Angular Momentum Addition

Dec. 1, 2016

Final Exam: Dec. 8 (Thursday) 8–11am.
ClassEval: Dec. 5 (M) 8am ends

We discussed both spin $\vec{S}$ and orbital angular momentum $\vec{L}$.
In realistic case, we often have to deal with both $\vec{S}$ and $\vec{L}$.
Typically, we get the total angular momentum $\vec{J} = \vec{S} + \vec{L}$.
In general, we need to discuss how one may add two different angular momenta, $\vec{J}_1$ and $\vec{J}_2$, $\boxed{\vec{J} = \vec{J}_1 + \vec{J}_2}$ as Total Ang. Mom.

<u>Ex</u>. electron-positron system with $H = A\,\vec{S}^{(e^-)} \cdot \vec{S}^{(e^+)}$.

$$j_1 \otimes j_2 = \underbrace{j_1 + j_2,\ j_1 + j_2 - 1,\ \cdots\ |j_1 - j_2|}_{\text{Total } j \text{ allowed}}$$

$$\frac{1}{2} \otimes \frac{1}{2} = \underbrace{1 \oplus 0}_{\text{irreducible representations of the total Hilbert space.}}$$

$\otimes$: outer product of two Hilbert spaces.

e^- (electron)	e^+ (positron)
$\lvert \frac{1}{2} + \frac{1}{2} \rangle = \lvert + \rangle$	$\lvert \frac{1}{2} + \frac{1}{2} \rangle = \lvert + \rangle$
$\lvert \frac{1}{2} - \frac{1}{2} \rangle = \lvert - \rangle$	$\lvert \frac{1}{2} - \frac{1}{2} \rangle = \lvert - \rangle$

$(2j_1 + 1) \times (2j_2 + 1) = 2 \times 2 = 4$

$\lvert ++ \rangle, \lvert +- \rangle, \lvert -+ \rangle, \lvert -- \rangle$: 4 degrees of freedom.

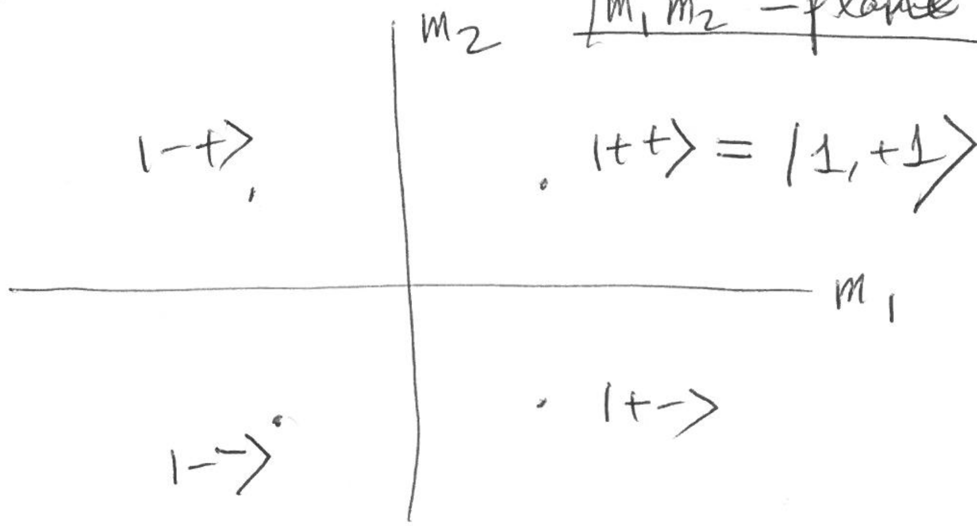
Irreducible representation of total space. ②

$j_1+j_2=1 \qquad |j_1-j_2|=0$

$$\underbrace{\text{Triplet}\begin{cases}|1\,1\,+1\rangle\\ |1\,1\,0\rangle\\ |1\,1\,-1\rangle\end{cases} \qquad |0,0\rangle \Big\}\text{ singlet}}_{\text{4 degrees of freedom.}}$$

$$\sum_{j=|j_1-j_2|}^{j_1+j_2}(2j+1)=\sum_{j=0}^{1}(2j+1)=1+3=4.$$

$\boxed{m_1 m_2 \text{ -plane}}$



Convention
positive real coefficient.
1

Since $\vec{J}=\vec{J}_1+\vec{J}_2$, $\quad J_z=J_{1z}+J_{2z}$ or $m=m_1+m_2$

$J_\pm=J_{1\pm}+J_{2\pm}$.

