Wavefunctions in Position and Momentum Space

Sep. 22, 2016

> 1st Exam, Sep. 27
> HW solutions & Lecture notes on Moodle

We introduced continuous Hilbert spaces of position and ~~momentum~~ momentum

ket $\{|x'\rangle\}$, bra $\{\langle x'|\}$; ket $\{|p'\rangle\}$, bra $\{\langle p'|\}$.

We got $\langle x'|p'\rangle$ as $\boxed{\langle x'|p'\rangle = N e^{\frac{ip'x'}{\hbar}}}$

as it satisfies the first-order diff. eq

$$\boxed{\frac{\hbar}{i}\frac{d}{dx'}\langle x'|p'\rangle = p'\langle x'|p'\rangle}$$

$$\langle x'|p|p'\rangle \to p'\langle x'|p'\rangle$$

$$\frac{\hbar}{i}\frac{d}{dx'}\langle x'|p'\rangle = \langle x'|p|p'\rangle$$

Let's find the normalization factor N.

$$\langle x'|\left[\int_{-\infty}^{\infty} dp'\, |p'\rangle\langle p'|\right]|x''\rangle = \langle x'|x''\rangle = \delta(x'-x'')$$

$$= \int_{-\infty}^{+\infty} dp'\, \langle x'|p'\rangle\langle p'|x''\rangle$$

$$= \int_{-\infty}^{\infty} dp'\, N e^{\frac{ip'x'}{\hbar}} N^* e^{\frac{-ip'x''}{\hbar}}$$

$$= |N|^2 \int_{-\infty}^{\infty} dp'\, e^{\frac{ip'(x'-x'')}{\hbar}}$$

Computation of $\int_{-\infty}^{\infty} dp' \, e^{\frac{ip'(x'-x'')}{\hbar}}$

$$\lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda} dp' \, e^{\frac{ip'(x'-x'')}{\hbar}} = \lim_{\Lambda\to\infty}\left[\frac{\hbar}{i(x'-x'')}\, e^{\frac{ip'(x'-x'')}{\hbar}}\right]_{-\Lambda}^{\Lambda}$$

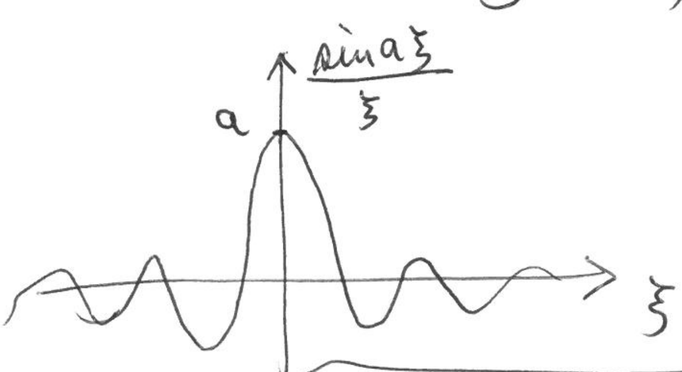
$$= \lim_{\Lambda\to\infty} \frac{\hbar}{i(x'-x'')}\left(e^{\frac{i\Lambda(x'-x'')}{\hbar}} - e^{\frac{-i\Lambda(x'-x'')}{\hbar}}\right)$$

$$= \lim_{\Lambda\to\infty} \frac{2\hbar}{x'-x''} \, \frac{e^{\frac{i\Lambda(x'-x'')}{\hbar}} - e^{\frac{-i\Lambda(x'-x'')}{\hbar}}}{2i}$$

$$= \lim_{\Lambda\to\infty} \frac{2\hbar \sin\left[\frac{\Lambda(x'-x'')}{\hbar}\right]}{x'-x''} = 2\pi\hbar\,\delta(x'-x'')$$

If we set $x'-x'' \equiv \xi$ and $\frac{\Lambda}{\hbar} \equiv a$, then the relevant function is given by

$$f_a(\xi) = \frac{\sin a\xi}{\xi}, \quad \text{where } \int_{-\infty}^{\infty} dp' e^{\frac{ip'\cancel{(x'-x'')}\,\xi}{\hbar}} = 2\hbar \lim_{a\to\infty} f_a(\xi)$$



$$\int_{-\infty}^{\infty} \frac{\sin a\xi}{\xi} d\xi = \int_{-\infty}^{\infty} \frac{\sin \xi}{\xi} d\xi$$

$$= 2\, S_i(\infty) = \pi$$

Sin Integral

$$\boxed{\lim_{a\to\infty} f_a(\xi) = \pi\,\delta(\xi)} \qquad \boxed{S_i(\infty) = \int_0^{\infty} \frac{\sin\xi}{\xi} d\xi = \frac{\pi}{2}}$$

