MTH9855 Homework Three

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Problem 3.1. Find the explicit solution for x^* by applying the block matrix inversion formula to (3.10). Which matrices must you assume are invertible for this formula to apply? If P is positive semi-definite but has a non-trivial null space, what can you say about existence/uniqueness of solutions to the original equality-constrained optimization problem that lead to (3.10)?

Solution 3.1. a)

The KKT conditions for this equality constrained convex quadratic minimization problem are

$$\begin{pmatrix} P & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x^* \\ v^* \end{pmatrix} = \begin{pmatrix} -q \\ b \end{pmatrix}$$

Applying the block matrix inversion formula to above, we get

$$\begin{pmatrix} x^* \\ v^* \end{pmatrix} = \begin{pmatrix} -P^{-1}q + P^{-1}A^T(AP^{-1}A^T)^{-1}AP^{-1}q + P^{-1}A^T(AP^{-1}A^T)^{-1}b \\ -(AP^{-1}A^T)^{-1}AP^{-1}q - (AP^{-1}A^T)^{-1}b \end{pmatrix}$$

In consequence, the explicit solution for x^* is

$$x^* = -P^{-1}q + P^{-1}A^T(AP^{-1}A^T)^{-1}AP^{-1}q + P^{-1}A^T(AP^{-1}A^T)^{-1}b$$

We assume matrices P and $AP^{-1}A^T$ to be invertible for this formula to apply.

b)

There are two equations to satisfy if this equality constrained convex quadratic minimization problem solution exists.

$$Ax^* = b\ ,\ \ Px^* + q + A^T\nu^* = 0$$

A is a $m \times n$ matrix. The existence of solution to the first equation depends on if $rank(A \mid b) = rank(A)$. If $rank(A \mid b) = rank(A)$, there is a solution.

To make the second equations hold, we must have

$$A^T\nu^* = -Px' - q$$

The existence of solution to the first equation depends on if $rank(A^T \mid -Px' - q) = rank(A^T)$. If $rank(A^T \mid -Px' - q) = rank(A^T)$, there is a solution.

If P is positive semi-definite but has a non-trivial null space, which means P is nonsingular and not invertible, then

$P\tilde{x} = 0$ has infinitely many non-trivial solution $\tilde{x}$

If we assume there is a solution x^* originally for the below equation

$$\begin{pmatrix} P & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x^* \\ v^* \end{pmatrix} = \begin{pmatrix} -q \\ b \end{pmatrix}$$

Then, If we have $A\tilde{x} = 0$, then we have

$$\begin{pmatrix} P & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x^* + \tilde{x} \\ v^* \end{pmatrix} = \begin{pmatrix} -q \\ b \end{pmatrix}$$

Which means the problem solution doesn't have uniqueness if A and P have the same non-trivial null space.

Problem 3.2. Suppose now that $p^*(u, v)$ is differentiable at $u = 0$, $v = 0$. Then, provided strong duality holds, show that

$$\lambda_i^* = -\frac{\partial p^*}{\partial u_i}(0, 0),\ \nu_j^* = -\frac{\partial p^*}{\partial v_j}(0, 0).$$

Solution 3.2. Provided strong duality, we know that for unpertutbed problem, following inequality holds

$$p^*(0, 0) = g(\lambda^*, \nu^*) = \inf_x L(x, \lambda^*, \nu^*) \leq f_0(x) + \sum_i \lambda_i^* f_i(x) + \sum_j \nu_j^* h_j(x)$$

For perturbed problem, suppose that x is any feasible point for the perturbed problem, then we have

$$f_i(x) \leq u_i,\ h_j(x) = v_j$$

Plugging into the above inequality with $\lambda \succeq 0$, we have

$$\begin{aligned} p^*(0, 0) = g(\lambda^*, \nu^*) = \inf_x L(x, \lambda^*, \nu^*) &\leq f_0(x) + \sum_i \lambda_i^* f_i(x) + \sum_j \nu_j^* h_j(x) \\ &\leq f_0(x) + \sum_i \lambda_i^* u_i + \sum_j \nu_j^* v_j \end{aligned}$$

In consequence, we conclude for any feasible point in the perturbed problem, we have

$$f_0(x) \geq p^*(0,0) - \sum_i \lambda_i^* u_i - \sum_j \nu_j^* v_j$$

which means

$$p^*(u,v) \geq p^*(0,0) - \sum_i \lambda_i^* u_i - \sum_j \nu_j^* v_j$$

Let $u_i = t$ and other $u_k = 0$ for all $k \neq i$ and $v_j = 0$ for all j, then we get

$$p^*(te_i, 0) \geq p^*(0,0) - \lambda_i^* t$$

where e_i is the i^{th} unit vector.

In other words, for $t > 0$

$$-\frac{p^*(te_i, 0) - p^*(0,0)}{t} \leq \lambda_i^*$$

taking the limit $t \to 0$, we get

$$-\frac{\partial p^*(0,0)}{\partial u_i} \leq \lambda_i^*$$

For $t < 0$, we have

$$-\frac{p^*(te_i, 0) - p^*(0,0)}{t} \geq \lambda_i^*$$

taking the limit $t \to 0$, we get

$$-\frac{\partial p^*(0,0)}{\partial u_i} \geq \lambda_i^*$$

Combining the situation of $t > 0$ and $t < 0$, we conclude

$$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial u_i}$$

In the same way, we get

$$\nu_j^* = -\frac{\partial p^*(0,0)}{\partial v_j}$$