Correlation Amplitude and Energy-Time Uncertainty ①

$\boxed{\text{Oct. 6, Fall Break}}$ Oct. 4, 2016

Let's discuss the time evolution of a system using Hamiltonian for a spin-magnetic moment interacting with a time-independent magnetic field.

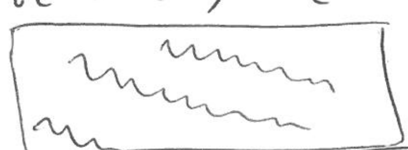
$$H = -\vec{\mu}\cdot\vec{B}, \text{ where } \vec{B} = B\hat{z}.$$

constant (or uniform) homogeneous field.

As $\vec{\mu} = g\frac{e}{2m_e c}\vec{S}$ $(g=2,\ e<0)$ for an electron,

$$H = -\left(\frac{eB}{m_e c}\right) S_z,$$

For muon, $m_e \to m_\mu$. (See slide)



$\omega = \frac{|e|B}{m_\mu c}$

and

$$H|S_z\pm\rangle = E_\pm|S_z\pm\rangle, \text{ where } E_\pm = \mp\frac{e\hbar B}{2m_e c}.$$

Define $\omega = \frac{|e|B}{m_e c} > 0$, then

$$H = \omega S_z \quad \text{and} \quad E_\pm = \pm\frac{\hbar\omega}{2}.$$

Time-evolution operator is given by

$$\mathcal{U}(t,t_0) = e^{\frac{-iH(t-t_0)}{\hbar}} = e^{\frac{-i\omega S_z(t-t_0)}{\hbar}}$$

Base kets $|+\rangle$ and $|-\rangle$ lead to $\underbrace{|\alpha\rangle}_{|\alpha, t_0=0\rangle} = C_+|+\rangle + C_-|-\rangle$,

where $C_+ = \langle +|\alpha\rangle$ and $C_- = \langle -|\alpha\rangle$.

$$|\alpha, t\rangle = \mathcal{U}(t, t_0 = 0)|\alpha\rangle$$

$$= C_+ e^{-i\frac{\omega}{2}t}|+\rangle + C_- e^{i\frac{\omega}{2}t}|-\rangle$$

due to $H|\pm\rangle = \pm\frac{\hbar\omega}{2}|\pm\rangle$.

i) $C_+ = 1,\ C_- = 0$; stationary state ($|\alpha\rangle = |+\rangle$)

$$|\alpha, t\rangle = e^{-i\frac{\omega t}{2}}|+\rangle$$ always spin-up state.

overall phase

ii) $C_+ = C_- = \frac{1}{\sqrt{2}}$ or $|\alpha\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = |S_x+\rangle$.

$$|\alpha, t\rangle = \frac{1}{\sqrt{2}}e^{\frac{-i\omega t}{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{\frac{i\omega t}{2}}|-\rangle$$

Probability to find $|S_x\pm\rangle$ state at time t :

$$|\langle S_x \pm|\alpha, t\rangle|^2 = \left|\left(\frac{1}{\sqrt{2}}\langle +| \pm \frac{1}{\sqrt{2}}\langle -|\right)\left(\frac{1}{\sqrt{2}}e^{\frac{-i\omega t}{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{\frac{i\omega t}{2}}|-\rangle\right)\right|^2$$

$$= \left|\frac{1}{2}e^{\frac{-i\omega t}{2}} \pm \frac{1}{2}e^{\frac{i\omega t}{2}}\right|^2$$

$$= \begin{cases} \cos^2\frac{\omega t}{2} & \text{for } |S_x+\rangle \\ \sin^2\frac{\omega t}{2} & \text{for } |S_x-\rangle \end{cases}$$

Note the unitarity and conservation of Probability sum.

$$|\langle S_x+|\alpha, t\rangle|^2 + |\langle S_x-|\alpha, t\rangle|^2 = \cos^2\frac{\omega t}{2} + \sin^2\frac{\omega t}{2} = 1.$$