$$\boxed{J_-|1,+1\rangle=(J_{1-}+J_{2-})|++\rangle}$$

$\hbar$ factors are cancelled between left and right sides.

$$\sqrt{1(1+1)-1(1-1)}\,|1,0\rangle=\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|-+\rangle$$
$$+\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|+-\rangle$$

Thus, one gets

$$|1,0\rangle = \frac{1}{\sqrt{2}}|-+\rangle + \frac{1}{\sqrt{2}}|+-\rangle$$

Clebsch-Gordon Coefficient (CG)

$$\boxed{J_-|1,0\rangle = (J_{1-}+J_{2-})\left(\frac{1}{\sqrt{2}}|-+\rangle + \frac{1}{\sqrt{2}}|+-\rangle\right)}$$

$$\sqrt{1(1+1)-0(0-1)}\,|1,-1\rangle = \frac{1}{\sqrt{2}}\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|--\rangle$$

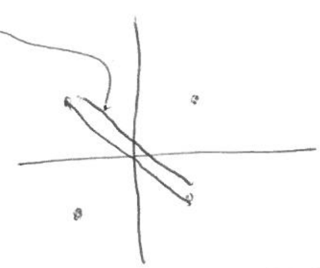
$$+\frac{1}{\sqrt{2}}\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|--\rangle$$

or $\sqrt{2}\,|1,-1\rangle = 2\times\frac{1}{\sqrt{2}}|--\rangle$

i.e. $|1,-1\rangle = |--\rangle$ as expected!

Now, how do we get a singlet state?

$$|00\rangle = a|+-\rangle + b|-+\rangle$$



$$\boxed{J_-|00\rangle = (J_{1-}+J_{2-})(a|+-\rangle + b|-+\rangle)}$$

$$0 = a\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|--\rangle + b\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|--\rangle$$

$$= (a+b)|--\rangle$$

or $a+b=0$; If we applied $J_+|00\rangle = 0$, then we get $(a+b)|++\rangle = 0$.

In any case, using the normalization $a^2+b^2=1$, we get

$a = \frac{1}{\sqrt{2}} > 0, \quad b = -\frac{1}{\sqrt{2}}$

Condon - Shortley's convention.

i.e. $|00\rangle = \frac{1}{\sqrt{2}}|+-\rangle - \frac{1}{\sqrt{2}}|-+\rangle$

Note that $\langle 10|00\rangle = 0$.

The essential factor in Hamiltonian $H = A\vec{S}_1 \cdot \vec{S}_2$ is given by

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2}{2} = \frac{s(s+1) - \frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)}{2}\hbar^2$$

$$= \left[\frac{s(s+1)}{2} - \frac{3}{4}\right]\hbar^2$$

Triplet states; $s = 1$, $E = \frac{A}{4}\hbar^2$

Singlet state; $s = 0$, $E = -\frac{3}{4}A\hbar^2$ (lower energy) $\boxed{\text{more stable}}$

<u>Ex.</u> $j_1 = \frac{3}{2} > j_2 = 1$; $j = \underset{\frac{5}{2}}{j_1+j_2},\ \underset{\frac{3}{2}}{j_1+j_2-1},\ \cdots,\ \underset{\frac{1}{2}}{j_1-j_2}$.

