Measurements and Observables

$\boxed{\text{Homework \#3 - Due by Sep. 20}}$

Sep. 13, 2016

We represented the spin observables by the Pauli matrices

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad (\text{See Eq. (3.2-32), Chapt. 3})$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = (\sigma_1, \sigma_2, \sigma_3)$

with $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$

Here, we note the hermiticity of the spin operator, as $\sigma_i^\dagger = \sigma_i$, $\vec{S}^\dagger = \vec{S}$

It means that the eigenvalues of the spin observables must be real, e.g. S_z measurement yields $\pm\frac{\hbar}{2}$ eigenvalues for the spin-$\frac{1}{2}$ system and $\vec{S}^2$ measurement yields $\frac{3}{4}\hbar^2$ eigenvalue.

Note that $[\vec{S}^2, S_z] = 0$, i.e. compatible observables.

Simultaneous ~~e~~eigenkets can be denoted by

$|+\rangle \equiv |\frac{1}{2}, +\frac{1}{2}\rangle$ and $|-\rangle \equiv |\frac{1}{2}, -\frac{1}{2}\rangle$.

In general, for spin-s system,

$$\vec{S}^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$$

$$S_z|s,m\rangle = m\hbar|s,m\rangle$$

$$\boxed{\begin{aligned} \vec{S}^2 &= \frac{\hbar^2}{4}\vec{\sigma}^2 \\ &= \frac{\hbar^2}{4}\sum_{i=1}^{3}\sigma_i^2 \\ &= \frac{3\hbar^2}{4}I. \end{aligned}}$$

In general, for two observables A and B, $A^\dagger = A$ and $B^\dagger = B$: ②

If $[A, B] = 0$, then A and B are compatible observables

$$\left.\begin{aligned} A|a,b\rangle &= a\,|a,b\rangle \\ B|a,b\rangle &= b\,|a,b\rangle \end{aligned}\right\} \ |a,b\rangle \text{ are simultaneous eigenkets.}$$

However, if $[A, B] \neq 0$, then A and B are incompatible observables.

For the incompatible observables, we may define the "fuzziness" of an observable in a given state: e.g.

$$S_z\,|\tfrac{1}{2}\tfrac{1}{2}\rangle = \frac{\hbar}{2}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle \quad \text{and} \quad \vec{S}^2\,|\tfrac{1}{2}\tfrac{1}{2}\rangle = \frac{3}{4}\hbar^2\,|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$\vec{S}^2$ and S_z ~~are~~ "sharp" for $|\tfrac{1}{2}\tfrac{1}{2}\rangle$ state

However, $S_x\,|\tfrac{1}{2}\tfrac{1}{2}\rangle \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\hbar}{2}\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$

S_x is "fuzzy" for $|\tfrac{1}{2}\tfrac{1}{2}\rangle$ state.

To quantify the fuzziness, we define the deviation of observable A for a given state by dispersion: $(\Delta A)^2$,

where

$\Delta A = A - \langle A \rangle$ ← expectation value of an observable A for a given state.

Expectation value of A observable for a given state $|\alpha\rangle$: (3)

$$\langle A\rangle = \langle\alpha|A|\alpha\rangle$$

If the operator is sharp for $|\alpha\rangle$, then

$$\langle A\rangle = \langle\alpha|A|\alpha\rangle = \alpha\langle\alpha|\alpha\rangle = \alpha$$, i.e.

Expectation value = Eigen value;

e.g. $\langle\tfrac{1}{2}\tfrac{1}{2}|S_z|\tfrac{1}{2}\tfrac{1}{2}\rangle = \frac{\hbar}{2}$

However the fuzzy operator won't give the expectation value same as the eigenvalue:

$$\langle\tfrac{1}{2}\tfrac{1}{2}|S_x|\tfrac{1}{2}\tfrac{1}{2}\rangle = \langle\tfrac{1}{2}\tfrac{1}{2}|\tfrac{\hbar}{2}|\tfrac{1}{2}-\tfrac{1}{2}\rangle = \frac{\hbar}{2}\langle\tfrac{1}{2}\tfrac{1}{2}|\tfrac{1}{2}-\tfrac{1}{2}\rangle = 0.$$