Correlation Amplitude is a quantitative measure of the "resemblance" between the state kets at different times. ③

$$C(t) = \langle \alpha, t_0=0 \mid \alpha, t \rangle$$

$$= \langle \alpha, t_0=0 \mid \mathcal{U}(t,0) \mid \alpha, t_0=0 \rangle$$

(Expectation value of the time evolution operator)

$$= \left(C_+^* \langle + | + C_-^* \langle - | \right) \left(C_+ e^{\frac{-i\omega t}{2}} |+\rangle + C_- e^{\frac{i\omega t}{2}} |-\rangle \right)$$

$$= |C_+|^2 e^{\frac{-i\omega t}{2}} + |C_-|^2 e^{\frac{i\omega t}{2}}$$

$$|C(t)|^2 = C^*(t)\, C(t)$$

$$= \left(|C_+|^2 e^{\frac{i\omega t}{2}} + |C_-|^2 e^{\frac{-i\omega t}{2}} \right) \left(|C_+|^2 e^{\frac{-i\omega t}{2}} + |C_-|^2 e^{\frac{i\omega t}{2}} \right)$$

$$= |C_+|^4 + |C_-|^4 + |C_+|^2 |C_-|^2 \underbrace{\left(e^{i\omega t} + e^{-i\omega t} \right)}_{2\cos\omega t}$$

$$= \underbrace{\left(|C_+|^2 + |C_-|^2 \right)}_{1}{}^2 - 2|C_+|^2|C_-|^2 + 2|C_+|^2|C_-|^2 \cos\omega t$$

$$= 1 - 2|C_+|^2|C_-|^2 (1 - \cos\omega t)$$

$$= 1 - 4|C_+|^2|C_-|^2 \sin^2\frac{\omega t}{2}$$

$C_+ = 1,\ C_- = 0$

$|C_+|^2 = 1$

$C_+ = C_- = \frac{1}{\sqrt{2}}$

$|C(t)|^2 = 1 - \sin^2\frac{\omega t}{2}$

$= \cos^2\frac{\omega t}{2}$

$$(\Delta E)_t^2 = \langle \alpha, t | (\Delta H)^2 | \alpha, t \rangle$$

$$= \langle \alpha, t | H^2 | \alpha, t \rangle - \langle \alpha, t | H | \alpha, t \rangle^2$$

$$= \left(C_+^* e^{\frac{i\omega t}{2}} \langle + | + C_-^* e^{-\frac{i\omega t}{2}} \langle - | \right) \left(\omega^2 S_z^2 \right) \left(C_+ e^{-\frac{i\omega t}{2}} |+\rangle + C_- e^{\frac{i\omega t}{2}} |-\rangle \right)$$

$$- \Big\{ \underbrace{\left(C_+^* e^{\frac{i\omega t}{2}} \langle + | + C_-^* e^{-\frac{i\omega t}{2}} \langle - | \right) (\omega S_z) \left(C_+ e^{-\frac{i\omega t}{2}} |+\rangle + C_- e^{\frac{i\omega t}{2}} |-\rangle \right)}_{\omega \langle S_z \rangle_t} \Big\}^2$$

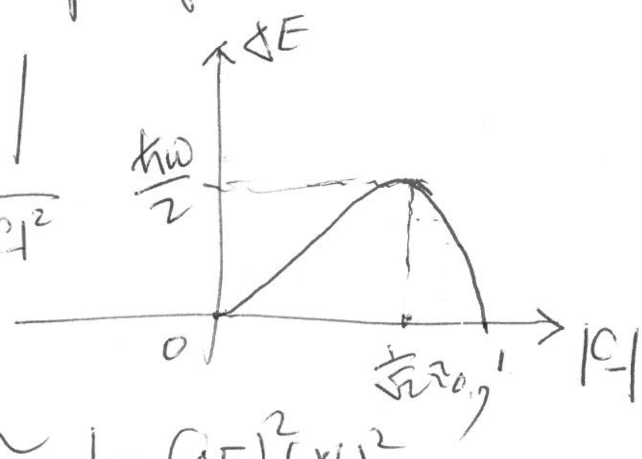
$$= \frac{\hbar^2}{4} \omega^2 \left(|C_+|^2 + |C_-|^2 \right)$$

$$- \omega^2 \left\{ \frac{\hbar}{2} \left(|C_+|^2 - |C_-|^2 \right) \right\}^2$$

