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Brief paper

Rigid body attitude estimation based on the Lagrange–d'Alembert principle[☆]

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ABSTRACT

Estimation of rigid body attitude and angular velocity without any knowledge of the attitude dynamics model is treated using the Lagrange–d'Alembert principle from variational mechanics. It is shown that Wahba's cost function for attitude determination from two or more non-collinear vector measurements can be generalized and represented as a Morse function of the attitude estimation error on the Lie group of rigid body rotations. With body-fixed sensor measurements of direction vectors and angular velocity, a Lagrangian is obtained as the difference between a kinetic energy-like term that is quadratic in the angular velocity estimation error and an artificial potential obtained from Wahba's function. An additional dissipation term that depends on the angular velocity estimation error is introduced, and the Lagrange–d'Alembert principle is applied to the Lagrangian with this dissipation. A Lyapunov analysis shows that the state estimation scheme so obtained provides stable asymptotic convergence of state estimates to actual states in the absence of measurement noise, with an almost global domain of attraction. These estimation schemes are discretized for computer implementation using discrete variational mechanics. A first order Lie group variational integrator is obtained as a discrete-time implementation. In the presence of bounded measurement noise, numerical simulations show that the estimated states converge to a bounded neighborhood of the actual states.

1. Introduction

Estimation of rigid body attitude motion is essential in its applications to spacecraft, unmanned aerial and underwater vehicles as well as formations and networks of such vehicles. In this work, we consider the state estimation problem for attitude and angular velocity of a rigid body, assuming that known inertial directions and angular velocity of the body are measured with body-fixed sensors. The number of direction vectors measured by the body may vary over time. For most of the theoretical developments in this paper, it is assumed that at least two directions are measured at any given instant; this assumption ensures that the attitude can be uniquely determined from the measured directions at each instant. The state estimation schemes presented here have the following important properties: (1) the attitude is represented globally over the configuration space of rigid body attitude motion without using local coordinates or quaternions; (2) the schemes developed do not assume any statistics (Gaussian or otherwise) on the measurement noise; (3) no knowledge of the attitude dynamics model is assumed; and (4) the continuous and discrete-time filtering schemes presented here are obtained by applying the Lagrange–d'Alembert principle or its discretization (Marsden & West, 2001) to a Lagrangian function that depends on the state estimate errors obtained from vector measurements for attitude and angular velocity measurements. In the development of the attitude and angular velocity estimation schemes presented here, it is assumed that measurements of direction vectors and angular velocity are available in continuous time, or at a suitably high sampling frequency. In such a measurement-rich estimation process, one need not use a dynamics model for propagation of state estimates between measurements. The filtering schemes presented here are obtained by applying the Lagrange–d'Alembert principle from variational mechanics (Goldstein, 1980; Greenwood, 1987) to a Lagrangian constructed from errors between measured states and estimated states. This is an idea that has never been applied before to state estimation of a mechanical system. The application of this idea to attitude state estimation of a rigid body using the framework of geometric mechanics, leads to a systematic approach

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to analyzing stability and convergence properties of the filtering schemes obtained, as is shown in this paper. Besides, application of this idea leads to a *physically intuitive* approach to designing filter gains based on the desired rate of convergence, as is also shown in this work.

The earliest solution to the attitude determination problem from two vector measurements is the so-called TRIAD algorithm, which dates from the early 1960s (Black, 1964). This was followed by developments in the problem of attitude determination from a set of three or more vector measurements, which was set up as an optimization problem called Wahba's problem (Wahba, 1965). This problem has been solved by different methods in the prior literature, a sample of which can be obtained in Farrell, Stuelpnagel, Wessner, Velman, and Brock (1966), Markley (1988) and Sanyal (2006). A group of nonlinear state estimation schemes for rigid body attitude motion that seek to minimize the stored "energy" in state estimation errors have been proposed (Aguiar & Hespanha, 2006; Hua, Zamani, Trumpf, Mahony, & Hamel, 2011; Zamani, 2013; Zamani, Trumpf, & Mahony, 2011). These schemes can be obtained by applying Hamilton–Jacobi–Bellman (HJB) theory (Kirk, 1970) to the state space of attitude motion, as shown in Zamani (2013) and Zamani et al. (2011). This work is related to "minimum energy" filtering schemes in the sense that it constructs a Lagrangian function of the state estimate errors on the state space of rigid body attitude motion. However, unlike the filtering schemes based on Hamilton–Jacobi–Bellman theory, the estimation schemes obtained here are based on a familiar formulation of variational mechanics that can be explicitly solved and have proven stability and domain of convergence.

The structure of this paper is as follows. In Section 2, the attitude determination problem from vector measurements is formulated on SO(3). Wahba's cost function is generalized in two ways: by choosing a symmetric matrix of weights instead of scalar weight factor for individual vector measurements, and by making the resulting cost function an argument of a continuously differentiable increasing scalar-valued function. It is shown that this generalization of Wahba's function is a Morse function on SO(3) under certain easily satisfiable conditions on the weight matrix, which can be chosen appropriately to satisfy these desirable conditions. Section 3 formulates the attitude estimation problem for continuous-time measurements of direction vectors and angular velocity on the state space of rigid body attitude motion, using the formulation of variational mechanics. A Lagrangian is constructed from the measurement residuals (between measured and estimated states) for the angular velocity measurements and attitude estimates obtained from the vector measurements. The Lagrange–d'Alembert applied to this Lagrangian, with a dissipative term linearly dependent on the angular velocity estimate error, leads to the state estimation scheme. This estimation scheme, when applied in the absence of measurement errors, is shown to provide almost global asymptotic stability of the actual attitude and angular velocity states, with a domain of attraction that is almost global over the state space. In fact, this domain of attraction is shown to be equivalent to that of the almost global asymptotic stabilization scheme for attitude dynamics in Chaturvedi, Sanyal, and McClamroch (2011). Section 4 of this paper applies the discrete Lagrange–d'Alembert principle (Marsden & West, 2001) to a discretization of the Lagrangian obtained in Section 3, to give rise to a first order discrete-time implementations of the continuous-time state estimator in the presence of measurement noise. Using this discrete time estimator as a Lie group variational integrator, the convergence of the estimated states to a bounded neighborhood of the true states in the presence of bounded measurement noise has been numerically verified in Section 5. Finally, Section 6 gives concluding remarks with contributions and possible future extensions of the work presented in this paper.

