MTH9855 Homework Two

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Problem 2.1. Show that if (2.2) holds for every p, p' and α, then U must have an expected utility form.

$$U(\alpha p + (1-\alpha)p') = \alpha U(p) + (1-\alpha)U(p') \ for \ \alpha \in [0,1].$$

Solution 2.1. Consider the finite set of lotteries which correspond to deterministic outcomes. Specifically, to each outcome x_i we associate a lottery L_i in which outcome x_i is attained with certainty, or in other words the probability vector is (0,0,...,0,1,0,...,0) where the 1 appears in the i-th position.

For some specific utility function $u : \mathcal{X} \to \mathbb{R}$, we know

$$U(L_i) = u(x_i) \ for \ i = 1 : n$$

At the same time, any arbitrary lottery p can be expressed as a linear combination of the extremal lotteries:

$$p = \sum_{i=1}^{n} p_i L_i \ for \ all \ p \in \mathcal{P}$$

According to (2.2) holds for every p,p' and α, we have

$$U(p) = \sum_{i=1}^{n} p_i U(L_i) = \sum_{i=1}^{n} p_i u(x_i) = E_p[u(x)] \ for \ all \ p \in \mathcal{P}$$

then U must have an expected utility form.

Problem 2.2. Show that if the independence axiom is violated by a given decision-maker′s preferences, then there is a dutch book the decision-maker would agree to (ie. an arbitrage they would be the source of), even if the other three axioms are satisfied by the decision-maker′s preferences. Hint: In practice the way you construct a dutch book is by finding a sequence of trades that the agent would agree to, and which in the end, gives you an arbitrage. If they strictly prefer lottery A to lottery B then they would pay something (however small, doesn′t matter) to trade B for A. If A $\sim$ B, then they would exchange them for free, and so forth.

Solution 2.2. Assume that there are two lotteries L_1 and L_2, and a given decision-maker have preferences for L_1 over L_2(ie. $L_1 \succ L_2$). If the independence axiom is violated, then there is one other lotteries denote as L such that $L_1 \succ L \succ L_2$, and we can denote two lotteries between L_1 and L_2 as $\alpha L_1 + (1-\alpha)L$ and $\alpha L_2 + (1-\alpha)L$, and the given decision-maker prefer $\alpha L_2 + (1-\alpha)L$ to $\alpha L_1 + (1-\alpha)L$, ie.

$$L_1 \succ \alpha L_2 + (1-\alpha)L \succ \alpha L_1 + (1-\alpha)L \succ L_2 \ for\ some\ \alpha \in (0,1)$$

First, assume the cash equivalent of $\alpha L_1 + (1-\alpha)$ is X, and the given decision-maker would pay X dollar to play lottery $\alpha L_1 + (1-\alpha)L$. Then he would agree to pay you a small amount, let's say one dollar, to exchange the lottery to $\alpha L_2 + (1-\alpha)L$ since he prefer $\alpha L_2 + (1-\alpha)L$ to $\alpha L_1 + (1-\alpha)L$.

Second, suppose the compound lottery have two stage, during the first stage, you have probability α to take lottery L_2 and (1-α) probability to take lottery L. If the outcome is the first one, then I can offer him to exchange the lottery L_2 to L_1 since he prefer L_1 to L_2. He would be willing to pay some small amount, let's say one dollar, to take the exchange. If the outcome is the second one, we don't change anything.

After that, let's sum up the result. The given decision-maker paid X+2 dollar to pay a game which is actually $\alpha L_1 + (1-\alpha)L$ because he have α probability to get lottery L_1 and (1-α) probability to get lottery L.

Finally, since the cash equivalent of $\alpha L_1 + (1-\alpha)L$ is X, we can sell this lottery to some other agent and get X dollar. So after all, we get two dollar for free and there is a dutch book.

Problem 2.3. Suppose given a complete and transitive preference relation $\succeq$ on $\mathcal{P}$. Show that, in the context of Theorem 2.1, if U is an expected utility representation of $\succeq$, then $\succeq$ must satisfy continuity and independence.

Solution 2.3. a) We first prove the propterty of continuous:

By the definition of expected utility representation (Definition 2.3),

$$U(p) = E_p[u(x)] = \sum_{i=1}^{n} p_i u(x_i) \ for\ all\ p \in \mathcal{P}$$

And the complete and transitive preference relation $\succeq$'s comparisons among lotteries are described by this preference function U. That is, for any lotteries p and q with relation $\succeq$,

$$p \succeq q \Leftrightarrow U(p) \geq U(q)$$

Suppose we have p, p' and p'' with relationship

$$p \succeq p' \succeq p''$$

so we have

$$U(p) \geq U(p') \geq U(p'')$$

Because $U(p)$, $U(p')$ and $U(p'')$ are numbers on the $\mathbb{R}$, and $\mathbb{R}$ is continuous, so we can find some α between [0,1] such that

$$U(p') = \alpha U(p) + (1-\alpha)U(p'')$$

According to (2.2),

$$\alpha U(p) + (1-\alpha)U(p'') = U(\alpha p + (1-\alpha)p'') = U(p')$$

By the definition of expected utility representation, we know

$$\alpha p + (1-\alpha)p'' \sim p' \ for \ some \ \alpha \in [0,1]$$

In consequence, the preference relation $\succeq$ is said to be continous.

b) now we prove independence property:

Suppose we have p, p' and p'' with relationship

$$p \succeq p', \ p, \ p' \ and \ p'' \in \mathcal{P}$$

so we have

$$U(p) \geq U(p')$$

Because $\alpha \geq 0$ then we must have for any $U(p'') \in \mathbb{R}$,

$$\alpha U(p) + (1-\alpha)U(p'') \geq \alpha U(p') + (1-\alpha)U(p'') \ for \ all \ \alpha \in [0,1]$$

According to (2.2), we have

$$U(\alpha p + (1-\alpha)p'') = U(\alpha p' + (1-\alpha)p'')$$

By the definition of expected utility representation, we know

$$p \succeq p' \ \Leftrightarrow \ \alpha p + (1-\alpha)p'' \sim \alpha p' + (1-\alpha)p'' \ for \ all \ \alpha \in [0,1]$$

In conclusion, if U is an expected utility representation of $\succeq$, then $\succeq$ must satisfy continuity and independence.