MTH9855 Homework One

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2018 Spring

February 9, 2018

Problem 1.1. Prove that an agent is risk-averse, ie. inequality

$$E[u(w+\tilde{z})] \leq \; u(w+E[\tilde{z}])$$

holds for all w and $\tilde{z}$, if and only if u() is concave.

Solution 1.1. According to Jensen Inequality, i.e.

$$E[\phi(\tilde{y})] \leq \; \phi(E[\tilde{y}])$$

for any random variable $\tilde{y}$ if and only if $\phi(\cdot)$ is concave.

let $\tilde{y}$ = w+$\tilde{z}$ and $\phi(\cdot)$ = u$(\cdot)$, then we have the rewriting form of Jensen Inequality

$$E[u(w+\tilde{z})] \leq \; u(w+E[\tilde{z}])$$

holds for all w and $\tilde{z}$, if and only if u$(\cdot)$ is concave.

Problem 1.2. Show that when u(w)=-exp(-kw)/k and $\tilde{w}$ is normally distributed with mean μ and variance σ^2, then the Arrow-Pratt approximation is exact.

Solution 1.2. According to the definition of risk premium Π, i.e.

$$E[u(\tilde{w})] = u(\mu - \Pi)$$

The Arrow-Pratt approximation is

$$\Pi \approx \frac{1}{2}\sigma^2 A(w) \; where \; A(w) := -\frac{u''(w)}{u'(w)}$$

In this exercise, u(w) = -exp(-kw)/k, so we have

$$A(w) = k \; for \; all \; w$$

It is required to show that $\Pi = \frac{1}{2}\sigma^2 k$, plugging into the risk premium definition, it is equivalent to show that

$$E[u(\tilde{w})] = u(\mu - \frac{1}{2}\sigma^2 k)$$

where $\tilde{w}$ is normally distributed with mean μ and variance σ^2.

$$
\begin{aligned}
E[u(\tilde{w})] &= \int_{-\infty}^{\infty} -\frac{e^{-k\tilde{w}}}{k} \frac{e^{-\frac{(\tilde{w}-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} d\tilde{w} \\
&= -\frac{1}{k} \int_{-\infty}^{\infty} \frac{e^{-\frac{(\tilde{w}-\mu)^2+2k\sigma^2\tilde{w}}{2\sigma^2}}}{\sqrt{2\pi}\sigma} d\tilde{w} \\
&= -\frac{1}{k} \int_{-\infty}^{\infty} \frac{e^{-\frac{(\tilde{w}-\mu+k\sigma^2)^2+2k\sigma^2\mu-k^2\sigma^4}{2\sigma^2}}}{\sqrt{2\pi}\sigma} d\tilde{w} \\
&= -\frac{e^{-k\mu+\frac{1}{2}\sigma^2k^2}}{k} \int_{-\infty}^{\infty} \frac{e^{-\frac{(\tilde{w}-\mu+k\sigma^2)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} d\tilde{w} \\
&= -\frac{e^{-k\mu+\frac{1}{2}\sigma^2k^2}}{k} \\
&= u(\mu - \frac{1}{2}\sigma^2 k)
\end{aligned}
$$

So we have $\Pi = \frac{1}{2}\sigma^2 k$, which means the Arrow-Pratt approximation is exact.

Problem 1.3. Prove the following three conditions are equivalent:

(a) Agent v is more risk-averse than agent u,

(b) For all w, $A_v(w) \geq A_u(w)$.

