Model Theory

JLH

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Chapter 1

Pure Model Theory

1.1 Basics

This section follows Marker's book on Model Theory [2] with more emphasis on where things are happening, i.e. what signature we are working in, and some more general statements such as working with embeddings rather than subsets. For a quick introduction to model theory we recommend reading up to where morphisms and completeness are introduced. Then it is worth following routes to particular results such as quantifier elimination in algebraically closed fields or Morley Rank being Krull dimension, which may help to motivate the rest of the theory.

1.1.1 Signatures

Definition – First order language
We assume we have a tuple $\mathcal{L} = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathcal{V}, \{\neg\,, \vee\,, \forall, =, \top\})$ such that

- $|\mathcal{C}|, |\mathcal{F}|, |\mathcal{R}|$ each sufficiently large (say they have cardinality $\aleph_5$ or something).
- $|\mathcal{V}| = \aleph_0$. We index $\mathcal{V} = \{v_0, v_1, \dots\}$ using $\mathbb{N}$.
- $\mathcal{C}$,$\mathcal{F}$,$\mathcal{R}$, $\mathcal{V}$, $\{\neg\,, \vee\,, \forall, =\}$ do not overlap.

We call $\mathcal{L}$ the language and only really use it to get symbols to work with. Whenever we introduce new symbols to create larger signatures, we are pulling them out of this box.

Definition – Signature
In a language $\mathcal{L}$, a tuple $\Sigma = (C, F, n_\star, R, m_\star)$ is a signature[†] when

- $C \subseteq \mathcal{C}$. We call C the set of constant symbols.
- $F \subseteq \mathcal{F}$ and $n_\star : F \to \mathbb{N}$, which we call the function arity. We call F the set of function symbols.
- $R \subseteq \mathcal{R}$ and $m_\star : R \to \mathbb{N}$, which we call the relation arity. We call T the set of relation symbols.

Given a signature Σ, we may refer to its constant, function and relation symbol sets as $\Sigma_{\text{con}}, \Sigma_{\text{fun}}, \Sigma_{\text{rel}}$. We will always denote function arity using $n_\star$ and relation arity using $m_\star$.

[†] Many authors call Σ the language, but I have chosen this way of defining things instead.

Example. *The signature of rings will be used to define the theory of rings, the theory of integral domains, the theory of fields, and so on. The signature of binary relations will be used to define the theory of partial orders with the interpretation of the relation as $<$, to define the theory of equivalence relations with the interpretation of the relation as , and to define the theory* ZFC *with the relation interpreted as* $\in$.

Definition – Σ-terms

Given Σ a signature, its set of terms Σ_{ter} is inductively defined using three constructors:

| If $c \in \Sigma_{\text{con}}$ then c is a Σ-term.

| If $v_i \in \mathcal{V}$, v_i is a Σ-term.

| Given $f \in \Sigma_{\text{fun}}$ and a n_f-tuple of Σ-terms $t \in (\Sigma_{\text{ter}})^{n_f}$ then the symbol $f(t)$ is a Σ-term.

The terms v_i are called variables and will be referred to as elements of Σ_{var}.

Example. *In the signature of rings, terms will be multivariable polynomials over $\mathbb{Z}$ since they are sums and products of constant symbols $0, 1$ and variable symbols. In the signature of binary relations there are no constant or function symbols so the only terms are variables.*

Definition – Atomic formula, quantifier free formula

Given Σ a signature, its set of atomic Σ-formulas is defined as

| $\top$ is an atomic Σ-formula

| Given $t, s \in \Sigma_{\text{ter}}$, the string $(t = s)$ is an atomic Σ-formula.

| Given $r \in \Sigma_{\text{rel}}, t \in (\Sigma_{\text{ter}})^{m_r}$, the string $r(t)$ is an atomic Σ-formula.

This will be the Σ-formulas that are not built 'inductively'.

The set of quantifier free Σ-formulas is defined inductively:

| Given ϕ an atomic Σ-formula, ϕ is a quantifier free Σ-formula.

| Given ϕ a quantifier free Σ-formula, the string $\neg\phi$ is a quantifier free Σ-formula.

| Given ϕ, ψ both quantifier free Σ-formulas, the string $\phi \vee \psi$ is a quantifier free Σ-formula.

Notice quantifier free Σ-formulas are indeed Σ-formulas

Definition – Σ-formula, free variable

Given Σ a signature, its set of Σ-formulas Σ_{for} is inductively defined:

| Given ϕ a quantifier free Σ-formula, ϕ is a Σ-formula.

| Given $\phi \in \Sigma_{\text{for}}$ and $v_i \in \Sigma_{\text{var}}$, we take the replace all occurrences of v_i with an unused symbol such as z in the string $(\forall v_i, \phi)$ and call this a Σ-formula.

Shorthand for some Σ-formulas include

- $\bot := \neg\top$
- $\phi \wedge \psi := \neg((\neg\phi) \vee (\neg\psi))$
- $\phi \to \psi := (\neg\phi) \vee \psi$
- $\exists v, \phi := \neg(\forall v, \neg\phi)$

> The symbol z is meant to be a 'bounded variable', and will not be considered when we want to evaluate variables in formulas. Not bound variables are called 'free variables'.

Remark. There are two different uses of the symbol '$=$' from now on, and context will allow us to tell them apart. Similarly, logical symbols might be used in our 'higher language' and will not be confused with symbols from formulas.

Formulas should be thought of as propositions with some bits loose, namely the free variables, since it does not make any sense to ask if $x = x \vee x = a$ without saying what x you are taking (where x is a variable as a is a constant symbol, say). When there are no free variables we get what intuitively looks like a proposition, and we will call these particular formulas sentences.

For the sake of learning the theory the following two lemmas should be skipped. They are essentially algorithms that tell us what the variables in terms and what the free variables in formulas are. We include them for formality.

> **Proposition – Terms have finitely many variables**
> For any term $t \in \Sigma_{\text{ter}}$ there exists a finite subset $S \subseteq \mathbb{N}$ indexing the variables v_i occuring in t.

Proof. If there exists a finite subset T of $\mathbb{N}$ such that if v_i occurs in t then $i \in T$ then we can take the intersection of all such sets and have the finite set S we're interested in.

We prove existence of such a T using the inductive definition of t:

- If $t = c$ a constant symbol, then $T = \varnothing$ satisfies the above.
- If $t = v_i$ a variable symbol, then $T = \{i\}$ satisfies the above.
- If $t = f(t_0, \dots, t_{n_f})$, then by our induction hypothesis we have a T_i satisfying the condition for each t_i. Then $\cup_i T_i$ satisfies the condition for t.

□

> **Proposition – Formulas have finitely many *free* variables**
> Given $\phi \in \Sigma_{\text{for}}$, there exists a finite $S \subseteq \mathbb{N}$ indexing the *free* variables v_i occuring in ϕ.

Proof. Like in the terms case, we only need to show that there exists a T If v_i occurs freely in ϕ then $i \in T$. We induct on what ϕ is, noting that until the last case there are no quantifiers being considered so the variables in question are free:

- If ϕ is $\top$ then it has no variables.
- If ϕ is $t = s$, then we have S_t, S_s indexing the (free) variables of t and s by the previous proposition, and so we can pick $T = S_t \cup S_s$.
- If ϕ is $r(t_0, \dots, t_{m_r})$, then for each t_i we have S_i indexing the variables of t_i. Hence we can pick $T = \cup_i S_i$.
- If ϕ is $\neg\,\psi$, then by the induction hypothesis we have T satisfying the above conditions for ψ. Pick this T for ϕ.
- If ϕ is $\psi \vee \chi$ then by the induction hypothesis we have T_ψ, T_χ satisfying the above conditions for ψ and χ. We take T to be the union of indexing sets for ψ and χ.
- If ϕ is $\forall v_i, \psi$ with v_i substituted for z, then by the induction hypothesis we have T_ψ satisfying the above conditions for ψ. Take $T = T_\psi \setminus \{v_i\}$. In fact taking T_ψ itself works as well.

□

NOTATION (SUBSTITUTING TERMS FOR VARIABLES). *If a Σ-formula ϕ has a free variable v_i then to remind ourselves of the variable we can write $\phi = \phi(v_i)$ instead.*

If ϕ has S indexing its free variables and $t \in (\Sigma_{\text{ter}})^S$, then we write $\phi(t)$ to mean ϕ with t_i substituted for each v_i. We can show by induction on terms and formulas that this is still a Σ-formula.

Definition – Σ-structure, interpretation

Given a signature Σ, a set M and interpretation functions

- $\star^{\mathcal{M}}_{\Sigma_{\text{con}}} : \Sigma_{\text{con}} \to M$
- $\star^{\mathcal{M}}_{\Sigma_{\text{fun}}} : \Sigma_{\text{fun}} \to (M^{n_\star} \to M)$
- $\star^{\mathcal{M}}_{\Sigma_{\text{rel}}} : \Sigma_{\text{rel}} \to \mathcal{P}(M^{m_\star})$

we say that $\mathcal{M} := (M, \star^{\mathcal{M}}_{\Sigma})$ is a Σ-structure. The latter functions two are dependant types since the powers $n_\star, m_\star$ depend on the function and relation symbols given. The class (or set or whatever) of Σ-structures is denoted Σ_{str}. Given only the Σ-structure $\mathcal{M}$, we call its underlying carrier set M as $\mathcal{M}_{\text{car}}$.

We write $\star^{\mathcal{M}}_{\Sigma}$ to represent any of the three interpretation functions when the context is clear. Given $c \in \Sigma_{\text{con}}, f \in \Sigma_{\text{fun}}, r \in \Sigma_{\text{rel}}$, we might write the 'interpretations' of these symbols as any of the following

$$c^{\mathcal{M}}_{\Sigma_{\text{con}}} = c^{\mathcal{M}}_{\Sigma} = c^{\mathcal{M}} \qquad f^{\mathcal{M}}_{\Sigma_{\text{fun}}} = f^{\mathcal{M}}_{\Sigma} = f^{\mathcal{M}} \qquad r^{\mathcal{M}}_{\Sigma_{\text{rel}}} = r^{\mathcal{M}}_{\Sigma} = r^{\mathcal{M}}$$

The structures in a signature will become the models which we are interested in, in particular structures will be a models of theories. For example $\mathbb{Z}$ is a structure in the signature of rings, and models the theory of rings but not the theory of fields. In the signature of binary relations, $\mathbb{N}$ with the usual ordering $\leq$ is a structure that models of the theory of partial orders but not the theory of equivalence relations.

Definition – Interpretation of terms

Given a signature Σ, a Σ-structure $\mathcal{M}$ and a Σ-term t, let S be the unique set indexing the variables of t. Then there exists a unique induced map $t^{\mathcal{M}}_{T} : \mathcal{M}^S_{\text{car}} \to \mathcal{M}_{\text{car}}$, that commutes with the interpretation of constants and functions†. We then refer to this map as *the* interpretation of the term t. This in turn defines a dependant Π-type

$$\star^{\mathcal{M}}_{T} : \Sigma_{\text{ter}} \to (\mathcal{M}^{S_\star}_{\text{car}} \to \mathcal{M}_{\text{car}})$$

†See this more precisely stated in the proof

Proof. To define a map $t^{\mathcal{M}}_{T} : M^S \to M$ for each t we use the inductive definition of $t \in \Sigma_{\text{ter}}$. If M is empty we define $t^{\mathcal{M}}_{T}$ as the empty function. Otherwise let $a \in M^S$:

- If $t = c \in \Sigma_{\text{con}}$ then define $t^{\mathcal{M}}_{T} : a \mapsto c^{\mathcal{M}}_{C}$, the constant map. This type checks since $S = \varnothing$ therefore $t^{\mathcal{M}}_{T} : M^0 \to M$.
- If $t = v_i \in \Sigma_{\text{var}}$ then define $t^{\mathcal{M}}_{T} : a \mapsto a$, the identity. This type checks since $|S| = 1$.
- If $t = f(s)$ for some $f \in \Sigma_{\text{fun}}$ and $s \in (\Sigma_{\text{var}})^{n_f}$ then define $t^{\mathcal{M}}_{T} : a \mapsto f^{\mathcal{M}}_{F}(s^{\mathcal{M}}_{T}(a))$. This type checks since s has the same number of variables as t.

By definition, this map commutes with the interpretation of constants and functions, i.e.

$$\begin{tikzcd}
\Sigma_{\text{con}} \arrow[r, "\subseteq"] \arrow[dr, "\star^{\mathcal{M}}_{\Sigma_{\text{con}}}"'] & \Sigma_{\text{ter}} \arrow[d, "\star^{\mathcal{M}}_{T}"] \\
& (\mathcal{M}^{S_\star} \to \mathcal{M})
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod_{f \in \Sigma_{\text{fun}}} (\Sigma_{\text{ter}})^{n_f} \arrow[rrr, bend left] \arrow[drrr, bend right] & s \in (\Sigma_{\text{ter}})^{n_f} \arrow[r, mapsto] \arrow[dr] & f(s) & \Sigma_{\text{ter}} \arrow[d, "\star^{\mathcal{M}}_{T}"] \\
& & f^{\mathcal{M}}_{\Sigma_{\text{fun}}}(s^{\mathcal{M}}_{T}(\star)) & (\mathcal{M}^{S_\star} \to \mathcal{M})
\end{tikzcd}$$

The map is clearly unique. □

Where there is no ambiguity, we write $t^{\mathcal{M}}_{T} = t^{\mathcal{M}}$. Furthermore, if we have a tuple $t \in (\Sigma_{\text{ter}})^k$, then we write $t^{\mathcal{M}}_{T} := (t^{\mathcal{M}}_0, \cdots, t^{\mathcal{M}}_k)$

Definition – Sentences and satisfaction

Let Σ be a signature and ϕ a Σ-formula. Let $S \subseteq \mathbb{N}$ index the free variables of ϕ. We say $\phi \in \Sigma_{\text{for}}$ is a Σ-sentence when S is empty.

If Σ has no constant symbols then $\varnothing$ is a Σ-structure by interpreting functions and relations as $\varnothing$. Then given a Σ-sentence (not any Σ-formula) ϕ we want to define $\varnothing \vDash_\Sigma \phi$ using the inductive definition of ϕ:

- If ϕ is $\top$ then $\varnothing \vDash_\Sigma \phi$.
- ϕ cannot be $t = s$ since this contains a constant symbol or a variable.

- If ϕ is $\neg\psi$ for some $\psi \in \Sigma_{\text{for}}$, then $\varnothing \vDash_\Sigma \phi$ when $\varnothing \nvDash_\Sigma \psi$
- If ϕ is $(\psi \vee \chi)$, then $\varnothing \vDash_\Sigma \phi$ when $\varnothing \vDash_\Sigma \psi$ or $\varnothing \vDash_\Sigma \chi$.
- If ϕ is of the form $\forall v, \psi$, then $\varnothing \vDash_\Sigma \phi$.

Let $\mathcal{M}$ be a Σ-structure with non-empty carrier set. Then given $a \in (\mathcal{M}_{\text{car}})^S$, we want to define $\mathcal{M} \vDash_\Sigma \phi(a)$:

- If ϕ is $\top$ then $\mathcal{M} \vDash_\Sigma \phi$.[†]
- If ϕ is $t = s$ then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)$.
- If ϕ is $r(t)$, where $r \in \Sigma_{\text{rel}}$ and $t \in (\Sigma_{\text{ter}})^{m_r}$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $t^{\mathcal{M}}(a) \in r^{\mathcal{M}}$.

- If ϕ is $\neg\psi$ for some $\psi \in \Sigma_{\text{for}}$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $\mathcal{M} \nvDash_\Sigma \psi(a)$
- If ϕ is $(\psi \vee \chi)$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ when $\mathcal{M} \vDash_\Sigma \psi(a)$ or $\mathcal{M} \vDash_\Sigma \chi(a)$.
- If ϕ is $(\forall v, \psi(a)) \in \Sigma_{\text{for}}$, then $\mathcal{M} \vDash_\Sigma \phi(a)$ if for any $b \in \mathcal{M}_{\text{car}}$, $\mathcal{M} \vDash_\Sigma \psi(a)(b)$.

We say $\mathcal{M}$ satisfies $\phi(a)$.

[†] We can omit the a when there are no free variables. Formally this a is the unique element in $\mathcal{M}^{\varnothing}$ given by the empty set.

Remark. Any Σ-structure $\mathcal{M}$ satisfies $\top$ and does not satisfy $\bot$. The empty set satisfies things of the form $\forall, \ldots$ but not $\exists, \ldots$, as we would expect. Note that for c a tuple of constant symbols $\mathcal{M} \vDash_\Sigma \phi(c)$ is the same thing as $\mathcal{M} \vDash_\Sigma \phi(c^{\mathcal{M}})$.

1.1.2 Theories and Models

Definition – Σ-theory

T is an Σ-theory when it is a subset of Σ_{for} such that all elements of T are Σ-sentences. We denote the set of Σ-theories as Σ_{the}.

Definition – Models

Given an Σ-structure $\mathcal{M}$ and Σ-theory T, we write $\mathcal{M} \vDash_\Sigma T$ and say $\mathcal{M}$ *is a Σ-model of T* when for all $\phi \in T$ we have $\mathcal{M} \vDash_\Sigma \phi$.

Example (The empty signature and theory). *$\Sigma_\varnothing = (\varnothing, \varnothing, n_\star, \varnothing, m_\star)$ is the empty signature. (We pick the empty functions for $n_\star, m_\star$.) The empty $\Sigma_\varnothing$-theory is given by $\varnothing$. Notice that any set is a $\Sigma_\varnothing$-structure and moreover a $\Sigma_\varnothing$-model of the empty $\Sigma_\varnothing$-theory.*

Example. *In the signature of rings, the rings axioms will be the theory of rings and structures satisfying the theory will be rings. The theory of ZFC consists of the ZFC axioms and a model of ZFC would be thought of as the 'class of all sets'.*

Definition – Consequence

Given a Σ-theory T and a Σ-sentence ϕ, we say ϕ is a consequence of T and say $T \vDash_\Sigma \phi$ when for all Σ-models $\mathcal{M}$ of T, we have $\mathcal{M} \vDash_\Sigma \phi$.

Remark. We have to be a bit careful when we go from something like $\mathcal{M} \vDash_\Sigma \phi(a)$ to deducing something about T. This is because there might not exist a Σ-constant c such that $c^{\mathcal{M}} = a$, it only makes sense to write $T \vDash_\Sigma \phi$ if ϕ is a *sentence*.

We have included the empty structure in this definition, often this will not make a difference since the signature could have constant symbols or the theory T could have an 'existential' sentence in it.

Exercise (Logical consequence). *Let T be a Σ-theory and ϕ and ψ be Σ-sentences. Show that the following are equivalent:*

- $T \vDash_\Sigma \phi \to \psi$
- $T \vDash_\Sigma \phi$ *implies* $T \vDash_\Sigma \psi$.

Definition – Consistent theory

A Σ-theory T is consistent if either of the following equivalent definitions hold:

- There does not exists a Σ-sentence ϕ such that $T \vDash_\Sigma \phi$ and $T \vDash_\Sigma \neg\phi$.
- There exists a Σ-model of T.

Thus the definition of consistent is intuitively 'T does not lead to a contradiction'.

A theory T is finitely consistent if all finite subsets of T are consistent. This will turn out to be another equivalent definition, given by the compactness theorem.

Proof. We show that the two definitions are equivalent. ($\Rightarrow$) Suppose no model exists. Take ϕ to be the Σ-sentence $\top$. Hence all Σ-models of T satisfy $\top$ and $\bot$ (there are none) so $T \vDash_\Sigma \top$ and $T \vDash_\Sigma \bot$. ($\Leftarrow$) Suppose T has a Σ-model $\mathcal{M}$ and $\mathcal{M} \vDash_\Sigma \phi$ and $\mathcal{M} \vDash_\Sigma \neg\phi$. This implies $\mathcal{M} \vDash_\Sigma \phi$ and $\mathcal{M} \nvDash_\Sigma \phi$, a contradiction. $\square$

1.1.3 The Compactness Theorem

Read ahead for the statement of the Compactness Theorem. The first two parts of the theorem are easy to prove. This chapter focuses on proving the final part.

Definition – Witness property

Given a signature Σ and a Σ-theory T, we say that Σ satisfies the witness property when for any Σ-formula ϕ with exactly one free variable v, there exists $c \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma (\exists v, \phi(v)) \to \phi(c)$.

This says that if for all Σ-model $\mathcal{M}$ of T, there exists an element $a \in \mathcal{M}$ such that $\mathcal{M} \vDash_\Sigma \phi(m)$ then there exists a constant symbol c of Σ such that $\phi(c^{\mathcal{M}})$ is true.

Definition – Maximal theory

A Σ theory T is Σ-maximal if for any Σ-formula ϕ, if ϕ is a Σ-sentences then $\phi \in T$ or $\neg\phi \in T$.

Proposition – Maximum property

Given a Σ-maximal and finitely consistent theory T and a Σ-sentence ϕ,

$$T \vDash_\Sigma \phi \text{ if and only if } \phi \in T \text{ if and only if } \neg\phi \notin T \text{ if and only if } \nvDash_\Sigma \neg\phi$$

Proof. First note that by maximality and finite consistency if $\phi, \neg\phi \in T$ then we have a finite subset $\{\phi, \neg\phi\} \subseteq T$, which is false. Hence

$$\phi \in T \Leftrightarrow \neg\phi \notin T$$

We prove the first if and only if and deduce the third by replacing ϕ with $\neg\phi$. ($\Rightarrow$) Suppose $T \vDash_\Sigma \phi$. Since T is Σ-maximal, we have $\phi \in T$ or $\neg\phi \in T$. If $\neg\phi \in T$ then we have a finite subset $\{\phi, \neg\phi\} \subseteq T$. Hence T is not finitely consistent, thus the second case is false. ($\Leftarrow$) Suppose $\phi \in T$. Case on $T \vDash_\Sigma \phi$ or $T \nvDash_\Sigma \phi$. If $T \nvDash_\Sigma \phi$ then there exists $\mathcal{N}$ a Σ-model of T such that $N \nvDash_\Sigma \phi$. But $\mathcal{N} \vDash_\Sigma \phi$ since $\phi \in T$. Thus the second case is false. ☐

Notation (Ordering signatures). *We write $\Sigma \leq \Sigma(*)$ for two signatures if $\Sigma_{\text{con}} \subseteq \Sigma(*)_{\text{con}}$, $\Sigma_{\text{fun}} \subseteq \Sigma(*)_{\text{fun}}$ and $\Sigma_{\text{rel}} \subseteq \Sigma(*)_{\text{rel}}$.*

For the sake of formality we include the following two lemmas, neither of which are particularly inspiring or significant, but they do allow us to move freely between signatures.

Lemma – Moving models down signatures

Given two signatures such that $\Sigma \leq \Sigma(*)$ and $\mathcal{N}$ a $\Sigma(*)$-structure we can make $\mathcal{M}$ a Σ-structure such that

1. $\mathcal{M}_{\text{car}} = \mathcal{N}_{\text{car}}$
2. They have the same interpretation on Σ.
3. For any Σ-formula ϕ with free variables indexed by S and any $a \in \mathcal{M}^S$

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$

4. If T is a Σ-theory and $T(*)$ is a $\Sigma(*)$-theory such that $T \subseteq T(*)$ and $\mathcal{N}$ a $\Sigma(*)$-model of $T(*)$, then $\mathcal{M}$ is a Σ-model of T.

Technically the new structure is not $\mathcal{N}$, but for convenience we will write $\mathcal{N}$ to mean either of the two and let subscripts involving Σ and $\Sigma(*)$ describe which one we mean.

Proof. We let the carrier set be the same and define $\star_\Sigma^\mathcal{M}$ by restriction:

- $\star_{\Sigma_{\text{con}}}^\mathcal{M}$ is the restriction of $\star_{\Sigma(*)_{\text{con}}}^\mathcal{N}$ to Σ_{con}
- $\star_{\Sigma_{\text{fun}}}^\mathcal{M}$ is the restriction of $\star_{\Sigma(*)_{\text{fun}}}^\mathcal{N}$ to Σ_{fun}
- $\star_{\Sigma_{\text{rel}}}^\mathcal{M}$ is the restriction of $\star_{\Sigma(*)_{\text{rel}}}^\mathcal{N}$ to Σ_{rel}

We will need that for any Σ-term t with variables indexed by S, the interpretation of terms is equal: $t_\Sigma^\mathcal{M} = t_{\Sigma(*)}^\mathcal{N}$. Indeed:

- If t is a constant then $t_\Sigma^\mathcal{M} = c_\Sigma^\mathcal{M} = c_{\Sigma(*)}^\mathcal{N} = t_{\Sigma(*)}^\mathcal{N}$
- If t is a variable then $t_\Sigma^\mathcal{M} = \text{id}_\mathcal{M} = \text{id}_\mathcal{N} = t_{\Sigma(*)}^\mathcal{N}$
- If t is $f(s)$ then by induction $t_\Sigma^\mathcal{M} = f_\Sigma^\mathcal{M}(s_\Sigma^\mathcal{M}) = f_{\Sigma(*)}^\mathcal{N}(s_{\Sigma(*)}^\mathcal{N}) = t_{\Sigma(*)}^\mathcal{N}$

Let ϕ be a Σ-formula with variables indexed by $S \subseteq \mathbb{N}$. Let a be in $\mathcal{M}^S$. Case on ϕ to show that $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$:

- If ϕ is $\top$ then both satisfy ϕ.
- If ϕ is $t = s$ then
$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow t_\Sigma^\mathcal{M} = s_\Sigma^\mathcal{M} \Leftrightarrow t_{\Sigma(*)}^\mathcal{M} = s_{\Sigma(*)}^\mathcal{M} \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$
Since the interpretation of terms are equal from above.
- If ϕ is $r(t)$ then by how we defined $r_\Sigma^\mathcal{M}$ and since interpretation of terms are equal
$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow t_\Sigma^\mathcal{M}(a) \in r_\Sigma^\mathcal{M} \Leftrightarrow t_{\Sigma(*)}^\mathcal{N}(a) \in r_{\Sigma(*)}^\mathcal{N} \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$
- If ϕ is $\neg\psi$ then using the induction hypothesis
$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \nvDash_\Sigma \psi(a) \Leftrightarrow \mathcal{N} \nvDash_{\Sigma(*)} \psi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$
- If ϕ is $\psi \vee \chi$ then using the induction hypothesis
$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \vDash_\Sigma \psi(a) \text{ or } \mathcal{M} \vDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \psi(a) \text{ or } \mathcal{N} \vDash_{\Sigma(*)} \chi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$$
- If ϕ is $\forall v, \psi$ then
$$\begin{aligned}\mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow \forall b \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \psi(a,b) \\ &\Leftrightarrow \forall b \in \mathcal{M}, \mathcal{N} \vDash_{\Sigma(*)} \psi(a,b) && \text{by the induction hypothesis} \\ &\Leftrightarrow \forall b \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma(*)} \psi(a,b) && \text{by the induction hypothesis} \\ &\Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)\end{aligned}$$

Hence $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(a)$.

Suppose $T \subseteq T(*)$ are respectively Σ and $\Sigma(*)$-theories and $\mathcal{N} \vDash_{\Sigma(*)} T(*)$. If $\phi \in T \subseteq T(*)$ then by the previous part, $\mathcal{N} \vDash_{\Sigma(*)} \phi$ implies $\mathcal{M} \vDash_\Sigma \phi$. Hence $\mathcal{M} \vDash_\Sigma T$. $\square$

Lemma – Moving models and theories up signatures
Suppose $\Sigma \leq \Sigma(*)$.

1. Suppose $\mathcal{M}$ is a Σ-model of Σ-theory T. Then if there exists $\mathcal{M}(*)$ a $\Sigma(*)$-structure whose carrier set is the same as $\mathcal{M}$ and whose interpretation agrees with $\star_\Sigma^\mathcal{M}$ on constants, functions and relations of Σ, then $\mathcal{M}(*)$ is a $\Sigma(*)$-model of T.

2. Suppose T is a Σ-theory and ϕ is a Σ-sentence such that $T \vDash_\Sigma \phi$. Then $T \vDash_{\Sigma(*)} \phi$.

Again, if we have constructed such a $\mathcal{M}(*)$ from $\mathcal{M}$ we tend to just refer to it as $\mathcal{M}$ and let subscripts involving Σ and $\Sigma(*)$ describe which one we mean.

Proof.

1. Suppose $\mathcal{M} \vDash_\Sigma T$. Let $\phi \in T$. To show that $\mathcal{M}(*) \vDash_{\Sigma(*)} \phi$ we first we prove a useful claim: if $t \in \Sigma_{\text{ter}}$ with no variables then $t^{\mathcal{M}(*)}_{\Sigma(*)} = t^{\mathcal{M}}_{\Sigma}$. Case on what t is:
   - If t is a constant symbol c in Σ_{con}, then since $\star^{\mathcal{M}(*)}_{\Sigma(*)} = \star^{\mathcal{M}}_{\Sigma}$ on Σ,
   $$t^{\mathcal{M}(*)}_{\Sigma(*)} = c^{\mathcal{M}(*)}_{\Sigma(*)} = c^{\mathcal{M}}_{\Sigma} = t^{\mathcal{M}}_{\Sigma}$$
   - If t is a variable then it has one variable, thus false.
   - If t is $f(s)$ then since $\star^{\mathcal{M}(*)}_{\Sigma(*)} = \star^{\mathcal{M}}_{\Sigma}$ on Σ,
   $$t^{\mathcal{M}(*)}_{\Sigma(*)} = f(s)^{\mathcal{M}(*)}_{\Sigma(*)} = f^{\mathcal{M}(*)}_{\Sigma(*)}(s^{\mathcal{M}(*)}_{\Sigma(*)}) = f^{\mathcal{M}}_{\Sigma}(s^{\mathcal{M}}_{\Sigma}) = f(s)^{\mathcal{M}}_{\Sigma} = t^{\mathcal{M}}_{\Sigma}$$

   Case on what ϕ is (ϕ has no variables):
   - If ϕ is $\top$ then it is satisfied.
   - If ϕ is $t = s$, then by the claim above,
   $$t^{\mathcal{M}(*)}_{\Sigma(*)} = t^{\mathcal{M}}_{\Sigma} = s^{\mathcal{M}}_{\Sigma} = s^{\mathcal{M}(*)}_{\Sigma(*)}$$
   - If ϕ is $r(t)$, then by the claim above and the fact that relations are interpreted the same way,
   $$t^{\mathcal{M}(*)}_{\Sigma(*)} = t^{\mathcal{M}}_{\Sigma} \in r^{\mathcal{M}}_{\Sigma} = r^{\mathcal{M}(*)}_{\Sigma(*)}$$
   - If ϕ is $\neg\psi$ then using the induction hypothesis on ψ,
   $$\mathcal{M} \vDash_\Sigma \phi \;\Leftrightarrow\; \mathcal{M} \nvDash_\Sigma \psi \;\Leftrightarrow\; \mathcal{M}(*) \nvDash_{\Sigma(*)} \psi \;\Leftrightarrow\; \mathcal{M}(*) \vDash_{\Sigma(*)} \phi$$
   - If ϕ is $\psi \vee \chi$ then using the induction hypothesis on ψ and χ,
   $$\mathcal{M} \vDash_\Sigma \phi \Leftrightarrow \mathcal{M} \vDash_\Sigma \psi \text{ or } \mathcal{M} \vDash_\Sigma \chi \Leftrightarrow \mathcal{M}(*) \vDash_{\Sigma(*)} \psi \text{ or } \mathcal{M}(*) \vDash_{\Sigma(*)} \chi \Leftrightarrow \mathcal{M}(*) \vDash_{\Sigma(*)} \phi$$
   - If ϕ is $\forall v, \psi(v)$ and $a \in \mathcal{M}(*) = \mathcal{M}$ then using the induction hypothesis on ψ, $\mathcal{M} \vDash_\Sigma \psi(a) \Rightarrow \mathcal{M}(*) \vDash_{\Sigma(*)} \psi(a)$. Hence $\mathcal{M}(*) \vDash_{\Sigma(*)} \phi$.

