

# Mechanical Equations for Systems with Constraints

Contact-aware Control, Lecture 3

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In classical mechanics, *Newton equations* are accepted as an axiom and are given (for a system of points):

$$m_i \ddot{\mathbf{r}}_i = \mathbf{f}_i, \quad i = 1, \dots, m \quad (1)$$

Let us introduce new compound vectors  $\mathbf{r}$  and  $\mathbf{f}$ :

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_1 \\ \dots \\ \mathbf{r}_m \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \dots \\ \mathbf{f}_m \end{bmatrix}, \quad \mathbf{M} = \text{diag}(m_1, m_1, m_1, \dots, m_m) \quad (2)$$

# Encoding: generalized coordinates

Let us introduce *generalized coordinates*  $\mathbf{q}$ , such there is an encoding for all admissible  $\mathbf{r}$ :

$$\mathbf{r} = \mathbf{r}(\mathbf{q}) \quad (3)$$

Let us observe two useful (but obvious) properties of this construct.

Lemma 1

$$\frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}} = \frac{\partial}{\partial \dot{\mathbf{q}}} \left( \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \dot{\mathbf{q}} \right) = \frac{\partial \mathbf{r}}{\partial \mathbf{q}}$$

Lemma 2

$$\frac{d}{dt} \frac{\partial \mathbf{r}}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}} \left( \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right) \dot{\mathbf{q}} = \frac{\partial}{\partial \mathbf{q}} (\dot{\mathbf{r}})$$

# Lagrange equations - derivation

## Part 1: right-hand-side

Multiplying Newton eq. (1) by the gen. coordinates jacobian  $\partial \mathbf{r} / \partial \mathbf{q}$  we get:

$$\left( \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^{\top} \mathbf{M} \ddot{\mathbf{r}} = \left( \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^{\top} \mathbf{f} \quad (4)$$

We can define generalized forces as:

$$\tau = \left( \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^{\top} \mathbf{f} \quad (5)$$

And we end with a very simple right-hand-side:

$$\left( \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^{\top} \mathbf{M} \ddot{\mathbf{r}} = \tau \quad (6)$$

# Lagrange equations - derivation

## Part 2: left-hand-side

Let us rewrite the left-hand-side:

$$\left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^{\top} \mathbf{M} \ddot{\mathbf{r}} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^{\top} \frac{d}{dt}(\mathbf{M} \dot{\mathbf{r}})$$

Let us notice that:

$$\frac{d}{dt} \left( \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^{\top} \mathbf{M} \dot{\mathbf{r}} \right) = \frac{d}{dt} \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^{\top} \mathbf{M} \dot{\mathbf{r}} + \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^{\top} \frac{d}{dt}(\mathbf{M} \dot{\mathbf{r}}) \quad (7)$$

Then:

$$\left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^{\top} \frac{d}{dt}(\mathbf{M} \dot{\mathbf{r}}) = \frac{d}{dt} \left( \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^{\top} \mathbf{M} \dot{\mathbf{r}} \right) - \frac{d}{dt} \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^{\top} \mathbf{M} \dot{\mathbf{r}}$$

# Lagrange equations - derivation

## Part 3: left-hand-side

Use both lemmas:

$$\begin{aligned} \frac{d}{dt} \left( \left( \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^\top \mathbf{M} \dot{\mathbf{r}} \right) - \frac{d}{dt} \left( \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^\top \mathbf{M} \dot{\mathbf{r}} = \\ \frac{d}{dt} \left( \left( \frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}} \right)^\top \mathbf{M} \dot{\mathbf{r}} \right) - \left( \frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}} \right)^\top \mathbf{M} \dot{\mathbf{r}} \end{aligned}$$

Using the fact that  $\mathbf{M}$  is symmetric:

$$\frac{\partial (\dot{\mathbf{r}}^\top \mathbf{M} \dot{\mathbf{r}})}{\partial \dot{\mathbf{q}}} = 2 \left( \frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}} \right)^\top \mathbf{M} \dot{\mathbf{r}},$$

and:

$$\frac{\partial (\dot{\mathbf{r}}^\top \mathbf{M} \dot{\mathbf{r}})}{\partial \mathbf{q}} = 2 \left( \frac{\partial \dot{\mathbf{r}}}{\partial \mathbf{q}} \right)^\top \mathbf{M} \dot{\mathbf{r}}$$

