Feynman's Path Integral Formulation ①

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$\boxed{\text{2nd Exam Solutions on Moodle} \quad \langle \text{Average} \rangle = 72}$

We introduced the propagator in wave mechanics

$$\psi(\vec{x}'',t) = \int d^3\vec{x}' \; K(\vec{x}'',t\,;\vec{x}',t_0)\; \psi(\vec{x}',t_0)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\langle \vec{x}'' | \alpha, t\rangle \qquad \underbrace{\langle \vec{x}'' | e^{\frac{-iH(t-t_0)}{\hbar}} | \vec{x}' \rangle} \qquad \langle \vec{x}' | \alpha, t_0 \rangle$$

$\downarrow$ change to Heisenberg picture.

$$\underbrace{\langle \vec{x}'' | e^{\frac{-iHt}{\hbar}}} \; \underbrace{e^{\frac{iHt_0}{\hbar}} | \vec{x}' \rangle}$$

$$\langle \vec{x}'', t | \alpha \rangle \qquad \langle \vec{x}'', t | \vec{x}', t_0 \rangle \qquad \langle \vec{x}', t_0 | \alpha \rangle$$

base ket

state ket

Thus, the propagator or the kernel can be reinterpreted as

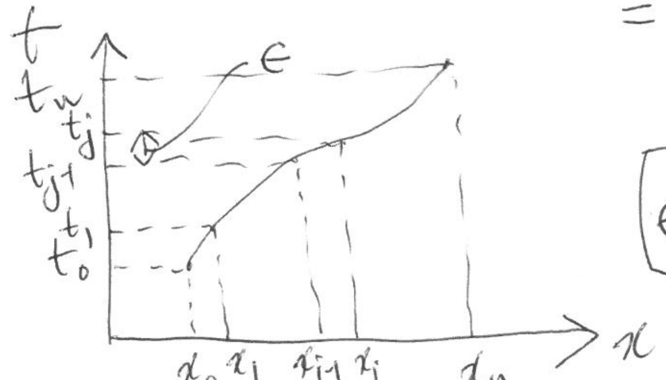
$$K(\vec{x}'',t\,;\vec{x}',t_0) = \langle \vec{x}'', t | \vec{x}', t_0 \rangle$$

As the extension to 3-dim is straightforward, let's consider 1-dim:

$$K(x_n,t_n\,;x_0,t_0) = \langle x_n, t_n | x_0\, t_0 \rangle$$

$$= \int dx_1 \int dx_2 \cdots \int dx_{n-1} \langle x_n, t_n | x_{n-1}, t_{n-1} \rangle \cdots$$

$$\cdots \langle x_j, t_j | x_{j-1}, t_{j-1} \rangle \cdots \langle x_1, t_1 | x_0, t_0 \rangle$$



$$\boxed{\epsilon \equiv \frac{t_n - t_0}{n}}$$

We find

$$\langle x_j, t_j | x_{j-1}, t_{j-1} \rangle = \langle x_j | e^{-\frac{iH\epsilon}{\hbar}} | x_{j-1} \rangle$$

$$= \langle x_j | e^{-\frac{i\epsilon}{\hbar}\left[\frac{p^2}{2m} + V(x)\right]} | x_{j-1} \rangle$$

$$\approx e^{-\frac{i\epsilon}{\hbar} V\left(\frac{x_j + x_{j-1}}{2}\right)} \underbrace{\langle x_j | e^{-\frac{i\epsilon}{\hbar}\frac{p^2}{2m}} | x_{j-1} \rangle}$$

See this derivation in the Appendix of this lecture note at the end $\rightarrow$ $\sqrt{\frac{m}{2\pi i\hbar\epsilon}}\, e^{\frac{i}{\hbar}\frac{m(x_j - x_{j-1})^2}{2\epsilon}}$

$$= \sqrt{\frac{m}{2\pi i\hbar\epsilon}}\, \exp\left[\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_j - x_{j-1}}{\epsilon}\right)^2 - V\left(\frac{x_j + x_{j-1}}{2}\right)\right\}\epsilon\right]$$

Thus, we get

$$K(x_n, t_n; x_0 t_0) = \underbrace{\lim_{\epsilon \to 0}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{\frac{n}{2}} \int_{-\infty}^{\infty} dx_1\, dx_2 \cdots dx_{n-1}}_{\text{Path Integral; } \int_{x_0}^{x_n} \mathcal{D}[x(t)]}\, e^{\frac{i}{\hbar}\underbrace{\sum_{j=1}^{n}\left\{\frac{1}{2}m\left(\frac{x_j - x_{j-1}}{\epsilon}\right)^2 - V\left(\frac{x_j + x_{j-1}}{2}\right)\right\}\epsilon}}$$

$$\int_{t_0}^{t_n} dt' \underbrace{\left\{\frac{1}{2}m\dot{x}^2(t') - V(x(t'))\right\}}_{L(x,\dot{x})}$$

$$= \int_{x_0}^{x_n} \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar}\int_{t_0}^{t_n} dt\, L(x,\dot{x})}$$

$$= \int_{x_0}^{x_n} \mathcal{D}[x(t)]\, e^{\frac{i\, S[x(t)]}{\hbar}}$$

$$\boxed{\begin{array}{l}\text{Action} \\ \hline \int dt\, L(x,\dot{x}) = S\end{array}}$$

cf. Least Action principle in Classical Mechanics.