   Thus $\mathcal{M}(*)$ is a $\Sigma(*)$-model of T.
2. Suppose $T \vDash_\Sigma \phi$. If $\mathcal{M}(*) \vDash_{\Sigma(*)} T$ then by moving $\mathcal{M}(*)$ down to Σ, we have a corresponding $\mathcal{M} \vDash_\Sigma T$ whose carrier set is the same as $\mathcal{M}(*)$. Hence $\mathcal{M} \vDash_\Sigma \phi$. Naturally the models agree on interpretation of constants, functions and relations of Σ and so $\mathcal{M}(*) \vDash_{\Sigma(*)} \phi$ by the previous part.

$\square$

The following lemma is the bulk of the proof of the compactness theorem.

Lemma – Henkin construction

Let Σ be a signature. Let $0 < \kappa$ be a cardinal such that $|\Sigma_{\text{con}}| \leq \kappa$. If a Σ-theory T

- has the witness property
- is Σ-maximal
- is finitely consistent

then it has a non-empty Σ-model $\mathcal{M}$ such that $|\mathcal{M}| \leq \kappa$.

Proof. Without loss of generality Σ is non-empty, since we can add a constant symbol to Σ, see T as a theory in that signature, and then take the model we make of T back down to being a Σ-model of T. **The Σ-structure**: Consider quotienting Σ_{con} by the equivalence relation $c \sim d$ if and only if $T \vDash_\Sigma c = d$. Let $\pi : \Sigma_{\text{con}} \to \Sigma_{\text{con}}/\sim$. This defines a non-empty Σ-structure $\mathcal{M}$ in the following way:

1. We let the carrier set be the image of the quotient. We let the constant symbols be interpreted as their equivalence classes: $\star^{\mathcal{M}}_{\Sigma_{\text{con}}} = \pi$. We now have the desired cardinality for $\mathcal{M}$:[1] $|\mathcal{M}| \leq |\Sigma_{\text{con}}| \leq \kappa$ and Σ_{con} is non-empty so $\mathcal{M}$ is non-empty.

2. To interpret functions we must use the witness property. Given $f \in \Sigma_{\text{fun}}$ and $\pi(c) \in \mathcal{M}^{n_f}$ (π is surjective), we obtain by the witness property some $d \in \Sigma_{\text{con}}$ such that
$$T \vDash_\Sigma (\exists v, f(c) = v) \to (f(c) = d)$$
We define $f^{\mathcal{M}}$ to map $\pi(c) \mapsto \pi(d)$.

   To show that $f^{\mathcal{M}}$ is well-defined, let $c_0, c_1 \in (\Sigma_{\text{con}})^{n_f}$ such that $\pi(c_0) = \pi(c_1)$ and suppose $\pi(d_0), \pi(d_1)$ are their images. It suffices that $\pi(d_0) = \pi(d_1)$, i.e. $T \vDash_\Sigma d_0 = d_1$. Indeed, let Σ-structure $\mathcal{N}$ be a model of T. Then
$$\mathcal{N} \vDash_\Sigma (\exists v, f(c_0) = v) \Rightarrow \mathcal{N} \vDash_\Sigma f(c_0) = d_0$$
Similarly $f(c_1) = d_1$. Hence
$$d_0^{\mathcal{N}} = f^{\mathcal{N}}(c_0^{\mathcal{N}}) = f^{\mathcal{N}}(c_1^{\mathcal{N}}) = d_1^{\mathcal{N}}$$
and $N \vDash_\Sigma d_0 = d_1$. Hence $T \vDash_\Sigma d_0 = d_1$.

3. Let $r \in \Sigma_{\text{rel}}$. We define $r^{\mathcal{M}} := \{\pi(c) \mid c \in (\Sigma_{\text{con}})^{m_r} \wedge T \vDash_\Sigma r(c)\}$

Hence $\mathcal{M}$ is a Σ-structure. We want to show that $\mathcal{M}$ is a Σ-model of T.

Terms: to show that $\mathcal{M}$ satisfies formulas of T we first need that interpretation of terms is working correctly. Claim: if $t \in \Sigma_{\text{ter}}$ with variables indexed by S, $d \in \Sigma_{\text{con}}$ and c is in $(\Sigma_{\text{con}})^S$ then $T \vDash_\Sigma t(c) = d$ if and only if $t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}}$. i.e. $\mathcal{M} \vDash_\Sigma t(c) = d$. Case on what t is:

- If t is a constant symbol, $T \vDash_\Sigma t(c) = d$ if and only if $T \vDash_\Sigma t = d$ if and only if $\pi(t) = \pi(d)$ if and only if $t^{\mathcal{M}} = t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}}$.

- Suppose $t \in \Sigma_{\text{var}}$, then it suffices to show that $T \vDash_\Sigma c = d$ if and only if $c^{\mathcal{M}} = d^{\mathcal{M}}$, which we have already done above.

- With the induction hypothesis, suppose $t = f(s)$, where $f \in \Sigma_{\text{fun}}$ and $s \in (\Sigma_{\text{ter}})^{n_f}$. ($\Rightarrow$) If we can find $e = (e_1, \dots e_{n_f}) \in (\Sigma_{\text{con}})^{n_f}$ such that each $T \vDash_\Sigma s_i(c) = e_i$ and $f(e)^{\mathcal{M}} = d^{\mathcal{M}}$, then we have $s_i^{\mathcal{M}}(c^{\mathcal{M}}) = e_i^{\mathcal{M}}$ and so
$$t^{\mathcal{M}}(c^{\mathcal{M}}) = (f(s))^{\mathcal{M}}(c^{\mathcal{M}}) = f^{\mathcal{M}}(s^{\mathcal{M}}(c^{\mathcal{M}})) = f^{\mathcal{M}}(e^{\mathcal{M}}) = f(e)^{\mathcal{M}} = d^{\mathcal{M}}$$
Indeed, using the witness property n_f times we can construct e. Suppose we have by induction $e_1, \dots, e_{i-1} \in \Sigma_{\text{con}}$ such that for each $j < i$, they satisfy
$$T \vDash_\Sigma \exists x_{j+1}, \dots, \exists x_{n_f}, f(e_1, \dots, e_j, x_{j+1}, \dots, x_{n_f}) = d \wedge e_j = s_j(c)$$
For each i let ϕ_i be the formula
$$\exists x_{i+1}, \dots, \exists x_{n_f}, f(e_1, \dots, e_{i-1}, v, x_{i+1}, \dots, x_{n_f}) = d \wedge v = s_i(c)$$
with a single variable v. Then by the witness property, there exists an $e_i \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma \exists v, \phi_i(v) \to \phi_i(e_i)$. To complete the induction we show that $T \vDash_\Sigma \phi_i(e_i)$. Then we will be done since each $\phi_i(e_i)$ will give us $T \vDash_\Sigma s_i(c) = e_i$ and the last ϕ_{n_f} gives $f(e)^{\mathcal{M}} = d^{\mathcal{M}}$.

[1] From this point onwards when it is obvious what we mean we write $\mathcal{M}$ for $\mathcal{M}_{\text{car}}$.

To this end, let $\mathcal{N}$ be a model of T. Then $\mathcal{N} \vDash_\Sigma \exists v, \phi_i(v) \to \phi_i(e_i)$. By assumption $T \vDash_\Sigma t(c) = d$ so $t^{\mathcal{N}}(c^{\mathcal{N}}) = d^{\mathcal{N}}$, hence

$$f^{\mathcal{N}}(s_1^{\mathcal{N}}(c), \dots, s_{n_f}^{\mathcal{N}}(c)) = d^{\mathcal{N}}$$

Taking v to be $s_i(c)$ and x_{i+k} to be $s_{i+k}(c)$ we have that $\mathcal{N} \vDash_\Sigma \exists v, \phi_i(v)$. Hence $\mathcal{N} \vDash_\Sigma \phi_i(e_i)$. Thus we have $T \vDash_\Sigma \phi_i(e_i)$.

($\Leftarrow$) Note that each $(s_i(c))^{\mathcal{M}} = e_i^{\mathcal{M}}$ for some $e_i \in \Sigma_{\text{con}}$ since π is surjective. By the induction hypothesis for each i we have $T \vDash_\Sigma s_i(c) = e_i$. Hence

$$f(e)^{\mathcal{M}} = f^{\mathcal{M}}(e^{\mathcal{M}}) = f^{\mathcal{M}}((s(c))^{\mathcal{M}}) = f^{\mathcal{M}}(s^{\mathcal{M}}(c^{\mathcal{M}})) = t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}}$$

Hence $\pi(f(e)) = \pi(d)$ and $T \vDash_\Sigma f(e) = d$. It follows that $T \vDash_\Sigma t(c) = d$.

Thus $T \vDash_\Sigma t(c) = d \Leftrightarrow t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}}$.

Formulas: now we can show that $\mathcal{M} \vDash_\Sigma T$. Since for all $\phi \in T$ we have $T \vDash_\Sigma \phi$, it suffices to show that for all Σ-sentences $\phi, T \vDash_\Sigma \phi$ implies $\mathcal{M} \vDash_\Sigma \phi$. We prove a stronger statement which will be needed for the induction: for all Σ-formulas ϕ with variables indexed by S and $c \in (\Sigma_{\text{con}})^S$,

$$T \vDash_\Sigma \phi(c) \Leftrightarrow \mathcal{M} \vDash_\Sigma \phi(c^{\mathcal{M}})$$

We case on what ϕ is:

- Case ϕ is $\top$: all Σ-structures satisfy $\top$.
- Case ϕ is $t = s$: ($\Rightarrow$) Apply the witness property to $t(c) = v$, obtaining $d \in \Sigma_{\text{con}}$ such that $T \vDash_\Sigma (\exists v, t(c) = v) \to t(c) = d$. Since clearly $T \vDash_\Sigma \exists v, t(c) = v$ (take any model and it has an interpretation of $t(c)$) we have $T \vDash_\Sigma t(c) = d$. Also by assumption $T \vDash_\Sigma t(c) = s(c)$ so it follows that $T \vDash_\Sigma s(c) = d$. Using the claim from before for terms, we obtain $t^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}} = s^{\mathcal{M}}(c^{\mathcal{M}})$. Hence $\mathcal{M} \vDash_\Sigma t(c^{\mathcal{M}}) = s(c^{\mathcal{M}})$.

  ($\Leftarrow$) If $\mathcal{M} \vDash_\Sigma t(c^{\mathcal{M}}) = s(c^{\mathcal{M}})$ then since π is surjective, there exists $d \in \Sigma_{\text{con}}$ such that

  $$t^{\mathcal{M}}(c^{\mathcal{M}}) = s^{\mathcal{M}}(c^{\mathcal{M}}) = d^{\mathcal{M}}$$

  Using the claim for terms we obtain $T \vDash_\Sigma t(c) = d$ and $T \vDash_\Sigma s(c) = d$. It follows that $T \vDash_\Sigma t(c) = s(c)$.
- Case ϕ is $r(t)$: ($\Rightarrow$) Suppose $T \vDash_\Sigma r(t(c))$. By induction, apply the witness property m_r times to the formulas

  $$\exists x_{i+1}, \dots, \exists x_{m_r}, r(\dots, e_{i-1}, v, x_{i+1}, \dots) \wedge v = t_i(c)$$

  each time obtaining $e_i \in \Sigma_{\text{con}}$ satisfying the formula. The result is $T \vDash_\Sigma r(e)$ and each $T \vDash_\Sigma t_i(c) = e_i$. Using the claim for terms and how we interpreted relations in $\mathcal{M}$ this implies $t^{\mathcal{M}}(c^{\mathcal{M}}) = e^{\mathcal{M}} \in r^{\mathcal{M}}$, and hence $\mathcal{M} \vDash_\Sigma r(t(c))$.

  ($\Leftarrow$) Suppose $\mathcal{M} \vDash_\Sigma r(t(c))$. Since π is surjective, there exists $e \in \Sigma_{\text{con}}$ such that $e^{\mathcal{M}} = t^{\mathcal{M}}(c^{\mathcal{M}}) \in r^{\mathcal{M}}$. Using the claim for terms again we obtain $T \vDash_\Sigma t(c) = e$ and using how $\mathcal{M}$ interprets relations, $T \vDash_\Sigma r(e)$. It follows that $T \vDash_\Sigma r(t(c))$.
- Case ϕ is $\neg \chi$: Using the maximal property of T for the first $\Leftrightarrow$ and the induction hypothesis for the second $\Leftrightarrow$ we have

  $$T \vDash_\Sigma \neg \chi(c) \;\Leftrightarrow\; T \nvDash_\Sigma \chi(c) \;\Leftrightarrow\; \mathcal{M} \nvDash_\Sigma \chi(c) \;\Leftrightarrow\; \mathcal{M} \vDash_\Sigma \neg \chi(c)$$
- Case ϕ is $\chi_0 \vee \chi_1$

  $$\begin{aligned} \mathcal{M} \vDash_\Sigma \chi_0(c^{\mathcal{M}}) \vee \chi_1(c^{\mathcal{M}}) &\Leftrightarrow \mathcal{M} \vDash_\Sigma \chi_0(c^{\mathcal{M}}) \text{ or } \mathcal{M} \vDash_\Sigma \chi_1(c^{\mathcal{M}}) \\ &\Leftrightarrow T \vDash_\Sigma \chi_0(c) \text{ or } T \vDash_\Sigma \chi_1(c) \quad \text{by the induction hypothesis} \end{aligned}$$

  Hence it suffices to show that

  $$T \vDash_\Sigma \chi_0(c) \text{ or } T \vDash_\Sigma \chi_1(c) \;\Leftrightarrow\; T \vDash_\Sigma \chi_0(c) \vee \chi_1(c)$$

($\Rightarrow$) Suppose $T \vDash_\Sigma \chi_0(c)$ or $T \vDash_\Sigma \chi_1(c)$. For $\mathcal{N}$ a Σ-model of T,

$$\mathcal{N} \vDash_\Sigma \chi_0(c^{\mathcal{N}}) \text{ or } \mathcal{N} \vDash_\Sigma \chi_1(c^{\mathcal{N}}) \Rightarrow \mathcal{N} \vDash_\Sigma \chi_0(c^{\mathcal{N}}) \vee \chi_1(c^{\mathcal{N}})$$

Thus $T \vDash_\Sigma \chi_0(c) \vee \chi_1(c)$.

($\Leftarrow$) Suppose $T \vDash_\Sigma \chi_0(c) \vee \chi_1(c)$. For $\mathcal{N}$ a Σ-model of T,

$$\mathcal{N} \vDash_\Sigma \chi_0(c^{\mathcal{N}}) \vee \chi_1(c^{\mathcal{N}}) \Rightarrow \mathcal{N} \vDash_\Sigma \chi_0(c^{\mathcal{N}}) \text{ or } \mathcal{N} \vDash_\Sigma \chi_1(c^{\mathcal{N}})$$

Thus $T \vDash_\Sigma \chi_0(c)$ or $T \vDash_\Sigma \chi_1(c)$.

- Case ϕ is $\forall v, \chi(v)$ ($\Rightarrow$) Let $d \in \mathcal{M}$, then since π surjective $\exists e \in \Sigma_{\text{con}}$ such that $\pi(e) = d$. Since $T \vDash_\Sigma \forall v, \chi(c, v)$ it follows that $T \vDash_\Sigma \chi(c, e)$ and by induction $\mathcal{M} \vDash_\Sigma \chi(c^{\mathcal{M}}, d)$. Hence $\mathcal{M} \vDash_\Sigma \forall v, \chi(c^{\mathcal{M}}, v)$.

  ($\Leftarrow$) We show the contrapositive. If $T \nvDash_\Sigma \forall v, \chi(c)(v)$, then by the maximal property of T, $T \vDash_\Sigma \exists v, \neg\chi(c, v)$. Applying the witness property to $\neg\chi(c, v)$, there exists $e \in \Sigma_{\text{con}}$ such that

  $$T \vDash_\Sigma (\exists v, \neg\chi(c)(v)) \to (\neg\chi(c)(v)) \quad \Rightarrow \quad T \vDash_\Sigma \neg\chi(c)(v)$$

  Thus $T \nvDash_\Sigma \chi(c)(v)$ by the maximal property of T and $\mathcal{M} \nvDash_\Sigma \chi(c^{\mathcal{M}})(v)$ by the induction hypothesis. Hence $\mathcal{M} \nvDash_\Sigma \forall v, \chi(c^{\mathcal{M}})(v)$.

Thus $T \vDash_\Sigma \phi \Leftrightarrow \mathcal{M} \vDash_\Sigma \phi$ and we are done. □

Proposition – Giving Theories the Witness Property
Suppose $\Sigma(0)$-theory $T(0)$ is finitely consistent. Then there exists a signature $\Sigma(*)$ and $\Sigma(*)$-theory $T(*)$ such that

1. $\Sigma(0)_{\text{con}} \subseteq \Sigma(*)_{\text{con}}$ and they share the same function and relation symbols.
2. $|\Sigma(*)_{\text{con}}| = |\Sigma(0)_{\text{con}}| + \aleph_0$
3. $T(0) \subseteq T(*)$
4. $T(*)$ is finitely consistent
5. Any $\Sigma(*)$-theory T' such that $T(*) \subseteq T'$ has the witness property

Proof. Again without loss of generality $\Sigma(0)$ is non-empty. We want to define $\Sigma(i), T(i)$, for each $i \in \mathbb{N}$. By induction, we assume we have $\Sigma(i)$ non-empty a signature and $T(i)$ a Σ-theory such that

1. $\Sigma(0)_{\text{con}} \subseteq \Sigma(i)_{\text{con}}$ and they share the same function and relation symbols.
2. $|\Sigma(i)_{\text{con}}| = |\Sigma(0)_{\text{con}}| + \aleph_0$
3. $T(0) \subseteq T(i)$
4. $T(i)$ is finitely consistent

Let
$$W(i) := \{\phi \in \Sigma(i)_{\text{for}} \mid \phi \text{ has exactly one free variable}\}$$

We construct $\Sigma(i+1)$ by adding constant symbols c_ϕ for each $\phi \in W(i)$ and keeping the same function and relation symbols of $\Sigma(i)$:
$$\Sigma(i+1)_{\text{con}} := \Sigma(i)_{\text{con}} \sqcup \{c_\phi \mid \phi \in W(i)\}$$

We create a witness formula $w(\phi)$ for each formula $\phi \in W$:

$$\begin{aligned} w : W(i) &\to \Sigma(i+1)_{\text{for}} \\ \phi &\mapsto ((\exists v, \phi(v)) \to \phi(c_\phi)) \end{aligned}$$

Then let

$$T(i+1) := T(i) \cup w(W(i))$$

Certainly $T(i+1)$ is a $\Sigma(i+1)$-theory such that $T(0) \subseteq T(i+1)$, $\Sigma(0)_{\text{con}} \subseteq \Sigma(i+1)_{\text{con}}$ where the function and relation symbols are unchanged. Since $W(i)$ is countibly infinite, $|\Sigma(i+1)_{\text{con}}| = |\Sigma(i)_{\text{con}}| + \aleph_0 = |\Sigma(0)_{\text{con}}| + \aleph_0$. We need to check that $T(i+1)$ is finitely consistent. Take a finite subset of $T(i+1)$. It is a union of two finite sets $\Delta_T \subseteq T(i)$ and $\Delta_w \subseteq w(W(i))$. Since $T(i)$ is finitely consistent there exists a $\Sigma(i)$-model $\mathcal{M}(i)$ of Δ_T. Let $\mathcal{M}(i+1)$ be defined to have carrier set $\mathcal{M}(i)$. There exists some $b \in \mathcal{M}(i) = \mathcal{M}(i+1)$ since $\Sigma(i)$ is non-empty. To define interpretation for $\mathcal{M}(i+1)$, let $c \in \Sigma(i+1)_{\text{con}}$:

$$c_{\Sigma(i+1)}^{\mathcal{M}(i+1)} := \begin{cases} c_{\Sigma(i)}^{\mathcal{M}(i)} & \text{when } c \in \Sigma(i)_{\text{con}} \\ a & \text{when } c = c_\phi \text{ and } \exists a \in \mathcal{M}(i), \mathcal{M}(i) \vDash_{\Sigma(i)} \phi(a) \\ b & \text{when } c = c_\phi \text{ and } \forall a \in \mathcal{M}(i), \mathcal{M}(i) \nvDash_{\Sigma(i)} \phi(a) \end{cases}$$

Then $\mathcal{M}(i+1)$ is a well defined $\Sigma(i+1)$-structure. We check is is a $\Sigma(i+1)$-model of $\Delta_T \cup \Delta_w$. Since $\star_{\Sigma(i+1)}^{\mathcal{M}(i+1)}$ agrees with $\star_{\Sigma(i)}^{\mathcal{M}(i)}$ for constants, functions and relations from $\mathcal{M}(i)$ - a $\Sigma(i)$-model of Δ_T - it is a $\Sigma(*)$-model of Δ_T. If $\psi \in \Delta_w$ then it is $\exists v, \phi(v) \to \phi(c_\phi)$ for some $\phi \in W(i)$. Supposing that $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \exists v, \phi(v)$ it suffices to show $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \phi(c_\phi)$. Then there exists $a \in \mathcal{M}(i+1) = \mathcal{M}(i)$ such that $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \phi(a)$. Hence $\mathcal{M}(i) \vDash_{\Sigma(i)} \phi(a)$ and so c_ϕ is interpreted as a in $\mathcal{M}(i+1)$. Hence $\mathcal{M}(i+1) \vDash_{\Sigma(i)} \phi(c_\phi)$. Thus the induction is complete.

Let $\Sigma(*)$ be the signature such that its function and relations are the same as $\Sigma(0)$ and $\Sigma(*)_{\text{con}} = \bigcup_{i \in \mathbb{N}} \Sigma(i)_{\text{con}}$. Then

$$|\Sigma(*)_{\text{con}}| = |\bigcup_{i \in \mathbb{N}} \Sigma(i)_{\text{con}}| = \aleph_0 \times (\aleph_0 + \Sigma(0)_{\text{con}}) = \aleph_0 + \Sigma(0)_{\text{con}}$$

Let $T(*) = \bigcup_{i \in \mathbb{N}} T(i)$. Any finite subset of $T(*)$ is a subset of some $T(i)$, hence has a non-empty $\Sigma(i)$-model $\mathcal{M}$. Checking the relevant conditions for moving models up signatures, we have $\mathcal{M}(*)$ a $\Sigma(*)$-model of the finite subset (by interpreting the new constant symbols as the element of the non-empty carrier set.). Hence $T(*)$ is finitely consistent.

If T' is a $\Sigma(*)$-theory such that $T(*) \subseteq T'$, and ϕ is a $\Sigma(*)$-formula of exactly one variable. There exists an $i \in \mathbb{N}$ such that $\phi \in \Sigma(i)_{\text{for}}$. Since $c_\phi \in \Sigma(i+1)$ satisfies $T(i) \vDash_{\Sigma(i+1)} (\exists v, \phi(v)) \to \phi(c_\phi)$, by moving the logical consequence up to $\Sigma(*)$, we have $T(i) \vDash_{\Sigma(*)} (\exists v, \phi(v)) \to \phi(c_\phi)$. If $\mathcal{N}$ is a $\Sigma(*)$-model of T' then it is a $\Sigma(*)$-model of $T(i)$, then $\mathcal{N} \vDash_{\Sigma(*)} (\exists v, \phi(v)) \to \phi(c_\phi)$. Hence $T' \vDash_{\Sigma(*)} (\exists v, \phi(v)) \to \phi(c_\phi)$, satisfying the witness property. □

Lemma – Adding Formulas to Consistent Theories

If T is a finitely consistent Σ-theory and ϕ is a Σ-sentence then at least one of $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely consistent.

Proof. We show that for any finite $\Delta \subseteq T \cup \{\phi\}$ and for any finite $\Delta_\neg \subseteq T \cup \{\neg\phi\}$, one of Δ or $\Delta_\neg$ is consistent. The finite subset

$$(\Delta \setminus \{\phi\}) \cup (\Delta_\neg \setminus \{\neg\phi\}) \subseteq T$$

is consistent by finite consistency of T. Let $\mathcal{M}$ be the model of $(\Delta \setminus \{\phi\}) \cup (\Delta_\neg \setminus \{\neg\phi\})$. Case on whether $\mathcal{M} \vDash_\Sigma \phi$ or not. In the first case $\mathcal{M} \vDash_\Sigma \Delta$ and in the second $\mathcal{M} \vDash_\Sigma \Delta_\neg$. Hence $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely consistent. □

Exercise. *Find a signature Σ, a consistent Σ-theory T and Σ-sentence ϕ such that $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are both consistent.*

Proposition – Extending a finitely consistent theory to a maximal theory (Zorn)
Given a finitely consistent Σ-theory $T(0)$ there exists a Σ-theory $T(*)$ such that

1. $T(0) \subseteq T(*)$
2. $T(*)$ is finitely consistent.
3. $T(*)$ is Σ-maximal.

Proof. We use Zorn's Lemma. Let

$$Z := \{T \in \Sigma_{\text{the}} \mid T \text{ finitely consistent and } T(0) \subseteq T\}$$

be ordered by inclusion. Let $T(0) \subseteq T(1) \subseteq \cdots$ be a chain. Then $\bigcup_{i \in \mathbb{N}} T(i)$ is a Σ-theory such that any finite subset is a subset of some $T(i)$, hence is consistent by finite consistency of $T(i)$. Zorn's lemma implies there exists a $T(*) \in Z$ that is maximal (in the order theory sense). Since $T(*)$ is in Z, we have that it is a finitely consistent Σ-theory containing $T(0)$.

To show that it is Σ-maximal we take a Σ-sentence ϕ. By the previous result, $T(*) \cup \{\phi\}$ or $T(*) \cup \{\neg\phi\}$ is finitely consistent. Hence $T(*) \cup \{\phi\} = T(*)$ or $T(*) \cup \{\neg\phi\} = T(*)$ by (order theoretic) maximality, so $\phi \in T(*)$ or $\neg\phi \in T(*)$. □

NOTATION (CARDINALITIES OF SIGNATURES AND STRUCTURES). *Given a signature Σ, we write $|\Sigma| := |\Sigma_{\text{con}}| + |\Sigma_{\text{fun}}| + |\Sigma_{\text{rel}}|$ and call this the cardinality of the signature Σ.*

Proposition – The compactness theorem
If T is a Σ-theory, then the following are equivalent:

1. T is finitely consistent.
2. T is consistent
3. For any infinite cardinal κ such that $|\Sigma| \leq \kappa$, there exists a non-empty Σ-model of T with cardinality $\leq \kappa$.

Proof. 3. implies 2. and 2. implies 1. are both obvious. A proof of 1. implies 3. follows.

Suppose an $\Sigma(0)$-theory $T(0)$ is finitely consistent. Let κ be an infinite cardinal such that $|\Sigma(0)| \leq \kappa$. Then $|\Sigma(0)_{\text{con}}| \leq |\Sigma(0)| \leq \kappa$. We wish to find a Σ-model of T with cardinality $\leq \kappa$. We have shown that there exists a signature $\Sigma(1)$ and $\Sigma(1)$-theory $T(1)$ such that

1. $\Sigma(0)_{\text{con}} \subseteq \Sigma(1)_{\text{con}}$ and they share the same function and relation symbols.
2. $|\Sigma(1)_{\text{con}}| = |\Sigma(0)_{\text{con}}| + \aleph_0$
3. $T(0) \subseteq T(1)$
4. $T(1)$ is finitely consistent.
5. Any $\Sigma(1)$-theory T such that $T(1) \subseteq T$ has the witness property.

$T(1)$ is finitely consistent so there exists a $\Sigma(1)$-theory $T(2)$ such that

6. $T(1) \subseteq T(2)$
7. $T(2)$ is finitely consistent.
8. $T(2)$ is $\Sigma(1)$-maximal.

Furthermore, $T(2)$ has the witness property due to point 5. Since $T(2)$ has the witness property, is $\Sigma(1)$-maximal and finitely consistent, $T(2)$ has a non-empty $\Sigma(1)$-model $\mathcal{M}$ such that $|\mathcal{M}| \leq \kappa$ by Henkin Construction. $\mathcal{M} \vDash_{\Sigma(1)} T(0)$ since $T(0) \subseteq T(1) \subseteq T(2)$. We can move $\mathcal{M}$ down to $\Sigma(0)$, obtaining $\mathcal{M} \vDash_{\Sigma(0)} T(0)$. □

1.1.4 The Category of Structures

Definition – Σ-morphism, Σ-embedding, Σ-isomorphism

Given Σ a signature, $\mathcal{M}, \mathcal{N}$ both Σ-structures, $A \subseteq \mathcal{M}_{\text{car}}$ and $\iota : A \to \mathcal{N}_{\text{car}}$, we call ι a partial Σ-morphism from $\mathcal{M}$ to $\mathcal{N}$ when

- For all $c \in C$ (such that $c^{\mathcal{M}} \in A$),
$$\iota(c^{\mathcal{M}}) = c^{\mathcal{N}}$$
- For all $f \in F$ and all $a \in M^{n_f}$ (such that $f^{\mathcal{M}}(a) \in A$),
$$\iota \circ f^{\mathcal{M}}(a) = f^{\mathcal{N}} \circ \iota(a)$$
- For all $r \in R$, for all $a \in M^{m_r} \cap A^{m_r}$,
$$a \in r^{\mathcal{M}} \Rightarrow \iota(a) \in r^{\mathcal{N}}$$

If in addition for relations we have

$$a \in r^{\mathcal{M}} \Leftarrow \iota(a) \in r^{\mathcal{N}} \quad \text{and} \quad \iota \text{ is injective,}$$

then ι is called a partial Σ-embedding (the word extension is often used interchangably with embedding).

In the case that $A = \mathcal{M}_{\text{car}}$ we write $\iota : \mathcal{M} \to \mathcal{N}$ and call ι a Σ-morphism.

The notion of morphisms here will be the same as morphisms in the algebraic setting. For example in the signature of monoids (groups), preserving interpretation of constant symbols says the identity is sent to the identity and preserving interpretation of function symbols says the multiplication is preserved.

Definition – Elementary Embedding

A partial Σ-embedding $\iota : A \to \mathcal{N}$ (for $A \subseteq \mathcal{M}$) is elementary if for any Σ-formula ϕ with variables indexed by S and $a \in A^S$,
$$\mathcal{M} \vDash_\Sigma \phi(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

The following is exactly what we expect - that terms are well behaved with respect to morphisms.