$$= \frac{\hbar^2 \omega^2}{4} \Big\{ \underset{(|C_+|^2 + |C_-|^2)^2}{\underset{\uparrow}{1}} - \left(|C_+|^2 - |C_-|^2 \right)^2 \Big\}$$

$$= \hbar^2 \omega^2 |C_+|^2 |C_-|^2 \quad \text{indep. of time}$$

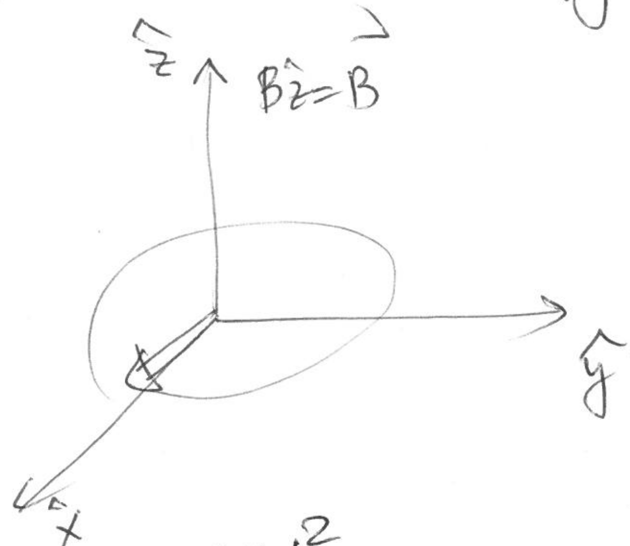
Thus, $\Delta E = \underbrace{\hbar\omega}_{E_+ - E_-} \underbrace{|C_+|}_{\sqrt{1 - |C_-|^2}} |C_-|$



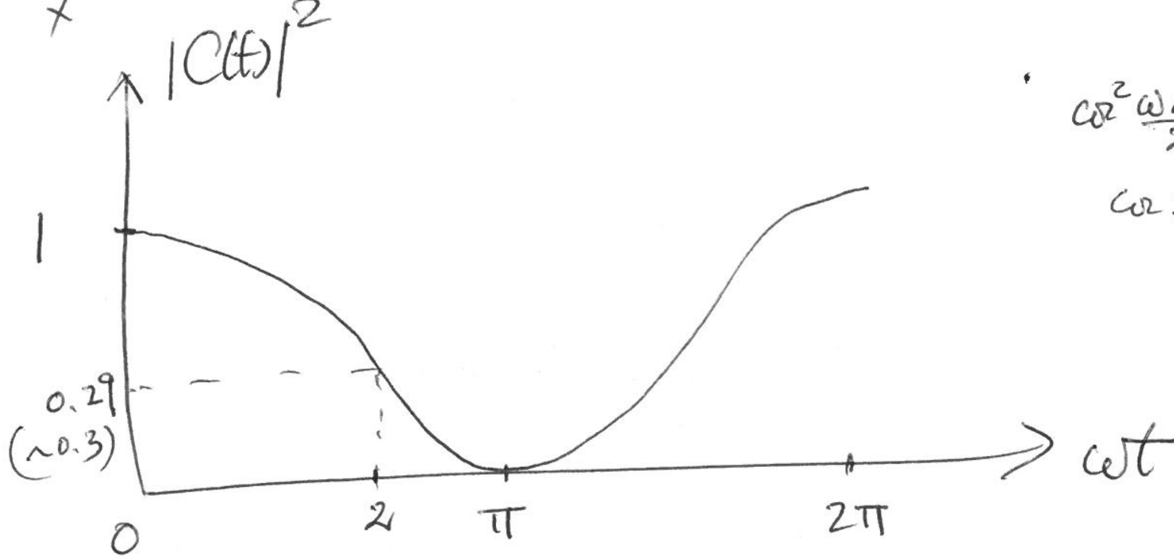
$$|C(t)|^2 = 1 - 4 \frac{(\Delta E)^2}{\hbar^2 \omega^2} \underbrace{\sin^2 \frac{\omega t}{2}}_{\left(\frac{\omega \Delta t}{2} \right)^2} \sim 1 - \frac{(\Delta E)^2 (\Delta t)^2}{\hbar^2}$$

$$\therefore \boxed{(\Delta E)(\Delta t) \approx \hbar}$$

For the maximum energy uncertainty, $C_+ = C_- = \frac{1}{\sqrt{2}}$.



