Heisenberg Equation

Oct. 11, 2018

$\boxed{\text{HW\#5 Due by Oct. 18}}$

In the last lecture, we discussed how the different picture of handling quantum dynamics could arise using the computation of expectation values in the spin precession problem; e.g.

$$\langle S_x \rangle_t = \langle \alpha, t | \underset{\substack{\uparrow \\ \text{stationary operator} \\ \text{in Schrödinger picture}}}{S_x} \underbrace{|\alpha, t\rangle}_{\substack{\text{time-evoluted} \\ \text{state ket} \\ \text{in Schrödinger picture.}}}$$

$$= \langle \alpha, t=0 | \underbrace{\mathcal{U}^\dagger(t) S_x \mathcal{U}(t)}_{\substack{\text{time-evoluted} \\ \text{operator in Heisenberg} \\ \text{picture}}} \underbrace{|\alpha, t=0\rangle}_{\substack{\text{initial stationary} \\ \text{state ket} \\ \text{as the Heisenberg} \\ \text{picture state ket.}}}$$

$$\equiv {}_H\langle \alpha | \underset{\substack{\uparrow \\ \text{Heisenberg operator}}}{S_x^H(t)} \underbrace{|\alpha\rangle_H}_{\substack{\text{Heisenberg} \\ \text{State ket.}}}$$

This works of course due to the Associativity of Multiplication.

The reason for this change of picture is to get a <u>close connection with the classical physics</u>, where the state kets & bras are not introduced at all. Physical observable are functions of time in C.M.

Thus, define $\boxed{|\alpha\rangle_H \equiv |\alpha, t=0\rangle_S}$

so that $|\alpha, t\rangle_S = \mathcal{U}(t)\, |\alpha\rangle_H$ (Unitary transf. of state kets)

Similarly, $\boxed{A^{(H)}(t) = \mathcal{U}^\dagger(t)\, A^{(S)}\, \mathcal{U}(t)}$.

This leads to the independence of pictures for the expectation value of an observable A.

$$
{}_S\langle \alpha, t|\, A^{(S)}\, |\alpha, t\rangle_S = {}_H\langle \alpha|\, A^{(H)}(t)\, |\alpha\rangle_H .
$$

$$
= \underbrace{{}_H\langle \alpha|}_{{}_S\langle \alpha, t=0|}\, \underbrace{\mathcal{U}^\dagger(t) A^{(S)} \mathcal{U}(t)}_{A^{(H)}(t)}\, \underbrace{|\alpha\rangle_H}_{|\alpha, t=0\rangle_S}
$$

Now, let's find the equation of motion for $A^{(H)}(t)$ by differentiating it over the time.

$$\frac{dA^{(H)}(t)}{dt} = \underbrace{\frac{\partial \mathcal{U}^\dagger(t)}{\partial t}}_{-\frac{1}{i\hbar}\mathcal{U}^\dagger(t) H} A^{(S)} \mathcal{U}(t) + \mathcal{U}^\dagger(t) A^{(S)} \underbrace{\frac{\partial \mathcal{U}(t)}{\partial t}}_{\frac{1}{i\hbar} H \mathcal{U}(t)}$$

from $i\hbar \frac{\partial \mathcal{U}(t)}{\partial t} = H\mathcal{U}(t)$

$$= \frac{1}{i\hbar}\left[\mathcal{U}^\dagger(t) A^{(S)} \underset{\uparrow \atop \mathcal{U}(t)\mathcal{U}^\dagger(t)}{} H \mathcal{U}(t) - \mathcal{U}^\dagger(t) H \underset{\uparrow \atop \mathcal{U}(t)\mathcal{U}^\dagger(t)}{} A^{(S)} \mathcal{U}(t)\right]$$

$$= \frac{1}{i\hbar}\left[A^{(H)}(t), H\right],$$

where one should note

$$H^{(H)} = \mathcal{U}^\dagger(t) H \mathcal{U}(t) = H$$

Heisenberg picture Hamiltonian $\left(\mathcal{U}(t) = e^{-\frac{iHt}{\hbar}}\right)$ original Hamiltonian in Schrödinger picture

Thus, we get Heisenberg Equation of motion.

same as $H(0)$

$$\frac{d A^{(H)}(t)}{dt} = \frac{1}{i\hbar}\left[A^{(H)}(t), H(t)\right]$$

(see Eq. (2.2.19) p. 83.)

