

Ranking and synchronization from pairwise measurements via SVD

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Abstract

Given a measurement graph $G = (V, E)$ and an unknown signal $r \in \mathbb{R}^n$, we investigate algorithms for recovering r from pairwise measurements of the form $r_i - r_j$; $\{i, j\} \in E$. This problem arises in a variety of applications, such as ranking teams in sports data and time synchronization of distributed networks. Framed in the context of ranking, the task is to recover the ranking of n teams (induced by r) given a small subset of noisy pairwise rank offsets. We propose a simple SVD-based algorithmic pipeline for both the problem of time synchronization and ranking. We provide a detailed theoretical analysis in terms of robustness against both sampling sparsity and noise perturbations with outliers, using results from matrix perturbation and random matrix theory. Our theoretical findings are complemented by a detailed set of numerical experiments on both synthetic and real data, showcasing the competitiveness of our proposed algorithms with other state-of-the-art methods.

Keywords: ranking, angular synchronization, spectral algorithms, matrix perturbation theory, singular value decomposition, random matrix theory, low-rank matrix completion.

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1. Introduction

Let $r = (r_1, \dots, r_n)^T \in \mathbb{R}^n$ be an unknown signal and $G = ([n], E)$ be an undirected measurement graph. Given a subset of noisy pairwise measurements of the form $r_i - r_j$ for each $\{i, j\} \in E$, the goal is to estimate the original vector r . Clearly, this is only possible only up to a global shift. Moreover, when measurements are exact without any measurement noise, one can recover the strength vector r if and only if the graph G is connected, by simply considering a spanning tree, fixing the value of the root node, and traversing the tree while propagating the information by summing the given offsets. For simplicity, we assume the graph is connected, otherwise it is not possible to estimate the offset values between nodes belonging to different connected components of the graph.

Instantiations of the above problem are ubiquitous in engineering, machine learning and computer vision, and have received a great deal of attention in the recent literature. There are two main classes of applications where this problem arises.

- *Time synchronization of wireless networks.* A popular application arises in engineering, and is known as time synchronization of distributed networks (Giridhar and Kumar, 2006; Karp et al., 2003), where clocks measure noisy time offsets $r_i - r_j$, and the goal is to recover $r_1, \dots, r_n \in \mathbb{R}$.
- *Ranking.* A fundamental problem in information retrieval is that of recovering the ordering induced by the latent strengths or scores $r_1, \dots, r_n \in \mathbb{R}$ of a set of n players, that is best reflected by the given set of pairwise comparisons $r_i - r_j$. We refer the reader to Cucuringu (2016) and references therein for a detailed overview.

A naive approach by sequentially propagating the information along spanning trees is doomed to fail in the presence of noise, due to accumulation of the errors. To this end, in order to increase the robustness to noise, one typically aims to simultaneously integrate all the pairwise measurements in a globally consistent framework. This line of thought appears in the literature in the context of the *group synchronization* problem (Singer, 2011), for recovering group elements from noisy pairwise measurements of their ratios. We briefly review the current literature in group synchronization in Section 2.1, which almost exclusively pertains to synchronization over compact groups. The problem we study in this paper can be construed as synchronization over the real line, hence a non-compact group. There exists a very rich literature on ranking, and it is beyond the scope of our work to provide an extensive review of it. Instead, in Section 2.2, we give a brief overview of the literature on ranking and time synchronization, and point out some of the methods that relate to our work. In particular, we focus on spectral methods and approaches that leverage low-rank matrix completion as a pre-processing step.

Contributions. We propose SVD-RS, a simple spectral algorithm for ranking and synchronization from pairwise comparisons, along with a normalized version denoted SVD-NRS¹. We provide a detailed theoretical consistency analysis for both algorithms for a random measurement model (see Section 3) in terms of robustness against sampling sparsity of

¹By normalized, we mean that the measurement matrix is normalized by the degree matrix of the graph, see Section 4.

the measurement graph and noise level. Additionally, we provide extensive numerical experiments on both synthetic and real data, showing that in certain noise and sparsity regimes, our proposed algorithms perform comparable or better than state-of-the-art methods.

On the theoretical side, our specific contributions can be summarized as follows (see also Section 3.3).

- For SVD-RS, we provide ℓ_2 and ℓ_∞ recovery guarantees for the **score** vector r (see Theorem 8). For instance, in the setting when $r_i = i$, $\Omega(n \log n)$ measurements suffice for ℓ_2 recovery, while $\Omega(n^{4/3}(\log n)^{2/3})$ measurements suffice for ℓ_∞ recovery.
- The ℓ_∞ analysis of SVD-RS leads to guarantees for **rank** recovery in terms of the maximum displacement error between the recovered ranking and the ground truth, as in Theorem 7. For e.g., when $r_i = i$, $\Omega(n^{4/3}(\log n)^{2/3})$ measurements suffice.
- For SVD-NRS, we provide in Theorem 10, ℓ_2 recovery guarantees for the **score** vector r , and leave the ℓ_∞ guarantees for future work, essentially by following a similar, though more intricate, pipeline. Similar to SVD-RS, for $r_i = i$, $\Omega(n \log n)$ measurements suffice for ℓ_2 recovery.

We remark that all the above recovery results hold with high probability.

Outline. The remainder of this paper is organized as follows. Section 2 is a brief survey of the relevant literature, with a focus on group synchronization and ranking. Section 3 starts with the formal setup of the problem, presents the gist of our SVD-based approach along with the two algorithms SVD-RS, SVD-NRS, and summarizes our main theoretical results. Section 4 details and interprets our theoretical results, with the main steps of the proofs outlined in Section 5. Section 6 discusses the low-rank matrix completion problem and its applicability in the setting of this paper. Section 7 contains numerical experiments on various synthetic and real data sets. Finally, Section 8 summarizes our findings along with future research directions. The Appendix contains additional technical details for the theoretical results, as well as further numerical experiments.

Notation. Vectors and matrices are denoted by lower case and upper case letters respectively. For a matrix $A \in \mathbb{R}^{m \times n}$ where $m \geq n$, we denote its singular values by $\sigma_1 \geq \dots \geq \sigma_n$ and the corresponding left (resp. right) singular vectors by u_i (resp. v_i). $\|A\|_2$ denotes the spectral norm (largest singular value), $\|A\|_*$ denotes the nuclear norm (sum of the singular values), and $\|A\|_{\max} := \max_{i,j} |A_{ij}|$ denotes the max-norm of A . $\mathcal{R}(A)$ denotes the range space of its columns. We denote e to be the all ones column vector. The symbol \mathcal{U} is used to denote the uniform distribution. For positive numbers a, b , we denote $a \lesssim b$ (resp. $a \gtrsim b$) to mean that there exists an absolute constant $C > 0$ such that $a \leq Cb$ (resp. $a \geq Cb$).

2. Related work

This section is a brief survey of the relevant literature for the tasks we consider out in this paper. The first part focuses on the group synchronization problem, a relevant instance of it being that of synchronization of clocks arising often in engineering, while the second part surveys the very rich ranking literature, with an emphasis on spectral methods and approaches that leverage low-rank matrix completion.

