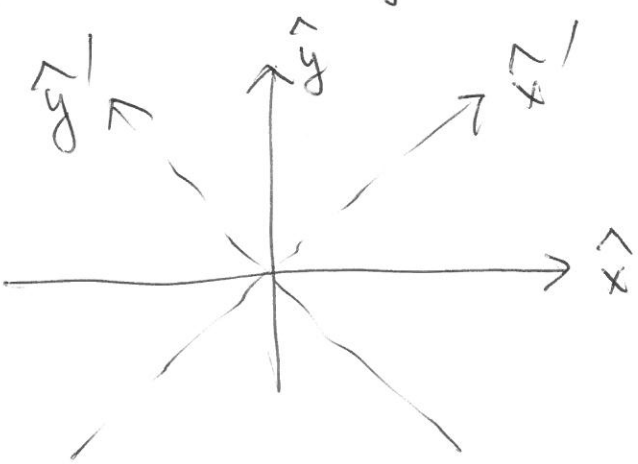
Kets, Bras and Operators

Aug. 23, 2016

$\boxed{\text{HW\#1 Due by Aug. 30}}$

$$\vec{E} = \begin{cases} E_0 \hat{x} \cos(kz-\omega t) = \mathrm{Re}\left[E_0 \hat{x} e^{i(kz-\omega t)}\right] \longleftrightarrow |S_z+\rangle \equiv |+\rangle \\ E_0 \hat{y} \cos(kz-\omega t) = \mathrm{Re}\left[E_0 \hat{y} e^{i(kz-\omega t)}\right] \longleftrightarrow |S_z-\rangle \equiv |-\rangle \end{cases}$$



$$\mathrm{Re}\left[E_0 \frac{\hat{x}+\hat{y}}{\sqrt{2}} e^{i(kz-\omega t)}\right] \longleftrightarrow |S_x+\rangle = \frac{|+\rangle+|-\rangle}{\sqrt{2}}$$

$$\mathrm{Re}\left[E_0 \frac{-\hat{x}+\hat{y}}{\sqrt{2}} e^{i(kz-\omega t)}\right] \longleftrightarrow |S_x-\rangle = \frac{-|+\rangle+|-\rangle}{\sqrt{2}}$$

Circularly polarized wave

$e^{i\frac{\pi}{2}} \longrightarrow \hat{y}\cos(kz-\omega t+\frac{\pi}{2})$

relative phase angle

$$\mathrm{Re}\left[E_0 \frac{\hat{x}+i\hat{y}}{\sqrt{2}} e^{i(kz-\omega t)}\right] \longleftrightarrow |S_y+\rangle = \frac{|+\rangle+i|-\rangle}{\sqrt{2}}$$

$$\mathrm{Re}\left[E_0 \frac{\hat{x}-i\hat{y}}{\sqrt{2}} e^{i(kz-\omega t)}\right] \longleftrightarrow |S_y-\rangle = \frac{|+\rangle-i|-\rangle}{\sqrt{2}}$$

<u>Complex vector space</u> must be introduced to cover the quantum space.

Dimension of complex vector space depends on the system.

e.g. ^{47}Ag : spin-$\frac{1}{2}$ system (dim = 2)

cf. ^{133}Cs (cesium atom) spin-4 system (dim = 9)

spin-s system;
dim = $2s+1$ $(-s, -s+1, \cdots 0, 1, \cdots +s)$ in case s is integer

Later, in Section 1.6, we consider the case of continuous spectra, e.g. position and momentum of a particle.

$$\boxed{\begin{array}{l}\text{Nondenumerably infinite dimensional complex vector space} \\ = \text{Hilbert space (Quantum Mechanical space).}\end{array}}$$

cf. Euclidean space

- denumerable (countable) ; N (natural number)
- nondenumerable (uncountable) ; real number (e.g. $\sqrt{2}$, $\sqrt{3}$, $\cdots$)

While we usually indicate a finite number of dimensions N, of the ket space, the results can immediately be generalized to <u>nondenumerably infinite dimensions</u>.

In Q.M., a physical state (e.g. ^{47}Ag atom) with a finite spin orientation is represented by a <u>state vector</u> in a <u>complex vector space</u>.