As H is independent on time here, one can think that this commutator at the same time for $A^{(H)}(t)$ and H, i.e. $H(t) = H(0)$.

Classical Correspondence

Q. M. $\qquad\qquad\qquad\qquad$ C. M

Poisson Bracket

$$\frac{[A, B]}{i\hbar} \longleftrightarrow [A, B]_{P.B.} \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial q}\frac{\partial A}{\partial p}$$

$$\frac{d A^{(H)}(t)}{dt} = \frac{1}{i\hbar}\left[A^{(H)}(t), H\right] \longleftrightarrow \frac{dA(q,p)}{dt} = [A, H]_{P.B.}$$

One may understand the C.M. equation of motion from the following consideration with Hamilton's equation of motion.

$$\frac{d A(q,p)}{dt} = \frac{\partial A}{\partial q}\dot{q} + \frac{\partial A}{\partial p}\dot{p}$$

$$= \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial A}{\partial p}$$

$$= [A, H]_{P.B.}$$

Hamilton's eq. of motion

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

If $A = q$, then

$$\frac{dq}{dt} = [q, H]_{P.B.} = \frac{\partial H}{\partial p}$$

Satisfying Poisson Bracket

Spin-precession in Heisenberg Picture

$$H = \omega S_z .$$

Here, we drop the "(H)" notation to denote Heisenberg picture for simplicity of notation.

$$\frac{dS_z}{dt} = \frac{1}{i\hbar}[S_z, H] = 0$$

Thus, $S_z(t) = S_z(0)$ and H is indep. of time as expected.

$$\frac{dS_x}{dt} = \frac{1}{i\hbar}[S_x, H]$$

$$= \frac{\omega}{i\hbar}\underbrace{[S_x, S_z]}_{i\hbar \underbrace{\epsilon_{132}}_{(-1)} S_y}$$

$$= -\omega S_y$$

$$\frac{dS_y}{dt} = \frac{1}{i\hbar}[S_y, H]$$

$$= \frac{\omega}{i\hbar}\underbrace{[S_y, S_z]}_{i\hbar \underbrace{\epsilon_{231}}_{(1)} S_x}$$

$$= \omega S_x .$$

In summary, we get

$$\frac{dS_x}{dt} = -\omega S_y \quad \text{and} \quad \frac{dS_y}{dt} = \omega S_x$$

while S_z is stationary.

To decouple S_x and S_y separately, let's raise the order of differentiation.

$$\frac{d^2 S_x}{dt^2} = -\omega \frac{dS_y}{dt} = -\omega^2 S_x$$

Similarly,

$$\frac{d^2 S_y}{dt^2} = -\omega^2 S_y .$$

One can derive these results using just commutation relations. See Baker-Hausdorf Lemma at the end of this lecture note.

Thus, the solutions are given by

$$S_x(t) = \cos\omega t\, S_x(0) - \sin\omega t\, S_y(0)$$

$$S_y(t) = \sin\omega t\, S_x(0) + \cos\omega t\, S_y(0)$$

We may represent these solutions in matrices.

$$S_x(t) \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & \cos\omega t + i\sin\omega t \\ \cos\omega t - i\sin\omega t & 0 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix}$$

$$S_y(t) \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & \sin\omega t - i\cos\omega t \\ \sin\omega t + i\cos\omega t & 0 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 & -ie^{i\omega t} \\ ie^{-i\omega t} & 0 \end{bmatrix}.$$ 17

Note that $S_x(0) \doteq \frac{\hbar}{2}\sigma_x$ and $S_y(0) \doteq \frac{\hbar}{2}\sigma_y$.

