Rotation and Angular Momentum

Review Lecture Notes Nov. 15 & 17 on moodle while I travel next week, HW8 will be given on Nov. 22

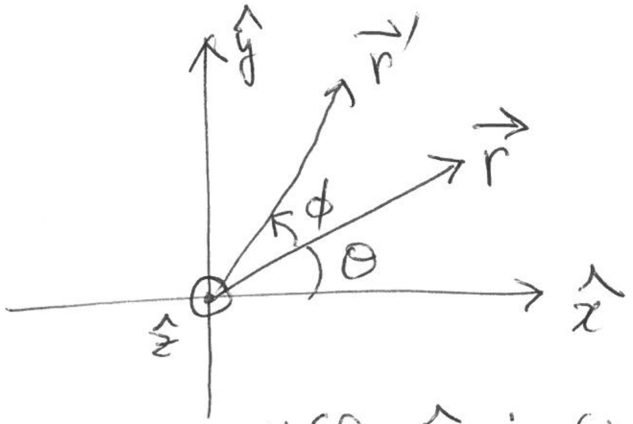
Nov. 10, 2016

The importance of angular momentum in modern physics can hardly be overemphasized. <u>A through understanding of angular momentum is essential in molecular, atomic and nuclear spectroscopy.</u> Angular momentum considerations play an important role in scattering and collision problems as well as in bound-state problems.

* Rotation involves "Non-inertial frame" (e.g. "Thomas precession" in spin-orbit coupling in atomic system)

Quantization shown in <u>spin</u> will be generalized through the group properties, e.g. $SO(3)$ & $SU(2)$, etc.

Consider a rotation of $\vec{r}$ to $\vec{r}\,'$ by an angle ϕ around $\hat{z}$-axis as shown below.



$$\vec{r} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y}$$

$$\vec{r}\,' = r\cos(\theta+\phi)\,\hat{x} + r\sin(\theta+\phi)\,\hat{y}$$

Let's take $r=1$ for simplicity without loss of any generality, and

use $\hat{x} \doteq \begin{pmatrix}1\\0\end{pmatrix}$ and $\hat{y} \doteq \begin{pmatrix}0\\1\end{pmatrix}$.

Then, $\vec{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \doteq \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

and

$$\vec{r}\,' \doteq \begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}}_{R_z(\phi)} \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}_{\vec{r}}$$

$R_z^T(\phi) = R_z^{-1}(\phi) = R_z(-\phi)$

or $R_z^T(\phi) R_z(\phi) = I$

Orthogonal!

$\mathrm{Det}\, R_z(\phi) = 1$; Special! $\rightarrow$ SO(3)

special orthogonal group in 3-dim.

or $\vec{r}\,' = R_z(\phi)\,\vec{r}$, where $R_z(\phi) \doteq \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

We may generalize to other axis by taking the element 1 to other diagonal element corresponding to the axis chosen. $R_x(\phi) \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$, $R_y(\phi) \doteq \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$

Note that finite rotations around different axes are not commutative or "Non-commutative".

e.g. $R_x\left(\frac{\pi}{2}\right) R_z\left(\frac{\pi}{2}\right) \neq R_z\left(\frac{\pi}{2}\right) R_x\left(\frac{\pi}{2}\right)$.

SO(3) group is a <u>non-abelian</u> group.

S: special, O: orthogonal, 3: 3-dim.

"non-commutative"

See Fig. 3.1

However, infinitesimal rotations about different axes <u>commute</u>!

$$R_x(\varepsilon)\,R_z(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\varepsilon^2}{2} & -\varepsilon \\ 0 & \varepsilon & 1-\frac{\varepsilon^2}{2} \end{pmatrix} \begin{pmatrix} 1-\frac{\varepsilon^2}{2} & -\varepsilon & 0 \\ \varepsilon & 1-\frac{\varepsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{O(\varepsilon^2)}{=} \begin{pmatrix} 1-\frac{\varepsilon^2}{2} & -\varepsilon & 0 \\ \varepsilon & 1-\varepsilon^2 & -\varepsilon \\ \varepsilon^2 & \varepsilon & 1-\frac{\varepsilon^2}{2} \end{pmatrix} \underset{\text{up to } O(\varepsilon)}{\approx} \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & -\varepsilon \\ 0 & \varepsilon & 1 \end{pmatrix}$$

