ASEN 5519 ALGORITHMIC MOTION PLANNING

HOMEWORK #2

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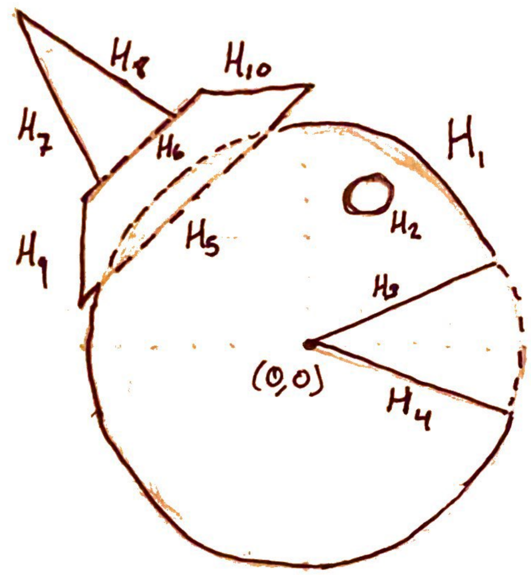
Exercise #1

Define a semi-algebraic set that models the shape



H_1 : $x^2+y^2 = r_1^2$

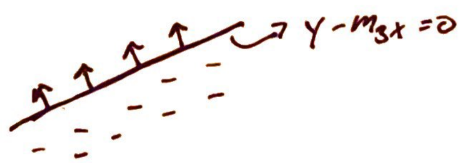
$$\boxed{x^2+y^2-r_1^2 \le 0}$$

H_2 : centered @ (x_2, y_2)

$$(x-x_2)^2+(y-y_2)^2 = r_2^2$$

$$\boxed{-\left((x-x_2)^2+(y-y_2)^2-r_2^2\right) \le 0}$$

H_3 : slope m_3



$$y = m_3 x$$

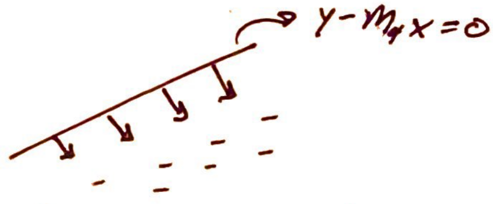
$$y - m_3 x \ge 0$$

$$\boxed{-(y - m_3 x) \le 0}$$

H_4 : slope m_4

$y = m_4 x$

$$\boxed{y - m_4 x \leq 0}$$



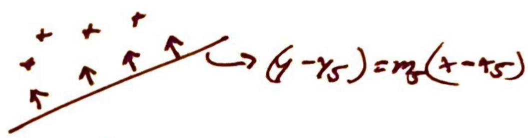
(pts below line)

H_5 : slope m_5, intersects point (x_5, y_5)

$(y - y_5) = m_5 (x - x_5)$

$(y - y_5) - m_5 (x - x_5) \geq 0$

$$\boxed{-\left((y - y_5) - m_5 (x - x_5)\right) \leq 0}$$

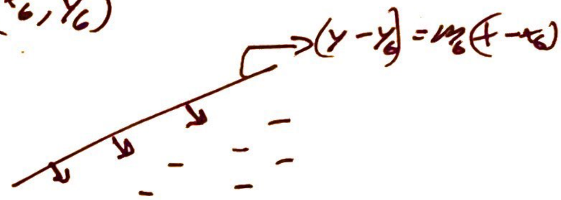


(pts above line)

H_6 : slope m_6, intersects point (x_6, y_6)

$(y - y_6) = m_6 (x - x_6)$

$$\boxed{(y - y_6) - m_6 (x - x_6) \leq 0}$$



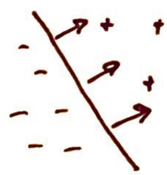
(pts below line)

H_7 : slope m_7, intersects point (x_7, y_7)