Following P.A.M. Dirac's notations, we call such a state vector a <u>ket</u> and denote it by $|\alpha\rangle$.

This state ket is postulated to contain complete information about the physical state, i.e. everything that we are allowed to ask about the state is contained in the <u>ket</u>.

A <u>observable</u> can be represented by an <u>operator</u> and denoted by A in the vector space in question.

Quite generally, an operator acts on a ket from the left:

$$A\left(|\alpha\rangle\right) = A|\alpha\rangle,$$

which is yet another ket.

In general, $A|\alpha\rangle$ is <u>not</u> a constant times $|\alpha\rangle$.

However, there are particular kets of importance, known as <u>eigenkets</u> of operator A denoted by $|a'\rangle, |a''\rangle \cdots$.

They have the property

$$A|a'\rangle = a'|a'\rangle, \quad A|a''\rangle = a''|a''\rangle, \quad \cdots,$$

where $a', a'', \cdots$ are just numbers (eigenvalues),

e.g.

$$S_z|S_z+\rangle = \frac{\hbar}{2}|S_z+\rangle$$

$$S_z|S_z-\rangle = -\frac{\hbar}{2}|S_z-\rangle$$

observable　　eigenket　　eigenvalue

($c|\alpha\rangle$ and $|\alpha\rangle$ represent the same physical state when $c \neq 0$. $\boxed{\text{cf. } c=0 \text{ case; null ket.}}$
However, distinguish this case from the eigenvalue $a'=0$ case)

Any arbitrary ket $|\alpha\rangle$ can be written as

$$|\alpha\rangle = \sum_{a'} C_{a'} |a'\rangle,$$

with $a', a'', \cdots,$ up to $a^{(N)}$, where $C_{a'}$ is a complex coefficient.

e.g. $|S_y \pm\rangle = \frac{1}{\sqrt{2}}|S_z +\rangle \pm \frac{i}{\sqrt{2}}|S_z -\rangle.$

At this point, it is convenient to introduce a <u>dual space</u> called the <u>Bra</u> space to the <u>ket</u> space.

$$\boxed{\text{Ket Space}} \xleftrightarrow{\text{Dual Correspondence}} \boxed{\text{Bra Space}}$$

Complex number

$$c \xleftrightarrow{\text{Complex conjugation}} c^*$$

State vector

$$|\alpha\rangle \longleftrightarrow \langle\alpha|$$

$$c|\alpha\rangle \longleftrightarrow c^*\langle\alpha|$$

cf. $\langle S_y \pm| = \frac{1}{\sqrt{2}}\langle S_z +| \mp \frac{i}{\sqrt{2}}\langle S_z -|.$

$|\alpha\rangle = \sum_{\alpha'} C_{a'}|a'\rangle \longrightarrow \boxed{C_{a'} = \langle a'|\alpha\rangle}$ Coefficients as innerproducts of bra & ket.

Inner product of a bra and a ket

$$\langle \beta | \alpha \rangle = \langle \alpha | \beta \rangle^* \neq \langle \alpha | \beta \rangle$$

e.g. $\langle S_y - | S_z - \rangle = \frac{i}{\sqrt{2}} = \langle S_z - | S_y - \rangle^* = \left(-\frac{i}{\sqrt{2}}\right)^*$

cf. Euclidean space; $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

$$\boxed{\begin{array}{l}\text{Theorem; } \langle \alpha | \alpha \rangle \text{ is real} \\ \text{proof: } \langle \alpha | \alpha \rangle = \langle \alpha | \alpha \rangle^*\end{array}}$$

Postulate of positive definite metric

$$\langle \alpha | \alpha \rangle \geq 0,$$

where $\langle \alpha | \alpha \rangle = 0$ holds only if $|\alpha\rangle$ is a null ket.

(cf. $\vec{a} \cdot \vec{a} = |\vec{a}|^2 \geq 0$)

From a physicist's point of view, this postulate is essential for the probabilistic interpretation of quantum mechanics.

