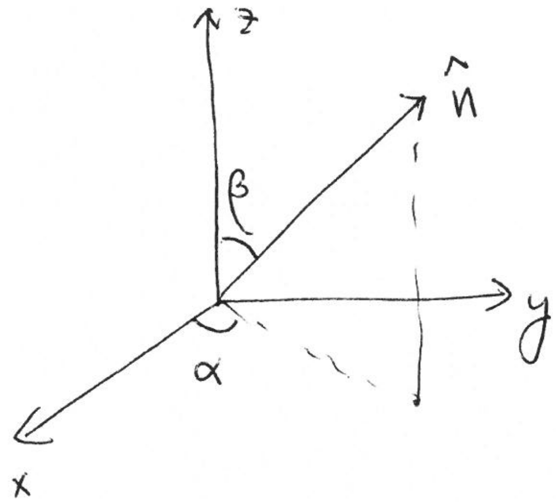
# SO(3) and SU(2)



November 17, 2016




$$\vec{S}\cdot\hat{n}\,|S_n+\rangle = \frac{\hbar}{2}|S_n+\rangle$$

$$|S_n+\rangle = \underbrace{\mathcal{D}(\hat{z},\alpha)}_{e^{-i\sigma_z\frac{\alpha}{2}}}\underbrace{\mathcal{D}(\hat{y},\beta)}_{e^{-i\sigma_y\frac{\beta}{2}}}|+\rangle$$

$$= e^{-i\frac{\alpha}{2}}\Big[\underbrace{\cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle}\Big]$$

even overall phase ~~add~~ was found

Prob. 9 of Chapt. 1 eigenvalue problem.

Let's look into the Drehung more carefully.

e.g. $\mathcal{D}(\hat{y},\beta) = e^{-i\sigma_y\frac{\beta}{2}} = \cos\frac{\beta}{2} - i\sigma_y\sin\frac{\beta}{2} \doteq \begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}$.

Since $|+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}$, we get

$$\mathcal{D}(\hat{y},\beta)|+\rangle \doteq \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} \text{ or } \mathcal{D}(\hat{y},\beta)|+\rangle = \cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}$$

Now, $\beta = \frac{\pi}{2}$ ; $\mathcal{D}(\hat{y},\frac{\pi}{2})|+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle = |S_x+\rangle$

$\longrightarrow \pi$ ; $\mathcal{D}(\hat{y},\pi)|+\rangle = |-\rangle$

$\frac{3\pi}{2}$ ; $\mathcal{D}(\hat{y},\frac{3\pi}{2})|+\rangle = -\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle = -|S_x-\rangle$

$\longrightarrow 2\pi$ ; $\mathcal{D}(\hat{y},2\pi)|+\rangle = -|+\rangle$ "Note the opposite phase"

To go back to the same state with the same phase, ②
$\beta$ must be $4\pi$ rather than $2\pi$.

SO(3): $R_x(\phi), R_y(\phi), R_z(\phi)$

$e^{G_x \phi}$ ↻ $2\pi$ to original

$2\pi$: ↑
$4\pi$: ↑ single

SU(2): $e^{-i\sigma_x \frac{\phi}{2}}, \; e^{-i\sigma_y \frac{\phi}{2}}, \; e^{-i\sigma_z \frac{\phi}{2}}$

$2\pi$; ↓
$4\pi$; ↑ double

The correspondence is not isomorphism
but <u>homomorphism</u>: <u>one to two</u> correspondence.

$$\mathcal{D}(\hat{n}, \phi+2\pi) = \cos\frac{\phi+2\pi}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\phi+2\pi}{2}$$
$$= \cos\left(\frac{\phi}{2}+\pi\right) - i\vec{\sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}+\pi\right) = -\left[\cos\frac{\phi}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\phi}{2}\right]$$
$$= -\mathcal{D}(\hat{n}, \phi).$$

SO(3) ← in 3-dim
special orthogonal group

SU(2) ← in 2-dim
Special Unitary group

$\mathrm{Det}\, R_{\hat{n}}(\phi) = 1$, $R_{\hat{n}}^T(\phi) R_{\hat{n}}(\phi) = 1$. 3×3 matrix.
↕
$\mathrm{Tr}\, \vec{G}_{\phi} = 0$

$\mathrm{Det}\, \mathcal{D}(\hat{n},\phi) = 1$, $\mathcal{D}^\dagger(\hat{n},\phi)\mathcal{D}(\hat{n},\phi) = 1$.
or $\mathrm{Tr}\, \vec{\sigma} = 0$ 2×2 matrix

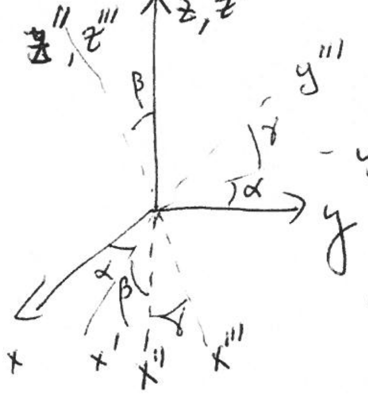
<u>Orthogonality is preserved under orthogonal group</u>

$\hat{r}_i \cdot \hat{r}_j = \delta_{ij}$, $\hat{r}_i' \cdot \hat{r}_j' = \delta_{ij}$
$(R\hat{r}_i)\cdot(R\hat{r}_j) = \hat{r}_i (R^T R) \hat{r}_j = \hat{r}_i \cdot \hat{r}_j = \delta_{ij}$.