$$(2j_1+1)\times(2j_2+1) = \sum_{j_1-j_2}^{j_1+j_2}(2j+1)$$

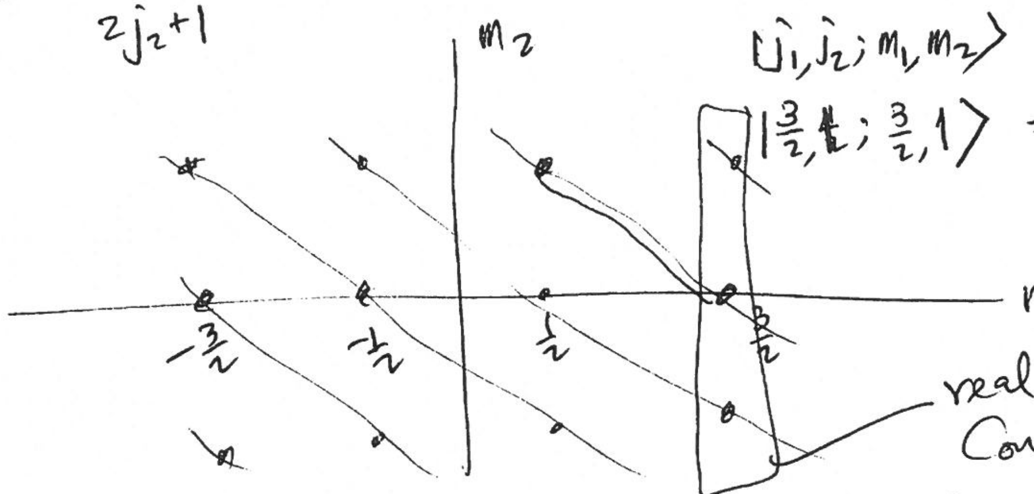
$$4 \times 3 = 6+4+2$$

<u>proof in general</u>

$$\underbrace{2\sum_{j_1-j_2}^{j_1+j_2} j}_{} + \underbrace{\sum_{j_1-j_2}^{j_1+j_2} 1}_{2j_2+1} = (2j_2+1)(2j_1) + (2j_2+1)$$

$$= (2j_2+1)(2j_1+1).$$

$$\begin{array}{rl} & (j_1-j_2)+(j_1-j_2+1)+\cdots+(j_1+j_2) \\ +) & (j_1+j_2)+(j_1+j_2-1)+\cdots+(j_1-j_2) \\ \hline & \underbrace{2j_1 + 2j_1 + \cdots + 2j_1}_{2j_2+1} \end{array}$$



$|j_1, j_2; m_1, m_2\rangle \qquad |j, m\rangle$

$|\frac{3}{2}, 1; \frac{3}{2}, 1\rangle = |\frac{5}{2}, \frac{5}{2}\rangle$

real and <u>positive</u>
Condon-Shortley's convention

$m_1+m_2 = \frac{3}{2}$; $|\frac{5}{2},\frac{3}{2}\rangle = \sqrt{\frac{2}{5}}\,|\frac{3}{2},1;\frac{3}{2},0\rangle + \sqrt{\frac{3}{5}}\,|\frac{3}{2},1;\frac{1}{2},1\rangle$

$|\frac{3}{2},\frac{3}{2}\rangle = \sqrt{\frac{3}{5}}\,|\frac{3}{2},1;\frac{3}{2},0\rangle - \sqrt{\frac{2}{5}}\,|\frac{3}{2},1;\frac{1}{2},1\rangle$.

$m_1 + m_2 = \frac{1}{2}$;

$$|\tfrac{5}{2}\,+\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{10}}\,|\tfrac{3}{2},1;\tfrac{3}{2}\,-1\rangle + \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},1;+\tfrac{1}{2},0\rangle + \sqrt{\tfrac{3}{10}}\,|\tfrac{3}{2},1;-\tfrac{1}{2}+1\rangle$$

$$|\tfrac{3}{2}\,+\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},1;\tfrac{3}{2}-1\rangle + \sqrt{\tfrac{1}{15}}\,|\tfrac{3}{2},1;+\tfrac{1}{2},0\rangle - \sqrt{\tfrac{8}{15}}\,|\tfrac{3}{2},1;-\tfrac{1}{2},+1\rangle$$

$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}}\,|\tfrac{3}{2},1;+\tfrac{3}{2}-1\rangle - \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2},1;+\tfrac{1}{2},0\rangle + \sqrt{\tfrac{1}{6}}\,|\tfrac{3}{2},1;-\tfrac{1}{2},+1\rangle$$

$m_1 + m_2 = -\frac{1}{2}$;

$$|\tfrac{5}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{3}{10}}\,|\tfrac{3}{2},1;+\tfrac{1}{2}-1\rangle + \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},1;-\tfrac{1}{2}\,0\rangle + \sqrt{\tfrac{1}{10}}\,|\tfrac{3}{2},1;-\tfrac{3}{2}+1\rangle$$