The dispersion of S_x is given by

$$\langle\tfrac{1}{2}\tfrac{1}{2}|(\Delta S_x)^2|\tfrac{1}{2}\tfrac{1}{2}\rangle = \langle\tfrac{1}{2}\tfrac{1}{2}|(S_x-\langle S_x\rangle)^2|\tfrac{1}{2}\tfrac{1}{2}\rangle = \langle\tfrac{1}{2}\tfrac{1}{2}|S_x^2-2\langle S_x\rangle S_x+\langle S_x\rangle^2|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= \underbrace{\langle\tfrac{1}{2}\tfrac{1}{2}|S_x^2|\tfrac{1}{2}\tfrac{1}{2}\rangle}_{\frac{\hbar^2}{4}I} \underbrace{- 2\langle S_x\rangle^2 + \langle S_x\rangle^2}_{-\langle S_x\rangle^2 \to 0}$$

$$= \frac{\hbar^2}{4} \neq 0.$$

cf. $\langle\tfrac{1}{2}\tfrac{1}{2}|(\Delta S_z)^2|\tfrac{1}{2}\tfrac{1}{2}\rangle = \langle S_z^2\rangle - \langle S_z\rangle^2 = \frac{\hbar^2}{4} - \frac{\hbar^2}{4} = 0$ "Sharp"!

In general, the mean square deviation or variance of A is given by

$$\langle(\Delta A)^2\rangle = \langle(A-\langle A\rangle)^2\rangle = \langle A^2-2A\langle A\rangle+\langle A\rangle^2\rangle = \langle A^2\rangle - \langle A\rangle^2.$$

In conclusion, the "fuzziness" is characterized (quantified) by the dispersion.

Uncertainty Relation

$$\boxed{\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}\left|\langle[A,B]\rangle\right|^2}$$

Proof.

$$|\alpha\rangle = (\Delta A)|\gamma\rangle \quad ; \quad \langle\alpha|\alpha\rangle = \langle\gamma|(\Delta A)^2|\gamma\rangle$$

$$|\beta\rangle = (\Delta B)|\gamma\rangle \quad ; \quad \langle\beta|\beta\rangle = \langle\gamma|(\Delta B)^2|\gamma\rangle$$

Lemma 1.1 Schwarz inequality

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq \left|\langle\alpha|\beta\rangle\right|^2$$

cf. $|\vec{a}|^2|\vec{b}|^2 \geq |\vec{a}\cdot\vec{b}|^2$

$$\left(\langle\alpha| - \langle\beta|\frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}\right)\left(|\alpha\rangle - \frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}|\beta\rangle\right)$$

$$= \langle\alpha|\alpha\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle} - \cancel{\frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle}} + \cancel{\frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle}} \geq 0$$

All kets in Hilbert space are positive definite.

or

$$\langle\alpha|\alpha\rangle \geq \frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle}, \quad \text{i.e.} \quad \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2.$$

Lemma 1.2. Expectation value of a Hermitian operator is <u>purely real</u>.

$$\langle\alpha|A|\alpha\rangle = \langle\alpha|A^\dagger|\alpha\rangle^* = \langle\alpha|A|\alpha\rangle^* \quad \#.$$

Lemma 1.3. Expectation value of an anti-Hermitian operator is <u>purely imaginary</u>. $(C = -C^\dagger)$

$$\langle\alpha|C|\alpha\rangle = \langle\alpha|C^\dagger|\alpha\rangle^* = -\langle\alpha|C|\alpha\rangle^* \quad \#.$$

Note that

$$(\Delta A)(\Delta B) = \underbrace{\frac{1}{2}[\Delta A, \Delta B]}_{\frac{1}{2}[A,B]} + \underbrace{\frac{1}{2}\{\Delta A, \Delta B\}}_{\text{Hermitian}}$$

$$\underbrace{\frac{1}{2}[A,B]}_{\text{anti-hemitian}} \qquad \{\Delta A, \Delta B\}^\dagger = \{\Delta A, \Delta B\}.$$

