4. Pricing Instruments with Credit Risk

Exercises

1. The term structure of zero rates is as follows:

t	rate (continuous)
1	0.03
2	0.035

(a) A bond paying annual coupons of 5% per year matures 1 year from today and currently trades at 101.25. What is the z-spread for this bond?
(b) A bond from a different issuer paying annual coupons of 5% per year matures 2 years from today and currently trades at 102. What is the constant z-spread for this bond?
(c) Suppose a second bond from the same issuer as in part (b) is found. This bond pays a 2% annual coupon and matures in 1 year; its price is 98.75. Use this bond and the one from part (b) to find the 2-year term structure of z-spreads for this issuer.

2. The z-spreads for a particular issuer at years 1 and 2 are:

t	z-spread
1	0.005
2	0.0068

For the purposes of these questions, assume no recovery in the event of default.

(a) What is the unconditional probability of default in the second year?
(b) What is the probability of default in the second year conditional on survival for the first year?
(c) What is the forward hazard rate for the second year?

3. A particular issuer has a constant hazard rate of 0.01. For the purposes of this question, assume that the risk-free interest rate is 0.05 to all maturities.

(a) A semiannual bond with maturity 10 years paying a coupon rate of 4% trades at a price of 89. What is the present value of the recovery on the bond?
(b) What is the recovery rate implied by this value?
(c) Suppose that the bond in question is backed by collateral posted by the issuer, and due to recent changes, that collateral has appreciated to such a degree that you now consider the recovery rate on the bond to be 100%. What is the price of the bond under this assumption?
(d) Compare your price above in (c) to the price this bond would have if it were free of credit risk. The two prices differ. Explain why one is greater. What change to the bond's coupon, if any, would cause the other to be greater?

Applications

Approximate Modeling of Correlated Credit Losses Using a Gaussian Copula

Modeling the behavior of loan portfolios ("pools") is of interest in several areas of finance. The following describes the assumptions and rationale behind one simple method that is sometimes used for large, homogeneous loan pools made up of small individual loans.

I. The Independent Case

Assume that the total pool balance B_Π is divided equally among N loans, each of which has the same probability p of defaulting. If we assume also that these defaults are uncorrelated, then the distribution of the number of defaults N_d is binomial: $b(N, p)$. As a consequence:

$$E[N_d] = Np$$
$$Var[N_d] = Np(1-p)$$

Since the total defaulted loan balance is directly related to the number of defaults under this assumption...

$$B_d = N_d \left(\frac{B_\Pi}{N}\right)$$

...it then follows that:

$$E[B_d] = E\left[N_d \left(\frac{B_\Pi}{N}\right)\right] = \frac{B_\Pi}{N} E[N_d] = \frac{B_\Pi}{N} Np = B_\Pi p$$
$$Var[B_d] = Var\left[N_d \left(\frac{B_\Pi}{N}\right)\right] = \left(\frac{B_\Pi}{N}\right)^2 Var[N_d] = \left(\frac{B_\Pi}{N}\right)^2 Np(1-p) = \frac{B_\Pi^2 p(1-p)}{N}$$

It is a well known result that the distribution of the number of successes with a binomially distributed random variable is well approximated with a normal distribution when the number of trials is large. At the limit, we see that if we allow the number of loans in the pool to approach infinity (i.e., the pool is infinitely divisible and consists of independent infinitesimal loans), then....

$$\lim_{N\to\infty} Var[B_d] = \lim_{N\to\infty} Var\left[\frac{B_\Pi^2 p(1-p)}{N}\right] = 0$$

That is, under this stylized view of a loan pool, all variation in credit loss outcomes vanishes, and the loss amount is certain to be:

$$B_d = B_\Pi p$$

II. Adding Correlation

The assumption that loan defaults are independent, however, is not realistic. Empirically, both corporate and consumer debts tend to default in clusters, due to the fact that all are exposed to the same general economic conditions. There is a need, then, to capture this effect in any model of a credit pool.

