Angular Momentum Operators and Eigenvalues

Last homework (HW#8) Due by Nov. 29
Class Eval Due by Dec. 5, 8am; Thanksgiving Break

Nov. 22, 2016

We discussed SO(3) and SU(2) for the rotations in the physical space and in the Hilbert space, respectively.

$$SO(3): \; R_{\hat{n}}(\phi) = e^{\vec{G}\cdot\hat{n}\phi} \quad (\hat{n} = \hat{x}, \hat{y}, \hat{z})$$

$$SU(2): \; D_{\hat{n}}(\phi) = e^{\frac{-i\vec{J}\cdot\hat{n}\phi}{\hbar}} \quad \left(\vec{G} \sim \frac{\vec{J}}{i\hbar}\right)$$

For spin $\frac{1}{2}$ system, $\vec{J} \to \vec{S} = \vec{\sigma}\left(\frac{\hbar}{2}\right) = \frac{\hbar}{2}\vec{\sigma}$

so that $D_{\hat{n}}(\phi) = e^{-i\vec{\sigma}\cdot\hat{n}\frac{\phi}{2}}$

We now extend the spin $\frac{1}{2}$ system in the Hilbert space to any angular momentum states in Q.M.

No matter what spin or angular momentum system, still there are three generators: J_x, J_y, J_z.

As $R_x(\varepsilon)R_y(\varepsilon) - R_y(\varepsilon)R_x(\varepsilon) = R_z(\varepsilon^2) - I$,

we have

$$D_x(\varepsilon)D_y(\varepsilon) - D_y(\varepsilon)D_x(\varepsilon) = D_z(\varepsilon^2) - I.$$

$$\left(I - i\frac{J_x \varepsilon}{\hbar}\right)\left(I - i\frac{J_y \varepsilon}{\hbar}\right) - \left(I - i\frac{J_y \varepsilon}{\hbar}\right)\left(I - i\frac{J_x \varepsilon}{\hbar}\right)$$

$$= \left(I - i\frac{J_z \varepsilon^2}{\hbar}\right) - I$$

$$\left(-\frac{i\varepsilon}{\hbar}\right)^2 \underbrace{[J_x, J_y]}_{i\hbar J_z} = -\frac{i\varepsilon^2}{\hbar} J_z$$

i.e. $[J_x, J_y] = i\hbar J_z$

$2^2 - 1 = 3$ generators

In general, $[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$. (SU(2) algebra)

Because J_x, J_y and J_z do not commute each other, one may choose only one of them (usually J_z) to be the observable.

However, $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ commute with every one of J_k $(k=1,2,3)$

i.e., $[\vec{J}^2, J_k] = 0$ ($\vec{J}^2$ is known as the Casimir's operator of SU(2))

Thus, one may look for simultaneous eigenstates of $\vec{J}^2$ and J_z.

eigenket $|jm\rangle$ (see pages 191–199 : Section 3.5)

Define the raising and lowering operators:

$$J_{\pm} = J_x \pm i J_y$$

so that

$$[J_+, J_-] = 2\hbar J_z$$

$$[J_z, J_{\pm}] = \pm\hbar J_{\pm}$$

Ladder operators

cf.

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i p}{m\omega}\right)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i p}{m\omega}\right)$$

$$[a, a^\dagger] = I$$

$$[N, a] = -a$$

$$[N, a^\dagger] = +a^\dagger$$

notice the correspondence

and $[\vec{J}^2, J_{\pm}] = 0$

$$[\vec{J}^2, J_x] = \underbrace{[J_y^2, J_x]}_{-i\hbar\{J_z, J_y\}} + \underbrace{[J_z^2, J_x]}_{+i\hbar\{J_y, J_z\}}$$

$$= 0$$

Similarly $[\vec{J}^2, J_y] = 0$ & $[\vec{J}^2, J_z] = 0$.

