Further Discussion on SHO

2nd Exam on Oct. 27
Chapt 2. SHO

Oct. 20, 2016

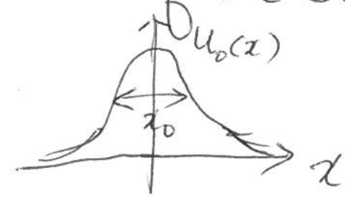
We discuss more in depth on the SHO system.

Let's consider $|\alpha, t=0\rangle_S = |\alpha\rangle_H = |0\rangle$, i.e. ground state, and note

$$\boxed{x = \frac{x_0}{\sqrt{2}}(a^\dagger + a), \quad p = \frac{i\hbar}{\sqrt{2}\,x_0}(a^\dagger - a)},$$

$$[x,p] = \frac{i\hbar}{2}[a^\dagger + a, a^\dagger - a] = \frac{i\hbar}{2}\Big(\underbrace{[a, a^\dagger]}_{+1} - \underbrace{[a^\dagger, a]}_{-1}\Big) = i\hbar$$

where $\boxed{x_0 = \sqrt{\frac{\hbar}{m\omega}}}$ is the length scale of the SHO, e.g. the size of Gaussian wave function.



Thus, $\langle x\rangle = 0$ and $\langle p\rangle = 0$

as $\langle a\rangle = \langle a^\dagger\rangle = 0$.

One can easily see that this result sustains for any time t using the Heisenberg operators;

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$$

$$p(t) = p(0)\cos\omega t - m\omega x(0)\sin\omega t$$

$$a(t) = e^{-i\omega t}a(0)$$

$$a^\dagger(t) = e^{i\omega t}a^\dagger(0),$$ i.e. $\boxed{\langle x\rangle_t = \langle p\rangle_t = \langle a\rangle_t = \langle a^\dagger\rangle_t = 0}$

In fact, this result holds for any energy eigenstate $|n\rangle$ because x, p, a and $a^\dagger$ change one quanta while the energy eigenstates are orthonormal to each other so that ~~all~~ the expectation values of the operators changing the number of quanta by one for any specific energy eigenstate must vanish.

How about $\langle x^2\rangle$ and $\langle p^2\rangle$?

$$\boxed{\begin{aligned} x^2 &= \frac{x_0^2}{2}(a^\dagger+a)(a^\dagger+a) = \frac{x_0^2}{2}\left((a^\dagger)^2+aa^\dagger+a^\dagger a+a^2\right) \\ p^2 &= -\frac{\hbar^2}{2x_0^2}(a^\dagger-a)(a^\dagger-a) = -\frac{\hbar^2}{2x_0^2}\left((a^\dagger)^2-aa^\dagger-a^\dagger a+a^2\right) \end{aligned}}$$

For $|n\rangle$, we get

$$\begin{aligned} \langle n|x^2|n\rangle &= \frac{x_0^2}{2}\langle n|(a^\dagger)^2+aa^\dagger+a^\dagger a+a^2|n\rangle \\ &= \frac{x_0^2}{2}\Big(\underbrace{\langle n|a}_{\langle n+1|\sqrt{n+1}}\,\underbrace{a^\dagger|n\rangle}_{\sqrt{n+1}\,|n+1\rangle} + \langle n|\underbrace{a^\dagger a}_{N}|n\rangle\Big) \\ &= \frac{x_0^2}{2}(n+1+n) = \frac{x_0^2}{2}(2n+1) \end{aligned}$$

$$\begin{aligned} \langle n|p^2|n\rangle &= \frac{\hbar^2}{2x_0^2}\left(\langle n|aa^\dagger|n\rangle + \langle n|a^\dagger a|n\rangle\right) \\ &= \frac{\hbar^2}{2x_0^2}(2n+1). \end{aligned}$$

Thus,

$$\boxed{\langle n|(\Delta x)^2|n\rangle \doteq \langle n|x^2|n\rangle = \frac{x_0^2}{2}(2n+1), \quad \langle n|(\Delta p)^2|n\rangle = \langle n|p^2|n\rangle = \frac{\hbar^2}{2x_0^2}(2n+1)}$$

$$\langle n|(\Delta x)^2|n\rangle\langle n|(\Delta p)^2|n\rangle = \frac{\hbar^2}{4}(2n+1)^2 \geq \frac{1}{4}|\langle n|\underbrace{[x,p]}_{i\hbar}|n\rangle|^2 = \frac{\hbar^2}{4}$$

(See Eq. (2.3.40))

For the ground state, $n=0$,

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2}{4} \quad \text{(minimum uncertainty)}$$

However, the uncertainty grows as n gets larger.

