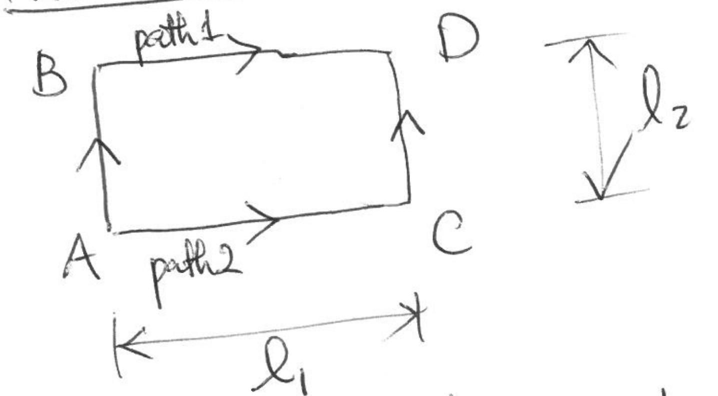
Ahronov - Bohm Effect

Nov. 8, 2016

$\boxed{\text{HW\#7 Due by Nov. 15 (Turn in Murat's mailbox) No classes on Nov. 15 \& 17 due to travel}}$

We've discussed the gravity-induced quantum interference as a simple application of path-integral formulation.

Reminder



$$|\Psi|^2 = 2\rho\{1+\cos(\phi_1-\phi_2)\},$$

where

$$\phi_1-\phi_2 = \frac{S_{ABD}-S_{ACD}}{\hbar}$$

As the classical action can be obtained by

$$S = \int \underbrace{L}_{\int dL} dt = \int\left[\int\left(\frac{\partial L}{\partial x}\right)dx\right]dt \xrightarrow{\text{via Lag. eq.}} = \int\left[\int \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{x}}\right)}_{p}\right]dx\,dt$$

$$= \int\Big(\underbrace{\int \frac{dp}{dt}\cdot dt}_{p}\Big)dx$$

$$= \int p\cdot dx .$$

Note that we had previously

$$\vec{j} = \frac{\rho\vec\nabla S}{m} \quad \text{for } \psi = \sqrt{\rho}\,e^{\frac{iS}{\hbar}}$$

$$\boxed{\vec{p} = \vec\nabla S}$$

As $E = \frac{p^2}{2m} + V(x)$, taking $V(x)=0$ for the path 2, we get

$$S_{ABD} - S_{ACD} = \left(\sqrt{2m(E-V)} - \underbrace{\sqrt{2mE}}_{p\,=\,\frac{\hbar}{\lambda\!\!\bar{}}}\right)\cdot l_1$$

$$S_{ABD} - S_{ACD} = \left(\sqrt{P^2 - 2mV} - P\right)\cdot l_1$$

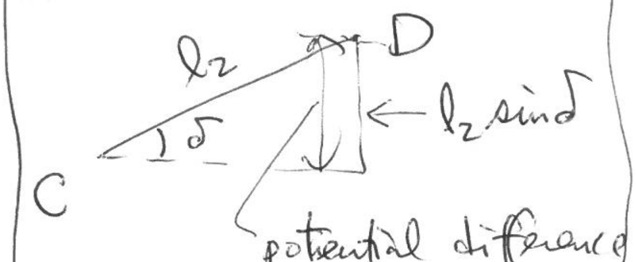
$$= \left\{ \underbrace{\left(1 - \frac{2mV}{P^2}\right)^{\frac{1}{2}}}_{\approx 1 - \frac{mV}{P^2}} - 1 \right\} \cdot P\, l_1$$

$$\simeq -\frac{m}{P} V \cdot l_1$$

$$= -\frac{m}{\left(\frac{\hbar}{\lambda}\right)} (mg\, l_2 \sin\delta)\, l_1$$

$$= -\frac{m^2 g\, l_1 l_2 \lambda \sin\delta}{\hbar}$$

$$V = mg\, l_2 \sin\delta$$



$V = mg\, l_2 \sin\delta$.

Thus, one gets

$$\phi_1 - \phi_2 = -\frac{m^2 g\, l_1 l_2 \lambda \sin\delta}{\hbar^2} \qquad \text{Eq. (2.7.17), p. 134.}$$

In the expt, $\lambda = 1.42\,\text{Å}$, $l_1 l_2 = 10\,\text{cm}^2$.

so that

$$\frac{m_n^2 g\, l_1 l_2 \lambda}{\hbar^2} \simeq 55.6.$$

$|\Psi|^2$ has 9 peaks of maxima from the term of

$$\cos(\phi_1 - \phi_2) = \cos(55.6 \sin\delta) = \cos\left(\underbrace{\frac{55.6}{2\pi}}_{\simeq 9} \cdot 2\pi \sin\delta\right)$$

as δ changes from $0°$ to $90°$.