<u>Unitarity is preserved under unitary group.</u>

$\langle\alpha|\alpha\rangle = \langle\alpha'|\alpha'\rangle$
$= \langle\alpha|\mathcal{D}^\dagger \mathcal{D}|\alpha\rangle$

# Euler Angles




$$R(\alpha,\beta,\gamma) = R_{z''}(\gamma) R_{y'}(\beta) R_z(\alpha)$$

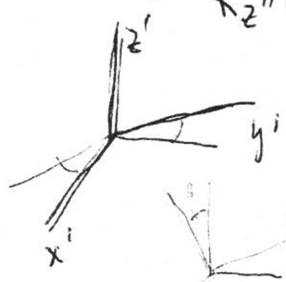
$$= R_z(\alpha) R_y(\beta) R_z(\gamma)$$

Demo: $R_{y'}(\beta) R_z(\alpha) = R_z(\alpha) R_y(\beta)$

$$R_{y'}(\beta) = R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha).$$

Similarly, $R_{z''}(\gamma) = R_{y'}(\beta) R_{z'}(\gamma) R_{y'}^{-1}(\beta).$

cf. $R_{z''}(\gamma) R_{y'}(\beta) = R_{y'}(\beta) R_{z'}(\gamma)$




If $\alpha = 0$, then

$R_{z'}(\gamma) R_y(\beta) = R_y(\beta) R_z(\gamma)$

e.g. $R_{z'}(90°) R_y(45°) = R_y(45°) R_z(90°)$

If $\alpha \neq 0$, then

$$R_{z''}(\gamma) R_{y'}(\beta) = R_{y'}(\beta) R_{z'}(\gamma)$$

or $R_{z''}(\gamma) = R_{y'}(\beta) R_{z'}(\gamma) R_{y'}^{-1}(\beta)$

Thus,

$$R(\alpha,\beta,\gamma) = R_{z''}(\gamma) R_{y'}(\beta) R_z(\alpha)$$

$$= R_{y'}(\beta) R_{z'}(\gamma) \underbrace{R_{y'}^{-1}(\beta) R_{y'}(\beta)}_{I} R_z(\alpha)$$

$$= \underbrace{R_{y'}(\beta)}_{R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha)} R_z(\alpha) R_z(\gamma)$$

$$= R_z(\alpha) R_y(\beta) R_z(\gamma) \quad \#.$$

$$\mathcal{D}(\alpha,\beta,\gamma) = \mathcal{D}(\hat{n},\theta)$$

$$\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$$

$$\mathcal{D}(\alpha,\beta,\gamma) = \mathcal{D}_z(\alpha)\,\mathcal{D}_y(\beta)\,\mathcal{D}_z(\gamma)$$

$$= e^{-i\sigma_3\frac{\alpha}{2}}\; e^{-i\sigma_2\frac{\beta}{2}}\; e^{-i\sigma_3\frac{\gamma}{2}}$$

cf. $e^{-i\sigma_2\frac{\beta}{2}} = \cos\frac{\beta}{2}\,\underset{\begin{pmatrix}1&0\\0&1\end{pmatrix}}{I} - i\underset{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}{\sigma_2}\sin\frac{\beta}{2}$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

$$= e^{-i\vec{\sigma}\cdot\hat{n}\frac{\theta}{2}} = \cos\frac{\theta}{2} - i\,(\vec{\sigma}\cdot\hat{n})\sin\frac{\theta}{2}$$

$$\vec{\sigma}\cdot\hat{n} \longrightarrow \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} - in_z\sin\frac{\theta}{2} & (-in_x - n_y)\sin\frac{\theta}{2} \\ (-in_x + n_y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z\sin\frac{\theta}{2} \end{bmatrix}$$

$$\mathrm{Tr}\,\mathcal{D}(\hat{n},\theta) = 2\cos\frac{\theta}{2}$$

$$\mathrm{Tr}\,\mathcal{D}(\alpha,\beta,\gamma) = 2\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\therefore\ \cos\frac{\theta}{2} = \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2} \quad \left(\text{or } \theta = 2\cos^{-1}\left(\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\right)\right)$$

$$2i\, n_z \sin\frac{\theta}{2} = \left(e^{i\frac{(\alpha+\gamma)}{2}} - e^{-i\frac{(\alpha+\gamma)}{2}}\right)\cos\frac{\beta}{2}$$