Lemma – Σ-morphisms commute with interpretation of terms

Given a Σ-morphism $\iota : \mathcal{M} \to \mathcal{N}$, we have that for any Σ-term t with variables indexed by S and $a \in \mathcal{M}^S$,
$$\iota(t^{\mathcal{M}}(a)) = t^{\mathcal{N}}(\iota(a))$$

Proof. We case on what t is:

- If $t = c \in \Sigma_{\text{con}}$ then
$$\iota(t^{\mathcal{M}}(a)) = \iota(c^{\mathcal{M}}) = c^{\mathcal{N}} = t^{\mathcal{N}}(\iota(a))$$
- If $t = v \in \Sigma_{\text{var}}$ then
$$\iota(t^{\mathcal{M}}(a)) = \iota(v^{\mathcal{M}}(a)) = \iota(a) = v^{\mathcal{N}}(\iota(a)) = t^{\mathcal{N}}(\iota(a))$$

- If $t = f(s)$ then
$$\iota(t^{\mathcal{M}}(a)) = \iota \circ f^{\mathcal{M}}(s^{\mathcal{M}}(a)) = f^{\mathcal{N}} \circ \iota(s^{\mathcal{M}}(a)) = f^{\mathcal{N}} \circ s^{\mathcal{N}}(\iota(a)) = t^{\mathcal{N}}(\iota(a))$$
Using the induciton hypothesis in the penultimate step.

□

It is worth knowing that the set of Σ-structures form a category:

Definition – The category of Σ-structures
Given a signature Σ, we have $\mathbb{M}\mathbf{od}(\Sigma)$ - a category where objects are Σ-structures and morphisms are Σ-morphisms.

Clearly for any $\mathcal{M}$, the identity exists and is a Σ-morphism. We show that composition of morphisms are morphisms. Furthermore, composition of embeddings are embeddings and composition of elementary embeddings are elementary. Thus we could also define morphisms between objects to be embeddings or elementary embeddings and obtain a subcategory.

Hence we inherit a notion of isomorphism of Σ-structures from category theory.

Proof. Let $\iota_1 : \mathcal{M}_0 \to \mathcal{M}_1$ and $\iota_2 : \mathcal{M}_1 \to \mathcal{M}_2$ be Σ-morphisms. We show that the composition is a Σ-morphism:

- If $c \in \Sigma_{\text{con}}$ then
$$\iota_1 \circ \iota_0(c^{\mathcal{M}_0}) = \iota_1(c^{\mathcal{M}_1}) = c^{\mathcal{M}_2}$$
- If $f \in \Sigma_{\text{fun}}$ and $a \in {\mathcal{M}_0}^{n_f}$ then
$$\iota_1 \circ \iota_0 \circ f^{\mathcal{M}_0}(a) = \iota_1 \circ f^{\mathcal{M}_1} \circ \iota_0(a) = f^{\mathcal{M}_2} \circ \iota_1 \circ \iota_0(a)$$
- If $r \in \Sigma_{\text{rel}}$ and $a \in {\mathcal{M}_0}^{m_f}$ then
$$a \in r^{\mathcal{M}_0} \Rightarrow \iota_1(a) \in r^{\mathcal{M}_1} \Rightarrow \iota_2 \circ \iota_1(a) \in r^{\mathcal{M}_2}$$

To show that embeddings compose to be embeddings we note that the composition of injective functions is injective and if $r \in \Sigma_{\text{rel}}$ and $a \in {\mathcal{M}_0}^{m_f}$ then
$$a \in r^{\mathcal{M}_0} \Leftrightarrow \iota_1(a) \in r^{\mathcal{M}_1} \Leftrightarrow \iota_2 \circ \iota_1(a) \in r^{\mathcal{M}_2}$$
To show that composition of elementary embeddings are elementary, let $\phi \in \Sigma_{\text{for}}$ and a in $\mathcal{M}_0$ be chosen suitably. Then
$$\mathcal{M}_0 \vDash_{\Sigma} \phi(a) \Leftrightarrow \mathcal{M}_1 \vDash_{\Sigma} \phi(\iota_1(a)) \Leftrightarrow \mathcal{M}_2 \vDash_{\Sigma} \phi(\iota_2 \circ \iota_1(a))$$

□

Example. *Given the category of structures from the signature of rings, we can take the subcategory whose objects are models of the theory of rings (namely rings), hence producing the category of rings. Similarly taking the subcategory whose objects are models of the theory of fields (namely fields) produces the category of fields.*

Proposition – Embeddings Preserve Satisfaction of Quantifier Free Formulas
Given $\iota : \mathcal{M} \to \mathcal{N}$ a Σ-embedding and ϕ a Σ-formula with variables indexed by S, and $a \in \mathcal{M}^S$,

1. If ϕ is $\top$ then it is satisfied by both.
2. If ϕ is $t = s$ then $\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma} \phi(\iota(a))$.
3. If ϕ is $r(s)$ then $\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma} \phi(\iota(a))$.

4. If ϕ is $\neg\chi$ and $\mathcal{M} \vDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi(\iota(a))$ then

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

5. If ϕ is $\chi_0 \vee \chi_1$ and $\mathcal{M} \vDash_\Sigma \chi_i(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_i(\iota(a))$ then

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

Thus from the above we can immediately conclude by induction that if ϕ is a quantifier free Σ-formula,

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

Note that our original result is stronger than this since we didn't assume the formula to be quantifier free.

Proof.

1. Trivial.
2. If ϕ is $t = s$ then

$$\begin{aligned} \mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a) \\ &\Leftrightarrow \iota(t^{\mathcal{M}}(a)) = \iota(s^{\mathcal{M}}(a)) && \text{by injectivity} \\ &\Leftrightarrow t^{\mathcal{N}}(\iota(a)) = s^{\mathcal{N}}(\iota(a)) && \text{morphisms commute with} \\ &\Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a)) && \text{interpretation of terms} \end{aligned}$$

3. If ϕ is $r(s)$ then

$$\begin{aligned} \mathcal{M} \vDash_\Sigma \phi(a) &\Leftrightarrow a \in r^{\mathcal{M}} \\ &\Leftrightarrow \iota(a) \in r^{\mathcal{N}} && \text{embeddings property} \\ &\Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a)) \end{aligned}$$

4. If ϕ is $\neg\chi$ and $\mathcal{M} \vDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi(\iota(a))$ then

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \nvDash_\Sigma \chi(a) \Leftrightarrow \mathcal{N} \nvDash_\Sigma \chi(\iota(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

5. If ϕ is $\chi_0 \vee \chi_1$ and $\mathcal{M} \vDash_\Sigma \chi_i(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_i(\iota(a))$

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \vDash_\Sigma \chi_0(a) \text{ or } \mathcal{M} \vDash_\Sigma \chi_1(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \chi_0(\iota(a)) \text{ or } \mathcal{N} \vDash_\Sigma \chi_1(\iota(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

□

Definition – Universal Formula, Universal Sentence
A Σ-formula is universal if it can be built inductively by the following two constructors:

| If ϕ is a quantifier free Σ-formula then it is a universal Σ-formula.

| If ϕ is a universal Σ-formula then $\forall v, \phi(v)$ is a universal Σ-formula.

In other words universal Σ-formulas are formulas that start with a bunch of 'for alls' followed by a quantifier free formula.

Proposition – Embeddings preserve satisfaction of universal formulas downwards
Given $\iota : \mathcal{M} \to \mathcal{N}$ a Σ-embedding and ϕ a universal Σ-formula with variables indexed by S (χ is quantifier free). For any $a \in \mathcal{M}^S$

$$\mathcal{N} \vDash_\Sigma \phi(\iota(a)) \quad \Rightarrow \quad \mathcal{M} \vDash_\Sigma \phi(a)$$

By taking the contrapositive we can show that embeddings preserve satisfaction of 'existential' Σ-formulas upwards.

Proof. We induct on ϕ:

- If ϕ is a quantifier free then since embeddings preserve satisfaction of quantifier free formulas, $\mathcal{N} \vDash_\Sigma \phi(\iota(a)) \Rightarrow \mathcal{M} \vDash_\Sigma \phi(a)$.
- If ϕ is $\forall v_i, \psi$ with S indexing the variables of ψ. Let $T := S \setminus \{i\}$ Assuming the inductive hypothesis: for any $a \in \mathcal{M}^T$ and $b \in \mathcal{M}$,

$$\mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b)) \quad \Rightarrow \quad \mathcal{M} \vDash_\Sigma \psi(a, b)$$

  Then for any $a \in \mathcal{M}^T$

$$\begin{aligned} & \mathcal{N} \vDash_\Sigma \phi(\iota(a)) \\ \Rightarrow\ & \forall b \in \mathcal{M}, \mathcal{N} \vDash_\Sigma \psi(\iota(a), \iota(b)) \\ \Rightarrow\ & \forall b \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \psi(\iota(a), \iota(b)) && \text{by the induction} \\ \Rightarrow\ & \mathcal{M} \vDash_\Sigma \phi(a) \end{aligned}$$

$\square$

Proposition – Isomorphisms are Elementary
If two Σ-structures $\mathcal{M}$ and $\mathcal{N}$ are Σ-isomorphic then the isomorphism is elementary.

Proof. Let $\iota : \mathcal{M} \to \mathcal{N}$ be a Σ-isomorphism. We case on what ϕ is:

- If ϕ is quantifier free, then each case is follows from applying embeddings preserve satisfaction of quantifier free formulas.
- If ϕ is $\forall v, \chi(v)$ then ($\Rightarrow$) Let $b \in \mathcal{N}_{\text{con}}$ then $\iota^{-1}(b) \in \mathcal{M}_{\text{con}}$ is well defined by surjectivity. Hence $\mathcal{M} \vDash_\Sigma \chi(\iota^{-1}(b), a)$ and so $\mathcal{N} \vDash_\Sigma \chi(b, \iota(a))$ by the induction hypothesis. Hence $\mathcal{N} \vDash_\Sigma \phi(\iota(a))$. ($\Leftarrow$) The same.

$\square$

1.1.5 Vaught's Completeness Test

Read ahead to the statement of Vaught's Completeness Test.

Definition – Finitely modelled, infinitely modelled
A Σ-theory T is finitely modelled when there exists a Σ-model of T with finite carrier set. It is infinitely modelled when there exists a Σ-model of T with infinite carrier set.

Finitely modelled is *not* the same as finitely consistent.

Proposition – Infinitely modelled theories have arbitrary large models

Given Σ a signature, T a Σ-theory that is infinitely modelled, and a cardinal κ such that $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$, there exists $\mathcal{M}$ a Σ-model of T such that $\kappa = |\mathcal{M}|$.

Proof. Enrich only the signature's constant symbols to create $\Sigma(*)$ a signature such that $\Sigma(*)_{\text{con}} = \Sigma_{\text{con}} \cup \{c_\alpha \,|\, \alpha \in \kappa\}$. Let $T(*) = T \cup \{c_\alpha \neq c_\beta \,|\, \alpha, \beta \in \kappa \wedge \alpha \neq \beta\}$ be a $\Sigma(*)$-theory.

Using the compactness theorem, it suffices to show that $T(*)$ is finitely consistent. Take a finite subset of $T(*)$. This is the union of a finite subset $\Delta_T \subseteq T$, and a finite subset of $\Delta_\kappa \subseteq \{c_\alpha \neq c_\beta \,|\, \alpha, \beta \in \kappa \wedge \alpha \neq \beta\}$. Let $\mathcal{M}$ be the Σ-model of T with infinite cardinality. We want to make $\mathcal{M}$ a $\Sigma(*)$-model of $\Delta_T \cup \Delta_\kappa$ by interpreting the new symbols of $\{c_\alpha \,|\, \alpha \in \kappa\}$ in a sensible way.

Since Δ_κ is finite, we can find a finite subset $I \subset \kappa$ that indexes the constant symbols appearing in Δ_κ. Since $\mathcal{M}$ is infinite and I is finite, we can find distinct elements of $\mathcal{M}$ to interpret the elements of $\{c_\alpha \,|\, \alpha \in I\}$. Interpret the rest of the new constant symbols however, for example let them all be sent to the same element, then $\mathcal{M} \vDash_{\Sigma *} \Delta_T \cup \Delta_\kappa$. Thus $T(*)$ is finitely consistent hence consistent.

Using the third equivalence of $T(*)$ being consistent, there exists $\mathcal{M}$ a $\Sigma(*)$-model of $T(*)$ with $|\mathcal{M}| \leq \kappa$. If $|\mathcal{M}| < \kappa$ then there would be c_α, c_β that are interpreted as equal, hence $\mathcal{M} \vDash_{\Sigma(*)} c_\alpha = c_\beta$ and $\mathcal{M} \nvDash_{\Sigma(*)} c_\alpha = c_\beta$, a contradiction. Thus $|\mathcal{M}| = \kappa$. Move $\mathcal{M}$ down a signature to make it a Σ-model of T. This doesn't change the cardinality of $\mathcal{M}$, so we have a Σ-model of T with cardinality κ. $\square$

Definition – Elementary equivalence

Let $\mathcal{M}$, $\mathcal{N}$ be Σ-structures. They are elementarily equivalent if for any Σ-sentence ϕ, $\mathcal{M} \vDash_\Sigma \phi$ if and only if $\mathcal{N} \vDash_\Sigma \phi$. We write $\mathcal{M} \equiv_\Sigma \mathcal{N}$.

Definition – Complete

A Σ-theory T is complete when either of the following equivalent definitions hold:

- For any Σ-sentence ϕ, $T \vDash_\Sigma \phi$ or $T \vDash_\Sigma \neg\phi$.
- All models of T are elementarily equivalent.

Proof. ($\Rightarrow$) Let $\mathcal{M}$ and $\mathcal{N}$ be models of T and ϕ be a Σ-sentence. If $\phi \in T$ then both satisfy ϕ. Otherwise $\neg\phi \in T$ and neither satisfy ϕ.

($\Leftarrow$) If ϕ is a Σ-sentence then suppose for a contradiction

$$T \nvDash_\Sigma \phi \text{ and } T \nvDash_\Sigma \neg\phi$$

Then there exist models of T such that $\mathcal{M} \nvDash_\Sigma \phi$ and $\mathcal{N} \nvDash_\Sigma \neg\phi$. By assumption they are elementarily equivalent and so $\mathcal{M} \vDash_\Sigma \neg\phi$ implies $\mathcal{N} \vDash_\Sigma \neg\phi$, a contradiction. $\square$

Note that if we have completeness and compactness, then for any Σ-sentence ϕ, *either* $T \vDash_\Sigma \phi$ or $T \vDash_\Sigma \neg\phi$.

Lemma – Not a consequence is consistent

Let T be a Σ-theory and ϕ is a Σ-sentence then $T \nvDash_\Sigma \phi$ if and only if $T \cup \{\neg\phi\}$ is consistent. Furthermore, $T \nvDash_\Sigma \neg\phi$ if and only if $T \cup \{\phi\}$ is consistent.

Proof. For the first statement: ($\Rightarrow$) Unfolding $T \nvDash_\Sigma \phi$, we have that there exists a Σ-model $\mathcal{M}$ of T such that $\mathcal{M} \nvDash_\Sigma \phi$. Hence $\mathcal{M} \vDash_\Sigma \neg\phi$ and we are done. The backward proof is straightward.

For the second statement, apply the first to $\neg\phi$ and obtain $T \nvDash_\Sigma \neg\phi$ if and only if $T \cup \{\neg\neg\phi\}$ is consistent. Note that for any Σ-structure $\mathcal{M}$, $\mathcal{M} \vDash_\Sigma \neg\neg\phi$ if and only if $\mathcal{M} \vDash_\Sigma \phi$. This completes the proof. $\square$

Proposition – Vaught's Completeness Test

Suppose that Σ-theory T is consistent, not finitely modelled, and κ-categorical for some cardinal satisfying $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$. Then T is complete.

Proof. Suppose not: If T is not complete then there exists Σ-formula ϕ such that $T \not\models_\Sigma \phi$ and $T \not\models_\Sigma \neg\phi$. These imply $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ are both consistent. Let $\mathcal{M}_\neg$ and $\mathcal{M}$ be models of $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ respectively. Then each are models of T so they are infinite and so $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ are infinitely modelled.

Since we have κ such that $|\Sigma_{\text{con}}| + \aleph_0 \leq \kappa$, there exists $\mathcal{N}_\neg, \mathcal{N}$ respectively Σ-models of $T \cup \{\neg\phi\}$ and $T \cup \{\phi\}$ such that $\kappa = |\mathcal{N}_\neg| = |\mathcal{N}|$. Since T is κ-categorical $\mathcal{N}$ and $\mathcal{N}_\neg$ are isomorphic by an elementary Σ-embedding. As ϕ has no free variables this implies that $\mathcal{N} \models_\Sigma \phi$ and $\mathcal{N} \models_\Sigma \neg\phi$, a contradiction. □

1.1.6 Elementary embeddings and diagrams of models

Proposition – Tarski-Vaught Elementary Embedding Test

Let $\iota : \mathcal{M} \to \mathcal{N}$ be a Σ-embedding, then the following are equivalent:

1. ι is elementary
2. For any $\phi \in \Sigma_{\text{for}}$ with free variables indexed by S, any $i \in S$ and any $a \in (\mathcal{M})^{S \setminus \{i\}}$,
$$\forall b \in \mathcal{M}, \mathcal{N} \models_\Sigma \phi(\iota(a), \iota(b)) \quad \Rightarrow \quad \forall c \in \mathcal{N}, \mathcal{N} \models_\Sigma \phi(\iota(a), c),$$
which we call the Tarski-Vaught condition.
3. For any $\phi \in \Sigma_{\text{for}}$ with free variables indexed by S, any $i \in S$ and any $a \in (\mathcal{M})^{S \setminus \{i\}}$,
$$\exists c \in \mathcal{N}, \mathcal{N} \models_\Sigma \phi(\iota(a), c) \quad \Rightarrow \quad \exists b \in \mathcal{M}, \mathcal{N} \models_\Sigma \phi(\iota(a), \iota(b))$$
This is essentially the contrapositive of the previous statement, and is included because it is more commonly version of the statement.

Proof. We only show the first two statements are equivalent and leave the third as an exercise. ($\Rightarrow$) First show that $\mathcal{M} \models_\Sigma \forall v, \phi(a, v)$. Let $b \in \mathcal{M}$, then by assumption $\mathcal{N} \models_\Sigma \phi(\iota(a), \iota(b))$, which is implies $\mathcal{M} \models_\Sigma \phi(a, b)$ as ι is an elementary embedding. Thus we indeed have $\mathcal{M} \models_\Sigma \forall v, \phi(a, v)$ which in turn implies $\mathcal{N} \models_\Sigma \forall v, \phi(\iota(a), v)$ and we are done.

($\Leftarrow$) We case on what ϕ is, though most of the work was already done before.

- If ϕ is quantifier free, then each case follows from applying embeddings preserve satisfaction of quantifier free formulas.
- The backwards implication follows from applying embeddings preserve satisfaction of universal formulas downwards.

  For the forwards implication we use the Tarski-Vaught condition (so far ι just needed to be a Σ-embedding)
$$\begin{aligned}
\mathcal{M} \models_\Sigma \forall v, \psi(a, v) &\Rightarrow \forall b \in \mathcal{M}, \mathcal{M} \models_\Sigma \psi(a, b) \\
&\Rightarrow \forall b \in \mathcal{M}, \mathcal{N} \models_\Sigma \psi(\iota(a), \iota(b)) && \text{by the induction hypothesis} \\
&\Rightarrow \forall c \in \mathcal{N}, \mathcal{N} \models_\Sigma \psi(\iota(a), c) && \text{by the Tarski-Vaught condition} \\
&\Rightarrow \mathcal{N} \models_\Sigma \phi
\end{aligned}$$

□

Again we have a technical detail which is not really worth spending too much time one. It is a sensible justification for arguments based on creating new signatures.

Lemma – Moving Morphisms Down Signatures

Suppose $\Sigma \leq \Sigma(*)$. If $\iota : \mathcal{M} \to \mathcal{N}$ is a $\Sigma(*)$-morphism then

1. ι can be made into a Σ-morphism.
2. If ι is an embedding then it remains an embedding.
3. If ι is an elementary embedding then it remains elementary.

Proof. 1. Move $\mathcal{M}$ and $\mathcal{N}$ down to being Σ structures (by picking $T(*) = T = \varnothing$). We show that the same set morphism $\iota : \mathcal{M} \to \mathcal{N}$ is a Σ-morphism.

- If $c \in \Sigma_{\text{con}}$ then since moving structures down signatures preserves interpretation on the lower signature, and since ι is a $\Sigma(*)$ embedding,

$$\iota(c_\Sigma^{\mathcal{M}}) = \iota(c_{\Sigma(*)}^{\mathcal{M}}) = c_{\Sigma(*)}^{\mathcal{N}} = c_\Sigma^{\mathcal{N}}$$

- If $f \in \Sigma_{\text{fun}}$ and $a \in (\mathcal{M})^{n_f}$ then similarly

$$\iota \circ f_\Sigma^{\mathcal{M}}(a) = \iota \circ f_{\Sigma(*)}^{\mathcal{M}}(a) = f_{\Sigma(*)}^{\mathcal{N}}(\iota(a)) = f_\Sigma^{\mathcal{N}}(\iota(a))$$

- If $r \in \Sigma_{\text{rel}}$ and $a \in (\mathcal{M})^{m_r}$ then

$$a \in r_\Sigma^{\mathcal{M}}(a) = r_{\Sigma(*)}^{\mathcal{M}} \Rightarrow \iota(a) \in r_{\Sigma(*)}^{\mathcal{N}} = r_\Sigma^{\mathcal{N}}$$

2. If we also have that it is an embedding in $\Sigma(*)$, then injectivity is preserved as it is a property of set morphisms. Given $r \in \Sigma_{\text{rel}}$ and $a \in (\mathcal{M})^{m_r}$,

$$\iota(a) \in r_\Sigma^{\mathcal{N}} = r_{\Sigma(*)}^{\mathcal{N}} \Rightarrow a \in r_{\Sigma(*)}^{\mathcal{M}}(a) = r_\Sigma^{\mathcal{M}}$$

3. If we also have that ι is elementary in $\Sigma(*)$ then we use the Tarski-Vaught Test: let $\phi \in \Sigma_{\text{for}}$ have free variables indexed by S, let $i \in S$ and let $a \in (\mathcal{M})^{S \setminus \{i\}}$. Then due to the construction in moving $\mathcal{M}$ and $\mathcal{N}$ down a signature we have that for any $b \in \mathcal{N}$,

$$\mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b)) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(*)} \phi(\iota(a), \iota(b))$$

and similarly for $\mathcal{M}$. Hence

$$\begin{aligned} & \forall b \in \mathcal{N}, \mathcal{N} \vDash_\Sigma \phi(\iota(a), \iota(b)) \\ \Rightarrow & \forall b \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma(*)} \phi(\iota(a), \iota(b)) \\ \Rightarrow & \forall c \in \mathcal{M}, \mathcal{M} \vDash_{\Sigma(*)} \phi(a, c) && \iota \text{ is elementary in } \Sigma(*) \\ \Rightarrow & \forall c \in \mathcal{M}, \mathcal{M} \vDash_\Sigma \phi(a, c) \end{aligned}$$

Hence ι is elementary in Σ.

□

NOTATION. *Let A be a set and Σ be a signature, enriching only the constant symbols of Σ we can create a signature $\Sigma(A)$ such that*

$$\Sigma(A)_{\text{con}} := \Sigma_{\text{con}} \cup \{c_a \mid a \in A\}$$

Definition – Diagram and the Elementary Diagram of a Structure

Let $\mathcal{M}$ be a Σ-structure, we move $\mathcal{M}$ up to the signature $\Sigma(\mathcal{M})$ by interpreting each new constant symbol c_a as a. ($\mathcal{M}$ satisfies the conditions of our lemma for moving models up signatures by choosing $T = \varnothing$). Thus we may treat $\mathcal{M}$ as a $\Sigma(\mathcal{M})$ structure. We define the atomic diagram of $\mathcal{M}$ over Σ:

| If ϕ is an atomic $\Sigma(\mathcal{M})$-sentence such that $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi$, then $\phi \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$.

| If $\phi \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$ then $\neg\phi \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$.

We define the elementary diagram of $\mathcal{M}$ over Σ as

$$\mathrm{ElDiag}(\Sigma, \mathcal{M}) := \left\{\phi \in \Sigma(\mathcal{M})\text{-sentences } \mid \mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi\right\}$$

The elementary diagram of $\mathcal{M}$ is a complete $\Sigma(\mathcal{M})$-theory with $\mathcal{M}$ as a model of it. It is not the same as the set of all Σ-sentences satisfied by $\mathcal{M}$, known as the theory of $\mathcal{M}$ in Σ.

Proposition – Models of the elementary diagram are elementary extensions

Given $\mathcal{M}$ a Σ-structure and $\mathcal{N}$ a $\Sigma(\mathcal{M})$-structure such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{AtDiag}(\Sigma, \mathcal{M})$, we can make $\mathcal{N}$ into a Σ-structure and find a Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$. Furthermore if $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$ then the embedding is elementary.

Conversely, given an elementary Σ-embedding from $\mathcal{M}$ into a Σ-structure $\mathcal{N}$, we can move $\mathcal{N}$ up to being a $\Sigma(\mathcal{M})$ structure such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$.

Proof. ($\Rightarrow$) Suppose $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{AtDiag}(\Sigma, \mathcal{M})$. Firstly we work in $\Sigma(\mathcal{M})$ to define the embedding: move $\mathcal{M}$ up a signature by taking the same interpretation as used in the definition of $\Sigma(\mathcal{M})$:

$$\star^{\mathcal{M}}_{\Sigma(\mathcal{M})} : c_a \mapsto a$$

and preserving the same interpretation for symbols of Σ. This makes $\star^{\mathcal{M}}_{\Sigma(\mathcal{M})_{\mathrm{con}}}$ surjective. Thus we write elements of $\mathcal{M}$ as $c^{\mathcal{M}}_{\Sigma(\mathcal{M})}$, for some $c \in \Sigma(\mathcal{M})_{\mathrm{con}}$.

Next we define the $\Sigma(\mathcal{M})$-morphism $\iota : \mathcal{M} \to \mathcal{N}$ such that $\iota : c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \to c^{\mathcal{N}}_{\Sigma(\mathcal{M})}$. To check that ι is well defined, take $c, d \in \Sigma(\mathcal{M})_{\mathrm{con}}$ such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} = d^{\mathcal{M}}_{\Sigma(\mathcal{M})}$.

$$\begin{aligned}
&\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} c = d\\
&\Rightarrow c = d \in \mathrm{AtDiag}(\Sigma, \mathcal{M})\\
&\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} c = d\\
&\Rightarrow c^{\mathcal{N}}_{\Sigma(\mathcal{M})} = d^{\mathcal{N}}_{\Sigma(\mathcal{M})}
\end{aligned}$$

Thus ι is well defined. In fact doing 'not' gives us injectivity in the same way: Take $c, d \in \Sigma(\mathcal{M})_{\mathrm{con}}$ such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \neq d^{\mathcal{M}}_{\Sigma(\mathcal{M})}$.

$$\begin{aligned}
&\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} c \neq d\\
&\Rightarrow c \neq d \in \mathrm{AtDiag}(\Sigma, \mathcal{M})\\
&\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} c \neq d\\
&\Rightarrow c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \neq d^{\mathcal{N}}_{\Sigma(\mathcal{M})}
\end{aligned}$$

Thus ι is injective. To check that ι is a $\Sigma(\mathcal{M})$-morphism, we check interpretation of functions and relations. Let $f \in \Sigma(\mathcal{M})_{\mathrm{fun}} = \Sigma_{\mathrm{fun}}$ and $c \in (\Sigma(\mathcal{M})_{\mathrm{con}})^{n_f}$. $\star^{\mathcal{M}}_{\Sigma(\mathcal{M})_{\mathrm{con}}}$ is surjective thus we can find $d \in \Sigma(\mathcal{M})_{\mathrm{con}}$ such that

$\mathcal{M} \vDash_{\Sigma(\mathcal{M})} f(c) = d$. Hence $f(c) = d \in \mathrm{AtDiag}(\Sigma, \mathcal{M})$. Hence $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} f(c) = d$.

$$\begin{aligned} \iota \circ f^{\mathcal{M}}_{\Sigma(\mathcal{M})}(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) &= \iota(d^{\mathcal{M}}_{\Sigma(\mathcal{M})}) \\ &= d^{\mathcal{N}}_{\Sigma(\mathcal{M})} \\ &= f^{\mathcal{N}}_{\Sigma(\mathcal{M})}(c^{\mathcal{N}}_{\Sigma(\mathcal{M})}) \\ &= f^{\mathcal{N}}_{\Sigma(\mathcal{M})} \circ \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) \end{aligned}$$

Let $r \in \Sigma(\mathcal{M})_{\mathrm{rel}} = \Sigma_{\mathrm{rel}}$ and $c \in (\Sigma(\mathcal{M})_{\mathrm{con}})^{m_r}$.

$$\begin{aligned} c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \in r^{\mathcal{M}}_{\Sigma(\mathcal{M})} &\Rightarrow \mathcal{M} \vDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow r(c) \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \in r^{\mathcal{N}}_{\Sigma(\mathcal{N})} \end{aligned}$$

To show that ι is an embedding it remains to show the backward implication for relations. Let $r \in \Sigma(\mathcal{M})_{\mathrm{rel}} = \Sigma_{\mathrm{rel}}$ and $c \in (\Sigma(\mathcal{M})_{\mathrm{con}})^{m_r}$.

$$\begin{aligned} c^{\mathcal{M}}_{\Sigma(\mathcal{M})} \notin r^{\mathcal{M}}_{\Sigma(\mathcal{M})} &\Rightarrow \mathcal{M} \nvDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow \neg\, r(c) \in \mathrm{AtDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \nvDash_{\Sigma(\mathcal{M})} r(c) \\ &\Rightarrow \iota(c^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c^{\mathcal{N}}_{\Sigma(\mathcal{M})} \notin r^{\mathcal{N}}_{\Sigma(\mathcal{N})} \end{aligned}$$

Assume furthermore that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$. We show that the embedding is elementary. Let ϕ be a $\Sigma(\mathcal{M})$-formula with variables indexed by S and $a \in (\mathcal{M})^S$. Let $c \in (\Sigma_{\mathrm{con}})^S$ be such that $c^{\mathcal{M}}_{\Sigma(\mathcal{M})} = a$.

$$\begin{aligned} \mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi(a) &\Rightarrow \phi(c) \in \mathrm{ElDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi(c) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi(\iota(a)) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{M} \nvDash_{\Sigma(\mathcal{M})} \phi(a) &\Rightarrow \neg\, \phi(c) \in \mathrm{ElDiag}(\Sigma, \mathcal{M}) \\ &\Rightarrow \mathcal{N} \vDash_{\Sigma(\mathcal{M})} \neg\, \phi(c) \\ &\Rightarrow \mathcal{N} \nvDash_{\Sigma(\mathcal{M})} \phi(\iota(a)) \end{aligned}$$

Hence ι is an elementary embedding. Moving $\iota : \mathcal{M} \to \mathcal{N}$ down to being a Σ-morphism of Σ-structures completes the proof.