$$|C(t)|^2 = 1 - \sin^2\frac{\omega t}{2}$$
$$= \cos^2\frac{\omega t}{2}$$
$$= \frac{1+\cos\omega t}{2}$$



$\cos^2\frac{\omega\Delta t}{2} = \cos^2 1 \approx 0.29$

$\cos 1 \approx 0.54$.

$$\Delta E = \frac{\hbar\omega}{2} \quad \text{cf. } (\Delta E)^2 = \frac{\hbar^2\omega^2}{4}. \ ; \quad E_+ - E_- = \hbar\omega$$

$$\Delta t \approx \frac{\hbar}{\Delta E} = \frac{2}{\omega} \quad (\Delta E \cdot \Delta t \approx \hbar)$$

Time-Energy uncertainty relation. p.80 Eq. (2-1.74).

This relation is different from the uncertainty relation between the two incompatible observables.

Note that <u>time</u> is <u>not an operator</u> <u>but a parameter</u>.

Expectation values of S_x, S_y and S_z.

$$\langle \alpha,t | S_z | \alpha,t\rangle$$

$$= \left(c_+^* e^{\frac{i\omega t}{2}} \langle +| + c_-^* e^{\frac{-i\omega t}{2}} \langle -| \right) \left(\frac{\hbar}{2} |+\rangle\langle +| - \frac{\hbar}{2} |-\rangle\langle -| \right) \left(c_+ e^{\frac{-i\omega t}{2}} |+\rangle + c_- e^{\frac{i\omega t}{2}} |-\rangle \right)$$

$$= \left(|c_+|^2 - |c_-|^2 \right) \frac{\hbar}{2}$$

$$\boxed{= \frac{\hbar}{2} \left(|c_+|^2 - |c_-|^2 \right)}$$

$\xrightarrow{|\alpha, t=0\rangle = |S_z +\rangle}$ $c_+ = 1$, $c_- = 0$; $\langle S_z \rangle_t = \frac{\hbar}{2}$

$\xrightarrow{|\alpha, t=0\rangle = |S_x +\rangle}$ $c_+ = c_- = \frac{1}{\sqrt{2}}$; $\langle S_z \rangle_t = 0$

$$\langle \alpha,t | S_x | \alpha,t\rangle$$

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \left(c_+^* e^{\frac{i\omega t}{2}} \langle +| + c_-^* e^{\frac{-i\omega t}{2}} \langle -| \right) \left(\frac{\hbar}{2} \right) \left(|+\rangle\langle -| + |-\rangle\langle +| \right) \left(c_+ e^{\frac{-i\omega t}{2}} |+\rangle + c_- e^{\frac{i\omega t}{2}} |-\rangle \right)$$

$$= \left(c_+^* c_- e^{i\omega t} + c_-^* c_+ e^{-i\omega t} \right) \frac{\hbar}{2}$$

$$\boxed{= \frac{\hbar}{2} \left(c_+^* c_- e^{i\omega t} + c_-^* c_+ e^{-i\omega t} \right)}$$

$\rightarrow$ $c_+ = 1$, $c_- = 0$; $\langle S_x \rangle_t = 0$

$\rightarrow$ $c_+ = c_- = \frac{1}{\sqrt{2}}$; $\langle S_x \rangle_t = \frac{\hbar}{2} \cos\omega t$

$$\langle \alpha,t | S_y | \alpha,t\rangle$$

$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

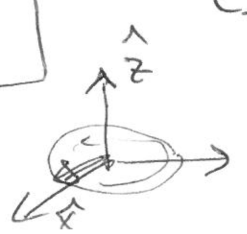
$$= \left(c_+^* e^{\frac{i\omega t}{2}} \langle +| + c_-^* e^{\frac{-i\omega t}{2}} \langle -| \right) \left(\frac{\hbar}{2} \right) \left(-i |+\rangle\langle -| + i |-\rangle\langle +| \right) \left(c_+ e^{\frac{-i\omega t}{2}} |+\rangle + c_- e^{\frac{i\omega t}{2}} |-\rangle \right)$$

$$= \left(-i c_+^* c_- e^{i\omega t} + i c_-^* c_+ e^{-i\omega t} \right) \frac{\hbar}{2}$$

$$\boxed{= \frac{\hbar}{2} (-i) \left(c_+^* c_- e^{i\omega t} - c_-^* c_+ e^{-i\omega t} \right)}$$

