Financial Econometrics

Lecture 11

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Introduction

We have introduced workhorse time series models of financial data in previous lectures. From now on, we turn to study the cross-sectional models for portfolio theory.

We start from briefly introducing:

1. Markowitz's (1952, 1959) work on portfolio optimization and asset pricing, and
2. Sharpe's (1964) Capital Asset Pricing Model (CAPM).

which are corner stones of the theory of portfolio choice and asset pricing.

Outline

Returns and Risks of Portfolios

Suppose that we have p risky assets at time t with returns $\{R_{i,t+1}\}_{i=1}^{p}$ in time $t+1$, and there is a riskless bond with interest $R_{0,t}$ known at time t.

A *portfolio* is characterized by an allocation vector $(a_0, a_1, ..., a_p)'$ with proportion a_i invested in security i. Denote $\alpha = (a_1, ..., a_p)'$ and assume $a_0 + a_1 + \cdots + a_p = 1$. Note that some a_i can be negative (short position).

Let $R_t = (R_{1,t}, ..., R_{p,t})'$. The return of the portfolio is

$$r_{t+1} = a_0 R_{0,t} + \alpha' R_{t+1}$$

The expected return and variance of the portfolio are

$$\mu_t(\alpha) = a_0 R_{0,t} + \alpha' E_t(R_{t+1})$$

$$\sigma_t^2(\alpha) = Var_t(r_{t+1}) = \alpha' Var_t(R_{t+1})\alpha$$

Returns and Risks of Portfolios

Let $Y_t \equiv R_{t+1} - R_{0,t}\mathbf{1}$ be the excess returns. Then,

$$r_{t+1} = R_{0,t} + \alpha' Y_{t+1}$$

$$\mu_t(\alpha) = R_{0,t} + \alpha' \xi_t$$
$$\sigma_t^2(\alpha) = \alpha' \Sigma_t \alpha$$

where $\xi_t \equiv E_t(Y_{t+1})$ and $\Sigma_t \equiv Var_t(Y_{t+1}) = Var_t(R_{t+1})$.

With the excess return formulation, it is clear that the expected return and variance depend only on the allocation vector α of the risky assets, which is (theoretically) unconstrained in $\mathbb{R}^p$.

We want to construct a portfolio which maximizes the trade-off of $\mu_t(\alpha)$ and $\sigma_t^2(\alpha)$. Obviously, this involves choosing a proper α.

Portfolio Optimization

Consider Markowitz's (1952, 1959) mean-variance optimization problem

$$\max_{\alpha \in \mathbb{R}^p} \left\{ \mu_t(\alpha) - \frac{A}{2}\sigma_t^2(\alpha) \right\} = \max_{\alpha \in \mathbb{R}^p} \left\{ R_{0,t} + \alpha' \xi_t - \frac{A}{2}\alpha' \Sigma_t \alpha \right\}$$

for some $A > 0$ representing the investor-specific risk aversion.

This problem is easy to solve as its first order condition (FOC) is just

$$\xi_t - A\Sigma_t \alpha = 0$$

Then we have

$$\alpha_t^* = \frac{1}{A}\Sigma_t^{-1}\xi_t$$

$$\alpha_{0,t}^* = 1 - \mathbf{1}'\alpha_t^* = 1 - \frac{1}{A}\mathbf{1}'\Sigma_t^{-1}\xi_t$$

Portfolio Optimization

It is easy to see that the larger the value of A, the smaller the proportion allocated on the risky assets in the optimal portfolio, i.e., A can be thought of as a measurement of risk aversion.

The portfolio with the allocation vector α^* is called the *efficient portfolio*.

Note that the optimal allocation above varies over time t, i.e., at the end of each period, the investor re-do the optimization above to find the optimal allocation for the next period.

From now on, we will drop the subscript t to simplify the notation as long as the dependence on t is clear from the context.