$(y - y_7) = m_7 (x - x_7)$

$(y - y_7) - m_7 (x - x_7) \geq 0$

$$\boxed{-\left((y - y_7) - m_7 (x - x_7)\right) \leq 0}$$

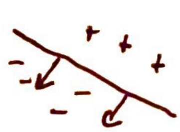


(pts above line)

H_8 : slope m_8, intersects point (x_8, y_8)

$(y - y_8) = m_8 (x - x_8)$

$$\boxed{(y - y_8) - m_8 (x - x_8) \leq 0}$$



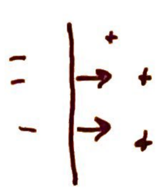
(pts below line)

H_9 : $x = x_9$, vertical line @ $x = x_9$

(pts right of line)

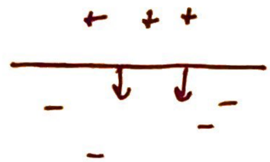
$x - x_9 \geq 0$

$$\boxed{-(x - x_9) \leq 0}$$



H_{10} : $y = y_{10}$, horizontal line @ $y = y_{10}$

$\boxed{y - y_{10} \leq 0}$ (pts below line)



$$\boxed{X = H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5 \cap H_6 \cap H_7 \cap H_8 \cap H_9 \cap H_{10}}$$

<u>Exercise #2</u>

$$R(\alpha, \beta, \gamma) = \overset{1}{R_z(\gamma)}\, \overset{2}{R_y(\beta)}\, \overset{3}{R_z(\alpha)}$$

(a) Determine R.

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$R_z(\gamma)\, R_y(\beta)\, R_z(\alpha) = R =$

$$\boxed{\begin{bmatrix} \cos(\gamma)\cos(\beta)\cos(\alpha) - \sin(\gamma)\sin(\alpha) & -\cos(\alpha)\sin(\gamma) - \cos(\gamma)\cos(\beta)\sin(\alpha) & \cos(\gamma)\sin(\beta) \\ \cos(\gamma)\sin(\alpha) + \cos(\beta)\cos(\alpha)\sin(\gamma) & \cos(\gamma)\cos(\alpha) - \cos(\beta)\sin(\gamma)\sin(\alpha) & \sin(\gamma)\sin(\beta) \\ -\sin(\beta)\cos(\alpha) & \sin(\beta)\sin(\alpha) & \cos(\beta) \end{bmatrix}}$$

(b) Show $R(\alpha, \beta, \gamma) = R(\alpha - \pi, -\beta, \gamma - \pi)$

R_{11}: $\cos(\gamma)\cos(\beta)\cos(\alpha) - \sin(\gamma)\sin\alpha \overset{?}{=} \cos(\gamma-\pi)\cos(-\beta)\cos(\alpha-\pi) - \sin(\gamma-\pi)\sin(\alpha-\pi)$

$\checkmark$ $= \cancel{(-1)}\cos(\gamma)\cos(\beta)\cancel{(-1)}\cos(\alpha) - \cancel{(-1)}\sin(\gamma)\cancel{(-1)}\sin(\alpha)$

R_{12}: $-\cos(\alpha)\sin(\gamma) - \cos(\gamma)\cos(\beta)\sin(\alpha) \overset{?}{=} -\cos(\alpha-\pi)\sin(\gamma-\pi) - \cos(\gamma-\pi)\cos(-\beta)\sin(\alpha-\pi)$

$\checkmark$ $= -\cancel{(-1)}\cos(\alpha)\cancel{(-1)}\sin(\gamma) - \cancel{(-1)}\cos(\gamma)\cos(\beta)\cancel{(-1)}\sin(\alpha)$

R_{13}: $\cos(\gamma)\sin(\beta) \overset{?}{=} \cos(\gamma-\pi)\sin(-\beta)$

$\checkmark$ $= \cancel{(-1)}\cos(\gamma)\cancel{(-1)}\sin(\beta)$

R_{21}: $\cos(\gamma)\sin(\alpha) + \cos(\beta)\cos(\alpha)\sin(\gamma) \overset{?}{=} \cos(\gamma-\pi)\sin(\alpha-\pi) + \cos(-\beta)\cos(\alpha-\pi)\sin(\gamma-\pi)$

