Financial Econometrics

Lecture 7

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Outline

Existence of Fourth Moments

Recall that to derive $\gamma(k)$ of $\{X_t^2\}$, we assume $E(X_t^4) < \infty$. However, this is not always true. To see this, first note that

$$E(X_t^4) = E(\sigma_t^4)E(\epsilon_t^4) = \frac{E(\epsilon_t^4)}{[E(\epsilon_t^2)]^2}E(\sigma_t^4) = \kappa_\epsilon E(\sigma_t^4)$$

For $\epsilon_t \overset{iid}{\sim} N(0,1)$, and so $\kappa_\epsilon < \infty$, $E(X_t^4) < \infty$ if and only if $E(\sigma_t^4) < \infty$.

Consider GARCH(1,1) model, where we have (with some algebra)

$$E(\sigma_t^4) = a_0^2 + (a_1^2\kappa_\epsilon + 2a_1b_1 + b_1^2)E(\sigma_t^4) + 2a_0(a_1 + b_1)E(\sigma_t^2)$$

which implies that $E(\sigma_t^4) < \infty$ if and only if $a_1^2\kappa_\epsilon + 2a_1b_1 + b_1^2 < 1$.

Note that this is stronger than the stationarity condition $a_1 + b_1 < 1$.

Outline

Estimating GARCH Model using its ARMA Representation?

To fit a GARCH(p, q) model, one may consider the ARMA representation for $\{X_t\}$:

$$X_t^2 = a_0 + \sum_{i=1}^{p \vee q} (a_i + b_i) X_{t-i}^2 + \eta_t - \sum_{j=1}^{q} b_j \eta_{t-j}$$

However, ARMA coefficients are determined by $\gamma(k)$. When $E(X_t^4) = \infty$, $\gamma(k)$ is not well-defined.

In practice, estimating GARCH models using ARMA representations is *not* recommended.

Estimating GARCH Model using its ARMA Representation?

Example: ARCH(1)

Consider ARCH(1) model with $(a_0, a_1) = (0.1, 0.9), \epsilon_t \sim N(0, 1)$.

$$X_t^2 = a_0 + a_1 X_{t-1}^2 + \eta_t$$

is stationary. But $E(X_t^4) = \infty$ as

$$a_1^2 \kappa_\epsilon + 2a_1 b_1 + b_1^2 = 0.9^2 \times 3 + 0 + 0 = 2.43 > 1$$

See R code and its output.

Conditional MLE for GARCH Models

Suppose that $X^T \equiv \{X_1, ..., X_T\}$ are observations from GARCH(p, q) model

$$X_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i X_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2$$

- Assume $\epsilon_t \sim$ IID F_ϵ (e.g. $N(0, 1)$, t_{ν_0}, generalized Gaussian).
- Assume $\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$.
- We want to estimate $\theta \equiv (a_0, a_1, ..., a_p, b_1, ..., b_q)$ for fixed (p, q).

The most often used estimators for θ are conditional likelihood estimator.

Conditional MLE for GARCH Models

Let f_ϵ be the PDF of ϵ_t. The conditional density function for $X_t|X^{t-1}$ is

$$f(X_t|X^{t-1};\theta) = f_\epsilon(X_t/\sigma_t)/\sigma_t$$

Let $\nu = p \vee q + 1$.

$$\begin{aligned} f(X_v,...,X_T|X^{\nu-1};\theta) &= f(X_T|X^{T-1};\theta)f(X_{T-1}|X^{T-2};\theta)\cdots f(X_\nu|X^{\nu-1};\theta) \\ &= \prod_{t=\nu}^{T} f(X_t|X^{t-1};\theta) = \prod_{t=\nu}^{T}[f_\epsilon(X_t/\sigma_t)/\sigma_t] \end{aligned}$$

The conditional MLE is defined as

$$\begin{aligned} \widehat{\theta} &= \arg\min_\theta \log f(X_v,...,X_T|X^{\nu-1};\theta) \\ &= \arg\min_\theta \sum_{t=\nu}^{T}[-\log\sigma_t + \log f_\epsilon(X_t/\sigma_t)] \end{aligned}$$

Conditional MLE for GARCH Models

Model Diagnostics

Let $\widehat{\sigma}_t = \sigma_t(\widehat{\theta})$ and $\widehat{\epsilon}_t = X_t/\widehat{\sigma}_t$.

1. Is $\{\widehat{\epsilon}_t\}$ an iid series? Draw ACF plot and perform Ljung-Box test.
2. Is $\{\widehat{\epsilon}_t^2\}$ an iid series? Draw ACF plot and perform Ljung-Box test.
3. Distribution of $\{\widehat{\epsilon}_t\}$: QQ plot and Jarque-Bera test (for normality).

Example: GARCH(1,1)

See R code and its output.

Outline

Fourth Moments

Estimation for GARCH Models

Forecasting Volatility

Application: S&P 500 Daily Return Data

Forecasting Volatility

Recall $E_T[\cdot] \equiv E[\cdot|X^T]$. Define $\sigma_T^2(k) = E_T(X_{T+k}^2)$ as the conditional volatility in k-period from time T.

Note that the ARMA representation for GARCH(p, q) gives

$$\sigma_T^2(k) = a_0 + \sum_{i=1}^{p \vee q} (a_i + b_i)\sigma_T^2(k-i) - \sum_{j=1}^{q} b_j \eta_T(k-j)$$

where $\sigma_T^2(m) = X_{T+m}^2$ and $\eta_T(m) = \eta_{T+m}$ if $m \leq 0$, $\eta_T(m) = 0$ if $m > 0$.

This formula can be used to forecast volatility σ_{T+k}^2 given observed X^T.

Forecasting Volatility

GARCH(1,1) Model

For a GARCH(1,1) model,

$$\sigma_T^2(k) = a_0 + (a_1 + b_1)\sigma_T^2(k-1)$$

By recursive substitution,

$$\sigma_T^2(k) = a_0 \frac{1-(a_1+b_1)^k}{1-(a_1+b_1)} + (a_1+b_1)^k \sigma_T^2$$

with

$$\sigma_t^2 = \frac{a_0}{1-b_1} + a_1(X_{t-1}^2 + b_1 X_{t-2}^2 + b_1^2 X_{t-3}^2 + \cdots)$$

Outline

Application: S&P 500 Daily Return Data

- See R code and its output.