Let's now turn to potentials that appear in electromagnetism.

Similar to $V(\vec{x}) \longrightarrow \tilde{V}(\vec{x}) = V(\vec{x}) + V_0$ discussed in gravity, the scalar and vector potentials in electromagnetism undergo the <u>local gauge transformations</u>:

$$\phi(\vec{x},t) \rightarrow \tilde{\phi}(\vec{x},t) = \phi(\vec{x},t) - \frac{1}{c}\frac{\partial \Lambda(\vec{x},t)}{\partial t}$$

$$\vec{A}(\vec{x},t) \longrightarrow \tilde{\vec{A}}(\vec{x},t) = \vec{A}(\vec{x},t) + \vec{\nabla}\Lambda(\vec{x},t).$$

Under these local gauge transformations, $\vec{E}$ and $\vec{B}$ fields are invariant and so the Maxwell's eqs given by $\vec{E}$ & $\vec{B}$ fields (more precisely, the space and time first order derivatives of $\vec{E}$ & $\vec{B}$).

$$\vec{E}(\vec{x},t) \rightarrow \tilde{\vec{E}}(\vec{x},t) = -\vec{\nabla}\tilde{\phi} - \frac{1}{c}\frac{\partial \tilde{\vec{A}}}{\partial t}$$

$$= -\vec{\nabla}\left(\phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\Lambda\right)$$

$$= -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$= \vec{E}(\vec{x},t)$$

$$\vec{E}(\vec{x},t) = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B}(\vec{x},t) \longrightarrow \tilde{\vec{B}}(\vec{x},t) = \vec{\nabla}\times\tilde{\vec{A}}$$

$$= \vec{\nabla}\times\left(\vec{A} + \vec{\nabla}\Lambda\right)$$

$$= \vec{\nabla}\times\vec{A}$$

$$= \vec{B}(\vec{x},t).$$

$$\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}$$

What happens to the quantum state $|\alpha,t\rangle$ under the local gauge transformation?

$$|\alpha,t\rangle \longrightarrow |\widetilde{\alpha,t}\rangle = e^{\frac{ie\Lambda(\vec{x},t)}{\hbar c}}|\alpha,t\rangle$$

$$\equiv U_\Lambda(\vec{x},t)|\alpha,t\rangle,$$

where $U_\Lambda(\vec{x},t) = e^{\frac{ie\Lambda(\vec{x},t)}{\hbar c}}$ is called $U(1)$ local gauge transformation:

$$\underline{U_\Lambda^\dagger U_\Lambda = I}$$

U (unitary) (1) (order 1) (single gauge degree of freedom $\Lambda(\vec{x},t)$)

cf. p. 139 Eq. (2.7.47)

$$|\tilde{\alpha}\rangle = \mathcal{G}|\alpha\rangle$$

While the expectation value of the position operator is invariant under this local gauge transformation, i.e.

$$\langle\alpha,t|\,\vec{x}\,|\alpha,t\rangle \longrightarrow \langle\widetilde{\alpha,t}|\,\vec{x}\,|\widetilde{\alpha,t}\rangle = \langle\alpha,t|\,\vec{x}\,|\alpha,t\rangle$$

the expectation value of the canonical momentum operator is <u>not</u> invariant under the same local gauge transformation:

$$\langle\alpha,t|\,\vec{p}\,|\alpha,t\rangle \longrightarrow \langle\widetilde{\alpha,t}|\,\vec{p}\,|\widetilde{\alpha,t}\rangle$$

$$= \langle\alpha,t|\,U_\Lambda^\dagger\,\underbrace{\vec{p}\,U_\Lambda}_{[\vec{p},\,U_\Lambda]+U_\Lambda\vec{p}}\,|\alpha,t\rangle$$

$$[\vec{p},U_\Lambda] = \frac{\hbar}{i}\vec{\nabla}U_\Lambda = \frac{\hbar}{i}\left(\frac{ie}{\hbar c}\vec{\nabla}\Lambda\right)U_\Lambda = \frac{e}{c}(\vec{\nabla}\Lambda)U_\Lambda$$

$$= \langle\alpha,t|\vec{p}|\alpha,t\rangle + \frac{e}{c}\langle\alpha,t|\vec{\nabla}\Lambda|\alpha,t\rangle$$

Thus, one may redefine the momentum operator as the so-called kinematical (or mechanical) momentum operator $\vec{\Pi}$ which is invariant under the local gauge transformation.

$$\boxed{\vec{\Pi} \equiv \vec{p} - \frac{e}{c}\vec{A}}$$

known as so-called "minimal substitution"