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$$

$$= \frac{J_+ J_- + J_- J_+}{2} + J_z^2 \quad \text{or} \quad = \frac{\{J_+, J_-\}}{2} + J_z^2$$

$$\{J_+, J_-\} = (J_x + iJ_y)(J_x - iJ_y) + (J_x - iJ_y)(J_x + iJ_y)$$

$$= J_x^2 + i[J_y, J_x] + J_y^2 + J_x^2 - i[J_y, J_x] + J_y^2$$

$$= 2(J_x^2 + J_y^2)$$

$$\{a^\dagger, a\} + N^2$$

Note

$$J_x J_y - J_y J_x = i\hbar J_z$$

$$[J_y^2, J_x] = J_y^2 J_x - J_x J_y^2$$

$$= J_y (J_y J_x) - (J_x J_y) J_y$$

$$J_x J_y - i\hbar J_z \qquad J_y J_x + i\hbar J_z$$

$$= J_y J_x J_y - i\hbar J_y J_z - J_y J_x J_y - i\hbar J_z J_y = -i\hbar\{J_z, J_y\}$$

Eigenvalues of J_z and $\vec{J}^2$

$$J_z|jm\rangle = m\hbar|jm\rangle \quad (m = \underbrace{-j, -j+1, \cdots, j-1, j}_{2j+1})$$

(regardless j is integer or half-integer)

Now,

$$J_z(J_\pm|jm\rangle) = (\underbrace{[J_z, J_\pm]}_{\pm\hbar J_\pm} + J_\pm J_z)|jm\rangle$$

$$= (m\pm1)\hbar(J_\pm|jm\rangle)$$

Thus, we find

$$J_\pm|jm\rangle = C_\pm|j\,m\pm1\rangle$$

Since $J_+|j,j\rangle = 0$,

we can find

$$\underbrace{J_-J_+}_{(J_x-iJ_y)(J_x+iJ_y)}|j,j\rangle = (\vec{J}^2 - J_z^2 - \hbar J_z)|j,j\rangle = 0$$

$$(J_x-iJ_y)(J_x+iJ_y) = \underbrace{J_x^2+J_y^2}_{\vec{J}^2-J_z^2} + i\underbrace{(J_xJ_y-J_yJ_x)}_{[J_x,J_y]=i\hbar J_z}$$

or $\vec{J}^2|j,j\rangle = j(j+1)\hbar^2|j,j\rangle$

Because $[\vec{J}^2, J_\pm]=0$,
$\vec{J}^2|jm\rangle = j(j+1)\hbar^2|jm\rangle$

e.g. $J_-\vec{J}^2|j,j\rangle = j(j+1)\hbar^2(\underbrace{J_-|j,j\rangle}_{C_-|j,j-1\rangle})$

$= \vec{J}^2(J_-|j,j\rangle) = \vec{J}^2 C_-|j,j-1\rangle$

Let's find $C_\pm$ using $J_- J_+ = \vec{J}^2 - J_z^2 - \hbar J_z$

$$\underbrace{\left(\langle j,m| J_+^\dagger \right)\left(J_+ |j,m\rangle \right)}_{|C_+|^2} = \underbrace{\langle j,m| \vec{J}^2 - J_z^2 - \hbar J_z |j,m\rangle}_{\{j(j+1) - m(m+1)\}\hbar^2}$$

$$\therefore C_+ = \sqrt{j(j+1) - m(m+1)}\,\hbar = \sqrt{(j-m)(j+m+1)}\,\hbar$$

$$j^2 + j - m^2 - m = (j+m)(j-m) + (j-m) = (j-m)(j+m+1)$$

Similarly, $J_+ J_- = \vec{J}^2 - J_z^2 + \hbar J_z$ yields

$$C_- = \sqrt{j(j+1) - m(m-1)}\,\hbar = \sqrt{(j+m)(j-m+1)}\,\hbar.$$

In summary, we get

$$\langle j',m'| J_\pm |j,m\rangle = \underbrace{\sqrt{j(j+1) - m(m\pm 1)}}_{\sqrt{(j\mp m)(j\pm m+1)}}\,\hbar\,\delta_{jj'}\delta_{m',m\pm 1}$$

Eq. (3.5.41), p. 196.