($\Leftarrow$) Sketch: Suppose $\iota : \mathcal{M} \to \mathcal{N}$ is an elementary embedding. Make $\mathcal{M}$ and $\mathcal{N}$ into $\Sigma(\mathcal{M})$-structures by $\star^{\mathcal{M}}_{\Sigma(\mathcal{M})} : c_a \to a$ and $\star^{\mathcal{N}}_{\Sigma(\mathcal{M})} : c_a \to \iota(a)$, where $a \in \mathcal{M}$. Show that ι is still an elementary embedding when moved up to $\Sigma(\mathcal{M})$. Then for any $\phi \in \mathrm{ElDiag}(\Sigma, \mathcal{M})$, $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \phi$ and so by the embedding being elementary $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \phi$. Hence $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$. □

1.1.7 Universal axiomatization

Definition – Axiomatization, universal theory, universal axiomatization

A Σ-theory A is an axiomatization of a Σ-theory T if for all Σ-structures $\mathcal{M}$,

$$\mathcal{M} \vDash_{\Sigma} T \Leftrightarrow \mathcal{M} \vDash_{\Sigma} A$$

If A is a set of universal Σ-sentences is called a universal Σ-theory. We are interested in universal axiomatizations of theories.

Lemma – Lemma on constants

Suppose $\Sigma_{\text{con}} \subseteq \Sigma(*)_{\text{con}}$, $T \in \Sigma_{\text{the}}$, $\phi \in \Sigma_{\text{for}}$ with variables indexed by $n \in \mathbb{N}$. Suppose there exists a list of constant symbols not from Σ, i.e. $c \in (\Sigma(*)_{\text{con}} \setminus \Sigma_{\text{con}})^n$ such that $T \vDash_{\Sigma(*)} \phi(c)$. Then

$$T \vDash_\Sigma \forall v, \phi(v)$$

Proof. If $n = 0$ then the result is clear as there are no quantifiers. Suppose $n \neq 0$. We prove the contrapositive. Suppose $T \nvDash_\Sigma \forall v, \phi(v)$ then there exists $\mathcal{M}$ a Σ-model of T such that $\mathcal{M} \nvDash_\Sigma \forall v, \phi(v)$. Thus there exists $a \in \mathcal{M}^n$ such that $\mathcal{M} \nvDash_\Sigma \phi(a)$.

We move $\mathcal{M}$ up a signature by extending the interpretation to the new constant symbols: if $d \in \Sigma(*)_{\text{con}} \setminus \Sigma_{\text{con}}$ then

$$d^{\mathcal{M}}_{\Sigma(\mathcal{M})} := \begin{cases} a_i & \text{, if for some } 0 \leq i < n \text{ we have } d = c_i, \\ a_0 & \text{, otherwise} \end{cases}$$

Then $\mathcal{M}$ is a $\Sigma(*)$-model of T such that $\mathcal{M} \nvDash_{\Sigma(*)} \phi(a)$, which by construction is equivalent to $\mathcal{M} \nvDash_{\Sigma(*)} \phi(c)$. □

Notation. *Universal consequences of T Let T be a Σ-theory, then*

$$T_\forall := \{\phi \text{ universal } \Sigma\text{-sentences} \mid T \vDash_\Sigma \phi\}$$

is called the set of universal consequences of T.

Proposition – Universal axiomatizations make substructures models

T a Σ-theory has a universal axiomatization if and only if for any Σ-model $\mathcal{N}$ of T and any Σ-embedding from some Σ-structure $\mathcal{M} \to \mathcal{N}$, $\mathcal{M}$ is a Σ-model of T.

Proof. ($\Rightarrow$) Suppose A is a universal axiomatization of T, $\mathcal{N}$ is a Σ-model of T and $\mathcal{M} \to \mathcal{N}$ is a Σ-embedding. Let $\phi \in T$. Then $\mathcal{N} \vDash_\Sigma T$ implies $\mathcal{N} \vDash_\Sigma A$ by definition of A. $\mathcal{N} \vDash_\Sigma A$ implies $\mathcal{M} \vDash_\Sigma A$ since embeddings preserve the satisfaction of quantifier free formulas downwards. Finally $\mathcal{M} \vDash_\Sigma A$ implies $\mathcal{M} \vDash_\Sigma T$ by definition of A.

($\Leftarrow$) We show that $T_\forall$ is a universal axiomatization of T. Let $\mathcal{M} \vDash_\Sigma T$ and let $\phi \in T_\forall$. Then by definition of $T_\forall$, $T \vDash_\Sigma \phi$. Hence $\mathcal{M} \vDash_\Sigma \phi$ and any Σ-model of T is a Σ-model of $T_\forall$.

Suppose $\mathcal{M} \vDash_\Sigma T_\forall$. We first show that $T \cup \operatorname{AtDiag}(\Sigma, \mathcal{M})$ is consistent. By the compactness theorem it suffices to show that for any subset Δ of $\operatorname{AtDiag}(\Sigma, \mathcal{M})$, $T \cup \Delta$ is consistent. Write $\Delta = \{\psi_1, \ldots, \psi_n\}$. Let $\psi = \bigwedge_{1 \leq i \leq n} \psi_i$. We can find S that indexes the constant symbols in $\Sigma(\mathcal{M})_{\text{con}} \setminus \Sigma_{\text{con}}$ that appear in ψ (in the same way as we made indexing sets of the variables). Then we can create $\phi \in \Sigma_{\text{for}}$ with variables indexed by S such that $\phi(c) = \psi$, where c is a list of constant symbols in $\Sigma(\mathcal{M})_{\text{con}} \setminus \Sigma_{\text{con}}$ indexed by S. Since $\Delta \subseteq \operatorname{AtDiag}(\Sigma, \mathcal{M})$ we have $\forall i, \mathcal{M} \vDash_\Sigma \psi_i$. Hence $\mathcal{M} \vDash_\Sigma \phi(c)$. Then $\mathcal{M} \vDash_\Sigma \exists v, \phi(v)$ and so $\mathcal{M} \nvDash_\Sigma \forall v, \neg \phi(v)$.

Since each ψ_i is from the the atomic diagram of $\mathcal{M}$ they are all quantifier free. Thus ϕ is a quantifier free Σ-formula and $\forall v, \neg \phi(v)$ is universal. Hence $T \nvDash_\Sigma \forall v, \neg \phi(v)$ by the definition of $T_\forall$. By the lemma on constants this implies that $T \nvDash_{\Sigma(\mathcal{M})} \neg \phi(c)$. Hence there exists a $\Sigma(\mathcal{M})$-model of $T \cup \phi(c)$. Then it follows that this is also a $\Sigma(\mathcal{M})$-model of $T \cup \Delta$. Thus $T \cup \Delta$ is consistent so $T \cup \operatorname{AtDiag}(\Sigma, \mathcal{M})$ is consistent.

Thus there exists $\mathcal{N}$ a Σ-model of $T \cup \operatorname{AtDiag}(\Sigma, \mathcal{M})$. This is a model of $\operatorname{AtDiag}(\Sigma, \mathcal{M})$ thus by there is a $\Sigma(\mathcal{M})$-embedding $\mathcal{M} \to \mathcal{N}$. We make this a Σ-embedding, hence using the theorem's hypothesis $\mathcal{M}$ is a Σ-model of T. □

The following result has doesn't come up at all until much later, but is included here as another demonstration of the lemma on constants in use. It appears as an exercise in the second chapter of Marker's book [2].

Corollary – Amalgamation

Let $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures, and suppose we have partial elementary Σ-embeddings $\iota_{\mathcal{M}} : A \to \mathcal{M}$ and $\iota_{\mathcal{N}} : A \to \mathcal{N}$, for $\varnothing \neq A \subseteq \mathcal{A}$. Then there exists a common elementary extension $\mathcal{P}$ of $\mathcal{M}$ and $\mathcal{N}$ such that the following commutes:

$$\begin{tikzcd} \mathcal{M} \arrow[r] & \mathcal{P} \\ A \arrow[u, "\iota_{\mathcal{M}}"] \arrow[r, "\iota_{\mathcal{N}}"] & \mathcal{N} \arrow[u] \end{tikzcd}$$

$\mathcal{P}$ is the 'amalgamation' of $\mathcal{M}$ and $\mathcal{N}$.

Proof. We show first that the theory $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \mathrm{ElDiag}(\Sigma, \mathcal{N})$ is consistent as a $\Sigma(\mathcal{M}, \mathcal{N})$-theory, where $\Sigma(\mathcal{M}, \mathcal{N})_{\mathrm{con}}$ is defined to be

$$\{c_a \mid a \in A\} \cup \{c_a \mid a \in \mathcal{M} \setminus \iota_{\mathcal{M}}(A)\} \cup \{c_a \mid a \in \mathcal{N} \setminus \iota_{\mathcal{N}}(A)\}$$

where terms and formulas from $\Sigma(A), \Sigma(\mathcal{M}), \Sigma(\mathcal{N})$ are interpreted in the natural way: the constants $c_{\iota_{\mathcal{M}}(a)} \mapsto c_a$ (similarly for $\mathcal{N}$). For the rest of the proof we identify $\Sigma(\mathcal{M})_{\mathrm{con}}$ with $\Sigma(A)_{\mathrm{con}} \cup \{c_a \mid a \in \mathcal{M} \setminus \iota_{\mathcal{M}}(A)\}$ (similarly with $\mathcal{N}$).

By the compactness theorem it suffices to show that for any finite subset $\Delta \subseteq \mathrm{ElDiag}(\Sigma, \mathcal{N})$, $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \Delta$ is consistent. Let ϕ be the $\Sigma(\mathcal{M})$-formula and $a \in \mathcal{N}^\star$ be such that [1]

$$\phi(a) = \bigwedge_{\psi \in \Delta} \psi$$

$\phi(a)$ is naturally a $\Sigma(\mathcal{M}, \mathcal{N})$-sentence such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M}, \mathcal{N})} \phi(a)$.

Suppose for a contradiction $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \Delta$ is inconsistent. Then any $\Sigma(\mathcal{M}, \mathcal{N})$-model of $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ is not a model of Δ, which implies it does not satisfy $\phi(a)$. Hence

$$\mathrm{ElDiag}(\Sigma, \mathcal{M}) \vDash_{\Sigma(\mathcal{M}, \mathcal{N})} \neg \phi(a)$$

By the lemma on constants applied to $\Sigma(\mathcal{M}) \leq \Sigma(\mathcal{M}, \mathcal{N})$, $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ and $a \in \Sigma(\mathcal{M}, \mathcal{N})_{\mathrm{con}} \setminus \Sigma(\mathcal{M})_{\mathrm{con}}$ we have

$$\mathrm{ElDiag}(\Sigma, \mathcal{M}) \vDash_{\Sigma(\mathcal{M})} \forall v, \neg \phi(v)$$

Noting that $\mathcal{M}$ is a Σ-model of its elementary diagram, and moving $\mathcal{M}$ down a signature we have that

$$\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \forall v, \neg \phi(v) \Rightarrow \mathcal{M} \vDash_{\Sigma(A)} \forall v, \neg \phi(v) \Rightarrow$$

Since $A \to \mathcal{M}$ and $A \to \mathcal{N}$ are partial elementary Σ-embeddings (and thus naturally $\Sigma(A)$-embeddings) we have that $\mathcal{A} \vDash_{\Sigma(A)} \forall v, \neg \phi(v)$ and so $\mathcal{N} \vDash_{\Sigma(A)} \forall v, \neg \phi(v)$. Move this up to $\Sigma(\mathcal{M}, \mathcal{N})$ we have a contradiction, by choosing v to be a: $\mathcal{N} \vDash_{\Sigma(\mathcal{M}, \mathcal{N})} \neg \phi(a)$, but we remarked before that $\mathcal{N} \vDash_{\Sigma(\mathcal{M}, \mathcal{N})} \phi(a)$.

Hence $\mathrm{ElDiag}(\Sigma, \mathcal{M}) \cup \mathrm{ElDiag}(\Sigma, \mathcal{N})$ is consistent as a $\Sigma(\mathcal{M}, \mathcal{N})$-theory. Let $\mathcal{P}$ be a $\Sigma(\mathcal{M}, \mathcal{N})$-model of this (and naturally a $\Sigma(\mathcal{M})$ or a $\Sigma(\mathcal{N})$ structure). Then there exist elementary $\Sigma(\mathcal{M})$ and $\Sigma(\mathcal{N})$-extensions $\lambda_{\mathcal{M}} : \mathcal{M} \to \mathcal{P}$ and $\lambda_{\mathcal{N}} : \mathcal{N} \to \mathcal{P}$ such that $\lambda_{\mathcal{M}}(c_{m}{}^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c_{m}{}^{\mathcal{P}}_{\Sigma(\mathcal{M})}$ for each constant symbol c_m for $m \in \mathcal{M} \setminus \iota_{\mathcal{M}}(A)$ and $\lambda_{\mathcal{M}}(c_{a}{}^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c_{a}{}^{\mathcal{P}}_{\Sigma(\mathcal{M})}$ for each constant symbol c_a for $a \in A$ (similarly with $\mathcal{N}$).

Naturally, we can move everything down to $\Sigma(A)$. Thus for any $a \in A$

$$\lambda_{\mathcal{M}} \circ \iota_{\mathcal{M}}(a) = \lambda_{\mathcal{M}}(c_{a}{}^{\mathcal{M}}_{\Sigma(A)}) = \lambda_{\mathcal{M}}(c_{a}{}^{\mathcal{M}}_{\Sigma(\mathcal{M})}) = c_{a}{}^{\mathcal{P}}_{\Sigma(\mathcal{M})} = c_{a}{}^{\mathcal{P}}_{\Sigma(A)}$$

By symmetry we have $\lambda_{\mathcal{M}} \circ \iota_{\mathcal{M}}(a) = c_{a}{}^{\mathcal{P}}_{\Sigma(A)} = \lambda_{\mathcal{N}} \circ \iota_{\mathcal{N}}(a)$. □

[1] Take out all the finitely many constants appearing from $\mathcal{N} \setminus \iota_{\mathcal{N}}(A)$ in Δ and make them into a tuple a, replacing them with free variables. What remains is a finite set of $\Sigma(A)$-formulas, which are naturally also $\Sigma(\mathcal{M})$-formulas. We take the 'and' of all of them to be ϕ.

1.1.8 The Löwenheim-Skolem Theorems

The results in this subsection aren't used until much later. It is worth skipping for now, but the material can be covered with the foundations made so far.

Proposition – Upward Löwenheim-Skolem Theorem
If $\mathcal{M}$ is an infinite Σ-structure and κ a cardinal such that $|\mathcal{M}| + |\Sigma_{\text{con}}| \leq \kappa$, there exists a Σ-structure $\mathcal{N}$ with cardinality κ as well as an elementary Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$.

Proof. Make $\mathcal{M}$ a $\Sigma(\mathcal{M})$ structure as in the definition of $\Sigma(\mathcal{M})$, then clearly $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \text{ElDiag}(\Sigma, \mathcal{M})$. Thus $\text{ElDiag}(\Sigma, \mathcal{M})$ is a $\Sigma(\mathcal{M})$-theory with an infinite model $\mathcal{M}$, hence there exists a $\Sigma(\mathcal{M})$-structure $\mathcal{N}$ of cardinality κ such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \text{ElDiag}(\Sigma, \mathcal{M})$. Hence we can make $\mathcal{N}$ a Σ-structure and find a Σ-embedding from $\mathcal{M}$ to $\mathcal{N}$. □

Definition – Skolem Functions
We say that a Σ-theory T *has built in Skolem functions* when for any Σ-formula ϕ that is not a sentence, with free variables indexed by S, there exists $f \in \Sigma_{\text{fun}}$ such that $n_f = k$ and

$$T \vDash_\Sigma \bigvee_{i \in S} w_i, (\exists v, \phi(v, w) \to \phi(f(w), w)),$$

Note that w can be length 0, in which case f has arity 0 and so would be interpreted as a constant map. We would have

$$T \vDash_\Sigma \exists v, \phi(v) \to \phi(f)$$

Proposition – Skolemization
Let $T(0)$ be a $\Sigma(0)$-theory, then there exists T a Σ theory such that

1. $|\Sigma_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$
2. $\Sigma(0)_{\text{fun}} \subseteq \Sigma_{\text{fun}}$, and they share the same constant and relation symbols
3. $T(0) \subseteq T$
4. T has built in Skolem functions
5. All models of $T(0)$ can be moved up to being models of T with interpretations agreeing on Σ.

We call T the Skolemization of $T(0)$.

Proof. Similarly to the Witness Property proof, we define $\Sigma(i), T(i)$ for each $i \in \mathbb{N}$. Suppose by induction that we have $T(i) \in \Sigma_{\text{the}}$, such that

1. $|\Sigma(i)_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$
2. $\Sigma(0)_{\text{fun}} \subseteq \Sigma(i)_{\text{fun}}$ and they share the same constant and relation symbols
3. $T(0) \subseteq T(i)$
4. All models of $T(0)$ can be moved up to being models of $T(i)$ with interpretations agreeing on Σ

Then define $\Sigma(i+1)$ such that only the function symbols are enriched:

$$\Sigma(i+1)_{\text{fun}} := \Sigma(i)_{\text{fun}} \cup \{f_\phi \mid \phi \in \Sigma(i)_{\text{for}} \text{ and } \phi \text{ is not a sentence}\}$$

extending the arity $n_\star$ to by having $n_{f_\phi} = |S| - 1$, where S is indexes the free variables of ϕ. There are countably infinite $\Sigma(i)$-formulas, thus $|\Sigma(i)_{\text{fun}}| = |\Sigma(0)_{\text{fun}}| + \aleph_0$.

Define $\Psi : \Sigma(i)_{\text{for}} \to \Sigma(i+1)_{\text{for}}$ mapping

$$\phi \mapsto \forall w, (\exists v, \phi(v,w)) \to (\phi(f_\phi(w), w)),$$

where w is a list of variables of the suitable length. We then define

$$T(i+1) := T(i) \cup \Psi(\Sigma(i)_{\text{for}})$$

which is a $\Sigma(i+1)$-theory because the image of Ψ has only Σ-sentences. Note that $T(0) \subseteq T(i) \subseteq T(i+1)$.

Let $\mathcal{M}(0)$ be a $\Sigma(0)$-model of $T(0)$, then we have $\mathcal{M}(i)$ a $\Sigma(i)$-model of $T(i)$ with the same carrier set and same interpretation on $\Sigma(0)$ as $\mathcal{M}(0)$. Let $\mathcal{M}(i+1)$ have the same carrier set as $\mathcal{M}(0)$. To extend interpretation to $\Sigma(i+1)$, we first deal with the case where $\mathcal{M}(i)$ is empty by simply interpreting all new function symbols as the empty function. Otherwise we have a $c \in \mathcal{M}(0)$. For $f_\phi \in \Psi(\Sigma(i)_{\text{for}})$ define

$$f_\phi^{\mathcal{M}(i+1)} : \mathcal{M}(i+1)^{n_{f_\phi}} \to \mathcal{M}(i+1)$$
$$a \mapsto \begin{cases} b & \text{, if } \exists b \in \mathcal{M}, \mathcal{M}(i) \vDash_{\Sigma(i)} \phi(b,a) \\ c & \text{, otherwise} \end{cases}$$

Then by construction, $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} \Psi(\Sigma(i)_{\text{for}})$. By checking the conditions on moving $\mathcal{M}(i)$ up to $\mathcal{M}(i+1)$, we can also conclude $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} T(i)$. Hence $\mathcal{M}(i+1) \vDash_{\Sigma(i+1)} T(i+1)$.

Let $\Sigma(*)$ be the signature such that its constants and relations agree with $\Sigma(0)$ and $\Sigma(*)_{\text{fun}} = \bigcup_{i \in \mathbb{N}} \Sigma(i)_{\text{fun}}$. Then

$$|\Sigma(*)_{\text{fun}}| = |\bigcup_{i \in \mathbb{N}} \Sigma(i)_{\text{fun}}| = \aleph_0 \times (\aleph_0 + \Sigma(0)_{\text{fun}}) = \aleph_0 + \Sigma(0)_{\text{fun}}$$

Let $T(*) = \bigcup_{i \in \mathbb{N}} T(i)$. We show that $T(*)$ has built in Skolem functions. Let ϕ be a $\Sigma(*)$-formula that is not a Σ-sentence. Then $\phi \in \Sigma(i)_{\text{for}}$ for some $i \in \mathbb{N}$. Thus $\Psi(\phi) \in T(i+1) \subseteq T(*)$, hence

$$T(*) \vDash_{\Sigma(*)} \forall w, (\exists v, \phi(v,w)) \to (\phi(f_\phi(w), w))$$

Thus $T(*)$ has built in Skolem functions.

If $\mathcal{M} \vDash_\Sigma T$ then let $\mathcal{M}(*) = \mathcal{M}$ and define the interpretation such that for all $i \in \mathbb{N}$, and $f \in T(i)$, $f_{\Sigma(*)}^{\mathcal{M}(*)} = f_{\Sigma(i)}^{\mathcal{M}(i)}$. Since all interpretations agree upon intersection this is well-defined. To show that $\mathcal{M}(*)$ is a $\Sigma(*)$-model of $T(*)$, let ϕ be in $T(*)$; there is some $i \in \mathbb{N}$ such that $\phi \in T(i)$. Using our lifted $\mathcal{M}(i)$ from before we have $\mathcal{M}(i) \vDash_{\Sigma(i)} \phi$. By checking the conditions on moving $\mathcal{M}(i)$ up to $\mathcal{M}(*)$, we can also conclude $\mathcal{M}(*) \vDash_{\Sigma(*)} \phi$ (by taking the $\Sigma(*)$ theory $\{\phi\}$). Hence $\mathcal{M}(*) \vDash_{\Sigma(*)} T(*)$. $\square$

Definition – Theory of a Structure

We define the theory of a Σ-structure $\mathcal{M}$ to be

$$\mathrm{Th}_{\mathcal{M}} := \{\phi \in \Sigma_{\text{for}} \mid \phi \text{ is a } \Sigma\text{-sentence and } \mathcal{M} \vDash_\Sigma \phi\}$$

Proposition – Downward Löwenheim-Skolem Theorem

Let $\mathcal{N}$ be a $\Sigma(0)$-structure and $M(0) \subseteq \mathcal{N}$. Then there exists a $\Sigma(0)$-structure $\mathcal{M}$ such that

- $M(0) \subseteq \mathcal{M} \subseteq \mathcal{N}$
- $|\mathcal{M}| \leq |M(0)| + |\Sigma(0)_{\text{fun}}| + \aleph_0$
- The inclusion $\subseteq : \mathcal{M} \to \mathcal{N}$ is an elementary embedding.

Proof. We first take the Skolemization of $\mathrm{Th}_{\mathcal{N}}$ and call the new signature and theories Σ and T. Since $\mathcal{N} \vDash_{\Sigma(0)} \mathrm{Th}_{\mathcal{N}}$, we can move it up to being a Σ-structure so that $\mathcal{N} \vDash_{\Sigma} T$.

We want to create the carrier set of $\mathcal{M}$, it has to be big enough so that interpreted functions are closed on $\mathcal{M}$. Given $M(i)$ such that $|M(i)| \leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0$, we inductively define $M(i+1)$:

$$M(i+1) := M(i) \cup \{ f^{\mathcal{N}}_{\Sigma}(a) \mid f \in \Sigma_{\text{fun}} \wedge a \in M(i)^{n_f} \}$$

Then

$$\begin{aligned} |M(i+1)| &\leq |M(i)| + |\Sigma_{\text{fun}}| \times |M(i)^{n_f}| \\ &\leq |M(i)| + |\Sigma_{\text{fun}}| \times (|M(i)| + \aleph_0) \\ &\leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0 + |\Sigma_{\text{fun}}| \times (|M(0)| + |\Sigma_{\text{fun}}| + \aleph_0) \\ &\leq |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0 \end{aligned}$$

Then $\mathcal{M} := \bigcup_i M(i)$ and $|\mathcal{M}| \leq |M(i)| \times \aleph_0 = |M(0)| + |\Sigma_{\text{fun}}| + \aleph_0 \leq |M(0)| + |\Sigma(0)_{\text{fun}}| + \aleph_0$.

We first interpret function symbols, which will give us a way to interpret constant symbols. For $f \in \Sigma_{\text{fun}}$ and $a \in (\mathcal{M})^{n_f}$, define $f^{\mathcal{M}}_{\Sigma}(a) = f^{\mathcal{N}}_{\Sigma}(a)$. This is well-defined as there exists $i \in \mathbb{N}$ such that $a \in (M(i))^{n_f}$,

$$f^{\mathcal{M}}_{\Sigma}(a) \in M(i+1) \subseteq \mathcal{M}$$

Then to interpret constant symbols, we consider for each $c \in \Sigma_{\text{con}}$ the formula $v = c$. Since T has built in Skolem functions and $\mathcal{N} \vDash_{\Sigma} T$, there exists f with arity $n_f = 0$ such $\mathcal{N} \vDash_{\Sigma} (\exists v, v = c) \to f = c$. Since $\mathcal{N} \vDash_{\Sigma} \exists v, v = c$, we have $f^{\mathcal{N}}_{\Sigma} = c^{\mathcal{N}}_{\Sigma}$. Since $f^{\mathcal{N}}_{\Sigma} = f^{\mathcal{M}}_{\Sigma} : (\mathcal{M})^0 \to \mathcal{M}$ we can define $c^{\mathcal{M}}_{\Sigma} = c^{\mathcal{N}}_{\Sigma} = f^{\mathcal{N}}_{\Sigma} = f^{\mathcal{M}}_{\Sigma} \in \mathcal{M}$.

Lastly define the interpretation of relations as $r^{\mathcal{M}}_{\Sigma} = (\mathcal{M})^{m_r} \cap r^{\mathcal{N}}_{\Sigma}$.

By construction the inclusion $\subseteq$ is a Σ-embedding. We check that it is elementary using the third equivalent condition in the Tarski Vaught Test: let $\phi \in \Sigma_{\text{for}}$ with free variables indexed by S, $i \in S$ and $a \in (\mathcal{M})^{S \setminus \{i\}}$. Suppose $\exists c \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma} \phi(a, c)$. T has built in Skolem functions, and $\mathcal{N} \vDash_{\Sigma} T$. Hence there exists $f \in \Sigma_{\text{fun}}$ such that

$$\mathcal{N} \vDash_{\Sigma} (\exists v, \phi(a, v)) \to \phi(a, f(a))$$

We can deduce $\mathcal{N} \vDash_{\Sigma} \phi(a, f(a))$. Noting that $f^{\mathcal{M}}_{\Sigma}(a) = f^{\mathcal{N}}_{\Sigma}(a)$ completes the Tarski Vaught Test. Hence $\subseteq$ is an elementary Σ-embedding.

We move $\subseteq : \mathcal{M} \to \mathcal{N}$ down a signature since by Skolemization we have $\Sigma(0) \leq \Sigma$. Then $\subseteq : \mathcal{M} \to \mathcal{N}$ is an elementary Σ-embedding. □

1.2 Types

This section mainly follows material from Tent and Ziegler's book [1].

1.2.1 Types on theories

Definition – $F(\Sigma, n)$ and formulas consistent with a theory
Let $v_1, \ldots v_n$ be variables, T be a Σ-theory. Let $F(\Sigma, n)$ be the set of Σ-formulas with at most $v_1, \ldots v_n$ as their free variables. For any $c \in \Sigma^n_{\text{con}}$, $p \subseteq F(\Sigma, n)$ we write

$$p(c) = \{ \phi(c) \mid \phi \in p \}$$

and if $\mathcal{M}$ is a Σ-structure with $a \in \mathcal{M}^n$ we write

$$\mathcal{M} \vDash_{\Sigma} p(a)$$

to mean for every $\phi \in p$, $\mathcal{M} \vDash_\Sigma \phi(a)$.

We say p is a maximal if for any $\phi \in F(\Sigma, n)$, $\phi \in p$ or $\neg\phi \in p$.

Definition – Consistency for types and compactness for types

Let T be a Σ-theory and p be a subset of $F(\Sigma, n)$. Let $c_1, \ldots, c_n$ be new constant symbols and let $\Sigma(c)$ be the signature with these added constant symbols. The following are equivalent:

1. $T \cup p(c)$ is a consistent $\Sigma(c)$-theory. (Where $p(c)$ is the formulas of p with the variables substituted by $c_1, \ldots, c_n$.)
2. (Consistent with T) There exists $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma p(a)$.
3. (Finitely consistent with T) For any finite subset $\Delta \subseteq p$, there exists $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma p(a)$.

If any of the above is true then we say p is consistent with T and say p is an n-type on T since there are up to n variables in the formulas of p. The second and third definitions being equivalent is the generalisation of compactness for n-types.

Proof. (1. $\Leftrightarrow$ 2.) ($\Rightarrow$) Suppose we have a $\Sigma(c)$-structure $\mathcal{M} \vDash_{\Sigma(c)} T \cup p(c)$. Then by taking the images of the interpretation of each c_i in $\mathcal{M}$ we obtain $a = c^{\mathcal{M}} \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_{\Sigma(c)} p(a)$. Moving this down to Σ preserves satisfaction of $p(a)$ as elements of $p(a)$ are Σ-formulas with values in $\mathcal{M}$ (and T for the same reason):

$$\mathcal{M} \vDash_\Sigma T \cup p(a)$$

and we have what we want.

($\Leftarrow$) Suppose we have $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma p(a)$. We can make $\mathcal{M}$ a $\Sigma(c)$-structure such that everything from Σ is interpreted in the same way and each constant symbol c_i is interpreted as a_i. Thus $\mathcal{M} \vDash_{\Sigma(c)} T$ and for any $\phi(c) \in p(c)$,

$$\mathcal{M} \vDash_\Sigma \phi(a) \Rightarrow \mathcal{M} \vDash_{\Sigma(c)} \phi(a) \Rightarrow \mathcal{M} \vDash_{\Sigma(c)} \phi(c)$$

as c is interpreted as a. Hence $\mathcal{M} \vDash_{\Sigma(c)} T \cup p(c)$ and $T \cup p(c)$ is consistent in $\Sigma(c)$.

(2. $\Leftrightarrow$ 3.)

$$\begin{aligned}
&p \text{ consistent with } T\\
&\Leftrightarrow T \cup p(c) \text{ consistent in } \Sigma(c) \quad \text{by (1. } \Leftrightarrow \text{ 2.)}\\
&\Leftrightarrow \text{for any finite } \Delta(c) \subseteq p(c), T \cup \Delta(c) \text{ consistent in } \Sigma(c) \quad \text{by compactness}\\
&\Leftrightarrow \text{for any finite } \Delta \subseteq p, T \cup \Delta(c) \text{ consistent in } \Sigma(c)\\
&\Leftrightarrow \text{for any finite } \Delta \subseteq p, \Delta \text{ consistent with } T \quad \text{by (1. } \Leftrightarrow \text{ 2.)}
\end{aligned}$$

□

Definition – Stone space of a theory

Let T be a Σ-theory. Let the stone space of T, $S_n(T)$ be the set of all maximal n-types on T. (The signature of the n-types of the on T is implicit, given by the signature of T.) We give a topology on $S_n(T)$ by specifying an open basis; $U \subseteq S_n(T)$ is an element of the basis when there exists $\phi \in F(\Sigma, n)$ such that

$$U = [\phi]_T := \{p \in S_n(T) \mid \phi \in p\}$$

Proposition – Extending to maximal n-types (Zorn)
Any n-type can be extended to a maximal n-type.