2.1 Synchronization literature

Much of the engineering literature has focused on least-squares approaches for solving the time synchronization problem. In its simplest terms, the approach can be formulated as follows. Let $m = |E|$ denote the number of edges in G , and B denote the edge-vertex incidence matrix of size $m \times n$. For $l = \{i, j\} \in E$, with $i < j$, the entries B_{li} and B_{lj} are given by $B_{li} = 1$, $B_{lj} = -1$. If $i' \notin \{i, j\}$ then $B_{li'} = 0$. Let $w \in \mathbb{R}^m$ encode the pairwise rank measurements C_{ij} , for all edges $\{i, j\} \in E$. The least-squares solution to the ranking problem can be obtained by solving

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \|Bx - w\|_2^2, \quad (2.1)$$

where B denotes the design matrix with two non-zero entries per row corresponding to the edge $\{i, j\}$ indexed by l . In a related line of work, Hirani et al. (2010) show that the problem of least-squares ranking on graphs has far-reaching rich connections with seemingly unrelated areas, such as spectral graph theory and multilevel methods for graph Laplacian systems, Hodge decomposition theory and random clique complexes in topology.

The problem we consider in this paper can also be framed in the context of the *group synchronization* problem, of finding group elements from noisy measurements of their ratios. For example, synchronization over the special orthogonal group $SO(d)$ consists of estimating a set of n unknown $d \times d$ matrices $R_1, \dots, R_n \in SO(d)$ from noisy measurements of a subset of the pairwise ratios $R_i R_j^{-1}$ via

$$\underset{R_1, \dots, R_n \in SO(d)}{\text{minimize}} \quad \sum_{\{i, j\} \in E} w_{ij} \|R_i^{-1} R_j - R_{ij}\|_F^2, \quad (2.2)$$

where $\|\cdot\|_F$ denotes the Frobenius norm, and w_{ij} are non-negative weights representing the confidence in the noisy pairwise measurements R_{ij} . Spectral and semidefinite programming (SDP) relaxations for solving an instance of the above synchronization problem were originally introduced and analyzed by Singer (2011), in the context of angular synchronization over the group $SO(2)$ of planar rotations. There, one is asked to estimate n unknown angles $\theta_1, \dots, \theta_n \in [0, 2\pi]$ given m noisy measurements of their offsets $\theta_i - \theta_j \bmod 2\pi$. The difficulty of the problem is amplified, on one hand, by the amount of **noise** in the offset measurements, and on the other hand by **sparsity** - the fact that $m \ll \binom{n}{2}$, i.e., only a very small subset of all possible pairwise offsets are measured. In general, one may consider other groups \mathcal{G} (such as $SO(d)$, $O(d)$) for which there are available noisy measurements g_{ij} of ratios between the group elements

$$g_{ij} = g_i g_j^{-1}, \quad g_i, g_j \in \mathcal{G}. \quad (2.3)$$

The set E of pairs $\{i, j\}$ for which a ratio of group elements is available can be realized as the edge set of a graph $G = (V, E)$, $|V| = n$, $|E| = m$, with vertices corresponding to the group elements g_1, \dots, g_n , and edges to the available pairwise measurements $g_{ij} = g_i g_j^{-1}$. As long as the group \mathcal{G} is compact and has a real or complex representation, one may construct a real or Hermitian matrix (which may also be a matrix of matrices) where the element in the position $\{i, j\}$ is the matrix representation of the measurement g_{ij} (possibly a matrix of size 1×1 , as is the case for \mathbb{Z}_2), or the zero matrix if there is no direct measurement for the ratio

of g_i and g_j . For example, the rotation group $\text{SO}(3)$ has a real representation using 3×3 rotation matrices, and the group $\text{SO}(2)$ of planar rotations has a complex representation as points on the unit circle.

The setting we consider in this paper, namely that of recovering points on the real line from a subset of noisy pairwise differences, is essentially synchronization over the non-compact group \mathbb{R} . In recent work in the context of ranking from pairwise cardinal and ordinal measurements (Cucuringu, 2016), the real line was compactified by wrapping it over the upper half of the unit circle, making the problem amenable to standard synchronization over the compact group $\text{SO}(2)$. The estimated solution allowed for the recovery of the player rankings, after a post-processing step of modding out the best circular permutation. Note that the proposed approach only focused on recovering the individual rankings, and not the magnitude (i.e., strength) of each player, as we propose to do in this paper.

Very recently, Ozyesil et al. (2018) proposed an approach that allows one to consider the synchronization problem in the setting of non-compact groups. The authors leverage a compactification process, that relies on a mapping from a non-compact domain into a compact one, for solving the synchronization problem. The contraction mapping enables one to transform measurements from a Cartan motion group to an associated compact group, whose synchronization solution provides a solution for the initial synchronization problem over the original domain.

2.2 Ranking literature

Fogel et al. introduced SERIAL-RANK (Fogel et al., 2014, 2016), a ranking algorithm that explicitly leverages the connection with seriation, a classical ordering problem that considers the setting where the user has available a similarity matrix (with ± 1 entries) between a set of n items, and assumes that there exists an underlying one-dimensional ordering such that the similarity between items decreases with their distance. The authors propose a spectral algorithm SERIAL-RANK, and show that under an Erdős-Renyi random graph model and a given noise model, it recovers the underlying ranking with $\Omega(n^{3/2} \log^4 n)$ comparisons where the estimation error is bounded in the ℓ_∞ norm.

A popular theme in the literature is the rank aggregation setting (where players meet in multiple matches), where there exists a latent probability matrix $P \in [0, 1]^{n \times n}$, wherein P_{ij} denotes the probability that player i defeats player j , with $P_{ij} + P_{ji} = 1$. For each pair (i, j) , one observes $Y_{ij} \in \{0, 1\} \sim \text{Bern}(P_{ij})$. A number of so-called Random Utility models have been considered in the literature, starting with the seminal work of Bradley-Terry-Luce (BTL) (Bradley and Terry, 1952), which is by far the most popular model considered. In the most basic version of the BTL model, the probability that player i beats player j is given by $P_{ij} = \frac{w_i}{w_i + w_j}$, where the vector $w \in \mathbb{R}_+^n$ is the parameter vector to be inferred from the data, with w_i being a proxy for the score or strength associated to player i . In the Thurstone model (Thurstone, 1927), $P_{ij} = \Phi(s_i - s_j)$, where Φ denotes the standard normal cumulative distribution function (CDF), and $s \in \mathbb{R}^n$ is the score vector. Negahban et al. proposed RANK-CENTRALITY in the context of the rank aggregation problem from multiple ranking systems, under a Bradley-Terry-Luce (BTL) model (Negahban et al., 2012, 2017). Rank-Centrality is an iterative algorithm that estimates the ranking scores from the stationary distribution of a certain random walk on the graph of players. Edges encode the outcome of

pairwise comparisons, and are directed $i \rightarrow j$ towards the winner, where the weight captures the proportion of times j defeated i in case of multiple direct matches. The authors show that under some assumptions on the connectivity of the underlying graph, their algorithm estimates the underlying score vector of the BTL model with $\Omega(npoly(\log n))$ comparisons. Very recently, Agarwal et al. (2018) propose Accelerated Spectral Ranking, a provably faster spectral ranking algorithm in the setting of the multinomial logit (MNL) and BTL models. The authors considered a random walk that has a faster mixing time than the random walks associated with previous algorithms (including Rank-Centrality (Negahban et al., 2017)), along with improved sample complexity bounds (of the order $npoly(\log n)$) for recovery of the MNL and BTL parameters.

