Financial Econometrics

Lecture 12

Fu Ouyang

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Outline

Econometric Model

The validity of CAPM relies on several assumptions, which may not be consistent with real-world data. This section introduces methods which can be used to validate the Sharpe-Linter version of CAPM.

Let $Y_t = (Y_{1t}, ..., Y_{Nt})$ be a vector of excess returns of N portfolios at time t, which are usually constructed from the p assets that from the market portfolio.

Let Y_t^m denote the excess return of the (proxy) market portfolio.

Recall the projection we considered in last lecture:

$$Y_t = \alpha + \beta Y_t^m + \epsilon_t$$

where $E(\epsilon_t) = 0$, $Var(\epsilon_t) = \Sigma$ and $Cov(Y_t^m, \epsilon_t) = 0$.

Econometric Model

When CAPM holds, $\alpha = 0$, and so we can fit the above model and test $H_0 : \alpha = 0$ vs. $H_1 : \alpha \neq 0$.

This framework has many other applications:

- Test whether a constructed portfolio is efficient among the assets used to construct the portfolio, in which Y_t is the vector of excess returns of all component assets and Y_t^m is the excess return of the constructed portfolio. If the construct portfolio is efficient, $\alpha = 0$.
- Test an analyst's stock picking ability. Among his/her recommended stocks, is there anyone having positive α?

Maximum Likelihood Estimation

Assume $\epsilon_t \overset{iid}{\sim} N(0, \Sigma)$. Then, we have

$$Y_t | Y_t^m \overset{iid}{\sim} N(\alpha + \beta Y_t^m, \Sigma)$$

and hence

$$\begin{aligned} & f(Y_1, ..., Y_t | Y_1^m, ..., Y_T^m) \\ = & \prod_{t=1}^{T} (2\pi)^{-\frac{N}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left[-\frac{1}{2} (Y_t - \alpha - \beta Y_t^m)' \Sigma^{-1} (Y_t - \alpha - \beta Y_t^m) \right] \end{aligned}$$

The log-likelihood function for parameter $\theta \equiv (\alpha, \beta, \Sigma)$ is

$$L(\theta) = -\frac{NT}{2} \log(2\pi) - \frac{T}{2} \log|\Sigma| - \frac{1}{2} \sum_{t=1}^{T} (Y_t - \alpha - \beta Y_t^m)' \Sigma^{-1} (Y_t - \alpha - \beta Y_t^m)$$

Maximum Likelihood Estimation

The maximum likelihood estimator is defined as

$$\widehat{\theta}_{MLE} = \arg\max_{\theta} L(\theta)$$

It is not hard to obtain that

$$\widehat{\alpha} = \bar{Y} - \widehat{\beta}\bar{Y}_m$$

$$\widehat{\beta} = \sum_{t=1}^{T}(Y_t - \bar{Y})(Y_t^m - \bar{Y}_m) / \sum_{t=1}^{T}(Y_t^m - \bar{Y}_m)^2$$

$$\widehat{\Sigma} = \sum_{t=1}^{T} \widehat{\epsilon}_t \widehat{\epsilon}_t' / T$$

where $\widehat{\epsilon} = Y_t - \widehat{\alpha} - \widehat{\beta} Y_t^m$.

Maximum Likelihood Estimation

Note that under the normality assumption, $(\widehat{\alpha}, \widehat{\beta})$ can be obtained via separately fitting

$$Y_{it} = \alpha_i + \beta_i Y_t^m + \epsilon_{it}$$

for all $i = 1, ..., N$ using OLS, and let $\widehat{\alpha} = (\widehat{\alpha}_1, ..., \widehat{\alpha}_N)$, $\widehat{\beta} = (\widehat{\beta}_1, ..., \widehat{\beta}_N)$.

It can be shown that

$$\widehat{\alpha} \overset{a}{\sim} N(\alpha, T^{-1}(1 + \bar{Y}_m^2/\widehat{\sigma}_m^2)\widehat{\Sigma})$$

where $\widehat{\sigma}_m^2 = \sum_{t=1}^T (Y_t^m - \bar{Y}_m)^2/(T-1)$ and $\overset{a}{\sim}$ means "approximately distributed".

