Pauli's Two-Component Formalism

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$R_x(\phi) = e^{\phi G_x}$, $R_y(\phi) = e^{\phi G_y}$, $R_z(\phi) = e^{\phi G_z}$ (SO(3) Group)

$\left(\vec{G} \longleftrightarrow \frac{\vec{J}}{i\hbar} \right)$ $[G_i, G_j] = \epsilon_{ijk} G_k$

$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$ includes $[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$
from $[x_i, p_j] = i\hbar \delta_{ij}$

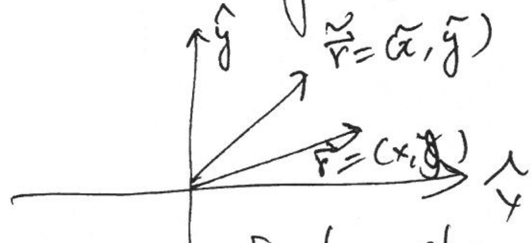
and $[S_i, S_j] = i\hbar\, \epsilon_{ijk} S_k$.

$$SO(3) \underset{\text{"correspondence"}}{\longleftrightarrow} SU(2)$$

In position Hilbert space, it provides

$$\underbrace{e^{\frac{-iL_z\phi}{\hbar}}}_{1 - \frac{i\phi}{\hbar} L_z + \cdots} |x, y, z\rangle = \underbrace{|x\cos\phi - y\sin\phi, x\cos\phi + y\sin\phi, z\rangle}_{|x,y,z\rangle + \phi\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)|x,y,z\rangle + \cdots}$$

$$L_z \to -\frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$



$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\phi\, x - \sin\phi\, y \\ \sin\phi\, x + \cos\phi\, y \end{bmatrix} = \begin{bmatrix} x - y\phi + \cdots \\ x\phi + y + \cdots \end{bmatrix}$$

cf.

$$\underbrace{e^{\frac{-ip_x l}{\hbar}}}_{1 - \frac{il}{\hbar} p_x + \cdots} |x,y,z\rangle = |x + l, y, z\rangle = |x,y,z\rangle + l\frac{\partial}{\partial x}|x,y,z\rangle + \cdots$$

$$p_x \to -\frac{\hbar}{i}\frac{\partial}{\partial x}$$

cf. $\langle x,y,z| p_x = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x,y,z|$

In general, we may apply to any state kets in Hilbert space: ②

$$|\hat{\alpha}\rangle = \underset{\substack{\uparrow \\ \text{"Drehung" rotation in German.}}}{\mathcal{D}}(R_{\hat{n}}(\phi))\,|\alpha\rangle$$

where $\mathcal{D}(R_{\hat{n}}(\phi)) = \mathcal{D}(\hat{n},\phi) = e^{\frac{-i\vec{J}\cdot\hat{n}\phi}{\hbar}}$.

For spin $\frac{1}{2}$ system without orbital motion, we may consider only the spin operator, $\vec{J} = \vec{S} = \frac{\hbar}{2}\vec{\sigma}$.

Note that $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k \longleftrightarrow [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$

Reminder of spin-$\frac{1}{2}$ case: p. 163–172

Section 3.2 "Spin $\frac{1}{2}$ systems and finite rotations".

$$\left.\begin{aligned}
S_x &= \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right) \doteq \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \frac{\hbar}{2}\sigma_x\\
S_y &= \frac{\hbar}{2}\left(-i|+\rangle\langle-| + i|-\rangle\langle+|\right) \doteq \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = \frac{\hbar}{2}\sigma_y\\
S_z &= \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right) \doteq \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \frac{\hbar}{2}\sigma_z
\end{aligned}\right\}\ \text{Eq. (3.2.1)}$$

where

$$\left.\begin{aligned}
&|+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix} \equiv \chi_+, \quad |-\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix} \equiv \chi_-\\
&\langle+| \doteq (1\ 0) = \chi_+^\dagger, \quad \langle-| \doteq (0\ 1) = \chi_-^\dagger
\end{aligned}\right\}\ \text{Eq. (3.2.26)}$$

Pauli matrices : Eq. (3.2.32) p. 169 ③

$$\sigma_x = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k$$

$$\begin{cases} \text{i) } i=j \; ; \; \sigma_i^2 = 1 \\ \text{ii) } i \neq j \; ; \; \sigma_i \sigma_j = -\sigma_j \sigma_i \text{ or } \sigma_i\sigma_j + \sigma_j\sigma_i = 0 \end{cases}$$

$$\hookrightarrow \{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \longleftrightarrow [S_i, S_j] = i\hbar \epsilon_{ijk} S_k$$