Then,

$$\langle \alpha,t | \vec{\Pi} | \alpha,t \rangle \longrightarrow \widetilde{\langle \alpha,t |} \vec{p} - \frac{e}{c}\tilde{\vec{A}} \,\widetilde{| \alpha,t \rangle}$$

$$= \langle \alpha,t | U_\Lambda^\dagger \vec{p}\, U_\Lambda | \alpha,t \rangle - \frac{e}{c}\langle \alpha,t | U_\Lambda^\dagger (\vec{A} + \vec{\nabla}\Lambda) U_\Lambda | \alpha,t \rangle$$

$$= \langle \alpha,t | \vec{p} | \alpha,t \rangle + \frac{e}{c}\cancel{\langle \alpha,t | \vec{\nabla}\Lambda | \alpha,t \rangle}$$

$$- \frac{e}{c}\langle \alpha,t | \vec{A} | \alpha,t \rangle - \frac{e}{c}\cancel{\langle \alpha,t | \vec{\nabla}\Lambda | \alpha,t \rangle}$$

$$= \langle \alpha,t | \vec{p} - \frac{e}{c}\vec{A} | \alpha,t \rangle .$$

or $\langle \alpha,t | \vec{\Pi} | \alpha,t \rangle \longrightarrow \widetilde{\langle \alpha,t |} \tilde{\vec{\Pi}} \widetilde{| \alpha,t \rangle} = \langle \alpha,t | \vec{\Pi} | \alpha,t \rangle$

invariant under the local gauge transformation.

or $U_\Lambda^\dagger \tilde{\vec{\Pi}} U_\Lambda = \vec{\Pi}$, $U_\Lambda^\dagger \tilde{\vec{\Pi}}^2 U_\Lambda = \vec{\Pi}^2$, etc.

Similarly, the minimal substitution applies to the Hamiltonian with the scalar potential, i.e.

$$H \longrightarrow H - e\phi$$

$$\langle \alpha,t | H - e\phi | \alpha,t \rangle \longrightarrow \widetilde{\langle \alpha,t |} H - e\tilde{\phi} \widetilde{| \alpha,t \rangle} = \langle \alpha,t | H - e\phi | \alpha,t \rangle$$

(*) see the proof in the next page.

invariant!

$$\langle \widetilde{\alpha,t} | H - e\tilde{\phi} | \widetilde{\alpha,t} \rangle$$

$$= \langle \alpha,t | U_\Lambda^\dagger H U_\Lambda | \alpha,t \rangle - e \langle \alpha,t | U_\Lambda^\dagger \tilde{\phi} U_\Lambda | \alpha,t \rangle$$

$$= \langle \alpha,t | U_\Lambda^\dagger \underbrace{i\hbar \frac{\partial}{\partial t} U_\Lambda | \alpha,t \rangle}_{U_\Lambda \left(-\frac{e}{c}\frac{\partial \Lambda}{\partial t} + \underbrace{i\hbar \frac{\partial}{\partial t}}_{H} \right) | \alpha,t \rangle} - e \langle \alpha,t | U_\Lambda^\dagger \left(\phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} \right) U_\Lambda | \alpha,t \rangle \qquad U_\Lambda = e^{\frac{ie\Lambda}{\hbar c}}$$

$$= \cancel{-\frac{e}{c} \langle \alpha,t | \frac{\partial \Lambda}{\partial t} | \alpha,t \rangle} + \langle \alpha,t | H | \alpha,t \rangle$$

$$- e \langle \alpha,t | \phi | \alpha,t \rangle + \cancel{\frac{e}{c} \langle \alpha,t | \frac{\partial \Lambda}{\partial t} | \alpha,t \rangle}$$

$$= \langle \alpha,t | H - e\phi | \alpha,t \rangle$$ as expected.

In summary, the minimal substitition saves the local gauge invariance on the quantum mechanical system under the influence of the electromagnetic fields.

$$\left.\begin{aligned} \frac{H - e\phi}{c} &= p^0 - \frac{e}{c} A^0 \\ \vec{\Pi} &= \vec{p} - \frac{e}{c}\vec{A} \end{aligned}\right\} \quad p^\mu - \frac{e}{c} A^\mu$$ (covariant notation)

Let's consider a particle of mass m and electric charge e under E&M fields. Then the corresponding Hamiltonian takes the minimal substitution from the free Hamiltonian:

$$\boxed{H = \frac{\vec{p}^2}{2m} \longrightarrow H - e\phi = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m}}$$

$$\boxed{H = \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} + e\phi}$$

Schrödinger Eq.

$$\left[\frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} + e\phi\right]|\alpha, t\rangle = i\hbar\frac{\partial}{\partial t}|\alpha, t\rangle$$

is invariant under the U(1) local gauge transformation.