$\checkmark$ $= \cancel{(-1)}\cos(\gamma)\cancel{(-1)}\sin(\alpha) + \cos(\beta)\cancel{(-1)}\cos\alpha\cancel{(-1)}\sin(\gamma)$

R_{22}: $\cos(\gamma)\cos(\alpha) - \cos(\beta)\sin(\gamma)\sin(\alpha) \overset{?}{=} \cos(\gamma-\pi)\cos(\alpha-\pi) - \cos(-\beta)\sin(\gamma-\pi)\sin(\alpha-\pi)$

$\checkmark$ $= \cancel{(-1)}\cos(\gamma)\cancel{(-1)}\cos(\alpha) - \cos(\beta)\sin(\gamma)\cancel{(-1)}\cancel{(-1)}\sin(\alpha)$

R_{23}: $\sin(\gamma)\sin(\beta) \overset{?}{=} \sin(\gamma-\pi)\sin(-\beta)$

$\checkmark$ $= \cancel{(-1)}\sin(\gamma)\cancel{(-1)}\sin(\beta)$

R_{31}: $-\sin(\beta)\cos(\alpha) \overset{?}{=} -\sin(-\beta)\cos(\alpha-\pi)$

$\checkmark$ $= (-1)(-1)\sin(\beta)(-1)\cos(\alpha)$

R_{32} : $\sin(\beta)\sin(\alpha) \overset{?}{=} \sin(-\beta)\sin(\alpha-\pi)$

$\overset{\checkmark}{=} (-1)\sin(\beta)(-1)\sin(\alpha)$

R_{33} : $\cos(\beta) \overset{?}{=} \cos(-\beta)$

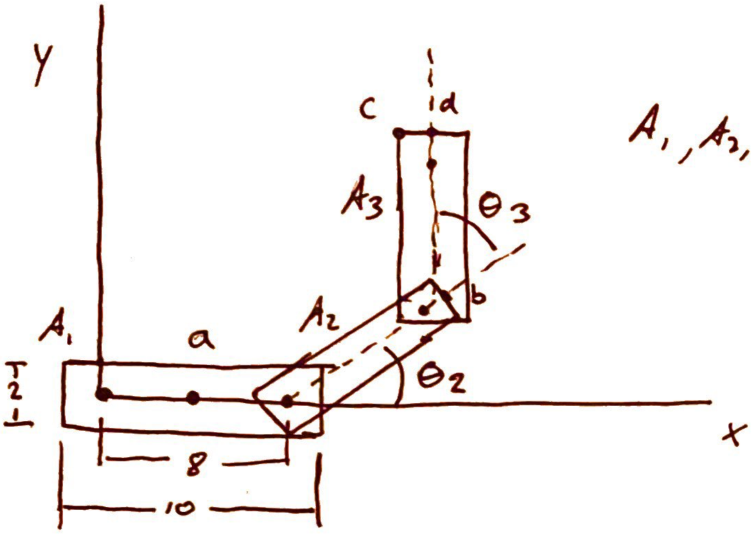
$= \cos(\beta)$

$$\boxed{\begin{array}{l} R(\alpha, \beta, \gamma) = \\ R(\alpha-\pi, -\beta, \gamma-\pi) \\ \text{Prooved using: } \cos(-\theta) = \cos(\theta),\ \sin(-\theta) = -\sin\theta, \\ \cos(\theta-\pi) = -\cos(\theta),\ \sin(\theta-\pi) = -\sin\theta \end{array}}$$

(c) Given rotation matrix R' determine α, β, γ in terms of elements of R'

$$\boxed{\begin{array}{l} \alpha = \operatorname{atan2}\left(\frac{R_{32}}{R_{31}}\right) \\ \beta = \operatorname{acos}(R_{33}) \\ \gamma = \operatorname{atan2}\left(\frac{R_{23}}{R_{13}}\right) \end{array}}$$

