Change of Basis and Eigenvalue Problem

Sep. 15, 2016

We discussed the compatible and incompatible observables and obtained the uncertainty relations between the two observables

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A,B] \rangle|^2 + \frac{1}{4} |\langle \{\Delta A, \Delta B\} \rangle|^2$$

$$\geq \frac{1}{4} |\langle [A,B] \rangle|^2$$

For a given state, the "sharp" operator's dispersion is zero and the given state is typically the eigenstate of the "sharp" operator.

e.g. $A = S_z$ is diagonal "sharp" in $|+\rangle$ & $|-\rangle$ basis

$$S_z \doteq \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The "fuzzy" operator is off-diagonal in the same basis.

e.g. $B = S_x \doteq \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The "sharpness" and "fuzziness" depend on the basis chosen. Let's consider how one can change the basis in general.

$$S_z = \frac{\hbar}{2}|+\rangle\langle+| - \frac{\hbar}{2}|-\rangle\langle-| \doteq \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S_x = \frac{\hbar}{2}|S_x+\rangle\langle S_x+| - \frac{\hbar}{2}|S_x-\rangle\langle S_x-|$$

$$\boxed{\begin{aligned} |S_x+\rangle &= U|+\rangle \\ |S_x-\rangle &= U|-\rangle \end{aligned}}$$

Find U operator that changes the basis from $|\pm\rangle$ to $|S_x\pm\rangle$.

$$|S_x+\rangle\langle+| = U|+\rangle\langle+|$$
$$|S_x-\rangle\langle-| = U|-\rangle\langle-| \quad +$$

$$|S_x+\rangle\langle+| + |S_x-\rangle\langle-| = U\underbrace{\left(|+\rangle\langle+| + |-\rangle\langle-|\right)}_{\text{Identity (closure relation)}}$$

$$\therefore \quad U = |S_x+\rangle\langle+| + |S_x-\rangle\langle-|$$

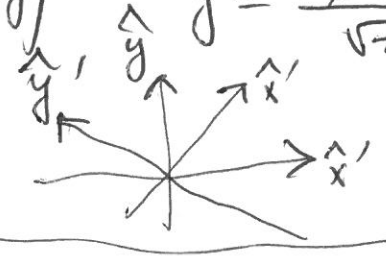
$$= \left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right)\langle+| + \left(-\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right)\langle-|$$

$$= \frac{1}{\sqrt{2}}|+\rangle\langle+| + \frac{1}{\sqrt{2}}|-\rangle\langle+| - \frac{1}{\sqrt{2}}|+\rangle\langle-| + \frac{1}{\sqrt{2}}|-\rangle\langle-|$$

$$\doteq \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

recall Polaroid analogy cf. $\hat{x}' = \frac{\hat{x} + \hat{y}}{\sqrt{2}}$

$\hat{y}' = \frac{-\hat{x} + \hat{y}}{\sqrt{2}}$



$$U^\dagger \doteq \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$UU^\dagger = U^\dagger U = I \quad (\text{Unitary!})$$

$$S_x = \frac{\hbar}{2}|S_x+\rangle\langle S_x+| - \frac{\hbar}{2}|S_x-\rangle\langle S_x-|$$

$$= U\left[\frac{\hbar}{2}|+\rangle\langle+| - \frac{\hbar}{2}|-\rangle\langle-|\right]U^\dagger$$

$$= U\, S_z\, U^\dagger$$

cf. $$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \underbrace{\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\sigma_z} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\sigma_x}$$

Conversely, $$S_z = U^\dagger S_x U$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{off-diagonal}} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{diagonal.}}$$

<u>diagonalization!</u>

In general, the equivalent dynamic variables are related to each other by a unitary transformation

$$A = \sum_i a_i |a^{(i)}\rangle\langle a^{(i)}|$$

$$B = \sum_i a_i |b^{(i)}\rangle\langle b^{(i)}|$$

same eigenvalues as A

$$|b^{(i)}\rangle = U|a^{(i)}\rangle$$

$$\langle b^{(i)}| = \langle a^{(i)}|U^\dagger$$

$$\langle b^{(i)}|b^{(i)}\rangle = \langle a^{(i)}|\underbrace{U^\dagger U}_{I}|a^{(i)}\rangle = \langle a^{(i)}|a^{(i)}\rangle$$

$$U = \sum_i |b^{(i)}\rangle\langle a^{(i)}|$$

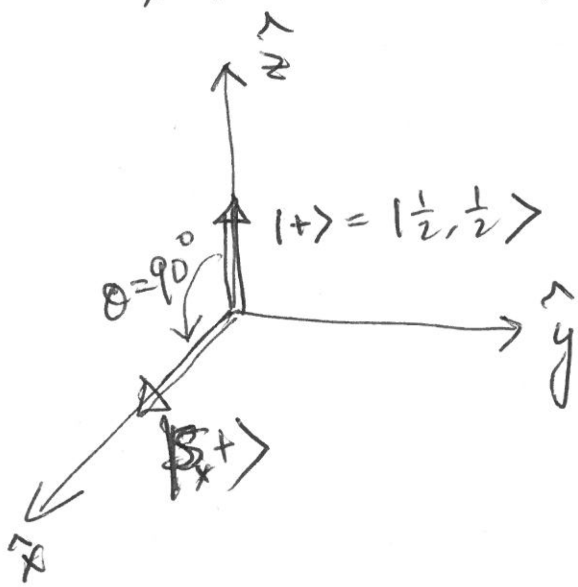
$$B = UAU^\dagger \quad \text{or} \quad A = U^\dagger B U$$

diagonal off-diagonal

Diagonalization with unitary transformation.