2. Attitude determination from vector measurements

Rigid body attitude is determined from $k \in \mathbb{N}$ known inertial vectors measured in a coordinate frame fixed to the rigid body. Let these vectors be denoted as u_i^m for $i = 1, 2, \ldots, k$, in the body-fixed frame. The assumption that $k \geq 2$ is necessary for instantaneous three-dimensional attitude determination. When $k = 2$, the cross product of the two measured vectors is considered as a third measurement for applying the attitude estimation scheme. Denote the corresponding known inertial vectors as seen from the rigid body as e_i, and let the true vectors in the body frame be denoted as $u_i = R^{\mathrm{T}} e_i$, where R is the rotation matrix from the body frame to the inertial frame. This rotation matrix provides a coordinate-free, global and unique description of the attitude of the rigid body. Define the matrix composed of all k measured vectors expressed in the body-fixed frame as column vectors,

$$U^m = [u_1^m \; u_2^m \; u_1^m \times u_2^m] \quad \text{when } k = 2, \text{ and}$$
$$U^m = [u_1^m \; u_2^m \; \ldots u_k^m] \in \mathbb{R}^{3\times k} \quad \text{when } k > 2, \tag{1}$$

and the corresponding matrix of all these vectors expressed in the inertial frame as

$$E = [e_1 \; e_2 \; e_1 \times e_2] \quad \text{when } k = 2, \text{ and}$$
$$E = [e_1 \; e_2 \; \ldots e_k] \in \mathbb{R}^{3\times k} \quad \text{when } k > 2. \tag{2}$$

Note that the matrix of the actual body vectors u_i corresponding to the inertial vectors e_i is given by

$$U = R^{\mathrm{T}} E = [u_1 \; u_2 \; u_1 \times u_2] \quad \text{when } k = 2, \text{ and}$$
$$U = R^{\mathrm{T}} E = [u_1 \; u_2 \; \ldots u_k] \in \mathbb{R}^{3\times k} \quad \text{when } k > 2.$$

2.1. Generalization of Wahba's cost function for instantaneous attitude determination from vector measurements

The optimal attitude determination problem for a set of vector measurements at a given time instant is to find an estimated rotation matrix $\hat{R} \in \mathrm{SO}(3)$ such that a weighted sum of the squared norms of the vector errors

$$s_i = e_i - \hat{R} u_i^m \tag{3}$$

are minimized. This attitude determination problem is known as Wahba's problem and is the problem of minimizing the value of

$$\mathcal{U}^0(\hat{R}, U^m) = \frac{1}{2} \sum_{i=1}^{k} w_i (e_i - \hat{R} u_i^m)^{\mathrm{T}} (e_i - \hat{R} u_i^m) \tag{4}$$

with respect to $\hat{R} \in \mathrm{SO}(3)$, where the weights $w_i > 0$. Defining the trace inner product on $\mathbb{R}^{m\times n}$ as

$$\langle A_1, A_2 \rangle = \mathrm{trace}(A_1^{\mathrm{T}} A_2), \tag{5}$$

we can re-express Eq. (4) for Wahba's cost function as

$$\mathcal{U}^0(\hat{R}, U^m) = \frac{1}{2} \langle E - \hat{R} U^m, (E - \hat{R} U^m) W \rangle, \tag{6}$$

where U^m is given by Eq. (1), E is given by (2), and $W = \mathrm{diag}(w_i)$ is the positive diagonal matrix of the weight factors for the measured directions.

From the expression (6), note that W may be generalized to be any positive definite matrix, not necessarily diagonal. Another generalization of Wahba's cost function is given by

$$\mathcal{U}(\hat{R}, U^m) = \Phi\Big(\frac{1}{2} \langle E - \hat{R} U^m, (E - \hat{R} U^m) W \rangle\Big), \tag{7}$$

where $\Phi : [0, \infty) \mapsto [0, \infty)$ is a C^2 function that satisfies $\Phi(0) = 0$ and $\Phi'(x) > 0$ for all $x \in [0, \infty)$. Furthermore, $\Phi'(\cdot) \leq \alpha(\cdot)$

where $\alpha(\cdot)$ is a Class-$\mathcal{K}$ function. Note that these properties of $\Phi(\cdot)$ ensure that the indices $\mathcal{U}^0(\hat{R}, U^m)$ and $\mathcal{U}(\hat{R}, U^m)$ have the same minimizer $\hat{R} \in \mathrm{SO}(3)$. In other words, minimizing the cost $\mathcal{U}$, which is a generalization of the cost $\mathcal{U}^0$, is equivalent to solving Wahba's problem. Here, W is positive definite (not necessarily diagonal), and E and U^m are assumed to be of rank 3, which is true under the assumption that $k \geq 2$ vectors are measured. The solution to Wahba's problem is given in Markley (1988) and Sanyal (2006).

2.2. Properties of Wahba's cost function in the absence of measurement errors

In the absence of measurement errors, $U^m = U = R^{\mathrm{T}}E$, and let $Q = R\hat{R}^{\mathrm{T}} \in \mathrm{SO}(3)$ denote the attitude estimation error. The following lemmas give the structure of Wahba's cost function in this case.

Lemma 2.1. *Let rank$(E) = 3$, where E is as defined in (2). Let the singular value decomposition of E be given by*

$$E := U_E \Sigma_E V_E^{\mathrm{T}} \quad \text{where } U_E \in \mathrm{O}(3),\ V_E \in \mathrm{O}(m),$$
$$\Sigma_E \in \mathrm{Diag}^+(3, m), \tag{8}$$

and $\mathrm{Diag}^+(n_1, n_2)$ is the vector space of $n_1 \times n_2$ matrices with positive entries along the main diagonal and all other components zero. Let $\sigma_1, \sigma_2, \sigma_3$ denote the main diagonal entries of Σ_E. Further, let the positive definite weight matrix W in the generalization of Wahba's cost function (7) be given by

$$W = V_E W_0 V_E^{\mathrm{T}} \quad \text{where } W_0 \in \mathrm{Diag}^+(m, m) \tag{9}$$

and the first three diagonal entries of W_0 are given by

$$w_1 = \frac{d_1}{\sigma_1^2}, \qquad w_2 = \frac{d_2}{\sigma_2^2}, \qquad w_3 = \frac{d_3}{\sigma_3^2}$$
$$\text{where } d_1, d_2, d_3 > 0. \tag{10}$$

Then $K = EWE^{\mathrm{T}}$ is positive definite and

$$K = U_E \Delta U_E^{\mathrm{T}} \quad \text{where } \Delta = \mathrm{diag}(d_1, d_2, d_3), \tag{11}$$

is its eigendecomposition. Moreover, if $d_i \neq d_j$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$, then $\langle I - Q, K\rangle$ is a Morse function whose critical points are

$$Q \in \{I, Q_1, Q_2, Q_3\} \quad \text{where } Q_i = 2U_E a_i a_i^{\mathrm{T}} U_E^{\mathrm{T}} - I, \tag{12}$$

and a_i is the ith column vector of the identity $I \in \mathrm{SO}(3)$.

Proof. It is straightforward to show that (11) holds given (8)–(10). It is shown here that $\langle I - Q, K\rangle$ has the isolated non-degenerate critical points given by (12). Consider a first differential in Q given by

$$\delta Q = Q\Sigma^\times, \tag{13}$$

where $\Sigma \in \mathbb{R}^3$ and $(\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3) \subset \mathbb{R}^{3\times 3}$ is the skew-symmetric cross-product operator that gives the vector space isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3)$:

$$v^\times = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}. \tag{14}$$

The first variation of $\langle I - Q, K\rangle$ with respect to Q is given by

$$\partial_Q \langle I - Q, K\rangle = \langle K, -\delta Q\rangle = \mathrm{trace}\Big(\frac{1}{2}(Q^{\mathrm{T}}K - KQ)\Sigma^\times\Big)$$
$$= \frac{1}{2}\langle KQ - Q^{\mathrm{T}}K, \Sigma^\times\rangle = S_K^{\mathrm{T}}(Q)\Sigma, \tag{15}$$

where

$$S_K(Q) = \mathrm{vex}\big(KQ - Q^{\mathrm{T}}K\big) \tag{16}$$

and $\mathrm{vex}(\cdot) : \mathfrak{so}(3) \to \mathbb{R}^3$ is the inverse of the $(\cdot)^\times$ map. The critical points of $\langle I - Q, K\rangle$ on SO(3) are therefore given by

$$S_K(Q) = 0 \Rightarrow KQ = Q^{\mathrm{T}}K. \tag{17}$$

Substituting the eigendecomposition of K given by (11) in Eq. (17), we obtain

$$U_E \Delta U_E^{\mathrm{T}} Q = Q^{\mathrm{T}} U_E \Delta U_E^{\mathrm{T}} \Rightarrow \Delta P = P^{\mathrm{T}}\Delta, \tag{18}$$

where $P = U_E^{\mathrm{T}} Q U_E \in \mathrm{SO}(3)$. Given that Δ is a positive diagonal matrix with distinct diagonal entries, the solution set for P that satisfies the condition (18) is

$$C_P = \big\{I, \mathrm{diag}(1, -1, -1), \mathrm{diag}(-1, 1, -1), \mathrm{diag}(-1, -1, 1)\big\}$$
$$= \big\{I, 2a_1a_1^{\mathrm{T}} - I, 2a_2a_2^{\mathrm{T}} - I, 2a_3a_3^{\mathrm{T}} - I\big\}. \tag{19}$$

