5. Hedging Instruments with Credit Risk

Exercises

1. For the following, assume that the risk-free interest rate is 2.5% (expressed with continuous compounding) to all maturities, and the z-spread is 50bp (also expressed continuously) to all maturities.

(a) What is the NPV per 100 face of a fixed-coupon bond from this issuer paying a 2.515% coupon semiannually for five years? Estimate its duration and spread duration via numerical approximation.
(b) What is the NPV per 100 face of an FRN from this issuer paying LIBOR flat quarterly for five years? Estimate its duration and spread duration via numerical approximation.
(c) Consider a portfolio consisting of a long position in the bond from (a) and a short position in the bond from (b), with both positions having 100 face. Calculate the NPV and estimate the duration and spread duration of the portfolio. (Note: Express your durations by dividing the central finite difference estimate of the derivative by $-N$.)
(d) Calculate the NPV and estimate the duration as in (c) of a swap with semiannual payments on the fixed leg and quarterly payments on the float leg paying 2.515% fixed for 5 years. The portfolio above in (c) can be thought of as this same swap contract entered into with a risky counterparty. Does credit risk make an appreciable difference to the value or interest rate sensitivity of the contract? How does this result change if we consider a 6.5% fixed coupon on the risky and risk-free swaps instead of the original coupon?

2. For the following questions, assume the risk-free rate is 2% (expressed with continuous compounding) and that the recovery rate on the issuer is 40%.

(a) What is the par CDS spread on a 3Y contract with the premium paid quarterly if the issuer has a hazard rate of 0.01? To simplify your calculation, assume that default in any premium period occurs at the *end* of the period—that is, if default occurs in a particular period, the protection buyer owes the full period's premium and is compensated for credit loss at that time.
(b) Using your result in (a), determine the arbitrage-free minimum par spread for a 5Y contract on the same issuer.

Applications

1. *Building a CDS pricer*
The goal is to build a fast and accurate CDS pricing function. As inputs, it will take the deal notional N (with a positive notional indicating a protection buy), the number of years to maturity T, the deal premium s, the premium payment frequency m, the recovery rate R, along with term structures for interest rates and default probabilities. The term structure of interest rates will be given as an array of maturities and discount factors, (t, P_t); the term structure of default probabilities will be given as an array of maturities and survival probabilities (t, S_t). The function will interpolate in these values assuming

piecewise constant forwards (i.e., log-linear interpolation). Your function should be able to return the following characteristics of the deal:

1. the NPV of the deal
2. the value of the premium leg
3. the value of the protection leg
4. the value of the risky annuity (sometimes called the risky duration)
5. the fair par spread of the deal

Since producing either 1 or 5 requires essentially that all of 2-4 be calculated, the most convenient way to do this is to implement all of the analytics in a single function, and to pass as its final argument an array indicating which analytics should be returned; the function should then return an array with the results appearing in the same order as the input array.

To price a CDS, you need to price the two legs, consisting of three components:

the periodic premium payments:

$$V_{strip} = \sum_{i=1}^{n} N \frac{s}{m} P_{t_i} S_{t_i}$$

the payment of the accrued premium at default time:

$$V_{accrued} = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} Ns(t - t_{i-1}) P_t(-S'_t) dt$$

and the protection leg:

$$V_{protection} = \int_{0}^{T} N(1-R) P_t(-S'_t) dt$$

The value of the premium leg is then $V_{premium} = V_{strip} + V_{accrued}$. Many methods will serve for approximation of the integrals. For this particular application, however, consider the following:

Given times t_1 and t_2, we can use whatever interpolation method we like to determine the discount factors at these two times as well as the survival probabilities. Suppose we assume that both the instantaneous forward rate and the instantaneous hazard rate are constant on the interval (t_1, t_2). Then the forward rate r_f can be calculated as…

$$r_f = -\frac{\ln P_{t_2} - \ln P_{t_1}}{t_2 - t_1}$$

…and the hazard rate λ_f similarly as:

$$\lambda_f = -\frac{\ln S_{t_2} - \ln S_{t_1}}{t_2 - t_1}$$

Under these assumptions, the value of a contract that pays the amount $a(t - t_1) + b$ at the time of default if it occurs on this interval can be shown to be:

$$\int_{t_1}^{t_2} \left[a(t - t_1) + b\right] P_t(-S'_t) dt = \frac{\lambda_f}{r_f + \lambda_f} \left[-a(t_2 - t_1) P_{t_2} S_{t_2} + \left(\frac{a}{r_f + \lambda_f} + b \right) \left(P_{t_1} S_{t_1} - P_{t_2} S_{t_2} \right) \right]$$

Clearly, the integrals needed for CDS pricing can be approximated arbitrarily well using this method for any suitably fine discretization of the interval $[0, T]$, no matter what (reasonable) interpolation method is used to determine the discount factors and survival probabilities at the endpoints of each interval in the discretization.

In this case, since we are using piecewise constant forward interpolation to determine all discount factors and survival probabilities, the above provides an *exact* calculation for any interval on which no knot points in the discount factor or survival curves occur, and on which the payoff function is also linear.

For example, for a two-year deal, if the discount factor curve and the survival curve have nodes only at the 1-year and 2-year points, then the protection leg can be priced exactly by breaking the domain into two 1-year intervals. If premium payments occur quarterly, then the value of the accrued premium payment in the event of default can be determined exactly by breaking the domain into 8 quarterly intervals. A function implemented using this method will run more quickly and produce a better value than one using the more generic rectangle, trapezoidal, or Simpson's rules.

(a) Testing your function:
Given the complexity of the implementation, you should validate it by verifying the following:

(i) Set $N = 100$, $R = 0$, and choose an arbitrary maturity T. Set interest rates to 0. Choose your survival curve so that it contains a single point at T with value S_T. What should the value of the protection leg alone be? Show that your code produces this value. What should it return when R is nonzero? Show a second example that demonstrates it produces this value for the protection leg.

(ii) Repeat the examples in (i), this time by giving your survival curve a term structure including at least one additional point at some time greater than zero but less than T. What should your code produce? Show that it in fact does.

(iii) Set $N = 100$ and choose an arbitrary frequency m, maturity T, and spread s. Set interest rates to 0 and hazard rates to a very small number. (Setting both to zero will cause the function to fail.) What should the value of the premium strip be? Show that your code produces this value. What if the maturity is not an integer number of coupon periods from today?

(iv) Set $N = 100$, $T = 5$ years, $m = 4$, $R = 0$ and interest rates to zero. Generate a survival curve to 5 years assuming a constant hazard rate of 0.05. What par spread does your code produce with these inputs? What should the result be? How about if R is nonzero?

(b) Determine the NPV, the PV of each leg, the value of the annuity, and the fair par spread for the following deal…

$N = 100$, $s = 0.02$, $m = 4$, $T = 5$
…using a recovery rate of 40% with the following discount factor and survival curves:

Interest Rates:

t	Pt	r
0	1	0.005
0.5	0.997503122	0.005
1	0.991536023	0.0085
2	0.980198673	0.01
3	0.955997482	0.015
5	0.904837418	0.02

Survival Rates:

t	St	lambda
0	1	0
1	0.990049834	0.01
3	0.927743486	0.025
5	0.869358235	0.028

(c) Reprice the deal above by applying the following shocks:

- interest rates up and down 1bp
- recovery rate up and down 1bp
- lambda up and down 1bp

Use your results to generate a central finite difference approximation of the dollar sensitivities $dV / d\Delta x$ for each of the three risk factors x. Briefly explain the sign of each of your results.

(d) For each of your dollar sensitivities above, rescale the result by dividing it by $-N$. Do the relative sizes of these sensitivities make sense?