

# Financial Econometrics

## Lecture 7

Fu Ouyang

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# Outline

## Fourth Moments

## Estimation for GARCH Models

## Forecasting Volatility

## Application: S&P 500 Daily Return Data

# Existence of Fourth Moments

Recall that to derive  $\gamma(k)$  of  $\{X_t^2\}$ , we assume  $E(X_t^4) < \infty$ . However, this is not always true. To see this, first note that

$$E(X_t^4) = E(\sigma_t^4)E(\epsilon_t^4) = \frac{E(\epsilon_t^4)}{[E(\epsilon_t^2)]^2} E(\sigma_t^4) = \kappa_\epsilon E(\sigma_t^4)$$

For  $\epsilon_t \stackrel{iid}{\sim} N(0, 1)$ ,  $\kappa_\epsilon < \infty$ , then  $E(X_t^4) < \infty$  if and only if  $E(\sigma_t^4) < \infty$ .

Consider GARCH(1,1) model, where we have (with some algebra)

$$E(\sigma_t^4) = a_0^2 + (a_1^2 \kappa_\epsilon + 2a_1 b_1 + b_1^2) E(\sigma_t^4) + 2a_0(a_1 + b_1) E(\sigma_t^2)$$

which implies that  $E(\sigma_t^4) < \infty$  if and only if  $a_1^2 \kappa_\epsilon + 2a_1 b_1 + b_1^2 < 1$ .

Note that this is stronger than the stationarity condition  $a_1 + b_1 < 1$ .

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# Estimating GARCH Model using its ARMA Representation?

To fit a GARCH( $p, q$ ) model, one may consider the ARMA representation for  $\{X_t^2\}$ :

$$X_t^2 = a_0 + \sum_{i=1}^{p \vee q} (a_i + b_i) X_{t-i}^2 + \eta_t - \sum_{j=1}^q b_j \eta_{t-j}$$

However, ARMA coefficients are determined by  $\gamma(k)$ . When  $E(X_t^4) = \infty$ ,  $\gamma(k)$  is not well-defined.

In practice, estimating GARCH models using ARMA representations is *not* recommended.

# Estimating GARCH Model using its ARMA Representation?

## Example: ARCH(1)

Consider ARCH(1) model with  $(a_0, a_1) = (0.1, 0.9)$ ,  $\epsilon_t \sim N(0, 1)$ .

$$X_t^2 = a_0 + a_1 X_{t-1}^2 + \eta_t$$

is stationary. But  $E(X_t^4) = \infty$  as

$$a_1^2 \kappa_\epsilon + 2a_1 b_1 + b_1^2 = 0.9^2 \times 3 + 0 + 0 = 2.43 > 1$$

See R code (Part 1).

# Conditional MLE for GARCH Models

Suppose that  $X^T \equiv \{X_1, \dots, X_T\}$  are observations from GARCH( $p, q$ ) model:

$$X_t = \sigma_t \epsilon_t$$
$$\sigma_t^2 = a_0 + \sum_{i=1}^p a_i X_{t-i}^2 + \sum_{j=1}^q b_j \sigma_{t-j}^2$$

- Assume  $\epsilon_t \sim \text{IID } F_\epsilon$  (e.g.  $N(0, 1)$ ,  $t_{\nu_0}$ , generalized Gaussian).
- Assume  $\sum_{i=1}^p a_i + \sum_{j=1}^q b_j < 1$ .
- We want to estimate  $\theta \equiv (a_0, a_1, \dots, a_p, b_1, \dots, b_q)$  for fixed  $(p, q)$ .

The most often used estimators for  $\theta$  are conditional maximum likelihood estimator (MLE).

# Conditional MLE for GARCH Models

Let  $f_\epsilon$  be the PDF of  $\epsilon_t$ . The conditional density function for  $X_t|X^{t-1}$  is

$$f(X_t|X^{t-1}; \theta) = f_\epsilon(X_t/\sigma_t)/\sigma_t$$

Let  $\nu = p \vee q + 1$ .

$$\begin{aligned} f(X_\nu, \dots, X_T|X^{\nu-1}; \theta) &= f(X_T|X^{T-1}; \theta) \cdots f(X_\nu|X^{\nu-1}; \theta) \\ &= \prod_{t=\nu}^T f(X_t|X^{t-1}; \theta) = \prod_{t=\nu}^T [f_\epsilon(X_t/\sigma_t)/\sigma_t] \end{aligned}$$

The conditional MLE is defined as

$$\begin{aligned} \hat{\theta} &= \arg \max_{\theta} \log f(X_\nu, \dots, X_T|X^{\nu-1}; \theta) \\ &= \arg \max_{\theta} \sum_{t=\nu}^T [-\log \sigma_t + \log f_\epsilon(X_t/\sigma_t)] \end{aligned}$$



# Conditional MLE for GARCH Models

## Model Diagnostics

Let  $\hat{\sigma}_t = \sigma_t(\hat{\theta})$  and  $\hat{\epsilon}_t = X_t/\hat{\sigma}_t$ .

1. Is  $\{\hat{\epsilon}_t\}$  an iid series? Draw ACF plot and perform Ljung-Box test.
2. Is  $\{\hat{\epsilon}_t^2\}$  an iid series? Draw ACF plot and perform Ljung-Box test.
3. Distribution of  $\{\hat{\epsilon}_t\}$ : QQ plot and Jarque-Bera test (for normality).

## Example: GARCH(1,1)

See R code (Part 2).

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# Forecasting Volatility

Recall  $E_T[\cdot] \equiv E[\cdot|X^T]$ . Define  $\sigma_T^2(k) = E_T(X_{T+k}^2)$  as the conditional volatility in  $k$ -period from time  $T$ .

Note that the ARMA representation for GARCH( $p, q$ ) gives

$$\sigma_T^2(k) = a_0 + \sum_{i=1}^{p \vee q} (a_i + b_i) \sigma_T^2(k-i) - \sum_{j=1}^q b_j \eta_T(k-j)$$

where  $\sigma_T^2(m) = X_{T+m}^2$  and  $\eta_T(m) = \eta_{T+m}$  if  $m \leq 0$ ,  $\eta_T(m) = 0$  if  $m > 0$ .

This formula can be used to forecast volatility  $\sigma_{T+k}^2$  given observed  $X^T$ .

# Forecasting Volatility

## Example: GARCH(1,1)

For a GARCH(1,1) model,

$$\sigma_T^2(k) = a_0 + (a_1 + b_1)\sigma_T^2(k-1)$$

By recursive substitution,

$$\sigma_T^2(k) = a_0 \frac{1 - (a_1 + b_1)^k}{1 - (a_1 + b_1)} + (a_1 + b_1)^k \sigma_T^2$$

with

$$\sigma_t^2 = \frac{a_0}{1 - b_1} + a_1(X_{t-1}^2 + b_1 X_{t-2}^2 + b_1^2 X_{t-3}^2 + \cdots)$$

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- See R code (Part 3)