Nevertheless, all the expectation values for the energy eigenstate $|n\rangle$ do not vary with time: i.e.

$$\langle n|x^2(t)|n\rangle = \frac{x_0^2}{2}(2n+1)$$

$$\langle n|p^2(t)|n\rangle = \frac{\hbar^2}{2x_0^2}(2n+1)$$

Note:

$$\langle n|x^2(t)|n\rangle = \langle n|e^{\frac{iHt}{\hbar}}x^2(0)e^{-\frac{iHt}{\hbar}}|n\rangle$$

$$= e^{\frac{iE_nt}{\hbar}}\cdot e^{-\frac{iE_nt}{\hbar}}\langle n|x^2(0)|n\rangle$$

$$= \langle n|x^2(0)|n\rangle$$

The uncertainty remain same independent of time.

Note that

$$\langle n|\frac{p^2(t)}{2m} + \frac{1}{2}m\omega^2x^2(t)|n\rangle$$

$$= \frac{1}{2m}\cdot\frac{\hbar^2}{2\left(\frac{\hbar}{m\omega}\right)}(2n+1) + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}(2n+1)$$

$$= \frac{\hbar\omega}{4}(2n+1) + \frac{\hbar\omega}{4}(2n+1)$$

$$= \hbar\omega\left(n+\frac{1}{2}\right) = E_n \quad \text{as } \langle n|H|n\rangle = E_n$$

and

$$\langle n|\frac{p^2(t)}{2m}|n\rangle = \langle n|\frac{1}{2}m\omega^2x^2(t)|n\rangle = \frac{E_n}{2}$$

(cf. Virial Theorem) HW#5, prob. 7 of chpt 2

To observe oscillations reminiscent of the classical oscillator, we must look at a superposition of energy eigenstates.

Let's consider $\quad |\alpha, t=0\rangle_S = |\alpha\rangle_H = e^{\frac{-ipl}{\hbar}}|0\rangle,$

i.e. translated ground state.

$$\boxed{|\Omega\rangle \equiv e^{\frac{-ipl}{\hbar}}|0\rangle = e^{\frac{l}{\sqrt{2}x_0}(a^\dagger - a)}|0\rangle}$$ "coherent state"

As $\quad e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ if $[A,B]$ becomes c-number as in the case of $[a^\dagger, a] = -1$,

$$|\Omega\rangle = e^{\frac{l}{\sqrt{2}x_0}a^\dagger} e^{-\frac{l}{\sqrt{2}x_0}a} e^{-\frac{l^2}{4x_0^2}}|0\rangle$$

$$= e^{-\frac{l^2}{4x_0^2}} e^{\frac{l}{\sqrt{2}x_0}a^\dagger}|0\rangle$$

$$= e^{-\frac{l^2}{4x_0^2}} \sum_{n=0}^{\infty} |n\rangle\langle n| e^{\frac{l}{\sqrt{2}x_0}a^\dagger}|0\rangle$$

Note here

$$a e^{\lambda a^\dagger}|0\rangle = a\left(1 + \lambda a^\dagger + \frac{\lambda^2}{2!}(a^\dagger)^2 + \cdots\right)|0\rangle$$

$$= \left[a + \lambda \underbrace{a a^\dagger}_{a^\dagger a + 1} + \frac{\lambda^2}{2!}\underbrace{a (a^\dagger)^2}_{(a^\dagger a + 1)a^\dagger = a^\dagger(a^\dagger a + 1) + a^\dagger = (a^\dagger)^2 a + 2a^\dagger} + \cdots\right]|0\rangle$$

$$= \lambda\left[1 + \lambda a^\dagger + \cdots\right]|0\rangle$$

$$= \lambda e^{\lambda a^\dagger}|0\rangle,$$ i.e. $e^{\lambda a^\dagger}|0\rangle$ is the coherent state.

