Schrödinger Equation

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<Exam 1 ~~average~~ 72> Solutions posted	HW #4 Due by Oct. 11
	Fall Break Oct. 6

We now start Chapt. 2 : Quantum Dynamics.

How does the state change in time?

$$t_0 \longrightarrow t > t_0$$

$$|\alpha\rangle = |\alpha, t_0\rangle \longrightarrow \underbrace{|\alpha, t_0 ; t\rangle}_{\text{book}} \equiv |\alpha, t\rangle.$$

Time is just a parameter, not an operator

$\nexists$ Time operator , $\exists$ Postion operator

Time Evalution Operator

$$\underbrace{|\alpha, t\rangle} = \mathcal{U}(t, t_0)\underbrace{|\alpha, t_0\rangle}$$

$$|\alpha, t\rangle = \sum_{a'} C_{a'}(t)|a'\rangle \qquad |\alpha, t_0\rangle = \sum_{a'} |a'\rangle\underbrace{\langle a'|\alpha, t_0\rangle}_{C_{a'}(t_0)}$$

$$= \sum_{a'} C_{a'}(t_0)|a'\rangle$$

$$|C_{a'}(t)| \neq |C_{a'}(t_0)|$$

However, $\sum_{a'} |C_{a'}(t)|^2 = \sum_{a'} |C_{a'}(t_0)|^2 = 1$

because $\langle \alpha, t_0|\alpha, t_0\rangle = \langle \alpha, t|\alpha, t\rangle = 1$.

This requires

$$\mathcal{U}^\dagger(t,t_0)\,\mathcal{U}(t,t_0) = I,$$

since

$$\langle \alpha,t|\alpha,t\rangle = \langle \alpha,t_0|\underbrace{\mathcal{U}^\dagger(t,t_0)\,\mathcal{U}(t,t_0)}_{I}|\alpha,t_0\rangle$$
$$= \langle \alpha,t_0|\alpha,t_0\rangle$$

Unitarity of the time-evolution operator is intimately related to the conservation of the probability sum.

Reminder of the space translation operator

$$T_\ell = e^{-\frac{ip\ell}{\hbar}}, \quad T_\ell^\dagger = e^{\frac{ip\ell}{\hbar}}$$

$$T_\ell^\dagger T_\ell = I$$

Note also $\quad T_{\ell_2} T_{\ell_1} = T_{\ell_1+\ell_2}$

Similarly in time-evolution operator,

$$\mathcal{U}(t_2,t_1)\,\mathcal{U}(t_1,t_0) = \mathcal{U}(t_2,t_0)$$

with the time sequence $\quad t_2 > t_1 > t_0$.

As $T_{\delta l} = \underset{\nearrow}{I} - \dfrac{i p \delta l}{\hbar}$, where p (momentum) is the generator of space translation and $p = p^\dagger$,

indep. of displacement

$\mathcal{U}(t+\delta t, t) = \underset{\nearrow}{I} - \dfrac{i H \delta t}{\hbar}$, where H (Hamiltonian or energy) is the generator of time translation and $H = H^\dagger$.

indep of t

From $\mathcal{U}(t_2, t_1)\mathcal{U}(t_1, t_0) = \mathcal{U}(t_2, t_0)$,
take $t_2 = t + \delta t$ and $t_1 = t$.

Then, $\underbrace{\mathcal{U}(t+\delta t, t)}_{I - \frac{iH\delta t}{\hbar}} \mathcal{U}(t, t_0) = \mathcal{U}(t+\delta t, t_0)$

and thus we get

$$\mathcal{U}(t+\delta t, t_0) - \mathcal{U}(t, t_0) = -i\frac{H\delta t}{\hbar}\mathcal{U}(t, t_0)$$

or $$\underbrace{\frac{\mathcal{U}(t+\delta t, t_0) - \mathcal{U}(t, t_0)}{\delta t}}_{\frac{\partial \mathcal{U}(t,t_0)}{\partial t}} = \frac{-i}{\hbar} H \mathcal{U}(t, t_0)$$

$$\boxed{i\hbar \frac{\partial}{\partial t}\mathcal{U}(t, t_0) = H\mathcal{U}(t, t_0)}$$

Eq. (2.1.25) p. 69.