$$\left[\frac{(\vec{p} - \frac{e}{c}\tilde{\vec{A}})^2}{2m} + e\tilde{\phi}\right]|\widetilde{\alpha, t}\rangle = i\hbar\frac{\partial}{\partial t}|\widetilde{\alpha, t}\rangle$$

cf $\frac{d\vec{p}}{dt} = -\vec{\nabla}V \Leftrightarrow \frac{d\vec{p}}{dt} = -\vec{\nabla}\tilde{V}$

Ex $\tilde{\psi}(\vec{x}, t) = \langle\vec{x}|\widetilde{\alpha, t}\rangle = e^{\frac{ie\Lambda}{\hbar c}}\langle\vec{x}|\alpha, t\rangle = e^{\frac{ie\Lambda}{\hbar c}}\psi(\vec{x}, t)$

As $\psi(\vec{x}, t) = \sqrt{\rho}\, e^{\frac{iS}{\hbar}}$, $\tilde{\psi}(\vec{x}, t) = \sqrt{\rho}\, e^{\frac{i}{\hbar}(S + \frac{e}{c}\Lambda)}$

or $S \to \tilde{S} = S + \frac{e}{c}\Lambda$ (Eq. 2.7.56), p. 140.

$$\rho = |\tilde{\psi}(\vec{x}, t)|^2 = |\psi(\vec{x}, t)|^2$$

$$\vec{j} = \frac{\rho\vec{\nabla}S}{m} \to \tilde{\vec{j}} = \frac{\rho(\vec{\nabla}\tilde{S} - \frac{e}{c}\tilde{\vec{A}})}{m} = \frac{\rho(\vec{\nabla}(S + \frac{e}{c}\Lambda) - \frac{e}{c}(\vec{A} + \vec{\nabla}\Lambda))}{m}$$

$$= \frac{\rho(\vec{\nabla}S - \frac{e}{c}\vec{A})}{m}$$

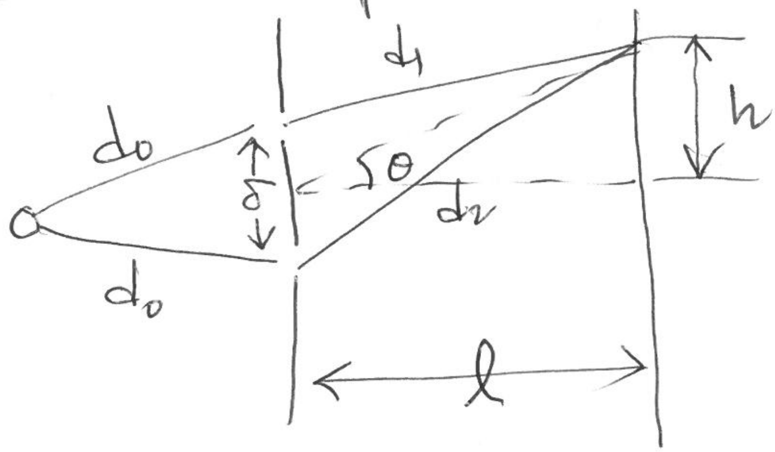
$$= \frac{\rho\vec{p}}{m}$$

$\vec{p} = \vec{\nabla}S \to \vec{\pi} = \vec{\nabla}S - \frac{e}{c}\vec{A}$

i.e. $\vec{j} = \frac{\rho\vec{\pi}}{m}$

under the influence of electromagnetic fields.

Double-slit Expt. without any electromagnetic field



$$d_1 = \sqrt{l^2 + (h - \tfrac{\delta}{2})^2} \approx \sqrt{l^2 + h^2} - \frac{\delta}{2} \cdot \frac{h}{\sqrt{l^2 + h^2}}$$

$$= \sqrt{l^2 + h^2} - \frac{\delta}{2} \sin\theta$$

$$d_2 = \sqrt{l^2 + (h + \tfrac{\delta}{2})^2} \approx \sqrt{l^2 + h^2} + \frac{\delta}{2} \sin\theta$$

Path integral

$$S = \int \vec{p} \cdot d\vec{x} \qquad p = \frac{h}{\lambda} = \frac{\hbar}{\left(\frac{\lambda}{2\pi}\right)} = \frac{2\pi\hbar}{\lambda}$$

$$S_1 = p(d_0 + d_1) = \frac{2\pi\hbar}{\lambda} d_0 + \frac{2\pi\hbar}{\lambda} d_1$$

$$S_2 = p(d_0 + d_2) = \frac{2\pi\hbar}{\lambda} d_0 + \frac{2\pi\hbar}{\lambda} d_2$$