See Eqs(2.3.52)—(2.3.54) pp.96-97.

Thus, using $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$,

or $\langle n| = \langle 0|\frac{a^n}{\sqrt{n!}}$,

we get

$$\langle n| e^{\frac{l}{\sqrt{2}x_0}a^\dagger}|0\rangle$$

$$= \langle 0|\frac{a^n}{\sqrt{n!}}\left(e^{\frac{l}{\sqrt{2}x_0}a^\dagger}|0\rangle\right)$$

$$= \frac{1}{\sqrt{n!}}\left(\frac{l}{\sqrt{2}x_0}\right)^n \underbrace{\langle 0| e^{\frac{l}{\sqrt{2}x_0}a^\dagger}|0\rangle}_{1}$$

so that

$$\boxed{|\Omega\rangle = \sum_{n=0}^{\infty} f(n)\,|n\rangle}$$

where

$$\boxed{f(n) = e^{-\frac{l^2}{4x_0^2}}\frac{1}{\sqrt{n!}}\left(\frac{l}{\sqrt{2}x_0}\right)^n}$$

or $|f(n)|^2 = \frac{\bar{n}^n}{n!}e^{-\bar{n}}$, with $\bar{n} = \frac{l^2}{2x_0^2}$.

See Eq. (12.3.54)) Poisson distribution

$$\sum_{n=0}^{\infty}|f(n)|^2 = e^{-\bar{n}}\sum_{n=0}^{\infty}\frac{(\bar{n})^n}{n!} = e^{-\bar{n}}e^{\bar{n}} = +1$$ as it must be.

In[110]:=
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DiscretePlot[Evaluate@Table[P[nbar, n], {nbar, {1, 2, 3}}],
  {n, 0, 10}, PlotRange → All]
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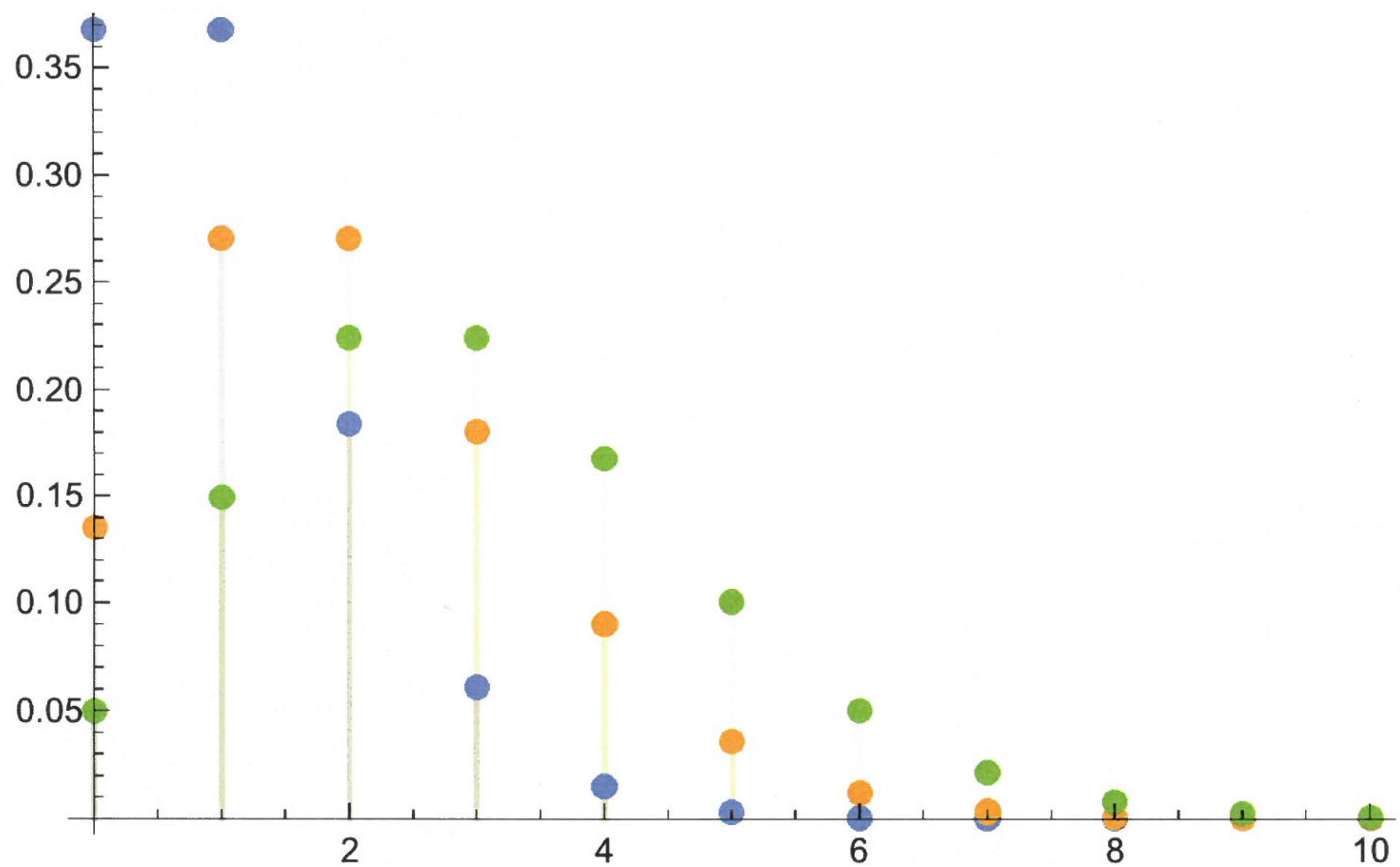
Out[110]=



