Manipulator Equations for Systems with Constraints

Contact-aware Control, Lecture 4

by Sergei Savin

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Remember Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} = \tau \tag{1}$$

Let us remember that in the general case of point masses, kinetic energy is:

$$T = \sum 0.5 m_i \dot{\mathbf{r}}_i^\top \dot{\mathbf{r}}_i, \tag{2}$$

and in the general case of rigid bodies, it is:

$$T = \sum 0.5 m_i \dot{\mathbf{r}}_i^\top \dot{\mathbf{r}}_i + \sum 0.5 \mathbf{w}_i^\top \mathbf{I}_i \mathbf{w}_i, \tag{3}$$

Where $\dot{\mathbf{r}}_i$ is the velocity of the center of mass of the i-th body, and $\mathbf{w}$ is the angular velocity of that body.

Using chain rule we can describe the velocity of the center of mass:

$$\dot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}_i^v \dot{\mathbf{q}} \tag{4}$$

This establishes the connection between $\dot{\mathbf{r}}_i$ and generalized velocities.

For the rotations, it is not as simple. We start by using a Poisson formula to connect rotation matrix $\mathbf{T}(\mathbf{q})$ of a body to angular velocity of a body:

$$\mathbf{W}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{T}\dot{\mathbf{T}}, \quad \dot{\mathbf{T}}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial \mathbf{T}}{\partial \mathbf{q}}\dot{\mathbf{q}} \tag{5}$$

where $\mathbf{W}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & -\omega_1 & \omega_2 \\ \omega_1 & 0 & -\omega_3 \\ -\omega_2 & \omega_3 & 0 \end{bmatrix}$.

We can create notation:

$$\mathbf{w}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -\mathbf{W}_{1,2} \\ \mathbf{W}_{1,3} \\ -\mathbf{W}_{2,3} \end{bmatrix} \tag{6}$$

Homework 1

Prove that $\mathbf{w}(\mathbf{q}, \dot{\mathbf{q}})$ is linear with respect to $\dot{\mathbf{q}}$.

Now we can *find angular velocity Jacobian* of (6) w.r.t. $\dot{\mathbf{q}}$:

$$\mathbf{J}_i^w = \frac{\partial \mathbf{w}_i}{\partial \dot{\mathbf{q}}} \tag{7}$$

Since $\mathbf{w}_i$ is linear w.r.t. $\dot{\mathbf{q}}$, we can represent it as:

$$\mathbf{w}_i = \mathbf{J}_i^w \dot{\mathbf{q}} \tag{8}$$

Therefor me can rewrite the kinetic energy in terms of generalized velocity:

$$T = \sum 0.5\dot{\mathbf{q}}^\top (\mathbf{J}_i^v)^\top m_i \mathbf{J}_i^v \dot{\mathbf{q}} + \sum 0.5\dot{\mathbf{q}}^\top (\mathbf{J}_i^w)^\top \mathbf{I}_i \mathbf{J}_i^w \dot{\mathbf{q}} \tag{9}$$

Kinetic energy is a *quadratic form* of the generalized velocities. We can define the matrix of the quadratic form:

$$\mathbf{H} = \sum (\mathbf{J}_i^v)^\top m_i \mathbf{J}_i^v + \sum (\mathbf{J}_i^w)^\top \mathbf{I}_i \mathbf{J}_i^w \tag{10}$$

And therefor: $T = 0.5\dot{\mathbf{q}}^\top \mathbf{H} \dot{\mathbf{q}}$

We can find derivatives of the kinetic energy (remembering that $T = T^\top$, and therefore $\mathbf{H} = \mathbf{H}^\top$):

$$\frac{\partial T}{\partial \dot{\mathbf{q}}} = \mathbf{H}\dot{\mathbf{q}} \tag{11}$$

$$\frac{\partial T}{\partial \mathbf{q}} = 0.5\dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} \tag{12}$$

Notice that it is very tempting to say that $0.5\dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} = 0.5\dot{\mathbf{q}}^\top \dot{\mathbf{H}}$ but it is *not* the case. $\frac{\partial \mathbf{H}}{\partial \mathbf{q}}$ is a three dimensional tensor, symmetric along the first and second dimension (so, transposing along these two dimensions doesn't change the products of the tensor with matrices or vectors). Multiplication by $\dot{\mathbf{q}}$ happens is along the first and second dimensions, while the partial differentiation happens along the third dimension, therefore the result is not necessarily equals to $0.5\dot{\mathbf{q}}^\top \dot{\mathbf{H}}$.