Testing Statistics: Wald Test

Under $H_0 : \alpha = 0$, we can easily construct the *Wald test* which is given by

$$T_0 = T(1 + \bar{Y}_m^2/\widehat{\sigma}_m^2)^{-1}\widehat{\alpha}'\widehat{\Sigma}^{-1}\widehat{\alpha} \overset{a}{\sim} \chi_N^2$$

Furthermore, under the normal model, it can be shown that

$$T_1 = \frac{T - N - 1}{NT} T_0 \sim F_{N,T-N-1}$$

Note that this is the exact distribution of the test statistic T_1 (and also T_0).

Testing Statistics: Likelihood Ratio Test

Note that

$$\sum_{t=1}^{T} \widehat{\epsilon}_t' \widehat{\Sigma}^{-1} \widehat{\epsilon}_t = \sum_{t=1}^{T} tr(\widehat{\epsilon}_t' \widehat{\Sigma}^{-1} \widehat{\epsilon}_t) = \sum_{t=1}^{T} tr(\widehat{\Sigma}^{-1} \widehat{\epsilon}_t \widehat{\epsilon}_t')$$
$$= tr(\widehat{\Sigma}^{-1} \sum_{t=1}^{T} \widehat{\epsilon}_t \widehat{\epsilon}_t') = tr(T \widehat{\Sigma}^{-1} \widehat{\Sigma}) = NT$$

and so

$$L(\widehat{\theta}) = -\frac{NT}{2} \log(2\pi) - \frac{T}{2} \log |\widehat{\Sigma}| - \frac{1}{2} \sum_{t=1}^{T} \widehat{\epsilon}_t' \widehat{\Sigma}^{-1} \widehat{\epsilon}_t$$
$$= -\frac{NT}{2} \log(2\pi) - \frac{T}{2} \log |\widehat{\Sigma}| - \frac{NT}{2}$$

Testing Statistics: Likelihood Ratio Test

Under $H_0 : \alpha = 0$, fit the model

$$Y_t = \beta Y_t^m + \epsilon_t$$

and obtain

$$\widehat{\beta}_{(r)} = \sum_{t=1}^{T} Y_t Y_t^m / \sum_{t=1}^{T} (Y_t^m)^2$$

$$\widehat{\Sigma}_{(r)} = \sum_{t=1}^{T} \widehat{\epsilon}_{(r)t} \widehat{\epsilon}'_{(r)t} / T$$

where $\widehat{\epsilon}_{(r)t} = Y_t - \widehat{\beta}_{(r)} Y_t^m$. The according log-likelihood is

$$L(\widehat{\theta}_{(r)}) = -\frac{NT}{2} \log(2\pi) - \frac{T}{2} \log |\widehat{\Sigma}_{(r)}| - \frac{NT}{2}$$

where $\widehat{\theta}_{(r)} = (0, \widehat{\beta}_{(r)}, \widehat{\Sigma}_{(r)})$.

Testing Statistics: Likelihood Ratio Test

The *likelihood ratio test* is then given by

$$T_2 = 2(L(\widehat{\theta}) - L(\widehat{\theta}_{(r)})) = -T(\log|\widehat{\Sigma}| - \log|\widehat{\Sigma}_{(r)}|) \overset{a}{\sim} \chi^2_N$$

A better approximation can be obtained by

$$T_3 = \frac{T - N/2 - 2}{T} T_2 \overset{a}{\sim} \chi^2_N$$

In fact,

$$T_2 = T \log\left(1 + \frac{NT_1}{T - N - 1}\right)$$

Hence, T_0, T_1, T_2, T_3 are asymptotically equivalent, but they have different finite sample performance.

Testing Statistics: Some Practical Issues

Selection of test statistics:

- When the normal distribution assumption holds, T_1 is preferred as it has an exact distribution (instead of an approximation).
- When the normality of ϵ_t might not hold, T_3 provides better approximation than others.

Data

- Most empirical research uses monthly data. Recall the aggregational Gaussianity of financial returns, which makes the normal model more reasonable.
- Use portfolio returns instead of returns of single stocks for Y_t. The latter exhibit large variances, which make test statistics less powerful in detecting small deviations from CAPM.

Illustration with Real-World Data

See R markdown file and code therein.