Taking $|\alpha, t=0\rangle_S = |\alpha\rangle_H = |\Omega\rangle$,

let's compute $\langle x\rangle_t$ and $\langle p\rangle_t$.

As

$$\langle x(t)\rangle = \langle x(0)\rangle \cos\omega t + \frac{\langle p(0)\rangle}{m\omega}\sin\omega t$$

and

$$\langle p(t)\rangle = \langle p(0)\rangle\cos\omega t - m\omega\langle x(0)\rangle\sin\omega t,$$

we need to compute $\langle x(0)\rangle$ and $\langle p(0)\rangle$.

Now, $\langle x(0)\rangle = \langle\Omega|\,x\,|\Omega\rangle$

$$= \langle 0|\underbrace{e^{\frac{ipl}{\hbar}}\, x\, e^{\frac{-ipl}{\hbar}}}_{x+l}|0\rangle$$

$$= \langle 0|x|0\rangle + l$$

$$= l$$

$$\left(\because\ \left[x, e^{\frac{-ipl}{\hbar}}\right] = i\hbar\frac{\partial}{\partial p}e^{\frac{-ipl}{\hbar}} = l\,e^{\frac{-ipl}{\hbar}}\right.$$

or $$x e^{\frac{-ipl}{\hbar}} - e^{\frac{-ipl}{\hbar}}x = l\,e^{\frac{-ipl}{\hbar}}$$

or $$\left. e^{\frac{ipl}{\hbar}}\, x\, e^{\frac{-ipl}{\hbar}} = x + l\right)$$

and

$$\langle p(0)\rangle = \langle\Omega|p|\Omega\rangle$$

$$= \langle 0|e^{\frac{ipl}{\hbar}}\,p\,e^{\frac{-ipl}{\hbar}}|0\rangle$$

$$= \langle 0|p|0\rangle = 0$$

Thus, we find

$$\langle x \rangle_t = l \cos\omega t$$

and $$\langle p \rangle_t = -m\omega l \sin\omega t = m \frac{d\langle x \rangle_t}{dt},$$

as we expect from classical mechanics.

How about $\langle x^2(t) \rangle$ and $\langle p^2(t) \rangle$?