Let's check the relationship between the two pictures: $A^{(H)}(t) = \mathcal{U}^\dagger(t)\, A^{(S)}\, \mathcal{U}(t)$.

Note here
$$\mathcal{U}(t) = e^{-\frac{iHt}{\hbar}}$$
$$= e^{-\frac{i\omega S_z t}{\hbar}}$$
$$= e^{-\frac{i\omega t\,\sigma_z}{2}}$$
$$\doteq \begin{bmatrix} e^{-\frac{i\omega t}{2}} & 0 \\ 0 & e^{\frac{i\omega t}{2}} \end{bmatrix}$$

Now,
$$S_x^{(H)}(t) = \mathcal{U}^\dagger(t)\, S_x\, \mathcal{U}(t)$$
$$\doteq \begin{bmatrix} e^{\frac{i\omega t}{2}} & 0 \\ 0 & e^{-\frac{i\omega t}{2}} \end{bmatrix} \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{i\omega t}{2}} & 0 \\ 0 & e^{\frac{i\omega t}{2}} \end{bmatrix}$$
$$= \frac{\hbar}{2}\begin{bmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix}$$ as obtained previously.

Heisenberg Operators as Outer Products of Bras & Kets

As we've done in Schrödinger picture, we may write

$$S_x = \frac{\hbar}{2}|S_x+\rangle\langle S_x+| - \frac{\hbar}{2}|S_x-\rangle\langle S_x-|$$
$$= \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right),$$

where $|S_x\pm\rangle = \pm\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$.

In Heisenberg picture,

$$S_x(t) = \mathcal{U}^\dagger(t)\, S_x\, \mathcal{U}(t)$$
$$= \frac{\hbar}{2}\left[\left(\mathcal{U}^\dagger(t)|S_x+\rangle\right)\left(\langle S_x+|\mathcal{U}(t)\right) - \left(\mathcal{U}^\dagger(t)|S_x-\rangle\right)\left(\langle S_x-|\mathcal{U}(t)\right)\right]$$
$$= \frac{\hbar}{2}\left[|S_x+,t\rangle_H\, {}_H\langle S_x+,t| - |S_x-,t\rangle_H\, {}_H\langle S_x-,t|\right]$$

where the <u>base kets and bras in Heisenberg picture changes in time</u> although the <u>state ket does not</u>.

As $|S_x+, t\rangle_H = \mathcal{U}^\dagger(t)|S_x+\rangle$,

$$i\hbar\frac{\partial}{\partial t}|S_x+, t\rangle_H = i\hbar\frac{\partial}{\partial t}\mathcal{U}^\dagger(t)\,|S_x+\rangle$$