In summary, we get

$$\langle x'|x''\rangle = |N|^2 \int_{-\infty}^{\infty} dp'\, e^{\frac{ip'(x'-x'')}{\hbar}} = |N|^2\, 2\hbar \underbrace{\lim_{a\to\infty} f_a(x'-x'')}_{\pi\delta(x'-x'')}$$

$$= \delta(x'-x'')$$

Thus, $N = \frac{1}{\sqrt{2\pi\hbar}}$ or $\boxed{\langle x'|p'\rangle = \frac{e^{ip'x'/\hbar}}{\sqrt{2\pi\hbar}}}$

Relation between the position and momentum wave functions

$$|\alpha\rangle$$

position wavefunction

momentum wavefunction

$$|\alpha\rangle = \int dx'\, |x'\rangle \langle x'|\alpha\rangle$$

$$|\alpha\rangle = \int dp'\, |p'\rangle \langle p'|\alpha\rangle$$

$$\psi_\alpha(x') = \langle x'|\alpha\rangle = \int dp' \langle x'|p'\rangle\langle p'|\alpha\rangle = \int dp' \langle x'|p'\rangle \underbrace{\phi_\alpha(p')}_{\phi_\alpha(p')}$$

$$\langle\alpha|\alpha\rangle = 1 = \int dx''\, \psi_\alpha^*(x'')\langle x''| \int dx'\, |x'\rangle \psi_\alpha(x')$$

$$= \int dx'\, dx''\, \psi_\alpha^*(x'')\psi_\alpha(x')\,\delta(x''-x')$$

$$= \int dx' \left|\psi_\alpha(x')\right|^2$$

Probability density of finding the state $|\alpha\rangle$ at position x'.

Similarly,

$$\int_{-\infty}^{+\infty} dp' \left|\phi_\alpha(p')\right|^2 = 1$$

Momentum distribution of probability

$$\psi_\alpha(x') = \langle x'|\alpha\rangle$$

$$= \langle x'|\left[\int dp'\, |p'\rangle\langle p'|\right]|\alpha\rangle$$

$$= \int_{-\infty}^{+\infty} dp'\, \langle x'|p'\rangle\langle p'|\alpha\rangle$$

$$\boxed{\psi_\alpha(x') = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} dp'\, e^{\frac{ip'x'}{\hbar}}\, \phi_\alpha(p')}$$ "Fourier Transf."

$$\phi_\alpha(p') = \langle p'|\alpha\rangle$$

$$= \langle p'|\left[\int dx'\, |x'\rangle\langle x'|\right]|\alpha\rangle$$

$$= \int_{-\infty}^{+\infty} dx'\, \langle p'|x'\rangle\langle x'|\alpha\rangle$$

$$\boxed{\phi_\alpha(p') = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} dx'\, e^{\frac{-ip'x'}{\hbar}}\, \psi_\alpha(x')}$$

cf. $$\psi_\alpha(x') = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp'\, e^{\frac{ip'x'}{\hbar}}\, \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx''\, e^{\frac{-ip'x''}{\hbar}}\, \psi_\alpha(x'')$$

$$= \frac{1}{2\pi\hbar}\underbrace{\int_{-\infty}^{\infty} dp' \int_{-\infty}^{\infty} dx''\, e^{\frac{ip'(x'-x'')}{\hbar}}}_{\int_{-\infty}^{\infty} dx''\, 2\pi\hbar\, \delta(x'-x'')}\, \psi_\alpha(x'') = \frac{\cancel{2\pi\hbar}}{\cancel{2\pi\hbar}}\int_{-\infty}^{+\infty} dx''\, \delta(x'-x'')\, \psi_\alpha(x'')$$

$$= \psi_\alpha(x')$$ as it must be!

