8. Options II

Exercises

1. We wish to price options on an underlying with the following attributes:
$S_0 = 45$
$q = 0.005$

The risk-free rate r is 2.5%, expressed with continuous compounding, to all maturities.

(a) What is the Black-Scholes price of a European call on this asset with expiry 0.25 years, strike price 50, and volatility 35%?
(b) What is the Black-Scholes price of a European put on this asset with expiry 0.25 years, strike price 40, and volatility 45%?
(c) The implied volatility of an option is the volatility that, if used in the appropriate option pricing formula, causes the price returned by the formula to equal the market price. If the market price of a European call option on this asset with strike 50 and expiry 0.5 years is 2.45, then what is the implied volatility of the option?
(d) If the market price of a European put option on this asset with strike 30 and expiry 0.5 years is 0.80, then what is the implied volatility of the option?

2. Use the risk-neutral distribution of the terminal spot price under the Black-Scholes assumptions to derive formulas for the present values of the following derivatives in terms of:
S_0: the spot price
r: the constant continuously compounded risk-free rate
q: the constant continuously compounded dividend rate
T: the expiry of the option
σ: the volatility of the underlying

(a) A cash-or-nothing call, often also called a binary or digital call, with strike price K and payment amount N: This option pays the fixed amount N to the holder of the option if the spot price at expiry is greater than the strike, and nothing otherwise.
(b) An asset-or-nothing call with strike price K: This option delivers the underlying asset to the holder of the option if its spot price at expiry is greater than the strike, and nothing otherwise.
(c) A squared power call option with strike price K: This option delivers the difference between the square of the spot price at expiry and the strike price if that value is positive, and nothing otherwise.
(d) A log-forward contract with strike price K: The payoff of this derivative is the natural log of the ratio between the spot price at expiry and the strike price. In particular, what is the value of this contract if the strike K is equal to the forward price of the asset?

3. The risk-free rate expressed with continuous compounding is 2.25% to all maturities. An asset has spot price 36, its continuously compounded dividend rate is 80 basis points to all maturities, and its volatility is 40%. Determine the price of the option, along with

the Greeks delta, gamma, vega, theta, and rho under Black-Scholes assumptions for the following options on the asset:

(a) A European call option struck at 36 with expiry in 3 months.
(b) A European put option struck at 45 with expiry in 6 months.

4. At a time when the risk-free rate is 5% to all maturities (expressed with continuous compounding), you are short 100 European call options with expiry 3 months and strike 60 on an asset priced at 55. The asset's volatility is 45%, and it pays no dividends.

For the purposes of the below, assume that all assets can be traded in any quantity, and that Black-Scholes assumptions hold; further, assume that the asset's volatility is constant through time.
(a) Suppose you wish to delta-hedge your exposure. How many shares of the underlying must you add to the portfolio in order for it to be delta-neutral?
(b) Suppose you wish to hedge your exposure to changes in the asset's volatility, as well as delta-hedging your exposure. You select a European put option with strike 50 and expiry 6 months for the purposes of vega-hedging. What quantity of this option and underlying must you add to the portfolio in order for it to be delta- and vega-neutral?
(c) What is the gamma of your initial short option position? What is the gamma of the hedged portfolio you constructed in (b)?
(d) You wish to hedge your gamma exposure as well. An additional asset—a European call option struck at 55 with expiry 1 month—is available for this purpose. Create a portfolio hedging the original option exposure which is delta-, vega-, and gamma-neutral.

Applications
1. *Put-call symmetry*
The concept of put-call symmetry relates the prices of puts and calls in terms of the ratio of their strike prices. The goal of this problem is to derive the classic put-call symmetry result in the Black-Scholes model.

(a) Show that, for European puts and calls struck at the underlying's forward price (usually called ATMF options) with the same expiry on the same underlying, $V_{call} = V_{put}$. (Note that this does not require Black-Scholes assumptions to show, and is true for any arbitrage-free model.)
(b) Show that, for any positive constant k, if we denote the forward price of the underlying at expiry by F, then the Black-Scholes value of a European call option struck at kF is equal in value to k European put options on the same underlying struck at F / k with the same expiry.
(c) Conclude that, under the Black-Scholes model, the price of a European call on an asset struck at K is equal to the price of K / F European put options on the same asset with the same expiry struck at F^2 / K.