$$= -H\,\mathcal{U}^\dagger(t)|S_x+\rangle$$

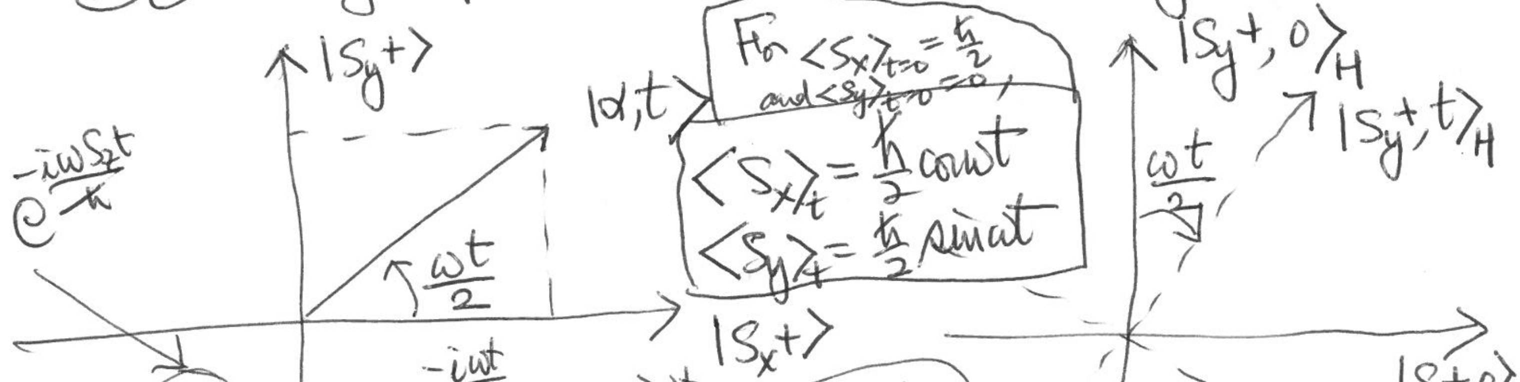
$$= -H\,|S_x+, t\rangle_H ,$$

where $\mathcal{U}^\dagger(t) = e^{+\frac{iHt}{\hbar}}$ and the negative sign in front of H appears in Heisenberg picture equation for the base ket.

In general, $\mathcal{U}^\dagger(t)|a\rangle_S = |a,t\rangle_H$.

and $i\hbar\frac{\partial}{\partial t}|a,t\rangle_H = -H\,|a,t\rangle_H$.

Schrödinger Picture $\qquad\qquad$ Heisenberg Picture



$|\alpha,t\rangle_S = \mathcal{U}(t)|S_x+\rangle = \frac{1}{\sqrt{2}}e^{\frac{-i\omega t}{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{\frac{i\omega t}{2}}|-\rangle$ (cf. $|\alpha, t=\frac{\pi}{\omega}\rangle_S \simeq -\frac{i}{\sqrt{2}}|+\rangle + \frac{i}{\sqrt{2}}|-\rangle = |S_y+\rangle$)

Appendix : Baker-Hausdorff Lemma

(See Eq. (2.3.47) p. 95.)

$$e^{iG\lambda} A e^{-iG\lambda} = A + i\lambda[G, A] + \frac{i^2\lambda^2}{2!}[G,[G,A]] + \cdots + \frac{i^n\lambda^n}{n!}[G,[G,[G,\cdots[G,A]]\cdots]] + \cdots$$

Example

$$S_x(t) = \mathcal{U}^\dagger(t) S_x(0)\, \mathcal{U}(t)$$

$$= e^{\frac{iHt}{\hbar}} S_x(0)\, e^{-\frac{iHt}{\hbar}} \qquad (H = \omega S_z)$$