"SU(2) Algebra"

$$D(\hat{n}, \phi) = e^{\frac{-i\vec{S}\cdot\hat{n}\phi}{\hbar}} = e^{-i\vec{\sigma}\cdot\hat{n}\left(\frac{\phi}{2}\right)}$$

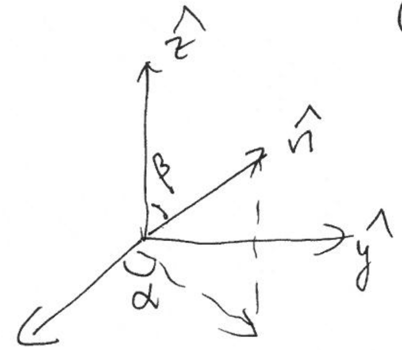
$$= I - i\vec{\sigma}\cdot\hat{n}\left(\frac{\phi}{2}\right) + \frac{1}{2!}\underbrace{(-i\vec{\sigma}\cdot\hat{n})^2}_{I}\left(\frac{\phi}{2}\right)^2 + \frac{1}{3!}\underbrace{(-i\vec{\sigma}\cdot\hat{n})^3}_{-i\vec{\sigma}\cdot\hat{n}}\left(\frac{\phi}{2}\right)^3 + \cdots$$

$$\boxed{\begin{aligned} \text{cf. } (\vec{\sigma}\cdot\hat{n})^2 &= \sum_i \sigma_i n_i \sum_j \sigma_j n_j \\ &= \sum_{i,j} \sigma_i \sigma_j n_i n_j = \sum_{i,j}\left(\delta_{ij} + i\sum_k \epsilon_{ijk}\sigma_k\right)(n_i n_j) \\ &= \sum_i n_i^2 + i \sum_k \left(\sum_{i,j}\epsilon_{ijk} n_i n_j\right)\sigma_k \\ &= \hat{n}^2 + i(\hat{n}\times\hat{n})\cdot\vec{\sigma} = I \end{aligned}}$$

$$= \left\{I - \frac{1}{2!}\left(\frac{\phi}{2}\right)^2 + \cdots\right\} - i\vec{\sigma}\cdot\hat{n}\left\{\frac{\phi}{2} - \frac{1}{3!}\left(\frac{\phi}{2}\right)^3 + \cdots\right\}$$

$$= \cos\frac{\phi}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\phi}{2}.$$

Ex. Prob. 9 of Chapt. 1



$$\vec{S}\cdot\hat{n}\,|S_n\pm\rangle = \pm\frac{\hbar}{2}\,|S_n\pm\rangle$$

$$\vec{\sigma}\cdot\hat{n}\,\chi_\pm = \pm\chi_\pm$$

$$\hat{n} = \sin\beta\cos\alpha\,\hat{x} + \sin\beta\sin\alpha\,\hat{y} + \cos\beta\,\hat{z}$$

$$\vec{\sigma}\cdot\hat{n} = \sin\beta\cos\alpha\,\sigma_x + \sin\beta\sin\alpha\,\sigma_y + \cos\beta\,\sigma_z$$

$$= \begin{bmatrix} \cos\beta & e^{-i\alpha}\sin\beta \\ e^{i\alpha}\sin\beta & -\cos\beta \end{bmatrix}$$

$$\boxed{\vec{\sigma}\cdot\hat{n}\,\chi = \lambda\chi}$$

Eigenvalue problem:
find λ and corresponding $\chi = \begin{bmatrix} a \\ b \end{bmatrix}$.

$$\begin{bmatrix} \cos\beta-\lambda & e^{-i\alpha}\sin\beta \\ e^{i\alpha}\sin\beta & -(\cos\beta+\lambda) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Characteristic eq:

$$\begin{vmatrix} \cos\beta-\lambda & e^{-i\alpha}\sin\beta \\ e^{i\alpha}\sin\beta & -(\cos\beta+\lambda) \end{vmatrix} = 0 \Rightarrow \lambda^2-1=0$$

or $\lambda = \pm 1$

For $\lambda = 1$; χ satisfies

$$\begin{bmatrix} \cos\beta-1 & e^{-i\alpha}\sin\beta \\ e^{i\alpha}\sin\beta & -(\cos\beta+1) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} a(\cos\beta-1) + b e^{-i\alpha}\sin\beta = 0 \rightarrow -2a\sin^2\frac{\beta}{2} + 2b e^{-i\alpha}\sin\frac{\beta}{2}\cos\frac{\beta}{2} = 0 \\ e^{i\alpha}\sin\beta\, a - (\cos\beta+1)\,b = 0 \rightarrow 2a e^{i\alpha}\sin\frac{\beta}{2}\cos\frac{\beta}{2} - 2b\cos^2\frac{\beta}{2} = 0 \end{array}$$

$$\therefore\ a\sin\frac{\beta}{2} = b e^{-i\alpha}\cos\frac{\beta}{2}$$

$$\chi_+ = \begin{bmatrix} \cos\frac{\beta}{2} \\ e^{i\alpha}\sin\frac{\beta}{2} \end{bmatrix}$$