Left-hand side of the Lagrange equations can be re-written as:

$$\frac{d}{dt}\left(\mathbf{H}\dot{\mathbf{q}}\right) - 0.5\dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}}\dot{\mathbf{q}} = \tau \tag{13}$$

We can expand the derivative of a product:

$$\mathbf{H}\ddot{\mathbf{q}} + \dot{\mathbf{H}}\dot{\mathbf{q}} - 0.5\dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}}\dot{\mathbf{q}} = \tau \tag{14}$$

Expression $\dot{\mathbf{H}}\dot{\mathbf{q}} - 0.5\dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}}\dot{\mathbf{q}}$ is often cast as a linear form. $\mathbf{C}\dot{\mathbf{q}}$ The classic formula for calculating $\mathbf{C}\dot{\mathbf{q}}$ uses Christoffel symbols.

Christoffel symbols-based formula for the $\mathbf{C}\dot{\mathbf{q}}$ is:

$$\mathbf{C}\dot{\mathbf{q}} = \begin{bmatrix} \sum_{j,k}^{n} \Gamma_{1,j,k}\dot{q}_j\dot{q}_k \\ ... \\ \sum_{j,k}^{n} \Gamma_{n,j,k}\dot{q}_j\dot{q}_k \end{bmatrix} \tag{15}$$

where Christoffel symbols $\Gamma_{i,j,k}$ are given as:

$$\Gamma_{i,j,k} = \frac{1}{2}\left(\frac{\partial H_{i,j}}{\partial q_k} + \frac{\partial H_{i,k}}{\partial q_j} - \frac{\partial H_{k,j}}{\partial q_i}\right) \tag{16}$$

My apologies for not providing a derivation

Sometimes we need to find matrix $\mathbf{C}$ specifically, rather than linear form $\mathbf{C}\dot{\mathbf{q}}$. This can be achieved using Christoffel symbols as well.

$$C_{i,j} = \sum_{k}^{n} \Gamma_{i,j,k} \dot{q}_k \tag{17}$$

Generalized inertia

Part 5, Alternative formulation

If you use auto-differentiation, you can consider directly using expression (14) to find $\mathbf{C}\dot{\mathbf{q}}$:

$$\mathbf{C}\dot{\mathbf{q}} = \dot{\mathbf{H}}\dot{\mathbf{q}} - 0.5\frac{\partial \dot{\mathbf{q}}^\top \mathbf{H}\dot{\mathbf{q}}}{\partial \mathbf{q}} \tag{18}$$

In Matlab code it looks like:

```
C_times_v = reshape(dHdt*v, length(v), 1) - reshape(jacobian((0.5*v' * H * v), q), length(v), 1);
```

Alternatively, you can use the following formula:

$$\mathbf{C}\dot{\mathbf{q}} = \dot{\mathbf{H}}\dot{\mathbf{q}} - 0.5\frac{\partial \mathrm{vec}(\mathbf{H})}{\partial \mathbf{q}}(\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}) \tag{19}$$

where $\otimes$ is a Kronecker product, and vec() is vectorization of matrix (representing all its elements as a vector.

