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Homework 6

Question 1.

First, by the problem statement let $L_1 \leq pL_2$ and $L_2 \leq pL_3$. This indicates that there exists some polynomial time achievable reduction function such that $f_1 : \{0,1\}* \rightarrow \{0,1\}*$ and $f_2 : \{0,1\}* \rightarrow \{0,2\}*$. This indicates that $x \in L_1 \iff f_1(x) \in L_2$ and $x \in L_2 \iff f_2(x) \in L_2$.

Second, suppose that $f_3 = f_1(f_2(x))$. Then L_3 is also a polynomial-time achievable function $f_3 : \{0,1\}* \rightarrow \{0,1\}*$ and $x \in L_1 \iff f_3(x) \in L_3$ holds.

By the previous two conditions, we get that $L_1 \leq pL_3$.

Question 2.

We begin by recalling that clique is a complete graph. Let's assume that the size of the clique is k. We also know that for a complete graph, the chromatic number is n. This gives that while coloring the clique, we need at least k colors. The problem explicitly states that the clique is maximal in nature. Because of this condition, at least k colors are needed to properly color the largest clique. Therefore, the chromatic number of the graph must be equal to or larger than k.

By these conditions, the chromatic number of the graph G is never less than the size of maximal clique of G.

Question 3.

Given a graph G and a number k, does G contain a vertex cover of size at most k? (Recall that a vertex cover $V_0 \subset V$ is a set of vertices such that every edge $e \in E$ has at least one of its endpoints in V_0)? The solution to this problem is in NP, since given a set of k counselors, we can check that they cover all of the sports.

Suppose we had an algorithm A that solves Efficient Recruiting; here is how we would solve an instance of Vertex-Cover. Given a graph $G = (V, E)$ and an integer k, we would define a sport Se for each edge e and a counselor Cv for each vertex v. Cv is qualified in sport Se if e has an endpoint equal to v. Now if there are k counselors that are qualified in all sports, the corresponding vertices in G have the property that each edge has an end in at least one of them; so they define a vertex cover of size k. Conversely, if there is a vertex cover of size k, then this set of counselors has the property that each sport is contained in the list of qualifications of at least one of them.

Question 4.

Let us begin with the assumption that for a subset S of customers to be diverse, no two of the customers in S have bought the same product. In other words, for each product, at most one of the customers in S has ever bought it. By this logic, a diverse subset problem in this context would be given by an $m \times n$ array A as defined in the problem statement, with a number $k \leq m$. The question is whether there is a subset k of the customers that are diverse.

The problem is NP-Complete because we can exhibit a set of k customers, and in polynomial time we can check that no two bought any product in common. In order, to show NP-Completeness, we need to show then that the Independent Set is less than the Diverse Subset.

Suppose that given a graph G and a number k, we construct a customer for each node of G, and a product for each edge of G. We can then build an array that says customer v bought product e if edge e is incident to node v. Finally, we can ask whether this array has a diverse subset of size k.

This should hold if G has an independent set of size k. If there is a diverse subset of size k, the corresponding set of nodes would present that no two are connected by the same edge and it is therefore an independent set of size k. On the other hand, should there be an independent set of size k, the corresponding set of two particular customers would not have bought the same product. This indicates that it is diverse.