Thus, the set of critical points of $\langle I - Q, K\rangle$ is given by

$$C_Q = U_E C_P U_E^{\mathrm{T}} = \big\{I, Q_1, Q_2, Q_3\big\}, \tag{20}$$

where Q_1, Q_2 and Q_3 are as given by (12). These critical points are clearly isolated. To show that they are non-degenerate, we evaluate the second variation of $\langle I - Q, K\rangle$ at $Q \in C_Q \subset \mathrm{SO}(3)$, as follows:

$$\partial_Q^2 \langle I - Q, K\rangle = -\langle Q^{\mathrm{T}}K, \delta\Sigma^\times\rangle + \langle \Sigma^\times Q^{\mathrm{T}}K, \Sigma^\times\rangle.$$

Since $Q^{\mathrm{T}}K$ is symmetric at the critical points according to (17), and since $\delta\Sigma^\times$ is clearly skew-symmetric, the first term on the right-hand side of the above expression vanishes, as symmetric and skew-symmetric matrices are orthogonal under the trace inner product. Therefore the second variation of $\langle I - Q, K\rangle$ evaluated at the critical points $Q \in C_Q$ is given by

$$\partial_Q^2 \langle I - Q, K\rangle = \langle \Sigma^\times Q^{\mathrm{T}}K, \Sigma^\times\rangle = -\langle Q^{\mathrm{T}}K, (\Sigma^\times)^2\rangle. \tag{21}$$

Since $(\Sigma^\times)^2$ is symmetric, the second variation vanishes for arbitrary non-zero $\Sigma^\times$ if and only if $Q^{\mathrm{T}}K = 0$ for $Q \in C_Q$. However, that possibility would contradict the positive definiteness of K, which we have already established. Therefore, the critical points of $\langle I - Q, K\rangle$ are non-degenerate and isolated, which makes this a Morse function on SO(3) (Milnor, 1963). □

Note that this lemma specifies the weight matrix W according to the SVD of the matrix E and selected eigenvalues $d_1, d_2, d_3 > 0$ for the matrix $K = EWE^{\mathrm{T}}$. As the following lemma shows, these eigenvalues play a special role in determining the overall properties of Wahba's cost function and its generalization.

Note that since $\langle I - Q, K\rangle$ is a Morse function on SO(3) by Lemma 2.1, by the properties of the function Φ, one can conclude that $\Phi(\langle I - Q, K\rangle) : \mathrm{SO}(3) \to \mathbb{R}$ is also a Morse function with the same critical points as those of $\langle I - Q, K\rangle$. The following result gives the characteristics of the critical points of $\Phi(\langle I - Q, K\rangle)$.

Lemma 2.2. *Let $K = EWE^{\mathrm{T}}$ have the properties given by Lemma 2.1. Then the critical points of $\Phi(\langle I - Q, K\rangle) : \mathrm{SO}(3) \to \mathbb{R}$ given by (12) consist of a global minimum at the identity $I \in \mathrm{SO}(3)$, a global maximum, and two hyperbolic saddle points whose indices depend on the distinct eigenvalues d_1, d_2, and d_3 of K.*

Proof. The characteristics of these critical points are obtained from the second variation of $\Phi(\langle I - Q, K\rangle)$ with respect to $Q \in C_Q$, which was obtained in (21). We express (21) as follows:

$$\partial_Q^2 \langle I - Q, K\rangle = \langle \Sigma^\times Q^{\mathrm{T}}K, \Sigma^\times\rangle = F_K^{\mathrm{T}}(Q, \Sigma)\Sigma, \tag{22}$$

where $F_K(Q,\Sigma)=\mathrm{vex}\big(KQ\Sigma^\times+\Sigma^\times Q^{\mathrm{T}}K\big)$. To express $F_K(Q,\Sigma)$ as a vector in $\mathbb{R}^3$, the following identity is useful:

$$\mathrm{vex}\big(A^{\mathrm{T}}\Sigma^\times+\Sigma^\times A\big)=\big(\mathrm{trace}(A)I-A\big)\Sigma, \tag{23}$$

for $A\in\mathbb{R}^{3\times 3}$ and $\Sigma\in\mathbb{R}^3$. Using identity (23) in the expression (22), one obtains $F_K(Q,\Sigma)=H_K(Q)\Sigma$, where

$$H_K(Q)=\mathrm{trace}(Q^{\mathrm{T}}K)I-Q^{\mathrm{T}}K. \tag{24}$$

Note that $H_K(Q)$ corresponds to the Hessian matrix of $\langle I-Q,K\rangle$ for $Q\in C_Q$. Moreover, at the critical points Q_i $(i=1,2,3)$ defined by (12), $\Delta P_i=\Delta(2a_ia_i^{\mathrm{T}}-I)$ is a diagonal matrix that is not positive definite. The Hessian at these critical points is therefore evaluated to be:

$$\begin{aligned}
&H_K(Q_i)=U_E\Lambda_iU_E^{\mathrm{T}},\quad \Lambda_i=\mathrm{trace}(\Delta P_i)I-\Delta P_i,\\
&\quad i=1,2,3,\ \text{such that}\\
&\quad \Lambda_1=\mathrm{diag}(-d_2-d_3,d_1-d_3,d_1-d_2),\\
&\quad \Lambda_2=\mathrm{diag}(d_2-d_3,-d_3-d_1,d_2-d_1),\\
&\quad \text{and } \Lambda_3=\mathrm{diag}(d_3-d_2,d_3-d_1,-d_1-d_2).
\end{aligned} \tag{25}$$

Clearly, the indices of these critical points depend on the distinct eigenvalues d_1,d_2 and d_3. For example, if $d_1>d_2>d_3$, then the index of Q_1 is one, the index of Q_2 is two, and the index of Q_3 is three, which makes Q_3 the global maximum of $\langle I-Q,K\rangle:\mathrm{SO}(3)\to\mathbb{R}$. Note that the identity $I\in\mathrm{SO}(3)$ is the global minimum of this function since the Hessian evaluated at the identity is

$$\begin{aligned}
&H_K(I)=\mathrm{trace}(K)I-K=U_E\Lambda_0U_E^{\mathrm{T}},\\
&\quad \text{where } \Lambda_0=\mathrm{diag}(d_2+d_3,d_3+d_1,d_1+d_2),
\end{aligned} \tag{26}$$

and therefore the identity is a critical point with index zero. Finally, note that the second variation of $\Phi(\langle I-Q,K\rangle):\mathrm{SO}(3)\to\mathbb{R}$ evaluated at its critical points is given by

$$\begin{aligned}
&\partial_Q^2\Phi\big(\langle I-Q,K\rangle\big)=\Phi'\big(\langle I-Q,K\rangle\big)\partial_Q^2\langle I-Q,K\rangle\\
&\quad =\Phi'\big(\langle I-Q,K\rangle\big)\Sigma^{\mathrm{T}}H_K(Q)\Sigma \quad \text{for } Q\in C_Q.
\end{aligned} \tag{27}$$