Efficient Portfolios and Sharpe Ratio

For a given portfolio with allocation vector α on risky assets. The *Sharpe ratio* is defined as

$$S(\alpha) = \frac{\mu(\alpha) - R_0}{\sigma(\alpha)} = \frac{\alpha'\xi}{\sqrt{\alpha'\Sigma\alpha}}$$

The Sharpe ratio gives excess return per unit risk, which measures the efficiency of a portfolio and can be thought of as a *risk-adjusted return*.

Given the optimal allocation α^*, let $P \equiv \xi'\Sigma^{-1}\xi$, we have

$$\mu^* = R_0 + \frac{1}{A}\xi'\Sigma^{-1}\xi = R_0 + P/A$$

$$\sigma^{*2} = \frac{1}{A^2}\xi'\Sigma^{-1}\xi = P/A^2$$

Efficient Portfolios and Sharpe Ratio

As $A = \sqrt{P}/\sigma^*$,

$$\mu^* = R_0 + \sigma^* \sqrt{P}$$

and so the Sharpe ratio of the efficient portfolio is

$$S(\alpha^*) = \frac{\mu^* - R_0}{\sigma^*} = \frac{\sigma^* \sqrt{P}}{\sigma^*} = \sqrt{P}$$

which does *not* depend on A.

Efficient Frontiers

Note that for all α such that $\alpha'\Sigma\alpha = \sigma^{*2}$,

$$\alpha'\xi = \alpha'\xi - \frac{A}{2}\alpha'\Sigma\alpha + \frac{A}{2}\sigma^{*2} \leq \alpha^{*T}\xi - \frac{A}{2}\alpha^{*T}\Sigma\alpha^* + \frac{A}{2}\sigma^{*2} = \alpha^{*T}\xi$$

which implies that the efficient portfolio has the highest expected return among all portfolios having risk σ^*.

For any given portfolio α, there exists an $A(\alpha)$ such that

$$\sigma(\alpha) = \sqrt{\alpha'\Sigma\alpha} = \sqrt{P}/A(\alpha) = \sigma^*$$

Then, the result above implies that

$$\frac{\alpha'\xi}{\sigma(\alpha)} = \frac{\alpha'\xi}{\sigma^*} \leq \frac{\alpha^{*T}\xi}{\sigma^*} = S(\alpha^*)$$

Efficient Frontiers

This gives an efficient (portfolio) frontier with intercept R_0 and slope $S(\alpha^*)$

$$\mu - R_0 = S(\alpha^*)\sigma$$

For all α, their corresponding $(\mu(\alpha), \sigma(\alpha))$ must locate below this line (have smaller Sharpe ratio).

Outline

Optimizing Expected Utility Functions

The main drawback of Markowitz problem is that it is utility-free. It is often more reasonable to assume that the investment objective is to maximize the investor's expected utility of wealth w.

An ideal candidate of utility function $U(w)$ is an increasing and concave function of w.

$$U'(w) > 0, U''(w) < 0$$

Commonly used utility functions are

1. exponential utility: $U(w) = 1 - \exp(-Aw)$ for some $A > 0$,
2. power utility: $U(w) = w^{\gamma}$ for some $\gamma \in (0, 1]$.

Optimizing Expected Utility Functions

With the utility $U(w)$, the investor's optimization problem becomes

$$\max_{\alpha \in \mathbb{R}^p} E[U(w_0(1 + R_0 + \alpha' Y)]$$

where w_0 is the investor's initial wealth.

To solve this problem, we should know/assume the distribution of Y.

Outline

Market Portfolio

Suppose that each investor in the market trades at he mean-variance optimal portfolio, but the risk preference A_i and amount of wealth w_i are investor specific.

A representative investor i's demand on the risky and riskless assets are $\alpha_i^* = w_i \Sigma^{-1} \xi / A_i$ and $w_i - \mathbf{1}' \alpha_i^*$. Then, the total (market) demands on the risky assets are

$$\alpha^D = \sum_i \alpha_i^* = \sum_i \frac{w_i}{A_i} \Sigma^{-1} \xi$$

Without loss of generality, assume that each stock is normalized to have \$1 per share, then the j-th component of α^D, a_j^D, is the total demand for the j-th asset in terms of number of shares.

