# Foundations of Algorithms Spring 2023
# Homework #6



All members of the collaboration group are expected to participate fully in solving collaborative problems. Note, however, that each student is required to write up their solutions individually. Common solution descriptions from a collaboration group will not be accepted. Furthermore, to receive credit for a collaboration problem, each student in the collaboration group must actively and substantially contribute to the collaboration. This implies that no single student should post a complete solution to any problem at the beginning of the collaboration process.

## Problems for Grading

1. [20 points] CLRS 34.3-2: Show that the $\leq_P$ relation is a transitive relation on languages. That is, show that if $L_1 \leq_P L_2$ and $L_2 \leq_P L_3$, then $L_1 \leq_P L3$.

2. [20 points] Recall the definition of a complete graph $K_n$ is a graph with $n$ vertices such that every vertex is connected to every other vertex. Recall also that a clique is a complete subset of some graph. The *graph coloring problem* consists of assigning a color to each of the vertices of a graph such that adjacent vertices have different colors and the total number of colors used is minimized. We define the *chromatic number* of a graph $G$ to be this minimum number of colors required to color graph $G$. Prove that the chromatic number of a graph $G$ is no less than the size of the maximal clique of $G$.

3. [30 points] Suppose you're helping to organize a summer sports camp, and the following problem comes up. The camp is supposed to have at least one counselor who's skilled at each of the $n$ sports covered by the camp (baseball, volleyball, and so on). They have received job applications from $m$ potential counselors. For each of the $n$ sports, there is some subset of the $m$ applicants qualified in that sport. The question is "For a given number $k < m$, is is possible to hire at most $k$ of the counselors and have at least one counselor qualified in each of the $n$-sports?" We'll call this the *Efficient Recruiting Problem*. Prove that Efficient Recruiting is $NP$-complete.

4. [30 points] ***Collaborative Problem***: We start by defining the *Independent Set Problem (IS)*. Given a graph $G = (V, E)$, we say a set of nodes $S \subseteq V$ is *independent* if no two nodes in $S$ are joined by an edge. The Independent Set Problem, which we denote $IS$, is the following. Given $G$, find an independent set that is as large as possible. Stated as a decision problem, $IS$ answers the question: "Does there exist a set $S \subseteq V$ such that $|S| \geq k$?" Then set $k$ as large as possible. For this problem, you may take as given that $IS$ is $NP$-complete.

   A store trying to analyze the behavior of its customers will often maintain a table A where the rows of the table correspond to the customers and the columns (or fields) correspond to products the store sells. The entry $A[i, j]$ specifies the quantity of product $j$ that has been purchased by customer $i$. For example, Table 1 shows one such table.

   One thing that a store might want to do with this data is the following. Let's say that a subset $S$ of the customers is *diverse* if no two of the customers in $S$ have ever bought the same product (i.e., for each product, at most one of the customers in $S$ has ever bought it). A diverse set of customers can be useful, for example, as a target pool for market research.

   We can now define the *Diverse Subset Problem (DS)* as follows: Given an $m \times n$ array $A$ as defined above and a number $k \leq m$, is there a subset of at least $k$ customers that is diverse?

   Prove that $DS$ is $NP$-complete.

Table 1: Customer Tracking Table

| Customer | Detergent | Beer | Diapers | Cat Litter |
|---|---|---|---|---|
| Raj | 0 | 6 | 0 | 3 |
| Alanis | 2 | 3 | 0 | 0 |
| Chelsea | 0 | 0 | 0 | 7 |