Since Φ is a Class-$\mathcal{K}$ function, the critical points and their indices are identical for $\Phi(\langle I-Q,K\rangle)$ and $\langle I-Q,K\rangle$. □

3. Attitude state estimation based on the Lagrange–d'Alembert principle

Let $\Omega\in\mathbb{R}^3$ be the angular velocity of the rigid body expressed in the body-fixed frame. The attitude kinematics is given by Poisson's equation:

$$\dot R=R\Omega^\times. \tag{28}$$

In order to obtain attitude state estimation schemes from continuous-time vector and angular velocity measurements, we apply the Lagrange–d'Alembert principle to an action functional of a Lagrangian of the state estimate errors, with a dissipation term in the angular velocity estimate error. This section presents an estimation scheme obtained using this approach, as well as stability properties of this estimator.

3.1. Action functional of the Lagrangian of state estimate errors

The "energy" contained in the errors between the estimated and the measured inertial vectors is given by $\mathcal{U}(\hat R,U^m)$, where $\mathcal{U}:\mathrm{SO}(3)\times\mathbb{R}^{3\times k}\to\mathbb{R}$ is defined by (7) and depends on the attitude estimate. The "energy" contained in the vector error between the estimated and the measured angular velocity is given by

$$\mathcal{T}(\hat\Omega,\Omega^m)=\frac{m}{2}(\Omega^m-\hat\Omega)^{\mathrm{T}}(\Omega^m-\hat\Omega) \tag{29}$$

where m is a positive scalar. One can consider the Lagrangian composed of these "energy" quantities, as follows:

$$\begin{aligned}
\mathcal{L}(\hat R,U^m,\hat\Omega,\Omega^m)&=\mathcal{T}(\hat\Omega,\Omega^m)-\mathcal{U}(\hat R,U^m)\\
&=\frac{m}{2}(\Omega^m-\hat\Omega)^{\mathrm{T}}(\Omega^m-\hat\Omega)\\
&\quad -\Phi\Big(\frac12\langle E-\hat RU^m,(E-\hat RU^m)W\rangle\Big).
\end{aligned} \tag{30}$$

If the estimation process is started at time t_0, then the action functional of the Lagrangian (30) over the time duration $[t_0,T]$ is expressed as

$$\begin{aligned}
&\mathcal{S}(\mathcal{L}(\hat R,U^m,\hat\Omega,\Omega^m))=\int_{t_0}^{T}\big(\mathcal{T}(\hat\Omega,\Omega^m)-\mathcal{U}(\hat R,U^m)\big)\mathrm{d}t\\
&=\int_{t_0}^{T}\Big\{\frac{m}{2}(\Omega^m-\hat\Omega)^{\mathrm{T}}(\Omega^m-\hat\Omega)\\
&\quad -\Phi\Big(\frac12\langle E-\hat RU^m,(E-\hat RU^m)W\rangle\Big)\Big\}\mathrm{d}t.
\end{aligned} \tag{31}$$

3.2. Variational filtering scheme

Consider attitude state estimation in continuous time in the presence of measurement noise and initial state estimate errors. Applying the Lagrange–d'Alembert principle to the action functional $\mathcal{S}(\mathcal{L}(\hat R,U^m,\hat\Omega,\Omega^m))$ given by (31), in the presence of a dissipation term on $\omega:=\Omega^m-\hat\Omega$, leads to the following attitude and angular velocity filtering scheme.

Proposition 3.1. *The filter equations for a rigid body with the attitude kinematics (28) and with measurements of vectors and angular velocity in a body-fixed frame, are of the form*

$$\begin{cases}
\dot{\hat R}=\hat R\hat\Omega^\times=\hat R(\Omega^m-\omega)^\times,\\
m\dot\omega=-m\hat\Omega\times\omega+\Phi'\big(\mathcal{U}^0(\hat R,U^m)\big)S_L(\hat R)-D\omega,\\
\hat\Omega=\Omega^m-\omega,
\end{cases} \tag{32}$$

where D is a positive definite filter gain matrix, $\hat R(t_0)=\hat R_0$, $\omega(t_0)=\omega_0=\Omega_0^m-\hat\Omega_0$, $S_L(\hat R)=\mathrm{vex}\big(L^{\mathrm{T}}\hat R-\hat R^{\mathrm{T}}L\big)\in\mathbb{R}^3$, $L=EW(U^m)^{\mathrm{T}}$ and W is chosen to satisfy the conditions in Lemma 2.1.

Proof. In order to find a filter equation which reduces the measurement noise in the estimated attitude, one may take the first variation of the action functional (31) with respect to $\hat R$ and $\hat\Omega$. Consider the potential term $\mathcal{U}^0(\hat R,U^m)$ as defined by (6). Taking the first variation of this function with respect to $\hat R$ gives

$$\begin{aligned}
\delta\mathcal{U}^0&=\langle-\delta\hat RU^m,(E-\hat RU^m)W\rangle\\
&=\frac12\langle\Sigma^\times,U^mWE^{\mathrm{T}}\hat R-\hat R^{\mathrm{T}}EW(U^m)^{\mathrm{T}}\rangle,\\
&=\frac12\langle\Sigma^\times,L^{\mathrm{T}}\hat R-\hat R^{\mathrm{T}}L\rangle=S_L^{\mathrm{T}}(\hat R)\Sigma.
\end{aligned} \tag{33}$$

Now consider $\mathcal{U}(\hat R,U^m)=\Phi\big(\mathcal{U}^0(\hat R,U^m)\big)$. Then,

$$\delta\mathcal{U}=\Phi'\big(\mathcal{U}^0(\hat R,U^m)\big)\delta\mathcal{U}^0=\Phi'\big(\mathcal{U}^0(\hat R,U^m)\big)S_L^{\mathrm{T}}(\hat R)\Sigma. \tag{34}$$

