2. Yield and Duration

Notes
Interest rates are expressed with continuous compounding unless otherwise specified.

We will not concern ourselves with industry daycount or business-day conventions. To translate maturities into times in years…

For maturities expressed in days, assume a 365-day year (i.e., a 1D maturity corresponds to $T = 1 / 365$).

For maturities expressed in weeks, assume 52 weeks per year.

For maturities expressed in months, assume 12 months per year.

Exercises
1. The term structure of risk-free zero rates is as follows:

t	r (continuous)
0.5	0.011
1	0.015
2	0.022
3	0.024
5	0.029

(a) What is the present value of a risk-free bond with maturity 3 years paying annual coupons of 3.5%? Express your answer per 100 face amount.
(b) What is the 2-year semiannual par yield? Use linear interpolation in the continuously compounded risk-free zero rate.
(c) What is the yield (expressed as a continuously compounded rate) of a 1-year risk-free bond with a 5% coupon rate per year that pays semiannually?

2. The following problems describe risk-free bond prices or yields observed at the same time. All bond prices are expressed per 100 face amount.

(a) A zero-coupon bond with a 6-month maturity trades at 99.875. What is the risk-free zero rate, expressed with continuous compounding, to this bond's maturity?
(b) A bond paying a coupon of 4% per year semiannually with a maturity of 1 year has a yield of 50 basis points, expressed in the bond's natural compounding frequency. What is the risk-free zero rate, expressed with continuous compounding, to this bond's maturity? (A basis point is 1% of 1%, or 0.0001.)
(c) A bond paying a 2% coupon per year semiannually with a maturity of 2 years has a yield of 1.05%, expressed in the bond's natural compounding frequency. Using linear interpolation in the continuously compounded risk-free zero rates, find the risk-free zero rate, expressed with continuous compounding, to this bond's maturity.

3. A bond with a 5-year maturity pays a 4% coupon semiannually. At the moment, it trades with a yield of 2.5%, expressed with continuous compounding.

(a) What is the price of the bond? Express your answer assuming a 100 face amount.
(b) What is the (Macaulay) duration of the bond? What is its convexity?
(c) What is the modified duration of the bond?
(d) Use the discussion in the chapter to derive an expression for a bond's modified convexity, and calculate it for this bond.

4. The following bond prices are observed from the same risk-free issuer in the same currency:

maturity	maturity unit	bond price	coupon	coupon frequency (interest payments / year)
3	months	99.72537778	0	n/a
6	months	99.37694906	0	n/a
1	year	98.06888952	0	n/a
2	years	100.0902937	0.018	4
5	years	100.7173062	0.022	4
10	years	102.3713352	0.026	4
30	years	100.0691529	0.03	4

Bootstrap to determine the term structure of discount factors at each of these maturities. Interpolate by assuming piecewise constant forward rates between the maturities listed above.

Applications

1. *A Lower Bound on the Convexity of a Collection of Deterministic Positive Cash Flows*

In the chapter, it was asserted that for any collection of deterministic and strictly positive cash flows with a particular duration, a zero-coupon bond exhibits the least convexity. The goal of this problem is to formalize that assertion.

For the purposes of this problem, we will take advantage of the fact, discussed more thoroughly in a later chapter, that for a portfolio Π consisting of several instruments with values V_1, V_2,…, V_n having durations D_1, D_2, D_n and convexities C_1, C_2,…, C_n, we may say that…

$$V_\Pi = \sum_{i=1}^{n} V_i$$

$$D_\Pi = \frac{1}{V_\Pi} \sum_{i=1}^{n} V_i D_i$$

$$C_\Pi = \frac{1}{V_\Pi} \sum_{i=1}^{n} V_i C_i$$

…for certain suitable definitions of duration and convexity. For the moment, we will not concern ourselves with what those definitions are, but will simply use the result.

(a) Show that for an arbitrary portfolio Π consisting of n strictly positive cash flows occurring at different times, the following inequality holds:

$C_{\Pi} > D_{\Pi}^{2}$

Suggestion: This can easily be done by induction. Begin with the case of two deterministic positive cash flows at times T_1 and T_2, and show that…

$C_{\Pi} \geq D_{\Pi}^{2}$

…with equality holding if and only if $T_1 = T_2$.

Next, assume that we have a portfolio of n cash flows Π_n satisfying…

$C_{\Pi_n} > D_{\Pi_n}^{2}$

…and show that the addition of any deterministic positive cash flow to this portfolio results in a new portfolio Π_{n+1} that satisfies:

$C_{\Pi_{n+1}} > D_{\Pi_{n+1}}^{2}$

(b) Explain why your argument from above in (a) suffices to prove the desired result concerning zero-coupon bonds.

(c) Can the result be extended to portfolios in which some of the cash flows are negative—that is, levered portfolios or portfolios that also include deterministic liabilities? Explain the reasoning behind your answer.

2. *Approximation of Portfolio Yield from Yields of Its Constituent Instruments*

A portfolio consists of two fixed-coupon bonds: The first has maturity 5 years, pays interest semiannually at the rate of 4% per year, and has a yield (expressed with continuous compounding) of 2.5%. The second has maturity 3 years, pays interest annually at the rate of 1.5% per year, and has a yield (expressed with continuous compounding) of 5%.

(a) Compute the prices and durations of these bonds, expressing each price per 100 face amount.

(b) Suppose we have 100 total to invest. We construct a portfolio by choosing weight w to invest in the first bond, and therefore weight $1 - w$ to invest in the second. For w running from 0 to 1 inclusive, spaced 0.025 apart, calculate the quantity (face amount) held of each bond given w.

(c) Using your nonlinear solver of choice, for each of the portfolios above in part b, calculate the yield of the portfolio's cash flows, expressed with continuous compounding.

(d) For each of the portfolios whose yield you calculated above in c, compare your result to the following approximations:

$$\tilde{y}_1 = \frac{V_1 y_1 + V_2 y_2}{V_1 + V_2}$$

$$\tilde{y}_2 = \frac{V_1 D_1 y_1 + V_2 D_2 y_2}{V_1 D_1 + V_2 D_2}$$

...where V is the present value of each bond, D is its duration, and y is its yield. How does each approximation perform in this example? The first—a simple weighted average of yields—is very common for approximating the yield of a portfolio of bonds. Provide an intuitive explanation of why the second is a better approximation.

(e) Consider a portfolio Π of N fixed-coupon bonds, with known V_i, D_i, and y_i for each bond, $i = 1, 2, ..., N$. Using the first-order approximation of the bond's price at a given yield...

$$V_i(y) \approx V_i - D_i V_i (y - y_i)$$

...derive the approximation of the portfolio yield:

$$y_\Pi \approx \frac{\sum_{i=1}^{N} V_i D_i y_i}{\sum_{i=1}^{N} V_i D_i}$$