Suppose that each individual loan is driven by some performance factor F_i. This factor is not directly observable, but for some choice of distribution, we may select a cutoff value F^* so that:

$$P[F_i < F^*] = p$$

...where p is the probability of the loan's default, making the hidden performance factor into a continuous random variable whose realization determines the binary outcome (default / no default) for the loan.

A *copula* is a technique used to create a joint distribution of several random variables in cases where only the individual marginal distributions are known. The simplest (and, therefore, most commonly used) copula is a *Gaussian copula*, in which the variables are bound together using a multivariate normal distribution. This is appealing in part because the sum of independent normally distributed random variables is itself normal, making many otherwise complex calculations quite tractable.

To apply the technique here, we assume that each F_i has a standard normal distribution, and that each pair $F_i, F_j, i \neq j$ has the same correlation ρ, which must be nonnegative. The model "explains" these identical pairwise market correlations by defining a (not directly observable) global market factor F, also with a standard normal distribution, with which each individual F_i has correlation $\sqrt{\rho}$. Thus, the random variable F_i can be expressed as:

$$F_i = \sqrt{\rho}\, F + \sqrt{1-\rho}\, Z_i$$

...where Z_i is the idiosyncratic risk associated with just this loan, independent of all others, while F controls the effect of the market on it, and completely explains its correlation with the performance of other loans.

III. Putting It All Together

As we have seen, under the assumption of an infinitely divisible pool of infinitesimal loans, the effect on the pool loss of idiosyncratic individual loan risk can be disregarded, and we are certain to realize the expected pool loss amount, given the default probability p of each loan.

The formulation of our Gaussian model of the market factor makes the expected loss amount of the pool now conditional on the particular realization f of the market factor F. That is, for this infinitely divisible pool, the market becomes the only random component determining the pool loss amount, and thus the pairwise default correlation becomes the sole parameter controlling the uncertainty of pool losses.

(a) For a given individual loan with default probability p, what function determines the cutoff value $F^*(p)$ defined above? Calculate and graph this function for choices of p running from 0.05 to 0.95, spaced 0.05 apart.

(b) Given a particular realization f of the market factor F and a nonnegative correlation value ρ, what is the distribution of the individual loan performance random variable F_i? Calculate and graph as a surface the mean of F_i for choices of f running from -3 to 3 spaced 0.25 apart along one axis, and for choices of ρ running from 0 to 0.95 spaced 0.05 apart along a second axis.

(c) Create a simple calculator for which the user supplies the value of $p \in (0,1)$. Given this input, calculate the cutoff F^*, and then for the same table of values as above in part (b), calculate the conditional probability of loan default $P(F_i < F^*)$ given that particular choice of ρ and realization f of the market risk factor. Based on our reasoning above, the probability of an individual loan defaulting in this instance is also then the conditional expected loss of the entire pool, given that particular f.

(d) The formula you used to calculate the result in part (c) defines a function we can call $L_{(pool|f)}$, representing the pool loss given a particular realization f of the random market factor F. This function depends upon the parameters p, the individual loan default probability, and ρ, the pairwise loan correlation.

The fact that we have defined the market factor F to have a standard normal distribution means that we can then calculate the cumulative distribution function of the pool losses, based on the parameters p and ρ. That is, for a particular loss proportion $l \in (0,1)$, we can calculate the probability $P(L_{(pool|F)} < l)$ for any valid loss proportion. Derive the formula for this calculation. Create a dynamic worksheet where the user supplies the values of p and ρ, and the sheet graphs the probability that losses will be less than each of the l values ranging from 0.025 to 0.975, spaced 0.025 apart.

(e) Show the resulting graph from your calculator for the choice of default probability $p = 0.25$, with each of the correlation inputs $\rho \in \{0.1, 0.9\}$. How do you explain the differences in these two graphs? What financial meaning might this have for an investor in a pool of loans, especially if his or her investment's payoff depends heavily upon the pool loss? Keep in mind as you answer that both of these graphs represent pools whose overall expected default rate is 25%!