Cucuringu (2016) introduced SYNC-RANK, formulating the problem of ranking with incomplete noisy information as an instance of the group synchronization problem over the group $SO(2)$ of planar rotations. SYNC-RANK starts by making the ansatz that items are embedded on the upper-half of the unit circle in the complex plane, and transforms the pairwise input measurements into pairwise angle offsets. It then solves the spectral or SDP relaxation of synchronization, and finally chooses amongst the circular permutations of the ranking induced by the synchronized solution by minimizing the number of upsets. The method was shown to compare favorably with a number of state-of-the-art methods, including RANK-CENTRALITY (Negahban et al., 2012), SERIAL-RANK (Fogel et al., 2016), and a SVD-based ranking algorithm similar to the one we consider in this paper, but without the projection step and its normalized extension, and without any theoretical guarantees (see Remark 3 for details). Along the same lines, and inspired by the group synchronization framework, Fanuel and Suykens (2017) propose a certain deformation of the combinatorial Laplacian, in particular, the so-called dilation Laplacian whose spectrum is leveraged for ranking in directed networks of pairwise comparisons. The method is reported to perform well on both synthetic and real data, and enjoys the property that it can place an emphasis of the top- k items in the ranking.

More recently, De Bacco et al. (2018) proposed SPRING-RANK, an efficient physically-inspired algorithm, based on solving a linear system of equations, for the task of inferring hierarchical rankings in directed networks, which also comes with a statistical significance test for the inferred hierarchy. The model considered also incorporates the assumption that interactions are more likely to occur between individuals with similar ranks, which is sometimes the case in real applications, such as sport competitions. The authors compare SPRING-RANK to a wealth of algorithms, including the above-mentioned SYNC-RANK and SERIAL-RANK, in terms of an accuracy defined as the fraction of edges whose direction is consistent with the inferred ranking, and conclude that, on average across various synthetic and real data sets, SPRING-RANK and SYNC-RANK have the highest accuracy.

Volkovs and Zemel (2014) consider two instances (in the unsupervised and supervised settings) of the preference aggregation problem where the task is to combine multiple preferences (either in the form of binary comparisons or as score differences) over objects into a single consensus ranking. Both settings rely on a newly introduced Multinomial Preference model (MPM) which uses a multinomial generative process to model the observed preferences. Dalal et al. (2012) consider a multi-objective rank aggregation problem (where pairwise information is in the form of score differences), and rely on the framework of the combinatorial Hodge decomposition (Jiang et al., 2011). In particular, they first formulate

the problem of reducing global inconsistencies and then propose techniques for identifying local observations which can reduce such global inconsistencies.

Low-rank matrix completion approaches for ranking. Gleich and Lim (2011) proposed a new method for ranking a set of items from pairwise observations in the form $r_i - r_j$. They first use matrix completion to fill in a partial skew-symmetric matrix, followed by a simple row-sum to recover the rankings. Using standard recovery results from matrix completion literature, it follows that in the absence of noise, and under some coherence assumptions on the score vector r , the true ranking can be recovered exactly from a random subset of pairs of size $\Omega(n \text{poly}(\log n))$. In case the observations are given as a rating matrix (users and items), they show how to transform it into a pairwise comparison matrix involving score differences so that their method can be applied. Ye et al. (2012) proposed a rank-minimization approach to aggregate the predicted confidence scores of multiple models. The authors cast the score fusion problem as that of finding a shared rank-2 pairwise relationship matrix that renders each of the original score matrices from the different models to be decomposed into the common rank-2 matrix and the sparse error component.

The recent work of Levy et al. (2018) relies on matrix completion in the context of the rank aggregation problem from a small subset of noisy pairwise comparisons, wherein the user observes L repeated comparisons between the same set of players. The authors work in the setting of the Bradley-Terry-Luce model (BTL) that assumes that a set of latent scores (or strengths) underlies all players, and each individual pairwise comparison is a probabilistic outcome, as a function of the underlying scores. The resulting pipeline is reported to improve over state-of-the-art in both simulated scenarios and real data. Prior to that, Kang et al. (2016) also relied on matrix completion and proposed an algorithm for top- N recommender systems. The authors filled in the user-item matrix based on a low-rank assumption, by considering a nonconvex relaxation, as opposed to the usual nuclear norm, and argued that it provides a better rank approximation and empirically leads to an accuracy superior to that of any state-of-the-art algorithm for the top- N recommender problem, on a comprehensive set of experiments on real data sets.

Another relevant line of work is that of Massimino and Davenport (2013), who consider an adaptation of the one-bit matrix completion framework (Davenport et al., 2014) to the setting of pairwise comparison matrices. One observes measurements $Y_{ij} = \pm 1$ with probability $f(M_{ij})$, where $M_{ij} = r_i - r_j$, and $f(M_{ij}) = \mathbb{P}(M_{ij} > 0)$; for example, the authors consider $f(x) = (1 + e^{-x})^{-1}$. On a related note, Yang and Wakin (2015) considered the rank aggregation setting, extending the work of Gleich and Lim (2011) to the setup of non-transitive matrices, for which it does not necessarily hold true that $Y_{i,j} = Y_{i,k} + Y_{k,j}, \forall i, j, k$, as is the case in the special setting when $Y_{i,j} = r_i - r_j$ (and thus $Y = re^T - er^T$), for a score vector r . In particular, Yang and Wakin were interested in modeling and recovering Y itself as opposed to the one-dimensional ranking, and introduced a model for non-transitive pairwise comparisons $Y_{ij} = r_i a_j - r_j a_i$, where, for example, r_i could denote offensive strength of player i , and a_j defensive strength of player j , thus giving Y_{ij} the interpretation of the anticipated margin of victory for player i over player j . The authors then propose a low-rank matrix completion approach based on alternating minimization, along with recovery guarantees for the estimate \hat{Y} of the form $\|\hat{Y} - Y\|_F \leq \epsilon$, after $\log 1/\epsilon$ iterations.

