Financial Econometrics

Lecture 8

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Outline

Multivariate Time Series

Stationarity and Auto-Correlation Matrices

Vector Autoregressive Models

Multivariate Time Series Analysis

In practice, time series data on multiple subjects are recorded together.

• e.g., S&P 500 index, S&P 400 (MidCap) index and S&P 600 (SmallCap) index

Data is probably correlated not only over different times but also across different subjects. With multivariate time series, one can

- estimate the cross-subject correlation
- compare effects of a certain shock among different time series
- identify latent stationary properties (e.g., nonstationarity of single time series canceled out in a "system")

In this lecture, we study d-vector time series $\mathbf{X}_t \equiv (X_{t,1},...,X_{t,d})'$, where $X_{t,j}$ is the j-th component series.

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Stationarity of Multivariate Time Series

Definition: Weak Stationarity

A multivariate time series $\{X_t\}$ is *weakly stationary* (or simply *stationary*) if all its first and second moments are time-invariant, i.e.,

$$E(\mathbf{X}_t) = \mu = (\mu_1, ..., \mu_d)'$$

$$\mathbf{\Gamma}(k) \equiv E[(\mathbf{X}_{t+k} - \mu)(\mathbf{X}_t - \mu)'] = [\gamma_{ij}(k)]_{d \times d}, k = 0, \pm 1, \pm 2, \dots$$

are independent of t.

Note that if $\{\mathbf{X}_t\}$ is stationary, then all its component series are stationary univariate time series. However, the converse is *not* necessarily true, which requires the covariances $(\gamma_{ij}(k)$ with $i \neq j)$ between different series are also time-invariant.

Stationarity of Multivariate Time Series

Definition: Strict Stationarity

A multivariate time series $\{\mathbf{X}_t\}$ is *strictly stationary* if the joint distribution of $(\mathbf{X}_t, ..., \mathbf{X}_{t+k})$ is independent of t for all $k \geq 1$.

Strict stationarity \Rightarrow weak stationarity provided that $E(\|\mathbf{X}_t\|^2) < \infty$.

In what follows, we assume $\{X_t\}$ is weakly stationary (stationary) unless otherwise specified.

Auto-Covariance Matrix

The matrix-valued function $\Gamma(\cdot)$ is called the *auto-covariance matrix function* or *cross variance function*.

The (i,j)-element of $\Gamma(k)$, $\gamma_{ij}(k)$, is called the *cross covariance* between the i-th and j-th component series at time lag k for $i \neq j$, and *auto-covariance* for the j-th component series at time lag k for i = j, i.e.,

$$\gamma_{ij}(k) = \begin{cases} E[(X_{t+k,i} - \mu)(X_{t,j} - \mu)] = Cov(X_{t+k,i}, X_{t,j}) & i \neq j \\ E[(X_{t+k,i} - \mu)(X_{t,i} - \mu)] = Cov(X_{t+k,i}, X_{t,i}) & i = j \end{cases}$$

Auto-Covariance Matrix

Note that

- Auto-covariance is symmetric, i.e., $\gamma_{jj}(-k) = \gamma_{jj}(k)$.
- Cross covariance is *not* necessarily symmetric, i.e., typically, $\gamma_{ij}(k) \neq \gamma_{ij}(-k)$.
- $\Gamma(k)$ is typically *not* a symmetric matrix for $k \neq 0$, i.e., $\gamma_{ij}(k) \neq \gamma_{ji}(k)$ unless k = 0.
- But $\Gamma(-k) = \Gamma(k)'$ as

$$\gamma_{ij}(k) = E[(X_{t+k,i} - \mu)(X_{t,j} - \mu)]$$

$$= E[(X_{\tau,i} - \mu)(X_{\tau-k,j} - \mu)] (\tau \equiv t + k)$$

$$= E[(X_{t,i} - \mu)(X_{t-k,j} - \mu)]$$

$$= \gamma_{ii}(-k)$$

Auto-Correlation Matrix

The *auto-correlation matrix*, which is also called the *cross correlation matrix*, of \mathbf{X}_t is defined as

$$\mathbf{R}(k) = [\rho_{ij}(k)]_{d \times d} = \mathbf{D}^{-1/2} \mathbf{\Gamma}(k) \mathbf{D}^{-1/2}$$

where $\mathbf{D} = Diag(\gamma_{jj}(0))_{d\times d}$. Then the (i,j)-element of $\mathbf{R}(k)$ is

$$\rho_{ij}(k) = \frac{\gamma_{ij}(k)}{\sqrt{\gamma_{ii}(0)\gamma_{ij}(0)}} = Corr(X_{t+k,i}, X_{t,j})$$

which is called the cross correlation coefficient between the i-th and j-th component series at time lag k.

Auto-Correlation Matrix

Note that

- Auto-correlation $\rho_{jj}(k)$ is symmetric, i.e., $\rho_{jj}(k) = \rho_{jj}(-k)$.
- Cross correlation is not necessarily symmetric, i.e., typically, $\rho_{ij}(k) \neq \rho_{ij}(-k)$.
- In general, $\rho_{ij}(k) \neq \rho_{ji}(k)$ and $\mathbf{R}(k) \neq \mathbf{R}(k)'$.
- However, $\rho_{ii}(k) = \rho_{ii}(-k)$ and hence $\mathbf{R}(-k) = \mathbf{R}(k)'$.

Vector White Noise and Moving Average Processes

We say a vector series $\{\epsilon_t\}$ is *vector white noise*, denoted by $\epsilon_t \sim WN(\mathbf{a}, \Sigma_{\epsilon})$, if

$$E(\epsilon_t) = \mathbf{a}, Var(\epsilon_t) = \mathbf{\Sigma}_{\epsilon}, Cov(\epsilon_t, \epsilon_s) = 0, \forall s \neq t$$

In practice, a is often normalized to be a vector of all zeros.