Gaussian Wave Packets

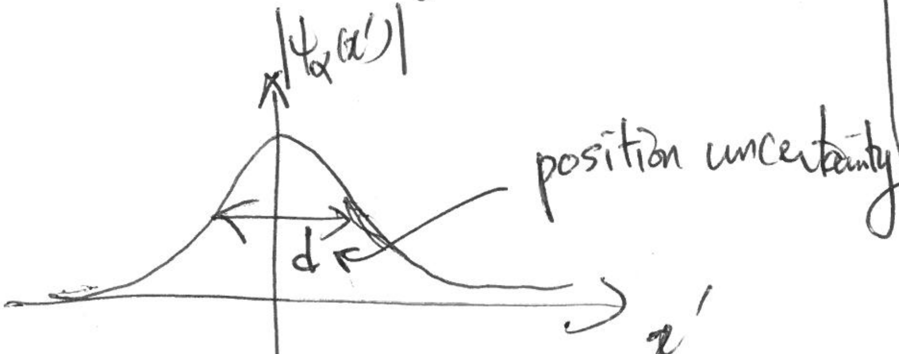
$$\psi_\alpha(x') = \frac{1}{\pi^{1/4}\sqrt{d}}\, e^{-\frac{x'^2}{2d^2}}$$

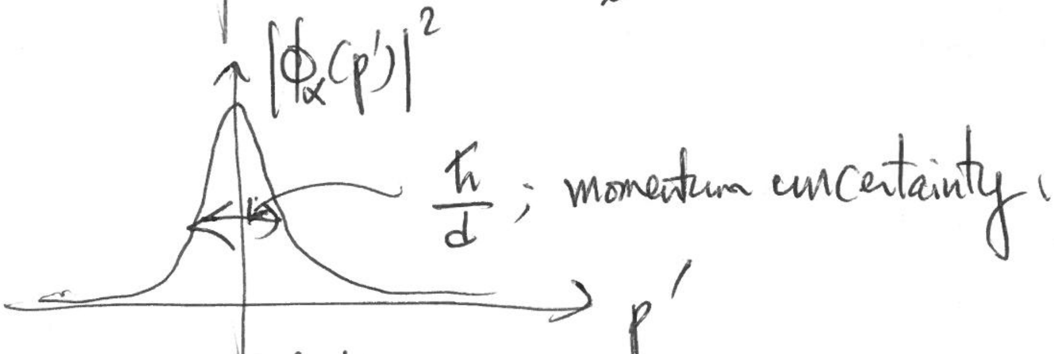
($k=0$ case in Eq. (1.7.35)
Ground state of S.H.O.)

$$\phi_\alpha(p') = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\underbrace{\int_{-\infty}^{\infty} dx'\, \exp\left(\frac{-ip'x'}{\hbar} - \frac{x'^2}{2d^2}\right)}_{\underbrace{\int_{-\infty}^{\infty} dx'\, e^{-\frac{\left(x' + \frac{ip'd^2}{\hbar}\right)^2}{2d^2}}}_{\sqrt{2\pi}\, d}\; e^{-\frac{p'^2 d^2}{2\hbar^2}}}$$

$$= \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\, e^{-\frac{p'^2 d^2}{2\hbar^2}}$$

$$\boxed{\text{cf } \int_0^{\infty} dx'\, e^{-r^2 x'^2} = \frac{\sqrt{\pi}}{2r}}$$





Uncertainty Relation

$$\underbrace{\langle(\Delta X)^2\rangle}_{d^2}\;\underbrace{\langle(\Delta p)^2\rangle}_{\frac{\hbar^2}{d^2}} \geq \frac{1}{4}\left|\langle\underbrace{[x,p]}_{i\hbar}\rangle\right|^2 + \frac{1}{4}\left|\langle\{\Delta X, \Delta p\}\rangle\right|^2$$

$$= \frac{\hbar^2}{4}.$$

For the above Gaussian wave packet,

$\frac{1}{4}|\langle\{x,p\}\rangle|^2$ due to $\langle x\rangle = \langle p\rangle = 0$

$= 0$ (as $\langle\alpha|xp|\alpha\rangle = \frac{i\hbar}{2}$, $\langle\alpha|px|\alpha\rangle = -\frac{i\hbar}{2}$)