A normalized ket,

$$|\tilde{\alpha}\rangle = \frac{1}{\sqrt{\langle \alpha | \alpha \rangle}} |\alpha\rangle, \quad \text{since } \langle \tilde{\alpha} | \tilde{\alpha} \rangle = 1.$$

$\sqrt{\langle \alpha | \alpha \rangle}$ is the norm of $|\alpha\rangle$.

(cf. analogous to $\sqrt{\vec{a} \cdot \vec{a}} = |\vec{a}|$ in Euclidean space.)

Since $|\alpha\rangle$ and $c|\alpha\rangle$ represent the same physical state, we may require that kets be normalized in the sense of

$$\langle\alpha|\alpha\rangle = 1. \quad \text{e.g. } \underline{\langle S_z+|S_z+\rangle = 1}$$

If $\langle\alpha|\beta\rangle = 0$, then $|\alpha\rangle$ and $|\beta\rangle$ are <u>orthogonal</u>.

e.g. $\langle S_z+|S_z-\rangle = 0$.

If normalized and orthogonal, then <u>orthonormal</u>.

Now, from $|\alpha\rangle = \sum_{a'} C_{a'}|a'\rangle$, we can get

$$\langle a'|\alpha\rangle = C_{a'} \quad \text{or} \quad |\alpha\rangle = \sum_{a'} \underbrace{|a'\rangle\langle a'|}_{\boxed{\text{outer product}} \text{ of a bra and a ket.}}\alpha\rangle$$

$$\boxed{\sum_{a'} |a'\rangle\langle a'| = I}$$

Closure or completeness relation.

<u>Ex.</u>

$$\langle\alpha|\alpha\rangle = \sum_{a'} \underbrace{\left|\langle a'|\alpha\rangle\right|^2}_{|C_{a'}|^2} = 1 \quad \text{or} \quad \sum_{a'}|C_{a'}|^2 = 1$$

$$\underbrace{\sum_{a'} |a'\rangle\langle a'| = I}_{\sum_{a'}\langle\alpha|a'\rangle\langle a'|\alpha\rangle}$$

i.e. Sum of probability is <u>unity</u>.

Properties of Operators (Read pp. 14-16)

1. Two operators are same (or equal); $X = Y$

   if $X|\alpha\rangle = Y|\alpha\rangle$ for any arbitrary ket $|\alpha\rangle$

   (e.g. $\sum_{a'} |a'\rangle\langle a'| = I$).

2. X is a <u>null operator</u> if $X|\alpha\rangle = 0$

   for any arbitrary ket $|\alpha\rangle$.

   (e.g. $\sum_{a'} |a'\rangle\langle a'| - I$ is null)

3. <u>Addition</u> of operators; $X + Y = Y + X$ (commutative)

   $X + (Y + Z) = (X + Y) + Z$ (associative)

4. Multiplication of operators $\boxed{XY = YX \text{ (commutative)} \leftarrow \text{compatible} \quad \text{vs. } XY \neq YX \text{ (noncommutative)} \leftarrow \text{incompatible}}$

   $X(YZ) = (XY)Z = XYZ$ (associative)

   $\boxed{\text{Compatible vs. Incompatible observables}}$

   $\boxed{\text{Commutator} \quad [X, Y] = 0 \text{ vs. } [X, Y] \neq 0}$

   $[X, Y] = XY - YX$ (noncommutative; $[X, Y] \neq 0$)

   <u>Anti commutator</u> $\{X, Y\} = XY + YX$.

5. <u>Adjoint of operator</u>

   $X \Leftrightarrow X^\dagger$

   Dual correspondence

   $X|\alpha\rangle \xleftrightarrow{D.C.} \langle\alpha|X^\dagger$

   If $X = X^\dagger$, then X is <u>Hermitian</u> or "Self-Adjoint".

<u>Theorem</u>

The eigenvalues of a Hermitian operator A are <u>real</u>; the eigenkets of A corresponding to different eigenvalues are <u>orthogonal</u>.