Exercise #3



A_1, A_2, A_3 are identical

(a) For configuration $(\theta_1, \theta_2, \theta_3) = \left(\frac{\pi}{4}, \frac{\pi}{2}, -\frac{\pi}{6}\right)$ find locations of a, b, c

Point a:

$$\begin{bmatrix} x_a \\ y_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\theta_1} & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

translation of frame

Position of a relative to A_1 frame

$$\begin{bmatrix} x_a \\ y_a \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2.8284 \\ 2.8284 \\ 1 \end{bmatrix} = \begin{bmatrix} x_a \\ y_b \\ 1 \end{bmatrix}}$$

Point b:

→ translation of the A_2 frame from A_1

→ position of b relative to A_2 frame

$$\begin{bmatrix} x_b \\ y_b \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\theta_1} & \boxed{P_{21}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{\theta_2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 8+1 \\ 0 \end{matrix}} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 8 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_b \\ y_b \\ 1 \end{bmatrix} = \begin{bmatrix} 1.636 \\ 6.364 \\ 1.0 \end{bmatrix}}$$

Point c:

→ translation of A_3 from A_2

→ pos of c relative to A_3

$$\begin{bmatrix} x_c \\ y_c \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\theta_1} & P_{21} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{\theta_2} & \boxed{P_{32}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{\theta_3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 9 \\ 1 \end{matrix}} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 8 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 8 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_c \\ y_c \\ 1 \end{bmatrix} = \begin{bmatrix} 10.3616 \\ 14.0914 \\ 1 \end{bmatrix}}$$

(b) Find the configuration(s) of the robot when d is at (0, 4)

$$(1)\quad X = 8\cos(\theta_1) + 8\cos(\theta_1+\theta_2) + 9\cos(\theta_1+\theta_2+\theta_3) = 0$$

$$(2)\quad y = \underbrace{8\sin(\theta_1)}_{\text{component from } A_1} + \underbrace{8\sin(\theta_1+\theta_2)}_{\text{component from } A_2} + \underbrace{9\sin(\theta_1+\theta_2+\theta_3)}_{\text{component from } A_3} = 4$$

SEE PYTHON IMPLEMENTATION

let $\gamma = (\theta_1+\theta_2+\theta_3)$ so (1), (2) can be rewritten as

$$(3)\quad x - 9\cos(\gamma) = 8\cos(\theta_1) + 8\cos(\theta_1+\theta_2) = x'$$
$$(4)\quad y - 9\sin(\gamma) = 8\sin(\theta_1) + 8\sin(\theta_1+\theta_2) = y'$$

Position of joint 3

Square both sides & rearrange & add

$$(x' - 8\cos\theta_1)^2 = [8\cos(\theta_1+\theta_2)]^2$$
$$(y' - 8\sin\theta_1)^2 = [8\sin(\theta_1+\theta_2)]^2$$

$$(x')^2 - 16x'\cos\theta_1 + (8\cos\theta_1)^2 = (8\cos(\theta_1+\theta_2))^2 \quad +$$
$$(y')^2 - 16y'\sin\theta_1 + (8\sin\theta_1)^2 = (8\sin(\theta_1+\theta_2))^2$$

$$(5)\quad (x')^2 + (y')^2 + \cancel{(8)^2 - (8)^2} - 16x'\cos\theta_1 - 16y'\sin\theta_1 = 0$$