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Computation of Expectation Values

$$\langle x \rangle = \langle \alpha | x | \alpha \rangle$$

$$= \langle \alpha | \left[\int_{-\infty}^{\infty} dx'\, x'\, |x'\rangle\langle x'| \right] |\alpha\rangle$$

$$= \int_{-\infty}^{+\infty} dx'\, x'\, \psi_\alpha^*(x')\, \psi_\alpha(x')$$

$$= \int_{-\infty}^{+\infty} dx'\, \underbrace{x'}_{\text{odd}}\, \underbrace{\left| \psi_\alpha(x') \right|^2}_{\frac{1}{\sqrt{\pi}\, d} e^{-\frac{x'^2}{d^2}} \;\leftarrow\; \text{even in } x' \leftrightarrow -x'.}$$

$$= 0$$

Similarly,

$$\langle p \rangle = \langle \alpha | p | \alpha \rangle$$

$$= \int_{-\infty}^{\infty} dp'\, \underbrace{p'}_{\text{odd}}\, \underbrace{\left| \phi_\alpha(p') \right|^2}_{\frac{d}{\hbar\sqrt{\pi}} e^{-\frac{p'^2 d^2}{\hbar^2}} \;\leftarrow\; \text{even}}$$

$$= 0$$

$$\langle \{\Delta x, \Delta p\} \rangle = \langle \{x, p\} \rangle = \langle xp + px \rangle = \langle xp \rangle + \langle xp \rangle^* = 0$$

because $\langle \alpha | xp | \alpha \rangle = \int_{-\infty}^{+\infty} dx'\, x'\, \psi_\alpha^*(x')\, \frac{\hbar}{i} \frac{d}{dx'} \psi_\alpha(x') = \frac{\hbar}{i} \frac{1}{\sqrt{\pi}\, d} \int_{-\infty}^{+\infty} dx'\, x'\, e^{-\frac{x'^2}{2d^2}} \frac{d}{dx'} e^{-\frac{x'^2}{2d^2}}$

"purely imaginary"

$$\langle\alpha|xp|\alpha\rangle = \frac{\hbar}{i}\frac{1}{\sqrt{\pi}d}\int_{-\infty}^{\infty} dx'\, x' e^{-\frac{x'^2}{2d^2}} \underbrace{\left(\frac{d}{dx'} e^{-\frac{x'^2}{2d^2}}\right)}_{-\frac{x'}{d^2} e^{-\frac{x'^2}{2d^2}}}$$

$$= -\frac{\hbar}{i\sqrt{\pi}d^3}\underbrace{\int_{-\infty}^{\infty} dx'\, x'^2 e^{-\frac{x'^2}{d^2}}}_{2\cdot\frac{\sqrt{\pi}}{4\left(\frac{1}{d}\right)^3} = \frac{\sqrt{\pi}d^3}{2}}$$

$$= \frac{i\hbar}{2},$$

$$\langle x^2\rangle = \int_{-\infty}^{+\infty} dx'\, x'^2 \left|\psi_\alpha(x')\right|^2$$

$$= \frac{1}{\sqrt{\pi}d}\underbrace{\int_{-\infty}^{\infty} dx'\, x'^2 e^{-\frac{x'^2}{d^2}}}_{\frac{\sqrt{\pi}d^3}{2}}$$

$$= \frac{d^2}{2}$$

$$\langle p^2\rangle = \int_{-\infty}^{+\infty} dp'\, p'^2 \left|\phi_\alpha(p')\right|^2 = \frac{d}{\sqrt{\pi}\hbar}\underbrace{\int_{-\infty}^{+\infty} dp'\, p'^2 e^{-\frac{p'^2 d^2}{\hbar^2}}}_{2\cdot\frac{\sqrt{\pi}}{4\left(\frac{d}{\hbar}\right)^3} = \frac{\sqrt{\pi}\hbar^3}{2d^3}}$$