<u>Proof</u> $A|a'\rangle = a'|a'\rangle$, $\underbrace{\langle a'|A^\dagger}_{\langle a'|A \text{ (Hermiticity)}} = a'^*\langle a'|$

$$\langle a'|A|a'\rangle = a' = a'^*$$

because $\langle a'|a'\rangle = 1$.

$$\left.\begin{aligned} A|a'\rangle &= a'|a'\rangle \\ A|a''\rangle &= a''|a''\rangle \end{aligned}\right\} \text{ Suppose } a' \neq a''.$$

Then, $\langle a''|A|a'\rangle = a'\langle a''|a'\rangle$

$$\langle a''|A^\dagger|a'\rangle = a''^*\langle a''|a'\rangle$$

Hemiticity, real

$$\langle a''|A|a'\rangle = a''\langle a''|a'\rangle \quad (-$$

$$0 = (a'-a'')\langle a''|a'\rangle$$

Since $a' \neq a''$, $\langle a''|a'\rangle = 0$ or $|a'\rangle$ and $|a''\rangle$ are <u>orthogonal</u>.

Observable as an outerproduct of bras & kets

$$A = \sum_{i=1}^{N} a^{(i)} \, |a^{(i)}\rangle\langle a^{(i)}|$$

because $A\,|a^{(i)}\rangle = a^{(i)}\,|a^{(i)}\rangle \quad (i = 1, 2, \cdots, N)$

so that $A\,|a^{(i)}\rangle\langle a^{(i)}| = a^{(i)}\,|a^{(i)}\rangle\langle a^{(i)}|$

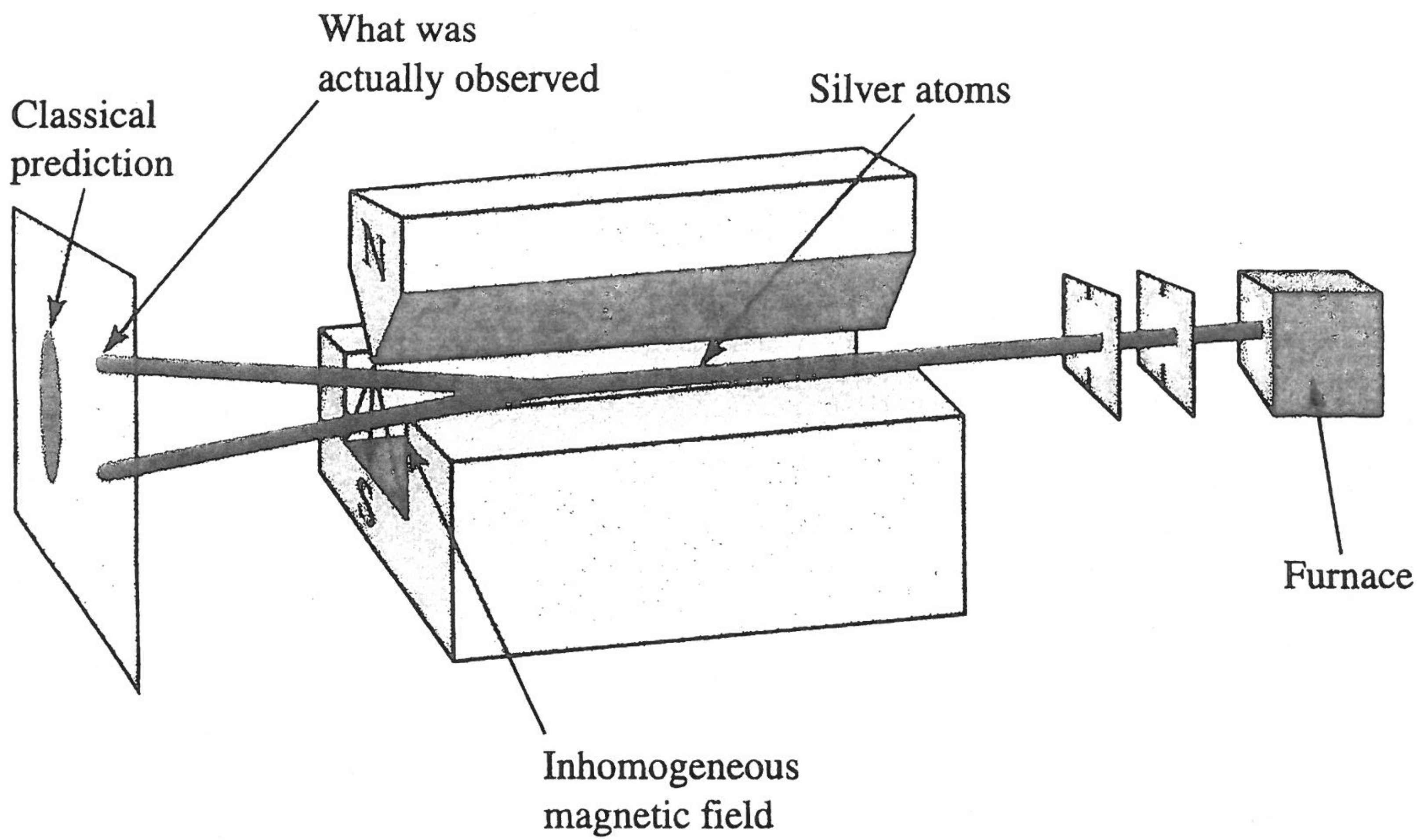
and $A \underbrace{\sum_{i=1}^{N} |a^{(i)}\rangle\langle a^{(i)}|}_{I} = \sum_{i=1}^{N} a^{(i)}\,|a^{(i)}\rangle\langle a^{(i)}|$