$$R_z(\varepsilon)\,R_x(\varepsilon) = \begin{pmatrix} 1-\frac{\varepsilon^2}{2} & -\varepsilon & 0 \\ \varepsilon & 1-\frac{\varepsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\varepsilon^2}{2} & -\varepsilon \\ 0 & \varepsilon & 1-\frac{\varepsilon^2}{2} \end{pmatrix}$$

$$\overset{O(\varepsilon^2)}{=} \begin{pmatrix} 1-\frac{\varepsilon^2}{2} & -\varepsilon & \varepsilon^2 \\ \varepsilon & 1-\varepsilon^2 & -\varepsilon \\ 0 & \varepsilon & 1-\frac{\varepsilon^2}{2} \end{pmatrix} \underset{\text{up to } O(\varepsilon)}{\approx} \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & -\varepsilon \\ 0 & \varepsilon & 1 \end{pmatrix}$$

Also, we note

$$R_z(\varepsilon)\,R_x(\varepsilon) - R_x(\varepsilon)\,R_z(\varepsilon) \overset{O(\varepsilon^2)}{\approx} \begin{pmatrix} 0 & 0 & +\varepsilon^2 \\ 0 & 0 & 0 \\ -\varepsilon^2 & 0 & 0 \end{pmatrix}$$

$$= \underbrace{R_y(\varepsilon^2)}_{\begin{pmatrix} \cos\varepsilon^2 & 0 & \sin\varepsilon^2 \\ 0 & 1 & 0 \\ -\sin\varepsilon^2 & 0 & \cos\varepsilon^2 \end{pmatrix}} - I \longleftarrow R_{\text{any}}(0)$$

$\boxed{\text{See p. 160, Eq. (3.1.7)}}$

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{\phi^2}{2!}+\frac{\phi^4}{4!}+\cdots & -\phi+\frac{\phi^3}{3!}-\cdots & 0 \\ \phi-\frac{\phi^3}{3!}+\cdots & 1-\frac{\phi^2}{2!}+\frac{\phi^4}{4!}+\cdots & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} + \phi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\phi^2}{2!} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \frac{\phi^3}{3!} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\phi^4}{4!} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots$$

$$= I + \phi G_z + \frac{\phi^2}{2!} G_z^2 + \frac{\phi^3}{3!} G_z^3 + \frac{\phi^4}{4!} G_z^4 + \cdots$$

$$= e^{\phi G_z}, \text{ where } G_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly,

$$R_x(\phi) = e^{\phi G_x}, \text{ where } G_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$R_y(\phi) = e^{\phi G_y}, \text{ where } G_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Note that

$$G_x G_y = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_y G_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } [G_x, G_y] = G_z$$

In general, $[G_i, G_j] = \epsilon_{ijk} G_k$.

Also, note that

$$e^{\varepsilon G_z} e^{\varepsilon G_x} - e^{\varepsilon G_x} e^{\varepsilon G_z} = e^{\varepsilon^2 G_y} - I \quad \text{for up to } \varepsilon^2 \text{ order},$$

as shown in Eq. (3.1.7) p.160.

Up to ε^2 order, one can see the above equality holds

$$\left(1+\varepsilon G_z + \frac{\varepsilon^2}{2!} G_z^2\right)\left(1+\varepsilon G_x + \frac{\varepsilon^2}{2!} G_x^2\right) - \left(1+\varepsilon G_x + \frac{\varepsilon^2}{2!} G_x^2\right)\left(1+\varepsilon G_z + \frac{\varepsilon^2}{2!} G_z^2\right)$$

$$= \varepsilon^2 \underbrace{[G_z, G_x]}_{G_y} = \varepsilon^2 G_y = \underbrace{\left(I+\varepsilon^2 G_y\right)}_{e^{\varepsilon^2 G_y} \text{ expansion up to } \varepsilon^2 \text{ order}} - I$$

Lie group; continuous transformation group "degree 2" (because only 3 generators, J_x, J_y, J_z)

