Fundamental Dynamical Variables: Position and Momentum

1st Exam, Sep. 27
Lectures till Thursday, Sep. 22, Chapt. 1 × HWs 1-3

Sep. 20, 2016

Discrete Hilbert Space | Continuous Hilbert Space

Hilbert Space = Nondenumerably infinite dimensional complex vector space

Spin $\frac{1}{2}$ system | Position

ket $\{|+\rangle, |-\rangle\}$ | $\{|x'\rangle\}$

bra $\{\langle +|, \langle -|\}$ | $\{\langle x'|\}$, where $-\infty < x' < +\infty$.

Incompatible observables
e.g. $[S_x, S_y] = i\hbar S_z$ | e.g. $[x, p] = i\hbar I$

Closure $|+\rangle\langle +| + |-\rangle\langle -| = I$ | $\int dx' |x'\rangle\langle x'| = I$

Operator eigenvalue eg:

$S_z |\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ | $X|x'\rangle = x'|x'\rangle$

Operator (Outer product of bra & ket)

$S_z = \frac{\hbar}{2}|+\rangle\langle +| - \frac{\hbar}{2}|-\rangle\langle -|$ | $X = \int dx' x' |x'\rangle\langle x'|$

Hermiticity

$S_z^\dagger = S_z$ | $X^\dagger = X$

Orthogonality

$\langle +|+\rangle = 1$
$\langle +|-\rangle = 0$ | $\langle x'|x''\rangle = \delta(x'-x'')$

Similarly; Momentum $\{|p'\rangle\}, \{\langle p'|\}$, where $-\infty < p' < \infty$
$P|p'\rangle = p'|p'\rangle$
$\int dp' |p'\rangle\langle p'| = I$, $P = \int dp' p' |p'\rangle\langle p'|$, $\langle p'|p''\rangle = \delta(p'-p'')$

Classical Mechanics

Physical observables are given by position and momentum.

$$A(x, p), \quad B(x, p)$$

x, p : fundamental dynamical variables

$$[A, B]_{PB} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$$

PB : Poisson Bracket

cf. $[\ ,\ ]_{PB} = \sum_i \left[\frac{\partial}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial}{\partial q_i}\right]$

$$[x, p]_{PB} = 1$$

Hamilton's eq. of motion ; $\dfrac{dq_i}{dt} = [q_i, H]_{PB} = \dfrac{\partial H}{\partial p_i}$

$$\frac{dp_i}{dt} = [p_i, H]_{PB} = -\frac{\partial H}{\partial q_i}$$

e.g.

$$H(x, p) = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

$$\frac{dx}{dt} = [x, H]_{PB} = \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$$

$$\frac{dp}{dt} = [p, H]_{PB} = -\frac{\partial H}{\partial x} = -kx = \dot{p}$$

Dirac's observation ; Classical correspondence in Q.M.

$$[A, B]_{PB} \longleftrightarrow \frac{[A, B]}{i\hbar}$$

Dirac Bracket or Commutator

i : antihermiticity of commutator, $[A, B]^\dagger = -[A, B]$

$\hbar$: to make the dimension right

In Q.M., x and p are incompatible observables,

$$[x,p]=i\hbar \quad \text{(fundamental commutation relation)}.$$

Ex.

$$\begin{aligned}[x,p^2] &= xp^2-p^2x \\ &= xp^2-pxp+pxp-p^2x \\ &= [x,p]p+p[x,p] \\ &= 2i\hbar p.\end{aligned}$$

$$\begin{aligned}[x,p^3] &= xp^3-p^3x \\ &= xp^3-p^2xp+p^2xp-p^3x \\ &= \underbrace{[x,p^2]}_{2i\hbar}p+p^2\underbrace{[x,p]}_{i\hbar} \\ &= 3i\hbar p^2\end{aligned}$$

$\vdots$

$$[x,p^n] = ni\hbar p^{n-1}$$

$$[x,f(p)] = i\hbar\frac{\partial f(p)}{\partial p}$$

cf. $$[x,f(p)]_{PB} = \underbrace{\frac{\partial x}{\partial x}}_{1}\frac{\partial f(p)}{\partial p} - \cancel{\frac{\partial x}{\partial p}\frac{\partial f(p)}{\partial x}}^{0} = \frac{\partial f(p)}{\partial p}.$$