e.g. $S_z = \frac{\hbar}{2} |S_z+\rangle\langle S_z+| - \frac{\hbar}{2} |S_z-\rangle\langle S_z-|$

$$= \frac{\hbar}{2} \left(|+\rangle\langle +| - |-\rangle\langle -| \right)$$

If $a^{(i)} = 1$ for all i, then $A = I$.

e.g. $|+\rangle\langle +| + |-\rangle\langle -| = I$


What was
actually observed
Classical
prediction
Silver atoms
N
S
Furnace
Inhomogeneous
magnetic field

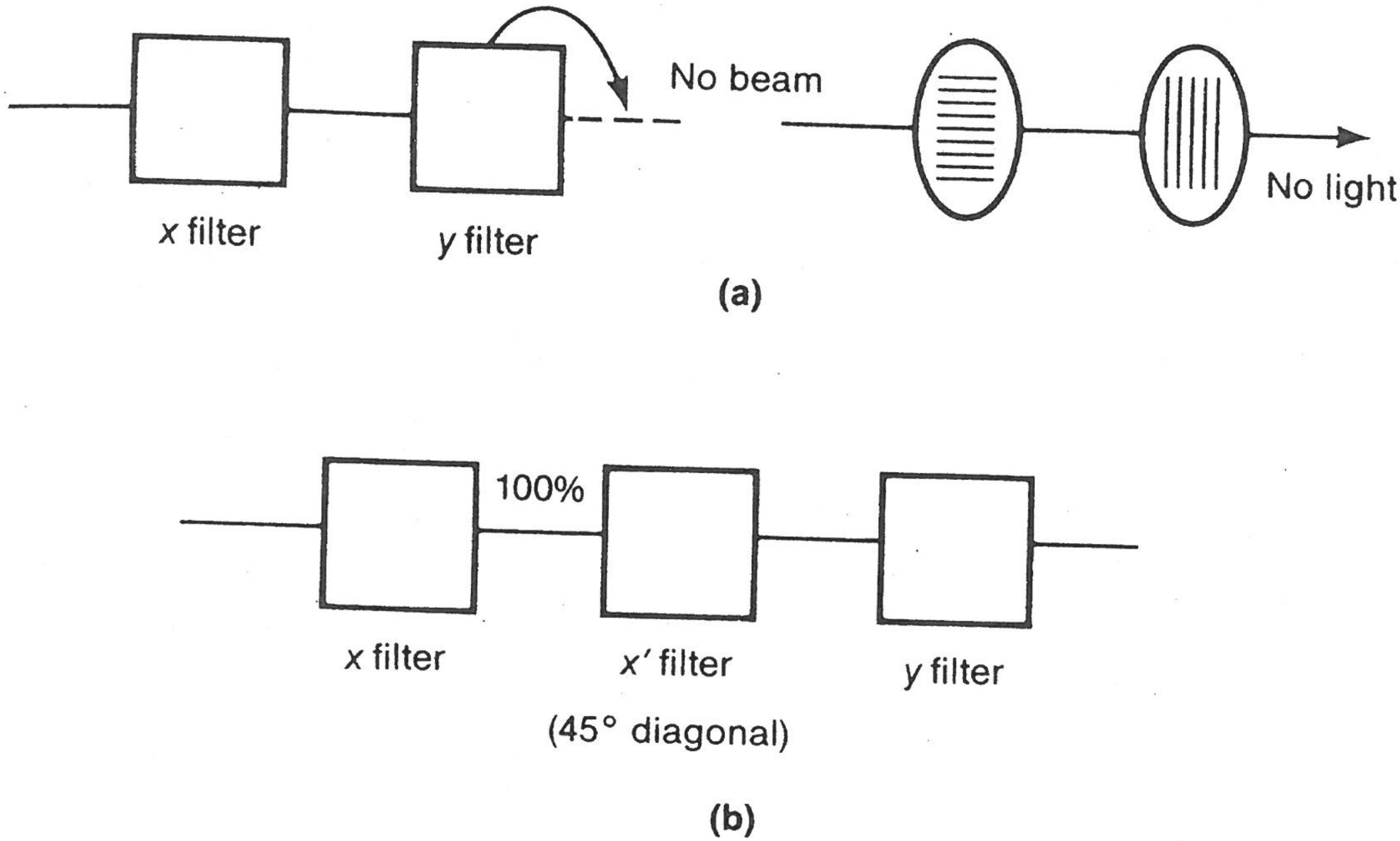


FIGURE 1.4. Light beams subjected to Polaroid filters.