$$\underbrace{S_z(t) = S_z(0)}_{\text{"Stationary"}}$$

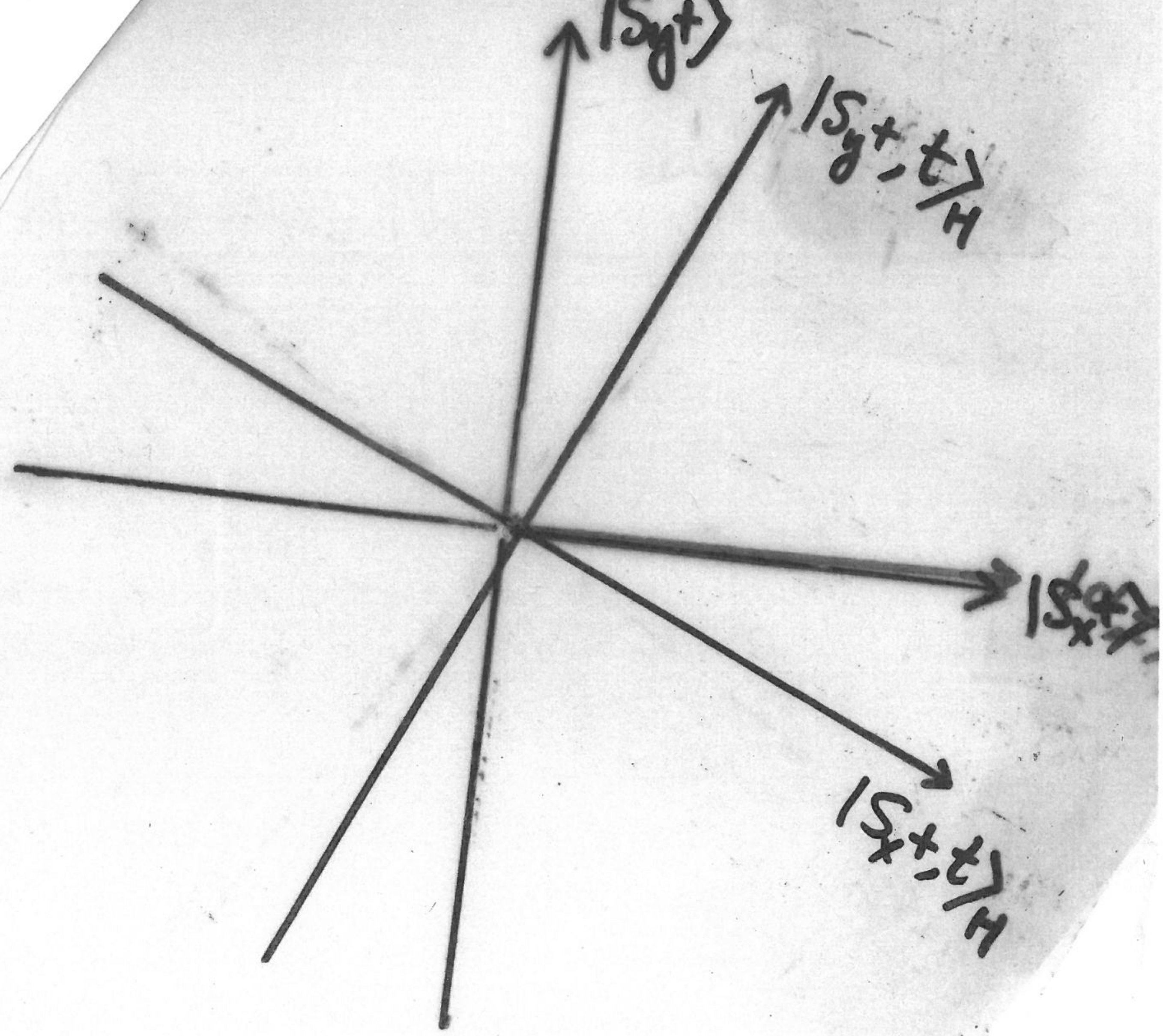
$$= e^{iS_z\left(\frac{\omega t}{\hbar}\right)} S_x(0)\, e^{-iS_z\left(\frac{\omega t}{\hbar}\right)}$$

$$= S_x(0) + i\left(\frac{\omega t}{\hbar}\right)\underbrace{[S_z(0), S_x(0)]}_{i\hbar\underbrace{\epsilon_{312}}_{(+1)} S_y(0)} + \frac{i^2\left(\frac{\omega t}{\hbar}\right)^2}{2!}\underbrace{[S_z(0),\underbrace{[S_z(0), S_x(0)]}_{i\hbar S_y(0)}]}_{(i\hbar)^2\underbrace{\epsilon_{321}}_{(-1)} S_x(0)}$$

$$+ \frac{i^3\left(\frac{\omega t}{\hbar}\right)^3}{3!}\underbrace{[S_z(0),\underbrace{[S_z(0),[S_z(0), S_x(0)]]}_{-(i\hbar)^2 S_x(0)}]}_{-(i\hbar)^3\underbrace{\epsilon_{312}}_{(+1)} S_y(0)} + \cdots$$

$$= S_x(0)\underbrace{\left(1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!} - \cdots\right)}_{\cos\omega t} - S_y(0)\underbrace{\left((\omega t) - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} - \cdots\right)}_{\sin\omega t}$$

\# $$= \cos\omega t\, S_x(0) - \sin\omega t\, S_y(0)$$


|Sy+⟩
|Sy+, t⟩H
|Sx+⟩
|Sx+, t⟩H