The recent seminal work of Rajkumar and Agarwal (2016) considered the question of recovering rankings from $O(n \log n)$ comparisons between the items. Their proposed pipeline starts by mapping the input binary entries of the comparison matrix to a low-rank matrix. More specifically, a *link function* is applied entry-wise to the input comparison matrix, which renders the resulting incomplete matrix to be of low rank. Standard low-rank matrix completion is applied to complete the matrix, followed by the inverse link function applied entry wise, which effectively maps the data back to binary pairwise measurements. Finally, the Copeland ranking procedure (Copeland, 1951) is applied, that ranks items by their Copeland scores, which is effectively given by the number of wins in the comparison matrix after thresholding the entries with respect to 0.5. This approach can be seen as a generalization of the pipeline proposed by Gleich and Lim (2011), in the sense that both lines of work rely on matrix completion followed by row-sum, the main difference being that Rajkumar and Agarwal (2016) uses the link function that maps the binary input matrix to a low-rank one. Borrowing tools from matrix completion, restricted strong convexity and prior work on Random Utility Models, Negahban et al. (2018) propose a nuclear norm regularized optimization problem for learning the parameters of the MultiNomial Logit (MNL) model that best explains the data, typically given as user preference in the form of choices and comparisons. The authors show that the convex relaxation for learning the MNL model is minimax optimal up to a logarithmic factor, by comparing its performance to a fundamental lower bound.

3. Problem setup and main results

Our formal setup is as follows. Consider an undirected graph $G = ([n], E)$ and an unknown vector $r \in \mathbb{R}^n$, where r_i is the score associated with node i . In particular, G is assumed to be a $G(n, p)$, i.e., the popular Erdős-Rényi random graph model, where edges between vertices are present independently with probability p . Moreover, we assume r_i to be bounded uniformly, i.e., $r_i \in [0, M]$ for each i . Hence $r_i - r_j \in [-M, M]$ for all i, j . Requiring $r_i \geq 0$ is only for convenience and w.l.o.g. Importantly, M is *not assumed* to be known to the algorithm.

For each $\{i, j\} \in E$, we are given noisy, independent measurements R_{ij} where

$$R_{ij} = \begin{cases} r_i - r_j; & \text{w.p } \eta \\ \sim \mathcal{U}[-M, M]; & \text{w.p } (1 - \eta). \end{cases} \quad (3.1)$$

The parameter $\eta \in [0, 1]$ controls the level of noise; we will denote the noise level explicitly by $\gamma = 1 - \eta$. It is important to note that the parameters η, p are not assumed to be known to the algorithm. This model will be referred to as the **Erdős-Rényi Outliers** model, or in short, ERO(n, p, η). It was also considered in previous works in ranking (Cucuringu, 2016) and angular synchronization (Singer, 2011), and is amenable to a theoretical analysis. Alternatives to this noise model include the multiplicative uniform noise model, as considered in Cucuringu (2016). Our goal is two fold - we would like to recover² the score vector r , and also the ranking π induced by r .

²Clearly, this is possible only up to a global shift.

Remark 1 *The above model is only for the purpose of theoretically analyzing the statistical performance of our methods. Other statistical models could also be considered of course, and we will see that our methods are completely model-independent.*

3.1 Main idea: SVD-based spectral algorithm

We start by forming the measurement matrix $H \in \mathbb{R}^{n \times n}$, where

- $H_{ii} = 0, \forall i = 1, \dots, n,$
- $H_{ij} = R_{ij}$ and $H_{ji} = -R_{ij}$, if $\{i, j\} \in E$, and
- $H_{ij} = 0$, if $\{i, j\} \notin E$.

If G is the complete graph and the measurements are noise free, it holds true that $H = re^T - er^T$, which is a rank 2 skew-symmetric matrix. Denoting $\alpha = \frac{r^T e}{n}$, one can verify that the two non-zero left singular vectors are $u_1 = e/\sqrt{n}, u_2 = \frac{r - \alpha e}{\|r - \alpha e\|_2}$ with equal non-zero singular values $\sigma_1 = \sigma_2 = \|r - \alpha e\|_2 \sqrt{n}$ (see Lemma 11). Therefore, given any orthonormal basis for $\text{span}\{u_1, u_2\}$, we can simply find a vector orthonormal to e/\sqrt{n} and which lies in $\text{span}\{u_1, u_2\}$; this will give us candidate solutions $\pm \frac{r - \alpha e}{\|r - \alpha e\|_2}$. Multiplying these candidates by σ_1/\sqrt{n} recovers the scale information of r , while the sign ambiguity is resolved by selecting the candidate which is most consistent with the measurements. If one is interested only in ranking the items, then there is of course no need to estimate the scale above.

If G is not complete (thus there are missing edges) and the measurements are noisy, then H will typically not be rank 2, but as will see shortly, it can be thought of as a perturbation of the rank-2 matrix $\eta p(re^T - er^T)$. Proceeding similarly to the noiseless case, we can find the top two singular vectors of H (denoted by \hat{u}_1, \hat{u}_2), project $u_1 = e/\sqrt{n}$ on to $\text{span}\{\hat{u}_1, \hat{u}_2\}$ to obtain \bar{u}_1 , and then find a unit vector in $\text{span}\{\hat{u}_1, \hat{u}_2\}$ which is orthonormal to \bar{u}_1 (call this \tilde{u}_2). If the noise level is not too large and we have sufficiently many edges in G , then one can imagine that \tilde{u}_2 will be close (up to a sign flip) to u_2 , and $\hat{\sigma}_i \approx \eta p \sigma_i$. Hence we can recover the ranking from \tilde{u}_2 after resolving the sign ambiguity (as mentioned earlier). This also implies that the centered version of $\frac{\hat{\sigma}_1 \hat{u}_2}{\eta p \sqrt{n}}$ will be close (up to a sign flip) to $r - \alpha e$. But since we do not know η, p , we will resort to other data-driven approaches for recovering the scale parameter, see Section 3.2.

The above approach is outlined formally as Algorithm 1, namely SVD-RS (SVD-Ranking and Synchronization) which is a spectral method for recovering the ranks and scores (up to a global shift) of a collection of n items. Additionally, we also consider a “normalized” version of SVD-RS, wherein H is replaced by $H_{ss} = \bar{D}^{-1/2} H \bar{D}^{-1/2}$ with \bar{D} being a diagonal matrix and $\bar{D}_{ii} = \sum_{j=1}^n |H_{ij}|$. Clearly, H_{ss} is also skew symmetric. Such a normalization step is particularly useful when the degree distribution is skewed, and is commonly employed in other problem domains involving spectral methods, such as clustering (see for eg. Kunegis et al. (2010)). The ensuing algorithm, namely SVD-NRS (SVD-Normalized Ranking and Synchronization), is outlined as Algorithm 2.