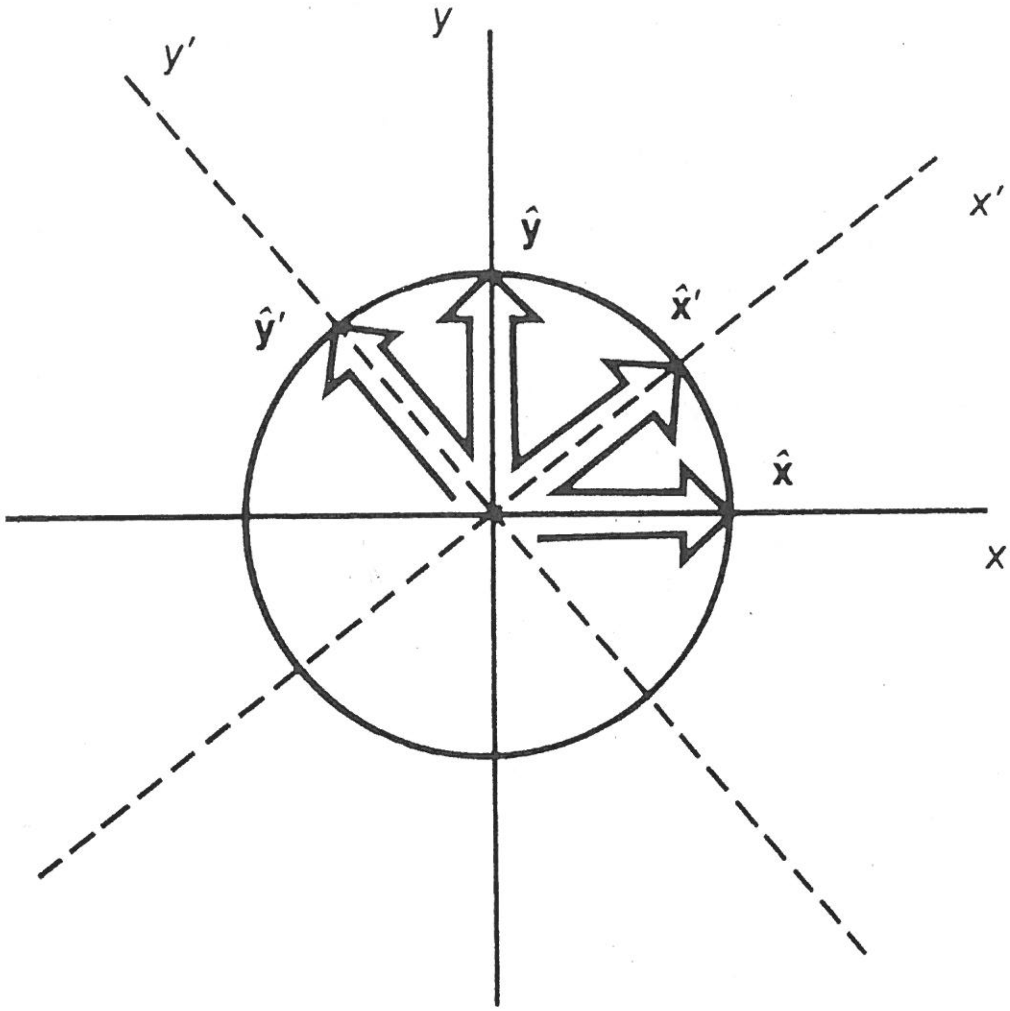


FIGURE 1.5. Orientations of the x'- and y'-axes.

assical electrodynamics. Using Figure 1.5 we obtain

$$
\begin{aligned}
E_0\hat{\mathbf{x}}'\cos(kz-\omega t) &= E_0\left[\frac{1}{\sqrt{2}}\hat{\mathbf{x}}\cos(kz-\omega t)+\frac{1}{\sqrt{2}}\hat{\mathbf{y}}\cos(kz-\omega t)\right],\\
E_0\hat{\mathbf{y}}'\cos(kz-\omega t) &= E_0\left[-\frac{1}{\sqrt{2}}\hat{\mathbf{x}}\cos(kz-\omega t)+\frac{1}{\sqrt{2}}\hat{\mathbf{y}}\cos(kz-\omega t)\right].
\end{aligned}
\tag{1.1.8}
$$