Thus, the relative phase difference between the two paths is given by

$$\phi_2 - \phi_1 = \frac{S_2 - S_1}{\hbar} = \frac{2\pi\cancel{\hbar}}{\lambda\cancel{\hbar}}(d_2 - d_1)$$

$$= \frac{2\pi}{\lambda} \delta \sin\theta .$$

$$|\Psi|^2 = 2\rho\{1+\cos(\phi_2-\phi_1)\}$$

$$= 2\rho\underbrace{\left(1+\cos\frac{2\pi\delta\sin\theta}{\lambda}\right)}_{2\cos^2\frac{\cancel{2}\pi\delta\sin\theta}{\cancel{2}\lambda}}$$

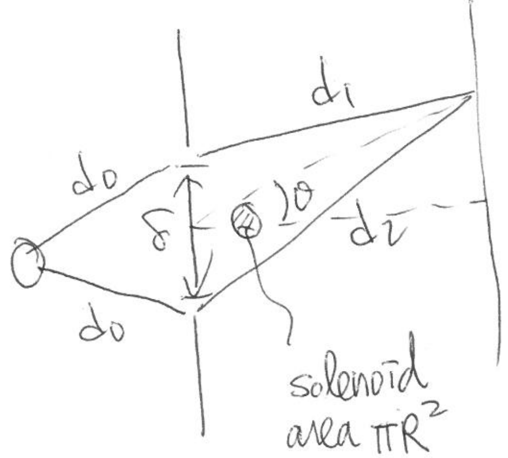
$$\boxed{I(\theta) = 4I_0\cos^2\frac{\pi\delta\sin\theta}{\lambda}}$$

Intensity observed without any Electromagnetic fields.

Now, under the influence of the electromagnetic fields, this interference pattern should change according the path integral due to the minimal substitution: i.e.

$$\vec{\Pi} = \vec{\nabla}S - \frac{e}{c}\vec{A} \qquad S = \int\vec{\Pi}\cdot d\vec{x} + \frac{e}{c}\int\vec{A}\cdot d\vec{x}$$

$$\phi_2-\phi_1 = \frac{S_2-S_1}{\hbar} = \frac{2\pi}{\lambda}\underbrace{\delta\sin\theta}_{d_2-d_1} + \frac{e}{\hbar c}\underbrace{\left[\int_{\text{path 2}} d\vec{x}\cdot\vec{A} - \int_{\text{path 1}} d\vec{x}\cdot\vec{A}\right]}$$



$$\oint d\vec{x}\cdot\vec{A} = \int(\vec{\nabla}\times\vec{A})\cdot d\vec{\sigma}$$

