MTH9855 Homework Nine

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Problem 12.1. Complete the proof of Theorem 12.2, in other words show that, under the hypotheses of the theorem, the ERC portfolio is proportional to inverse-volatility weighting.

Proof. (a) First, we show that a portfolio proportional to inverse-volatility weighting is the ERC portfolio.

we fix $x_i = \sigma 1_i$, for this choice of x, we have $Sx = \mathbf{1}$ hence $RSx = m \cdot \mathbf{1}$, hence

$$SRSx = mS\mathbf{1} = m(\sigma_1, ..., \sigma_n)^T$$

Finally, $x'SRSx = nm = \sigma^2(x)$. It follows that $x_i(\Sigma x)_i = m$ hence

$$\frac{x_i(\Sigma x)_i}{\sigma^2(x)} = \frac{m}{nm} = \frac{1}{n}$$

(b) Then, we show that the ERC portfolio is proportional to inverse-volatility weighting. The ERC condition is

$$\frac{x_i(\Sigma x)_i}{\sigma^2(x)} = \frac{1}{n} \text{ for } i = 1, 2, .., n$$

So we have

$$x_i(\Sigma x)_i = x_j(\Sigma x)_j \text{ for } i \neq j$$

Denote $R = (R_{ij})_{n\times n}$, we have

$$\sigma_i x_i(\sum_{k=1}^{n} \sigma_k x_k R_{ik}) = \sigma_j x_j(\sum_{k=1}^{n} \sigma_k x_k R_{jk})$$

So we get,

$$\frac{\sigma_i x_i}{\sigma_j x_j}(\sum_{k=1}^{n} \sigma_k x_k R_{ik}) = \sum_{k=1}^{n} \sigma_k x_k R_{jk}$$

Since $R\mathbf{1} = m \cdot \mathbf{1}$, which means

$$\sum_{k=1}^{n} R_{ik} = \sum_{k=1}^{n} R_{jk}$$

Since R_{ik} is arbitrary, the above two equations hold true if and only if

$$\sigma_i x_i = \sigma_j x_j = const\ for\ i \neq j$$

In consequence, the ERC portfolio is proportional to inverse-volatility weighting.

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