Gauge Transformation in Electromagnetism

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As we have derived Feynman's path integral formulation, we would like to apply it to physical examples. In particular, this approach is gratifying to discuss a novel electromagnetic effect, e.g. Ahronov-Bohm effect, from a conceptual point of view on gauge degrees of freedom. Thus, we will discuss the gauge transformation in electromagnetism for the discussion of Ahronov-Bohm effect in the next lecture. Before we get into immediately the discussion of the gauge degrees of freedom, let's first summarize what we obtained from the last lecture giving an explicit example of the application of Feynman's path integral formulation.

Ex. SHO in 1-dim has the Kernel

$$K(x_n,t_n;x_0,t_0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega(t_n-t_0)}}\exp\left[\frac{i}{\hbar}\frac{m\omega\{(x_n^2+x_0^2)\cos\omega(t_n-t_0)-2x_nx_0\}}{2\sin\omega(t_n-t_0)}\right]$$

We will show that this is nothing but

$$K(x_n,t_n;x_0,t_0) = J(t_n,t_0)\, e^{\frac{iS[x_{cl}(t)]}{\hbar}}$$ according to Feynman's path integral formulation

with $J(t_n,t_0)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega(t_n-t_0)}}$ and $\Big[S_{cl.}\Big]_{t_0}^{t_n} = \frac{m\omega\{(x_n^2+x_0^2)\cos\omega(t_n-t_0)-2x_nx_0\}}{2\sin\omega(t_n-t_0)}$

Let's show this result following path integral in 1-dim SHO. ②

As $x_{cl}(t) = A\sin(\omega t+\phi)$, velocity is given by

$$\dot{x}_{cl}(t) = A\omega\cos(\omega t+\phi).$$

The classical action is then

$$S_{cl} = \int_{t_0}^{t_n} dt\, L(x_{cl}, \dot{x}_{cl}) \qquad \boxed{L = T - V}$$

$$= \frac{m}{2}\int_{t_0}^{t_n} dt\,\left\{\dot{x}_{cl}^2(t) - \omega^2 x_{cl}^2(t)\right\}$$

$$= \frac{1}{2}mA^2\omega^2 \underbrace{\int_{t_0}^{t_n} dt\,\{\underbrace{\cos^2(\omega t+\phi) - \sin^2(\omega t+\phi)}_{\cos(2\omega t+2\phi)}\}}_{\left[\frac{1}{2\omega}\sin(2\omega t+2\phi)\right]_{t_0}^{t_n}}$$

$$= \frac{m\omega}{4}A^2\{\underbrace{\sin(2\omega t_n+2\phi)}_{\sin 2\omega t_n\cos 2\phi + \cos 2\omega t_n \sin 2\phi} - \underbrace{\sin(2\omega t_0+2\phi)}_{\sin 2\omega t_0\cos 2\phi + \cos 2\omega t_0\sin 2\phi}\}$$

Now, from $x_0 = A\sin(\omega t_0+\phi)$ and $x_n = A\sin(\omega t_n+\phi)$,

we get $A\cos\phi = \dfrac{x_n\cos\omega t_0 - x_0\cos\omega t_n}{\sin\omega(t_n-t_0)}$

and $A\sin\phi = \dfrac{x_0\sin\omega t_n - x_n\sin\omega t_0}{\sin\omega(t_n-t_0)}$

$$\boxed{\begin{array}{l} A^2\underbrace{\cos 2\phi}_{\cos^2\phi-\sin^2\phi} = (A\cos\phi)^2 - (A\sin\phi)^2 \\ A^2\underbrace{\sin 2\phi}_{2\cos\phi\sin\phi} = 2(A\cos\phi)(A\sin\phi) \end{array}}$$

Using these, we can get

$$S_{cl} = \frac{m\omega\{(x_n^2+x_0^2)\cos\omega(t_n-t_0) - 2x_n x_0\}}{2\sin\omega(t_n-t_0)}$$

and note that this is exactly what we obtained as the phase factor in the propagator of SHO, $K(x_n,t_n;x_0,t_0)$.