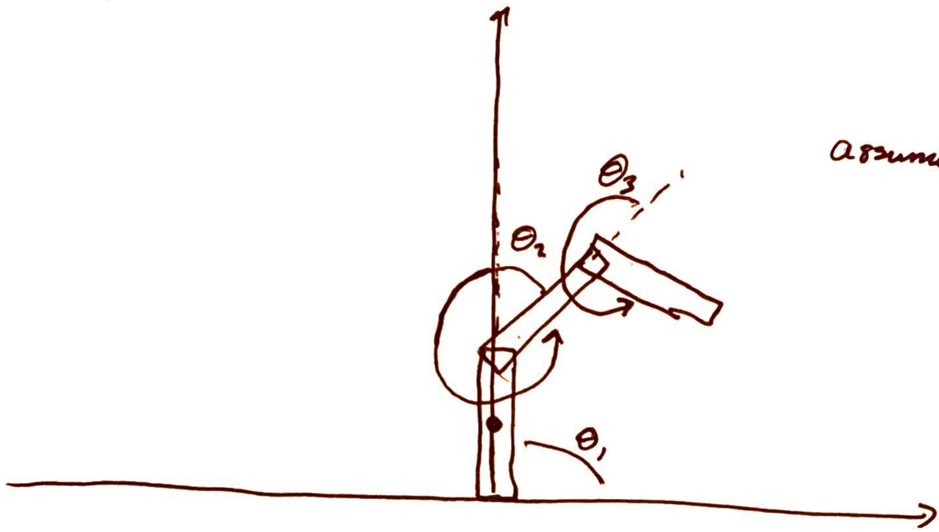
$R + P\cos\alpha + Q\sin\alpha = 0$ has solution of form

$$\theta_1 = \xi \pm \cos^{-1}\left[\frac{-(x')^2 + (y')^2}{16\sqrt{(x')^2 + (y')^2}}\right]$$

V. Kumar, "Robot Geometry and kinematics"

where $\xi = \operatorname{atan2}\left(\frac{-y'}{\sqrt{x'^2+y'^2}}, \frac{x'}{\sqrt{x'^2+y'^2}}\right)$

assume $\theta_1 \in [0, 180]$



Using discretized brute force approach:..

↳ Check all values of t_1, t_2, t_3

↳ Implemented in python.

SEE Homework2_Exercise3b_Solution.txt FOR SOLUTION VALUES

EXERCISE #4 Express config spaces of following systems in terms of cartesian product and determine their dimension

(a) Two trains on two tracks

- Only 1-D translation (forward - backward)

- A point in this c-space is the position of train 1 **AND** the position of train 2

$$\boxed{\mathbb{R}^1 \times \mathbb{R}^1 = C\text{-Space} \qquad \mathrm{Dim}(C\text{-space}) = 2}$$

(b) Spacecraft that can translate & rotate in 2D

Translation : $\mathbb{R}^2$ ("2D")

Rotation : S^1 ("2D rotation")

$$\boxed{\mathbb{R}^2 \times S^1 = SE(2) = C\text{-Space} \qquad \mathrm{Dim}(C\text{-space}) = 3}$$

(c) Two mobile robots translating & rotating in plane

EACH ROBOT $\begin{cases} \text{Translation: } \mathbb{R}^2 \\ \text{Rotation: } S^1 \end{cases}$

A point in C-space is the rotation & translation of both robots i.e. $[x_1, y_1, \theta_1, x_2, y_2, \theta_2]$

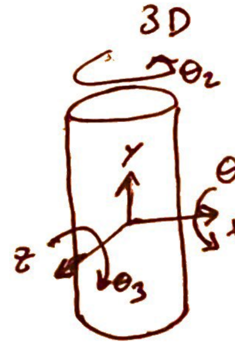
$$\boxed{\mathbb{R}^2 \times S^1 \times \mathbb{R}^2 \times S^1 = C\text{-Space} \qquad \mathrm{Dim}(C\text{-space}) = 6}$$

(d) Two translating & rotating planar mobile robots connected by a rigid bar

- Assume they're connected in such a way that the translation/rotation of 1 affects the translation/rotation of 2, i.e. orientation & position of 2 can be expressed in terms of orientation/position of 1
$\mathbb{R}^2 \times S^1 = C\text{-Space} \qquad \dim(C\text{-space}) = 3$

1)

(e) Cylindrical rod that can translate & rotate in 3D



Axially symmetric $\Rightarrow$ θ_1 and θ_3 can be used interchangably $\Rightarrow$ S^2

Translation: $\mathbb{R}^3$

Rotation: $SO(2)$

$\mathbb{R}^3 \times SO(2)$ = C-space Dim(C-space) = 5

(f) Spacecraft that can translate & rotate in 3D w/ 3 link operator arm (3 joints)