$$= \int\vec{B}\cdot d\vec{\sigma} = \pi R^2|\vec{B}|$$

$\frac{1+\cos(\phi_2-\phi_1)}{2}$

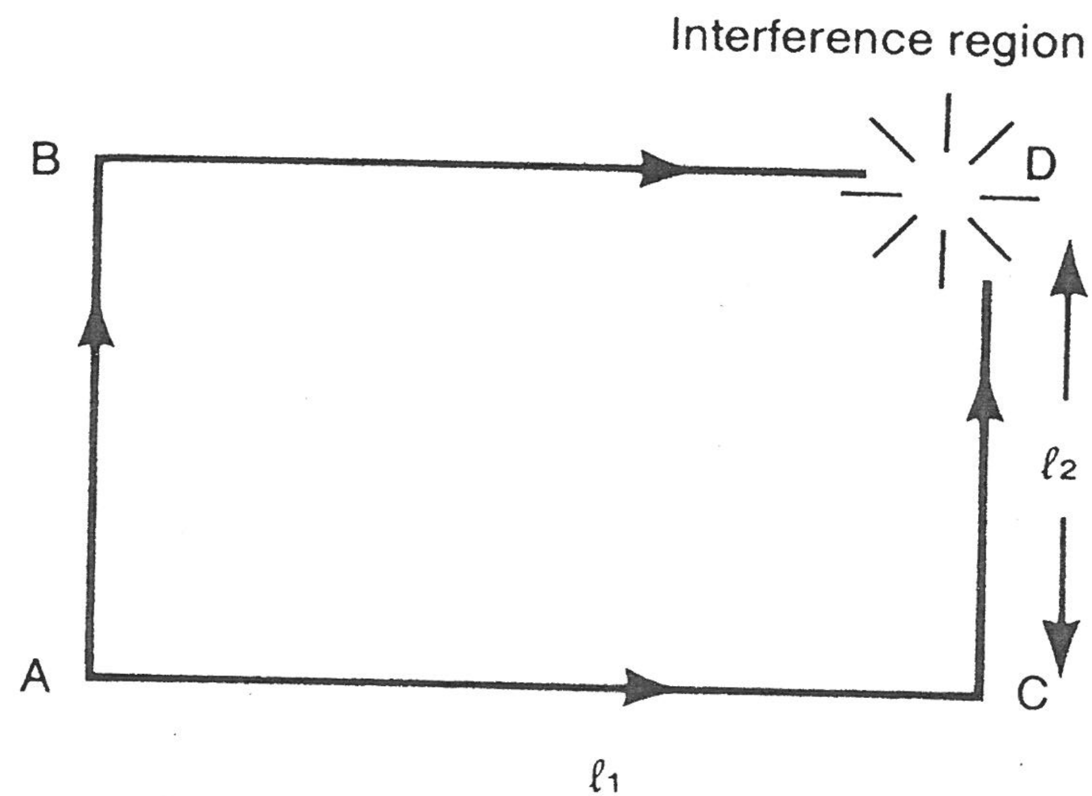
$\cos^2\frac{\phi_2-\phi_1}{2}$

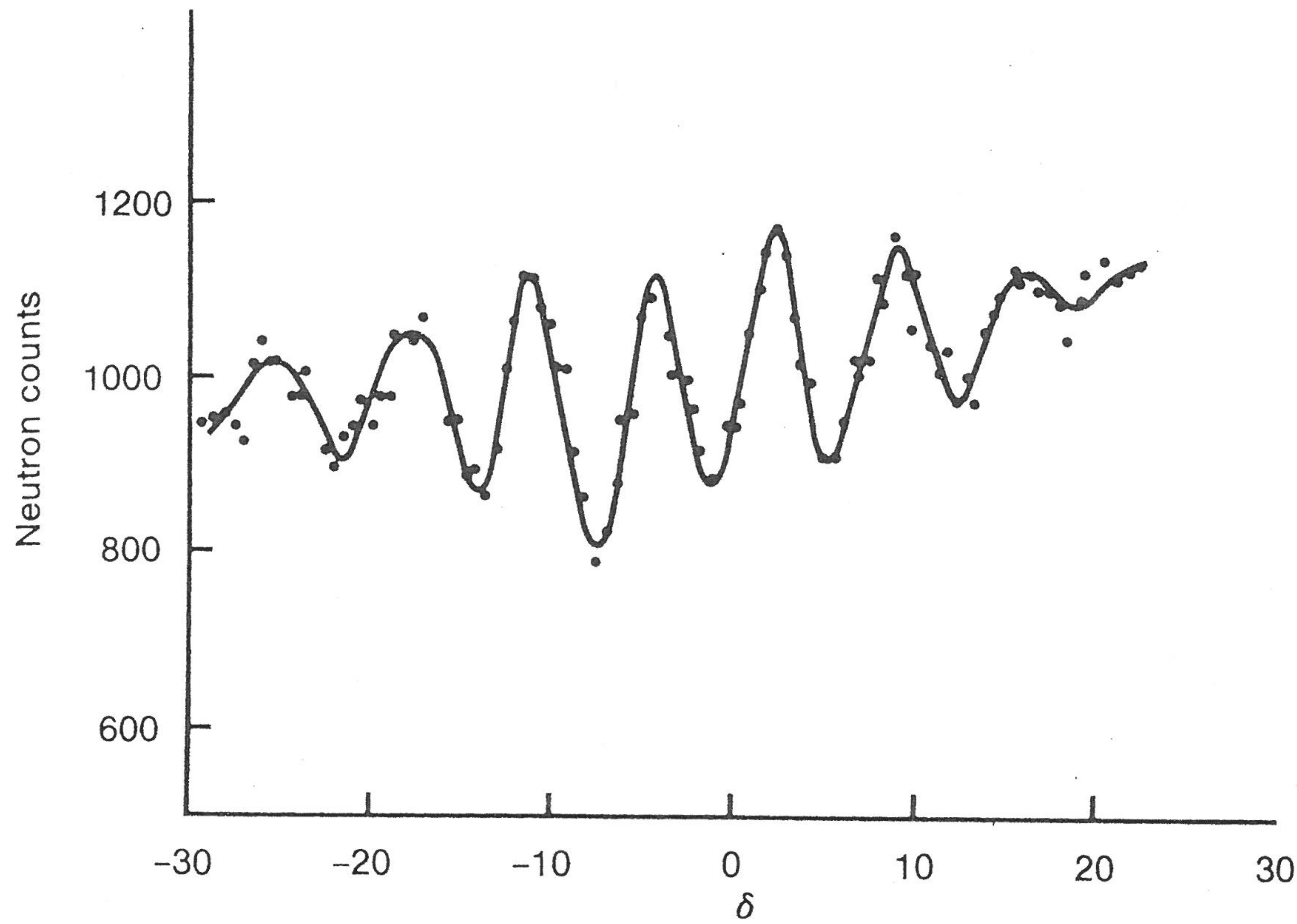
$$I(\theta) = 4I_0\cos^2\left(\frac{\pi\delta\sin\theta}{\lambda} + \frac{e|\vec{B}|\pi R^2}{2\hbar c}\right)$$