We can also find $J(t_n, t_0)$, noticing

$$J_n(t_n,t_0)\, e^{\frac{i S[x_{cl}(t)]_{t_0}^{t_n}}{\hbar}} = J(t_n,t_1)\,J(t_1,t_0)\int_{-\infty}^{\infty} dx_1\, e^{\frac{i S[x_{cl}(t)]_{t_1}^{t_n} + S[x_{cl}(t)]}{\hbar}}$$

from

$$\langle x_n, t_n | x_0, t_0\rangle = \int_{-\infty}^{\infty} dx_1\, \langle x_n, t_n | x_1, t_1\rangle \langle x_1, t_1 | x_0, t_0\rangle$$

Using S_{cl}, we get

$$\frac{J(t_n,t_0)}{J(t_n,t_1)\,J(t_1,t_0)} = \sqrt{\frac{2\pi i\hbar}{m\omega}\,\frac{\sin\omega(t_n-t_1)\sin\omega(t_1-t_0)}{\sin\omega(t_n-t_0)}}$$

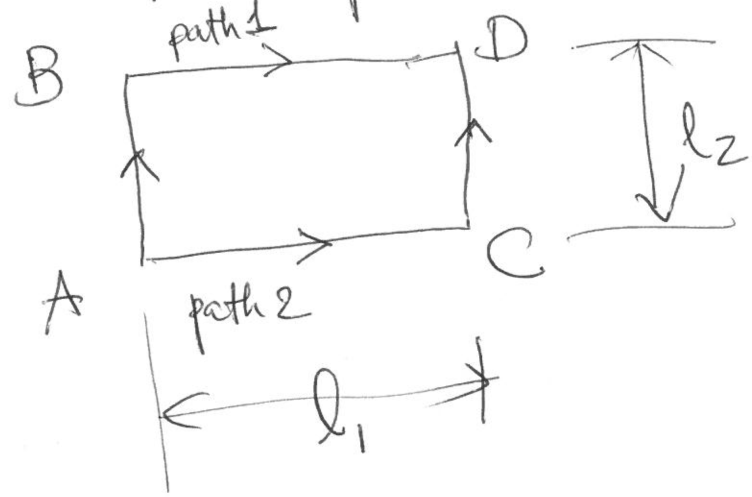
Note: As we can vary t_1, we can see that $J(t_n-t_0) \sim \frac{1}{\sqrt{\sin\omega(t_n-t_0)}}$

From this, we realize

$$J(t_n,t_0) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega(t_n-t_0)}}$$ as expected.

Thus, we found the SHO propagator using Feynman's path integral and obtained the same result that we obtained before. However, Feynman's path integral formulation is not too convenient to solve practical problems in Nonrelativistic quantum mechanics. Even for 1-dim SHO, it is rather cumbersome to evaluate explicitly the relevent path integral as we have shown here. Nevertheless, this approach is useful from a conceptual point of view.

As a preliminary to Aharonov-Bohm effect, let's discuss "Gravity-induced quantum interference" which was done by R. Colella, A. Overhauser and S. Werner in 1975 Expt.



As $\psi = \sqrt{\rho}\, e^{i\phi}$, the wavefunction at D position can be written as a sum of two wavefunctions which went through the two different paths.

$$
\begin{aligned}
\Psi &= \psi_1 + \psi_2 \\
&= \sqrt{\rho}\left(e^{i\phi_1} + e^{i\phi_2}\right) \\
&= \sqrt{\rho}\, e^{i\phi_2}\left(1 + e^{i(\phi_1 - \phi_2)}\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
|\Psi|^2 &= \rho\left|1 + e^{i(\phi_1-\phi_2)}\right|^2 \\
&= 2\rho\left\{1 + \cos(\phi_1 - \phi_2)\right\}.
\end{aligned}
$$

The phase difference $\phi_1 - \phi_2$ can be obtained from the path integral idea, i.e. the action difference in the two paths.

$$
\phi_1 - \phi_2 = \frac{S_1 - S_2}{\hbar} = \frac{S_{ABD} - S_{ACD}}{\hbar}
$$

As the action

$$S = \int L \cdot dt$$
$$= \int p \cdot dx$$
$$= \int dx \sqrt{2m(E - V(x))},$$

$$\boxed{\begin{aligned} \frac{\partial S}{\partial x} &= \int dt \, \frac{\partial L}{\partial x} \quad \text{use Lag. Eq.} \\ &\doteq \int dt \, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \\ &= \frac{\partial L}{\partial \dot{x}} = p \\ \text{or } S &= \int p \cdot dx \end{aligned}}$$

cf $\psi = \sqrt{\rho}\, e^{\frac{iS}{\hbar}}$; $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$, $\vec{J} = \rho \frac{\vec{\nabla} S}{m}$ or $\vec{p} = \vec{\nabla} S$

where $E = \frac{p^2}{2m} + V(x)$.