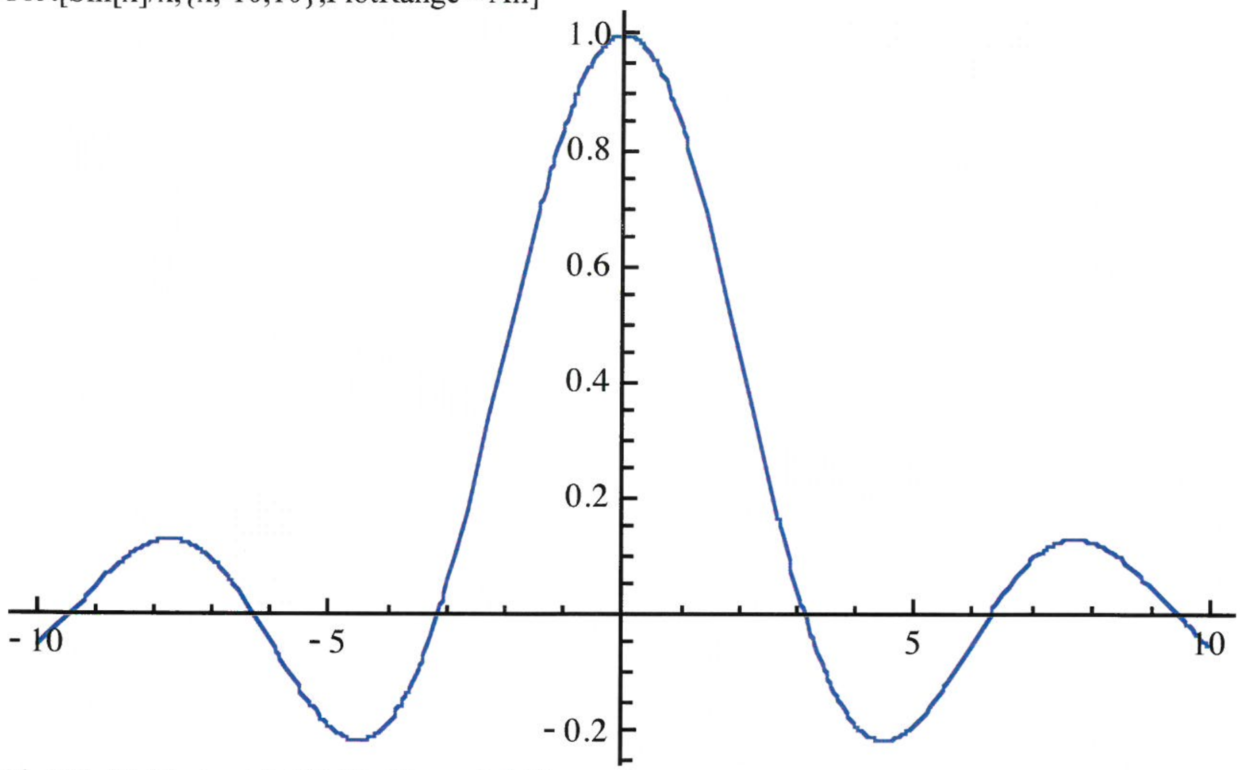
$$= \frac{\hbar^2}{2d^2}$$

as $\langle x\rangle = \langle p\rangle = 0$

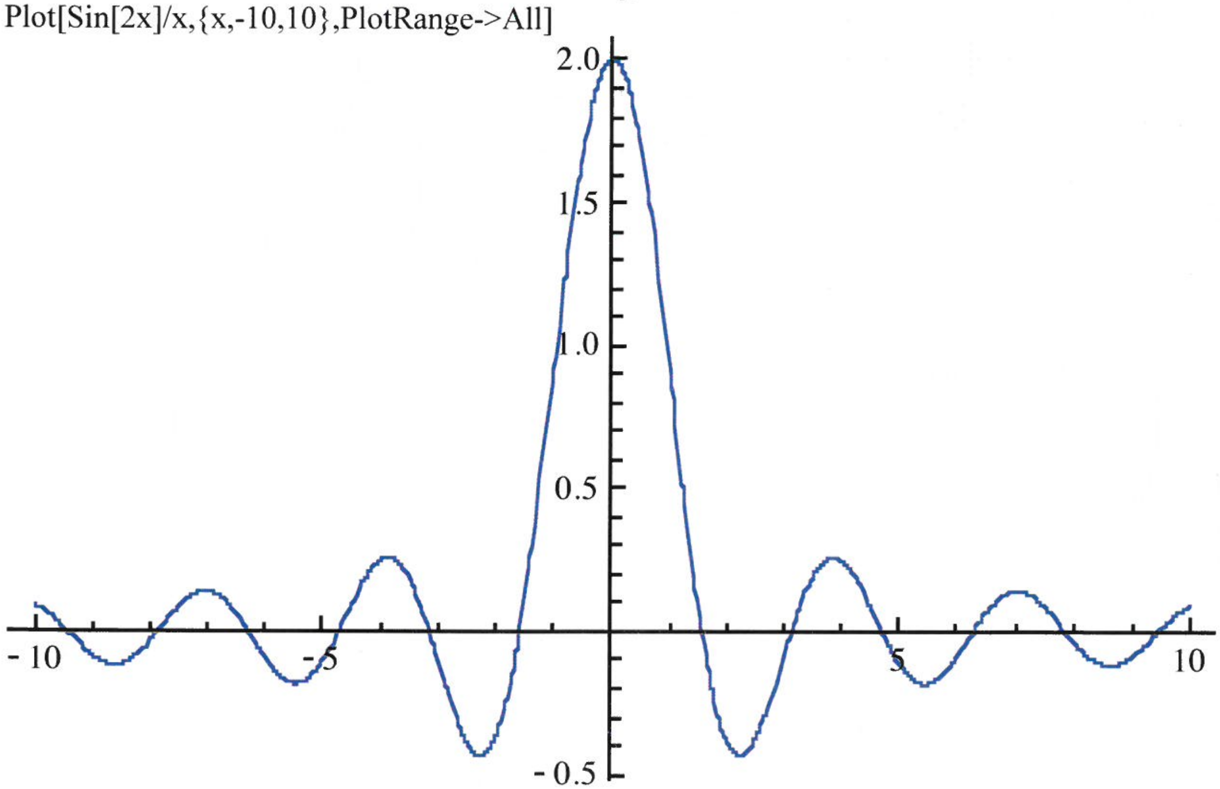
$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \langle x^2\rangle\langle p^2\rangle = \frac{d^2}{2}\cdot\frac{\hbar^2}{2d^2} = \frac{\hbar^2}{4} \geq \frac{1}{4}\left|\langle\underbrace{[x,p]}_{i\hbar}\rangle\right|^2$$

Minimum uncertainty.

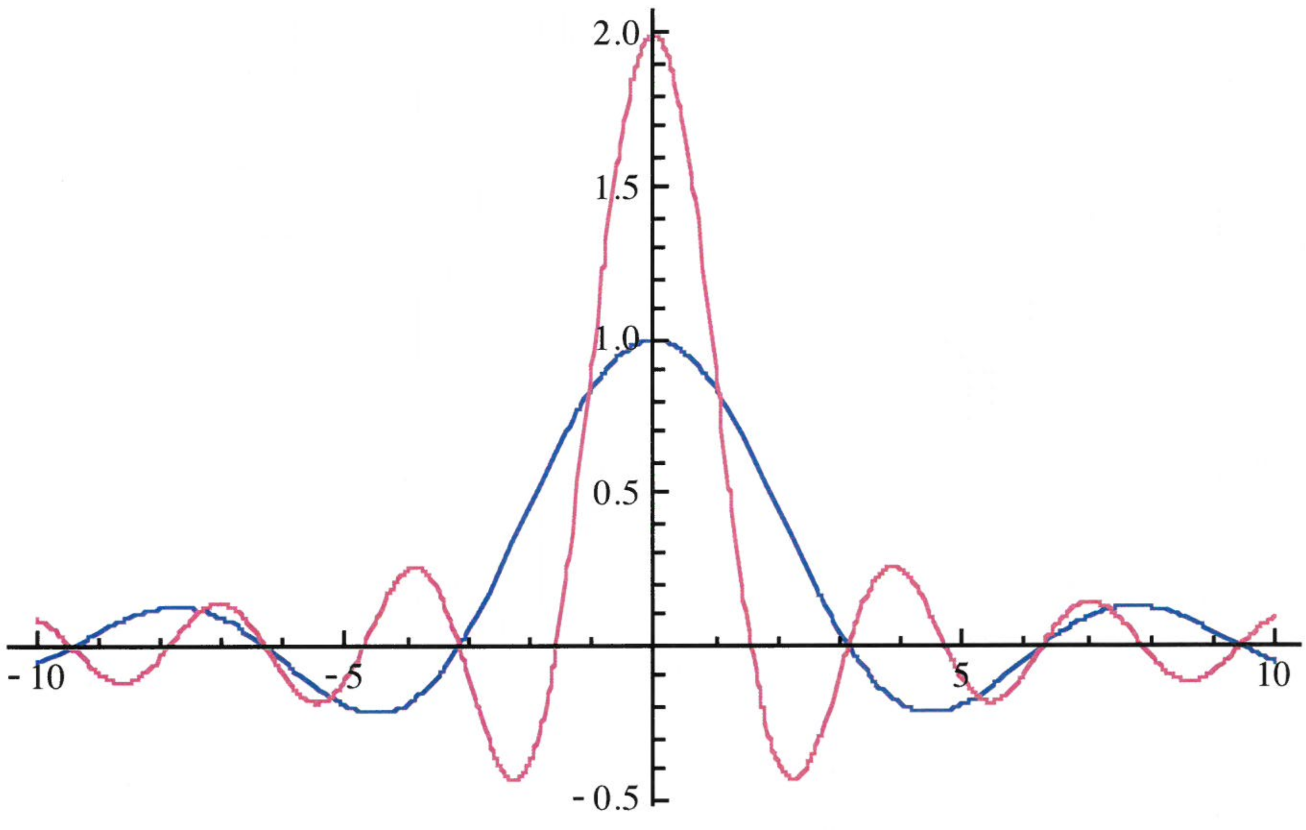
Plot[Sin[x]/x,{x,-10,10},PlotRange->All]