Since $$x^2(t) = X^2(0)\cos^2\omega t + \frac{1}{m\omega}\{X(0), p(0)\}\cos\omega t \sin\omega t + \frac{p^2(0)}{m^2\omega^2}\sin^2\omega t$$

and $$p^2(t) = p^2(0)\cos^2\omega t - m\omega\{X(0), p(0)\}\cos\omega t \sin\omega t + m^2\omega^2 X^2(0)\sin^2\omega t,$$

we need to compute $\langle X^2(0) \rangle$, $\langle \{X(0), p(0)\} \rangle$, and $\langle p^2(0) \rangle$.

$$\langle X^2(0) \rangle = \langle \Omega | X^2 | \Omega \rangle = \langle 0 | e^{\frac{ipl}{\hbar}} X^2 e^{-\frac{ipl}{\hbar}} | 0 \rangle$$
$$= \langle 0 | e^{\frac{ipl}{\hbar}} X e^{-\frac{ipl}{\hbar}} e^{\frac{ipl}{\hbar}} X e^{-\frac{ipl}{\hbar}} | 0 \rangle$$
$$= \langle 0 | (X + l)^2 | 0 \rangle = \langle 0 | X^2 | 0 \rangle + l^2 = \frac{x_0^2}{2} + l^2.$$

$$\langle \{x(0), p(0)\} \rangle = \langle \Omega | \{x, p\} | \Omega \rangle$$

$$= \langle 0 | e^{\frac{ipl}{\hbar}} \{x, p\} e^{\frac{-ipl}{\hbar}} | 0 \rangle$$

$$= \langle 0 | \{x + l, p\} | 0 \rangle$$

$$= \langle 0 | \underbrace{\{x, p\}}_{xp+px} | 0 \rangle + \langle 0 | \underbrace{\{l, p\}}_{lp+pl = 2lp} | 0 \rangle$$

$$\underbrace{xp+px}$$

$$\frac{i\hbar}{2}\{a^\dagger + a, a^\dagger - a\} = \frac{i\hbar}{2}\left[(a^\dagger + a)(a^\dagger - a) + (a^\dagger - a)(a^\dagger + a)\right]$$

$$= i\hbar\left((a^\dagger)^2 - a^2\right)$$

$$= i\hbar \langle 0 | (a^\dagger)^2 - a^2 | 0 \rangle + 2l \langle 0 | p | 0 \rangle$$

$$= 0$$

$$\langle p^2(0) \rangle = \langle \Omega | p^2 | \Omega \rangle = \langle 0 | p^2 | 0 \rangle = \frac{\hbar^2}{2x_0^2}$$

Thus, we get $\qquad \frac{\hbar}{m\omega} \qquad\qquad x_0^4$

$$\langle x^2 \rangle_t = \left(\frac{x_0^2}{2} + l^2\right)\cos^2\omega t + \frac{\hbar^2}{2x_0^2 m^2\omega^2}\sin^2\omega t$$

$$\frac{\hbar}{m\omega}$$

$$= \frac{x_0^2}{2} + l^2\cos^2\omega t$$

$$\langle p^2 \rangle_t = \frac{\hbar^2}{2x_0^2}\cos^2\omega t + m^2\omega^2\left(\frac{x_0^2}{2} + l^2\right)\sin^2\omega t = \frac{\hbar^2}{2x_0^2} + m^2\omega^2 l^2 \sin^2\omega t$$

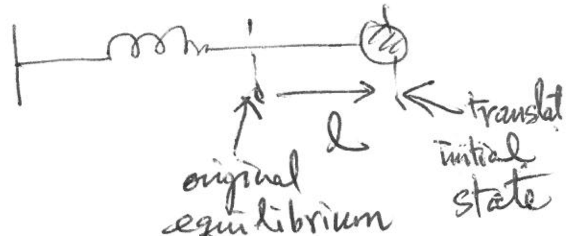
Note that

$$\langle\Omega|\frac{P^2(t)}{2m}+\frac{1}{2}m\omega^2X^2(t)|\Omega\rangle$$

$$= \frac{\hbar^2}{4m\underbrace{x_0^2}_{\frac{\hbar}{m\omega}}} + \frac{m}{2}\omega^2 l^2\sin^2\omega t + \frac{1}{2}m\omega^2\left(\frac{\overbrace{x_0^2}^{\frac{\hbar}{m\omega}}}{2}+l^2\cos^2\omega t\right)$$

$$= \frac{\hbar\omega}{4} + \frac{1}{2}m\omega^2l^2\sin^2\omega t + \frac{\hbar\omega}{4} + \frac{1}{2}m\omega^2l^2\cos^2\omega t$$