Remark 2 *Algorithms 1 and 2 are model-independent, and only require as input (a) the measurement graph G , and (b) the corresponding pairwise measurements R_{ij} .*

Algorithm 1 SVD-RS

- 1: **Input:** Measurement graph $G = ([n], E)$ and pairwise measurements R_{ij} for $\{i, j\} \in E$.
 - 2: **Output:** Rank estimates: $\hat{\pi}$ and score estimates $\hat{r} \in \mathbb{R}^n$.
 - 3: Form measurement matrix $H \in \mathbb{R}^{n \times n}$ using R_{ij} as outlined in Section 3.1.
 - 4: Find the top two left singular vectors (resp. singular values) of H , namely \hat{u}_1, \hat{u}_2 (resp. $\hat{\sigma}_1, \hat{\sigma}_2$).
 - 5: Obtain vector \bar{u}_1 as the orthogonal projection of $u_1 = e/\sqrt{n}$ on to $\text{span}\{\hat{u}_1, \hat{u}_2\}$.
 - 6: Obtain a unit vector $\tilde{u}_2 \in \text{span}\{\hat{u}_1, \hat{u}_2\}$ such that $\tilde{u}_2 \perp \bar{u}_1$.
 - 7: **Rank recovery:** Obtain ranking $\tilde{\pi}$ induced by \tilde{u}_2 (up to global sign ambiguity) and reconcile its global sign by minimizing the number of upsets. Output ranking estimate $\hat{\pi}$.
 - 8: **Score recovery:** Use \tilde{u}_2, H to recover the scale $\tau \in \mathbb{R}$ as in Section 3.2. Output $\hat{r} = \tau \tilde{u}_2 - \frac{e^T(\tau \tilde{u}_2)}{n} e$.
-

Algorithm 2 SVD-NRS

- 1: **Input:** Measurement graph $G = ([n], E)$ and pairwise measurements R_{ij} for $\{i, j\} \in E$.
 - 2: **Output:** Rank estimates: $\hat{\pi}$ and score estimates $\hat{r} \in \mathbb{R}^n$.
 - 3: Form $H_{ss} = \bar{D}^{-1/2} H \bar{D}^{-1/2}$ with H formed using R_{ij} as outlined in Section 3.1. \bar{D} is a diagonal matrix with $\bar{D}_{ii} = \sum_{j=1}^n |H_{ij}|$.
 - 4: Find the top two left singular vectors (resp. singular values) of H_{ss} , namely \hat{u}_1, \hat{u}_2 (resp. $\hat{\sigma}_1, \hat{\sigma}_2$).
 - 5: Obtain vector \bar{u}_1 as the orthogonal projection of $u_1 = \frac{\bar{D}^{-1/2} e}{\|\bar{D}^{-1/2} e\|_2}$ on to $\text{span}\{\hat{u}_1, \hat{u}_2\}$.
 - 6: Obtain a unit vector $\tilde{u}_2 \in \text{span}\{\hat{u}_1, \hat{u}_2\}$ such that $\tilde{u}_2 \perp \bar{u}_1$.
 - 7: **Rank recovery:** Obtain ranking $\tilde{\pi}$ induced by $\bar{D}^{1/2} \tilde{u}_2$ (up to global sign ambiguity) and reconcile its global sign by minimizing the number of upsets. Output ranking estimate $\hat{\pi}$.
 - 8: **Score recovery:** Use \tilde{u}_2, H to recover the scale $\tau \in \mathbb{R}$ as in Section 3.2. Output $\hat{r} = \tau \bar{D}^{1/2} \tilde{u}_2 - \frac{e^T(\tau \bar{D}^{1/2} \tilde{u}_2)}{n} e$.
-

Remark 3 Note that the SVD-based algorithm introduced in Cucuringu (2016) considers the four possible rankings induced by the singular vectors $\{\pm \hat{u}_1, \pm \hat{u}_2\}$, and chooses the one that minimizes the number of upsets. In contrast, SVD-RS first computes \bar{u}_1 as the orthogonal projection of the all ones vector e on to $\text{span}\{\hat{u}_1, \hat{u}_2\}$, finds a vector $\tilde{u}_2 \in \text{span}\{\hat{u}_1, \hat{u}_2\}$ perpendicular to \bar{u}_1 , and finally extracts the ranking induced by \tilde{u}_2 . Furthermore, unlike Cucuringu (2016), SVD-NRS introduces an additional normalization step of the measurement matrix H to alleviate potential issues arising from skewed degree distributions. Such a normalization step is fairly common for spectral methods in general. For example, this operator was considered in Cucuringu et al. (2012a) and Cucuringu et al. (2012b) in the context of angular synchronization (Singer, 2011) and the graph realization problem, and Singer and Wu (2012) who introduced Vector Diffusion Maps for nonlinear dimensionality reduction and explored the interplay with the Connection-Laplacian operator for vector fields over manifolds.

Computational complexity. Our proposed SVD-based methods rely on computing the top two singular vectors of the input matrix which in practice can be done very efficiently. In general, the leading singular values and singular vectors can be computed using iterative techniques at a typical cost of $O(pn^2)$, using for example, a simple power method (since all iterations only require a matrix-vector product at a cost of pn^2 , where p is the sampling probability). In the sparse setting, the computational cost is essentially linear in the number of nonzero entries in the matrix, thus making the approach scalable to large measurement graphs. The data-driven approach for scale recovery will also require $O(pn^2)$ cost, as will be seen in the next Section 3.2.

Remark 4 *The computational complexity of our proposed SVD-based algorithms is on par with that of other competitive methods, detailed and compared against in the numerical experiments Section 7. For example, the Serial-Rank approach (Fogel et al., 2014) first computes a shifted version of the similarity matrix HH^T , then considers the graph Laplacian, and computes its Fiedler eigenvector, much like our SVD-based approach, which can be done at a typical cost of $O(pn^2)$. The Row-Sum method computes the sum of the entries in each row of H , which has complexity $O(pn^2)$, though Gleich and Lim (2011) first rely on a matrix completion step to fill in a partial skew-symmetric matrix, before computing the row sums. However, performing a low-rank matrix completion step can be computationally expensive, depending on the approach used; in general, alternating minimization has been proven empirically to be one of the most accurate and efficient methods for matrix completion (Jain et al., 2013). In the numerical experiments reported in Section 7, we ran into computational issues when running matrix completion for values of n significantly larger than 1000, even for very sparse graphs (eg, $p = 0.01$).*

For the Least-Squared-based ranking approach, also considered in Cucuringu (2016), the setup is the following. Each (potentially noisy) pairwise comparison $r_i - r_j$ contributes with a row to a design matrix T of size $m \times n$, where m ($\approx pn^2$) denotes the number of edges in the graph G . Each row of T has only two non-zero elements, in particular, a +1 in column i , and a -1 in column j . To estimate the score vector s of length n , we solve the linear system $Ts = b$ in the least-squares sense, where b is a column vector with the outputs of the pairwise comparisons. One potential approach for solving such linear least-squares problems of the form $Ts = b$ is to use conjugate gradient iterations applied to the normal equations $T^T Ts = T^T b$ (which can be achieved without explicitly performing the expensive calculation of the matrix $T^T T$). Note that the rate of convergence of the gradient iterations is dictated by the condition number κ of the matrix $T^T T$, and the number of iterations required for convergence is $O(\sqrt{\kappa})$ (Trefethen and Bau, 1997). For matrices that are sparse or have exploitable structure (like the sparse matrix T with only two nonzero entries per row), each conjugate gradient iteration has complexity as low as $O(m)$.

3.2 Recovering the global scale: data-driven procedures

To recover the scaling factor $\tau \in \mathbb{R}$, we consider two possible approaches: a first one based on a median estimator, and a second one that relies on a least-squares estimator. We briefly review both methods, and remark that we only consider the median-based estimator throughout the rest of the paper, due to its additional robustness. We compare results on

two problem instances of varying levels of noise, and also compare with the ground truth, as detailed below.