Plot[Sin[2x]/x,{x,-10,10},PlotRange->All]

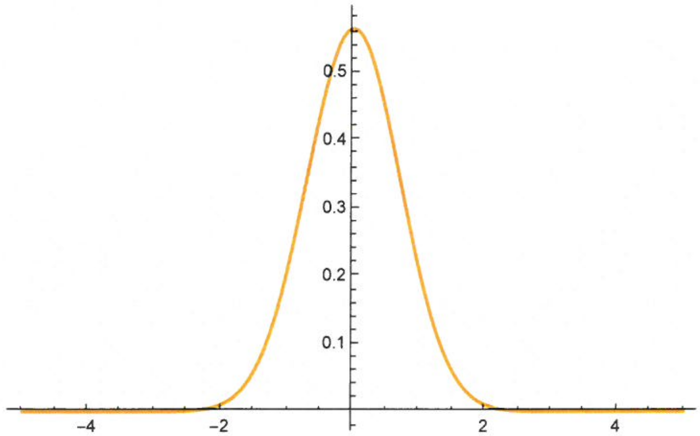


Plot[{Sin[x]/x,Sin[2x]/x},{x,-10,10},PlotRange->All]



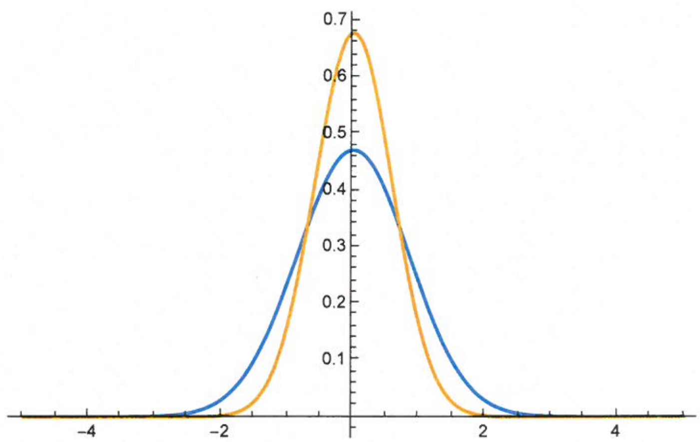
In[88]:= `Plot[{psi[xi, 1.0]^2, phi[xi, 1.0]^2}, {xi, -5, 5}, PlotRange → All]`

Out[88]=



In[90]:= `Plot[{psi[xi, 1.2]^2, phi[xi, 1.2]^2}, {xi, -5, 5}, PlotRange → All]`

Out[90]=



In[91]:= `Plot[{psi[xi, 1.4]^2, phi[xi, 1.4]^2}, {xi, -5, 5}, PlotRange → All]`

Out[91]=