MTH9855 Homework Six

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2018 Spring

March 24, 2018

Problem 6.1. Use the Sherman-Morrison-Woodbury matrix inversion lemma to derive a simple expression for the inverse of the covariance matrix in an APT model. In other words, derive an expression for Σ^{-1} where

$$\Sigma = \mathbb{V}[R] = XFX' + D$$

where D is diagonal and X is $n \times p$ and as usual we assume $p \ll n$. In your answer, any matrices being inverted should be either diagonal or $p \times p$.

Solution 6.1. Applying the Sherman-Morrison-Woodbury matrix inversion lemma, we have

$$\Sigma^{-1} = D^{-1} - D^{-1}X(F^{-1} + X'D^{-1}X)^{-1}X'D^{-1}$$

Problem 6.2. Show that, for any $n \times p$ real matrix X (not necessarily full rank) and n-vector Y, the following are equal:

(1) $\lim_{\delta \to 0+}(X'X + \delta)^{-1}X'Y$

(2) The smallest-norm element of $\text{argmin}_b \|Y - Xb\|$.

(3) $VS^+U'Y$ where $X = USV'$ is the SVD of X.

Solution 6.2. "(a)$\Leftrightarrow$(b)":

The expression in (1) is the solution to minimizing the following optimization problem,

$$\min_b \left(\|Y - Xb\|^2 + \delta \|b\|^2 \right)$$

That is,

$$(X'X + \delta I)^{-1}X'Y = \arg\min_b \left(\|Y - Xb\|^2 + \delta \|b\|^2 \right)$$

As $\delta \to 0^+$,

$$\lim_{\delta \to 0^+} (X'X + \delta I)^{-1}X'Y = \arg\min_b \|Y - Xb\|$$

"(b)$\Leftrightarrow$(c)":

By taking the first order condition of $\min_b \|Y - Xb\|^2$, we have

$$X'Y = X'Xb$$

where b is the smallest-norm element of $\operatorname{argmin}_b \|Y - Xb\|$.

Since the SVD of X is $X = USV'$ (U and V are orthogonal matrix), we have

$$VS'U'Y = VS'SV'b$$

Left multiply the above equation by V', then $(S'S)^{-1}$, and finally V, we get

$$V(S'S)^{-1}S'U'Y = b$$

Since we define S^+ by replacing every non-zero diagonal entry of S by its reciprocal and transposing the resulting matrix, i.e. $S^+ = (S'S)^{-1}S'$, we have

$$VS^+U'Y = V(S'S)^{-1}S'U'Y = b$$

Proof completed.