How do we find U?

e.g. $|S_x+\rangle = U|+\rangle$



$$U = e^{-i\frac{S_y\left(\frac{\pi}{2}\right)}{\hbar}}$$

$$= e^{-i\frac{\left(\frac{\hbar}{2}\sigma_y\right)\left(\frac{\pi}{2}\right)}{\hbar}}$$

$$= e^{-i\frac{\pi}{4}\sigma_y}$$

$$e^{i\vec{\sigma}\cdot\hat{n}\theta} = I + i\vec{\sigma}\cdot\hat{n}\theta + \frac{1}{2!}\left(i\vec{\sigma}\cdot\hat{n}\theta\right)^2 + \frac{1}{3!}\left(i\vec{\sigma}\cdot\hat{n}\theta\right)^3 + \cdots$$

cf $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$= I\left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right)$$
$$+ i\vec{\sigma}\cdot\hat{n}\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right)$$

$$= I\cos\theta + i\vec{\sigma}\cdot\hat{n}\sin\theta$$

$$U = e^{-i\frac{\pi}{4}\sigma_y} = I\cos\frac{\pi}{4} - i\sigma_y\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}I - \frac{i}{\sqrt{2}}\sigma_y = \begin{bmatrix}\frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}}\end{bmatrix} - \frac{i}{\sqrt{2}}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

Confirm:

$$U|+\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = |S_x+\rangle$$

$$U|-\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = |S_x-\rangle$$

Ex. Prob. 1.14

"same eigenvalue"
"Dynamically equivalent"

$$B = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B|b^{(i)}\rangle = a_i |b^{(i)}\rangle$$

$$\boxed{A = U^\dagger B U}$$

$$A|a^{(i)}\rangle = a_i |a^{(i)}\rangle$$

Eigenvalue Problem.

Solving the eigenvalue problem is equivalent to find the unitary transformation for the matrix diagonalization.

$$\boxed{U = \sum_i |b^{(i)}\rangle\langle a^{(i)}|}$$

$$(B - \lambda I)|b\rangle = 0$$

For the existence of nontrivial solution, the characteristic equation must be satisfied, i.e.

$$|B - \lambda I| = 0 \quad \text{or} \quad \det(B - \lambda I) = 0$$

"Dropping the overall factor $\frac{1}{\sqrt{2}}$ because the right-hand-side is zero."

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & -\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 1) + \lambda = \lambda^3 + 2\lambda$$

$$= \lambda(2 - \lambda^2) = 0$$

$$\therefore \lambda = 0, \pm\sqrt{2}$$

(i.e. a_i's are found, e.g. "putting the overall factor $\frac{1}{\sqrt{2}}$ back"

$a_1 = +\frac{\sqrt{2}}{\sqrt{2}} = 1$

$a_2 = \frac{0}{\sqrt{2}} = 0$

$a_3 = -\frac{\sqrt{2}}{\sqrt{2}} = -1$

i) $\lambda = +\sqrt{2}$, i.e. $a_1 = 1$

$$\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -\sqrt{2}\alpha + \beta &= 0 \\ \alpha - \sqrt{2}\beta + \gamma &= 0 \\ \beta - \sqrt{2}\gamma &= 0 \end{aligned}\right\} \quad \begin{aligned} &\alpha = \frac{\beta}{\sqrt{2}} = \gamma \\ &\alpha^2 + \beta^2 + \gamma^2 = \frac{\beta^2}{2} + \beta^2 + \frac{\beta^2}{2} = 2\beta^2 = 1 \end{aligned}$$

or $\beta = \pm\frac{1}{\sqrt{2}}$ choose $+\frac{1}{\sqrt{2}}$

$\alpha = \frac{1}{2}$, $\beta = \frac{1}{\sqrt{2}}$, $\gamma = \frac{1}{2}$ $\therefore |b^{(1)}\rangle \doteq \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$

ii) $\lambda = 0$, ie $a_2 = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\beta = 0$, $\alpha + \gamma = 0$ $\therefore \alpha = -\gamma = \frac{1}{\sqrt{2}}$

$$|b^{(2)}\rangle \doteq \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

iii) $\lambda = -\sqrt{2}$, ie $a_3 = -1$

$$\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} \sqrt{2}\alpha + \beta &= 0 \\ \alpha + \sqrt{2}\beta + \gamma &= 0 \\ \beta + \sqrt{2}\gamma &= 0 \end{aligned} \qquad |b^{(3)}\rangle \doteq \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

$\therefore \alpha = -\frac{\beta}{\sqrt{2}} = \gamma = \frac{1}{2}$

$$A\,|a^{(1)}\rangle = (+1)\,|a^{(1)}\rangle \qquad B\,|b^{(1)}\rangle = (+1)\,|b^{(1)}\rangle$$

$$A\,|a^{(2)}\rangle = (0)\,|a^{(2)}\rangle \qquad B\,|b^{(2)}\rangle = (0)\,|b^{(2)}\rangle$$

$$A\,|a^{(3)}\rangle = (-1)\,|a^{(3)}\rangle \qquad B\,|b^{(3)}\rangle = (-1)\,|b^{(3)}\rangle$$

$$|b^{(1)}\rangle \doteq \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}, \quad |b^{(2)}\rangle \doteq \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad |b^{(3)}\rangle \doteq \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

$$U = \sum_i |b^{(i)}\rangle\langle a^{(i)}|$$

$$\langle a^{(1)}| \doteq [1, 0, 0]$$
$$\langle a^{(2)}| \doteq [0, 1, 0]$$
$$\langle a^{(3)}| \doteq [0, 0, 1]$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

$$U^\dagger = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \qquad U^\dagger U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = U^\dagger B U$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}}_{\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

diagonalized as expected!