1. Discount Factors and Interest Rates

Notes

Interest rates are expressed with continuous compounding unless otherwise specified.

We will not concern ourselves with industry daycount or business-day conventions. To translate maturities into times in years…

For maturities expressed in days, assume a 365-day year (i.e., a 1D maturity corresponds to $T = 1 / 365$).

For maturities expressed in weeks, assume 52 weeks per year.

For maturities expressed in months, assume 12 months per year.

Exercises

1. At a given moment, you observe the following prices for various maturities of risk-free zero-coupon bonds:

maturity	bond price (per 100 face)
1D	99.9995
1W	99.994
1M	99.971
3M	99.9
6M	99.65
1Y	99.05

(a) What is the term structure of risk-free zero rates implied by these bond prices, expressed with continuous compounding?

(b) What is the term structure of single-period forward rates (i.e., the forwards between adjacent maturities), also expressed with continuous compounding?

(c) Verify that the 1Y zero rate is the time-weighted average of the single-period forward rates you calculated.

2. The USD LIBOR fixings for a particular day are as follows:

USD	**Rate (percent)**
o/n	0.12800
1w	0.16270
2w	0.17170
1m	0.19043
2m	0.22413
3m	0.25288
4m	0.29463
5m	0.35013
6m	0.40313
7m	0.45913
8m	0.51288

9m	0.56463
10m	0.61913
11m	0.67125
12m	0.72950

For all parts of this problem, recall that LIBOR is quoted as a rate of simple interest, and make the simplifying assumptions detailed above concerning daycount and business-day conventions. Treat the o/n (overnight) rate as a one-day rate.

(a) What is the term structure of discount factors implied by these fixings?
(b) What are the one-month forward rates, expressed as rates of simple interest, for months 1-11 implied by these fixings?
(c) What is the term structure of interest rates implied by these fixings, with all rates expressed with monthly compounding?

3. Using market data, you extract the following term structure of discount factors:

maturity	discount factor
0.00274	0.99999
0.01923	0.99989
0.08333	0.9995
0.25	0.99875
0.5	0.99576
1	0.98807

(a) Using linear interpolation in the continuously compounded spot rates, determine the discount factor at 4 months.
(b) Using the assumption of piecewise constant forward rates, determine the discount factor at 10 months.
(c) Under each of the interpolation methods used above, what is the six-month forward rate in four months, expressed with continuous compounding?

4. The risk-free zero rate to the maturity 1 year is 1.25%. The risk-free zero rate to the maturity 3 years is 1.6%. Under each of the following interpolation methods, determine the instantaneous forward rate at 1 year (with the limit taken from the right) and 3 years (with the limit taken from the left).

(a) Constant forward rate
(b) Linear zero rates
(c) Cubic spline in the zero rates, where the maturities at 1 and 3 years are connected by a segment with the equation:

$$r(t) = -0.001138889t^3 + 0.001638889t^2 + 0.01t + 0.002$$

Here r is the rate expressed simply as a number—i.e., 1% is 0.01.

Applications

Quartic forward spline

It has been shown[1] that the method giving the smoothest forward rates matching a set of zero-rate points is produced by a spline in the instantaneous forward rates in which the segments are quartic polynomials. The purpose of this problem is to derive equations pertinent to this method and employ them in a simple example.

We suppose we are given the set of points in the continuously compounded zero rate (t_i, r_i), $i = 0, 1,..., N$, with $t_0 = 0$ and r_0 thus the initial instantaneous rate. This set of $N + 1$ points defines N segments, each of which is represented by a quartic polynomial. For t in the interval $[t_{i-1}, t_i)$, the instantaneous forward is thus:

$$r^*(t) = Q_i(t) = c_{1,i}t^4 + c_{2,i}t^3 + c_{3,i}t^2 + c_{4,i}t + c_{5,i}$$

To fit a spline we then must determine $5N$ unknown coefficients. As in the simpler case of a cubic spline, we define a set of constraints on the polynomials to produce a system of linear equations that can be solved for these coefficients.

(a) For $i = 1, 2,..., N - 1$, t_i is an interior node of the curve—i.e., one which is a boundary point of both segments Q_i and Q_{i+1}. We enforce a constraint that our instantaneous forward rates be continuous at these points. Thus, expressed in terms that are amenable to representation in matrix form:

$$Q_i(t_i) - Q_{i+1}(t_i) = 0$$

$$c_{1,i}t_i^4 + c_{2,i}t_i^3 + c_{3,i}t_i^2 + c_{4,i}t_i + c_{5,i} - c_{1,i+1}t_i^4 - c_{2,i+1}t_i^3 - c_{3,i+1}t_i^2 - c_{4,i+1}t_i - c_{5,i+1} = 0$$

We also require that the instantaneous forward rates have 3 continuous derivatives. Write the equations for these additional constraints.

The continuity constraints provide $4(N - 1)$ of the $5N$ required equations.

(b) At each interior node, we additionally require that the *spot curve* arising from the instantaneous forward rates pass through the known points. Recall that, for some $t < T$:

$$P_T = P_t e^{-\int_t^T r^*(s)ds}$$

Show that the spot-curve constraint at the interior node t_i can be expressed as:

$$\frac{1}{5}c_{1,i}\left(t_i^5 - t_{i-1}^5\right) + \frac{1}{4}c_{2,i}\left(t_i^4 - t_{i-1}^4\right) + \frac{1}{3}c_{3,i}\left(t_i^3 - t_{i-1}^3\right) + \frac{1}{2}c_{4,i}\left(t_i^2 - t_{i-1}^2\right) + c_{5,i}\left(t_i - t_{i-1}\right) = r_i t_i - r_{i-1}t_{i-1}$$

This constraint provides $N - 1$ of the $5N$ required equations. Altogether, interior nodes account for $5(N - 1)$ constraints, leaving 5 remaining.

[1] For more detail on this method, see the work of Donald van Deventer, a principal exponent of it.

(c) We further require that the curve reproduce the known information at the left and right edges. Because the instantaneous rate at time zero is both the spot rate and the "forward" rate, trivially at the left edge we have:

$$Q_1(t_0) = r_0$$

$$Q_1(0) = r_0$$

$$c_{5,1} = r_0$$

Formulate the constraint for the right edge that ensures the *spot curve* passes through the point (t_N, r_N).

With these two constraints added, 3 remain.

(d) There are several possible choices for the 3 remaining constraints. For the sake of this problem, we choose:

$$Q_1''(t_0) = 0$$

$$Q_N''(t_N) = 0$$

$$Q_N'''(t_N) = 0$$

Write these constraints in a form consistent with the previous constraints.

(e) Suppose the term structure of known zero rates is as follows:

t	r
0	0.025
1	0.0265
5	0.0325
10	0.0315

Determine the fifteen unknown coefficients of the quartic forward spline entailed by this data.

(f) Show that for any positive t in the interval $[t_{i-1}, t_i)$, the zero rate at this time can be calculated as:

$$r_t = \frac{1}{t}\left[r_{i-1}t_{i-1} + \frac{1}{5}c_{1,i}\left(t^5 - t_{i-1}^5\right) + \frac{1}{4}c_{2,i}\left(t^4 - t_{i-1}^4\right) + \frac{1}{3}c_{3,i}\left(t^3 - t_{i-1}^3\right) + \frac{1}{2}c_{4,i}\left(t^2 - t_{i-1}^2\right) + c_{5,i}\left(t - t_{i-1}\right)\right]$$

(g) Taking times t from 0 to 10 years spaced 0.1 years apart, graph the zero rate to t and the instantaneous forward rate at t.