<u>Ex</u>. Prob. 26 of Chapt. 3

Consider a system with $j=1$ and explicitly write $\langle j=1, m'| J_y |j=1, m\rangle$ in 3×3 matrix form.

Since $J_y = \dfrac{J_+ - J_-}{2i}$, we get

$$\langle j=1, m' | J_y | j=1, m \rangle$$

$$= \langle j=1, m' | \frac{J_+ - J_-}{2i} | j=1, m \rangle$$

$$= \frac{1}{2i} \left(\langle j=1, m' | J_+ | j=1, m \rangle - \langle j=1, m' | J_- | j=1, m \rangle \right)$$

$$= \frac{1}{2i} \left(\sqrt{1(1+1) - m(m+1)}\, \hbar\, \delta_{m', m+1} - \sqrt{1(1+1) - m(m-1)}\, \hbar\, \delta_{m', m-1} \right)$$

$$\doteq \frac{\hbar}{2i} \begin{array}{c} \begin{array}{ccc} m'=1, m=1 & m'=1, m=0 & m'=1, m=-1 \end{array} \\ \begin{bmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{bmatrix} \begin{array}{l} m'=1 \\ m'=0 \\ m'=-1 \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ m=1 & m=0 & m=-1 \end{array} \end{array}$$

$$= \frac{i\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

cf. $G_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ Generator of the physical space rotation: (x, y, z) space

$J_y = \frac{i\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ Generator of the Hilbert space rotation: $(|1,1\rangle, |1,0\rangle, |1,-1\rangle)$ space

$$e^{-\frac{iJ_y\beta}{\hbar}} = 1 - \frac{i\beta}{\hbar}J_y + \frac{1}{2!}\left[\left(-\frac{i\beta}{\hbar}\right)J_y\right]^2$$

$$J_y^2 = \frac{(i\hbar)^2}{2}\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$+ \frac{1}{3!}\left[\left(-\frac{i\beta}{\hbar}\right)J_y\right]^3 + \cdots$$

$$J_y^3 = \frac{(i\hbar)^3}{2\sqrt{2}}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} = \hbar^2 J_y$$

$$= 1 - \frac{iJ_y}{\hbar}\underbrace{\left(\beta - \frac{\beta^3}{3!} + \cdots\right)}_{\sin\beta}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$- \left(\frac{J_y}{\hbar}\right)^2\underbrace{\left(\frac{\beta^2}{2!} - \frac{\beta^4}{4!} + \cdots\right)}_{1-\cos\beta}$$

$$= 1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta)$$

cf. $$e^{-\frac{iS_y\beta}{\hbar}} = e^{-i\sigma_y\frac{\beta}{2}} = \cos\frac{\beta}{2} - i\sigma_y\sin\frac{\beta}{2} = \cos\frac{\beta}{2} - i\left(\frac{S_y}{\left(\frac{\hbar}{2}\right)}\right)\sin\frac{\beta}{2}$$

$$\boxed{S_y = \frac{\hbar}{2}\sigma_y}$$

$$= \cos\frac{\beta}{2} - i\left(\frac{2S_y}{\hbar}\right)\sin\frac{\beta}{2}$$

$$J_y^3 = \hbar^2 J_y \qquad \frac{1}{3!}\left[\left(-\frac{i\beta}{\hbar}\right)J_y\right]^3 = \frac{1}{3!}\left(-\frac{i\beta}{\hbar}\right)^3\hbar^2 J_y = -\frac{iJ_y}{\hbar}\left(-\frac{\beta^3}{3!}\right)$$

$$J_y^4 = \hbar^2 J_y^2 \qquad \frac{1}{4!}\left[\left(-\frac{i\beta}{\hbar}\right)J_y\right]^4 = \frac{1}{4!}\left(-\frac{i\beta}{\hbar}\right)^4\hbar^2 J_y^2 = \frac{\beta^4}{4!}\left(\frac{J_y}{\hbar}\right)^2$$

$$J_y^5 = (\hbar^2)^2 J_y$$

$$\vdots$$