A basic stationary multivariate time series is the *vector moving average* process (with order q) defined as

$$\mathbf{X}_t = \mu + \epsilon_t + \mathbf{B}_1 \epsilon_{t-1} + \dots + \mathbf{B}_q \epsilon_{t-q}$$

where $\mathbf{B}_1,...,\mathbf{B}_q$ are coefficient matrices and $\epsilon_t \sim \text{WN}(\mathbf{0}, \Sigma_{\epsilon})$. We denote $\mathbf{X}_t \sim \text{MA}(q)$.

Vector White Noise and Moving Average Processes

It is easy to verify that $E(\mathbf{X}_t) = \mu$ and

$$\Gamma(k) = (\mathbf{B}_k \mathbf{\Sigma}_{\epsilon} \mathbf{I}_d + \mathbf{B}_{k+1} \mathbf{\Sigma}_{\epsilon} \mathbf{B}'_1 + \dots + \mathbf{B}_q \mathbf{\Sigma}_{\epsilon} \mathbf{B}'_{q-k}) \mathbf{1}[0 \le k \le q]$$

- Vector moving average process is stationary for all q.
- $\Gamma(k)$ and $\mathbf{R}(k)$ cut off at k = q.
- Overparameterization: the number of unknown coefficients is $O(d^2)$, need regularization or dimension-reduction in practice.

Sample Cross-Covariance/Correlation Matrices

With observations $X_1, ..., X_T$, we can calculate the sample cross-covariance matrix

$$\widehat{\mathbf{\Gamma}}(k) \equiv [\widehat{\gamma}_{ij}(k)]_{d \times d} = \frac{1}{T} \sum_{t=1}^{T-k} (\mathbf{X}_{t+k} - \widehat{\mu}) (\mathbf{X}_t - \widehat{\mu})'$$

where

$$\widehat{\mu} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{X}_{t}$$

Accordingly, the sample cross-correlation matrix can be obtained as

$$\widehat{\mathbf{R}}(k) \equiv [\widehat{\rho}_{ij}(k)]_{d \times d} = \widehat{\mathbf{D}}^{-1/2} \widehat{\mathbf{\Gamma}}(k) \widehat{\mathbf{D}}^{-1/2}$$

where $\widehat{\mathbf{D}} = Diag(\widehat{\gamma}_{ij}(0))_{d \times d}$.

Illustration with Real Data

Example: $X_t = \text{daily returns of } (S\&P 500, S\&P 400, S\&P 600)'$

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Vector Autoregressive Models

It is natural to extend univariate ARMA models to multivariate cases.

A d-vector autoregressive model with order p is of the form

$$\mathbf{X}_t = \mathbf{c} + \mathbf{A}_1 \mathbf{X}_{t-1} + \dots + \mathbf{A}_p \mathbf{X}_{t-p} + \epsilon_t$$

where $\epsilon_t \sim \text{WN}(\mathbf{0}, \mathbf{\Sigma}_{\epsilon})$, \mathbf{c} is a $d \times 1$ vector, $\mathbf{A}_1, \cdots, \mathbf{A}_p$ are $d \times d$ coefficient matrices. Then, we say $\mathbf{X}_t \sim \text{VAR}(p)$.

Example: VAR(1) Model

Example: VAR(1) Model

Consider the following model:

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \epsilon_t$$

By recursive substitution, we have

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \epsilon_t = \dots = \epsilon_t + \sum_{j=1}^{\infty} \mathbf{A}^j \epsilon_{t-j}$$

provided $\mathbf{A}^j \to \mathbf{0}_{d \times d}$ as $j \to \infty$. Then \mathbf{X}_t has a MA(∞) representation and hence weakly stationary (why?).

Let Λ be the Jordan norm form of A, i.e., $A = P\Lambda P^{-1}$, where P is an invertible matrix and Λ is an "almost" diagonal matrix.

Example: VAR(1) Model

Example: VAR(1) Model

Then

$$\mathbf{A}^j = \mathbf{P} \mathbf{\Lambda}^j \mathbf{P}^{-1}$$

Hence, \mathbf{A}^j converges to 0 as $j \to \infty$ if and only if all eigenvalues of \mathbf{A} are within the unit circle.

By multiplying \mathbf{X}_{t-k} on both sides and taking expectations, we have

$$\mathbf{\Gamma}(0) = \mathbf{A}\mathbf{\Gamma}(1)' + \mathbf{\Sigma}_{\epsilon}$$

$$\Gamma(k) = \mathbf{A}\Gamma(k-1), k \ge 1$$

where the second equation is the Yule-Walker equation. Hence, $(\mathbf{A}, \Sigma_{\epsilon})$ and $\{\Gamma(k), k=0,1,2,...\}$ uniquely determine each other.

Identification Issue

Consider the following VAR(1) model

$$\begin{pmatrix} X_{t,1} \\ X_{t,2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{t-1,1} \\ X_{t-1,2} \end{pmatrix} + \begin{pmatrix} \epsilon_{t,1} \\ \epsilon_{t,2} \end{pmatrix}$$

where

$$\mathbf{A} = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \mathbf{A}^2 = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$$

Hence, the VAR(1) model is also an MA(1) model

$$\begin{pmatrix} X_{t,1} \\ X_{t,2} \end{pmatrix} = \begin{pmatrix} \epsilon_{t,1} \\ \epsilon_{t,2} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \epsilon_{t-1,1} \\ \epsilon_{t-1,2} \end{pmatrix}$$

This won't happen in univariate cases. To avoid this identification issue, we consider autoregressive models only.