We obtained the operator equation for time-evolution.

$$\boxed{i\hbar \frac{\partial \mathcal{U}(t,t_0)}{\partial t} = H\,\mathcal{U}(t,t_0)}$$

Schrödinger Eq. for the time-evolution operator.

Applying this to the state $|\alpha\rangle = |\alpha, t_0\rangle$,

we get

$$i\hbar \frac{\partial}{\partial t} \underbrace{\mathcal{U}(t,t_0)\,|\alpha,t_0\rangle}_{|\alpha,t\rangle} = H\,\underbrace{\mathcal{U}(t,t_0)\,|\alpha,t_0\rangle}_{|\alpha,t\rangle}$$

or $$\boxed{i\hbar \frac{\partial}{\partial t}|\alpha,t\rangle = H|\alpha,t\rangle}$$ Eq. (2.1.27)

Schröding Eq. for the state $|\alpha,t\rangle$

In case H is indep. of time, then

$$\mathcal{U}(t,t_0) = e^{-\frac{iH(t-t_0)}{\hbar}}$$

since it satsifies the Schrödinger equation for the time-evolution operator

$$i\hbar \frac{\partial \mathcal{U}(t,t_0)}{\partial t} = H\,\mathcal{U}(t,t_0)$$

and is unitary $\mathcal{U}(t,t_0)^\dagger\,\mathcal{U}(t,t_0) = I$

as well as $\mathcal{U}(t_0,t_0) = I$ from the initial condition.

Time-Dependent Wave Equation

$$\langle \vec{x}' | \alpha, t \rangle = \psi(\vec{x}', t)$$

From $i\hbar \frac{\partial}{\partial t} |\alpha, t\rangle = H |\alpha, t\rangle$,

we get

$$i\hbar \frac{\partial}{\partial t} \langle \vec{x}' | \alpha, t \rangle = \langle \vec{x}' | H | \alpha, t \rangle .$$

Taking $H = \frac{\vec{p}^2}{2m} + V(\vec{x})$,

we have

$$\langle \vec{x}' | \frac{\vec{p}^2}{2m} | \alpha, t \rangle = -\frac{\hbar^2}{2m} \vec{\nabla}'^2 \langle \vec{x}' | \alpha, t \rangle$$

Note $\langle \vec{x}' | \vec{p} | \alpha, t \rangle = \frac{\hbar}{i} \vec{\nabla}' \langle \vec{x}' | \alpha, t \rangle$

and $\langle \vec{x}' | \vec{p}^2 | \alpha, t \rangle = -\hbar^2 \vec{\nabla}'^2 \langle \vec{x}' | \alpha, t \rangle$

and

$$\underbrace{\langle \vec{x}' | V(\vec{x})}_{V(\vec{x}') \langle \vec{x}' |} | \alpha, t \rangle = V(\vec{x}') \langle \vec{x}' | \alpha, t \rangle$$

With $\langle \vec{x}' | \alpha, t \rangle = \psi(\vec{x}', t)$, we get

$$\boxed{i\hbar \frac{\partial}{\partial t} \psi(\vec{x}', t) = -\frac{\hbar^2}{2m} \vec{\nabla}'^2 \psi(\vec{x}', t) + V(\vec{x}') \psi(\vec{x}', t)}$$

Time-Dependent Schrödinger Equation.

Wave-Mechanics: $|\psi(\vec{x}', t)|^2 = \rho(\vec{x}', t)$ probability density

$$\int d^3\vec{x}' \rho(\vec{x}', t) = \int d^3\vec{x}' |\psi(\vec{x}', t)|^2 = \int d^3\vec{x}' \langle \alpha, t | \vec{x}' \rangle \langle \vec{x}' | \alpha, t \rangle$$
$$= \langle \alpha, t | \alpha, t \rangle = \langle \alpha, t_0 | \alpha, t_0 \rangle = 1$$