Taking the first variation of the kinematic energy term associated with the artificial system (29) with respect to $\hat\Omega$ yields

$$\begin{aligned}
\delta\mathcal{T}&=-m(\Omega^m-\hat\Omega)^{\mathrm{T}}\delta\hat\Omega=-m(\Omega^m-\hat\Omega)^{\mathrm{T}}(\dot\Sigma+\hat\Omega\times\Sigma)\\
&=-m\omega^{\mathrm{T}}(\dot\Sigma+\hat\Omega\times\Sigma),
\end{aligned} \tag{35}$$

where $\omega = \Omega^m - \hat{\Omega}$. Applying Lagrange–d'Alembert principle leads to

$$\delta\mathcal{S} + \int_{t_0}^{T} \tau_D^{\mathrm{T}} \Sigma \mathrm{d}t = 0$$

$$\Rightarrow \int_{t_0}^{T} \Big\{ -m\omega^{\mathrm{T}}(\dot{\Sigma} + \hat{\Omega} \times \Sigma) - \Phi'\big(\mathcal{U}^0(\hat{R}, U^m)\big) S_L^{\mathrm{T}}(\hat{R})\Sigma + \tau_D^{\mathrm{T}}\Sigma \Big\} \mathrm{d}t = 0$$

$$\Rightarrow -m\omega^{\mathrm{T}}\Sigma\big|_{t_0}^{T} + \int_{t_0}^{T} m\dot{\omega}^{\mathrm{T}}\Sigma \mathrm{d}t = \int_{t_0}^{T} \Big\{ m\omega^{\mathrm{T}}\hat{\Omega}^{\times} + \Phi'\big(\mathcal{U}^0(\hat{R}, U^m)\big) S_L^{\mathrm{T}}(\hat{R}) - \tau_D^{\mathrm{T}} \Big\} \Sigma \mathrm{d}t, \tag{36}$$

where the first term on the left hand side vanishes, since $\Sigma(t_0) = \Sigma(T) = 0$, and after replacing the dissipation term $\tau_D = D\omega$ gives the second equation in (32). □

3.3. Stability of filter

Next consider the stability of the estimation scheme (filter) given by Proposition 3.1. The following result shows that this scheme is stable, with almost global convergence of the estimated states to the real states in the absence of measurement noise.

Theorem 3.2. *The filter presented in Proposition 3.1, with distinct positive eigenvalues for $K = EWE^{\mathrm{T}}$, is asymptotically stable at the estimation error state $(Q, \omega) = (I, 0)$ in the absence of measurement noise. Further, the domain of attraction of $(Q, \omega) = (I, 0)$ is a dense open subset of $\mathrm{SO}(3) \times \mathbb{R}^3$.*

Proof. In the absence of measurement noise, $U^m = U = R^{\mathrm{T}}E$ and therefore $\mathcal{U}^0(\hat{R}, U^m) = \frac{1}{2}\langle E - \hat{R}U, (E - \hat{R}U)W\rangle = \langle I - Q, K\rangle = \mathcal{U}^0(Q)$ where $K = EWE^{\mathrm{T}}$ and $Q = R\hat{R}^{\mathrm{T}}$. Therefore, $\Phi(\langle I - Q, K\rangle)$ is a Morse function on SO(3). The stability of this filter can be shown using the following candidate Morse–Lyapunov function, which can be interpreted as the total energy function (equal in value to the Hamiltonian) corresponding to the Lagrangian (30):

$$V(\hat{R}, \omega, U) = \Phi\Big(\frac{1}{2}\langle E - \hat{R}U, (E - \hat{R}U)W\rangle\Big) + \frac{m}{2}\omega^{\mathrm{T}}\omega = \Phi(\langle I - Q, K\rangle) + \frac{m}{2}\omega^{\mathrm{T}}\omega = V(Q, \omega). \tag{37}$$

Note that $V(Q, \omega) \geq 0$ and $V(Q, \omega) = 0$ if and only if $(Q, \omega) = (I, 0)$. Therefore, $V(Q, \omega)$ is positive definite on $\mathrm{SO}(3) \times \mathbb{R}^3$. Using (28) and (32)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi\big(\langle I - Q, K\rangle\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\langle I - R\hat{R}^{\mathrm{T}}, K\rangle) \\ &= \Phi'\big(\langle I - Q, K\rangle\big)\langle K, -R\Omega^{\times}\hat{R}^{\mathrm{T}} + R\hat{\Omega}^{\times}\hat{R}^{\mathrm{T}}\rangle \\ &= \Phi'\big(\langle I - Q, K\rangle\big)\Big(\frac{1}{2}\langle \hat{R}^{\mathrm{T}}KR - R^{\mathrm{T}}K\hat{R}, \omega^{\times}\rangle\Big) \\ &= -\Phi'\big(\langle I - Q, K\rangle\big) S_L^{\mathrm{T}}(\hat{R})\omega. \end{aligned} \tag{38}$$

Therefore, the time derivative of the candidate Morse–Lyapunov function is

$$\begin{aligned}\dot{V}(Q, \omega) &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\langle I - Q, K\rangle) + m\omega^{\mathrm{T}}\dot{\omega} \\ &= \omega^{\mathrm{T}}\Big(-\Phi'\big(\mathcal{U}^0(Q)\big)S_L(\hat{R}) - m\hat{\Omega} \times \omega \\ &\quad + \Phi'\big(\mathcal{U}^0(Q)\big)S_L(\hat{R}) - D\omega\Big). \end{aligned} \tag{39}$$

Noting that $m\omega^{\mathrm{T}}(\hat{\Omega} \times \omega) = 0$, this yields

$$\dot{V}(Q, \omega) = -\omega^{\mathrm{T}}D\omega. \tag{40}$$

Hence, the derivative of the Morse–Lyapunov function is negative semi-definite.

Note that the error dynamics for the attitude estimate error is given by

$$\dot{Q} = Q\psi^{\times} \quad \text{where } \psi = \hat{R}\omega, \tag{41}$$

while the error dynamics for the angular velocity estimate error ω is given by the second of Eqs. (32). Therefore, the error dynamics for (Q, ω) is non-autonomous, since they depend explicitly on $(\hat{R}, \hat{\Omega})$. Considering (37) and (40) and applying Theorem 8.4 in Khalil (2001), one can conclude that $\omega^{\mathrm{T}}D\omega \to 0$ as $t \to \infty$, which consequently implies $\omega \to 0$. Thus, the positive limit set for this system is contained in

$$\mathcal{E} = \dot{V}^{-1}(0) = \big\{(Q, \omega) \in \mathrm{SO}(3) \times \mathfrak{so}(3) : \omega \equiv 0\big\}. \tag{42}$$

Substituting $\omega \equiv 0$ in the filter equations (32), we obtain the positive limit set where $\dot{V} \equiv 0$ (or $\omega \equiv 0$) as the set

$$\begin{aligned}\mathcal{I} &= \big\{(Q, \omega) \in \mathrm{SO}(3) \times \mathbb{R}^3 : S_K(Q) \equiv 0, \omega \equiv 0\big\} \\ &= \big\{(Q, \omega) \in \mathrm{SO}(3) \times \mathbb{R}^3 : Q \in C_Q,\ \omega \equiv 0\big\}. \end{aligned} \tag{43}$$

Therefore, in the absence of measurement errors, all the solutions of this filter converge asymptotically to the set $\mathcal{I}$. Thus, the attitude estimate error converges to the set of critical points of $\langle I - Q, K\rangle$ in this intersection. The unique global minimum of this function is at $(Q, \omega) = (I, 0)$ (Lemma 2.2, see also Sanyal (2006) and Sanyal and Chaturvedi (2008)), so this estimation error is asymptotically stable.