Proof. Let T be a theory and p be a n-type. Order by inclusion the set

$$Z = \{q \in S_n(T) \,|\, q \text{ is an } n\text{-type and } p \subseteq q\}$$

This is non-empty as it contains p. Let $p_0 \subseteq p_1 \subseteq \ldots$ be a chain in Z. Then $m = \bigcup_{i \in \mathbb{N}} p_i$ is finitely consistent with T (by taking large enough i) and so is consistent with T. By Zorn we have the existence of a maximal element q in Z. To show that q is a maximal n-type let $\phi \in F(\Sigma, n)$. As q is consistent with T there exists a Σ-structure $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma q(a)$. In the case that $\mathcal{M} \vDash_\Sigma \phi(a)$ we have $q \cup \{\phi\}$ is consistent with T and so by maximality $\phi \in q$. In the other case $q \cup \{\neg \phi\}$ is consistent and so $\neg \phi \in q$. □

Proposition – Basic facts about the basis elements
Let T be a Σ-theory, $\phi, \psi \in F(\Sigma, n)$.

- $(\neg \phi) \in p$ if and only if $p \notin [\psi]_T$.
- $[\phi]_T = [\psi]_T$ if and only if ϕ and ψ are equivalent modulo T.

  The basis elements are closed under Boolean operations
- $[\top]_T = S_n(T)$
- $[\neg \phi]_T = S_n(T) \setminus [\phi]_T$
- $[\phi \vee \psi]_T = [\phi]_T \cup [\psi]_T$
- $[\bot]_T = \varnothing$
- $[\phi \wedge \psi]_T = [\phi]_T \cap [\psi]_T$

Proof.

- Suppose $(\neg \phi) \in p$. Then if $p \in [\phi]_T$ then since p is consistent with T there exists a model $\mathcal{M}$ and a from $\mathcal{M}$ such that $\mathcal{M} \vDash_\Sigma \phi(a)$ and $\mathcal{M} \nvDash_\Sigma \phi(a)$, a contradiction. For the other direction, $p \notin [\psi]_T$ and so $\psi \notin p$ and by maximality $\neg \phi \in p$.
- $(\Rightarrow)$ Suppose for a contradiction $T \nvDash_\Sigma \forall v, (\phi \Leftrightarrow \psi)$. then there exists $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma \phi \wedge \neg \psi$ or $\mathcal{M} \vDash_\Sigma (\neg \phi) \wedge \psi$. In the first case we have that $\{\phi, \neg \psi\}$ is consistent with T and so can be extended to a maximal n-type p. Thus $p \in [\phi]_T = [\psi]_T$ and $p \notin [\psi]_T$, a contradiction. $(\Leftarrow)$ Suppose $T \vDash_\Sigma \forall v, (\phi \Leftrightarrow \psi)$. Let $p \in [\phi]_T$. It suffices to show that $p \in [\psi]_T$. Since p is consistent with T there exists a Σ-structure $\mathcal{M} \vDash_\Sigma T$ and $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma p(a)$. By assumption $\mathcal{M} \vDash_\Sigma (\phi \Leftrightarrow \psi)(a)$ and $p \in [\phi]_T$ so $\mathcal{M} \vDash_\Sigma \psi(a)$. Suppose $p \notin \psi$, then $\neg \psi \in p$ hence we have a contradiction.
- For any maximal n-type p, either $\top$ or $\bot$ is in p and in the latter case we have a contradiction as p is consistent with T.
- $[\neg \phi]_T = S_n(T) \setminus [\phi]_T$ follows from the first point.
- $p \in [\phi \vee \psi]_T$ if and only if $(\phi \vee \psi) \in p$. Suppose $\phi \notin p$ then by maximality $(\neg \phi) \in p$ and so $p \in [\psi]_T$. In the other case $p \in [\phi]_T$. For the other direction $p \in [\phi]_T \cup [\psi]_T$ implies $\phi \in p$ or $\psi \in p$. In the first case we have $\mathcal{M} \vDash_\Sigma T$ such that $\mathcal{M} \vDash_\Sigma \phi$. Then $\mathcal{M} \vDash_\Sigma \phi \vee \psi$ and so $(\phi \vee \psi) \in p$.

We omit the last two parts. □

Proposition – Properties of the Stone space
Let T be a theory.

- Elements of the basis of $S_n(T)$ are clopen.
- $S_n(T)$ is Hausdorff.
- $S_n(T)$ is compact.

Proof.

- By maximality of each p the complement of U is also in the open basis:
$$\{p \in S_n(T) \,|\, \phi \notin p\} = \{p \in S_n(T) \,|\, (\neg \phi) \in p\}$$
Hence each element of the basis is clopen.
- Let $p, q \in S_n(T)$ and suppose $p \neq q$. By maximality and the fact that Σ_{for} is non-empty we can assume without loss of generality that there is $\phi \in p \setminus q$. Again by maximality $(\neg \phi) \in q$, and so $p \in [\phi], q \in [\neg \phi]_T$. These opens are disjoint: if $r \in [\phi]_T \cap [\neg \phi]_T$ then as r is consistent with T, there exists $\mathcal{M} \vDash_\Sigma T$ such that $\mathcal{M} \vDash_\Sigma \phi$ and $\mathcal{M} \vDash_\Sigma (\neg \phi)$ a contradiction.
- Let C be a collection of closed sets with finite intersection property. Then each closed set can be written as an intersection of basis elements (a finite union of closed sets is still a basis element since $[\phi] \cup [\psi]_T = [\phi \vee \psi]_T$):
$$C = \left\{ \bigcap_{\phi \in \alpha} [\phi]_T \,|\, \alpha \in I \right\}$$
Let
$$\Gamma = \{\phi \,|\, \phi \in \alpha \in I\} \quad \text{and} \quad [\Gamma]_T = \{[\phi]_T \,|\, \phi \in \alpha \in I\}$$
Then the intersection any finite subset of $[\Gamma]_T$ is non-empty as it contains a finite intersection of elements in C. Thus for any finite subset $\Delta \subseteq \Gamma$ there exists $p \in S_n(T)$ such that $\Delta \subseteq p$, and as p is consistent with T so is Δ. Hence Γ is finitely consistent with T and by Γ is consistent with T. Extending Γ to a maximal n-type q gives us $\phi \in q$ for every $\phi \in \Gamma$. Hence for all $\alpha \in I$ and for all $\phi \in \alpha$, $p \in [\phi]_T$ and the intersection of C is non-empty.

□

Stone space is meant to have a geometric interpretation as $\text{Spec}(\mathcal{M}[x_1, \ldots, x_n])$ when $\mathcal{M}$ is an algebraically closed field. We will this in a few results.

1.2.2 Types on structures

Definition – Realisation

Let $\mathcal{M}$ be a Σ-structure and $A \subseteq \mathcal{M}$. Let p be a subset of $F(\Sigma(A), n)$ (we will often be considering the n-types on $\text{ElDiag}(\Sigma, \mathcal{M})$, a special case of this where $A = \mathcal{M}$). Let $\mathcal{N}$ be a $\Sigma(A)$-structure.

- p is realised in $\mathcal{N}$ by $a \in \mathcal{N}^n$ over A if
$$\mathcal{N} \vDash_{\Sigma(A)} p(a)$$
We also just say p is realised in $\mathcal{N}$. If p is not realised in $\mathcal{N}$ then we say $\mathcal{N}$ omits p.
- p is finitely realised in $\mathcal{N}$ over A if for each finite subset $\Delta \subseteq p$ there exists $a \in \mathcal{N}^n$ such that Δ is realised in $\mathcal{M}$ by a.

Lemma – Finite realisation and embeddings

Let $\mathcal{M}$ be a Σ-structure, A a subset of $\mathcal{M}$ and p a subset of $F(\Sigma(A), n)$. Then the following are equivalent

- p is consistent with $\text{ElDiag}(\Sigma, \mathcal{M})$ (i.e. it is an n-type over $\text{ElDiag}(\Sigma, \mathcal{M})$).

- There exists an elementary embedding $\mathcal{M} \to \mathcal{N}$ and a $b \in \mathcal{N}^n$ such that p is realised by b in $\mathcal{N}$.
- There exists an elementary embedding $\mathcal{M} \to \mathcal{N}$ such that p is finitely realised in $\mathcal{N}$.
- p is finitely realised in $\mathcal{M}$.

The elementary embeddings can be seen as both Σ-embeddings or $\Sigma(A)$-embeddings for any subset $A \subseteq \mathcal{M}$.

Proof. (1. $\Rightarrow$ 2.) If there exists $\mathcal{N}$ and a $b \in \mathcal{N}^n$ such that $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} \mathrm{ElDiag}(\Sigma, \mathcal{M})$ and $\mathcal{N} \vDash_{\Sigma(\mathcal{M})} p(b)$. Then since models of the elementary diagram correspond to elementary extensions, there exists an elementary $\Sigma(\mathcal{M})$-embedding $\mathcal{M} \to \mathcal{N}$ and $b \in \mathcal{N}^n$ such that p is realised by b in $\mathcal{N}$. (This can be moved down to being a $\Sigma(A)$-embedding for any subset $A \subseteq \mathcal{M}$.)

(2. $\Rightarrow$ 3.) Let $\Delta \subseteq p$ be finite. Then for the same embedding into $\mathcal{N}$ we can see that Δ is realised by b in $\mathcal{N}$.

(3. $\Rightarrow$ 4.) Let $\Delta \subseteq p$ be finite. Then by assumption there exists an elementary $\Sigma(A)$-embedding $\iota : \mathcal{M} \to \mathcal{N}$ and $b \in \mathcal{N}^n$ such that $\mathcal{N} \vDash_{\Sigma(A)} \Delta(b)$. Choose the constant symbols $c_1, \dots, c_m$ from $\Sigma(\mathcal{M})$ such that $c_i^{\mathcal{N}} = b_i$ for each i. We take $a = (c_1^{\mathcal{M}}, \dots, c_n^{\mathcal{M}})$ to realise Δ. Note that since embeddings commute with interpretation of constants, $\iota(a) = \iota(c^{\mathcal{M}}) = c^{\mathcal{N}} = b$ and since the embedding is elementary

$$\mathcal{N} \vDash_{\Sigma(A)} \phi(b) \Rightarrow \mathcal{N} \vDash_{\Sigma(A)} \phi(\iota(a)) \Rightarrow \mathcal{M} \vDash_{\Sigma(A)\phi(a)}$$

(4. $\Rightarrow$ 1.) By compactness for types, it suffices to show that p is finitely consistent with the elementary diagram. Let $\Delta \subseteq p$ be finite. Then by assumption there is $a \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_{\Sigma(A)(a)} \Delta(a)$ and so $\mathcal{M} \vDash_{\Sigma(\mathcal{M})} \Delta(a)$. Clearly $\mathcal{M}$ is a model of its elementary diagram. Hence we have satisfied the conditions for 2. ☐

Definition – Type of an element

Let $\mathcal{M}$ be a Σ-structure containing $A \subseteq \mathcal{M}$ and $a \in \mathcal{M}^n$ Then

$$\mathrm{tp}^{\mathcal{M}}_{A,n}(a) := \left\{\phi \in F(\Sigma(A), n) \mid \mathcal{M} \vDash_{\Sigma(A)} \phi(a)\right\}$$

is the type of a in $\mathcal{M}$ over A. One can verify that this is a maximal n-type on T if $\mathcal{M}$ is a model of T.

Proposition

A type p of a Σ-theory T is realised by a in an extension $\mathcal{N}$ if and only if $p = \mathrm{tp}(a)^{\mathcal{N}}_{\varnothing,n}$. Any element of the Stone space is the type of an element.

Proof. If p is realised by a in $\mathcal{N}$ then $\mathcal{N} \vDash_{\Sigma} p(a)$ hence $p \subseteq \mathrm{tp}^{\mathcal{M}}_{\varnothing,n}(a)$ and by maximality of p they are equal.

Any element of the Stone space is realised in some Σ-structure due to consistency hence it is the type of an element by the above. ☐

Proposition – All maximal realised n-types are types of an element

Let $\mathcal{M}$ be a Σ-structure, A a subset of $\mathcal{M}$ and p a subset of $F(\Sigma(A), n)$. Let $a \in \mathcal{M}^n$. Then

- p is a maximal n-type on $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ that is realised by $a \in \mathcal{M}^n$ if and only if $p = \mathrm{tp}^{\mathcal{M}}_{A,n}(a)$.
- If $\mathcal{M} \subseteq \mathcal{N}$ is an elementary embedding then

$$\mathrm{tp}^{\mathcal{M}}_{A,n}(a) = \mathrm{tp}^{\mathcal{N}}_{A,n}(a)$$

Proof.

- ($\Rightarrow$) As p is realised by a, $p \subseteq \mathrm{tp}^{\mathcal{M}}_{A,n}(a)$. By maximality of p any formula in $\mathrm{tp}^{\mathcal{M}}_{A,n}(a)$ is either in p or its negation is in p. If its negation is in $p \subseteq \mathrm{tp}^{\mathcal{M}}_{A,n}(a)$ we have a contradiction as this would imply $\mathcal{M} \vDash_{\Sigma(A)} \phi(a)$ and $\mathcal{M} \nvDash_{\Sigma(A)} \phi(a)$. ($\Leftarrow$) If $p = \mathrm{tp}^{\mathcal{M}}_{A,n}(a)$ then clearly p is realised by a and so it is consistent with $\mathrm{ElDiag}(\Sigma, \mathcal{M})$ thus it is an n-type over $\mathrm{ElDiag}(\Sigma, \mathcal{M})$. For any $\phi \in F(\Sigma(A), n)$, $\mathcal{M} \vDash_{\Sigma(A)} \phi(a)$ or $\mathcal{M} \nvDash_{\Sigma(A)} \phi(a)$. Hence ϕ or $\neg\,\phi$ is in p and so it is maximal.

- $$\phi \in \mathrm{tp}^{\mathcal{M}}_{A,n}(a) \Leftrightarrow \mathcal{M} \vDash_{\Sigma(A)} \phi(a) \Leftrightarrow \mathcal{N} \vDash_{\Sigma(A)} \phi(a) \Leftrightarrow \phi \in \mathrm{tp}^{\mathcal{N}}_{A,n}(a)$$

□

1.2.3 Quantifier free types

Quantifier free types become important when we start making back and forth constructions. All the results carry through, thus there isn't much to say until they are relevant. Thus this section is worth skipping for now.

Definition – $QFF(\Sigma, n)$ and formulas consistent with a theory
We define $QFF(\Sigma, n)$ to be the subset of $F(\Sigma, n)$ of quantifier free formulas.

For subsets $p \subseteq QFF(\Sigma, n) \subseteq F(\Sigma, n)$ the definitions of consistency carry through and we can apply compactness for types on these subsets too. The definition of maximality is the same but restricted to $QFF(\Sigma, n)$.

Let T be a Σ-theory. Any subset $p \subseteq QFF(\Sigma, n)$ that is consistent with T is called a quantifier free n-type on T. Note that any quantifier free n-type is an n-type.

Definition – Quantifier free Stone space of a theory
Let T be a Σ-theory. Let the stone space of T, $S^{\mathrm{qf}}_n(T)$ be the set of all maximal quantifier free n-types on T. We give the same topology: $U \subseteq S_n(T)$ is an element of the basis when there exists $\phi \in QFF(\Sigma, n)$ such that
$$U = [\phi]^{\mathrm{qf}}_T := \left\{p \in S^{\mathrm{qf}}_n(T) \,|\, \phi \in p\right\}$$

Proposition – Extending to maximal quantifier free n-types
Any quantifier free n-type on a theory can be extended to a maximal quantifier free n-type.

Proof. Any quantifier free n-type is an n-type, hence can be extended to a maximal n-type. The intersection of a maximal n-type with QFF is a maximal quantifier free n-type. This intersection extends the quantifier free n-type and we are done. □

Proposition – Properties of the Stone space
Let T be a Σ-theory, $\phi, \psi \in QFF(\Sigma, n)$. Elementary properties:

- $(\neg\,\phi) \in p$ if and only if $p \notin [\psi]^{\mathrm{qf}}_T$.
- $[\phi]_T = [\psi]^{\mathrm{qf}}_T$ if and only if ϕ and ψ are equivalent modulo T.
- $[\top]^{\mathrm{qf}}_T = S^{\mathrm{qf}}_n(T)$
- $[\neg\,\phi]^{\mathrm{qf}}_T = S_n(T) \setminus [\phi]^{\mathrm{qf}}_T$
- $[\phi \vee \psi]^{\mathrm{qf}}_T = [\phi]^{\mathrm{qf}}_T \cup [\psi]^{\mathrm{qf}}_T$
- $[\bot]^{\mathrm{qf}}_T = \varnothing$

- $[\phi \wedge \psi]_T^{\mathrm{qf}} = [\phi]_T^{\mathrm{qf}} \cap [\psi]_T^{\mathrm{qf}}$

Topological properties:

- Elements of the basis of $S_n^{\mathrm{qf}}(T)$ are clopen.
- $S_n^{\mathrm{qf}}(T)$ is Hausdorff.
- $S_n^{\mathrm{qf}}(T)$ is compact.

Proof. The proofs of these are exactly the same as before, some of them can even be bypassed by using the previous results. □

Definition – Quantifier free type of an element

Let $\mathcal{M}$ be a Σ-structure, $A \subseteq \mathcal{M}$ and $a \in \mathcal{M}^n$ Then

$$\mathrm{qftp}_{A,n}^{\mathcal{M}}(a) := \left\{ \phi \in QFF(\Sigma(A), n) \,|\, \mathcal{M} \vDash_{\Sigma(A)} \phi(a) \right\}$$

is the quantifier free type of a in $\mathcal{M}$ over A. One can verify that this is a maximal quantifier free n-type on T if $\mathcal{M}$ is a model of T. We will often drop parts of the subscripts and superscripts when it is clear. In fact the n can be deduced by the length of a and serves only to explicitely spell things out.

1.3 Quantifier elimination and model completeness

Written whilst following section on algebraically closed fields.

1.3.1 Quantifier elimination

Definition – Equivalence modulo a theory

We say two Σ-formulas ϕ and ψ with free variables indexed by S are equivalent modulo a Σ-theory T if

$$T \vDash_\Sigma \forall v, (\phi \Leftrightarrow \psi)$$

where $v = (v_i)_{i \in S}$.

Definition – Quantifier elimination

Let T be a Σ-theory and ϕ a Σ-formula. We say the quantifiers of ϕ can be eliminated if there exists a quantifier free Σ-formula ψ that is equivalent to ϕ modulo T. We say ϕ is reduced to ψ.

We say T has quantifier elimination if the quantifiers of any Σ-formula can be eliminated.

Lemma – Deduction

Let T be a Σ-theory, Δ a finite Σ-theory and ψ a Σ-sentence. Then $T \cup \Delta \vDash_\Sigma \psi$ if and only if

$$T \vDash_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$$

Proof. We first case on if Δ is empty or not. If it is empty then $T \cup \Delta \vDash_\Sigma \psi$ if and only if $T \vDash_\Sigma \psi$ if and only if $T \vDash_\Sigma \top \to \psi$ if and only if

$$T \vDash_\Sigma \left(\bigwedge_{\phi \in \Delta} \phi \right) \to \psi$$

($\Rightarrow$) Suppose $\mathcal{M} \vDash_\Sigma T$ then we need to show $\mathcal{M} \vDash_\Sigma \left(\bigwedge_{\phi\in\Delta} \phi\right) \to \psi$. Indeed, suppose $\mathcal{M} \vDash_\Sigma \left(\bigwedge_{\phi\in\Delta} \phi\right)$ then by induction $\mathcal{M} \vDash_\Sigma T \cup \Delta$ and so by assumption that $T \cup \Delta \vDash_\Sigma \psi$ we have $\mathcal{M} \vDash_\Sigma \psi$. Hence $\mathcal{M} \vDash_\Sigma \left(\bigwedge_{\phi\in\Delta} \phi\right) \to \psi$.

($\Leftarrow$) Suppose $\mathcal{M} \vDash_\Sigma T \cup \Delta$ then $\mathcal{M} \vDash_\Sigma T$ thus by assumption that $T \vDash_\Sigma \left(\bigwedge_{\phi\in\Delta} \phi\right) \to \psi$ we have $\mathcal{M} \vDash_\Sigma \left(\bigwedge_{\phi\in\Delta} \phi\right) \to \psi$. By induction $\mathcal{M} \vDash_\Sigma \left(\bigwedge_{\phi\in\Delta} \phi\right)$ thus we have $\mathcal{M} \vDash_\Sigma \psi$. □

> **Lemma – Proofs are finite**
> Suppose T is a Σ-theory and ϕ a Σ-sentence such that $T \vDash_\Sigma \phi$. Then there exists a finite subset Δ of T such that $\Delta \vDash_\Sigma \phi$.

Proof. We show the contrapositive. Suppose for all finite subsets Δ of T, $\Delta \nvDash_\Sigma \phi$, then $\Delta \cup \{\phi\}$ is consistent and by compactness $T \cup \{\phi\}$ is consistent. Hence $T \nvDash_\Sigma \phi$. □

> **Proposition – Eliminating quantifiers of a formula**
> Let Σ be a signature such that $\Sigma_{\text{con}} \neq \varnothing$. Suppose T is a Σ-theory and ϕ is a Σ-formula with free-variables $v = (v_1, \ldots, v_n)$. Then the quantifiers of ϕ can be eliminated if and only if the following holds: for any two Σ-models $\mathcal{M}, \mathcal{N}$ of T and any Σ-structure $\mathcal{A}$ that with Σ-embeddings into both $\mathcal{M}$ and $\mathcal{N}$ $(\iota_\mathcal{M}, \iota_\mathcal{N})$, if $a \in \mathcal{A}^n$ then
> $$\mathcal{M} \vDash_\Sigma \phi(\iota_\mathcal{M}(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota_\mathcal{N}(a))$$

Proof. ($\Rightarrow$) Let $a \in \mathcal{A}^n$. By assumption there exists $\psi \in \Sigma_{\text{for}}$ such that $T \vDash_\Sigma \forall v, (\phi(v) \leftrightarrow \psi(v))$ Then $\mathcal{M} \vDash_\Sigma \phi(\iota_\mathcal{M}(a))$ if and only if $\mathcal{M} \vDash_\Sigma \psi(\iota_\mathcal{M}(a))$ if and only if $\mathcal{A} \vDash_\Sigma \psi(a)$, since embeddings preserve the satisfaction of quantifier free formulas. Similarly, this is if and only if $\mathcal{N} \vDash_\Sigma \phi(\iota_\mathcal{N}(a))$.

($\Leftarrow$) Let
$$\Gamma := \{\psi \text{ quantifier free } \Sigma_{\text{for}} \,|\, T \vDash_\Sigma \forall v, (\phi \to \psi)\}$$
and let $\Sigma(*)$ be such that $\Sigma(*)_{\text{con}} = \Sigma_{\text{con}} \cup \{d_1, \ldots, d_n\}$ for some new constant symbols d_i (indexed according to the free-variables of ϕ). We claim that $T \cup \{\psi(d) \,|\, \psi \in \Gamma\} \vDash_{\Sigma(*)} \phi(d)$. We first look at how this would complete the proof. If it is true then as proofs are finite we have a finite subsets $\Delta \subseteq \Gamma$ such that $T \cup \{\psi(d) \,|\, \psi \in \Delta\} \vDash_{\Sigma(*)} \phi(d)$. By deduction we have

$$T \vDash_{\Sigma(*)} \left(\bigwedge_{\psi\in\Delta} \psi(d)\right) \to \phi(d)$$

and by the lemma on constants

$$T \vDash_\Sigma \forall v, \left(\bigwedge_{\psi\in\Delta} \psi(v)\right) \to \phi(v)$$

where $\left(\bigwedge_{\psi\in\Delta} \psi(v)\right)$ is quantifier free. By the definition of Δ we have the other implication as well:

$$T \vDash_\Sigma \forall v, \left(\bigwedge_{\psi\in\Delta} \psi(v)\right) \leftrightarrow \phi(v)$$

hence the result.

Suppose for a contradiction $T \cup \{\psi(d) \,|\, \psi \in \Gamma\} \nvDash_{\Sigma(*)} \phi(d)$. Then there exists a model $\mathcal{M}$ of $T \cup \{\psi(d) \,|\, \psi \in \Gamma\}$ such that $\mathcal{M} \nvDash_\Sigma \phi(d)$.

Suppose for a second contradiction that the $\Sigma(*)(\mathcal{M})$-theory $T \cup \mathrm{AtDiag}(\Sigma(*), \mathcal{M}) \cup \{\phi(d)\}$ is inconsistent. Then by compactness some subset $T \cup \Delta \cup \{\phi(d)\}$ is inconsistent, where $\Delta \subseteq \mathrm{AtDiag}(\Sigma(*), \mathcal{M})$ is finite. This implies $T \cup \Delta \vDash_{\Sigma(*)(\mathcal{M})} \neg\,\phi(d)$. Hence by deduction we have

$$T \vDash_{\Sigma(*)(\mathcal{M})} \left(\bigwedge_{\psi(d) \in \Delta} \psi(d) \right) \to \neg\,\phi(d)$$

By the lemma on constants applied to $\Sigma_{\mathrm{con}} \subseteq \Sigma(*)(\mathcal{M})_{\mathrm{con}}$

$$T \vDash_{\Sigma} \forall v, \left[\left(\bigwedge_{\psi(d) \in \Delta} \psi(v) \right) \to \neg\,\phi(v) \right]$$

Taking the contrapositive,

$$T \vDash_{\Sigma} \forall v, \left[\phi(v) \to \left(\bigvee_{\psi(d) \in \Delta} \neg\,\psi(v) \right) \right]$$

Hence $\bigvee_{\psi(d) \in \Delta} \neg\,\psi(v) \in \Gamma$ and so $\mathcal{M} \vDash_{\Sigma(*)} \bigvee_{\psi(d) \in \Delta} \neg\,\psi(v)$ by definition of $\mathcal{M}$. However each $\Delta \subseteq \mathrm{AtDiag}(\Sigma(*), \mathcal{M})$ and so $\mathcal{M} \vDash_{\Sigma(*)} \bigwedge_{\psi(d) \in \Delta} \psi(v)$, a contradiction. Thus there exists a model

$$\mathcal{N} \vDash_{\Sigma(*)(\mathcal{M})} T \cup \mathrm{AtDiag}(\Sigma(*), \mathcal{M}) \cup \{\phi(d)\}$$

Since $\mathcal{N} \vDash_{\Sigma(*)(\mathcal{M})} \mathrm{AtDiag}(\Sigma(*), \mathcal{M})$ there exists a $\Sigma(*)(\mathcal{M})$ morphism $\iota : \mathcal{M} \to \mathcal{N}$. Move this morphism down to Σ, then by assumption with $\mathcal{A} := \mathcal{M}$, for any sentence χ

$$\mathcal{M} \vDash_{\Sigma} \chi \Leftrightarrow \mathcal{N} \vDash_{\Sigma} \chi$$

Since $\mathcal{N} \vDash_{\Sigma(*)(\mathcal{M})} \phi(d)$ by the lemma on constants $\mathcal{N} \vDash_{\Sigma} \forall v, \phi(v)$ and so $\mathcal{M} \vDash_{\Sigma} \forall v, \phi(v)$. Which is a contradiction because $\mathcal{M} \vDash_{\Sigma(*)} \neg\,\phi(d)$ and so by the lemma on constants $\mathcal{M} \vDash_{\Sigma} \forall v, \neg\,\phi(v)$. (We have $\mathcal{M} \vDash_{\Sigma} \phi(c^{\mathcal{M}}, \dots c^{\mathcal{M}})$ and $\mathcal{M} \nvDash_{\Sigma} \phi(c^{\mathcal{M}}, \dots c^{\mathcal{M}})$.) □

> **Lemma – Sufficient condition for quantifier elimination**
> Let T be a Σ-theory and suppose for any quantifier free Σ-formula ψ with at least one free variable w, the quantifier of $\forall w, \psi(w)$ can be eliminated. Then T has quantifier elimination.

Proof. Induct on what ϕ is.

- If ϕ is $\top$, an equality or a relation then it is already quantifier free.
- If ϕ is $\neg\,\chi$ and there exists a quantifier free Σ-formula ψ such that $T \vDash_{\Sigma} \forall v, \chi \leftrightarrow \psi$. Then $T \vDash_{\Sigma} \forall v, \neg\,\chi \leftrightarrow \neg\,\psi$. Hence ϕ can be reduced to $\neg\,\psi$ which is quantifier free.
- If ϕ is $\chi_0 \vee \chi_1$ and there exist respective reductions of these ψ_0 and ψ_1 then ϕ reduces to $\psi_0 \vee \psi_1$ which is quantifier free.
- If ϕ is $\forall w, \chi(w)$ and there exists quantifier free ψ such that

  $$T \vDash_{\Sigma} \forall w, \bigforall_{v \in S} v, (\chi \leftrightarrow \psi)$$

  where S indexes the rest of the free variables in χ and ψ. Then we can show that

  $$T \vDash_{\Sigma} \bigforall_{v \in S} (\phi \leftrightarrow (\forall w, \psi))$$

By assumption there exists ω a quantifier free Σ-formula such that

$$T \vDash_\Sigma \bigvee_{v \in S} (\omega \leftrightarrow (\forall w, \psi))$$

Hence ϕ can be reduced to ω.

□

Corollary – Improvement: Sufficient condition for quantifier elimination
If T be a Σ-theory if for any quanfier free Σ-formula ϕ with at least one free variable w (index the rest by S), for any $\mathcal{M}, \mathcal{N}$ Σ-models of T, for any Σ-structure $\mathcal{A}$ that embeds into $\mathcal{M}$ and $\mathcal{N}$ (via $\iota_\mathcal{M}, \iota_\mathcal{N}$) and any $a \in (\mathcal{A})^S$,

$$\mathcal{M} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{M}(a)) \Rightarrow \mathcal{N} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{N}(a))$$

then T has quantifier elimination.

Equivalently we can use the statement

$$\mathcal{M} \vDash_\Sigma \exists w, \phi(\iota_\mathcal{M}(a)) \Rightarrow \mathcal{N} \vDash_\Sigma \exists w, \phi(\iota_\mathcal{N}(a))$$

by negating ϕ.

Proof. To show that T has quantifier elimination it suffices to show that for any quanfier free Σ-formula ϕ with at least one free variable w (index the rest by S), the quantifiers of $\forall w, \phi$ can be eliminated. This is true if and only if for any $\mathcal{M}, \mathcal{N}$ Σ-models of T, for any Σ-structure $\mathcal{A}$ that injects into $\mathcal{M}$ and $\mathcal{N}$ (via $\iota_\mathcal{M}, \iota_\mathcal{N}$) and any $a \in (\mathcal{A})^S$,

$$\mathcal{M} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{M}(a)) \Rightarrow \mathcal{N} \vDash_\Sigma \forall w, \phi(\iota_\mathcal{N}(a))$$

By symmetry of $\mathcal{M}$ and $\mathcal{N}$ we only require one implication. Hence the proposition. □

Remark. For quantifier elimination it also suffices to show that for any Σ-models $\mathcal{M}$ of T, for any Σ-structure $\mathcal{A}$ that embeds into $\mathcal{M}$ (via $\iota_\mathcal{M}$) and any $a \in (\mathcal{A})^S$,

$$\mathcal{M} \vDash_\Sigma \forall w, \phi(\iota(a)) \Rightarrow \mathcal{A} \vDash_\Sigma \forall w, \phi(a)$$

since embeddings preserve satisfaction of universal formulas downwards. Equivalently we can use

$$\mathcal{A} \vDash_\Sigma \exists w, \phi(a) \Rightarrow \mathcal{M} \vDash_\Sigma \exists w, \phi(\iota(a))$$

1.3.2 Back and Forth

'Back and forth' is a technique used to determine elementary equivalence of models, quantifier elimination of theories and completeness of theories. This section draws together work from Poizat [5], OLP [3], and Pillay [4]. It is motivated by the example at the end, which should be looked at first.