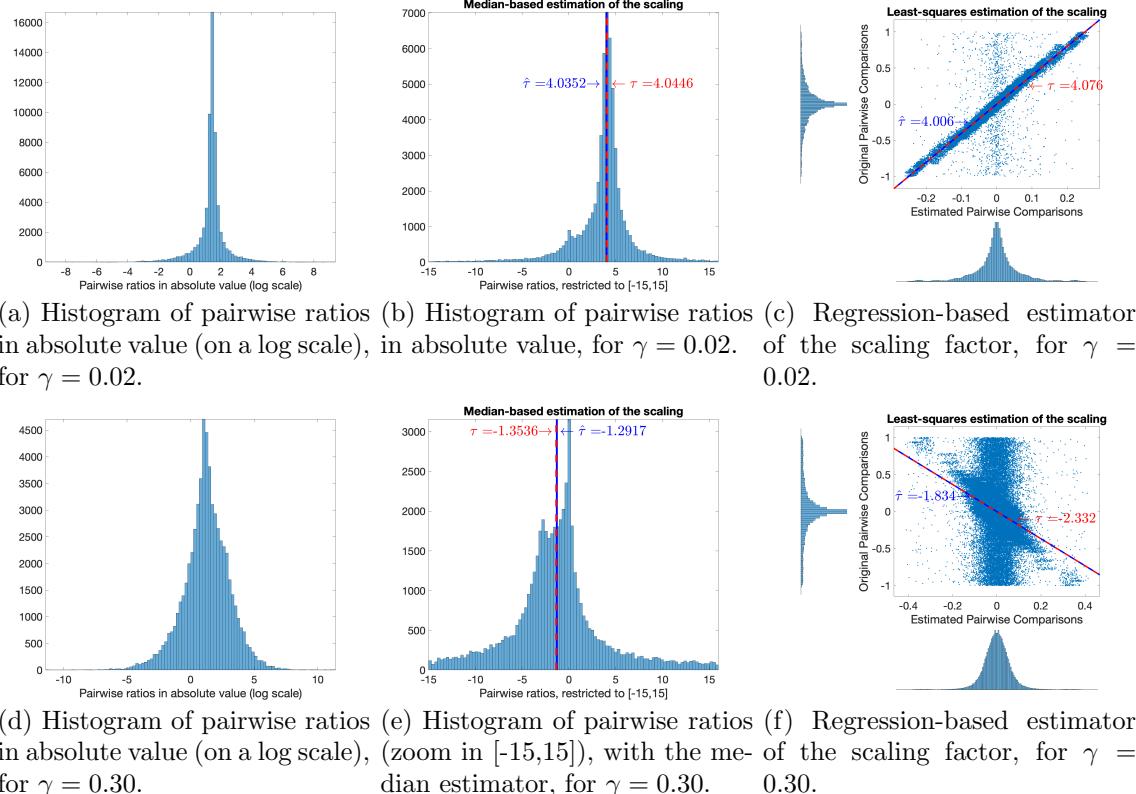


Figure 1: Two instances (top: low-noise regime $\gamma = 0.02$, bottom: high-noise regime $\gamma = 0.30$) showcasing the median and least-squares based estimators for scale recovery (shown in blue), along with their respective ground truth values (shown in red). Note that the bottom instance is also reconciliating the global sign. Scores are gamma distributed, with $n = 500$, and edge density $p = 0.25$.

To begin with, we define the entrywise ratio of offsets/comparisons as

$$\Pi_{i,j} = \begin{cases} \frac{H_{i,j}}{S_{i,j}}, & \{i,j\} \in E \\ 0; & \text{otherwise,} \end{cases} \quad (3.2)$$

where $H_{i,j}$ denotes the pairwise comparison measurement initially available, and $S_{i,j}$ denotes the recovered/estimated pairwise offset given by $S_{i,j} = s_i - s_j$, where s_i denotes the recovered score by either SVD-RS and SVD-NRS. In particular, $s_i = (\tilde{u}_2)_i$ for SVD-RS, and $s_i = (\bar{D}^{1/2}\tilde{u}_2)_i$ for SVD-NRS. This scale recovery method is broadly applicable to other ranking methods we compare against, and we further detail this in the experiments section. If $|E| = m$, we next consider the vector ρ of length m with the pairwise ratios, i.e. nonzero entries in the matrix $\Pi_{i,j}$ in (3.2) corresponding to the edges in the measurement graph G .

The median estimator is given by

$$\hat{\tau}^{Median} = \text{median}(\rho_1, \dots, \rho_m), \quad (3.3)$$

as illustrated in the middle plot in Figure 1. From a computational perspective, the median-based estimator only requires forming the pairwise ratios for the m edges in the graph, and computing the median element can be performed in linear time.

An alternative approach is to consider the least-squares formulation

$$\hat{\tau}^{LS} = \arg \min_{\tau \in \mathbb{R}} \sum_{\{i,j\} \in E} (H_{i,j} - \tau S_{i,j})^2 = \frac{\sum_{\{i,j\} \in E} H_{i,j}}{\sum_{\{i,j\} \in E} S_{i,j}}, \quad (3.4)$$

as shown in the rightmost column of Figure 1. In the low-noise regime, the least-squares estimator is close to the median-based estimator, and the discrepancy increases as the noise levels becomes larger. As shown in the left plot of Figure 1, which contains a histogram of the nonzero entries $\Pi_{i,j} \neq 0, \{i,j\} \in E$ on a log-scale, gross outliers can affect the regression formulation (3.4). As expected, we have observed in our simulations that the median estimator (3.3) leads to lower recovery errors compared to the regression-based estimator. In Figure 1, we show in red the recovered scaling when employing the ground truth data, in particular

$$\Pi_{i,j}^{(\text{ground truth})} = \begin{cases} \frac{r_i - r_j}{S_{i,j}}, & \{i,j\} \in E \\ 0; & \text{otherwise,} \end{cases}$$

Similarly to (3.3) and (3.4), we compute τ^{Median} , respectively, τ^{LS} , construed as the ground truth, and shown in red in the middle and right plots of Figure 1. Note that, in both instances, the median-based estimator is closer to the ground truth than the least-squares based estimator. In the low-noise regime $\gamma = 0.02$ (top plot of Figure 1), $\hat{\tau}^{Median}$ is within 0.23% from its ground true counterpart, while $\hat{\tau}^{LS}$ is within 1.7%. However, the performance gap widens as we increase the noise level. In the high noise regime $\gamma = 0.30$ (bottom plot of Figure 1), $\hat{\tau}^{Median}$ is off by 4.5% with respect to the ground truth, while $\hat{\tau}^{LS}$ is off by 27.1% from its ground truth counterpart. Note that in the latter problem instance, the recovered scaling is negative, emphasizing that the global sign has been reconciled. Due to its increased robustness, for the remainder of the numerical experiments in this paper, we henceforth rely only on the median-based estimator to perform the scale recovery step.