2. *Volatility Skew*
In reality, the Black-Scholes assumption of normally distributed log-returns at constant volatility does not hold. Equities, in particular, tend to exhibit large negative returns more

frequently than a normal model would predict, making OTM put options relatively more valuable than OTM call options.

The Black-Scholes model, however, remains useful for quantifying the difference in relative value in an intuitive way. For a given option price, the Black-Scholes model is commonly used to report the *implied volatility* of the option: This is the volatility that, if supplied to the Black-Scholes model, causes the model price to match the traded price. The typical finding from equity markets is that the implied volatilities exhibit a skew: OTM put options have higher implied volatilities than OTM call options.

For the following example, we choose the values:
$S_0 = 100$
$r = 0.01$
$q = 0$
$\sigma = 0.25$
$T = 0.25$

(a) For strikes running from 50 to 200, spaced 5 apart, compute the Black-Scholes value of the corresponding OTMf option (a put option if the strike is less than the forward, a call option otherwise). Implement a VBA function that takes a call / put price and solves for the implied volatility, and verify that your function returns the same volatility, up to estimation error, for all of your options.

(b) A simple way to replicate an equity's tendency to exhibit large negative returns is to augment the geometric Brownian diffusion of the Black-Scholes model with a Poisson jump process. We will consider a version of this model with fixed downward jumps of nonrandom size.

We choose a downward jump size m so that at the time immediately before the jump, if the spot price is S_{t-}, then after the jump the spot price becomes $(1 - m)\, S_{t-}$. We parameterize the Poisson process by choosing an expected wait time in years to the next jump τ and taking the parameter $\lambda = 1 / \tau$.

Under this choice, the expected number of jumps between now and expiry T is λT, making the number of observed jumps Y a Poisson random variable with this as its parameter. In order for our model to agree with the forward price, between jumps the asset must grow *faster* than the risk-free rate by the amount $m\lambda\, dt$, which is the average rate of loss on the equity that the jumps incur. The forward price of the asset under this model is therefore the forward price conditional on the number of jumps, weighted by the probability of observing that number of jumps:

$$F = \sum_{y=0}^{\infty} (1-m)^y S_0 e^{(r-q+m\lambda)T} P(Y=y) = \sum_{y=0}^{\infty} (1-m)^y S_0 e^{(r-q+m\lambda)T} \frac{(\lambda T)^y}{y!} e^{-\lambda T}$$

Using Excel, verify that this formula's result agrees with the forward price in the example from part a when we choose the downward jump size $m = 0.2$ and the expected wait time to the next jump $\tau = 4$ years.

(c) To price an option under this jump diffusion model, we must as above price conditional on the number of jumps and then weight each price by the probability of that number of jumps. However, the drift rate of the asset is no longer $r - q$, as in the usual Black-Scholes case, but instead $r - q + m\lambda$, meaning that we require modified versions of the Black-Scholes formulas.

Suppose that the terminal distribution of some asset S_T is given by:

$$S_T = S_0 e^{\left(r-q-\frac{\sigma^2}{2}+\delta\right)T+\sigma\sqrt{T}Z}$$

...where as usual Z is a standard normal random variable and σ the volatility, with an additional excess drift rate δ. Derive the modified pricing formula for a European call:

$$V_{\delta,call} = e^{-rT} E\left[\left(S_T - K\right)^+\right]$$

...under this distribution, and use put-call parity to derive the corresponding formula for a European put.

(d) Use the formulas you derived in c to price the vanilla options in a under the jump-diffusion model using the jump size and expected wait time parameters from part b. That is, for a call option you need to evaluate:

$$V_{call,jump} = \sum_{y=0}^{\infty} V_{\delta,call}\left(spot = (1-m)^y S_0\right) P\left(Y = y\right)$$

(e) Use these prices to back out the Black-Scholes implied volatilities of the options, and graph them. Comment on the result.