Translation: $\mathbb{R}^3$

Rotation: $SO(3)$

3-joint arm: $S^1 \times S^1 \times S^1 = T^3$

$\mathbb{R}^3 \times SO(3) \times T^3$ = C-space Dim(C-space) = 9

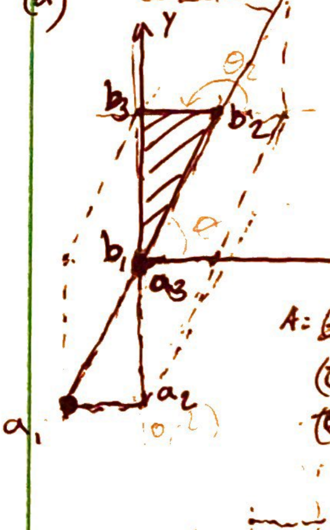
(g) Manipulator w/ 7 joints

- Assuming stationary (no translation)

Joints = $S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1 = T^7$

EXERCISE #5 Obstacle w/ vertices (0,0) (0,2) (1,2)

(a) and robot w/ same shape



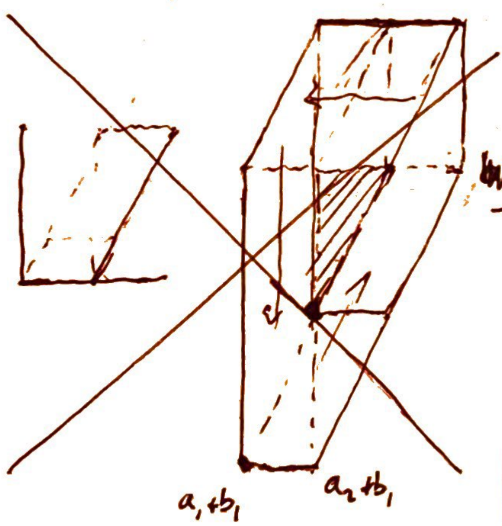
A: ~~(-1, -2)~~ (0, 0)
~~(0, -2)~~ (0, 1)
(0, 0) (1, 2)

$b_1 \xrightarrow{63^\circ} b_2 \xrightarrow{180^\circ} b_3 \xrightarrow{270} b_1$

$a_1 \xrightarrow{0^\circ} a_2 \xrightarrow{90} a_3 \xrightarrow{243} a_1$

(-2, -3)

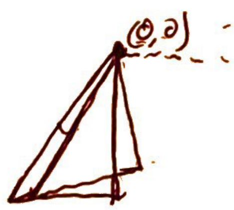
	$a_1 + b_1$	$i=0$ $j=0$
0°	$a_2 + b_1$	$i=1$ $j=0$
63°	$a_2 + b_2$	$i=1$ $j=1$
90°	$a_3 + b_2$	$i=2$ $j=1$
~~243°~~	~~$a_1 + b_2$~~	
180	$a_3 + b_3$	$i=2$ $j=2$
243	$a_1 + b_3$	$i=0$ $j=2$



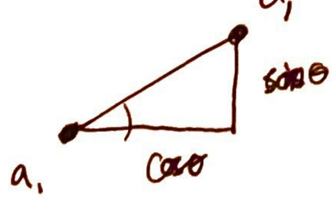
C-Obs = (-2, -1)
(0, -2)
(1, 0)
(1, 2)

(See Minkowski.py for implementation)

(b)



SEE Exercise_5_a.png and Exercise_5_b.png FOR RESULTS



$a_1'[0] = a_1[0] * \cos\theta$

$a_1'[1] = a_1[1] * \sin\theta$

EXERCISE #6 Consider $W \subseteq \mathbb{R}^n$ w/ convex obstacles Show that the c-space obstacles are also convex for convex robot w/ transitional motion in W

- A set A is convex if $\forall x, y \in A$ the line segment $\{tx + (1-ty) \mid t \in [0,1]\}$
- C-Space obstacles created using Minkowski difference & convex hull

Let A be the convex set of robot w/ $a, b \in A$ and B be convex set of obs w/ $c, d \in B$

Show $C = A + B$ has elements $e, f \in C$ s.t.