3.3 Summary of main theoretical results

Broadly speaking, the bulk of our analysis revolves around bounding the distance between the unit norm vectors $u_2 = \frac{r - \alpha e}{\|r - \alpha e\|_2}$ and \tilde{u}_2 (up to a global sign), the latter obtained in SVD-RS (Step 6) and SVD-NRS (Step 6). For SVD-RS, these error estimates are stated in Theorem 5 (resp. Theorem 6) for the ℓ_2 (resp. ℓ_∞) norm, while Theorem 9 states the corresponding ℓ_2 error bound for SVD-NRS. Below, we outline what this implies for rank and score recovery.

1. Theorem 7 gives guarantees for rank recovery (up to a global order reversal) for SVD-RS in terms of the maximum displacement error (defined in (4.6)) between

the ranking $\tilde{\pi}$ obtained in Step 7 of Algorithm 1, and the ground truth ranking π . Denoting $\alpha = \frac{r^T e}{n}$ to be the average score, it states that if

$$\|r - \alpha e\|_2 \gtrsim \frac{M}{\eta \sqrt{p}}, \quad p \gtrsim \frac{\log n}{n}, \quad (3.5)$$

and if n is large enough, then $\|\tilde{\pi} - \pi\|_\infty \lesssim \frac{\|r - \alpha e\|_2}{\rho} \Upsilon(n, M, \eta, p, \varepsilon, r)$ w.h.p. Here $\Upsilon(n, M, \eta, p, \varepsilon, r)$ is the bound on the ℓ_∞ error between \tilde{u}_2 and u_2 (up to a global sign) in Theorem 6. For concreteness, if $r_i = i$, we show in Example 3 that for a fixed $\delta \in (0, 1)$, if $p \gtrsim \frac{1}{\eta^2 \delta^{2/3}} \left(\frac{(\log n)^{2/3}}{n^{2/3}} \right)$ and n is large enough, then $\|\tilde{\pi} - \pi\|_\infty \lesssim \delta \frac{n}{\log n}$ w.h.p. Note that this implies that $\Omega(n^{4/3} (\log n)^{2/3})$ measurements suffice.

2. Theorem 8 provides ℓ_2 and ℓ_∞ bounds (up to a global sign and shift) for SVD-RS for the score vector r and a vector \tilde{r} (derived from \tilde{u}_2) where

$$\tilde{w} = \frac{\widehat{\sigma_1}}{\eta p \sqrt{n}} \tilde{u}_2; \quad \tilde{r} := \tilde{w} - \frac{e^T \tilde{w}}{n} e. \quad (3.6)$$

If one knew the product ηp , then (3.6) corresponds to choosing the scale $\tau = \frac{\widehat{\sigma_1}}{\eta p \sqrt{n}}$ in Step 8.

- (a) If the conditions in (3.5) hold, then for \tilde{r} as in (3.6), $\exists \beta \in \{-1, 1\}$ such that

$$\|\tilde{r} - \beta(r - \alpha e)\|_2 \leq \frac{M}{\eta \sqrt{p}} + \sqrt{\frac{M \|r - \alpha e\|_2}{\eta p^{1/2}}}$$

holds w.h.p. If $r_i = i$, we show in Example 4 that if $p \gtrsim \frac{\log n}{\eta^2 \delta^2 n}$ for a fixed $\delta \in (0, 1)$, then $\|\tilde{r} - \beta(r - \alpha e)\|_2 \lesssim \frac{n^{3/2}}{(\log n)^{1/4}} \sqrt{\delta}$ w.h.p. Hence $\Omega(n \log n)$ measurements suffice for ℓ_2 recovery.

- (b) If the conditions in (3.5) hold and n is large enough, then for \hat{r} as in (3.6), there exists $\beta \in \{-1, 1\}$ such that

$$\|\tilde{r} - \beta(r - \alpha e)\|_\infty \lesssim \|r - \alpha e\|_2 [C(n, M, \eta, p, r) + \sqrt{n} C^2(n, M, \eta, p, r)] + \frac{M(M - \alpha)}{\eta \sqrt{p} \|r - \alpha e\|_2},$$

holds with $C(n, M, \eta, p, r)$ as defined in (4.3). When $r_i = i$, we show in Example 4 that if $p \gtrsim \frac{1}{\eta^2 \delta^{2/3}} \left(\frac{(\log n)^{2/3}}{n^{2/3}} \right)$ for a fixed $\delta \in (0, 1)$, and n is large enough, then $\|\tilde{r} - \beta(r - \alpha e)\|_\infty \lesssim \frac{n}{\log n} \delta$ holds w.h.p.

3. In the same spirit as Theorem 8, Theorem 10 provides a ℓ_2 error bound (up to a global sign and shift) for SVD-NRS for the score vector r and a vector \tilde{r} (derived from \tilde{u}_2 in Step 6) where

$$\tilde{w} = \frac{\widehat{\sigma_1}}{\eta p \|\bar{D}^{-1/2} e\|_2} \bar{D}^{1/2} \tilde{u}_2; \quad \tilde{r} := \tilde{w} - \frac{e^T \tilde{w}}{n} e. \quad (3.7)$$

Again, if one knew the product ηp , then (3.7) corresponds to choosing the scale $\tau = \frac{\widehat{\sigma}_1}{\eta p \|\bar{D}^{-1/2}e\|_2}$ in Step 8. While the conditions on p are more convoluted than those in Theorem 8 – in large part due to the normalization – the essence is the same. In the setting where $r_i = i$, we can derive a simplified version of the theorem as shown in Example 6. Let us now denote $\alpha = \frac{r^T(\mathbb{E}[\bar{D}])^{-1}e}{e^T(\mathbb{E}[\bar{D}])^{-1}e}$ and $\alpha' = \frac{e^T(r-\alpha e)}{n}$. Theorem 10 states that if $p \gtrsim \frac{\log n}{\eta^2 \delta^2 n}$ for fixed $\delta \in (0, 1)$, then for \tilde{r} as in (3.7), there exists $\beta \in \{-1, 1\}$ such that w.h.p,

$$\|\tilde{r} - \beta(r - (\alpha + \alpha')e)\|_2 \lesssim n^{3/2} \eta^{1/4} \delta^{1/2} + \delta^{1/2} \frac{n^{3/2}}{(\log n)^{1/4}}.$$

The closest works to our setting are those of Gleich and Lim (2011) and Yang and Wakin (2015) which rely on matrix completion, followed by a row sum to compute rankings. In our experiments, we also consider matrix completion as a preprocessing step (see Section 6) before applying our SVD-based algorithms, and find that in general it does improve the performance of our methods. However, this comes at the cost of an additional computational overhead; our SVD-based methods rely on computing the top two singular vectors of the input matrix, which in practice can be done very efficiently. The theoretical guarantees we provide for our problem setting, are new to the best of our knowledge, and have been lacking in the ranking and synchronization literatures currently.

4. Theoretical results for SVD-RS and SVD-NRS

We begin by detailing the theoretical results for SVD-RS in Section 4.1, followed by those for SVD-NRS in Section 4.2. Throughout, we instantiate the main theorems for the special setting where $r_i = i$, for ease of interpretability.