# Lagrange equations - derivation

## Part 4: left-hand-side

Substitute:

$$\frac{d}{dt} \left( \left( \frac{\partial \dot{\mathbf{r}}}{\partial \dot{\mathbf{q}}} \right)^\top \mathbf{M} \dot{\mathbf{r}} \right) - \left( \frac{\partial \dot{\mathbf{r}}}{\partial \mathbf{q}} \right)^\top \mathbf{M} \dot{\mathbf{r}} = \frac{d}{dt} \left( \frac{\partial (0.5 \dot{\mathbf{r}}^\top \mathbf{M} \dot{\mathbf{r}})}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial (0.5 \dot{\mathbf{r}}^\top \mathbf{M} \dot{\mathbf{r}})}{\partial \mathbf{q}}$$

Now we can remember that kinetic energy is defined as:

$T = 0.5 \dot{\mathbf{r}}^\top \mathbf{M} \dot{\mathbf{r}}$ , and rewrite the left-hand-side:

$$\frac{d}{dt} \left( \frac{\partial (0.5 \dot{\mathbf{r}}^\top \mathbf{M} \dot{\mathbf{r}})}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial (0.5 \dot{\mathbf{r}}^\top \mathbf{M} \dot{\mathbf{r}})}{\partial \mathbf{q}} = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}}$$

Thus we find the classic form of the Lagrange equations:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \tau \quad (8)$$



# Lagrange equations with reaction forces

## Motivation

Assume we additionally have reaction force  $\lambda \in \mathbb{R}^3$ , acting on the point  $\mathbf{r}_1$ . We can easily see, that the equation (1) will attain form:

$$\mathbf{M}\ddot{\mathbf{r}} = \mathbf{f} + \begin{bmatrix} \mathbf{I}_{3 \times 3} \\ \mathbf{0}_{3(m-1) \times 3} \end{bmatrix} \lambda \quad (9)$$

After multiplying (9) by the jacobian  $\partial \mathbf{r} / \partial \mathbf{q}$  it takes the form:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \tau + \left( \frac{\partial \mathbf{r}_1}{\partial \mathbf{q}} \right)^\top \lambda \quad (10)$$

Notice that by the same logic we can show that for reactions force in  $\lambda \in \mathbb{R}^3$  (and indeed, any external force) applied to the point  $\mathbf{r}_j$ , Lagrange equations will take the form:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \tau + \left( \frac{\partial \mathbf{r}_j}{\partial \mathbf{q}} \right)^\top \lambda \quad (11)$$

# Lagrange equations with reaction forces

## General form

In general, for the reaction force that enforce a constraint  $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^k$ , the general form of the Lagrange equation is:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \tau + \left( \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \right)^{\top} \lambda \quad (12)$$

# Lagrange equations with constraints

## General form

In order to describe a system with explicit constraints  $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^k$ , we need to explicitly include them in the dynamics formulation:

$$\begin{cases} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \boldsymbol{\tau} + \left( \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \right)^\top \boldsymbol{\lambda} \\ \mathbf{g}(\mathbf{q}) = 0 \end{cases} \quad (13)$$

### Statement

A system with explicit constraint  $\mathbf{g}(\mathbf{q}) = 0$  can be replaced by an equivalent system with no explicit constraints, with a different choice of generalized coordinates.

You can read more about Lagrange equation derivation at:

- [Chapter 1. Lagrange's equations](#) - very simple and good.
- [Lecture 15 Lagrangian Dynamics: Derivations of Lagrange's Equations](#) - a different but also a classic approach
- [Chapter 6. The Lagrangian Method. David Morin](#) - a version less derivation, more examples, and with poetry

# Homework

Write derivation of dynamics equations for a three link mechanism with fixed end effector (see figure). It should have three generalized coordinates, two constraints. You can do the derivation on paper, in latex or using symbolic computations.



Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Contact-Aware-Control-Slides-Fall-2020](https://github.com/SergeiSa/Contact-Aware-Control-Slides-Fall-2020)

Check Moodle for additional links, videos, textbook suggestions.