Translation in position

Consider $f(p) = e^{\frac{-ipl}{\hbar}} \equiv T_l(p)$, $\boxed{\text{cf. } e^{\frac{-iS_y\beta}{\hbar}} = e^{-i\sigma_y\frac{\beta}{2}} \text{ rotation}}$

then $T_l^\dagger(p) = e^{\frac{ipl}{\hbar}}$. (Note; $p^\dagger = p$ Hermitian.)

$$T_l^\dagger(p)\, T_l(p) = I = T_l(p)\, T_l^\dagger(p) \; ; \; T_l(p) \text{ is } \underline{\text{unitary!}}$$

$$[x, T_l(p)] = i\hbar \frac{\partial T_l(p)}{\partial p}$$

$$= i\hbar\left(-\frac{il}{\hbar}\right) T_l(p) = l\, T_l(p).$$

$$x\, T_l(p) - T_l(p)\, x = l\, T_l(p)$$

$$\boxed{T_l^\dagger(p)\, x\, T_l(p) = x + l}$$

Momentum as a generator of translation

cf. $\underline{A = U^\dagger B U}$

$S_z = U^\dagger S_x U$

Since $T_l^\dagger(p)\, x\, T_l(p)\, |x'\rangle = (x+l)|x'\rangle = (x'+l)|x'\rangle$,

we get $x\left[T_l(p)|x'\rangle\right] = (x'+l)\left[T_l(p)|x'\rangle\right]$

and $T_l(p)|x'\rangle = |x'+l\rangle$ Translation operator

$$\langle x'| T_l^\dagger(p) = \langle x'+l|$$

Consider an arbitrary ket $|\alpha\rangle$ and define wave function,

$$\psi_\alpha(x') \equiv \langle x'|\alpha\rangle.$$

Then,
$$\langle x'| T_l^\dagger |\alpha\rangle = \langle x'+l|\alpha\rangle$$

or
$$\langle x'| e^{\frac{ipl}{\hbar}} |\alpha\rangle = \psi_\alpha(x'+l).$$

Now, suppose l is infinitesimal, say $l \to \delta l \ll 1$, then we may expand both sides;

$$\langle x'| I + \frac{i\delta l}{\hbar} p + \mathcal{O}((\delta l)^2) |\alpha\rangle$$

Taylor Series expansion

$$= \psi_\alpha(x') + \delta l \frac{d}{dx'} \psi_\alpha(x') + \mathcal{O}((\delta l)^2).$$

$$\langle x'|\alpha\rangle + \frac{i\delta l}{\hbar} \langle x'|p|\alpha\rangle + \mathcal{O}((\delta l)^2)$$

Thus, we find

$$\boxed{\langle x'|p|\alpha\rangle = \frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle.}$$

or
$$\boxed{\langle x'|p = \frac{\hbar}{i}\frac{d}{dx'}\langle x'| \quad ; \quad p|x'\rangle = i\hbar\frac{d}{dx'}|x'\rangle}$$

(6)

Momentum operator is represented as a derivative in position space

$$\underbrace{\langle x'|P|p'\rangle}_{p'|p'\rangle} = \frac{\hbar}{i}\frac{d}{dx'}\langle x'|p'\rangle$$

$$\therefore \boxed{\frac{\hbar}{i}\frac{d}{dx'}\langle x'|p'\rangle = p'\langle x'|p'\rangle}$$

or $\langle x'|p'\rangle = N\, e^{\frac{ip'x'}{\hbar}}$

To determine the normalization N, consider the orthonormality of the momentum bras & kets:

$$\langle p'|\left[\int_{-\infty}^{+\infty} dx'\,|x'\rangle\langle x'| = I\right]|p''\rangle = \langle p'|p''\rangle = \delta(p'-p'')$$

$$\shortparallel$$

$$\int_{-\infty}^{+\infty} dx'\,\langle p'|x'\rangle\langle x'|p''\rangle$$

$$\shortparallel$$

$$\int_{-\infty}^{\infty} dx'\, N^* e^{\frac{-ip'x'}{\hbar}}\, N e^{\frac{ip''x'}{\hbar}} = |N|^2 \underbrace{\int_{-\infty}^{\infty} dx'\, e^{\frac{-i(p'-p'')x'}{\hbar}}}_{2\pi\hbar\,\delta(p'-p'')}$$

$$\therefore \boxed{N = \frac{1}{\sqrt{2\pi\hbar}}}$$