In the Hilbert space, $G_k \longrightarrow \frac{J_k}{i\hbar} = -\frac{i}{\hbar} J_k$

so that $\boxed{SO(3);\ e^{G_k \phi}} \rightarrow \boxed{e^{\frac{-iJ_k\phi}{\hbar}};\ SU(2)}$ Det$\left[e^{\frac{-iJ_k\phi}{\hbar}}\right] = 1$; special "special" "unitary" $e^{\frac{-iJ_k\phi}{\hbar}}\left[e^{\frac{-iJ_k\phi}{\hbar}}\right]^\dagger = I$ unitary

$$[G_i, G_j] = \epsilon_{ijk} G_k \longrightarrow [J_i, J_j] = i\hbar \epsilon_{ijk} J_k$$

Note that we had already the spin algebra given by Eq. (1.4.20) p.28, i.e. $[S_i, S_j] = i\hbar \epsilon_{ijk} S_k$

See the lecture note on Nov. 15 "Pauli's Two-Component Formalism" $\boxed{\vec{S} = \frac{\hbar}{2}\vec{\sigma}}$

What we obtained here is the generalization of the spin algebra to any angular momentum algebra. (e.g. orbital angular momentum as well as any other spin angular momentum beyond spin-$\frac{1}{2}$)

Infinitesimal rotations in Q.M. Hilbert space introduce the angular momentum generators; J_x, J_y, J_z.

e.g. Orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$.

From $\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$, we have

$$L_x = y p_z - z p_y, \quad L_y = z p_x - x p_z, \quad L_z = x p_y - y p_x .$$

Using the Q.M. commutation relations between position and momentum, i.e. $[x_i, p_j] = i\hbar \delta_{ij}$, we get

e.g.

$$[L_x, L_y] = [y p_z - z p_y, \; z p_x - x p_z]$$

$$= \underbrace{[y p_z, z p_x]}_{y \underbrace{[p_z, z p_x]}_{\underbrace{[p_z, z]}_{-i\hbar} p_x + z \underbrace{[p_z, p_x]}_{0}} + [y, z p_x] p_z} - [y p_z, x p_z] - [z p_y, z p_x] + \underbrace{[z p_y, x p_z]}_{z[p_y, x p_z] + \underbrace{[z, x p_z]}_{\underbrace{[z, x]}_{0} p_z + x \underbrace{[z, p_z]}_{i\hbar}} p_y}$$

$$= i\hbar (x p_y - y p_x) + \boxed{[y, z p_x] p_z - [y p_z, x p_z] - [z p_y, z p_x] + z[p_y, x p_z]}$$

$$= i\hbar L_z$$

$$\downarrow$$

$$\cancel{y z p_x p_z} - \cancel{z p_x y p_z} - \cancel{y p_z x p_z} + \cancel{x p_z y p_z}$$
$$- \cancel{z p_y z p_x} + \cancel{z p_x z p_y} + \cancel{z p_y x p_z} - \cancel{z x p_z p_y}$$
$$\downarrow$$
$$0$$

or

$$[L_x, L_y] = i\hbar L_z$$

as it should be!

Define a rotation in Hilbert space as Drehung (meaning rotation in German) given by

$$|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle .$$

R ↑ rotated state

More specifically, Drehung by angle ϕ around $\hat{n}$ axis is denoted as

$$\boxed{\mathcal{D}(\hat{n},\phi) = e^{-\frac{i\vec{J}\cdot\hat{n}}{\hbar}\phi}}$$

Recall the transition operator $e^{-\frac{ipl}{\hbar}}$ to get

$$|x+l\rangle = e^{-\frac{ipl}{\hbar}}|x\rangle$$

$e^{-\frac{ipl}{\hbar}}|x\rangle = (1-\frac{ipl}{\hbar}+\cdots)|x\rangle = |x\rangle - \frac{ipl}{\hbar}|x\rangle$

$p|x\rangle = -\frac{\hbar}{i}\frac{\partial}{\partial x}|x\rangle$

$\langle x|p = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|$

As an example, $\mathcal{D}(\hat{z},\phi)$ rotates $|x,y,z\rangle$ to $|x',y',z\rangle$, where $x' = x\cos\phi - y\sin\phi$ and $y' = x\sin\phi + y\cos\phi$, i.e.

$$e^{-\frac{iJ_z\phi}{\hbar}}|x,y,z\rangle = |x\cos\phi - y\sin\phi,\, x\sin\phi + y\cos\phi,\, z\rangle$$

This can be verified ~~taking~~ J_z as $L_z = xp_y - yp_x$.