In Matlab code it looks like:

```
C_times_v = reshape(dHdt*v, [], 1) - 0.5*( jacobian(
    reshape(H, [], 1), q)'*kron(v, v) );
```

You can express generalized forces of all kinds using discussed previously multiplication by the Jacobian:

$$\tau = \left(\frac{\partial \mathbf{r}_{\text{application point}}}{\partial \mathbf{q}} \right)^\top \mathbf{f}_{\text{ext}} \tag{20}$$

For a torque ξ applied to a rigid body, the corresponding generalized force is:

$$\tau = \mathbf{J}_w^\top \xi \tag{21}$$

where $\mathbf{J}_w$ is the angular velocity Jacobian of that body. Note that both $\mathbf{J}_w$ and ξ need to be expressed in the same basis.

If the force is conservative, it is often easy to describe potential energy U associated with it. Then you can find the relevant generalized forces as:

$$\tau = -\frac{\partial U}{\partial \mathbf{q}} \tag{22}$$

Typically this is useful for gravitational forces and elastic forces.

The discussion of the reaction forces stays the same as for any other general case forces, we use Jacobians to transform them into a generalized form.

We can define constraint Jacobian $\mathbf{F}$ as:

$$\mathbf{F} = \frac{\partial \mathbf{g}(\mathbf{q})}{\partial \mathbf{q}} \tag{23}$$

Generalized reaction forces are found as:

$$\tau = \mathbf{F}^\top \lambda \tag{24}$$

Sometimes a constraint can be expressed not as an explicit function of the Cartesian coordinates, but as an implicit one; for example, it can be expressed as a function of generalized coordinates.

Example 1

$$\mathbf{g}(\mathbf{q}) = q_1 - q_3 + 2 = 0$$

Example 2

$$\mathbf{g}(\mathbf{q}) = q_1^2 + q_2^2 - 1 = 0$$

But the generalized reaction forces for this case are still found the same way, as:

$$\tau = \mathbf{F}^\top \lambda \tag{25}$$

Finally we can write the form of manipulator equations:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} = \tau \tag{26}$$

Another popular form specifically points out conservative forces $\mathbf{p}$:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{p} = \tau_{\text{non-conservative}} \tag{27}$$

The most concise and useful for this class form is:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{c} = \tau_{\text{non-conservative}} \tag{28}$$

where $\mathbf{c} = \mathbf{C}\dot{\mathbf{q}} + \mathbf{p}$.

Manipulator equations with reaction forces have the form:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} = \tau + \mathbf{F}^\top \lambda \tag{29}$$

The most concise and useful for this class form is:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{c} = \tau_{\text{non-conservative}} + \mathbf{F}^\top \lambda \tag{30}$$

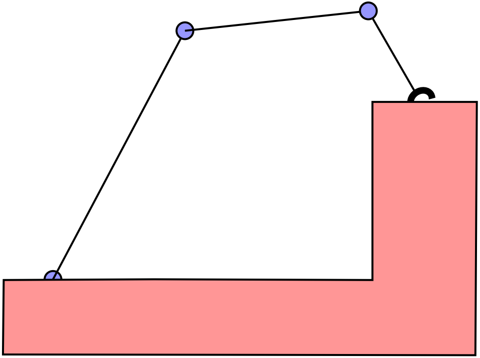
As usual, this is incomplete without the mention of the constraint:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{c} = \tau_{\text{non-conservative}} + \mathbf{F}^\top \lambda \\ \mathbf{g}(\mathbf{q}) = 0 \end{cases} \tag{31}$$

You can read more about Lagrange equation derivation at:

- Chapter 4. Robot Dynamics and Control - part 3.2, an interesting derivation.
- Robot Dynamics Lecture Notes. Robotic Systems Lab, ETH Zurich HS 2017:
  - 2.5 Angular Velocity
  - Chapter 3. Dynamics
  - 3.4.2 Kinetic Energy
  - 3.4.3 Potential Energy
  - 3.4.5 Additional Constraints (note some notational differences)
  - 3.5.2 Deriving Generalized Equations of Motion

Compute manipulator equations for a three link mechanism with fixed end effector (see figure). It should have tree generalized coordinates, two constraints. Preferably, use symbolic computations or auto-differentiation.



Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Contact-Aware-Control-Slides-Fall-2020

Check Moodle for additional links, videos, textbook suggestions.