Financial Econometrics I

Lecture 6

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Outline

GARCH Models

A GARCH can be expressed as:

$$X_t = \sigma_t \epsilon_t, \epsilon_t \sim \text{IID}(0,1)$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i X_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2$$

where $a_0 > 0$ and $a_i \geq 0$, $b_j \geq 0$. We say $X_t \sim \text{GARCH}(p,q)$.

The necessary and sufficient condition for a GARCH(p,q) model defining a unique strictly stationary process $\{X_t, t = 0, \pm 1, ...\}$ with $E(X_t^2) < \infty$ is

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$$

Furthermore, $E(X_t) = 0$ and

$$Var(X_t) = E(X_t^2) = \frac{a_0}{1 - \sum_{i=1}^{p} a_i - \sum_{j=1}^{q} b_j}$$

Outline

Estimating GARCH Model using its ARMA Representation?

To fit a GARCH(p, q) model, one may consider the ARMA representation for $\{X_t^2\}$:

$$X_t^2 = a_0 + \sum_{i=1}^{p \vee q} (a_i + b_i) X_{t-i}^2 + \eta_t - \sum_{j=1}^{q} b_j \eta_{t-j}$$

However, ARMA coefficients are determined by $\gamma(k)$. When $E(X_t^4) = \infty$, $\gamma(k)$ is not well-defined.

In practice, estimating GARCH models using ARMA representations is *not* recommended.

Estimating GARCH Model using its ARMA Representation?

Example: ARCH(1)

Consider ARCH(1) model with $(a_0, a_1) = (0.1, 0.9)$, $\epsilon_t \sim N(0,1)$.

$$X_t^2 = a_0 + a_1 X_{t-1}^2 + \eta_t$$

is stationary. But $E(X_t^4) = \infty$ as

$$a_1^2 \kappa_\epsilon + 2a_1 b_1 + b_1^2 = 0.9^2 \times 3 + 0 + 0 = 2.43 > 1$$

Conditional MLE for GARCH Models

Suppose that $X^T \equiv \{X_1, ..., X_T\}$ are observations from GARCH(p, q) model:

$$X_t = \sigma_t \epsilon_t$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i X_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2$$

- Assume $\epsilon_t \sim$ IID F_ϵ (e.g. $N(0,1)$, t_{ν_0}, generalized Gaussian).
- Assume $\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$.
- We want to estimate $\theta \equiv (a_0, a_1, ..., a_p, b_1, ..., b_q)$ for fixed (p, q).

The most often used estimators for θ are conditional maximum likelihood estimator (MLE).

Conditional MLE for GARCH Models

Let f_ϵ be the PDF of ϵ_t. The conditional density function for $X_t|X^{t-1}$ is

$$f(X_t|X^{t-1};\theta) = f_\epsilon(X_t/\sigma_t)/\sigma_t$$

Let $\nu = p \vee q + 1$.

$$\begin{aligned} f(X_v, ..., X_T|X^{\nu-1};\theta) &= f(X_T|X^{T-1};\theta)\cdots f(X_\nu|X^{\nu-1};\theta) \\ &= \prod_{t=\nu}^{T} f(X_t|X^{t-1};\theta) = \prod_{t=\nu}^{T}[f_\epsilon(X_t/\sigma_t)/\sigma_t] \end{aligned}$$

The conditional MLE is defined as

$$\begin{aligned} \widehat{\theta} &= \arg\max_\theta \log f(X_v, ..., X_T|X^{\nu-1};\theta) \\ &= \arg\max_\theta \sum_{t=\nu}^{T}[-\log\sigma_t + \log f_\epsilon(X_t/\sigma_t)] \end{aligned}$$

Conditional MLE for GARCH Models

Model Diagnostics

Let $\widehat{\sigma}_t = \sigma_t(\widehat{\theta})$ and $\widehat{\epsilon}_t = X_t/\widehat{\sigma}_t$.

1. Is $\{\widehat{\epsilon}_t\}$ an iid series? Draw ACF plot and perform Ljung-Box test.
2. Is $\{\widehat{\epsilon}_t^2\}$ an iid series? Draw ACF plot and perform Ljung-Box test.
3. Distribution of $\{\widehat{\epsilon}_t\}$: QQ plot and Jarque-Bera test (for normality).

Outline

Forecasting Volatility

Recall $E_T[\cdot] \equiv E[\cdot|X^T]$. Define $\sigma_T^2(k) = E_T(X_{T+k}^2)$ as the conditional volatility in k-period from time T.

Based on the ARMA representation for GARCH(p, q)

$$X_t^2 = a_0 + \sum_{i=1}^{p \vee q} (a_i + b_i) X_{t-i}^2 + \eta_t - \sum_{j=1}^{q} b_j \eta_{t-j}$$

We have

$$\sigma_T^2(k) = a_0 + \sum_{i=1}^{p \vee q} (a_i + b_i) \sigma_T^2(k-i) - \sum_{j=1}^{q} b_j \eta_T(k-j)$$

where $\sigma_T^2(m) = X_{T+m}^2$ and $\eta_T(m) = \eta_{T+m}$ if $m \leq 0$, $\eta_T(m) = 0$ if $m > 0$.

This formula can be used to forecast volatility σ_{T+k}^2 given observed X^T.