$$\left([A,B]^\dagger = (AB - BA)^\dagger = BA - AB = -[A,B] \right)$$

$$\langle \Delta A \Delta B \rangle = \underbrace{\frac{1}{2}\langle [A,B] \rangle}_{\text{purely imaginary}} + \underbrace{\frac{1}{2}\langle \{\Delta A, \Delta B\} \rangle}_{\text{purely real}}$$

$$|\langle \Delta A \Delta B \rangle|^2 = \frac{1}{4}|\langle [A,B] \rangle|^2 + \frac{1}{4}|\langle \{\Delta A, \Delta B\} \rangle|^2$$

$$\therefore \langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq |\langle \Delta A \Delta B \rangle|^2 = \frac{1}{4}|\langle [A,B] \rangle|^2 + \frac{1}{4}|\langle \{\Delta A, \Delta B\} \rangle|^2 \geq \frac{1}{4}|\langle [A,B] \rangle|^2$$

⑥

<u>Caution</u> Uncertainty ~~realation~~ applies ~~only~~ not only for the fuzzy operators, ~~also~~ but for any ~~sharp~~ sharp operator.

$$\langle \tfrac12 \tfrac12 | (\Delta S_z)^2 | \tfrac12, \tfrac12 \rangle = \langle \tfrac12 \tfrac12 | \underbrace{S_z^2}_{\frac{\hbar^2}{4} I} | \tfrac12 \tfrac12 \rangle - \underbrace{\langle \tfrac12 \tfrac12 | S_z | \tfrac12 \tfrac12 \rangle^2}_{\left(\frac{\hbar}{2}\right)^2} = 0$$

Then $\langle (\Delta S_z)^2 (\Delta S_x)^2 \rangle = 0 \geq \frac14 \left| \langle \underbrace{[S_z, S_x]}_{i\hbar S_y} \rangle \right|^2$

$$\frac{\hbar^2}{4} \langle S_y \rangle^2$$

$$\langle S_y \rangle = \langle \tfrac12 \tfrac12 | \tfrac{\hbar}{2} \sigma_y | \tfrac12 \tfrac12 \rangle = 0$$

$\langle \tfrac12 \tfrac12 | \sigma_y | \tfrac12 \tfrac12 \rangle$

$$= [1, 0] \underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{i \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$= i \cdot 0 = 0$

$\therefore \; 0 \geq 0.$

cf. $\frac14 \left| \langle \{\Delta S_z, \Delta S_x\} \rangle \right|^2 = 0$

$\langle \tfrac12 \tfrac12 | \Delta S_z = \langle \tfrac12 \tfrac12 | (S_z - \underbrace{\langle S_z \rangle}_{\frac{\hbar}{2}})$

$= \frac{\hbar}{2} \langle \tfrac12 \tfrac12 | - \frac{\hbar}{2} \langle \tfrac12 \tfrac12 |$

$= 0$

$\Delta S_z | \tfrac12 \tfrac12 \rangle = 0$

Ex

$$\langle \tfrac12 \tfrac12 | (\Delta S_x)^2 | \tfrac12 \tfrac12 \rangle = \frac{\hbar^2}{4}$$

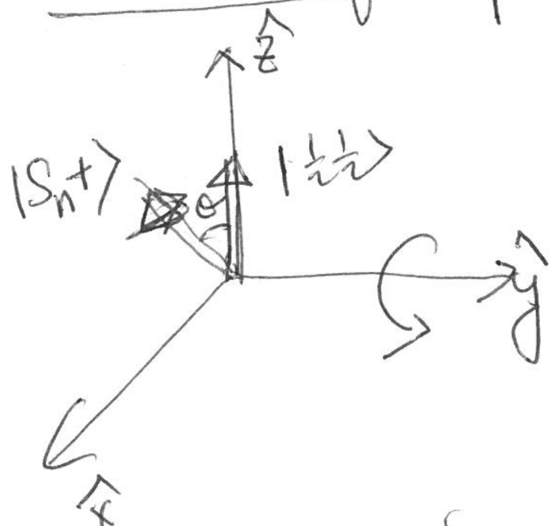
$$\langle \tfrac12 \tfrac12 | (\Delta S_y)^2 | \tfrac12 \tfrac12 \rangle = \frac{\hbar^2}{4}.$$