$\rightarrow$ $c_+ = 1$, $c_- = 0$; $\langle S_y \rangle_t = 0$

$\rightarrow$ $c_+ = c_- = \frac{1}{\sqrt{2}}$; $\langle S_y \rangle_t = \frac{\hbar}{2} \sin\omega t$



In summary,

$$\langle S_z \rangle_t = \frac{\hbar}{2}\left(|C_+|^2 - |C_-|^2\right)$$

$$\langle S_x \rangle_t = \frac{\hbar}{2}\left(C_+^* C_- e^{i\omega t} + C_-^* C_+ e^{-i\omega t}\right)$$

$$\langle S_y \rangle_t = \frac{\hbar}{2}(-i)\left(C_+^* C_- e^{i\omega t} - C_-^* C_+ e^{-i\omega t}\right).$$

We may relate $\langle S_{x,y,z} \rangle_t$ to $\langle S_{x,y,z} \rangle_{t=0}$.

As $\langle S_x \rangle_0 = \frac{\hbar}{2}\left(C_+^* C_- + C_-^* C_+ \right)$

$$\langle S_y \rangle_0 = \frac{\hbar}{2}(-i)\left(C_+^* C_- - C_-^* C_+ \right)$$

$$\langle S_z \rangle_0 = \frac{\hbar}{2}\left(|C_+|^2 - |C_-|^2 \right),$$

we find

$$\langle S_x \rangle_t = \frac{\hbar}{2}\left(C_+^* C_- (\cos\omega t + i\sin\omega t) + C_-^* C_+ (\cos\omega t - i\sin\omega t)\right)$$

$$= \underbrace{\frac{\hbar}{2}\left(C_+^* C_- + C_-^* C_+\right)}_{\langle S_x \rangle_0} \cos\omega t - \underbrace{\frac{\hbar}{2}(-i)\left(C_+^* C_- - C_-^* C_+\right)}_{\langle S_y \rangle_0} \sin\omega t ,$$

i.e. $\langle S_x \rangle_t = \langle S_x \rangle_0 \cos\omega t - \langle S_y \rangle_0 \sin\omega t$

Similarly,

$$\langle S_y \rangle_t = \langle S_x \rangle_0 \sin\omega t + \langle S_y \rangle_0 \cos\omega t$$

and $\langle S_z \rangle_t = \langle S_z \rangle_0 .$

This result can be found rather easily from the Heinsenberg picture that we will discuss after the break.

e.g.

$$\langle S_x\rangle_t = \underbrace{\langle \alpha, t|}_{\langle \alpha, t=0| \mathcal{U}^\dagger(t)} S_x \underbrace{|\alpha, t\rangle}_{\mathcal{U}(t)|\alpha, t=0\rangle} = \langle \alpha, t=0| \underbrace{\mathcal{U}^\dagger(t) S_x \mathcal{U}(t)}_{S_x^H(t)} |\alpha, t=0\rangle .$$

$$\frac{d S_x^H(t)}{dt} = \frac{1}{i\hbar}\left[S_x^H(t), H\right]$$ See Eq. (2.2.19) p. 83.

Using these operator equations, one can find

$$S_x(t) = S_x(0)\cos\omega t - S_y(0)\sin\omega t$$

$$S_y(t) = S_x(0)\sin\omega t + S_y(0)\cos\omega t$$

$$S_z(0) = S_z(0)$$

as expected.

