Financial Econometrics

Lecture 6

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Outline

Heteroscedastic Volatility Models

- *Volatility* measures the uncertainty of asset returns, which is usually defined as the conditional standard deviation of an asset return given all the available information up to the present time.
- ARMA models are for modeling the conditional mean of a time series, which facilitates the forecasting of asset returns. However, the white noise assumption says nothing about the volatility.
- To carry out statistical inference for ARMA models, we often assume IID innovations. But this is questionable as time-varying volatility (i.e., conditional *heteroscedasticity*) is often observed in real-world data (e.g., volatility clustering).
- In the following couple of lectures, we are going to introduce popular heteroscedasticity models such as ARCH, GARCH, and stochastic volatility models, which can be used to model time-varying volatility.

Outline

Heteroscedastic Volatility Models

ARCH Models

GARCH Models

ARCH Models

A standard model for asset returns is

$$r_t = \mu_t + X_t$$

where μ_t denotes the conditional mean of r_t, X_t is a diffusion term which may be modeled as

$$X_t = \sigma_t \epsilon_t, \epsilon_t \sim \mathrm{IID}(0, 1)$$

where $\sigma_t > 0$ is determined by the information available before time t.

In practice, often assume $\mu_t = E(r_t) = \mu$, and estimate μ by $\sum_{t=1}^{T} r_t/T$.

σ_t is time varying and often called a *volatility function*. A specification for σ_t is the *autoregressive conditional heteroscedastic* (ARCH) *model*:

$$\sigma_t^2 = a_0 + a_1 X_{t-1}^2 + \cdots + a_p X_{t-p}^2$$

where $a_0 > 0$ and $a_j \geq 0, 1 \leq j \leq p$ are constants. We say $X_t \sim \mathrm{ARCH}(p)$.

ARCH Models

Denote $E_t[\cdot] = E[\cdot|X^t]$ and $V_t[\cdot] = Var[\cdot|X^t]$. It is easy to see that

- $\{X_t\}$ is a martingale difference sequence and unpredictable.

$$E_{t-1}(X_t) = \sigma_t E_{t-1}(\epsilon_t) = 0$$

- σ_t is the conditional standard deviation of X_t given X^{t-1}.

$$V_{t-1}(X_t) = E_{t-1}(X_t^2) = \sigma_t^2$$

- Let $\eta_t \equiv \sigma_t^2(\epsilon_t^2 - 1)$.

$$X_t^2 = \sigma_t^2 + \eta_t = a_0 + a_1 X_{t-1}^2 + \cdots + a_p X_{t-p}^2 + \eta_t$$

$$E_{t-1}(\eta_t) = \sigma_t^2 E(\epsilon_t^2 - 1) = 0$$

i.e., $\{\eta_t\}$ is martingale difference sequence ($\Rightarrow$ WN) and $X_t^2 \sim \mathrm{AR}(p)$.

Example: ARCH(1) Model

ARCH(1) Model

$$X_t = \sigma_t \epsilon_t, \epsilon_t \sim \text{IID}(0,1)$$

$$\sigma_t = a_0 + a_1 X_{t-1}^2, a_0 > 0, a_1 \geq 0$$

Hence,

$$X_t^2 = a_0 + a_1 X_{t-1}^2 + \eta_t$$

Provided that $\{X_t\}$ is strictly stationary and $E(X_t^4) < \infty$, this AR(1) representation gives that $Corr(X_t^2, X_{t+k}^2) = a_1^{|k|}$.

- $\{X_t\}$ is martingale difference, compatible with the EMH.
- $\{X_t^2\}$ is auto-correlated $\Rightarrow$ predictable volatility.

Example: ARCH(1) Model

ARCH(1) Model

Note that

$$V_t(X_{t+k}) = E_t(X_{t+k}^2) = a_0 + a_1 E(X_{t+k-1}^2) = \frac{a_0(1-a_1^k)}{1-a_1} + a_1^k X_t^2$$

i.e., large $|X_t|$ leads to large volatilities in near future (volatility clustering).

Provided that $E(\epsilon_t^k) < \infty$ and $E(X_t^k) < \infty$ for some $k \geq 1$, by the LIE and the independence of ϵ_t and $\{X_t\}$

$$E(X_t^k) = E(\sigma_t^k \epsilon_t^k) = E[\sigma_t^k E_{t-1}(\epsilon_t^k)] = E(\sigma_t^k) E(\epsilon_t^k)$$

Kurtosis (κ) measures if the data are heavy-tailed (relative to $N(0,1)$), i.e., data sets with high kurtosis tend to have heavy tails, or outliers.

Example: ARCH(1) Model

ARCH(1) Model

By definition and the Cauchy–Schwarz inequality,

$$\kappa_X = \frac{E(X_t^4)}{[E(X_t^2)]^2} = \frac{E(\sigma_t^4)E(\epsilon_t^4)}{[E(\sigma_t^2)]^2[E(\epsilon_t^2)]^2} = \kappa_\epsilon \frac{E(\sigma_t^4)}{[E(\sigma_t^2)]^2} \geq \kappa_\epsilon$$

i.e., $\{X_t\}$ has heavier tails than $\{\epsilon_t\}$.

Outline

GARCH Models

Perhaps the most important extension of ARCH model is the *generalized autoregressive conditional heteroscedastic* (GARCH) *model*:

$$X_t = \sigma_t \epsilon_t, \epsilon_t \sim \text{IID}(0,1)$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i X_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2$$

where $a_0 > 0$ and $a_i \geq 0$, $b_j \geq 0$. We say $X_t \sim \text{GARCH}(p,q)$.

