MTH9855 Homework Four

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Problem 4.1. Under the conditions of Theorem 4.1, show that $d^2\mu/d\sigma^2 > 0$, or the rate at which an individual must be compensated for accepting greater σ (this rate is $d\mu/d\sigma$) increases as σ increases.

Solution 4.1. Under the conditions of Theorem 4.1, we can differentiate the equation below w.r.t σ

$$\mathbb{E}[u(x)] = \int u(\mu + \sigma z) g_1(z^2) dz$$

Then we get

$$0 = \int u'(\mu + \sigma z)(\frac{d\mu}{d\sigma} + z) g_1(z^2) dz$$

From this we get

$$\frac{d\mu}{d\sigma} = -\frac{\int_{\mathbb{R}} z\mu'(\mu + \sigma z) g_1(z^2) dz}{\int_{\mathbb{R}} \mu'(\mu + \sigma z) g_1(z^2) dz} > 0$$

Further differentiating the above equation w.r.t σ to get the $d^2\mu/d\sigma^2$

$$0 = \int_{\mathbb{R}} u'(\mu + \sigma z) \frac{d^2\mu}{d\sigma^2} g_1(z^2) dz + \int_{\mathbb{R}} u''(\mu + \sigma z)[\frac{d\mu}{d\sigma} + z]^2 g_1(z^2) dz$$

Because $[\frac{d\mu}{d\sigma} + z]^2 > 0$ and $u''(\mu + \sigma z) < 0$, we get

$$\int_{\mathbb{R}} u''(\mu + \sigma z)[\frac{d\mu}{d\sigma} + z]^2 g_1(z^2) dz < 0$$

and so we have

$$\frac{d^2\mu}{d\sigma^2} \int_{\mathbb{R}} u'(\mu + \sigma z) g_1(z^2) dz > 0$$

Because $\int_{\mathbb{R}} \mu'(\mu + \sigma z) g_1(z^2) dz > 0$, we get

$$\frac{d^2\mu}{d\sigma^2} > 0$$

Proof completed.

Problem 4.2. Prove (4.15), and then use (4.15) to show that for a parametric curve of the form (4.16), $d\mu/d\sigma > 0$ everywhere along the curve, but $d^2\mu/d\sigma^2$ changes sign. Verify this on a computer by plotting the relevant curves for various values of the parameters m, s.

Solution 4.2. With a log-normal distribution of wealth, we have

X=log(Y) where Y follows normal distribution N(m,s^2)

So the moments of X are

$$\mu = \mathbb{E}[X] = \mathbb{E}[\exp(Y)] = \exp(m + s^2/2)$$

and

$$\begin{aligned}\sigma^2 = Var[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \exp(2m + 2s^2) - \exp(2m + s^2) \\ &= \exp(2m + s^2)(\exp(s^2) - 1) \\ &= (\exp(m + s^2/2))^2(\exp(s^2) - 1)\end{aligned}$$

With logarithmic utility u(x) = log x, we get

$$\mathbb{E}[u(x)] = \mathbb{E}[\log(x)] = \mathbb{E}[Y] = m$$

Use μ and σ^2 to represent m and s^2, we have

$$\log(\mu) = m + s^2/2,\ \log(\sigma^2/\mu^2 + 1) = s^2$$

So

$$\mathbb{E}[u(x)] = \mathbb{E}[\log(x)] = \mathbb{E}[Y] = m = \log(\mu) - \frac{1}{2}\log(\sigma^2/\mu^2 + 1)$$

(4.15) proved.

Now, we show that $d\mu/d\sigma > 0$ everywhere along the curve. First, differentiate (4.15) above w.r.t. σ.

$$\begin{aligned}0 &= \frac{1}{\mu}\frac{d\mu}{d\sigma} - \frac{1}{2}\frac{1}{\frac{\sigma^2}{\mu^2}+1}\frac{d(\frac{\sigma^2}{\mu^2}+1)}{d\sigma} \\ &= \frac{1}{\mu}\frac{d\mu}{d\sigma} - \frac{1}{2}\frac{1}{\frac{\sigma^2}{\mu^2}+1}(\frac{2\sigma}{\mu^2} - 2\frac{\sigma^2}{\mu^3}\frac{d\mu}{d\sigma}) \\ &= \frac{1}{\mu}(\frac{d\mu}{d\sigma} - \frac{1}{\frac{\sigma^2}{\mu^2}+1}(\mu\sigma - \sigma^2\frac{d\mu}{d\sigma}))\end{aligned}$$

So we get

$$\frac{d\mu}{d\sigma} = \frac{\mu\sigma}{2\sigma^2 + \mu^2}$$

Figure 1: Indifference curve with Eu = 1.

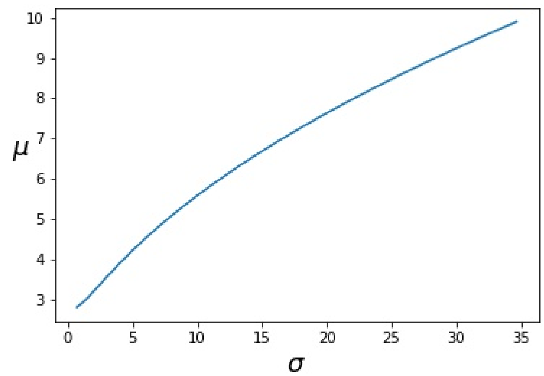
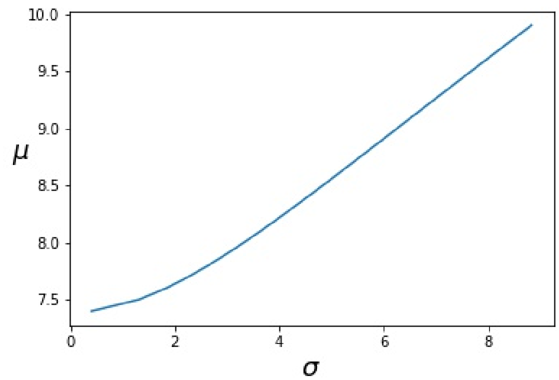


Figure 2: Indifference curve with Eu = 2.



Because $\mu > 0$ and $\sigma > 0$ as well as $2\sigma^2 + \mu^2 > 0$, we know

$$\frac{d\mu}{d\sigma} > 0$$

However, the second derivative's expression is as follow

$$\begin{aligned}\frac{d^2\mu}{d\sigma^2} &= \frac{d(\frac{d\mu}{d\sigma})}{d\sigma} \\ &= \frac{(2\sigma^2 + \mu^2)(\mu + \sigma\frac{d\mu}{d\sigma}) - \mu\sigma(4\sigma + 2\mu\frac{d\mu}{d\sigma})}{(2\sigma^2 + \mu^2)^2} \\ &= \frac{(2\sigma^2 - \mu^2)(\sigma\frac{d\mu}{d\sigma} - \mu)}{(2\sigma^2 + \mu^2)^2}\end{aligned}$$

which means $d^2\mu/d\sigma^2$ changes sign with different values of μ and σ.

Verify this by programming in python and the results are Figure 1 and Figure 2. Figure 1 is concave and Figure 2 is convex, so the sign of $d^2\mu/d\sigma^2$ is changing with different m and s.