Definition – Substructure generated by a subset
Let $\mathcal{M}$ be a Σ-structure. Let $A \subseteq \mathcal{M}$. Then the following are equal:

- The set $\langle A \rangle$ defined inductively: $A \subseteq \langle A \rangle$; if $c \in \Sigma_{\text{con}}$ then $c^\mathcal{M} \in \langle A \rangle$; if $f \in \Sigma_{\text{fun}}$ and $\alpha \in \langle A \rangle^{n_f}$ then $f^\mathcal{M}(\alpha) \in \langle A \rangle$.
- $\bigcap \{\mathcal{N} \text{ substructure of } \mathcal{M} \mid A \subseteq \mathcal{N}\}$

and define a substructure of $\mathcal{M}$. We say it is the 'substructure of $\mathcal{M}$ generated by A'.

We say a substructure is finitely generated if there exists a finite set A such that it is equal to $\langle A \rangle$.

Proof. We note show that $\langle A \rangle$ is a substructure of $\mathcal{M}$ containing A: It contains the interpretations of constant symbols from $\mathcal{M}$. By definition $f^{\langle A \rangle} := f^{\mathcal{M}}$ is well defined. Each relation r is naturally interpreted as the intersection of relations on $\mathcal{M}$ intersected with $\langle A \rangle^{m_r}$. Hence $\bigcap \mathcal{N} \subseteq \langle A \rangle$.

For the other direction note that if $a \in \langle A \rangle$ then it is in A, $c^{\mathcal{M}}$ or $f^{\mathcal{M}}(\alpha)$ for some $\alpha \in \langle A \rangle^{n_f}$. If it is in A then we are done. Any substructure of $\mathcal{M}$ contains the $c^{\mathcal{M}}$ for each constant symbol hence the first case is fine. Any substructure of $\mathcal{M}$ is closed under $f^{\mathcal{M}}$ and by induction $\alpha \in \mathcal{N}^{n_f}$ for any substructure $\mathcal{N}$. Hence $f(\alpha) \in \mathcal{N}$ for any substructure. Thus $\langle A \rangle \subseteq \bigcap \mathcal{N}$ and we are done. □

Proposition – Image of generators are generators of the image
The image of a substructure generated by a subset is a substructure generated by the image of a set. In particular, a finitely generated substructure has finitely generated image under a Σ-morphism given by the image of the generators.

Proof. Let $\iota : \langle A \rangle \to \mathcal{N}$ be a Σ-morphism. We show that $\langle \iota(A) \rangle = \iota(\langle A \rangle)$. If $b \in \langle \iota(A) \rangle$ then $b = c^{\mathcal{N}}$ or $b = f^{\mathcal{N}}(\iota(\alpha))$ for $al \in \langle A \rangle$. Hence $b = c^{\mathcal{N}} = \iota(c^{\mathcal{M}}) \in \iota(\langle A \rangle)$ or

$$b = f^{\mathcal{N}}(\iota(\alpha)) = \iota(f^{\mathcal{M}}(\alpha)) \in \iota(\langle A \rangle)$$

Thus $\langle \iota(A) \rangle \subseteq \iota(\langle A \rangle)$. The other direction is similar. □

Definition – Partial isomorphisms
Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. A partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$ is a Σ-isomorphism p with finitely generated domain in of $\mathcal{M}$ and codomain in $\mathcal{N}$.

Proposition – Equivalent definition of partial isomorphism
Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. Let $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$. The following are equivalent:

- There exists a partial isomorphism $p : \langle a \rangle \to \langle b \rangle$ such that $p(a) = b$.
- $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$

Proof. ($\Rightarrow$) We induct on terms to show that $t^{\mathcal{M}}(a) = t^{\langle a \rangle}(a)$ for each term t:

- If t is a constant symbol or a variable then by definition of the the substructure interpretation they are equal.
- If t is $f(s)$ and we have the inductive hypothesis $s^{\mathcal{M}}(a) = s^{\langle a \rangle}(a)$ then by definition of the substructure interpretation
$$t^{\mathcal{M}}(a) = f^{\mathcal{M}}(s^{\mathcal{M}}(a)) = f^{\mathcal{M}}(s^{\langle a \rangle}(a)) = f^{\langle a \rangle}(s^{\langle a \rangle}(a)) = t^{\langle a \rangle}(a)$$

Let ϕ be a quantifier free Σ-formula with up to n variables. We show by induction on ϕ that

$$\mathcal{M} \vDash_{\Sigma} \phi(a) \Leftrightarrow \langle a \rangle \vDash_{\Sigma} \phi(a)$$

- If ϕ is $\top$ it is trivial.
- If ϕ is $t = s$ then it is clear that
$$t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a) \Leftrightarrow t^{\langle a \rangle}(a) = s^{\langle a \rangle}(a)$$
by what we showed for terms.
- If ϕ is $r(t)$ then
$$(a_{i_1}, \dots, a_{i_m}) \in r^{\mathcal{M}} \Leftrightarrow (a_{i_1}, \dots, a_{i_m}) \in r^{\mathcal{M}} \cap \langle a \rangle = r^{\langle a \rangle}$$

- If ϕ is $\neg\psi$ or $\psi \vee \chi$ then it is clear by induction.

As p is an Σ-isomorphism, for any quantifier free Σ-formula with up to n variables,

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \langle a\rangle \vDash_\Sigma \phi(a) \Leftrightarrow \langle b\rangle \vDash_\Sigma \phi(b)\mathcal{N} \vDash_\Sigma \phi(b)$$

($\Leftarrow$) Suppose $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$. We define $p : \langle a\rangle \to \mathcal{N}$ by the following: if $\alpha \in \langle a\rangle$ then one can write α as a term t evaluated at a: $\alpha = t^{\mathcal{M}}(a)$; p maps a to $t^{\mathcal{N}}(b)$. To show that p is well-defined, note that if two terms t and s are such that $t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)$ then $t = s$ is a formula in $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$ and so $t^{\mathcal{N}}(b) = s^{\mathcal{N}}(b)$. it is injective because if two terms t and s are such that $t^{\mathcal{N}}(b) = s^{\mathcal{N}}(b)$ then $t = s$ is a formula in $\mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b) = \mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a)$ and so $t^{\mathcal{M}}(a) = s^{\mathcal{M}}(a)$.

By definition p commutes with the interpretation of constant symbols, function symbols, and relations. Furthermore, for each i, $p(a_i) = b_i$ by taking the term to be a variable and evaluating at a_i. The image of p is $\langle b\rangle$ as the image of a is b. Hence it is a partial isomorphism $\langle a\rangle \to \langle b\rangle$ such that $p(a) = b$. □

Proposition – Basic facts about partial isomorphisms

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures.

- The inverse of a partial isomorphism is a partial isomorphism.
- The restriction of a partial isomorphism is a partial isomorphism.
- The composition of partial isomorphisms is a partial isomorphism.

Definition – Partially isomorphic structures

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. A partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$ is said to have the back and forth property if

- (Forth) For each $a \in \mathcal{M}$ there exists a partial isomorphism q such that q extends p and $a \in \operatorname{dom} p$.
- (Back) For each $p \in I$ there exists a partial isomorphism q such that q extends p and $b \in \operatorname{codom} q$.

We say $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent when all partial isomorphisms from $\mathcal{M}$ to $\mathcal{N}$ have the back and forth property.

Proposition – Equivalent definition of back and forth property

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. Let $p : \langle a\rangle \to \langle b\rangle$ for $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ be a partial isomorphism such that $p(a) = b$. It has the back and forth property if and only if the two conditions hold

- (Forth) For any $\alpha \in \mathcal{M}$, there exists $\beta \in \mathcal{N}$ such that $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a, \alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b, \beta)$
- (Back) For any $\beta \in \mathcal{N}$, there exists $\alpha \in \mathcal{M}$ such that $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a, \alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b, \beta)$

Proof. ($\Rightarrow$) Suppose p has the back and forth property. We only show 'forth' as the 'back' case is similar. Let $\alpha \in \mathcal{M}$. By 'forth' there exists q a partial isomorphism extending p such that $\alpha \in \mathrm{dom}(q)$. By restriction and the fact that the image of generators generates the image, there exists $\beta \in \mathcal{N}$ such that

$$q\big|_{\langle a,\alpha\rangle \to \langle b,\beta\rangle}$$

is a local isomorphism. Using the the equivalent definition we obtain $\mathrm{qftp}(a, \alpha) = \mathrm{qftp}(b, \beta)$.

($\Leftarrow$) We show that p has the 'forth' property. Let $\alpha \in \mathcal{M}$. By assumption there exists $\beta \in \mathcal{M}$ such that

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a, \alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b, \beta)$$

Thus there exists $q : \langle a, \alpha\rangle \to \langle b, \beta\rangle$ such that $q(a) = b$ and $q(\alpha) = \beta$. Hence p is extended by q with α in its domain. □

Proposition – Quantifier elimination for types

Let T be a Σ-theory. T has quantifier elimination if and only if for any $n \in \mathbb{N}$, any two Σ-models of T and any $a \in \mathcal{M}^n, b \in \mathcal{N}^n$, if

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

then

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

Proof. ($\Rightarrow$) Let $\phi \in \mathrm{tp}(a)$. By quantifier elimination there exists quantifier free ψ such that they are equivalent modulo T. Then $\mathcal{M} \vDash_\Sigma \psi(a)$ and $\psi \in \mathrm{qftp}(a) = \mathrm{qftp}(b)$. Thus $\mathcal{N} \vDash_\Sigma \psi(b)$ and by equivalence modulo T $\mathcal{N} \vDash_\Sigma \phi(b)$. Hence $\phi \in \mathrm{tp}(b)$. The other inclusion is similar.

($\Leftarrow$) Let $n \in \mathbb{N}$. Define a map $f : S_n(T) \to S^{\mathrm{qf}}_n(T)$ that takes a maximal n-type p to $p \cap QFF(\Sigma, n)$. It is well-defined as the image is indeed a maximal n-type. It is a surjection as any quantifier free maximal n type is an n-type and therefore can be extended to a maximal n-type. To show injectivity we note that any two elements of $S_n(T)$ can be written as types of elements $\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a)$ and $\mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$. If their images are equal then

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

thus by assumption they are equal.

To show that f is continuous we show that elements of the clopen basis have clopen preimage. Let $[\phi]^{\mathrm{qf}}_T$ be in the clopen basis of $S^{\mathrm{qf}}_n(T)$. Then $p \in [\phi]_T$ if and only if $\phi \in p$ if and only if $\phi \in f(p)$ if and only if $f(p) \in [\phi]^{\mathrm{qf}}_T$. Hence the preimage is $[\phi]_T$ which is clopen.

A continuous bijection between Hausdorff compact spaces is a homeomorphism. Hence for any $\phi \in F(\Sigma, n)$ the image of the clopen set generated by ϕ is clopen: there exists $\psi \in QFF(\Sigma, n)$ such that $[\phi]_T = f^{-1}[\psi]^{\mathrm{qf}}_T = [\psi]_T$. $[\phi]_T = [\psi]_T$ if and only if they are equivalent modulo T. Thus we can eliminate quantifiers for any $\phi \in F(\Sigma, n)$ for any n. Thus T has quantifier elimination. □

Lemma – Back and forth equivalence implies quantifier elimination for types

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. If $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent and $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ are such that

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

then

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

Proof. Let $\phi \in F(\Sigma, n)$. If ϕ is quantifier free then $\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(b)$. By induction on formulas it suffices to show that if ϕ is the formula $\forall v, \psi$ and for any $\alpha \in \mathcal{M}$ there exists $\beta \in \mathcal{N}$ such that $\mathcal{M} \vDash_\Sigma \psi(a, \alpha) \Leftrightarrow \mathcal{N} \vDash_\Sigma \psi(b, \beta)$, then we have $\mathcal{M} \vDash_\Sigma \forall v, \psi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \forall v, \psi(b)$.

By the equivalent definition of partial isomorphisms, there exists $p : \langle a \rangle \to \langle b \rangle$ a partial isomorphism in p such that $p(a) = b$. Suppose $\mathcal{M} \vDash_\Sigma \forall v, \psi(a)$ and let $\beta \in \mathcal{N}$, then $\mathcal{M} \vDash_\Sigma \forall v, \psi(a, \alpha)$. By 'back' in the equivalent definition of the back and forth property there exists $\alpha \in \mathcal{M}$ such that $\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a, \alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b, \beta)$ Hence $\mathcal{N} \vDash_\Sigma \forall v, \psi(a, \alpha)$. The other direction is similar. □

Corollary – Back and forth implies elementary equivalence

Let $\mathcal{M}$ and $\mathcal{N}$ be Σ-structures. If $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent then they are elementarily equivalent.

Proof. Let ϕ be a quantifier free Σ-formula with 0 variables, i.e. a quantifier free sentence. As the empty set is a partial isomorphism. Thus by the equivalent definition of a partial isomorphism,

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing,0}(\varnothing) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing,0}(\varnothing)$$

By the fact that back and forth equivalence implies quantifier elimination for types,

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,0}(\varnothing) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,0}(\varnothing)$$

Thus for any Σ-sentence ϕ, $\mathcal{M} \vDash_\Sigma \phi$ if and only if $\phi \in \mathrm{tp}^{\mathcal{M}}_{\varnothing,0}(\varnothing) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,0}(\varnothing)$ if and only if $\mathcal{N} \vDash_\Sigma \phi$. □

Definition – ω-saturation
Let $\mathcal{M}$ be a Σ-structure. $\mathcal{M}$ is ω-saturated if for every finite subset $A \subseteq \mathcal{M}$, every $n \in \mathbb{N}$ and every $p \in S_n(\mathrm{Th}_{\mathcal{M}}(A))$, p is realised in $\mathcal{M}$.

See the general version κ-saturated here.

Proposition – ∞-equivalence
Let $\mathcal{M}$ and $\mathcal{N}$ be ω-saturated Σ-structures. If $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ satisfy

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,n}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,n}(b)$$

then

- (Forth) For any $\alpha \in \mathcal{M}$ there exists $\beta \in \mathcal{N}$ such that

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,n+1}(a,\alpha) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,n+1}(b,\beta)$$

- (Back) For any $\beta \in \mathcal{N}$ there exists $\alpha \in \mathcal{M}$ such that

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing,n+1}(a,\alpha) = \mathrm{tp}^{\mathcal{N}}_{\varnothing,n+1}(b,\beta)$$

If this property holds for any pair a, b related by a partial isomorphism we say $\mathcal{M}$ and $\mathcal{N}$ are ∞-equivalent.

Proof. Let $\alpha \in \mathcal{M}$ and consider

$$p(a,v) := \mathrm{tp}^{\mathcal{M}}_{a,1}(\alpha) \in S_1(\mathrm{Th}_{\mathcal{M}}(a))$$

Any formula in $p(a,v)$ can be written as a Σ-formula $\phi(w,v)$ with variables w replaced with elements of a (v represents a single variable to be replaced by α). Let

$$p(w,v) := \{\phi(w,v) \,|\, \phi(a,v) \in p(a,v)\}$$

We claim that

$$p(b,v) := \{\phi(b,v) \,|\, \phi \in p(w,v)\} \in S_1(\mathrm{Th}_{\mathcal{N}}(b))$$

To this end, we note that it is indeed a maximal subset of $F(\Sigma(b),1)$ since for any $\phi(b) \in F(\Sigma(b),1)$

$$\phi(a) \in p(a,v) \text{ or } \neg\phi(a) \in p(a) \Rightarrow \phi(b) \in p(b,v) \text{ or } \neg\phi(b) \in p(b)$$

We just need to show that it is consistent with $\mathrm{Th}_{\mathcal{N}}(b)$.

By compactness for types and noting that $\mathcal{N}$ is a $\Sigma(b)$-model of $\mathrm{Th}_{\mathcal{N}}(b)$, it suffices to show that for any finite subset $\Delta(w,v) \subseteq p(w,v)$ there exists $\beta \in \mathcal{N}^m$ such that $\mathcal{N} \vDash_{\Sigma(b)} \Delta(b,\beta)$.

$$\begin{aligned}
&\mathcal{M} \vDash_{\Sigma(a)} \bigwedge_{\phi\in\Delta} \phi(a,\alpha) \\
&\Rightarrow \mathcal{M} \vDash_{\Sigma} \exists v, \bigwedge_{\phi\in\Delta} \phi(a,v) \\
&\Rightarrow \left(\exists v, \bigwedge_{\phi\in\Delta} \phi(a,v)\right) \in \mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b) \\
&\Rightarrow \mathcal{N} \vDash_{\Sigma} \exists v, \bigwedge_{\phi\in\Delta} \phi(b,v) \\
&\Rightarrow \exists \beta \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma} \bigwedge_{\phi\in\Delta} \phi(b,\beta) \\
&\Rightarrow \exists \beta \in \mathcal{N}, \mathcal{N} \vDash_{\Sigma(b)} \Delta(b,\beta)
\end{aligned}$$

Thus $p(b,v) \in S_1(\mathrm{Th}_{\mathcal{N}}(b))$ and since $\mathcal{N}$ is ω-saturated $p(b,v)$ is realised in $\mathcal{N}$ by some β. Thus by maximality, $p(b,v) = \mathrm{tp}^{\mathcal{N}}_{b,1}(\beta)$.

Finally, for $\phi(v,w) \in F(\Sigma, n+1)$

$$\begin{aligned}
&\phi(v,w) \in \mathrm{tp}^{\mathcal{M}}_{\varnothing}(a,\alpha) \Leftrightarrow \qquad \mathcal{M} \vDash_{\Sigma} \phi(a,\alpha) \Leftrightarrow \mathcal{M} \vDash_{\Sigma(a)} \phi(a,\alpha) \\
&\Leftrightarrow \phi(a,v) \in \mathrm{tp}^{\mathcal{M}}_{a,1}(\alpha) = p(a,v) \\
&\Leftrightarrow \phi(b,v) \in p(b,v) = \mathrm{tp}^{\mathcal{M}}_{b,1}(\beta) \\
&\Leftrightarrow \mathcal{N} \vDash_{\Sigma(b)} \phi(b,\beta) \Leftrightarrow \mathcal{N} \vDash_{\Sigma} \phi(b,\beta) \\
&\Leftrightarrow \phi(w,v) \in \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b,\beta)
\end{aligned}$$

□

Proposition – Back and forth method for showing quantifier elimination

Let T be a Σ-theory. If T has quantifier elimination then for any two ω-saturated Σ-models of T are back and forth equivalent.

If any two Σ-models of T are back and forth equivalent then T has quantifier elimination. [†]

[†] We could also phrase this as T has quantifier elimination if and only if any two ω-saturated Σ-models of T are back and forth equivalent, but the saturation requirement becomes redundant in one direction.

Proof. ($\Rightarrow$) Let p be a partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$. By the equivalent definition of partial isomorphisms there exists $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$ such that $p(a) = b$ and

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

By quantifier elimination for types

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

The models are ω-saturated, hence by ∞-equivalence for any $\alpha \in \mathcal{M}$ there exists $\beta \in \mathcal{N}$ such that

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a,\alpha) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b,\beta)$$

Taking only the quantifier free elements, we obtain

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a,\alpha) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b,\beta)$$

and by the equivalent definition of the back and forth property we have that p has the back and forth property.

($\Leftarrow$) Let $n \in \mathbb{N}$, $\mathcal{M}$ and $\mathcal{N}$ be models of T, $a \in \mathcal{M}^n$ and $b \in \mathcal{N}^n$. By quantifier elimination for types it suffices to show that if

$$\mathrm{qftp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{qftp}^{\mathcal{N}}_{\varnothing}(b)$$

then

$$\mathrm{tp}^{\mathcal{M}}_{\varnothing}(a) = \mathrm{tp}^{\mathcal{N}}_{\varnothing}(b)$$

This is satisfied as any two models of T are back and forth equivalent. □

Corollary – Back and forth condition for completeness
Let T be a Σ-theory. If any two models of T are back and forth equivalent then T is complete.

Proof. If any two models are back and forth equivalent then any two non-empty models are elementarily equivalent (the non-empty is redundant information). Hence T is complete. □

We end this section with a nice example of all of this in action.

Example (Infinite infinite equivalence classes).

$$\Sigma_E := (\varnothing, \varnothing, n_f, \{E\}, m_r)$$

where $m_E = 2$ and n_f is the empty function, defines the signature of binary relations. We write for variables x and y, we write $x \sim y$ as notation for $E(x, y)$ The theory of equivalence relations ER *is set set containing the following formulas:*

Reflexivity - $\forall x, x \sim x$
Symmetry - $\forall x \forall y, x \sim y \to y \sim x$
Transitivity - $\forall x \forall y \forall z, (x \sim y \land y \sim z) \to x \sim z$

For $n \in \mathbb{N}_{>1}$ define

$$\phi_n := \exists_{i=1}^{n} x_i, \bigwedge_{i<j} x_i \nsim x_j$$
$$\psi_n := \forall x, \exists_{i=1}^{n} x_i, \bigwedge_{i=1}^{n} (x \sim x_i) \land \bigwedge_{i<j} (x_i \neq x_j)$$

Show that the theory $T = \mathrm{ER} \cup \phi_n, \psi_{n1<i}$ has quantifier elimination and is complete. (You may wonder if it is indeed a theory and what nasty induction must be done to show that its formulas can be constructed.)

Proof. We first define the projection into the quotient: if $\mathcal{M} \vDash_{\Sigma_E} T$ and $a \in \mathcal{M}$ then

$$\pi_{\mathcal{M}}(a) := \{b \in \mathcal{M} \,|\, \mathcal{M} \vDash_{\Sigma_E} a \sim b\}$$

If $A \subseteq \mathcal{M}$ we write $\pi_{\mathcal{M}}(A)$ to be the image

$$\{\pi_{\mathcal{M}}(a) \,|\, \exists a \in A\}$$

Note that the quotient is $\pi_{\mathcal{M}}(\mathcal{M})$.

Let $\mathcal{M}, \mathcal{N}$ be Σ_E-models of T and let p be a partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$. By the back and forth condition for quantifier elimination and the back and forth condition for completeness it suffices to show that p has the back and forth property.

We only show 'forth'. Let $\alpha \in \mathcal{M}$. Suppose $\pi_{\mathcal{M}}(\alpha) \cap \operatorname{dom} p$ is empty. We can show that $\pi_{\mathcal{N}}(\mathcal{N})$ is infinite whilst $\pi_{\mathcal{N}}(\operatorname{codom} p)$ is finite, hence there exists $\beta \in \mathcal{N}$ such that $\pi(\beta) \in \pi_{\mathcal{N}}(\mathcal{N}) \setminus \pi_{\mathcal{N}}(\operatorname{codom} p)$ is non-empty. Then define $q : \operatorname{dom} p \cup \{\alpha\} \to \operatorname{codom} p \cup \{\beta\}$ to agree with p on its domain and send α to β. Note that the domain and codomain of q are substructures as the language only contains a relation symbol (thus all subsets are substructures). We show that q is an isomorphism. It is clearly bijective, and to be an embedding it just needs to preserve interpretation of the relation. Let $a, b \in \operatorname{dom} q$, if both are in $\operatorname{dom} p$ then as p is a partial isomorphism

$$a \sim^{\mathcal{M}} b \Leftrightarrow p(a) \sim^{\mathcal{N}} p(b) \Leftrightarrow q(a) \sim^{\mathcal{N}} q(b)$$

Otherwise WLOG $a = \alpha$. If $b = \alpha$ then it is clear. If $b \in \operatorname{dom} p$ then by assumption $b \notin \pi_{\mathcal{M}}(\alpha) = \pi(a)$ hence $\neg\, a \sim^{\mathcal{M}} b$. By construction

$$q(a) = q(\alpha) = \beta \Rightarrow \pi_{\mathcal{N}}(q(a)) \notin \pi_{\mathcal{N}}(\operatorname{codom} p) \quad \text{and} \quad q(b) = p(b) \in \operatorname{codom} p$$

hence $\neg\, q(a) \sim^{\mathcal{N}} q(b)$. Thus q is a local isomorphism extending p.

Suppose $\pi_{\mathcal{M}}(\alpha) \cap \operatorname{dom} p$ is non-empty, i.e. there exists $a \in \operatorname{dom} p$ such that $\alpha \sim^{\mathcal{M}} a$ We can show that $\pi_{\mathcal{N}}(p(a))$ is infinite and $\operatorname{codom} p$ is finite hence there exists $\beta \in \pi_{\mathcal{N}}(p(a)) \setminus \operatorname{codom} p$. Then define $q : \operatorname{dom} p \cup \{\alpha\} \to \operatorname{codom} p \cup \{\beta\}$ to agree with p on its domain and send α to β. Again p is clearly a bijection on substructures, and we show that the relation is preserved. Let $b, c \in \operatorname{dom} q$. If $b, c \in \operatorname{dom} p$ then it is clear as p is an isomorphism, it is also clear if $b, c = \alpha$. Otherwise WLOG $c = \alpha$ and $b \in \operatorname{dom} p$. Then $c = \alpha \sim^{\mathcal{M}} a$ and by construction of β

$$q(c) = q(\alpha) = \beta \sim^{\mathcal{N}} p(a)$$

Noting $a \sim^{\mathcal{M}} b$ if and only if $p(a) \sim^{\mathcal{N}} p(b)$ as p is a partial isomorphism thus $c \sim^{\mathcal{M}} a \sim^{\mathcal{M}} b$ if and only if $q(c) \sim^{\mathcal{N}} p(a) \sim^{\mathcal{N}} p(b) = q(b)$. Hence q is a local isomorphism extending p. Thus p has the 'forth' property (and similarly the 'back' property). □

Proposition – Countable back and forth equivalent structures are isomorphic

Let $\mathcal{M}$ and $\mathcal{N}$ be countably infinite Σ-structures. If $\mathcal{M}$ and $\mathcal{N}$ are back and forth equivalent then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Proof. Write $\mathcal{M} = \{a_i\}_{i \in \mathbb{N}}$ and $\mathcal{N} = \{b_i\}_{i \in \mathbb{N}}$. Inductively define partial isomorphisms p_n for $n \in \mathbb{N}$:

- Take p_0 to be the empty function.
- If $n+1$ is odd then ensure $a_{n/2}$ is in the domain: by the 'forth' property of p there exists p_{n+1} extending p_n such that $a_{n/2} \in \operatorname{dom}(p_{n+1})$.
- If $n+1$ is even then ensure $b_{(n+1)/2}$ is in the codomain: by the 'back' property of p there exists p_{n+1} extending p_n such that $b_{(n-1)/2} \in \operatorname{codom}(p_{n+1})$.

We claim that p, the union of the partial isomorphisms p_n for each $n \in \mathbb{N}$, is an isomorphism. Note that it is well-defined and has image $\mathcal{N}$ as the p_i are nested and for any $a_i \in \mathcal{M}$ and $b_i \in \mathcal{N}$, $a_i \in \operatorname{dom}(p_{2i+1})$ and $b_i \in \operatorname{dom}(p_{2i+2})$. It is injective: if $a_i, a_j \in \mathcal{M}$ and $p(a_i) = p(a_j)$ then $p_{2i+2}(a_i) = p_{2i+2}(a_j)$ and so $a_i = a_j$ as p_{2i+2} is a partial isomorphism. One can show that it is an Σ-embedding. □

1.3.3 Model completeness

Definition – Model Completeness

We say a Σ-theory T is model complete when given two Σ-models of T and a Σ-embedding $\iota : \mathcal{M} \to \mathcal{N}$, the embedding is elementary.

Remark. Any Σ-theory T with quantifier elimination is model complete. If ϕ is a Σ-formula and $a \in (\mathcal{M})^S$. Then given two Σ-models of T and a Σ-embedding $\iota : \mathcal{M} \to \mathcal{N}$ we can take ψ a quantifier free formula such that $T \vDash_\Sigma \forall v, \phi \leftrightarrow \psi$. Since embeddings preserve satisfaction of quantifier free formulas

$$\mathcal{M} \vDash_\Sigma \phi(a) \Leftrightarrow \mathcal{M} \vDash_\Sigma \psi(a) \Leftrightarrow \mathcal{N} \vDash_\Sigma \psi(\iota(a)) \Leftrightarrow \mathcal{N} \vDash_\Sigma \phi(\iota(a))$$

Thus the extension is elementary.

Chapter 2

Appendix

2.1 Boolean Algebras, Ultrafilters and the Stone Space

2.1.1 Boolean Algebras

There is a very detailed wikipedia page [6] on Boolean algebras.

Definition – Partially ordered set

The signature of partially ordred sets Σ_{PO} consists of $(\varnothing, \varnothing, n_f, \{\leq\}, m_f)$, where $n_{\leq} = 2$. The theory of partially ordered sets PO consists of

| Reflexivity: $\forall x, x \leq x$ (this is just notation for $\leq (x, x)$)

| Antisymmetry: $\forall x \forall y, (x \leq y \wedge y \leq x) \to x = y$

| Transitivity: $\forall x \forall y \forall z, (x \leq y \wedge y \leq z) \to x \leq z$

Definition – Boolean algebra

The signature of Boolean algebras Σ_{BLN} consists of $(\{1, 0\}, \{\leq, \sqcap, \sqcup, -\}, n_f, \varnothing, m_f)$, where $n_{\leq} = 2$, $n_{\sqcap} = n_{\sqcup} = 2$ and $n_{-} = 1$. The theory of Boolean algebras BLN consists of the theory of partially ordered sets[†] PO together with the formulas

| Assosiativity of adjunction: $\forall x \forall y \forall z, (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$

| Assosiativity of disjunction: $\forall x \forall y \forall z, (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$

| Identity for adjunction: $\forall x, x \sqcap 1 = x$

| Identity for disjunction: $\forall x, x \sqcup 0 = x$

| Commutativity of adjunction: $\forall x \forall y, x \sqcap y = y \sqcap x$

| Commutativity of disjunction: $\forall x \forall y, x \sqcup y = y \sqcup x$

| Distributivity of adjunction: $\forall x \forall y \forall z, x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$

| Distributivity of disjunction: $\forall x \forall y \forall z, x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$

| Negation on adjunction: $\forall x, x \sqcap -x = 0$

| Negation on disjunction: $\forall x, x \sqcup -x = 1$

| Order on adjunction: $\forall x \forall y, (x \sqcap y) \leq x$

| Maximal property of adjunction: $\forall x \forall y \forall z, (x \leq y) \wedge (x \leq z) \rightarrow (x \leq y \sqcap z)$

| Order on disjunction: $\forall x \forall y, x \leq (x \sqcup y)$

| Minimal property of disjunction: $\forall x \forall y \forall z, (x \leq z) \wedge (y \leq z) \rightarrow (x \sqcup y \leq z)$

Often 'absorption' is also included, but it can be deduced from the other axioms. I have not used the usual logical symbols due to obvious clashes with our notation, and we will be using this in the context of sets anyway. If B is a Σ_{BLN}-model of BLN we call it a Boolean algebra.

† Often the ordering is defined afterwards as $a \leq b$ if and only if $a = a \sqcap b$.