Note that $\langle y|p_y = \frac{\hbar}{i}\frac{\partial}{\partial y}\langle y|$ or $p_y|y\rangle = -\frac{\hbar}{i}\frac{\partial}{\partial y}|y\rangle$, etc.

$$|x\cos\phi - y\sin\phi,\, x\sin\phi + y\cos\phi,\, z\rangle = |x,y,z\rangle + \phi\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)|x,y,z\rangle + \cdots$$

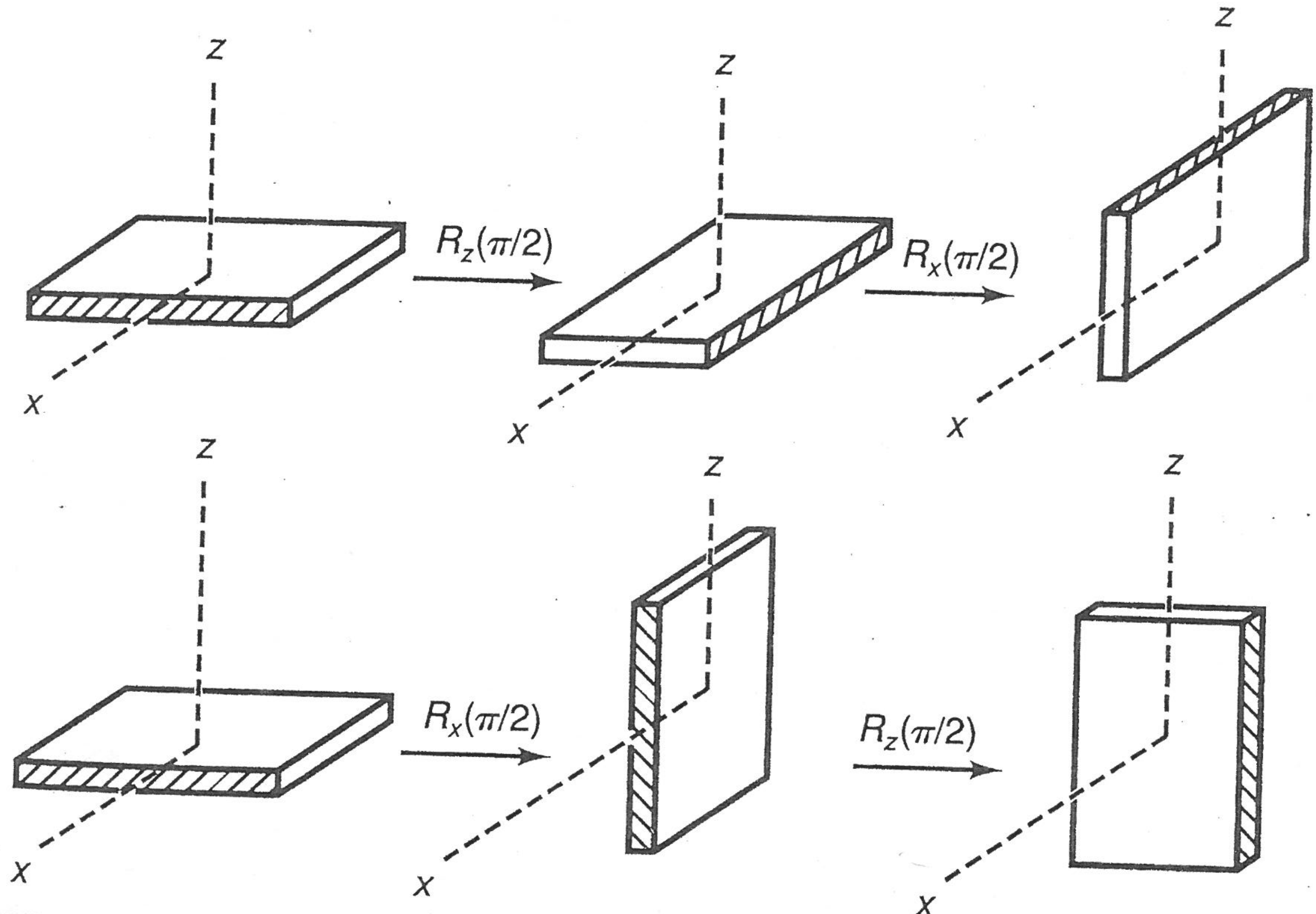
$$= \left(1 - \frac{i\phi}{\hbar}L_z + \cdots\right)|x,y,z\rangle = e^{-\frac{iL_z\phi}{\hbar}}|x,y,z\rangle .$$