$$= 2i \sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\therefore\ n_z = \frac{\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}}{\sin\frac{\theta}{2}} = \frac{\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}}{\sqrt{1-\cos^2\left(\frac{\alpha+\gamma}{2}\right)\cos^2\frac{\beta}{2}}}$$

$$-2i\, n_x \sin\frac{\theta}{2} = \left(e^{i\frac{(\alpha-\gamma)}{2}} - e^{-i\frac{(\alpha-\gamma)}{2}}\right)\sin\frac{\beta}{2}$$

$$= 2i \sin\frac{\alpha-\gamma}{2}\sin\frac{\beta}{2}.$$

$$\therefore\ n_x = -\frac{\sin\left(\frac{\alpha-\gamma}{2}\right)\sin\frac{\beta}{2}}{\sqrt{1-\cos^2\left(\frac{\alpha+\gamma}{2}\right)\cos^2\frac{\beta}{2}}}$$

$$2 n_y \sin\frac{\theta}{2} = \left(e^{i\left(\frac{\alpha-\gamma}{2}\right)} + e^{-i\left(\frac{\alpha-\gamma}{2}\right)}\right)\sin\frac{\beta}{2}$$

$$= 2\cos\left(\frac{\alpha-\gamma}{2}\right)\sin\frac{\beta}{2}$$

$$\therefore\ n_y = \frac{\cos\left(\frac{\alpha-\gamma}{2}\right)\sin\frac{\beta}{2}}{\sqrt{1-\cos^2\frac{\alpha+\gamma}{2}\cos^2\frac{\beta}{2}}}$$

Note: $n_x^2 + n_y^2 + n_z^2 = 1$.

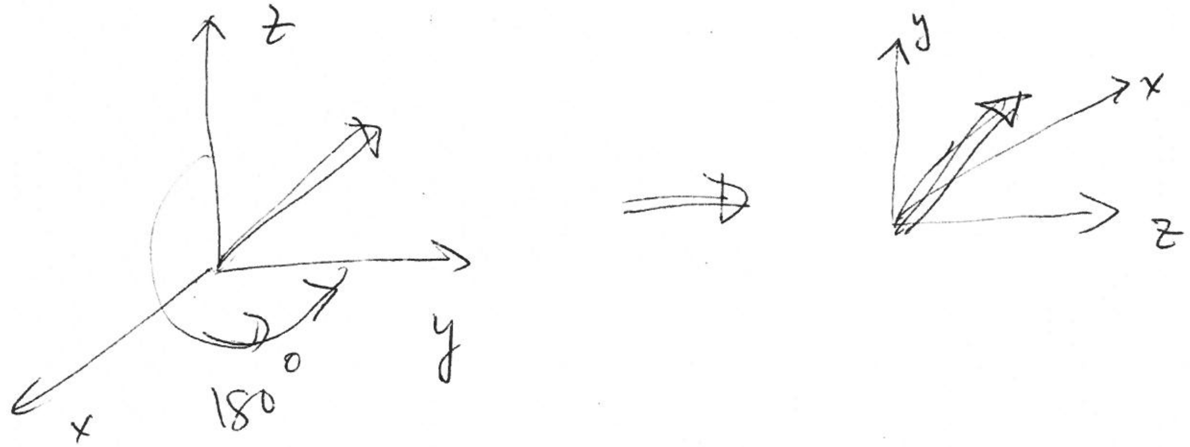
e.g. $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$, $\gamma = \frac{\pi}{2}$.

$$\cos\frac{\theta}{2} = \cos\frac{\pi}{2}\cos\frac{\pi}{4} = 0 ; \quad \frac{\theta}{2} = \frac{\pi}{2} \text{ or } \theta = \pi .$$

$$n_z = \frac{\sin\frac{\pi}{2}\cos\frac{\pi}{4}}{\sin\frac{\pi}{2}} = \frac{1}{\sqrt{2}}$$

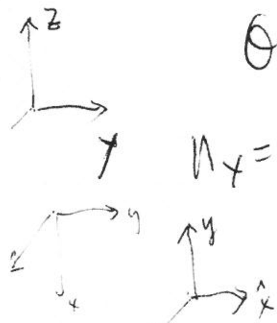
$$n_x = 0$$

$$n_y = \frac{\cos 0 \sin\frac{\pi}{4}}{\sin\frac{\pi}{2}} = \frac{1}{\sqrt{2}}$$




e.g. $\alpha = 0$, $\beta = \frac{\pi}{2}$, $\gamma = \frac{\pi}{2}$

$\theta = 120^\circ$




$n_x = n_y = n_z = \frac{1}{\sqrt{3}}$.

e.g.

$\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$, $\gamma = 0$

$\theta = 120^\circ$

$n_x = -\frac{1}{\sqrt{3}}$

$n_y = \frac{1}{\sqrt{3}}$

$n_z = \frac{1}{\sqrt{3}}$