2) $\mathcal{D}(\hat{z},\alpha) = e^{-i\sigma_z\left(\frac{\alpha}{2}\right)} = \cos\frac{\alpha}{2} - i\sigma_z \sin\frac{\alpha}{2} = \begin{bmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{bmatrix}$ (5)

$\mathcal{D}(\hat{z},\alpha)\left[\mathcal{D}(\hat{y},\beta)|+\rangle\right] \doteq \begin{bmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{bmatrix} \begin{bmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{bmatrix}$

$\therefore \mathcal{D}(\hat{z},\alpha)\left[\mathcal{D}(\hat{y},\beta)|+\rangle\right] = e^{-i\frac{\alpha}{2}}\Big[\underbrace{\cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle}_{|S_n+\rangle}\Big]$

(We can even get the overall phase!

$|S_n+\rangle$ prob. 9 in Chapt. 1

Drehung can determine the precise phase of the state which couldn't be found by solving the eigenvalue problem.

$SU(2)$ ← 2-dim.

unimodular, unitary

(HW#7, prob. 3 of Chapt. 3)

$$\left.\begin{array}{l} \mathcal{D}^\dagger(\hat{n},\phi)\mathcal{D}(\hat{n},\phi) = I \\ \det \mathcal{D}(\hat{n},\phi) = 1 \\ \quad \mathrm{Tr}\,\vec{\sigma} = 0 \end{array}\right\} e^{-i\vec{\sigma}\cdot\hat{n}\frac{\phi}{2}}$$

$SO(3)$ ← 3-dim.

unimodular, orthogonal

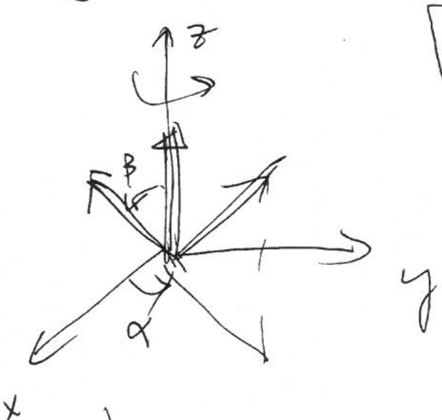
$$\left.\begin{array}{l} \mathrm{Det}\,R_{\hat{n}}(\phi) = 1 \\ R_{\hat{n}}^T(\phi)\,R_{\hat{n}}(\phi) = I \\ \quad \mathrm{Tr}\,\vec{J} = 0 \\ \text{or } \mathrm{Tr}\,\vec{G} = 0. \end{array}\right\} R_{\hat{n}}(\phi) = e^{-i\frac{\vec{J}\cdot\hat{n}\phi}{\hbar}} = e^{\vec{G}\cdot\hat{n}\phi}.$$

$|S_n+\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle$ (See page 59, prob. 9 in Chapt. 1)

Now, we may solve this problem using the Drehung,

$$D(\hat{n},\phi) = e^{-i\vec{\sigma}\cdot\hat{n}\frac{\phi}{2}} = \cos\frac{\phi}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\phi}{2}$$

Fig. 3.3.



$$\boxed{|S_n+\rangle = D(\hat{z},\alpha)D(\hat{y},\beta)|+\rangle}$$

1) $D(\hat{y},\beta)|+\rangle = \left(\cos\frac{\beta}{2} - i\sigma_y\sin\frac{\beta}{2}\right)|+\rangle$

$$= \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{bmatrix} \sim \underline{D(\hat{y},\beta)|+\rangle = \cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}|-\rangle}$$

e.g. $\beta=\frac{\pi}{2}$; $e^{-i\sigma_y\frac{\pi}{4}}|+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle = |S_x+\rangle$

$\beta=\pi$; $e^{-i\sigma_y\frac{\pi}{2}}|+\rangle = |-\rangle$

$\beta=\frac{3\pi}{2}$; $e^{-i\sigma_y\frac{3\pi}{4}}|+\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle = |S_x-\rangle$

$\beta=2\pi$; $e^{-i\sigma_y\pi}|+\rangle = -|+\rangle$ "Möbius strip";

note that opposite phase