Let's take $V(x) = 0$ for the path 2, while the potential energy in the path 1 is larger than zero; i.e.

~~for path~~ 2, $E = \frac{P^2}{2m}$ with $P = \frac{\hbar}{\lambda}$

for path 1, $E = \frac{p^2}{2m} + V$ and $p = \sqrt{2m(E-V)}$.

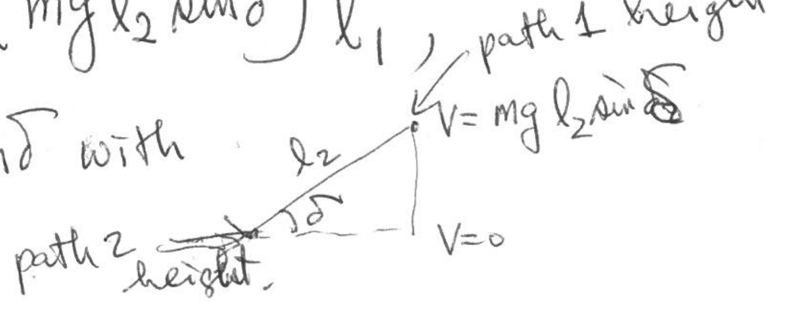
Now, $S_{path 1} - S_{path 2} = (p - P) \cdot l_1$

$$= l_1\left(\sqrt{P^2 - 2mV} - P\right) = \Big\{\underbrace{\left(1 - \frac{2mV}{P^2}\right)^{\frac{1}{2}}}_{\approx 1 - \frac{mV}{P^2}} - 1\Big\} P l_1$$

$$\approx -\frac{m}{P} V l_1$$

$$= -\frac{m\lambda}{\hbar}(mg l_2 \sin\delta)\, l_1,$$

where $V = mg l_2 \sin\delta$ with



Thus, one gets

$$\phi_1 - \phi_2 = -\frac{m^2 g l_1 l_2 \lambda \sin\delta}{\hbar^2} \qquad \text{Eq. (2.7.17), p. 134.}$$

In the expt, $\lambda = 1.42\,\text{Å}$, $l_1 l_2 = 10\,\text{cm}^2$,
comparable to interatomic spacing in silicon.

so that
$$\frac{m_n^2 g l_1 l_2 \lambda}{\hbar^2} \simeq 55.6$$

$$\overbrace{\frac{55.6 \sin\delta}{2\pi}}^{9} \cdot 2\pi$$
$$\|$$
$$\cos(\phi_1 - \phi_2) = \cos\left[55.6 \sin\delta\right]$$

As $\dfrac{55.6}{2\pi} \simeq 9$,

max for $\delta = 0$ for $\delta = \frac{\pi}{2}$

$$0 \times 2\pi \leftarrow \boxed{(9 \sin\delta)\, 2\pi} \rightarrow 9 \times 2\pi$$

For $0 < \delta < \frac{\pi}{2}$ or $-\frac{\pi}{4} < \delta < \frac{\pi}{4}$,
one may expect 9 peaks.

Note that the gauge invariance here is realized by the invariance of the equation of motion, i.e.

$$\frac{d\vec{p}}{dt} = -\vec{\nabla} V \text{ is invariant under}$$

$$V(\vec{x}) \rightarrow \tilde{V}(\vec{x}) = V(\vec{x}) + V_0$$

gauge transf.   constant potential difference "gauge degree of freedom"

In E&M, gauge invariance is realized as the conservation of electric charge.