While I travel next week,
please review the lecture notes on Moodle:

Nov. 15: Pauli's Two-Component Formalism
(Remind the spin algebra with Pauli matrices)

Nov. 17: $SO(3)$ and $SU(2)$
(Study the Euler angles with respect to Lab frame vs. Body frame)

Homework #8 (Last Homework) will be
given on Nov. 22.


z
x
$R_z(\pi/2)$
z
x
$R_x(\pi/2)$
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z
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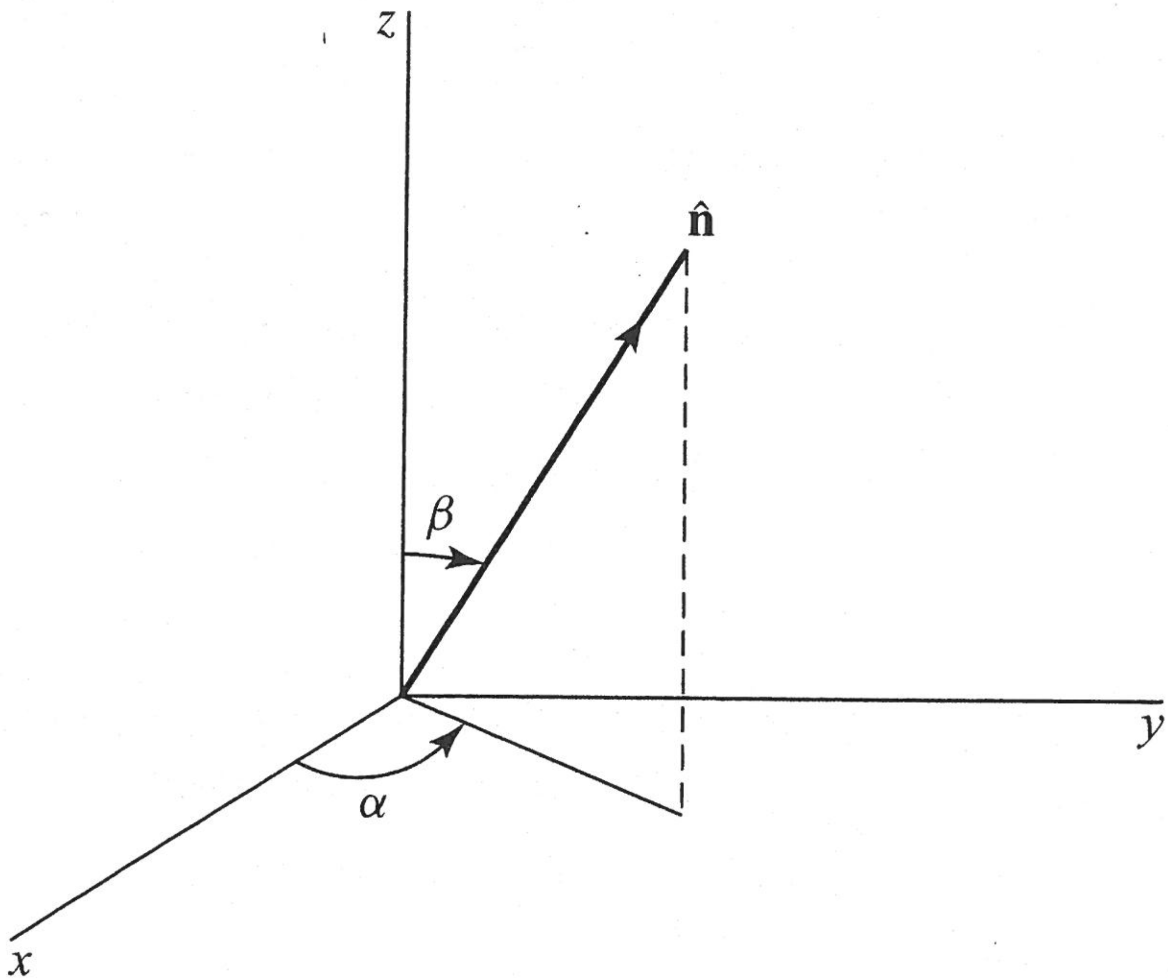
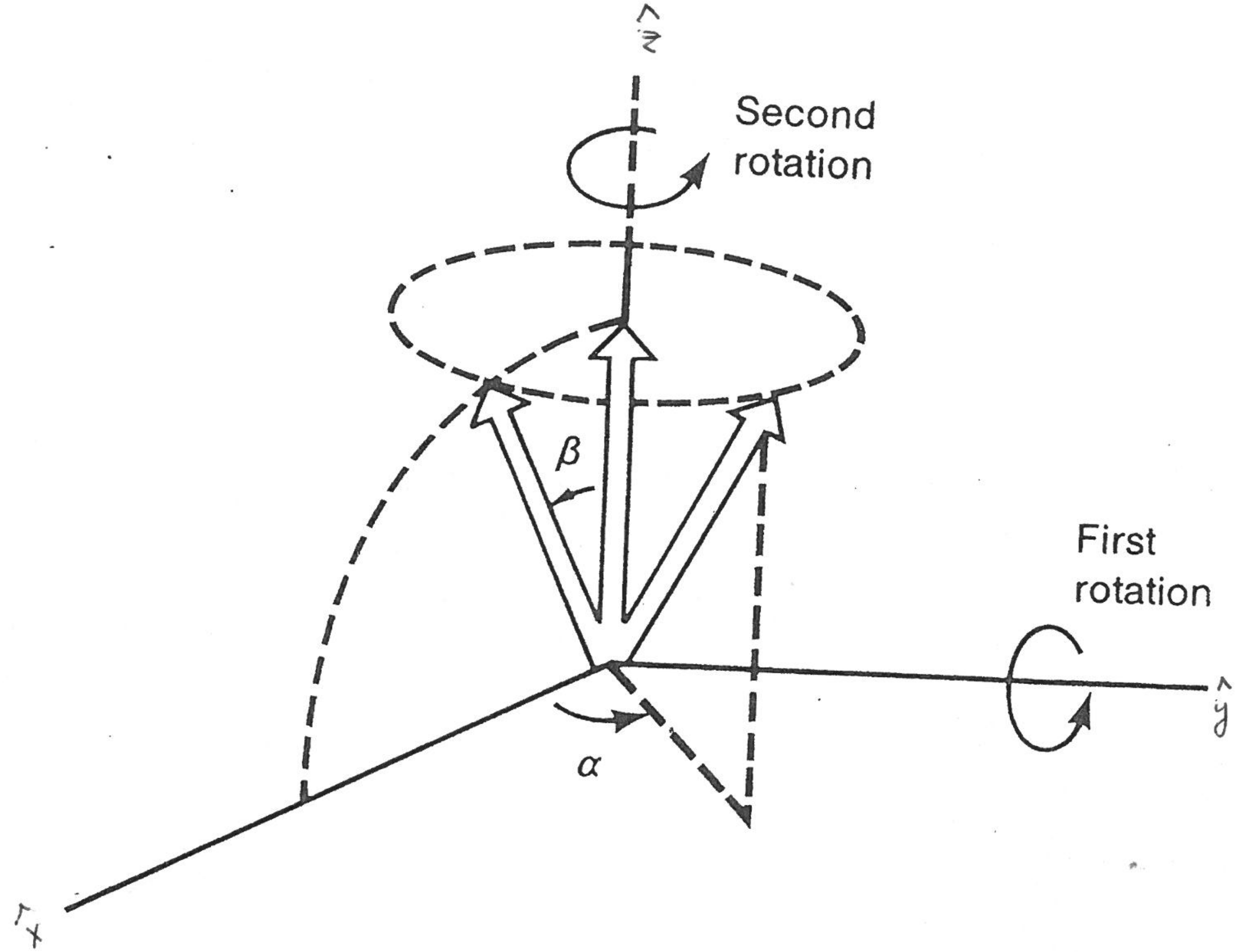