$$= \frac{\hbar\omega}{2} + \frac{1}{2}m\omega^2l^2$$



$$\langle(\Delta X)^2\rangle_t = \langle X^2\rangle_t - \langle X\rangle_t^2$$

$$= \frac{x_0^2}{2} + l^2\cos^2\omega t - l^2\cos^2\omega t = \frac{x_0^2}{2}$$

$$\langle(\Delta p)^2\rangle_t = \langle p^2\rangle_t - \langle p\rangle_t^2$$

$$= \frac{\hbar^2}{2x_0^2} + m^2\omega^2l^2\sin^2\omega t - (-m\omega l\sin\omega t)^2$$

$$= \frac{\hbar^2}{2x_0^2}$$

$$\langle(\Delta X)^2\rangle_t\langle(\Delta p)^2\rangle_t = \frac{x_0^2}{2}\cdot\frac{\hbar^2}{2x_0^2} = \frac{\hbar^2}{4} \;\underset{=\frac{\hbar^2}{4}}{\geq \frac{1}{4}\left|\langle\Omega|\underbrace{[X(t),P(t)]}_{i\hbar}|\Omega\rangle\right|^2}$$

Discussion on Hermite polynomials.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx'^2}u_n(x') + \frac{1}{2}m\omega^2 x'^2 u_n(x') = E_n u_n(x'),$$

where $E_n = (n+\frac{1}{2})\hbar\omega$.

$$\frac{2}{\hbar\omega} \times \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx'^2}u_n(x') + \cdots\right]$$

$$-\underbrace{\frac{\hbar}{m\omega}}_{x_0^2}\frac{d^2}{dx'^2}u_n(x') + \underbrace{\frac{m\omega}{\hbar}}_{\frac{1}{x_0^2}}x'^2 u_n(x') = \underbrace{\frac{2E_n}{\hbar\omega}}_{\varepsilon_n = 2n+1}u_n(x')$$

$\varepsilon_n = 2n+1$ (dimensionless energy)

$x' \equiv x_0 y$, where y is dimensionless position ($y = \frac{x'}{x_0}$).

$$\boxed{\frac{d^2}{dy^2}u_n(y) + (\varepsilon_n - y^2)u_n(y) = 0}$$

For $y \to \pm\infty$, the solution must go to zero. Otherwise, the wavefunction won't be normalizable and hence unphysical.

e.g. $\varepsilon_0 = 1$, $u_0''(y) + (1-y^2)u_0(y) = 0$; $u_0(y) \propto e^{-\frac{y^2}{2}}$

$u_0'(y) \propto -y\,u_0(y)$ $\quad u''(y) \propto -u_0(y) + y^2 u_0(y)$

If $\varepsilon_0 = -1$, then $\tilde{u}_0''(y) - (1+y^2)\tilde{u}_0(y) = 0$ and $\tilde{u}_0(y) \propto e^{+\frac{y^2}{2}}$ (unphysical)

Let's take $u_n(y) = H_n(y) e^{-\frac{y^2}{2}}$,

then
$$\frac{du_n(y)}{dy} = \frac{dH_n(y)}{dy} e^{-\frac{y^2}{2}} + H_n(y)\left(-y e^{-\frac{y^2}{2}}\right)$$
$$= \left(\frac{dH_n}{dy} - yH_n\right) e^{-\frac{y^2}{2}}$$

and
$$\frac{d^2u_n(y)}{dy^2} = \left(\frac{d^2H_n}{dy^2} - H_n - y\frac{dH_n}{dy}\right) e^{-\frac{y^2}{2}} + \left(\frac{dH_n}{dy} - yH_n\right)\left(-y e^{-\frac{y^2}{2}}\right)$$
$$= \left\{\frac{d^2H_n}{dy^2} - 2y\frac{dH_n}{dy} + (y^2-1)H_n\right\} e^{-\frac{y^2}{2}}$$

Thus, we get
$$\frac{d^2H_n}{dy^2} - 2y\frac{dH_n}{dy} + \underbrace{(\varepsilon_n - 1)}_{2n} H_n = 0,$$

where H_n must be the polynomial that terminates at the order n. $H_n'' \sim n(n-1)y^{n-2}$; $H_n' \sim ny^{n-1}$, $H_n \sim y^n$.

From the generating function of $H_n(y)$, one can see that $H_n(y)$ satisfies the above second order differential eq.