(c) Function v is a concave transformation of function u, meaning:

$\exists\ \phi$ with $\phi' < 0$ and $\phi'' < 0$ such that $v(w) = \phi(u(w))$

Solution 1.3. (a)$\Leftrightarrow$(c):

By the definition of risk-averse, Agent v is more risk-averse than agent u means,

$$(a) \Leftrightarrow \Pi_v \geq \Pi_u$$

Since utility function is strictly increasing function, we have

$$\Pi_v \geq \Pi_u \Leftrightarrow u(w_0 - \Pi_u) \geq u(w_0 - \Pi_v)$$

According to the definition of risk premium, for all $\phi(\cdot)$ such that $\phi' > 0$, we have

$$u(w_0 - \Pi_u) \geq u(w_0 - \Pi_v) \Leftrightarrow E[u(w)] \geq u(w_0 - \Pi_v) \Leftrightarrow \phi(E[u(w)]) \geq \phi(u(w_0 - \Pi_v))$$

Using Jensen's Inequality theorem and the condition of (c), we know that

$$\phi(E[u(w)]) \geq E[\phi(u(w))]) = E[v(w)] = \phi(u(w_0 - \Pi_v)) \Leftrightarrow (c)\ \phi(\cdot) \text{ is concave}$$

So we have $(a) \Leftrightarrow (c)$.

(b)⇔(c):

By the definition of $A_v(w) = \frac{-v''(w)}{v'(w)}$,

$$(b) \Leftrightarrow A_v(w) \geq A_u(w) \Leftrightarrow \frac{-v''(w)}{v'(w)} \geq \frac{-u''(w)}{u'(w)} \Leftrightarrow \frac{v''(w)}{v'(w)} \leq \frac{u''(w)}{u'(w)}$$

Given that $v(w)$ and $u(w)$ are strictly increasing functions, which means $u'(\cdot) > 0$

If there exists a $\phi(\cdot)$ such that $v(w) = \phi(u(w))$, then $\phi'(\cdot) > 0$. Then we have,

$$\frac{v''(w)}{v'(w)} \leq \frac{u''(w)}{u'(w)} \Leftrightarrow \frac{v''(w)}{v'(w)} \leq \frac{\phi'(u(w))u''(w)}{\phi'(u(w))u'(w)}$$

Since $v'(w) = \phi'(u(w))u'(w), v''(w) = \phi''(u(w))(u'(w))^2 + \phi'(u(w))u''(w)$ and $(u'(w))^2$ >0, we have

$$\frac{v''(w)}{v'(w)} \leq \frac{\phi'(u(w))u''(w)}{\phi'(u(w))u'(w)} \Leftrightarrow \phi''(u(w)) < 0 \Leftrightarrow (c)$$

In conclusion, $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.

Problem 1.4. Consider a function $v(\cdot)$ such that $v(x) = a + bu(x)$ for all x, for some pair of scalars a and b, where $b > 0$. Show that a decision-maker with utility function $v(\cdot)$ makes the same decisions and has the same certainty-equivalents as a decision maker with utility function $u(\cdot)$.

Solution 1.4. According to the definition of certainty-equivalents,

$$E[u(\tilde{w})] = u(\mu + e)$$

where $\tilde{w}$ is a random variable with mean μ.

Since $v(x) = a + bu(x)$ for all x, then we have

$$E[v(\tilde{w})] = E[a + bu(\tilde{w})] = a + bu(\mu + e) = v(\mu + e)$$

So a decision-maker with utility function $v(\cdot)$ has the same certainty-equivalents as a decision-maker with utility function $u(\cdot)$.

Since $v'(w) = bu'(w), v''(w) = bu''(w)$ and $b > 0$, then $v''(\cdot)$ has the same sign with $u''(\cdot)$.

If $u(\cdot)$ is concave, then a decision-maker with utility function $u(\cdot)$ is risk-averse, and so is a decision-maker with utility function $v(\cdot)$.(because if $u''(\cdot) < 0$ then $v''(\cdot) < 0$)

If $u(\cdot)$ is concex, then a decision-maker with utility function $u(\cdot)$ is risk-tolerant, and so is a decision-maker with utility function $v(\cdot)$.(because if $u''(\cdot) > 0$ then $v''(\cdot) > 0$)

All in all, a decision-maker with utility function $v(\cdot)$ makes the same decisions and has the same certainty-equivalents as a decision maker with utility function $u(\cdot)$ if $v(x) = a + bu(x)$ for $b > 0$.