Lemma – Facts about Boolean algebras

Let B be a Boolean algebra, let $\mathcal{F}$ be an ultrafilter on B, let $a, b \in B$ and let $f : B \to C$ be a morphism.

- $a \sqcap a = a$ and $a \sqcup a = a$.
- $a \sqcup 1 = 1$ and $a \sqcap 0 = 0$.
- If $a \sqcap b = 0$ and $a \sqcup b = 1$ then $a = -b$ (negations are unique.)
- $-(a \sqcup b) = (-a) \sqcap (-b)$ and its dual. (De Morgan)
- $(a \in \mathcal{F}$ or $b \in \mathcal{F})$ if and only if $a \sqcup b \in \mathcal{F}$.
- Morphisms are order preserving.
- Morphisms commute with negation.
- $a \sqcap b = 1$ if and only if $a = 1$ and $b = 1$. Similarly for $\sqcup$ with 0.
- If $a \sqcap b = 0$ then $b \leq -a$.

Proof.

- We prove only $a \sqcap a = a$ as the other has the same proof.

$$a \sqcap a = (a \sqcap a) \sqcup 0 = (a \sqcap a) \sqcup (a \sqcap -a) = a \sqcap (a \sqcup -a) = a \sqcap 1 = a$$

- Again, we prove only $a \sqcup 1 = 1$. Using the previous part,

$$a \sqcup 1 = a \sqcup (a \sqcup -a) = (a \sqcup a) \sqcup -a = a \sqcup -a = 1$$

-

$$\begin{aligned} a &= a \sqcup 0 = a \sqcup (b \sqcap -b) = (a \sqcup b) \sqcap (a \sqcup -b) = 1 \sqcap (a \sqcup -b) \\ &= 0 \sqcup (a \sqcup -b) = (b \sqcap -b) \sqcup (a \sqcup -b) = (b \sqcup a) \sqcup -b = 0 \sqcup -b \\ &= -b \end{aligned}$$

- By the previous part it suffices to show that

$$((-a) \sqcap (-b)) \sqcap (a \sqcup b) = 0 \text{ and } ((-a) \sqcup (-b)) \sqcup (a \sqcup b) = 1$$

  These are clear.

- $(\Rightarrow)$ In the case that $a \in \mathcal{F}$ we have $a \leq a \sqcup b \in \mathcal{F}$ as $\mathcal{F}$ is under superset. The other case is the same. $(\Leftarrow)$ Suppose $a \notin \mathcal{F}$ and $b \notin \mathcal{F}$ then $-a \in \mathcal{F}$ and $-b \in \mathcal{F}$ hence $-a \sqcap -b \in \mathcal{F}$ by closure under intersection. By the previous part this is equal to $-(a \sqcup b)$, which implies $(a \sqcup b) \notin \mathcal{F}$ as $\mathcal{F}$ is an ultrafilter.

- Suppose $b \leq a$. We show that $f(a) \leq f(b)$. By the minimal property of disjunction,
$$b \leq a \wedge a \leq a \Rightarrow b \sqcup a \leq a$$
Clearly $a \leq b \sqcup a$ and so $a = b \sqcup a$ Hence $f(b) \leq f(b) \sqcup f(a) = f(b \sqcup a) = f(a)$.
- Suppose $f(a) = b$. Then $f(a) \wedge f(-a) = f(a \wedge -a) = f(0) = 0$ and $f(a) \vee f(-a) = f(a \vee -a) = f(1) = 1$. As negations are unique (shown above) this gives us that $f(-a) = -f(a)$.
- Suppose $a \sqcap b = 1$ then $1 \leq a \sqcap b \leq a$ hence $1 = a$ and similarly $1 = b$.
-
$$\begin{aligned}
&a \sqcup -a = 1 \\
\Rightarrow &b \sqcap (a \sqcup -a) = b \sqcap 1 = b \\
\Rightarrow &(b \sqcap a) \sqcup (b \sqcap -a) = b \\
\Rightarrow &0 \sqcup (b \sqcap -a) = b \\
\Rightarrow &b = b \sqcap -a \leq -a
\end{aligned}$$

□

Definition – Filters and ultrafilters on Boolean algebras
Let B be a Boolean algebra. A subset $\mathcal{F}$ of B is an filter on B if

- $1 \in \mathcal{F}$.
- For any two members $a, b \in \mathcal{F}$ the adjunction $a \sqcap b$ is in $\mathcal{F}$. (Closure under finite intersection.)
- If $a \in \mathcal{F}$ then any $b \in B$ such that $a \leq b$ is also a member of $\mathcal{F}$. (Closure under superset.)

We say a filter on B is proper if it does not contain 0. A proper filter $\mathcal{F}$ on B is an ultrafilter (maximal filter) when for any filter $\mathcal{G}$ on B, if $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G} = \mathcal{F}$ or $\mathcal{G} = B$.

Proposition – Equivalent definition of ultrafilter
Let B be a boolean algebra. Let $\mathcal{F}$ be a proper filter on B. The following are equivalent:

1. $\mathcal{F}$ is an ultrafilter over B.
2. If $a \sqcup b \in \mathcal{F}$ then $a \in \mathcal{F}$ or $b \in \mathcal{F}$. '$\mathcal{F}$ is prime'.
3. For any a of B, $a \in \mathcal{F}$ or $(-a) \in \mathcal{F}$.

For 2. the or is in fact 'exclusive or' since if both a and its negation were in $\mathcal{F}$ then $\mathcal{F}$ would not be proper.

Proof. (1. $\Rightarrow$ 2.) Suppose $a_0 \sqcup a_1 \in \mathcal{F}$.
$$\mathcal{G}_{a_0} = \{b \in B \,|\, \exists c \in \mathcal{F}, c \sqcap a_0 \leq b\}$$
Then clearly $\mathcal{G}_{a_0}$ is a filter containing $\mathcal{F}$ and a_0. Similarly have $\mathcal{G}_{a_1}$. Since $\mathcal{F}$ is an ultrafilter, we can case on whether $\mathcal{G}_{a_i}$ is $\mathcal{F}$ or B. If either is $\mathcal{F}$ then $a_i \in \mathcal{F}$ and we are done. If $\mathcal{G}_{a_0} = \mathcal{G}_{a_1} = B$ then both contain 0 and so there exist $c_i \in \mathcal{F}$ such that $c_i \sqcap a_i = 0$. Hence $c_i \leq -a_i$, and so $(-a_i) \in \mathcal{F}$. Thus $-(a_0 \sqcup a_1) = -a_0 \sqcap -a_1 \in \mathcal{F}$, and so $0 \in \mathcal{F}$, a contradiction.

(2. $\Rightarrow$ 3.) Let $a \in \mathcal{F}$. Then we have that $a \sqcup -a = 1$, which is by definition a member of $\mathcal{F}$. By assumption this implies that $a \in \mathcal{F}$ or $-a \in \mathcal{F}$.

(3. $\Rightarrow$ 1.) Let $\mathcal{G}$ be a proper filter such that $\mathcal{F} \subseteq \mathcal{G}$. It suffices to show that $\mathcal{G} = \mathcal{F}$. Then let $a \in \mathcal{G}$. Suppose $a \notin \mathcal{F}$. Then $-a \in \mathcal{F}$ and so $-a \in \mathcal{G}$. Thus $0 = (a \sqcap -a) \in \mathcal{G}$ thus $\mathcal{G}$ is not proper, a contradiction. Hence $\mathcal{G} = \mathcal{F}$. □

Proposition – Extending filters to ultrafilters

Definition
The category of Boolean algebras consists of Boolean algebras as the objects and for any two Boolean algebras B, C morphisms $f : B \to C$ such that for any $a, b \in B$

$$f(0) = 0, f(1) = 1, f(a \sqcup b) = f(a) \sqcup f(b), f(a \sqcap b) = f(a) \sqcap f(b)$$

Definition
The category of Stone spaces has 0-dimensional, compact and Hausdorff topological spaces as objects and continuous maps as morphisms.

Proposition – Contravariant functor from Boolean algebras to Stone spaces [1]
The map $S(\star)$ that sends a Boolean algebra B to the Stone space

$$S(B) := \{\mathcal{F} \subseteq B \,|\, \mathcal{F} \text{ is an ultrafilter}\}$$

and a Boolean algebra morphism $f : A \to B$ to a continuous map of Stone spaces:

$$S(f) := f^{-}1(\star) : S(B) \to S(A)$$

is a contravariant functor from Boolean algebras to the category of Stone spaces.

Proof. We give the topology on $S(B)$ the topology generated by the clopen sets: Let $b \in B$, then an element of the clopen basis is given by

$$[b] := \{\mathcal{F} \subseteq B \,|\, b \in \mathcal{F}\}$$

Thus $S(B)$ is 0-dimensional by definition. It is Hausdorff because if $\mathcal{F}, \mathcal{G} \in S(B)$ are not equal then without loss of generality there exists $b \in \mathcal{F}$ such that $b \notin \mathcal{G}$. As ultrafilters are proper, $[b] \cap [-b] = \varnothing$ and so we have obtained disjoint open sets such that $\mathcal{F} \in [b]$ and $\mathcal{G} \in [-b]$.

Next we show that it is compact. To do this, we will look at $S(B)$ as a subspace of the power set of B which is isomorphic to 2^B, the set of functions from B to 2. We endow $2 = \{0, 1\}$ with the discrete topology and note that it is compact. Then $2^B = \prod_{a \in B} 2$ has an induced product topology which is compact by Tychonoff's Theorem. The isomorphism from 2^B to the power set of B is given by

$$2^B \to \mathcal{P}(B) := f \mapsto f^{-1}(1)$$

We take the topology on $\mathcal{P}(B)$ induced by this isomorphism. We must show that

- $S(B)$ is the image of $\mathrm{Mor}(B, 2) := \left\{f \in 2^B \,|\, f \text{ is a Boolean algebra morphism}\right\}$ under this isomorphism.
- $\mathrm{Mor}(B, 2)$ is a closed subset of 2^B and therefore compact.
- The induced topology on $S(B)$ is the same as its original topology.

With these facts we see that $S(B)$ is also compact.

Let $f : B \to 2$ be a Boolean algebra morphism. We must show that $\mathcal{F} := f^{-1}(1)$ is an ultrafilter. Since $f(1) = 1$, $1 \in \mathcal{F}$. If $a, b \in \mathcal{F}$ then $f(a) = f(b) = 1$ so $f(a \sqcap b) = f(a) \sqcap f(b) = 1$ thus $\mathcal{F}$ is closed under intersection. If $a \subseteq b$ and $a \in \mathcal{F}$ then as f is order preserving $f(a) \subseteq f(b)$. It is a proper filter as

$f(0) = 0 \neq 1$. To show it is an ultrafilter we use the equivalent definition: let $a \in \mathcal{P}(B)$. If $f(a) = 1$ then we are done, otherwise

$$f(-a) = -f(a) = -0 = 1 \Rightarrow -a \in \mathcal{F}$$

and we are done. To show that this is a surjection we use the inverse:

$$\mathcal{F} \mapsto \left(a \to \begin{cases} 1, a \in \mathcal{F} \\ 0, a \notin \mathcal{F} \end{cases} \right)$$

To show that this is a morphism we note that $\mathcal{F}$ is proper and contains 1 thus $f(0) = 0$ and $f(1) = 1$. Also

$$\begin{aligned} & f(a \sqcap b) = 1 \Leftrightarrow a \sqcap b \in \mathcal{F} \\ & \Leftrightarrow a \in \mathcal{F} \text{ and } b \in \mathcal{F} \quad \text{by closure under finite intersection and superset} \\ & \Leftrightarrow f(a) = 1 \text{ and } f(b) = 1 \\ & \Leftrightarrow f(a) \sqcap f(b) = 1 \quad \text{as proven before} \end{aligned}$$

Hence $f(a \sqcap b) = f(a) \sqcap f(b)$. Lastly since

$$-f(a) = 1 \Leftrightarrow f(a) = 0 \Leftrightarrow a \notin \mathcal{F} \Leftrightarrow -a \in \mathcal{F} \Leftrightarrow f(-a) = 1$$

thus by De Morgan and uniqueness of negations we have

$$f(a \sqcup b) = f(-(-a \sqcap -b)) = -[-f(a) \sqcap -f(b)] = (- - f(a)) \sqcup - - f(b) = f(a) \sqcup f(b)$$

Thus the inverse map gives back a Boolean algebra morphism and $\operatorname{Mor}(B, 2) \cong S(B)$ under the isomorphism.

To show that $\operatorname{Mor}(B, 2)$ is a closed subset we write it as

$$\operatorname{Mor}(B, 2) = \{f \mid f(0) = 0\} \sqcap \{f \mid f(1) = 1\} \sqcap \{f \mid f \text{ commutes with } \sqcap\} \sqcap \{f \mid f \text{ commutes with } \sqcup\}$$

and show that all of these four sets are closed. Call them $\min, \max, C_\sqcap, C_\sqcup$ respectively and for each $a \in B$ call the projection map $\pi_a : 2^B \to 2$ (these send $f \mapsto f(a)$ such that $\pi_a(f) = f(a)$) and note that by definition of the product topology each π_a is continuous; the closed sets of the product are generated by preimages of closed sets.

$$f(0) = 0 \Leftrightarrow \pi_0(f) = 0 \Leftrightarrow f \in \pi_0^{-1}(0)$$

Hence $\min = \pi_0^{-1}(0)$ is closed as $\{0\}$ is closed in the discrete topology on 2. Similarly $\max = \pi_1^{-1}(1)$ is closed.

$$C_\sqcap = \bigcap_{a,b \in B} \{f \mid f(a \sqcap b) = f(a) \sqcap f(b)\} = \bigcap_{a,b \in B} \{f \mid \pi_{a \sqcap b}^{-1}(f(a) \sqcap f(b))\}$$

Thus $C_\sqcap$ is an arbitrary intersection of preimages of closed sets since each $f(a) \sqcap f(b)$ is closed in the discrete toipology on 2, hence $C_\sqcap$ is closed. Similarly

$$C_\sqcup = \bigcap_{a,b \in B} \{f \mid \pi_{a \sqcup b}^{-1}(f(a) \sqcup f(b))\}$$

is closed and so $\operatorname{Mor}(B, 2)$ is closed.

With regards to compactness it remains to show that the topologies on $S(B)$ are the same. It suffices to show that any (closed) basis element of each can be written as a closed set in the other. Let $[b]$ be an element of the basis for $S(B)$ under the Stone topology. Then this is the image of the closed subset $\pi_b^{-1}(1) \subseteq \operatorname{Mor}(B, 2)$ under the isomorphism:

$$\operatorname{iso}(\pi_b^{-1}(1)) = \left\{ f^{-1}(1) \mid f(b) = 1 \right\} = \{\mathcal{F} \text{ ultrafilter} \mid b \in \mathcal{F}\}$$

Conversely, any element of the closed basis for $\operatorname{Mor}(B, 2)$ is of the form $\pi_b^{-1}(X)$ where $b \in B$ and $X \subseteq 2$. Hence

$$\operatorname{iso}(\pi_b^{-1}(X)) = \left\{ f^{-1}(1) \mid f(b) \in X \right\}$$

We can case on if $X = \varnothing, \{0\}, \{1\}, 2$ and deduce that respectively $\mathrm{iso}(\pi_b^{-1}(X))$ becomes $\varnothing, [-b], [b], S(B)$, all of which are closed in the Stone topology. Thus the topologies are the same under this isomorphism and hence $S(B)$ is compact.

To show that $S(\star)$ is a contravariant functor we need to check that the morphism map

$$S(f) := f^-1(\star) : S(B) \to S(A)$$

is a well-defined, respects the identity and composition. We show that $S(f)$ is continuous: it suffices that preimages of clopen elements are clopen. Let $[b] \subseteq S(A)$ be clopen.

$$\begin{aligned} &S(f)^{-1}[b] \\ =&\left\{\mathcal{F} \in S(B) \mid f^{-1}(\mathcal{F}) \in [b]\right\} \\ =&\{\mathcal{F} \in S(B) \mid f(b) \in \mathcal{F}\} \\ =&[f(b)] \end{aligned}$$

which is clopen. □

Proposition – Stone Duality

There is an equivalence between the category of Stone spaces and the category of Boolean algebras. Given by the functor $\mathcal{B}_\star$ sending any topological space X to the set of its clopen subsets (this is a basis of X as it is 0 dimensional):

$$\mathcal{B}_X := \{a \subseteq X \mid a \text{ is clopen}\}$$

and its inverse $S(\star)$.

Proof. Let X be a 0-dimensional compact Hausdorff topological space. There is an obvious Boolean algebra to take on

$$\mathcal{B}_X := \{a \subseteq X \mid a \text{ is clopen}\}$$

which is interpreting 0 to be $\varnothing$, 1 to be X, $\leq$ as $\subseteq$, adjunction as intersection, disjunction as union and negation to be taking the complement in X. One can check that this is a Boolean algebra.

We make this a contravariant functor by taking any continuous map $f : X \to Y$ to an induced map $f^\diamond := f^{-1}(\star) : \mathcal{B}_Y \to \mathcal{B}_X$. One can check that this is a well-defined functor.

Lastly we prove the equivalence of categories by giving natural transformations $S(\mathcal{B}_\star) \to \mathrm{id}_\star$ in the category of topological spaces and $\mathrm{id}_\star \to \mathcal{B}_{S(\star)}$ in the category of Stone spaces. □

2.1.2 Isolated points of the Stone space

Definition – Isolated point

Let X be a topological space and $x \in X$. We say x is isolated if $\{x\}$ is open.

Definition – Derived set

Let X be a topological space. The derived set of X is defined as

$$X' := X \setminus \{x \in X \mid x \text{ isolated}\}$$

Exercise (Equivalent definition of derived set). *Let U be a subspace of X a topological space. $x \in U$ is not isolated (in the subspace topology) if and only if for any open set O_x containing x, $O_x \cap U \setminus \{x\}$ is non-empty. (it is a limit point of U.)*

Definition – Atom
Let B be a Boolean algebra. We say $a \in B$ is an atom if it is non-zero and for any $b \in B$ if $b \leq a$ then $b = a$ or $b = 0$.

Definition – Principle filter
Let B be a Boolean algebra and $a \in B$. Suppose a is non-zero. Then principle filter of a is defined as

$$a^{\uparrow} := \{b \in B \,|\, a \leq b\}$$

One should check that this is a proper filter.

Proposition
Let $a \in B$ a Boolean algebra. Then $a^{\uparrow}$ is an ultrafilter if and only if a is atomic.

Proof. ($\Rightarrow$) Suppose $b \leq a$. As $a^{\uparrow}$ is ultrafilter either $b \in a^{\uparrow}$ or $-b \in a^{\uparrow}$. In the first case $a \leq b$ hence $a = b$. If $-b \in a^{\uparrow}$ then $b \leq a \leq -b$ and so

$$1 = b \sqcup -b \leq -b \sqcup -b = -b$$

and so $1 = -b$ and $b = 0$.

($\Leftarrow$) Suppose a is an atom. We show that for any $b \in B$, it is in $a^{\uparrow}$ or its negation is in $a^{\uparrow}$ Since $1 \in a^{\uparrow}$, $b \sqcup -b \in a^{\uparrow}$ and so $a \leq b \sqcup -b$. Thus

$$a = a \sqcap (b \sqcup -b) = (a \sqcap b) \sqcup (a \sqcap -b)$$

Since a is non-zero, either $a \sqcap b$ or $a \sqcap -b$ is non-zero. If $a \sqcap b \neq 0$ then together with the fact that $a \sqcap b \leq a$ we conclude that $a \sqcap b = a$ as a is atomic. Hence $a \leq b$ and $b \in a^{\uparrow}$. Similarly the other case results in $-b \in a^{\uparrow}$. □

Proposition – Correspondence between atoms and isolated points
Let $a \in B$ a Boolean algebra. If a is an atom if and only if $[a]$ is a singleton in $S(B)$. Hence $a^{\uparrow}$ is an isolated point in $S(B)$.

Proof. ($\Rightarrow$) If a is an atom then $[a] = \{a^{\uparrow}\}$ is the only ultrafilter containing a is the principle filter of a: Let $\mathcal{F}$ be an ultrafilter containing a. Then for any $b \in \mathcal{F}$, $a \sqcap b \in \mathcal{F}$ and non-zero. Therefore $a \sqcap b = a$ as a is an atom and $a \leq b$. Hence $b \in a^{\uparrow}$. By maximality $\mathcal{F} = a^{\uparrow}$. Hence $[a]$ is a singleton.

($\Leftarrow$) Suppose a is not atomic. Then there exists $b \leq a$ such that $b \neq 0$ and $b \neq a$.

$$\begin{aligned} b \sqcup (a \sqcap -b) &= (a \sqcap b) \sqcup (a \sqcap -b) \\ &= a \sqcup (b \sqcap -b) = a \sqcup 0 = a \end{aligned}$$

There exist ultrafilters extending the principle filters of b and $a \sqcap -b$. These are not equal since b and $-b$ cannot be in the same proper filter. Both filters contain a. Hence $[a]$ is not a singleton. □

2.1.3 Ultraproducts and Łos's Theorem

This section introduces ultrafilters and ultraproducts and uses Łos's Theorem to prove the compactness theorem. Łos's theorem appears as an exercise in Tent and Ziegler's book [1].

Definition – Filters on sets

Let X be a set. The power set of X is a Boolean algebra with 0 interpreted as $\varnothing$, 1 interpreted as X and $\leq$ interpreted as $\subseteq$. We say $\mathcal{F}$ is a filter on X if it is a filter on the power set. The definition translates to:

- $X \in \mathcal{F}$.
- For any two members of $\mathcal{F}$ their intersection is in $\mathcal{F}$.
- If $a \in \mathcal{F}$ then any b in the power set of X such that $a \subseteq b$ is also a member of $\mathcal{F}$.

Translating definitions over we have that a filter on X is proper if and only if it does not contain the empty set, if and only if the filter is not equal to the power set. Furthermore a proper filter $\mathcal{F}$ is an ultrafilter if and only if for any filter $\mathcal{G}$, if $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{F} = \mathcal{G}$ or $\mathcal{G}$ is the power set of X.

Definition – Ultraproduct

Let $\mathcal{F}$ be an ultrafilter on X. We define a relation on $\prod_{x \in X} x$ by

$$(a_x)_{x \in X} \sim (b_x)_{x \in X} := \{x \in X \mid a_x = b_x\} \in \mathcal{F}$$

This is an equivalence relation as

- $(a_x)_{x \in X} \sim (a_x)_{x \in X} \Leftrightarrow \{x \in X \mid a_x = a_x\} = X \in \mathcal{F}$
- Symmetry is due to symmetry of $=$.
- If $\{x \in X \mid a_x = b_x\} \in \mathcal{F}$ and $\{x \in X \mid b_x = c_x\} \in \mathcal{F}$ then $\{x \in X \mid a_x = b_x = c_x\}$ is their intersection and so is in $\mathcal{F}$. Thus its superset $\{x \in X \mid a_x = c_x\}$ is in $\mathcal{F}$.

We define the ultraproduct of X over $\mathcal{F}$:

$$\prod X/\mathcal{F} := \prod_{x \in X} x/\sim$$

Proposition – Equivalent definition of ultrafilter (translated to the power set)

Let X be a set. Let $\mathcal{F}$ be a proper filter on X. $\mathcal{F}$ is an ultrafilter over X if and only if for every subset $U \subseteq X$, *either* $U \in \mathcal{F}$ or $X \setminus U \in \mathcal{F}$.

Proof. Follows immediately from the equivalent definition of an ultrafilter. □

Proposition – Łos's Theorem

Let $\mathfrak{M} \subseteq \Sigma_{\text{str}}$ where Σ is a signature such that each carrier set is non-empty. Suppose $\mathcal{F}$ is an ultrafilter on $\mathfrak{M}$ (i.e. an ultrafilter on the Boolean algebra $P(\mathfrak{M})$). Then we want to make $\mathcal{N} := \prod \mathfrak{M}/\mathcal{F}$ into a Σ-structure. Let π be the natural surjection $\prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M} \to \prod \mathfrak{M}/\mathcal{F}$. If $a = (a_1, \ldots, a_n) \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ then write $a_{\mathcal{M}} := ((a_1)_{\mathcal{M}}, \ldots, (a_n)_{\mathcal{M}})$.

- Constant symbols $c \in \Sigma_{\text{con}}$ are interpreted as

$$c^{\mathcal{N}} := \pi(c^{\mathcal{M}})_{\mathcal{M} \in \mathfrak{M}}$$

- Any function symbol $f \in \Sigma_{\text{fun}}$ is interpreted as the function

$$f^{\mathcal{N}} : \left(\prod \mathfrak{M}/\mathcal{F}\right)^n \to \prod \mathfrak{M}/\mathcal{F} := \pi\left((a_{\mathcal{M}})_{\mathcal{M} \in \mathfrak{M}}\right) \mapsto \pi(f^{\mathcal{M}}(a_{\mathcal{M}}))_{\mathcal{M} \in \mathfrak{M}}$$

where $\pi\left((a_{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}}\right) = (\pi((a_i)_{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}})_{i=1}^n$.

- Any relation symbols $r \in \Sigma_{\text{rel}}$ is interpreted as the subset such that

$$\pi(a) \in r^{\mathcal{N}} \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \mid a_{\mathcal{M}} \in r^{\mathcal{M}}\} \in \mathcal{F}$$

where $a = a_1, \ldots, a_m$ and $\pi(a) = (\pi(a_i))_{i=1}^m$.

Then for any Σ-formula ϕ with free variables indexed by S, If $a = (a_1, \ldots, a_n) \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ then

$$\mathcal{N} \vDash_\Sigma \phi(\pi(a)) \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_\Sigma \phi(a_{\mathcal{M}})\} \in \mathcal{F}$$

Proof. We show that the interpretation of functions is well defined. Let $a, b \in \left(\prod_{\mathcal{M}\in\mathfrak{M}}\right)^{n_f}$. Suppose for each $i \in \{1, \ldots, n_f\}$, $a_i \sim b_i \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$. Then

$$\begin{aligned}
&\text{for each } i, \{\mathcal{M} \in \mathfrak{M} \mid (a_i)_{\mathcal{M}} = (b_i)_{\mathcal{M}}\} \in \mathcal{F} \\
&\Rightarrow \left\{\mathcal{M} \in \mathfrak{M} \mid \bigwedge_{i=1}^n (a_i)_{\mathcal{M}} = (b_i)_{\mathcal{M}}\right\} \in \mathcal{F} \text{ by closure under finite adjunction} \\
&\Rightarrow \{\mathcal{M} \in \mathfrak{M} \mid a_{\mathcal{M}} = b_{\mathcal{M}}\} \in \mathcal{F} \\
&\Rightarrow \{\mathcal{M} \in \mathfrak{M} \mid f^{\mathcal{M}}(a_{\mathcal{M}}) = f^{\mathcal{M}}(b_{\mathcal{M}})\} \in \mathcal{F} \text{ by closure under superset} \\
&\Rightarrow \pi(f^{\mathcal{M}}(a_{\mathcal{M}})) = \pi(f^{\mathcal{M}}(b_{\mathcal{M}})) \text{ by definition of the quotient}
\end{aligned}$$

We use the following claim: If t is a term with variables S and $a \in (\prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M})^S$ then there exists $b \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ such that

$$t^{\mathcal{N}} \circ \pi(a) = \pi(b) \text{ and } \forall \mathcal{M} \in \mathfrak{M}, t^{\mathcal{M}}(a_{\mathcal{M}}) = b_{\mathcal{M}}$$

We prove this by induction on t:

- If t is a constant symbol c then pick $b := (c^{\mathcal{M}})_{\mathcal{M}\in\mathfrak{M}}$.
- If t is a variable then let $a \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ (only one varible), pick $b := a$.
- If t is a $f(s)$ then let $a \in (\prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M})^S$ by the inducition hypothesis there exists $c \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ such that

  $$s^{\mathcal{N}} \circ \pi(a) = \pi(c) \text{ and } \forall \mathcal{M} \in \mathfrak{M}, s^{\mathcal{M}}(a_{\mathcal{M}}) = c_{\mathcal{M}}$$

  Then we can take $b = (f^{\mathcal{M}}(c_{\mathcal{M}}))_{\mathcal{M}\in\mathfrak{M}}$. Thus

  $$t^{\mathcal{N}} \circ \pi(a) = f^{\mathcal{N}}(s^{\mathcal{N}} \circ (\pi(a))) = f^{\mathcal{N}}(\pi(c)) = \pi(f^{\mathcal{M}}(c_{\mathcal{M}}))_{\mathcal{M}\in\mathfrak{M}} = \pi(b)$$

  and for any $\mathcal{M} \in \mathfrak{M}$,

  $$t^{\mathcal{M}}(a_{\mathcal{M}}) = f^{\mathcal{M}} \circ s^{\mathcal{M}}(a_{\mathcal{M}}) = f^{\mathcal{M}}(c_{\mathcal{M}}) = b_{\mathcal{M}}$$

We now induct on ϕ to show that for any appropriate a,

$$\mathcal{N} \vDash_\Sigma \phi(\pi(a)) \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_\Sigma \phi(a_{\mathcal{M}})\} \in \mathcal{F}$$

- The case where ϕ is $\top$ is trivial (noting that anything models $\top$ and $\mathfrak{M} \in \mathcal{F}$).
- If ϕ is $s = t$ then it suffices to show that

  $$s^{\mathcal{N}} \circ \pi(a) = t^{\mathcal{N}} \circ \pi(a) \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_\Sigma t^{\mathcal{M}}(a_{\mathcal{M}}) = s^{\mathcal{M}}(a_{\mathcal{M}})\} \in \mathcal{F}$$

  ($\Rightarrow$) If for two terms s, t we have $s^{\mathcal{N}} \circ \pi(a) = t^{\mathcal{N}} \circ \pi(a)$ then by the claim there exists $b \in \prod_{\mathcal{M}\in\mathfrak{M}} \mathcal{M}$ such that

  $$\mathfrak{M} = \{\mathcal{M} \in \mathfrak{M} \mid s^{\mathcal{M}}(a_{\mathcal{M}}) = b_{\mathcal{M}} = t^{\mathcal{M}}(a_{\mathcal{M}})\} = \{\mathcal{M} \in \mathfrak{M} \mid t^{\mathcal{M}}(a_{\mathcal{M}}) = s^{\mathcal{M}}(a_{\mathcal{M}})\}$$

which is therefore in the filter $\mathcal{F}$. ($\Leftarrow$) If for two terms s, t we have $s^{\mathcal{N}} \circ \pi(a) \neq t^{\mathcal{N}} \circ \pi(a)$ then by the claim there exist $b \neq c \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ such that

$$\{\mathcal{M} \in \mathfrak{M} \mid t^{\mathcal{M}}(a_{\mathcal{M}}) = s^{\mathcal{M}}(a_{\mathcal{M}})\} = \{\mathcal{M} \in \mathfrak{M} \mid b_{\mathcal{M}} = c_{\mathcal{M}}\} = \varnothing$$

which is not in the filter $\mathcal{F}$ as it is proper.

- If ϕ is $r(t)$ then by the claim we have $b \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ with the desired properties. It suffices to show that
$$\pi(b) \in r^{\mathcal{N}} \Leftrightarrow \{\mathcal{M} \in \mathcal{M} \mid b_{\mathcal{M}} \in r^{\mathcal{M}}\} \in \mathcal{F}$$
This follows from our definition of interpretation of relation symbols.