$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{8}{15}}\,|\tfrac{3}{2},1;+\tfrac{1}{2}-1\rangle - \sqrt{\tfrac{1}{15}}\,|\tfrac{3}{2},1;-\tfrac{1}{2}\,0\rangle - \sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},1;-\tfrac{3}{2}+1\rangle$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{6}}\,|\tfrac{3}{2},1;+\tfrac{1}{2}-1\rangle - \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2},1;-\tfrac{1}{2}\,0\rangle + \sqrt{\tfrac{1}{2}}\,|\tfrac{3}{2},1;-\tfrac{3}{2}+1\rangle$$

$m_1 + m_2 = -\frac{3}{2}$;

$$|\tfrac{5}{2},-\tfrac{3}{2}\rangle = \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},1;-\tfrac{1}{2}-1\rangle + \sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},1;-\tfrac{3}{2},0\rangle$$

$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = \sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},1;-\tfrac{1}{2}-1\rangle - \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},1;-\tfrac{3}{2},0\rangle$$

$m_1 + m_2 = -\frac{5}{2}$;

$$|\tfrac{5}{2},-\tfrac{5}{2}\rangle = |\tfrac{3}{2},1;-\tfrac{3}{2},-1\rangle$$

Formal Theory of Angular-Momentum Addition

pp. 221–224 (Section 3.8)

$$|j_1 j_2; jm\rangle = \underbrace{\sum_{m_1}\sum_{m_2} |j_1 j_2; m_1 m_2\rangle \overbrace{\langle j_1 j_2; m_1 m_2|j_1 j_2; jm\rangle}^{\text{CG-Coefficients}}}_{\text{Closure Relation}}$$

Convention: ① $j_1 \geq j_2$

② $\langle j_1 j_2; jm|j_1 j_2; m_1 m_2\rangle = \langle j_1 j_2; m_1 m_2|j_1 j_2; jm\rangle$

↑ real

$m_1 = j_1$ case is taken to be positive

Condon-Shortley's convention

Orthogonality conditions:

$$\sum_j \sum_m \langle j_1 j_2; m_1 m_2|j_1 j_2; jm\rangle\langle j_1 j_2; m_1' m_2'|j_1 j_2; jm\rangle = \delta_{m_1 m_1'}\delta_{m_2 m_2'}$$

$$\sum_{m_1}\sum_{m_2} \langle j_1 j_2; m_1 m_2|j_1 j_2; jm\rangle\langle j_1 j_2; m_1 m_2|j_1 j_2; j'm'\rangle = \delta_{jj'}\delta_{mm'}$$

e.g. $\sum_{m_1}\sum_{m_2} |\langle j_1 j_2; m_1 m_2|j_1 j_2; jm\rangle|^2 = 1.$

$RR^T = I$

Orthogonal Matrix

$$\begin{array}{cc|ccc} & j & +5/2 & 3/2 & 1/2 \\ m_1 & m_2 \quad m & +1/2 & +1/2 & +1/2 \\ +3/2 & -1 & & & \\ +1/2 & 0 & & & \\ -1/2 & +1 & & & \end{array}$$

$$\begin{bmatrix} \sqrt{\frac{1}{10}} & \sqrt{\frac{2}{5}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{3}{5}} & \sqrt{\frac{1}{15}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{3}{10}} & -\sqrt{\frac{8}{15}} & \sqrt{\frac{1}{6}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{10}} & \sqrt{\frac{3}{5}} & \sqrt{\frac{3}{10}} \\ \sqrt{\frac{2}{5}} & \sqrt{\frac{1}{15}} & -\sqrt{\frac{8}{15}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{6}} \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

Recursion Relations Among the CG-Coefficients

$$J_\pm |j_1 j_2; jm\rangle = (J_{1\pm} + J_{2\pm}) \sum_{m_1}\sum_{m_2} |j_1 j_2; m_1 m_2\rangle\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle$$

$\hbar$ cancells

$$\sqrt{j(j+1)-m(m\pm1)}\,|j_1 j_2; j\, m\pm1\rangle$$
$$= \sum_{m_1'}\sum_{m_2'}\Big(\sqrt{j_1(j_1+1)-m_1'(m_1'\pm1)}\,|j_1 j_2; m_1'\pm1, m_2'\rangle$$
$$+ \sqrt{j_2(j_2+1)-m_2'(m_2'\pm1)}\,|j_1 j_2; m_1', m_2'\pm1\rangle\Big)$$
$$\times\langle j_1 j_2; m_1' m_2' | j_1 j_2; jm\rangle$$