Summary of Gauge Transformations in E&M

Just like $V(\vec{x}) \rightarrow \tilde{V}(\vec{x}) = V(\vec{x}) + V_0$ for the potential energy, the scalar and vector potentials in E&M undergo the local gauge transfs:

$$\phi(\vec{x},t) \longrightarrow \tilde{\phi}(\vec{x},t) = \phi(\vec{x},t) - \frac{1}{c}\frac{\partial \Lambda(\vec{x},t)}{\partial t}$$

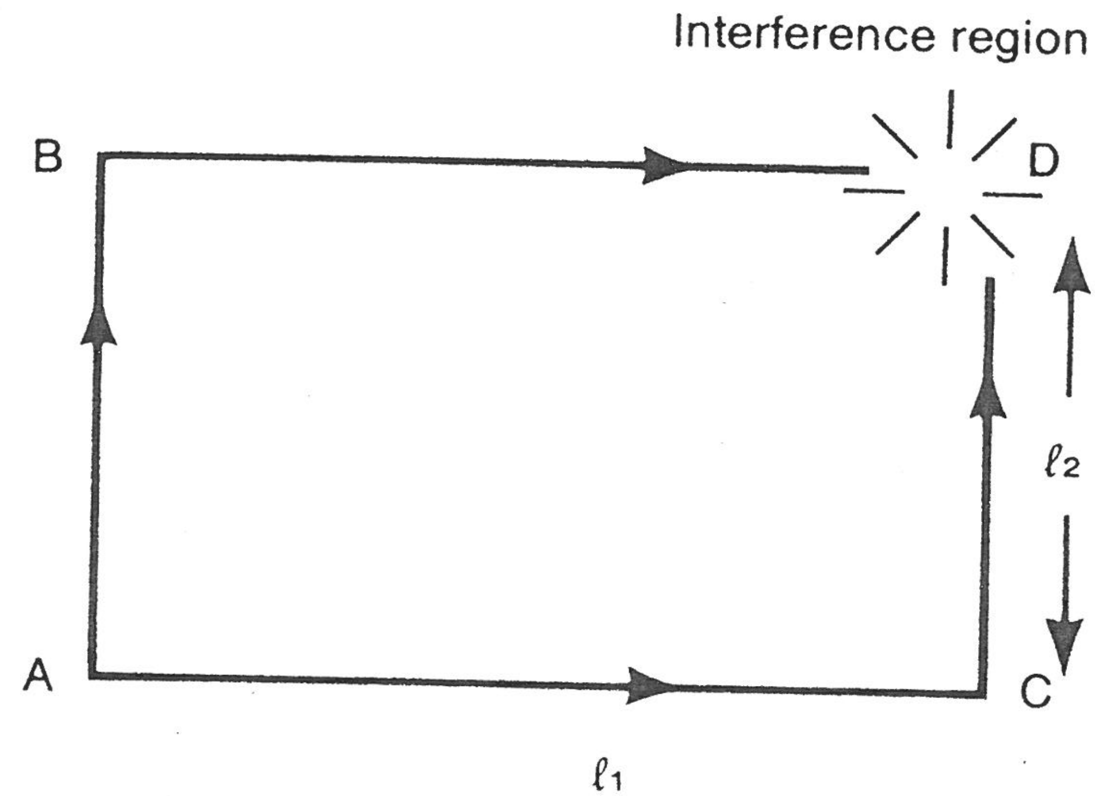
$$\vec{A}(\vec{x},t) \longrightarrow \tilde{\vec{A}}(\vec{x},t) = \vec{A}(\vec{x},t) + \vec{\nabla}\Lambda(\vec{x},t)$$

Just like $\frac{d\vec{p}}{dt} = -\vec{\nabla} V$ is invariant under $V \rightarrow \tilde{V}$, Maxwell's eqs are invariant under these local gauge transfs:

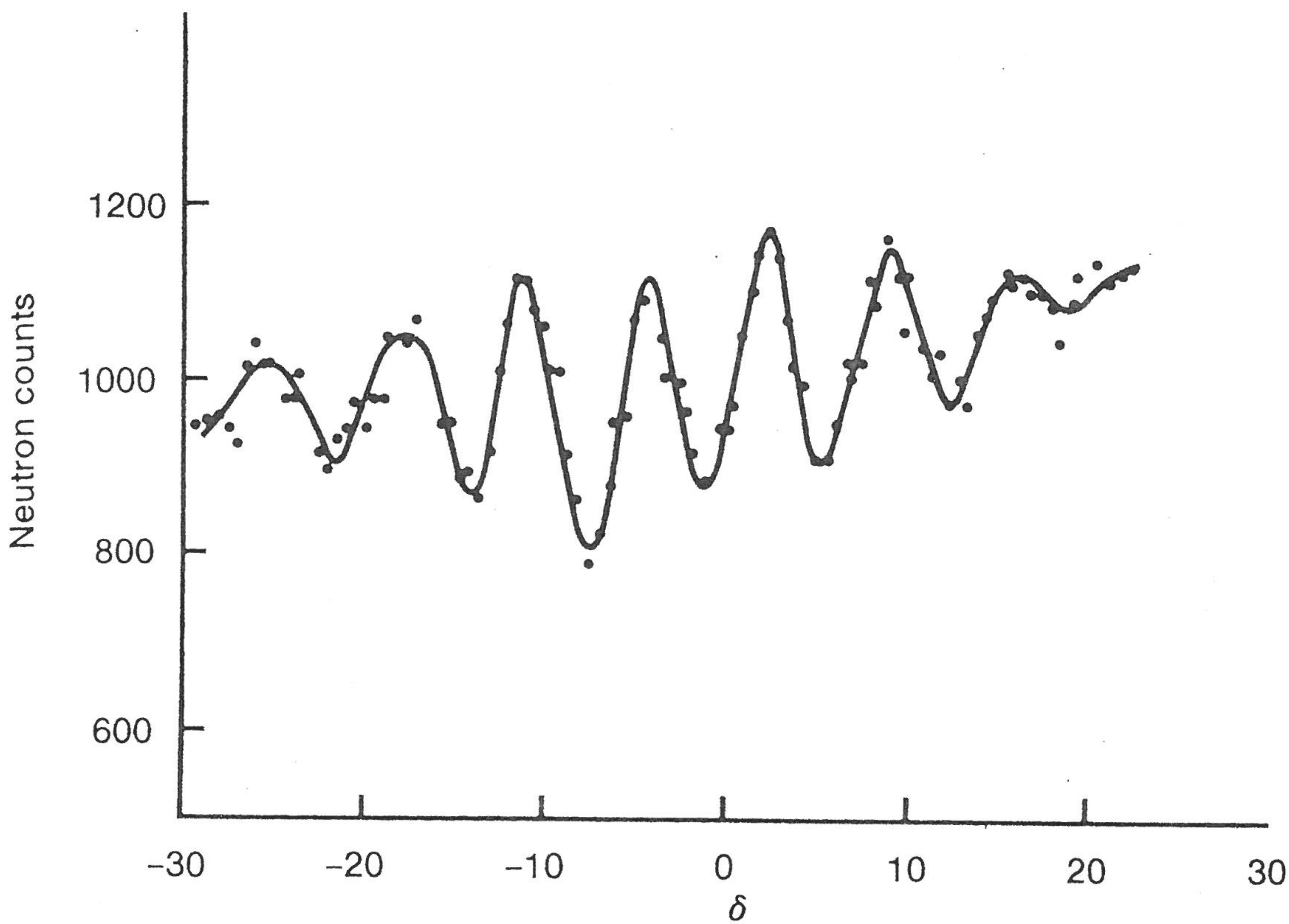
$$\begin{aligned}
\vec{E} \longrightarrow \tilde{\vec{E}} &= -\vec{\nabla}\tilde{\phi} - \frac{1}{c}\frac{\partial \tilde{\vec{A}}}{\partial t} \\
&= -\vec{\nabla}\left(\phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\Lambda\right) \\
&= -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\
&= \vec{E}
\end{aligned}$$

$$\begin{aligned}
\vec{B} \longrightarrow \tilde{\vec{B}} &= \vec{\nabla} \times \tilde{\vec{A}} \\
&= \vec{\nabla} \times \left(\vec{A} + \vec{\nabla}\Lambda\right) \\
&= \vec{\nabla} \times \vec{A} \\
&= \vec{B}
\end{aligned}$$

As $\vec{E}$ and $\vec{B}$ are invariant under gauge transf, Maxwell's eqs are invariant under gauge transf.



Interference region
B
D
A
C
ℓ_2
ℓ_1


Neutron counts
1200
1000
800
600
−30
−20
−10
0
10
20
30
δ