Now consider the set

$$\mathcal{C} = \mathcal{I} \setminus (I, 0), \tag{44}$$

which consists of all stationary states that the estimation errors may converge to, besides the desired estimation error state $(I, 0)$. Note that all states in the stable manifold of a stationary state in $\mathcal{C}$ will converge to this stationary state. From the properties of the critical points $Q_i \in C_Q \setminus (I)$ of $\Phi(\langle K, I - Q\rangle)$ given in Lemma 2.2, we see that the stationary points in $\mathcal{I} \setminus (I, 0) = \big\{(Q_i, 0) : Q_i \in C_Q \setminus (I)\big\}$ have stable manifolds whose dimensions depend on the index of Q_i. Since the angular velocity estimate error ω converges globally to the zero vector, the dimension of the stable manifold $\mathcal{M}_i^S$ of $(Q_i, 0) \in \mathrm{SO}(3) \times \mathbb{R}^3$ is

$$\dim(\mathcal{M}_i^S) = 3 + (3 - \text{ index of } Q_i) = 6 - \text{ index of } Q_i. \tag{45}$$

Therefore, the stable manifolds of $(Q, \omega) = (Q_i, 0)$ are three-dimensional, four-dimensional, or five-dimensional, depending on the index of $Q_i \in C_Q \setminus (I)$ according to (45). Moreover, the value of the Lyapunov function $V(Q, \omega)$ is non-decreasing (increasing when $(Q, \omega) \notin \mathcal{I}$) for trajectories on these manifolds when going backwards in time. This implies that the metric distance between error states (Q, ω) along these trajectories on the stable manifolds $\mathcal{M}_i^S$ grows with the time separation between these states, and this property does not depend on the choice of the metric on $\mathrm{SO}(3) \times \mathbb{R}^3$. Therefore, these stable manifolds are embedded (closed) submanifolds of $\mathrm{SO}(3) \times \mathbb{R}^3$ and so is their union. Clearly, all states starting in the complement of this union, converge to the stable equilibrium $(Q, \omega) = (I, 0)$; therefore the domain of attraction of this equilibrium is

$$\mathrm{DOA}\{(I, 0)\} = \mathrm{SO}(3) \times \mathbb{R}^3 \setminus \big\{\cup_{i=1}^{3} \mathcal{M}_i^S\big\},$$

which is a dense open subset of $\mathrm{SO}(3) \times \mathbb{R}^3$. □

4. Discrete-time variational estimator

In order to numerically implement the filtering scheme introduced in this paper, a discrete-time version is obtained to estimate the attitude states from vector measurements and angular velocity measurements. It is assumed that these measurements are obtained in discrete-time at a sufficiently high but constant sample rate. In this section, a discrete-time version of the filter introduced in Proposition 3.1 is obtained in the form of a Lie group variational integrator (LGVI). A variational integrator works by discretizing the (continuous-time) variational mechanics principle that leads to the equations of motion, rather than discretizing the equations of motion directly. A good background on variational integrators is given in the excellent treatise (Marsden & West, 2001). The correspondence between variational integrators and symplectic integrators (for conservative systems) is given in the book (Hairer, Lubich, & Wanner, 2002). Lie group variational integrators are variational integrators for mechanical systems whose configuration spaces are Lie groups, like rigid body systems. In addition to maintaining properties arising from the variational principles of mechanics, like energy and momenta, LGVI schemes also maintain the geometry of the Lie group that is the configuration space of the system (Lee, Leok, & McClamroch, 2005).

4.1. Discrete-time Lagrangian

As a first step to obtaining the LGVI that discretizes the filter in Proposition 3.1, a discrete-time counterpart of the (continuous-time) Lagrangian expressed in (30) is obtained. Consider an interval of time $[t_0, T] \in \mathbb{R}^+$ separated into N equal-length subintervals $[t_i, t_{i+1}]$ for $i = 0, 1, \ldots, N$, with $t_N = T$ and $t_{i+1} - t_i = h$ is the time step size. Let $(\hat{R}_i, \hat{\Omega}_i) \in \mathrm{SO}(3) \times \mathbb{R}^3$ denote the discrete state estimate at time t_i, such that $(\hat{R}_i, \hat{\Omega}_i) \approx (\hat{R}(t_i), \hat{\Omega}(t_i))$ where $(\hat{R}(t), \hat{\Omega}(t))$ is the exact solution of the continuous-time filter at time $t \in [t_0, T]$.

It is assumed that $k \geq 2$ known inertial vectors are measured in the body frame, as in Proposition 3.1. The term encapsulating the "energy" in the attitude estimate error, given by (7), is discretized as follows:

$$\mathcal{U}(\hat{R}_i, U_i^m) = \Phi\Big(\frac{1}{2}\langle E_i - \hat{R}_i U_i^m, (E_i - \hat{R}_i U_i^m)W_i\rangle\Big), \tag{46}$$

where $E_i \in \mathbb{R}^{3\times k}$ is the set of inertial vectors and $U_i^m \in \mathbb{R}^{3\times k}$ is the corresponding set of measured body vectors observed at time t_i, and W_i is the corresponding diagonal matrix of weight factors. The term containing the "energy" in the angular velocity estimate error is discretized as

$$\mathcal{T}(\hat{\Omega}_i, \Omega_i^m) = \frac{m}{2}(\Omega_i^m - \hat{\Omega}_i)^{\mathrm{T}}(\Omega_i^m - \hat{\Omega}_i), \tag{47}$$

which is the discrete-time version of Eq. (29).

As with the continuous-time state estimation process in Sections 2 and 3, one can express these "energy" terms in the state estimate errors for the case that perfect measurements (with no measurement noise) are available. In this case, these "energy" terms can be expressed in terms of the state estimate errors $Q_i = R_i\hat{R}_i^{\mathrm{T}}$ and $\omega_i = \Omega_i - \hat{\Omega}_i$ as follows:

$$\begin{aligned}\mathcal{U}(Q_i) &= \Phi\Big(\frac{1}{2}\langle E_i - Q_i^{\mathrm{T}}E_i, (E_i - Q_i^{\mathrm{T}}E_i)W_i\rangle\Big)\\ &= \Phi\big(\langle I - Q_i, K_i\rangle\big) \quad \text{where } K_i = E_i W_i E_i^{\mathrm{T}},\\ &\quad \text{and } \mathcal{T}(\omega_i) = \frac{m}{2}\omega_i^{\mathrm{T}}\omega_i \text{ where } m > 0.\end{aligned} \tag{48}$$

The weights in W_i can be chosen such that K_i is always positive definite with distinct (perhaps constant) eigenvalues, as in the continuous-time filter given by Proposition 3.1. Using these "energy" terms in the state estimate errors, the discrete-time Lagrangian can be expressed as:

$$\begin{aligned}\mathcal{L}(Q_i, \omega_i) &= \mathcal{T}(\omega_i) - \mathcal{U}(Q_i)\\ &= \frac{m}{2}\omega_i^{\mathrm{T}}\omega_i - \Phi\big(\langle I - Q_i, K_i\rangle\big).\end{aligned} \tag{49}$$

4.2. Discrete-time attitude state estimation based on the discrete Lagrange–d'Alembert principle

The following statement gives the discrete-time filter equations, in the form of a Lie group variational integrator, corresponding to the continuous-time filter given by Proposition 3.1.