In more general situation, where the base kets $\{|a'\rangle\}$ are many, we have $|\alpha\rangle = \sum_{a'} C_{a'} |a'\rangle$.

$$C(t) = \left(\sum_{a'} C_{a'}^* \langle a'| \right) \left(\sum_{a''} C_{a''} e^{\frac{-iE_{a''}t}{\hbar}} |a''\rangle \right)$$

$$= \sum_{a'} |C_{a'}|^2 e^{\frac{-iE_{a'}t}{\hbar}}$$

p. 79 Eq. (2.1.68).

For continuous spectrum,

$$\sum_{a'} \longrightarrow \int dE \, \rho(E)$$ ($\rho(E)$ characterizes the density of energy eigenstates.)

$$C_{a'} \longrightarrow g(E)\Big|_{E \simeq E_{a'}}$$

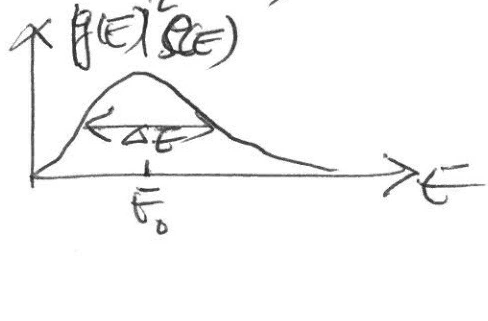
$$C(t) = \sum_{a'} |C_{a'}|^2 e^{\frac{-iE_{a'}t}{\hbar}} \longrightarrow \int dE \, |g(E)|^2 \rho(E) e^{\frac{-iEt}{\hbar}}.$$

Normalization condition: $\int dE \, |g(E)|^2 \rho(E) = 1$

$\left(\text{cf. } \sum_{a'} |C_{a'}|^2 = 1 \quad \text{or} \quad C(0) = 1 \right)$

In a realistic physical situation $|g(E)|^2 \rho(E)$ may be peaked around $E = E_0$ with a width ΔE, i.e.

$$|E - E_0| \ll \Delta E \sim \frac{\hbar}{\delta t}$$



Writing $C(t) = e^{-\frac{iE_0 t}{\hbar}} \int dE\, |g(E)|^2 \rho(E)\, e^{-\frac{i(E-E_0)t}{\hbar}}$

we see that as t becomes large, the integrand oscillates rapidly unless the energy interval $|E-E_0| < \frac{\hbar}{t} < \Delta E$.

If the interval for which $|E-E_0| \simeq \frac{\hbar}{t}$ holds is much narrower than ΔE, we get essentially no contribution to $C(t)$ because of strong cancellations.

The characteristic time at which the modulus of the correlation amplitude starts becoming appreciably different from 1 is given by $t \approx \frac{\hbar}{\Delta E}$.

As a result of time evolution, the state ket of a physical system ceases to retain its original form after a time interval of order $\frac{\hbar}{\Delta E}$.

$$\boxed{\Delta t \cdot \Delta E \approx \hbar}$$

Time-Energy uncertainty.