(1) $\{te + (1-tf) \mid t \in [0,1]\}$

$C = A + B \Rightarrow e = a + c, \quad f = b + d$

(1) can be written as

$\{t(a+c) + 1 - t(b+d) \mid t \in [0,1]\}$

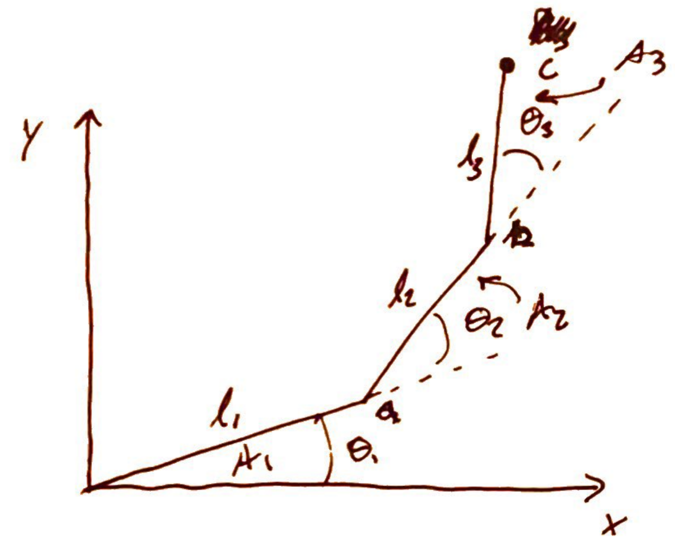
$ta + tc + (1-t)b + (1-t)d$

$\{\underbrace{ta + (1-t)b}_{\text{must be true because } A\ B \text{ convex}} + \underbrace{tc + (1-t)d}_{\text{must be true because } B \text{ is convex}} \mid t \in [0,1]\}$

Thus, $C = A + B$ is convex

Exercise #7 Implement kinematic model for planar 3-link manipulator

- User can specify length of each link as well as configuration
- Output is visual display of configuration as well as exact location of end point on final link



A_1 rel to global:

$$\begin{bmatrix}X_a\\Y_a\\1\end{bmatrix} = T_1\begin{bmatrix}l_1\\0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}\cos\theta_1 & -\sin\theta_1 & 0\\ \sin\theta_1 & \cos\theta_1 & 0\\ 0 & 0 & 1\end{bmatrix} = T_1$$

A_2 rel to ~~global~~ A_1

$$\cancel{\begin{bmatrix}X_b\\Y_b\\1\end{bmatrix}} \cancel{=} \cancel{\begin{bmatrix}R_{\theta_1} & l_1\\0 & 1\end{bmatrix}} \begin{bmatrix}R_{\theta_2} & \cancel{l_2}\\0 & 1\end{bmatrix} \cancel{\begin{bmatrix}l_2\\0\\1\end{bmatrix}} = T_2$$

A_3 rel to ~~global~~ A_2

$$\begin{bmatrix}X_c\\Y_c\\1\end{bmatrix} = \begin{bmatrix}R_{\theta_1} & l_1\\0 & 1\end{bmatrix}\begin{bmatrix}R_{\theta_2} & l_2\\0 & 1\end{bmatrix}\begin{bmatrix}R_{\theta_3} & l_2\\0 & 1\end{bmatrix} = T_3$$

Pos of c relative to A_3

$$T_4 = \begin{bmatrix} 1 & 0 & l_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} X_c \\ Y_c \\ 1 \end{bmatrix} = T_1 T_2 T_3 T_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

NOTES ON PERFORMANCE:

- Fwd kinematic version is nearly instant to execute
- Reverse kinematic version takes several minutes to executed due to implementation
  - ↳ Reverse model also is not very accurate due to discretization of theta