Forecasting Volatility

Example: GARCH(1,1)

For a GARCH(1,1) model,

$$\sigma_t^2 = a_0 + a_1 X_{t-1}^2 + b_1 \sigma_{t-1}^2$$

$$\sigma_T^2(k) = a_0 + (a_1 + b_1)\sigma_T^2(k-1)$$

By recursive substitution,

$$\sigma_T^2(k) = a_0 \frac{1-(a_1+b_1)^k}{1-(a_1+b_1)} + (a_1+b_1)^k \sigma_T^2$$

with

$$\sigma_T^2 = \frac{a_0}{1-b_1} + a_1(X_{T-1}^2 + b_1 X_{T-2}^2 + b_1^2 X_{T-3}^2 + \cdots)$$

Outline

ARMA-GARCH Models

If we assume the innovations in an ARMA(p, q) model follow a GARCH(p_1, q_1) structure, we have the ARMA(p, q)-GARCH(p_1, q_1) specification

$$r_t = \mu + \beta_1 r_{t-1} + \cdots + \beta_p r_{t-p} + X_t + \alpha_1 X_{t-1} + \cdots + \alpha_q X_{t-q}$$

where

$$\begin{aligned} X_t &= \sigma_t \epsilon_t \\ \sigma_t^2 &= a_0 + \sum_{i=1}^{p_1} a_i X_{t-1}^2 + \sum_{j=1}^{q_1} b_j \sigma_{t-j}^2 \end{aligned}$$

ARMA-GARCH models capture both time-varying conditional mean $E_{t-1}(r_t) \equiv \mu_t = \mu + \beta_1 r_{t-1} + \cdots + \beta_p r_{t-p}$ and time-varying conditional variance $V_{t-1}(r_t) = \sigma_t^2$.

ARMA-GARCH Models

To estimate $(\mu, \beta_1, ..., \beta_p, \alpha_1, ..., \alpha_q, a_0, a_1, ..., a_{p_1}, b_1, ..., b_{q_1})$, we can use the maximum likelihood procedure with the conditional likelihood function

$$\mathcal{L} = \prod_{t=v+1}^{T} f_\epsilon((r_t - \mu_t)/\sigma_t)/\sigma_t$$

where $v = p \vee p_1$. Note that the observations now are $r_1, ..., r_T$, and $\{X_t, \sigma_t\}_{t=1}^T$ can be calculated using recursive formulas.

Stochastic Volatility Models

The following *stochastic volatility model* offers an alternative to GARCH

$$X_t = g(h_t)\epsilon_t$$
$$h_t = c + \sum_{j=1}^{p} b_j h_{t-j} + e_t$$

where $\epsilon_t \sim \text{IID}(0, 1)$, $e_t \sim \text{IID}(0, \sigma_e^2)$, $\{\epsilon_t\}$ and $\{e_t\}$ are independent, and $g(\cdot) > 0$ is a known function.

- Like ARCH/GARCH, p is typically small, e.g., 1 or 2.
- A stochastic volatility model is driven by two indepent noise processes (i.e., $\{\epsilon_t\}$ and $\{e_t\}$), while ARCH/GARCH is driven by only one noise process.
- $\{h_t\}$ may represent information that is more complex than merely lagged X_t.

Stochastic Volatility Models

A popular specification of stochastic volatility model is

$$X_t = \exp(h_t/2)\epsilon_t$$
$$h_t = c + bh_{t-1} + e_t$$

- Heavy tails: $E(X_t^k) = E(\epsilon_t^k)E[\exp(kh_t/2)]$.
- Leverage effects can be captured by assuming $Cov(\epsilon_t, e_t) < 0$.
- "ARMA(1,1)" $\log(X_t^2)$:

$$\begin{aligned}\log(X_t^2) &= h_t + \log(\epsilon_t^2) = c + bh_{t-1} + \log(\epsilon_t^2) + e_t \\ &= c + b[\log(X_{t-1}^2) - \log(\epsilon_{t-1}^2)] + \log(\epsilon_t^2) + e_t\end{aligned}$$

Outline

Values-at-Risk

Value at Risk (VaR) is an important concept in financial risk management. Let X_t be the return of an asset with

$$X_t = \sigma_t \epsilon_t$$

VaR is defined as the critical value such that a loss exceeding this critical value is of the (conditional) probability not greater than self-specified level α (e.g., $\alpha = 0.01, 0.05$).

More precisely, the VaR at time $t+1$ at level α is

$$VaR_{\alpha,t+1} \equiv \max\{L : P(X_{t+1} \leq L | X^t) \leq \alpha\} = \sigma_{t+1} Q_{\alpha,\epsilon}$$

where $Q_{\alpha,\epsilon}$ denotes the α-th quantile of ϵ_{t+1}.

Values-at-Risk

In practice, we can estimate VaR by

$$\widehat{VaR}_{\alpha,t+1} = \widehat{\sigma}_{t+1}\widehat{Q}_{\alpha,\epsilon}$$

- $\widehat{\sigma}_{t+1}$ may be the volatility forecast obtained from GARCH.
- $\widehat{Q}_{\alpha,\epsilon}$ is the α-th quantile of $\widehat{\epsilon}_t = X_t/\widehat{\sigma}_t$.

Application: S&P 500 Daily Return Data

- See R markdown