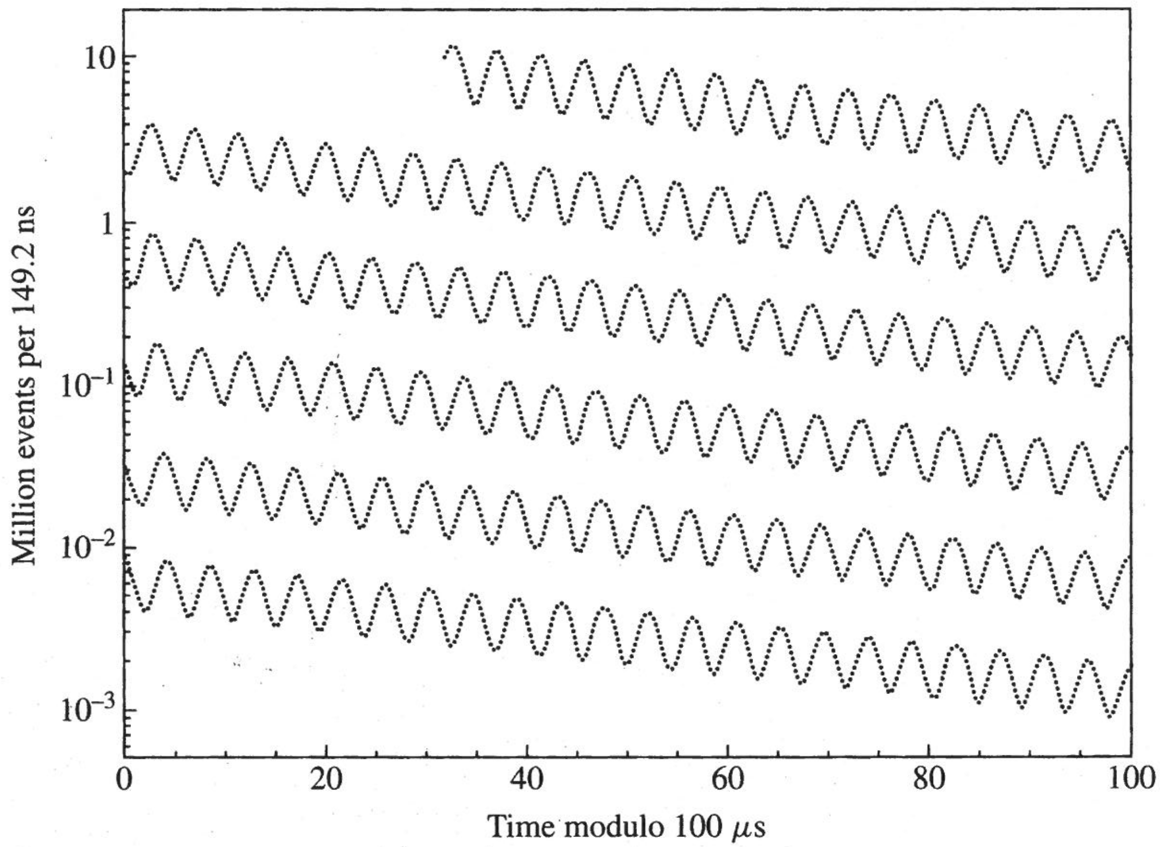


FIGURE 2.1 Observations of the precession of muon spin by G. W. Bennett et al., *Phys. Rev.* D **73** (2006) 072003. Data points are wrapped around every 100 μs. The size of the signal decreases with time because the muons decay.

In[10]:= `Plot[Sqrt[1 - x^2] x, {x, 0, 1}]`

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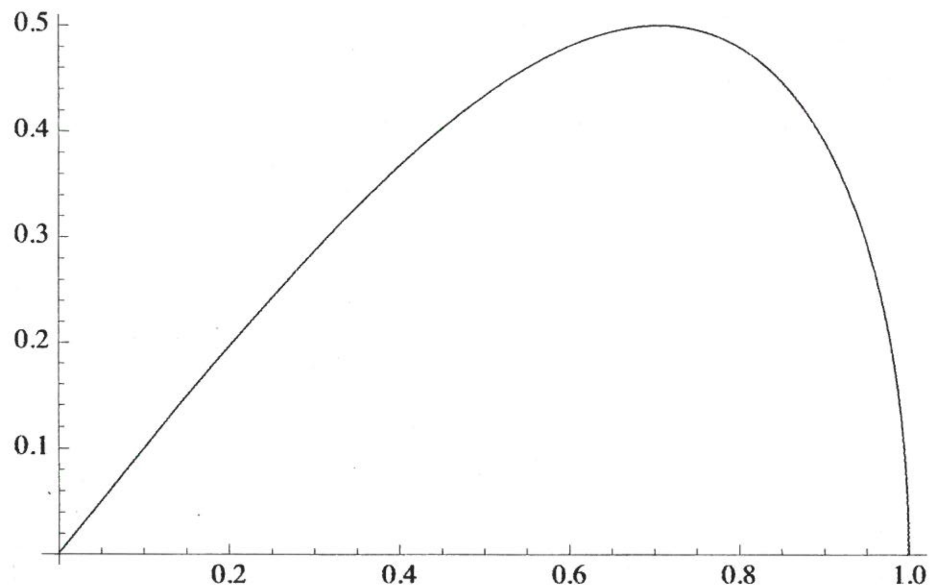

$X = |C_-|$