so

$$|A[x]| \leq \aleph_0 \times |A| \leq |A| + |S| + |\aleph_0|$$

If S is of the form $T \sqcup x$ we can apply the above:

$$|A[S]| = |A[T][x]| = |A[T]| + |\{x\}| + \aleph_0$$

By the induction hypothesis on T we have the above has cardinality below

$$|A| + |T| + |\{x\}| + \aleph_0 = |A| + |S| + \aleph_0$$

If S bijects with a limit ordinal we index S by this bijection, which induces a well-ordering $<$ on S. We can then write

$$A[S] = \bigcup_{x \in S} A[S_{<x}]$$

Where the set $S_{<x}$ is the set of elements $s \in S$ such that $s < x$. By induction, for each $x \in S$ we have

$$|A[S_{<x}]| \leq |A| + |S_{<x}| + \aleph_0 \leq |A| + |S| + \aleph_0$$

Hence

$$|A[S]| \leq |S| \times (|A| + |S| + \aleph_0) \leq |A| + |S| + \aleph_0$$

□

Lemma – Cardinality of algebraically closed fields
If L is an algebraically closed field then it has cardinality $\aleph_0 + \mathrm{t.deg}(L)$.

Proof. Let S be a transcendence basis and call the minimal subfield K. Since L is algebraically closed it splits the seperable polynomials $x^n - 1$ for each n. Hence L is infinite. Also $S \subseteq L$ and so $\aleph_0 + \mathrm{t.deg}(L) \leq |L|$. For the other direction, since L is algebraic over $K(S)$ we have

$$L = \bigcup_{f \in I} \{a \in L \mid f = \min(a, K(S))\}$$

where $I \subseteq K(S)[x]$ is the set of irreducible monic polynomials over $K(S)$. Applying the following respectively: polynomials have finitely many roots in a field; $I \subseteq K(S)[x]$; the cardinality of polynomial rings; $K[S] \times K[S]$ surjects onto $K(S)$; case on whether or not $K[S]$ is infinite; the cardinality of polynomial rings again;

$$\begin{aligned}
|L| &\leq |I| \times \aleph_0 \leq |K(S)[x]| \times \aleph_0 \\
&\leq |K(S)| \times \aleph_0 \leq |K[S]| \times |K[S]| \times \aleph_0 \\
&= |K[S]| \times \aleph_0 \\
&\leq |K| + |S| + \aleph_0
\end{aligned}$$

Lastly $K = \mathbb{Q}$ or $\mathbb{F}_p$ so K is at most countable. and the whole cardinality is below $|S| + \aleph_0$. By Schröder–Bernstein we conclude $\aleph_0 + \mathrm{t.deg}(L) = |L|$. □

4.2 Locally finite fields and polynomial maps

Definition – Locally finite
We say that a field is locally finite if the minimal subfield generated by any *finite* subset is finite.

Proposition – Ax-Grothendieck for locally finite fields
Let L be a locally finite field. Then any injective polynomial map from $L^n \to L^n$ is surjective.

Proof. Let $b = (b_1, \ldots, b_n) \in L^n$. We find a subfield K such that $b \in K^n = f(K^n)$ Writing $f = (f_1, \ldots, f_n)$ for $f_i \in \Omega[x_1, \ldots, x_n]$ we can find $A \subseteq L$, the set of all the coefficients of all of the f_i. $A \cup \{b_1, \ldots, b_n\}$ is finite, so the subfield K generated by it is also finite.

The restriction $f\big|_{K^n}$ is injective and has image inside K^n since each polynomial has coefficients in K and is evaluated at an element of K^n. Hence $f\big|_{K^n}$ is an injective endomorphism of a finite set thus is surjective. We conclude that $b \in K^n = f(K^n)$. □

Lemma – Equivalences for locally finite over prime characteristic [4]
Let K be a field of characteristic p a prime. Then the following are equivalent:

1. K is locally finite.
2. $\mathbb{F}_p \to K$ is algebraic.
3. K embeds into an algebraic closure of $\mathbb{F}_p$.

Proof. 1. $\Rightarrow$ 2. Let $a \in K$. Then $\mathbb{F}_p(a)$ is the minimal subfield generated by a, and is finite by assumption. Finite field extensions are algebraic hence $F_p(a)$ (in particular a) is algebraic over $\mathbb{F}_p$.

2. $\Rightarrow$ 1. We show by induction that K is locally finite. Let S be a finite subset of K. If S is empty then $\mathbb{F}_p(S) = \mathbb{F}_p$ and so it is finite. If $S = T \cup s$ and $\mathbb{F}_p(T)$ is finite, then $s \in K$ is algebraic so by some basic field theory we can take the quotient by the minimal polynomial of s giving

$$\mathbb{F}_p(T)[x]/\min(s, \mathbb{F}_p(T)) \cong \mathbb{F}_p(S)$$

The left hand side is finite because it is a finite dimensional vector space over a finite field. Hence K is locally finite.

2. $\Leftrightarrow$ 3. These are the properties of algebraic closures. □

Corollary – Ax-Grothendieck for algebraic closure of finite fields
If Ω is an algebraic closure of a finite field then any injective polynomial map over Ω is surjective.

Proof. It suffices that Ω is locally finite. Any finite field is an algebraic extension over $\mathbb{F}_p$ where p is its prime characteristic. Hence its algebraic closure Ω is an algebraic extension over $\mathbb{F}_p$ and so it is locally finite. □

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