Taking ~~an~~ inner product with $\langle j_1 j_2; m_1, m_2|$ yields

From $\langle j_1 j_2; m_1 m_2 | j_1 j_2; m_1'\pm1, m_2'\rangle = \delta_{m_1, m_1'\pm1}\delta_{m_2 m_2'}$ we get $m_1'\pm1 = m_1$ or $m_1' = m_1 \mp 1$ and $m_2 = m_2'$

$$\sqrt{j(j+1)-m(m\pm1)}\,\langle j_1 j_2; m_1 m_2 | j_1 j_2; j\, m\pm1\rangle$$
$$= \sqrt{j_1(j_1+1)-m_1(m_1\mp1)}\,\langle j_1 j_2; m_1\mp1, m_2 | j_1 j_2; jm\rangle$$
$$+ \sqrt{j_2(j_2+1)-m_2(m_2\mp1)}\,\langle j_1 j_2; m_1, m_2\mp1 | j_1 j_2; jm\rangle$$

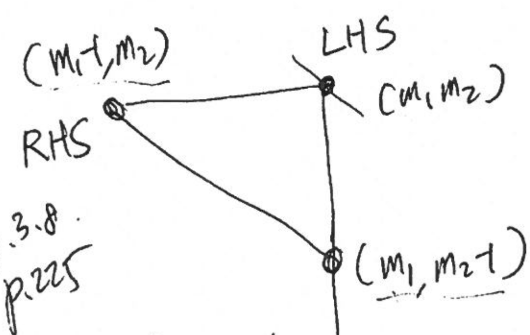
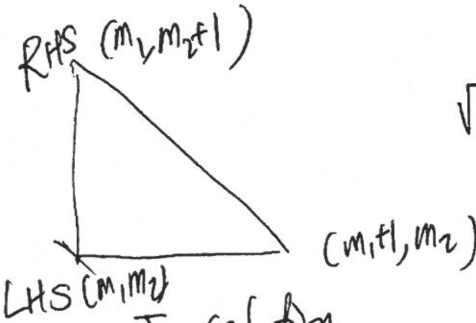


Fig. 3.8
p.225

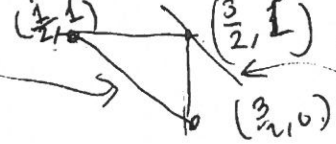
J_+ relation



J_- relation

$$\sqrt{5}\times 1 = \sqrt{3}\times\sqrt{\tfrac{3}{5}} + \sqrt{2}\times\sqrt{\tfrac{2}{5}} = \frac{3+2}{\sqrt{5}} = \sqrt{5}.$$

$j=\frac{5}{2}, m=\frac{3}{2}$



$j=\frac{5}{2}, m+1=\frac{5}{2}$

$$\sqrt{\tfrac{5}{2}(\tfrac{5}{2}+1)-\tfrac{3}{2}(\tfrac{3}{2}+1)}\,\langle \tfrac{3}{2}, 1; \tfrac{3}{2}, 1 | \tfrac{3}{2}, 1; \tfrac{5}{2}, \tfrac{5}{2}\rangle$$
$$= \sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{3}{2}(\tfrac{3}{2}-1)}\,\langle \tfrac{3}{2}, 1; \tfrac{3}{2}-1, 1 | \tfrac{3}{2}, 1; \tfrac{5}{2}, \tfrac{3}{2}\rangle$$
$$+ \sqrt{1(1+1)-1(1-1)}\,\langle \tfrac{3}{2}, 1; \tfrac{3}{2}, 0 | \tfrac{3}{2}, 1; \tfrac{5}{2}, \tfrac{3}{2}\rangle$$

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	…
		M	M	…
m_1	m_2			
m_1	m_2	Coefficients		
⋮	⋮			

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2×1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

3/2×1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

2×1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

3/2×1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2×3/2

		3
		+3
+3/2	+3/2	1

		3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

		3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

		3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

		3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

		3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

		3
		−3
−3/2	−3/2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

		7/2
		+7/2
+2	+3/2	1

		7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

		7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

		7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

		7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

		7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

		7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

		7/2
		−7/2
−2	−3/2	1

2×2

		4
		+4
+2	+2	1

		4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

		4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

		4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

		4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

		4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

		4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

		4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

		4
		−4
−2	−2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.