Proposition 4.1. *Let two or more vector measurements be available, along with angular velocity measurements in discrete-time, at time intervals of length h. Further, let the weight matrix W_i for the set of vector measurements E_i be chosen such that $K_i = E_i W_i E_i^{\mathrm{T}}$ satisfies the eigendecomposition condition (11) of Lemma 2.1. A discrete-time filter that approximates the continuous-time filter of Proposition 3.1 to first order in h is*

$$\begin{cases}\hat{R}_{i+1} = \hat{R}_i \exp(h\hat{\Omega}_i^{\times}) = \hat{R}_i \exp\big(h(\Omega_i^m - \omega_i)^{\times}\big),\\ m\omega_{i+1} = \exp(-h\hat{\Omega}_{i+1}^{\times})\Big\{(mI_{3\times 3} - hD)\omega_i\\ \quad + h\Phi'\big(\mathcal{U}^0(\hat{R}_{i+1}, U_{i+1}^m)\big)S_{L_{i+1}}(\hat{R}_{i+1})\Big\},\\ \hat{\Omega}_i = \Omega_i^m - \omega_i,\end{cases} \tag{50}$$

where $S_{L_i}(\hat{R}_i) = \mathrm{vex}(L_i^{\mathrm{T}}\hat{R}_i - \hat{R}_i^{\mathrm{T}}L_i) \in \mathbb{R}^3$, $L_i = E_i W_i (U_i^m)^{\mathrm{T}} \in \mathbb{R}^{3\times 3}$ and $(\hat{R}_0, \hat{\Omega}_0) \in \mathrm{SO}(3) \times \mathbb{R}^3$ are initial estimated states.

Proof. The action functional in expression (31) is replaced by the discrete-time action sum as follows:

$$\begin{aligned}&\mathcal{S}_d(\mathcal{L}(\hat{R}_i, U_i^m, \hat{\Omega}_i, \Omega_i^m))\\ &= h\sum_{i=0}^{N}\Big\{\frac{m}{2}(\Omega_i^m - \hat{\Omega}_i)^{\mathrm{T}}(\Omega_i^m - \hat{\Omega}_i) - \Phi\big(\mathcal{U}^0(\hat{R}_i, U_i^m)\big)\Big\}.\end{aligned} \tag{51}$$

Discretize the kinematics of the attitude estimate as

$$\hat{R}_{i+1} = \hat{R}_i \exp(h\hat{\Omega}_i^{\times}), \tag{52}$$

and consider a first variation in the discrete attitude estimate, R_i, of the form

$$\delta\hat{R}_i = \hat{R}_i \Sigma_i^{\times}, \tag{53}$$

where $\Sigma_i \in \mathbb{R}^3$ gives a variation vector for the discrete attitude estimate. For fixed end-point variations, we have $\Sigma_0 = \Sigma_N = 0$. Further, a first order approximation is to assume that $\hat{\Omega}_i^{\times}$ and $\delta\hat{\Omega}_i^{\times}$ commute. With this assumption, taking the first variation of the discrete kinematics (52) and substituting from (53) gives:

$$\begin{aligned}\delta\hat{R}_{i+1} &= \delta\hat{R}_i \exp(h\hat{\Omega}_i^{\times}) + \hat{R}_i\delta\big(\exp(h\hat{\Omega}_i^{\times})\big)\\ &= \hat{R}_i\Sigma_i^{\times}\exp(h\hat{\Omega}_i^{\times}) + h\hat{R}_i\exp(h\hat{\Omega}_i^{\times})\delta\hat{\Omega}_i^{\times}\\ &= \hat{R}_{i+1}\Sigma_{i+1}^{\times}.\end{aligned} \tag{54}$$

Eq. (54) can be re-arranged to obtain:

$$\begin{aligned}h\delta\hat{\Omega}_i^{\times} &= \exp(-h\hat{\Omega}_i^{\times})\hat{R}_i^{\mathrm{T}}\big[\delta\hat{R}_{i+1} - \hat{R}_i\Sigma_i^{\times}\exp(h\hat{\Omega}_i^{\times})\big]\\ &= \exp(-h\hat{\Omega}_i^{\times})\hat{R}_i^{\mathrm{T}}\hat{R}_{i+1}\Sigma_{i+1}^{\times} - \mathrm{Ad}_{\exp(-h\hat{\Omega}_i^{\times})}\Sigma_i^{\times}\\ &= \Sigma_{i+1}^{\times} - \mathrm{Ad}_{\exp(-h\hat{\Omega}_i^{\times})}\Sigma_i^{\times}.\end{aligned} \tag{55}$$

This in turn can be expressed as an equation on $\mathbb{R}^3$ as follows:

$$h\delta\hat{\Omega}_i = \Sigma_{i+1} - \exp(-h\hat{\Omega}_i^\times)\Sigma_i, \tag{56}$$

since $\mathrm{Ad}_R\Omega^\times = R\Omega^\times R^{\mathrm{T}} = (R\Omega)^\times$.

Applying the discrete Lagrange–d'Alembert principle (Marsden & West, 2001), one obtains

$$\begin{aligned}
&\delta\mathfrak{s}_d + h\sum_{i=0}^{N-1}\tau_{D_i}^{\mathrm{T}}\Sigma_i = 0\\
&\Rightarrow h\sum_{i=0}^{N-1} m(\hat{\Omega}_i - \Omega_i^m)^{\mathrm{T}}\delta\hat{\Omega}_i\\
&\quad - \Big\{\Phi'\big(\mathcal{U}^0(\hat{R}_i, U_i^m)\big)S_{L_i}^{\mathrm{T}}(\hat{R}_i) - \tau_{D_i}^{\mathrm{T}}\Big\}\Sigma_i = 0.
\end{aligned} \tag{57}$$

Substituting (53) and (56) into Eq. (57), one obtains

$$\begin{aligned}
&\sum_{i=0}^{N-1}\Big\{m(\hat{\Omega}_i - \Omega_i^m)^{\mathrm{T}}\big(\Sigma_{i+1} - \exp(-h\hat{\Omega}_i^\times)\Sigma_i\big)\\
&\quad - h\Phi'\big(\mathcal{U}^0(\hat{R}_i, U_i^m)\big)S_{L_i}^{\mathrm{T}}(\hat{R}_i)\Sigma_i + h\tau_{D_i}^{\mathrm{T}}\Sigma_i\Big\} = 0.
\end{aligned} \tag{58}$$

For $0 \le i < N$, the expression (58) leads to the following one-step first-order LGVI for the discrete-time filter:

$$\begin{aligned}
&m(\Omega_{i+1}^m - \hat{\Omega}_{i+1})^{\mathrm{T}}\exp(-h\hat{\Omega}_{i+1}^\times) + h\tau_{D_{i+1}}^{\mathrm{T}}\\
&\quad - h\Phi'\big(\mathcal{U}^0(\hat{R}_{i+1}, U_{i+1}^m)\big)S_{L_{i+1}}^{\mathrm{T}}(\hat{R}_{i+1}) + m(\hat{\Omega}_i - \Omega_i^m)^{\mathrm{T}} = 0\\
&\Rightarrow m\exp(h\hat{\Omega}_{i+1}^\times)(\Omega_{i+1}^m - \hat{\Omega}_{i+1}) = m(\Omega_i^m - \hat{\Omega}_i)\\
&\quad + h\Big(\Phi'\big(\mathcal{U}^0(\hat{R}_{i+1}, U_{i+1}^m)\big)S_{L_{i+1}}(\hat{R}_{i+1}) - \tau_{D_{i+1}}\Big),
\end{aligned} \tag{59}$$

which after substituting $\omega_i = \Omega_i^m - \hat{\Omega}_i$ and $\tau_{D_{i+1}} = D\omega_i$ gives the discrete-time filter presented in (50). □

The stability and convergence properties of this discrete-time filter are not shown here directly. Since this filter is a first-order discretization of the continuous-time filter in Proposition 3.1, its solution will be a first-order (in h) approximation to the continuous-time filter.