Let $\eta_t \equiv \sigma_t^2(\epsilon_t^2 - 1)$, $a_{p+j} = b_{q+j} = 0 \forall j \geq 1$ and $p \vee q \equiv \max\{p,q\}$. We have

$$\begin{aligned} X_t^2 = \sigma_t^2 + \eta_t &= a_0 + \sum_{i=1}^{p} a_i X_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2 + \eta_t \\ &= a_0 + \sum_{i=1}^{p \vee q} (a_i + b_i) X_{t-i}^2 + \eta_t - \sum_{j=1}^{q} b_j \eta_{t-j} \end{aligned}$$

Stationarity of GARCH Models

It is easy to verify that $\{\eta_t\}$ is a sequence of martingale differences, and so $\{X_t^2\}$ is an ARMA($p \vee q, q$) process.

If $\{X_t^2\}$ is stationary, provided that $E(X_t^2) < \infty$,

$$\begin{aligned} E(X_t^2) &= E(a_0 + \sum_{i=1}^{p} a_i X_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2 + \eta_t) \\ &= a_0 + \sum_{i=1}^{p} a_i E(X_t^2) + \sum_{j=1}^{q} b_j E(X_t^2) \end{aligned}$$

Solving this gives the *long-run variance*:

$$Var(X_t) = E(X_t^2) = \frac{a_0}{1 - \sum_{i=1}^{p} a_i - \sum_{j=1}^{q} b_j} \Rightarrow \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$$

Stationarity of GARCH Models

In fact, $\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$ is also sufficient for $\{X_t^2\}$ being stationary.

To see this, note that the characteristic polynomial

$$a(z) = 1 - \sum_{i=1}^{p \vee q} (a_i + b_i)\, z^i$$

has all roots outside the unit circle when $\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$ since for all $|z| \leq 1$

$$|a(z)| \geq 1 - \sum_{i=1}^{p \vee q} (a_i + b_i)\, |z^i| \geq 1 - \sum_{i=1}^{p \vee q} (a_i + b_i) > 0$$

Then, this gives us the following general results:

Stationarity of GARCH Models

Stationarity of GARCH Models

The necessary and sufficient condition for a GARCH(p, q) model defining a unique strictly stationary process $\{X_t, t = 0, \pm 1, ...\}$ with $E(X_t^2) < \infty$ is

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$$

Furthermore, $E(X_t) = 0$ and

$$Var(X_t) = E(X_t^2) = \frac{a_0}{1 - \sum_{i=1}^{p} a_i - \sum_{j=1}^{q} b_j}$$

Stationarity of GARCH Models

- The necessary and sufficient condition for a ARCH(p) model being unique strictly stationary is

$$\sum_{i=1}^{p} a_i < 1$$

- Why weak stationarity $\Rightarrow$ strict stationarity? Note that

$$\sigma_t^2 = b(B)^{-1}(a_0 + \sum_{i=1}^{p} a_i X_{t-i}^2)$$

where $b(z) = 1 - \sum_{j=1}^{q} b_j z^j$.

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1 \Rightarrow \sum_{j=1}^{q} b_j < 1 \Rightarrow \sigma_t^2 \sim \text{ARCH}(\infty)$$

Example: GARCH(1,1) Model

GARCH(1,1) Model

Consider GARCH(1,1) model

$$X_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = a_0 + a_1 X_{t-1}^2 + b_1 \sigma_{t-1}^2, a_0 > 0, a_1 \geq 0, b_1 \geq 0$$

Note that for stationary GARCH model, $a_1 + b_1 < 1$, and so

$$\begin{aligned}
\sigma_t^2 &= (1 - b_1 B)^{-1}(a_0 + a_1 X_{t-1}^2) \\
&= (1 + b_1 B + b_1^2 B^2 + \cdots)(a_0 + a_1 X_{t-1}^2) \\
&= \frac{a_0}{1 - b_1} + a_1 (X_{t-1}^2 + b_1 X_{t-2}^2 + b_1^2 X_{t-3}^2 + \cdots)
\end{aligned}$$

Then GARCH(1,1) has an ARCH(∞) representation.

Example: GARCH(1,1) Model

GARCH(1,1) Model

Recall that GARCH(1,1) model also has an ARMA(1,1) representation

$$X_t^2 = a_0 + (a_1 + b_1)X_{t-1}^2 + \eta_t - b_1\eta_{t-1}$$

which gives us a convenient way to compute $\gamma(k)$.

Assume $E(X_t^4) < \infty$and define

$$\sigma^2 = E(\eta_t^2) = E[(\epsilon_t^2 - 1)^2]E(\sigma_t^4)$$

Then for $\{X_t^2\}$

$$\gamma(0) = \frac{1 + b_1^2 - 2(a_1 + b_1)b_1}{1 - (a_1 + b_1)^2}\sigma^2$$

Example: GARCH(1,1) Model

GARCH(1,1) Model

$$\gamma(1) = \frac{-b_1 + (a_1 + b_1)[1 + b_1^2 - (a_1 + b_1)b_1]}{1 - (a_1 + b_1)^2}\sigma^2$$

and

$$\gamma(k) = (a_1 + b_1)^{k-1}\gamma(1), k \geq 1$$