$$S[x(t)] = S[x_{cl}(t) + \delta x(t)]$$

$$= S[x_{cl}(t)] + \delta x \left.\frac{\delta S}{\delta x}\right|_{x=x_{cl}} + \frac{1}{2!}(\delta x)^2 \frac{\delta^2 S}{\delta x^2} + \cdots$$

$\downarrow$

0 due to the stability of classical trajectory.

$$= S[x_{cl}(t)] + S[\delta x(t)]$$

Note here that any slight modification in the path gives very different phase due to the smallness of $\hbar$. Thus, there is a tendency of cancellation among various contributions from neighboring paths. Exception is the classical path due to $\left.\frac{\delta S}{\delta x}\right|_{x=x_{cl}} = 0$ and as $\hbar \to 0$ only the classical trajectory contributes/survives.

Thus, we get

$$K(x_n, t_n; x_0, t_0) = e^{\frac{i S[x_{cl}(t)]}{\hbar}} \underbrace{\int_{x_0}^{x_n} \mathcal{D}[\underset{\delta x(t)}{x(t)}]\, e^{\frac{i S[\delta x(t)]}{\hbar}}}_{K(\underset{0}{\delta x_n}, t_n; \underset{0}{\delta x_0}, t_0) \equiv J(t_n, t_0)}$$

$$= J(t_n, t_0)\, e^{\frac{i S[x_{cl}(t)]}{\hbar}}$$

Cf. $\psi(\vec{x}, t) = \sqrt{\rho(\vec{x}, t)}\, e^{\frac{i S(\vec{x}, t)}{\hbar}}$

Continuity eq: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$, where $\vec{J} = \rho\left(\frac{\overset{\vec{p}}{\vec{\nabla} S}}{m}\right)$

$$\boxed{\begin{aligned} S &= \int d\vec{x} \cdot \vec{\nabla} S \qquad \left(\frac{d\vec{p}}{dt} = -\vec{\nabla} V = \vec{\nabla} L\right) \\ &= \int d\vec{x} \cdot \vec{p} \\ &= \int dt \int d\vec{x} \cdot \vec{\nabla} L \\ &= \int dt \int dL = \int dt\, L \end{aligned}}$$

Appendix : Calculation of matrix element $\langle x_j | e^{-\frac{i\epsilon}{\hbar}\frac{p^2}{2m}} | x_{j-1} \rangle$ ④

$$\langle x_j | e^{-\frac{i\epsilon}{\hbar}\frac{p^2}{2m}} | x_{j-1} \rangle$$

$$= \langle x_j | \int_{-\infty}^{+\infty} dp' |p'\rangle\langle p'| \, e^{-\frac{i\epsilon}{\hbar}\frac{p^2}{2m}} \int_{-\infty}^{+\infty} dp'' |p''\rangle\langle p''| x_{j-1} \rangle$$

$$= \int_{-\infty}^{+\infty} dp' \underbrace{\langle x_j | p' \rangle}_{\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i x_j p'}{\hbar}}} e^{-\frac{i\epsilon}{\hbar}\frac{p'^2}{2m}} \underbrace{\langle p' | x_{j-1} \rangle}_{\frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i x_{j-1} p'}{\hbar}}} \quad \left(\text{using } \int_{-\infty}^{+\infty} dp'' \delta(p'-p'') = 1 \right)$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp' \, e^{\frac{i}{\hbar}\left\{ \underbrace{p' \cdot (x_j - x_{j-1}) - \frac{p'^2}{2m}\epsilon}_{-\frac{\epsilon}{2m}\left\{ p' - \frac{m(x_j - x_{j-1})}{\epsilon} \right\}^2 + \frac{m(x_j - x_{j-1})^2}{2\epsilon}} \right\}}$$

$$= \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar}\frac{m(x_j - x_{j-1})^2}{2\epsilon}} \underbrace{\int_{-\infty}^{+\infty} dp'' e^{-\frac{i}{\hbar}\frac{\epsilon}{2m} p''^2}}_{\sqrt{\frac{2\pi\hbar\pi}{i\epsilon}}} \quad \left(\text{cf. } \int_{-\infty}^{+\infty} dx \, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \quad \left(\text{Take } \alpha = \frac{i\epsilon}{2\pi\hbar} \right) \right)$$

$$= \sqrt{\frac{m}{2\pi i \hbar \epsilon}} \, e^{\frac{i}{\hbar}\frac{m(x_j - x_{j-1})^2}{2\epsilon}}$$

Note here $\frac{x_j - x_{j-1}}{\epsilon} = \frac{\Delta x}{\Delta t} = v$ and $\frac{m(x_j - x_{j-1})^2}{2\epsilon} = \left(\frac{1}{2} m v^2\right) \cdot \epsilon$

Note also the sign change before and after taking the position representation:
Before $e^{\frac{-iH\epsilon}{\hbar}} = e^{\frac{-i\epsilon}{\hbar}\frac{p^2}{2m}}$; After $e^{+\frac{i\epsilon}{\hbar}\left(\frac{1}{2}mv^2\right)}$ with the normalization factor in front.