- If ϕ is $\neg\psi$ then $\mathcal{N} \vDash_{\Sigma} \phi(\pi(a))$ if and only if $\{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \psi(a_{\mathcal{M}})\} \notin \mathcal{F}$ by induction. This holds if and only if its complement is in the filter: $\{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \nvDash_{\Sigma} \psi(a_{\mathcal{M}})\} \in \mathcal{F}$ which is if and only if $\{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \phi(a_{\mathcal{M}})\} \notin \mathcal{F}$

- Without loss of generality we can use $\wedge$ instead of $\vee$ to make things simpler (replacing this comes down to dealing with a couple of $\neg$ statements). If ϕ is $\psi \wedge \chi$ then one direction follows filters being closed under intersection:
$$\begin{aligned}
&\mathcal{N} \vDash_{\Sigma} \phi(\pi(a)) \\
&\Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \psi(a_{\mathcal{M}})\} \in \mathcal{F} \text{ and } \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \chi(a_{\mathcal{M}})\} \in \mathcal{F} \\
&\Rightarrow \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \psi(a_{\mathcal{M}})\} \cap \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \chi(a_{\mathcal{M}})\} \in \mathcal{F} \\
&\Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \psi(a_{\mathcal{M}}) \wedge \chi(a_{\mathcal{M}})\} \in \mathcal{F}
\end{aligned}$$
To make second implication a double implication we note that each of the two sets
$$\{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \psi(a_{\mathcal{M}})\} \in \mathcal{F} \text{ and } \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \chi(a_{\mathcal{M}})\} \in \mathcal{F}$$
are supersets of the intersection which is in $\mathcal{F}$.

- Without loss of generality we can use $\exists$ instead of $\forall$ to make things simpler. ($\Rightarrow$) Suppose $\mathcal{N} \vDash_{\Sigma} \exists v, \psi(\pi(a), v)$. Then there exists $b \in \prod_{\mathcal{M} \in \mathfrak{M}} \mathcal{M}$ such that $\mathcal{N} \vDash_{\Sigma} \psi(\pi(a), \pi(b))$. Then by induction
$$\{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \psi(a_{\mathcal{M}}, b_{\mathcal{M}})\} \in \mathcal{F}$$
This is a subset of
$$\{\mathcal{M} \in \mathfrak{M} \mid \exists c \in \mathcal{M}, \mathcal{M} \vDash_{\Sigma} \psi(a_{\mathcal{M}}, c)\} = \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \exists v, \psi(a_{\mathcal{M}}, v)\}$$
($\Leftarrow$) Suppose $Y := \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_{\Sigma} \exists v, \psi(a_{\mathcal{M}}, v)\} \in \mathcal{F}$. Then by the axiom of choice we have for each $\mathcal{M}$
$$\begin{cases} b_{\mathcal{M}} \in \mathcal{M}, \mathcal{M} \vDash_{\Sigma} & \text{if } \mathcal{M} \in Y \\ b_{\mathcal{M}} \in \mathcal{M} & \mathcal{M} \notin Y \end{cases}$$
since each $\mathcal{M}$ is non-empty. By induction we have $\mathcal{N} \vDash_{\Sigma} \psi(\pi(a), \pi(b))$ and so $\mathcal{N} \vDash_{\Sigma} \exists v, \psi(\pi(a), v)$

□

Corollary – The Compactness Theorem
A Σ-theory is consistent if and only if it is finitely consistent.

Proof. Suppose T is finitely consistent. For each finite subset $\Delta \subseteq T$ we let $\mathcal{M}_{\Delta}$ be the given non-empty model of Δ, which exists by finite consistency. We generate an ultrafilter $\mathcal{F}$ on $\mathfrak{M} := \{\mathcal{M}_{\Delta} \mid \Delta \in I\}$ and use Łos's Theorem to show that $\prod \mathfrak{M}/\mathcal{F}$ is a model of T. Let

$$I = \{\Delta \subseteq T \mid \Delta \text{ finite}\} \quad \text{and } [\star] : I \to \mathcal{P}(I) := \Delta \mapsto \{\Gamma \in I \mid \Delta \subseteq \Gamma\}$$

Writing $[I]$ for the image of I, we claim that $\mathcal{F} := \{U \in \mathcal{P}(I) \mid \exists V \in [I], V \subseteq U\}$ forms an ultrafilter on I (i.e. an ultrafilter on the Boolean algebra $\mathcal{P}(I)$). Indeed

- $\varnothing \in I$ thus $I = \{[\varnothing] \in [I] \subseteq \mathcal{F}\}$.
- Suppose $\varnothing \in \mathcal{F}$ then $\varnothing \in [I]$ and so there exists $\Delta \in I$ such that $[\Delta] = \varnothing$. This is a contradiction as $\Delta \in [\Delta]$.
- If $U, V \in \mathcal{F}$ then there exist $\Delta_U, \Delta_V \in I$ such that $[\Delta_U] \subseteq U$ and $[\Delta_V] \subseteq V$.

$$\begin{aligned}[\Delta_U] \cap [\Delta_V] &= \{\Gamma \in I \mid \Delta_0 \subseteq \Gamma \text{ and } \Delta_1 \subseteq \Gamma\}\\ &= \{\Gamma \in I \mid \Delta_0 \cup \Delta_1 \subseteq \Gamma\}\\ &= [\Delta_0 \cup \Delta_1] \in [I] \subseteq \mathcal{F}\end{aligned}$$

- Closure under superset is clear.

We identify each $\mathcal{M}_\Delta \in \mathfrak{M}$ with $\Delta \in I$ and generate the same filter (which we will still call $\mathcal{F}$) on $\mathfrak{M}$ (this is okay as the power sets are isomorphic as Boolean algebras.) By Łos's Theorem $\prod \mathfrak{M}/\mathcal{F}$ is a well-defined Σ-structure such that for any Σ-sentence ϕ

$$\prod \mathfrak{M}/\mathcal{F} \vDash_\Sigma \phi \Leftrightarrow \{\mathcal{M} \in \mathfrak{M} \mid \mathcal{M} \vDash_\Sigma \phi\} \in \mathcal{F}$$

Let $\phi \in T$, then $\{\Delta \in I \mid \{\phi\} \subseteq \Delta\} \in \mathcal{F}$ and so

$$\{\Delta \in I \mid \{\phi\} \subseteq \Delta\} \subseteq \{\Delta \in I \mid \phi \in \Delta\} \subseteq \{\Delta \in I \mid \mathcal{M} \vDash_\Sigma \phi\} \in \mathcal{F}$$

The image of this under the isomorphism is $\{\mathcal{M}_\Delta \in X \mid \mathcal{M} \vDash_\Sigma \phi\}$ thus is in $\mathcal{F}$ and so $\prod \mathfrak{M}/\mathcal{F} \vDash_\Sigma \phi$. □

2.1.4 Stone Duality

This section of the appendix gives a more algebraic way of constructing Stone spaces.

Definition – A Boolean algebra on $F(\Sigma, n)$
Let T be a Σ-theory. We quotient out $F(\Sigma, n)$ by the equivalence relation

$$\phi \sim \psi \quad := \quad \phi \text{ and } \psi \text{ equivalent modulo } T := T \vDash_\Sigma \forall v, (\phi \leftrightarrow \psi)$$

Call the projection into the quotient π and the quotient $F(\Sigma, n)/T$. We make $F(\Sigma, n)/T$ into a Boolean algebra by interpreting 0 as $\pi(\bot)$, 1 as $\pi(\top)$, $\pi(\phi) \sqcap \pi(\psi)$ as $\pi(\phi \wedge \psi)$, $\pi(\phi) \sqcup \pi(\psi)$ as $\pi(\phi \vee \psi)$, $-\pi(\phi)$ as $\pi(\neg \phi)$ and $\pi(\phi) \leq \pi(\psi)$ as

$$\{(\pi(\phi), \pi(\psi)) \mid T \vDash_\Sigma \forall v, (\phi \to \psi)\}$$

One can verify that these are well-defined and satisfy the axioms of a Boolean algebra. Notice we need T (potentially chosen to be the empty set) to make $\to$ look like $\leq$ and that we had to quotient modulo T to make $\leq$ satisfy antisymmetry. Antisymmetry in this context looks very much like 'propositional extensionality'. Thus it makes sense to consider the Stone space of this Boolean algebra $S(F(\Sigma, n)/T)$.

Lemma
If $p \subseteq F(\Sigma, n)$ is a maximal subset $(\forall \phi \in F(\Sigma, n), \phi \in p \text{ or } \neg \phi \in p)$ then $\pi(\phi) \in \pi(p)$ in the quotient implies $\phi \in p$.

Proof. If $\pi(\phi) \in \pi(p)$ then there exists $\psi \in p$ such that ψ is equivalent to ϕ modulo T. By consistency with T there exists a non-empty Σ-model $\mathcal{M}$ of T and $b \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma p(b)$, in particular $\mathcal{M} \vDash_\Sigma \psi(b)$. Equivalence modulo T then gives us that $\mathcal{M} \vDash_\Sigma \phi(b)$. By maximality of p, ϕ or $\neg \phi$ is in p but the latter would lead to $\mathcal{M} \nvDash_\Sigma \phi(b)$, a contradiction. □

Proposition – The Stone space is a set of ultrafilters
The Stone space of a Σ-theory T is homeomorphic to the set of ultrafilters from $S(F(\Sigma,n)/T)$ that have preimage consistent with T with the subspace topology. In other words, if π is the projection to the quotient then $S_n(T) \cong X$, where

$$X := \{\mathcal{F} \in S(F(\Sigma,n)/T) \mid \pi^{-1}(\mathcal{F}) \text{ consistent with } T\}$$

Proof. Warning: this proof uses π to be three different things, the quotient $F(\Sigma,n) \to F(\Sigma,n)/T$, the image map (of the quotient) $S_n(T) \to X$, and the map of clopen sets (the image map of the image map) $\mathcal{P}(S_n(T)) \to \mathcal{P}(X)$. The second will be a homeomorphism and the third will be a map between subsets of the topologies (in particular the clopen subsets).

We show that sending $p \in S_n(T)$ to its image under the projection to the quotient $\pi(p)$ is a homeomorphism. To show that it is well-defined it suffices to show that for any p, a maximal n-type over T, $\pi(p)$ is an ultrafilter of $F(\Sigma,n)/T$ with preimage consistent with T. Preimage being consistent with T follows from the definition of n-types over theories. To show that it is a proper filter:

- $\top \in p$ by consistency and maximality. Hence $\pi(\top) \in \pi(p)$.
- If $\pi(\bot) \in \pi(p)$ then $\bot \in p$ which is a contradiction with consistency.
- If $\pi(\phi), \pi(\psi) \in \pi(p)$ then $\phi, \psi \in p$ and so $\phi \wedge \psi \in p$ thus by definition of the Boolean algebra $F(\Sigma,n)/T$,

$$\pi(\phi) \sqcap \pi(\psi) = \pi(\phi \wedge \psi) \in \pi(p)$$

- If $\pi(\phi) \in \pi(p)$ and $\pi(\phi) \leq \pi(\psi)$ then $\phi \in p$ and by definition of $\leq$,

$$T \vDash_\Sigma \forall v, (\phi \to \psi)$$

  Since p is consistent with T there exists a non-empty Σ-structure $\mathcal{M}$ and $b \in \mathcal{M}^n$ such that $\mathcal{M} \vDash_\Sigma \phi(b) \to \psi(b)$ and $\mathcal{M} \vDash_\Sigma \phi(b)$. Hence $\mathcal{M} \vDash_\Sigma \psi(b)$ and by maximality of p we have $\psi \in p$ and $\pi(\psi) \in \pi(p)$.

The image $\pi(p)$ is an ultrafilter by the equivalent definition: if $\pi(\phi) \in F(\Sigma,n)/T$ then either $\phi \in p$ or $\neg\phi \in p$ by maximality of p, hence $\pi(\phi) \in \pi(p)$ or $\pi(\neg\phi) \in \pi(p)$. Thus we have π is a map into

$$\{\mathcal{F} \in S(F(\Sigma,n)/T) \mid \pi^{-1}(\mathcal{F}) \text{ consistent with } T\}$$

Injectivity: if $p, q \in S_n(T)$ and $\pi(p) = \pi(q)$ then if $\phi \in p$, we have $\pi(\phi) \in \pi(p) = \pi(q)$ and by our claim above $\phi \in q$. Surjectivity: let $\mathcal{F} \in S(F(\Sigma,n)/T)$ have its preimage consistent with T. Then its preimage is an n-type. If its preimage is a maximal n-type then we have surjectivity. Indeed since $\mathcal{F}$ is an ultrafilter if $\phi \in F(\Sigma,n)$ then $\pi(\phi) \in \mathcal{F}$ or $\pi(\neg\phi) = -\pi(\phi) \in \mathcal{F}$, hence $\phi \in \pi^{-1}(\mathcal{F})$ or $\neg\phi \in \pi^{-1}(\mathcal{F})$.

To show that the map is continuous in both directions, it suffices to show that images of clopen sets are clopen and preimages of clopen sets are clopen, as each topology is generated by their clopen sets. For $\phi \in F(\Sigma,n)$ since $\pi : S_n(T) \to X$ is a bijection we have that

$$\pi([\phi]_T) = \{\mathcal{F} \in X \mid \phi \in \pi^{-1}(\mathcal{F})\} = \{\mathcal{F} \in F(\Sigma,n)/T \mid \pi(\phi) \in \mathcal{F} \text{ and } \mathcal{F} \in X\} = [\pi(\phi)] \sqcap X$$

and similarly $\pi^{-1}([\pi(\phi)] \sqcap X) = [\phi]_T$. Hence there is a correspondence between clopen sets. □

Lemma – Topological consistency
Let $\mathcal{F} \in S(F(\Sigma,n)/T)$ and T be a Σ-theory. $\pi^{-1}(\mathcal{F})$ is consistent with T if and only if

$$\pi^{-1}(\mathcal{F}) \in \bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T$$

if and only if

$$\bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T \text{ is non empty}$$

Proof. (1. $\Rightarrow$ 2. $\Rightarrow$ 3.) Suppose $\pi^{-1}(\mathcal{F})$ is consistent with T. Then $\pi^{-1}(\mathcal{F}) \in S_n(T)$ thus for any $\phi \in \pi^{-1}(\mathcal{F})$, $\pi^{-1}(\mathcal{F}) \in [\phi]_T$. Hence

$$\pi^{-1}(\mathcal{F}) \in \bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T$$

and it is non-empty.

(3. $\Rightarrow$ 1.) Suppose

$$p \in \bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T$$

then $\forall \phi \in \pi^{-1}(\mathcal{F}), \phi \in p$. As $\mathcal{F}$ is an ultrafilter, for any $\phi \in p$,

$$\phi \notin \pi^{-1}(\mathcal{F}) \Rightarrow \neg\phi \in \pi^{-1}(\mathcal{F}) \Rightarrow \neg\phi \in p \text{ a contradiction}$$

Hence $p = \pi^{-1}(\mathcal{F})$. Hence $\pi^{-1}(\mathcal{F}) \in S_n(T)$ and thus is consistent with T. □

Proposition – Topological compactness implies compactness for types
Let $\mathcal{F} \in S(F(\Sigma, n)/T)$ and T be a Σ-theory. Then $\pi^{-1}(\mathcal{F})$ is consistent with T if and only if $\pi^{-1}(\mathcal{F})$ if finitely consistent with T.

Proof. By definition $\pi^{-1}(\mathcal{F})$ is finitely consistent with T if and only if any finite subset of $\pi^{-1}(\mathcal{F})$ is consistent with T. Translating this to the topology, this is if and only if for any finite subset $\Delta \subseteq \pi^{-1}(\mathcal{F})$,

$$\bigcap_{\phi \in \Delta} [\phi]_T \text{ is non empty}$$

By topological compactness of $S_n(T)$ this is if and only if

$$\bigcap_{\phi \in \pi^{-1}(\mathcal{F})} [\phi]_T \text{ is non empty}$$

Translating this back to model theory this is if and only if $\pi^{-1}(\mathcal{F})$ is consistent with T. □

Chapter 3

Model Theory of Fields

3.1 Ax-Grothendieck

This section studies the theories of fields in the language of rings, with particular focus on algebraically closed fields.

3.1.1 Language of Rings

We introduce rings and fields and construct the field of fractions of integral domains to see the models in action.

Definition – Signature of rings, theory of rings
We define $\Sigma_{\mathrm{RNG}} := (\{0,1\}, \{+,-,\cdot\}, n_\star, \varnothing, m_\star)$ to be the signature of rings, where $n_+ = n_\cdot = 2$, $n_- = 1$ and $m_\star$ is the empty function.

Using the obvious abbreviations $x + (-y) = x - y, x \cdot y = xy$ and so on, we define the theory of rings RNG as the set containing:

| Assosiativity of addition: $\forall x \forall y \forall z, (x+y)+z = x+(y+z)$

| Identity for addition: $\forall x, x + 0 = x$

| Inverse for addition: $\forall x, x - x = 0$

| Commutativity of addition: $\forall x \forall y, x + y = y + x$

| Assosiativity of multiplication: $\forall x \forall y \forall z, (x \cdot y) \cdot z = x \cdot (y \cdot z)$

| Identity for multiplication: $\forall x, x \cdot 0 = x$

| Commutativity of multiplication: $\forall x \forall y, x \cdot y = y \cdot x$

| Distributivity: $\forall x \forall y \forall z, x \cdot (y + z) = x \cdot y + x \cdot z$

Note that we don't have axioms for closure of functions and existence or uniqueness of inverses as it is encoded by interpretation of $+, -, \cdot$ being well-defined. Note that the theory of rings is universal.

Definition – Theory of integral domains and fields
We define the Σ_{RNG}-theory of integral domains

$$\mathrm{ID} := \mathrm{RNG} \cup \{0 \neq 1, \forall x \forall y, xy = 0 \to (x = 0 \lor y = 0)\}$$

and the Σ_{RNG}-theory of fields

$$\mathrm{FLD} := \mathrm{RNG} \cup \{\forall x, x = 0 \vee \exists y, xy = 1\}$$

Note that the theory of integral domains is universal but the theory of fields is not.

Proposition – Field of fractions

Suppose $\mathcal{A} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$. Then there exists an Σ_{RNG}-embedding $\iota : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{B} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$. We call $\mathcal{B}$ the field of fractions.

Proof. We construct $X = \left\{(x, y) \in \mathcal{A}^2 \,|\, y \neq 0\right\}$ and and equivalence relation $(x, y) \sim (v, w) \Leftrightarrow xw = yv$. (Use $\mathcal{A} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ to show that this is an equivalence relation.) Let $\mathcal{B} = X/\sim$ with $\pi : X \to \mathcal{B}$ as the quotient map. Denote $\pi(x, y) := \frac{x}{y}$, interpret $0^{\mathcal{B}} = \frac{0^{\mathcal{A}}}{1^{\mathcal{A}}}$ and $1^{\mathcal{B}} = \frac{1^{\mathcal{A}}}{1^{\mathcal{A}}}$. Interpret $+$ and $\cdot$ as standard fraction addition and multiplication and use $\mathcal{A} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ to check that these are well defined.

Check that $\mathcal{B}$ is an Σ_{RNG} structure and that $\mathcal{B} \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$. Define $\iota : \mathcal{A} \to \mathcal{B} := a \mapsto \frac{a}{1}$ and show that this well defined and injective. Check that ι is a Σ_{RNG}-morphism and note that since there are no relation symobls in Σ_{RNG} it is also an embedding. $\square$

Proposition – Universal property of field of fractions

Suppose $A \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ID}$ and K its field of fractions. Then if $L \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{FLD}$ and there exists a Σ_{RNG}-embedding $\iota_L : A \to L$, then there exists a unique Σ_{RNG}-embedding $K \to L$ that commutes with the other embeddings:

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & K \arrow[d, dashed] \\ & L \end{tikzcd}$$

Proof. Define the map $\iota : K \to L$ sending $\frac{a}{b} \mapsto \frac{\iota_L(a)}{\iota_L(b)}$. Check that this is well-defined and a Σ_{RNG}-morphism. It is injective because ι_L is injective:

$$\frac{\iota_L(a)}{\iota_L(b)} = 0 \Rightarrow \iota_L(a) = 0 \Rightarrow a = 0$$

Thus it is an embedding.

It is unique: suppose $\phi : K \to L$ is a Σ_{RNG}-embedding that commutes with the diagram. Then for any $a \in K$, $\phi(\frac{a}{1}) = \iota_L(a) = \iota(\frac{a}{1})$. Since both ϕ, ι are embeddings they commute with taking the inverse for $a \neq 0$: $\phi(\frac{1}{a}) = \iota(\frac{1}{a})$. Since any element of K can be written as $\frac{a}{b}$, we have shown that $\phi = \iota$. $\square$

3.1.2 Algebraically closed fields

Definition – Theory of algebraically closed fields

We define the Σ_{RNG} theory of algebraically closed fields

$$\mathrm{ACF} := \mathrm{FLD} \cup \left\{ \bigvee_{i=0}^{n-1} a \exists x,\ x^n + \sum_{i=0}^{n-1} a_i x^i = 0 \,|\, n \in \mathbb{N}_{>0}, a \in {\Sigma_{\mathrm{RNG}}}_{\mathrm{var}}^{n-1} \right\}$$

Unlike the theories $\mathrm{RNG}, \mathrm{ID}, \mathrm{FLD}$ this theory is countably infinite.

Proposition
ACF is not complete.

Proof. Take the Σ_{RNG}-formula $\forall x, x + x = 0$. This is satisfied by the algebraic closure of $\mathbb{F}_2$ but not by that of $\mathbb{F}_3$, since field embeddings preserve characteristic. □

Definition – Algebraically closed fields of characteristic p
For $p \in \mathbb{Z}_{>0}$ prime define

$$\phi_p := \forall x, \sum_{i=1}^{p} x = 0$$

and let $\mathrm{ACF}_p := \mathrm{ACF} \cup \{\phi_p\}$. Furthermore, let

$$\mathrm{ACF}[0] := \mathrm{ACF} \cup \{\neg\, \phi_p \mid p \in \mathbb{Z}_{>0} \text{ prime}\}$$

An important fact about algebraically closed fields of characteristic p:

Proposition – Transcendence degree and characteristic determine algebraically closed fields of characteristic p up to isomorphism
If K_0, K_1 are fields with same characteristic and transcendence degree over their minimal subfield ($\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$) then they are (non-canonically) isomorphic.

Proof. See appendix. □

Notation. *If $K \vDash_{\Sigma_{\mathrm{RNG}}} ACF[p]$, write $\mathrm{t.deg}(K)$ to mean the transcendence degree over its minimal subfield ($\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$).*

Lemma – Cardinality of algebraically closed fields
If L is an algebraically closed field then it has cardinality $\aleph_0 + \mathrm{t.deg}(L)$.

Proof. Let S be a transcendence basis and call the minimal subfield K. Since L is algebraically closed it splits the seperable polynomials $x^n - 1$ for each n. Hence L is infinite. Also $S \subseteq L$ and so $\aleph_0 + \mathrm{t.deg}(L) \leq |L|$. For the other direction note that

$$M = \bigcup_{f \in I} \{a \in M \mid f = \min(a, K)\}$$

where $I \subseteq K(S)[x]$ is the set of monic and irreducible polynomials over $K(S)$. Thus

$$\begin{aligned}
|M| &\leq |I| \times \aleph_0 \leq |K(S)[x]| \times \aleph_0 \\
&\leq |K(S)| \times \aleph_0 \leq |K[S]| \times |K[S]| \times \aleph_0 \\
&= |K[S]| \times \aleph_0 \leq \left| \bigcup_{n \in \mathbb{N}} (K \cup S)^n \right| \times \aleph_0 \\
&= \left| \bigcup_{n \in \mathbb{N}} K \cup S \right| \times \aleph_0 = |K \cup S| \times \aleph_0 \\
&= |S| \times \aleph_0
\end{aligned}$$

Noting that $K = \mathbb{Q}$ or $\mathbb{F}_p$ and so is at most countable. By Schröder–Bernstein we have $\aleph_0 + \mathrm{t.deg}(L) = |L|$. □

Proposition

ACF_p is κ-categorical for uncountable κ, consistent and complete.

Proof. Suppose $K, L \vDash_{\Sigma_{\mathrm{RNG}}} \mathrm{ACF}_p$ and $|K| = |L| = \kappa$. Then $\mathrm{t.deg}(K) + \aleph_0 = |K| = \kappa$ and so $\mathrm{t.deg}(K) = \kappa$ (as κ is uncountable). Similarly $\mathrm{t.deg}(L) = \kappa$ and so $\mathrm{t.deg}(K) = \mathrm{t.deg}(L)$. Thus K and L are isomorphic.

ACF_p is consistent due to the existence of the algebraic closures for any characteristic, it is not finitely modelled and is $\aleph_1$-categorical with $\Sigma_{\mathrm{RNG}\,\mathrm{con}} + \aleph_0 \leq \aleph_1$, hence it is complete by Vaught's test. $\square$

3.1.3 Ax-Grothendieck

Proposition – Lefschetz principle

Let ϕ be a Σ_{RNG}-sentence. Then the following are equivalent:

1. There exists a Σ_{RNG}-model of ACF_0 that satisfies ϕ. (If you like $\mathbb{C} \vDash_{\Sigma_{\mathrm{RNG}}} \phi$.)
2. $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \phi$
3. There exists $n \in \mathbb{N}$ such that for any prime p greater than n, $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \phi$
4. There exists $n \in \mathbb{N}$ such that for any prime p greater than n there exists a non-empty Σ_{RNG}-model of ACF_p that satisfies ϕ.

Proof.

1\. $\Rightarrow$ 2. If $\mathbb{C} \vDash_{\Sigma_{\mathrm{RNG}}} \phi$ then since ACF_0 is complete $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \phi$ or $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \neg\phi$. In the latter case we obtain a contradiction.

2\. $\Rightarrow$ 3. Suppose $\mathrm{ACF}[0] \vDash_{\Sigma_{\mathrm{RNG}}} \phi$ then since 'proofs are finite' there exists a finite subset Δ of $\mathrm{ACF}[0]$ such that $\Delta \vDash_{\Sigma_{\mathrm{RNG}}} \phi$. Let n be maximum of all $q \in \mathbb{N}$ such that $\neg\phi_q \in \Delta$. By uniqueness of characteristic, if p is prime and greater than n and q is prime such that $\neg\phi_q \in \Delta$ then $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \neg\phi_q$. Thus if $\mathcal{M}$ is a Σ_{RNG}-model of ACF_p then $\mathcal{M} \vDash_{\Sigma_{\mathrm{RNG}}} \Delta$ and so $\mathcal{M} \vDash_{\Sigma_{\mathrm{RNG}}} \phi$. Hence for all primes p greater than n, $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \phi$.

3\. $\Rightarrow$ 4. ACF_p is consistent thus there exists a non-empty Σ_{RNG}-model of ACF_p. Our hypothesis implies it satisfies ϕ.

4\. $\Rightarrow$ 1. Let $n \in \mathbb{N}$ such that for any prime p greater than n there exists a non-empty Σ_{RNG}-model of ACF_p that satisfies ϕ. Then because ACF_p is complete $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \phi$. Suppose for a contradiction $\mathrm{ACF}_0 \nvDash_{\Sigma_{\mathrm{RNG}}} \phi$. Then by completeness $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \neg\phi$. Hence by the above we obtain there exists m such that for all p greater than m, $\mathrm{ACF}_p \vDash_{\Sigma_{\mathrm{RNG}}} \neg\phi$. Then since there are infinitely many primes, take p greater than both m and n, then ACF_p is inconsistent, a contradiction. Hence $\mathrm{ACF}_0 \vDash_{\Sigma_{\mathrm{RNG}}} \phi$ and in particular $\mathbb{C} \vDash_{\Sigma_{\mathrm{RNG}}} \phi$.

$\square$

Lemma – Ax-Grothendieck for algebraic closures of finite fields

If Ω is an algebraic closure of a finite field then any injective polynomial map over Ω is surjective.

Proof. See appendix. $\square$

Lemma – Construction of Ax-Grothendieck formula

There exists a Σ_{RNG}-sentence $\Phi_{n,d}$ such that for any field K, $K \vDash_{\Sigma} \Phi_{n,d}$ if and only if for all $d, n \in \mathbb{N}$ any injective polynomial map $f : K^n \to K^n$ of degree less than or equal to d is surjective.

Proof. We first need to be able to express polynomials in n varibles of degree less than or equal to d in an elementary way. We first note that for any $n, d \in \mathbb{N}$ there exists a finite set S and powers $r_{s,j} \in \mathbb{N}$ (for each $(s, j) \in S \times \{1, \ldots, n\}$). such that any polynomial $f \in K[x_1, \ldots, x_n]$ can be written as

$$\sum_{s \in S} \lambda_s \prod_{j=1}^{n} x_j^{r_{s,j}}$$

for some $\lambda_s \in K$. Now we have a way of quantifying over all such polynomials, which is by quantifying over all the coefficients. We define $\Phi_{n,d}$:

$$\Phi_{n,d} := \bigforall_{i=1}^{n} \bigforall_{s \in S} \lambda_{s,i}, \left[\bigforall_{j=1}^{n} x_j \bigforall_{j=1}^{n} y_j, \bigwedge_{i=1}^{n} \left(\sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} x_j^{r_{s,j}} = \sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} y_j^{r_{s,j}} \right) \longrightarrow \bigwedge_{i=1}^{n} x_i = y_i \right]$$
$$\longrightarrow \bigforall_{j=1}^{n} x_j, \bigexists_{i=1}^{n} z_i, \bigwedge_{i=1}^{n} \left(z_i = \sum_{s \in S} \lambda_{s,i} \prod_{j=1}^{n} x_j^{r_{s,j}} \right)$$

At first it quantifies over all of the coefficients of all the f_i. The following part says that if the polynomial map is injective then it is surjective. Thus $K \vDash_{\Sigma} \Phi_{n,d}$ if and only if for all $d, n \in \mathbb{N}$ any injective polynomial map $f : K^n \to K^n$ of degree less than or equal to d is surjective. □

> **Proposition – Ax-Grothendieck**
> If K is an algebraically closed field of characterstic 0 then any injective polynomial map over K is surjective. In particular injective polynomial maps over $\mathbb{C}$ are surjective.

Proof. We show an equivalent statement: for any $n, d \in \mathbb{N}$, any injective polynomial map $f : K^n \to K^n$ of degree less than or equal to d is surjective. This is true if and only if $K \vDash_{\Sigma_{\mathrm{RNG}}} \Phi_{n,d}$ (by construction of the A-G formula) which is true if and only if for all p prime greater than some natural number there exists an algebraically closed field of characteristic p that satisfies $\Phi_{n,d}$, by the Lefschetz principle. Indeed, take the natural 0 and let p be a prime greater than 0. Take Ω an algebraic closure of $\mathbb{F}_p$, which indeed models ACF_p. $\Omega \vDash_{\Sigma_{\mathrm{RNG}}} \Phi_{n,d}$ if and only if for any $n, d \in \mathbb{N}$, any injective polynomial map $f : \Omega^n \to \Omega^n$ of degree less than or equal to d is surjective (by construction of the A-G formula). The final statement is true due to A-G for algebraic closures of finite fields. □

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