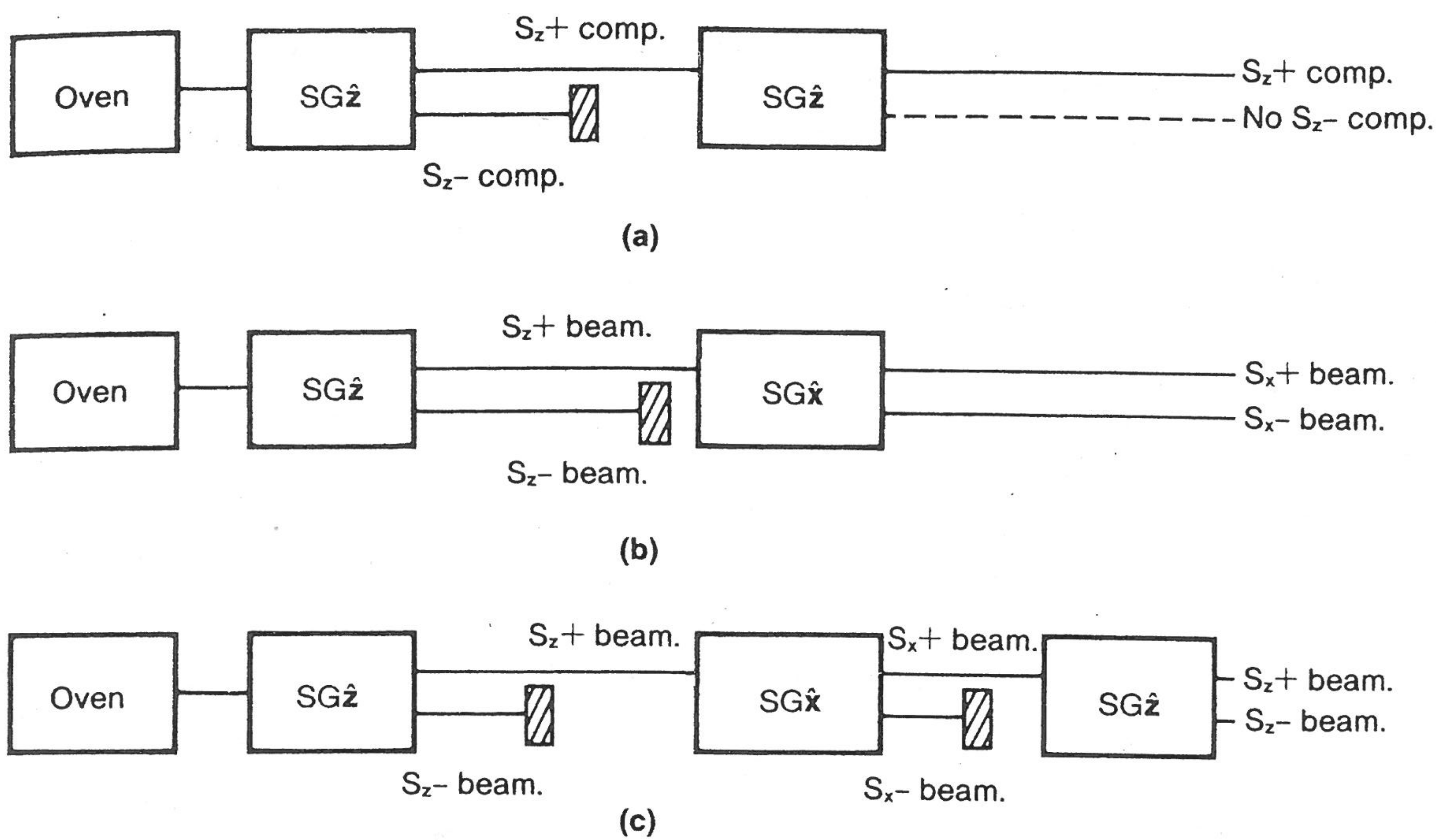
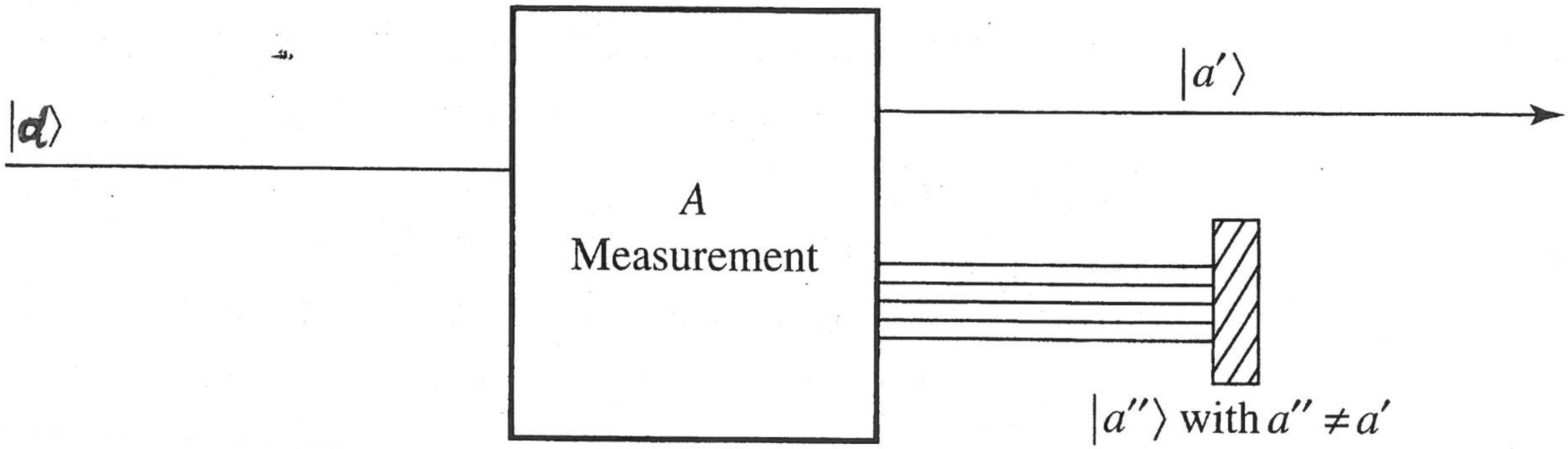


FIGURE 1.3. Sequential Stern-Gerlach experiments.



|d⟩
A
Measurement
|a′⟩
|a″⟩ with a″ ≠ a′