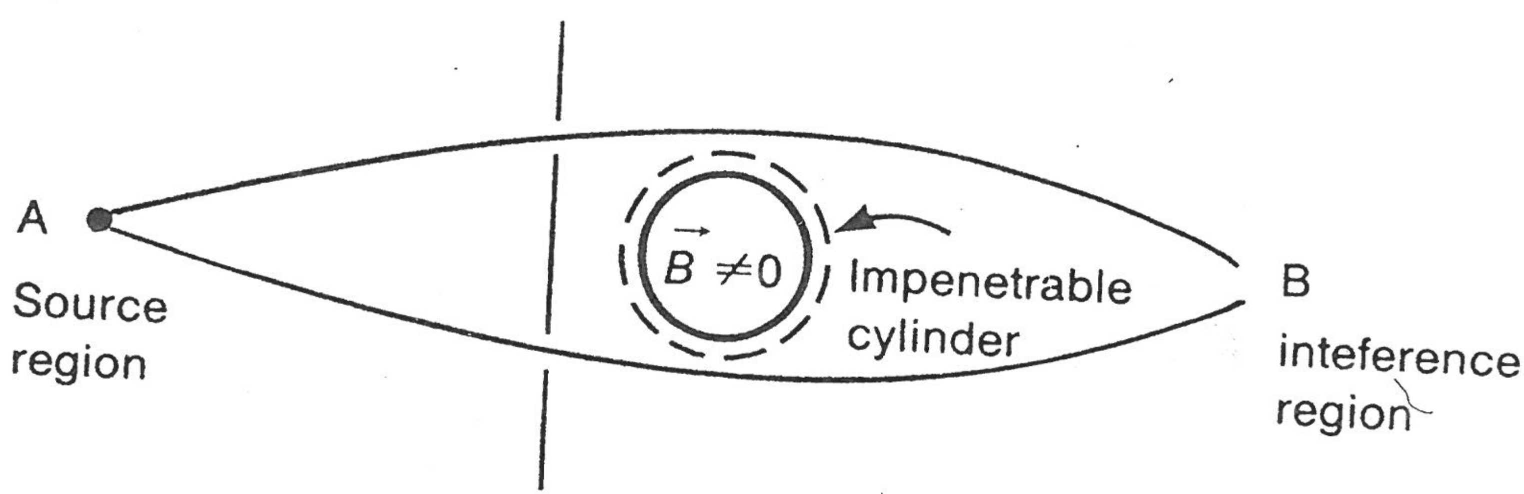
Φ_B magnetic flux inside solenoid.