5. Numerical simulations

This section presents numerical simulation results of the discrete time estimator presented in Section 4, which is a first order Lie group variational integrator. The estimator is simulated over a time interval of $T = 300$ s, with a time stepsize of $h = 0.01$ s. The rigid body is assumed to have an initial attitude and angular velocity given by,

$$R_0 = \mathrm{expm}_{\mathrm{SO(3)}}\left(\left(\frac{\pi}{4}\times\begin{bmatrix}\frac{3}{7} & \frac{6}{7} & \frac{2}{7}\end{bmatrix}^{\mathrm{T}}\right)^\times\right),$$

$$\text{and}\quad \Omega_0 = \frac{\pi}{60}\times[-2.1\ 1.2\ -1.1]^{\mathrm{T}}\ \mathrm{rad/s}.$$

The inertia scalar gain is $m = 100$ and the dissipation matrix is selected as the following positive definite matrix:

$$D = \mathrm{diag}\big([12\ 13\ 14]^{\mathrm{T}}\big).$$

$\Phi(\cdot)$ could be any C^2 function with the properties described in Section 2, but is selected to be $\Phi(x) = x$ here. W is selected based on the measured set of vectors E at each instant, such that it satisfies the conditions in Lemma 2.1. The initial estimated states have the following initial estimation errors:

$$Q_0 = \mathrm{expm}_{\mathrm{SO(3)}}\left(\left(\frac{\pi}{2.5}\times\begin{bmatrix}\frac{3}{7} & \frac{6}{7} & \frac{2}{7}\end{bmatrix}^{\mathrm{T}}\right)^\times\right),$$

$$\text{and}\quad \omega_0 = [0.001\ 0.002\ -0.003]^{\mathrm{T}}\ \mathrm{rad/s}. \tag{60}$$

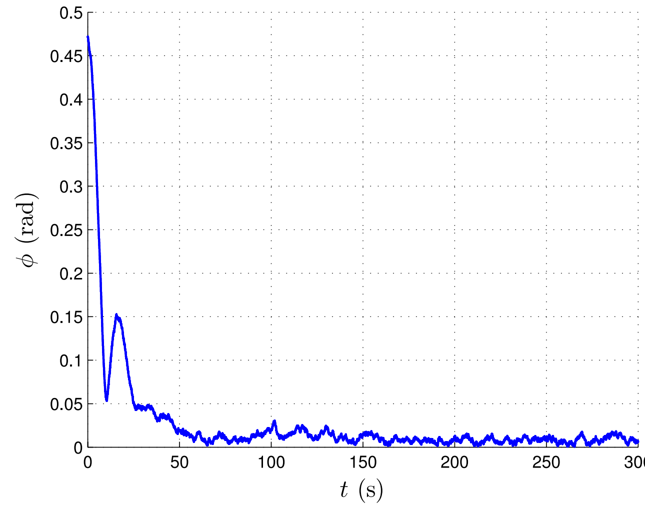

Fig. 1. Principal angle of the attitude estimation error.

We assume that there are at most 9 inertially known directions which are being measured by the sensors fixed to the rigid body at a constant sample rate. The number of observed directions is taken to be variable over different time intervals. The dynamics equations produce the true states of the rigid body, assuming that a sinusoidal force is applied to it. These true states are used to simulate the observed directions in the body-fixed frame, as well as the comparison between true and estimated states. Bounded zero mean noises are considered to be added to the true quantities to generate each measured component. A summation of three sinusoidal matrix functions is added to the matrix $U = R^{\mathrm{T}}E$, to generate a measured U^m with measurement noise. The frequency of the noise signals are 1, 10 and 100 Hz, with different phases and amplitudes up to 2.4°, based on coarse attitude sensors like sun sensors and magnetometers. Similarly, two sinusoidal noise signals of 10 and 200 Hz frequencies are added to Ω to form the measured Ω^m. These signals also have different phases and their magnitude is up to 0.97°/s, which is close to the real noise levels for coarse rate gyros. In order to integrate the implicit set of equations in (50) numerically, the first equation is solved at each sampling step, then the result for R_{i+1} is substituted in the second one. Using the Newton–Raphson method, the resulting equation is solved with respect to ω_{i+1} iteratively. The root of this nonlinear equation with a specific accuracy along with the $\hat{R}_{i+1}$ is used for the next sampling time instant. This process is repeated to the end of the simulation time. Using the aforementioned quantities and the integration method, the simulation is carried out. The principal angle ϕ corresponding to the rigid body's attitude estimation error Q is depicted in Fig. 1. Components of the estimation error ω in the rigid body's angular velocity are shown in Fig. 2. All the estimation errors are seen to converge to a neighborhood of $(Q, \omega) = (I, 0)$, where the size of this neighborhood depends on the bounds of the measurement noise.

6. Conclusion

This work obtains an attitude and angular velocity estimation scheme on the Lie group of rigid body rotational motion, assuming that measurements of inertial vectors and angular velocity are available in continuous-time or at a high sample rate in discrete-time. It is shown that Wahba's cost function for attitude determination from vector measurements can be generalized and cast as a Morse function on the Lie group of rigid body rotations. This Morse function can also be considered as an artificial potential function.

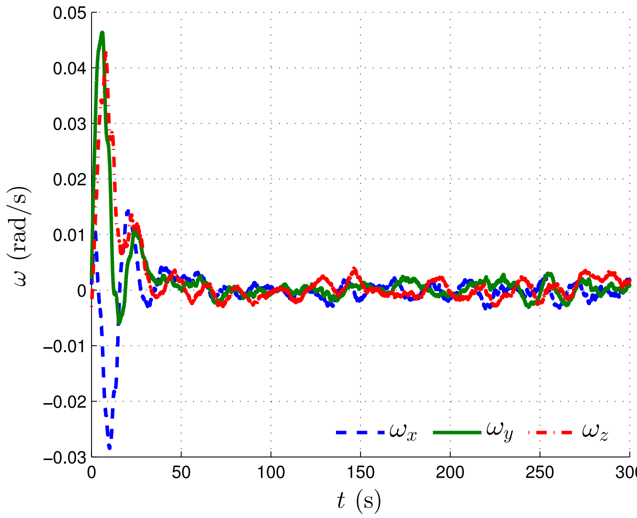

Fig. 2. Angular velocity estimation error.

A kinetic energy-like term, quadratic in the angular velocity estimation errors, can be used along with this artificial potential to construct a Lagrangian dependent on state estimation errors. The estimator is obtained by applying the Lagrange–d'Alembert principle and its discretization to this Lagrangian and a dissipation term dependent on the angular velocity estimation error. This estimation scheme is shown to be almost globally asymptotically stable, with estimates converging to actual states in a domain of attraction that is open and dense in the state space. In the presence of bounded measurement noise, the numerical results show that state estimates converge to a bounded neighborhood of the actual states. A first order discrete-time version of the continuous-time estimation algorithm is obtained by applying the discrete Lagrange–d'Alembert principle. Using a realistic set of data for a rigid body, numerical simulations show that the estimation errors in attitude and angular velocities converge to a bounded neighborhood of $(I, 0)$ in the presence of a bounded measurement noise. Future extensions of this work will include implementations of similar filtering schemes in the presence of measurements of direction vectors and angular velocities at different rates, use of a dynamics model for propagation of state estimates when measurements are available at low sampling rates, and design of state-varying or time-varying filter gains for faster convergence of state estimates.

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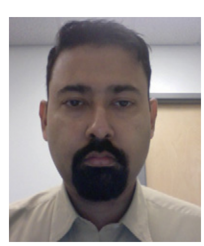

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