z
n̂
β
y
α
x



$\hat{z}$
Second rotation
β
First rotation
$\hat{y}$
α
$\hat{x}$

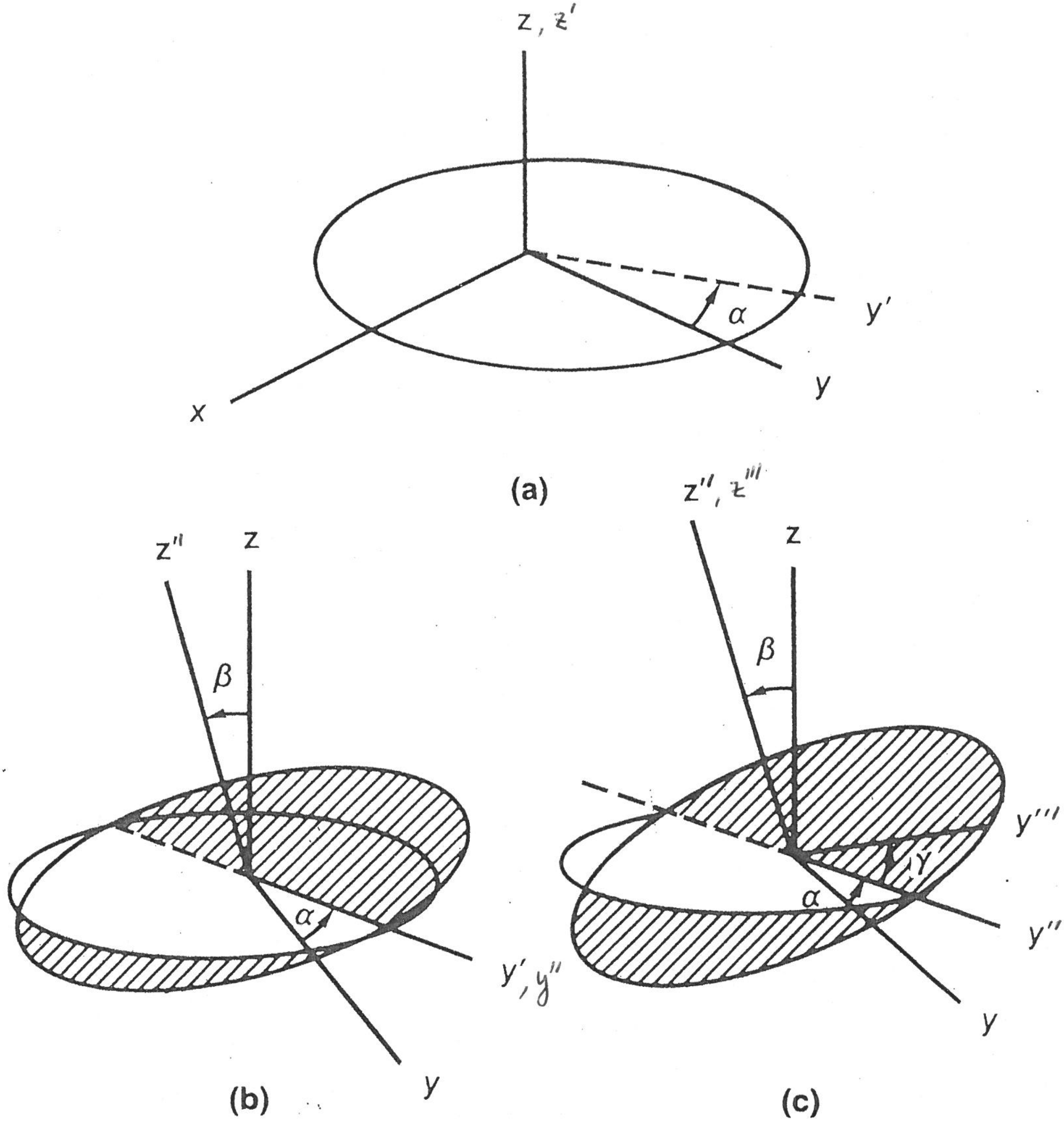


FIGURE 3.4. Euler rotations.