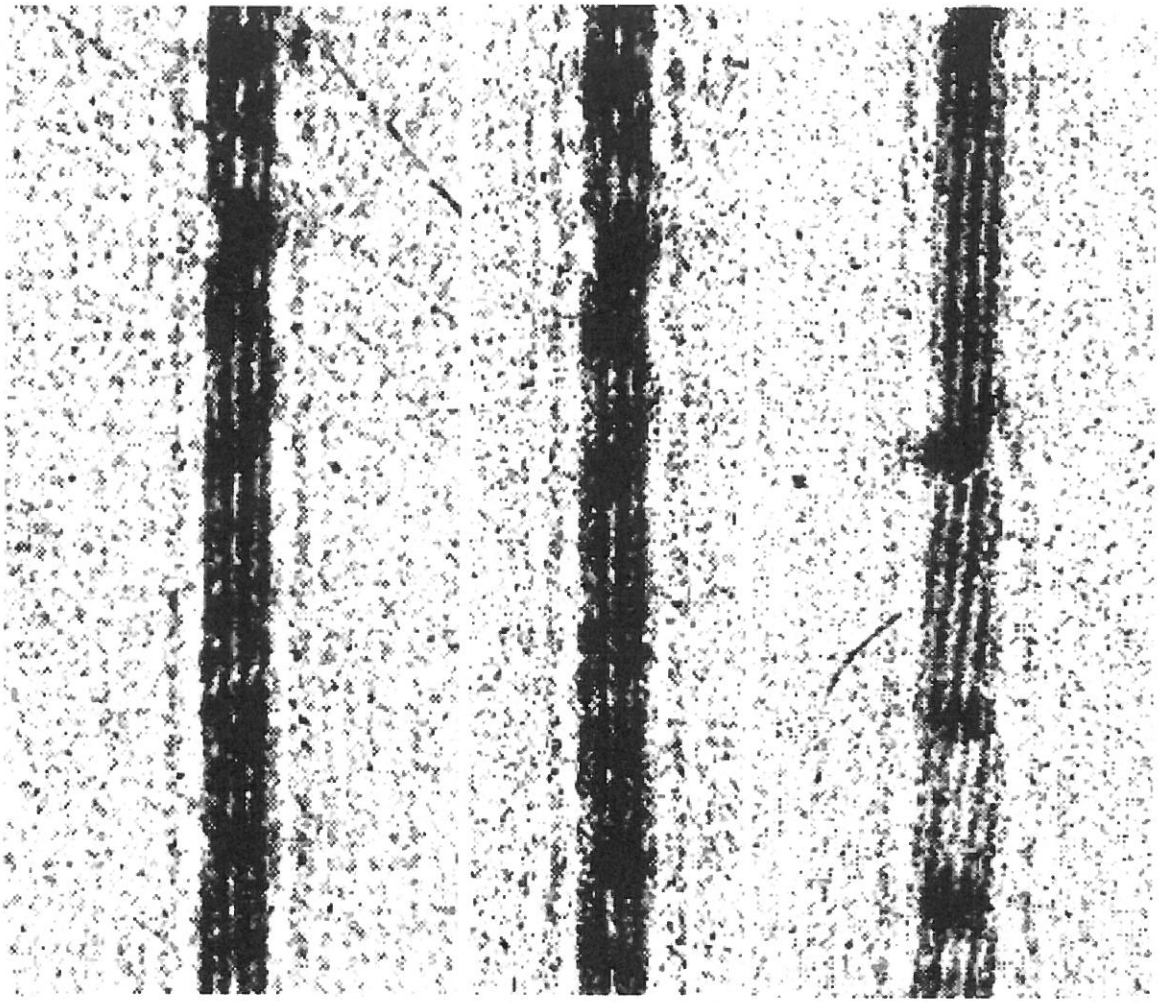
$\frac{e\Phi_B}{2\hbar c}$

Fundamental unit of magnetic flux: $\frac{2\pi\hbar c}{|e|} = 4.135\times10^{-7}$ gauss-cm^2.


Interference region
B
D
A
C
ℓ_2
ℓ_1


Neutron counts
1200
1000
800
600
−30
−20
−10
0
10
20
30
δ

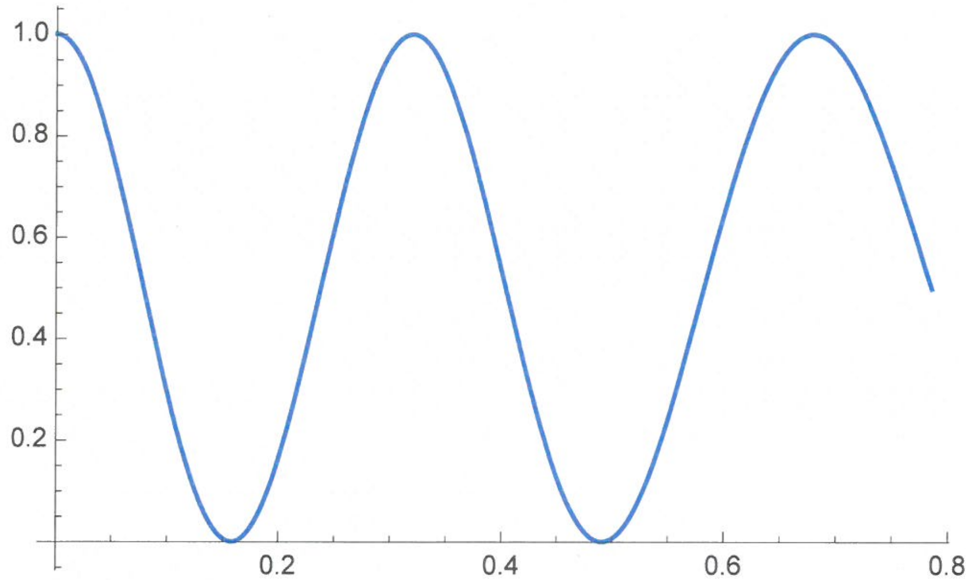

A
Source
region
$\vec{B} \neq 0$
Impenetrable
cylinder
B
inteference
region



R. G. Chambers, PRL 5, 3 (1960)

In[67]:= `Plot[Cos[10 Sin[th]] ^ 2, {th, 0, Pi / 4}]`

Out[67]=



In[72]:= `Plot[Cos[10 Sin[th] + .4] ^ 2, {th, 0, Pi / 4}]`

Out[72]=