Market Portfolio

At the market equilibrium,

$$\alpha^S = \alpha^D = \sum_i \frac{w_i}{A_i} \Sigma^{-1} \xi$$

The market portfolio α^S consists of all shares of all tradeable financial assets, and its allocation vector can be defined as

$$b = \alpha^S / \sum_i w_i = \left(\sum_i \frac{w_i}{A_i} / \sum_i w_i \right) \Sigma^{-1} \xi = \frac{1}{A} \Sigma^{-1} \xi$$

where $A^{-1} = (\sum_i w_i / A_i) / \sum_i w_i$ is a weighted harmonic average of A_i. The excess return of the market portfolio is $Y^m = b'Y$.

Treating A as a risk aversion measurement, we know that the market portfolio b is mean-variance efficient (optimal) .

Market Portfolio

Recall that for investor i, $\alpha_i^* = w_i \Sigma^{-1}\xi / A_i$ and $w_i - \mathbf{1}'\alpha_i^*$. The *two fund separation theorem* says that each investor should invest only on the market portfolio and riskless asset. To see this, note that

$$\alpha_i^*/w_i = \frac{1}{A_i}\Sigma^{-1}\xi = \frac{A}{A_i}\frac{1}{A}\Sigma^{-1}\xi = \frac{A}{A_i}b = s_i b$$

i.e., investor can hold proportion $s_i \mathbf{1}'b$ of w_i on the market portfolio with expected return r_m and the rest on riskless asset with interest rate r_f.

Then, the Sharpe ratio of this portfolio is

$$\frac{s_i \mathbf{1}'b r_m + (1 - s_i \mathbf{1}'b)\, r_f - r_f}{s_i \mathbf{1}'b \sigma_m} = \frac{r_m - r_f}{\sigma_m}$$

and so this portfolio is on the efficient frontier.

Capital Asset Pricing Model

Consider the following projection

$$\min_{a,b} E[(Y - a - bY^m)'(Y - a - bY^m)]$$

where Y is the excess return vector of all risky assets, Y^m is the excess return of the market portfolio.

Suppose that the solution is (α, β). Let $\epsilon \equiv Y - \alpha - \beta Y^m$. The FOC tells that Y can be decomposed as

$$Y = \alpha + \beta Y^m + \epsilon$$

with $E(\epsilon) = 0$ and $E(\epsilon Y^m) = Cov(\epsilon, Y^m) = 0$. The ϵ is an idiosyncratic noise not correlated with the market. The parameters α and β are called respectively the *market* α and *market* β.

Capital Asset Pricing Model

Theorem (Capital Asset Pricing Model)

In the decomposition, $Y = \alpha + \beta Y^m + \epsilon$

1. $\alpha = 0$, i.e., $Y = \beta Y^m + \epsilon$.
2. $\beta = Cov(Y, Y^m)/Var(Y^m)$.

The above theorem is the *capital asset pricing model* (CAPM) derived by Sharpe (1964) and Lintner (1965), which quantifies the relation between the expected return and risk.

Capital Asset Pricing Model

For a risk asset i, CAPM tells us that

$$E(Y_i) = \beta_i E(Y^m)$$

$$Var(Y_i) = \beta_i^2 Var(Y^m) + Var(\epsilon_i)$$

where $E(Y^m)$ is called the *market risk premium*. The β_i is the sensitivity of an asset to the market portfolio. Large β_i implies higher return $E(Y_i)$ and higher risk $Var(Y_i)$.

In practice, S&P 500 index or CRSP index is used as a proxy of the market portfolio, the US treasury bill is taken as the riskless bond. Each (α_i, β_i) associated with risky asset i can be estimated from

$$Y_{it} = \alpha_i + \beta_i Y_t^m + \epsilon_{it}, t = 1, ..., T$$

when T is sufficiently large.