$$[S_x, S_y] = i\hbar S_z$$

$$\underbrace{\langle \tfrac12 \tfrac12 | (\Delta S_x)^2 | \tfrac12 \tfrac12 \rangle \langle \tfrac12 \tfrac12 | (\Delta S_y)^2 | \tfrac12 \tfrac12 \rangle}_{\frac{\hbar^2}{4} \times \frac{\hbar^2}{4} = \frac{\hbar^4}{16}} \geq \underbrace{\frac14 \left| \langle \tfrac12 \tfrac12 | \underbrace{[S_x, S_y]}_{i\hbar S_z} | \tfrac12 \tfrac12 \rangle \right|^2}$$

$$= \frac14 \left| i \frac{\hbar^2}{2} \right|^2 = \frac{\hbar^4}{16}.$$

Rotation of a spin-$\frac12$ state, $|\frac12 \frac12\rangle$:



$$|S_n+\rangle = e^{\frac{-i S_y \theta}{\hbar}} |\tfrac12 \tfrac12\rangle$$

$$= e^{-i\sigma_y \frac{\theta}{2}} |\tfrac12 \tfrac12\rangle$$

$$= \left(\cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2}\right) |\tfrac12 \tfrac12\rangle$$

$$= \cos\frac{\theta}{2} |\tfrac12 \tfrac12\rangle + \sin\frac{\theta}{2} |\tfrac12 -\tfrac12\rangle$$

$$\sigma_y |\tfrac12 \tfrac12\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} = i |\tfrac12 -\tfrac12\rangle$$

$$\langle \tfrac12 \tfrac12 | (\Delta S_y)^2 | \tfrac12 \tfrac12 \rangle = \underbrace{\langle \tfrac12 \tfrac12 | S_y^2 | \tfrac12 \tfrac12 \rangle}_{\frac{\hbar^2}{4}} - \langle \tfrac12 \tfrac12 | \underbrace{S_y | \tfrac12 \tfrac12 \rangle}_{\frac{\hbar}{2} \underbrace{\sigma_y | \frac12 \frac12 \rangle}_{i | \frac12 -\frac12 \rangle}}{}^2$$

$$\left(i \frac{\hbar}{2} \langle \tfrac12 \tfrac12 | \tfrac12 -\tfrac12 \rangle \right)^2 = 0$$

$$| \tfrac12 -\tfrac12 \rangle = e^{-i \frac{S_y(\pi)}{\hbar}} | \tfrac12 \tfrac12 \rangle$$

$$= \left(\cos\frac{\pi}{2} - i \sigma_y \sin\frac{\pi}{2} \right) | \tfrac12 \tfrac12 \rangle$$

$$= -i \underbrace{\sigma_y | \tfrac12 \tfrac12 \rangle}_{i | \frac12 -\frac12 \rangle} = | \tfrac12 -\tfrac12 \rangle.$$

$$e^{i \frac{\vec{S}\cdot\hat{n}}{\hbar} \theta} = e^{i \vec{\sigma}\cdot\hat{n} \frac{\theta}{2}}$$

$$= I + \left(i \vec{\sigma}\cdot\hat{n} \frac{\theta}{2} \right) + \frac{1}{2!} \left(i \vec{\sigma}\cdot\hat{n} \frac{\theta}{2} \right)^2 + \frac{1}{3!} \left(i \vec{\sigma}\cdot\hat{n} \frac{\theta}{2} \right)^3 + \cdots$$

$$= I \left(1 - \frac{1}{2!} \left(\frac{\theta}{2}\right)^2 + \frac{1}{4!} \left(\frac{\theta}{2}\right)^4 - \cdots \right)$$

$$+ i \vec{\sigma}\cdot\hat{n} \left(\frac{\theta}{2} - \frac{1}{3!} \left(\frac{\theta}{2}\right)^3 + \cdots \right)$$

$$= I \cos\frac{\theta}{2} + i \vec{\sigma}\cdot\hat{n} \sin\frac{\theta}{2}.$$

cf. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\sigma_i \sigma_j = \delta_{ij} I + i \epsilon_{ijk} \sigma_k$

$\{\sigma_i, \sigma_j\} = 2 \delta_{ij} I$

$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$