4.1 Analysis of Algorithm 1 (SVD-RS)

We will assume throughout this section that for $\{i, j\} \in E$, the measurement R_{ij} corresponds to $i < j$. This is clearly without loss of generality. From (3.1), we can write the entry H_{ij} (for $i < j$) as the following mixture model

$$H_{ij} = \begin{cases} r_i - r_j & \text{w.p } \eta p \\ N_{ij} \sim \mathcal{U}[-M, M] & \text{w.p } (1 - \eta)p \\ 0 & \text{w.p } 1 - p, \end{cases}$$

with $(H_{ij})_{i < j}$ being independent random variables. Since $H_{ii} = 0$ and $H_{ij} = -H_{ji}$ (by construction of H), therefore for all $i, j \in [n]$, we obtain $\mathbb{E}[H_{ij}] = \eta p(r_i - r_j)$. In particular, we have that

$$\mathbb{E}[H] = \eta p C, \tag{4.1}$$

where $C = re^T - er^T$ is a skew-symmetric matrix of rank 2. We can decompose H as

$$H = \mathbb{E}[H] + Z = \eta p C + Z,$$

where Z is a random noise matrix with $Z_{ii} = 0, \forall i = 1, \dots, n$. For $1 \leq i < j \leq n$ the entries of Z are defined by the following mixture model

$$Z_{ij} = \begin{cases} (r_i - r_j) - \eta p(r_i - r_j) & ; \text{ w.p } \eta p \\ N_{ij} - \eta p(r_i - r_j) & ; \text{ w.p } (1 - \eta)p \\ -\eta p(r_i - r_j) & ; \text{ w.p } (1 - p). \end{cases} \quad (4.2)$$

Note that $(Z_{ij})_{i < j}$ are independent³ random variables and $Z_{ij} = -Z_{ji}$. Hence the matrices H, C, Z are all skew-symmetric. We now proceed to show in Theorem 5 that, provided p is large enough, it holds true that \tilde{u}_2 (obtained in Step 6 of SVD-RS) is close to $u_2 = \frac{r - \alpha e}{\|r - \alpha e\|_2}$ in the ℓ_2 norm (up to a global sign) with high probability. The proof is outlined in Section 5.1.

Theorem 5 (ℓ_2 recovery, ERO model) *Denoting $\alpha = \frac{r^T e}{n}$, for a given $0 < \varepsilon \leq 1/2$, let $\|r - \alpha e\|_2 \geq \frac{24M}{\eta} \sqrt{\frac{5}{3p}}(2 + \varepsilon)$. Let $\tilde{u}_2 \in \mathbb{R}^n$ be the vector obtained in Step 6 of SVD-RS (with $\|\tilde{u}_2\|_2 = 1$) and $u_2 = \frac{r - \alpha e}{\|r - \alpha e\|_2}$. Then, there exists $\beta \in \{-1, 1\}$ and a constant $c_\varepsilon > 0$ depending only on ε such that*

$$\|\tilde{u}_2 - \beta u_2\|_2^2 \leq \frac{120M}{\eta} \sqrt{\frac{5}{3p}} \frac{(2 + \varepsilon)}{\|r - \alpha e\|_2},$$

with probability at least $1 - 2n \exp\left(-\frac{20pn}{3c_\varepsilon}\right)$.

Theorem 5 says that for $\delta \in (0, 1)$, if $p \gtrsim \max\left\{\frac{M^2 \log n}{\eta^2 \delta^2 \|r - \alpha e\|_2^2}, \frac{\log n}{n}\right\}$, then with high probability, we have

$$\|\tilde{u}_2 - \beta u_2\|_2^2 \lesssim \frac{\delta}{\sqrt{\log n}}.$$

Note that the bounds involved are invariant to the scaling of r , and essentially depend on the variance of the normalized entries of r , wherein each $\frac{r_i}{M} \in [0, 1]$. We can see that as $\frac{\|r - \alpha e\|_2^2}{M^2}$ becomes small, then the corresponding condition on p becomes more stringent.

Example 1 Consider the case where $r_i = i$ for $i = 1, \dots, n$. Then $M = n$, $\alpha = \frac{n+1}{2}$ and $\|r - \alpha e\|_2 = \Theta(n^{3/2})$. Hence Theorem 5 now says that if $p \gtrsim \frac{\log n}{\eta^2 \delta^2 n}$, then with high probability, we have $\|\tilde{u}_2 - \beta u_2\|_2^2 \lesssim \frac{\delta}{\sqrt{\log n}}$.

If in addition, n is also large enough, then we show in Theorem 6 that \tilde{u}_2 is close to u_2 (up to a global sign) in the ℓ_∞ norm as well. The proof is outlined in Section 5.2.

Theorem 6 (ℓ_∞ recovery, ERO model) *With the same notation as in Theorem 5, for a given $0 < \varepsilon \leq 1/2$, let $\|r - \alpha e\|_2 \geq \frac{24M}{\eta} \sqrt{\frac{5}{3p}}(2 + \varepsilon)$. Assume $p \geq \max\left\{\frac{1}{2n}, \frac{2 \log n}{15n}\right\}$. Choose $\xi > 1$, $0 < \kappa < 1$ and define $\mu = \frac{2}{\kappa+1}$. Let n satisfy $\frac{16}{\kappa} \leq (\log n)^\xi$. Then, there exists $\beta \in \{-1, 1\}$ and constants $C_\varepsilon, c_\varepsilon > 0$ depending only on ε such that with probability at least*

$$1 - 2n \exp\left(-\frac{20pn}{3c_\varepsilon}\right) - \frac{4}{n} - 2n^{1-\frac{1}{4}(\log_\mu n)^{\xi-1}(\log_\mu e)^{-\xi}}$$

³Since $Z_{ii} = 0$, clearly $(Z_{ij})_{i \leq j}$ are independent as well.

we have that

$$\|\tilde{u}_2 - \beta u_2\|_\infty \leq 4(2 + \sqrt{2})C(n, M, \eta, p, \varepsilon, r) + 4\sqrt{n}C^2(n, M, \eta, p, \varepsilon, r),$$

where

$$C(n, M, \eta, p, \varepsilon, r) = C_\varepsilon \left[\left(\frac{M\sqrt{\log n}}{\eta\sqrt{p}\|r - \alpha e\|_2} + \frac{M^2(\log n)^{2\xi}}{\eta^2 p \|r - \alpha e\|_2^2} \right) \left(\frac{1}{\sqrt{n}} + \frac{M - \alpha}{\|r - \alpha e\|_2} \right) + \frac{M^3}{\eta^3 p^{3/2} \|r - \alpha e\|_2^3} \right]. \quad (4.3)$$

Let us look at bounding the admittedly complicated looking term $C(n, M, \eta, p, \varepsilon, r)$. For $\delta \in (0, 1)$, say p satisfies

$$p \gtrsim \frac{M^2}{\|r - \alpha e\|_2^2} \max \left\{ \frac{n \log^3 n}{\eta^2 \delta^2} \left(\frac{1}{\sqrt{n}} + \frac{M - \alpha}{\|r - \alpha e\|_2} \right)^2, \frac{\sqrt{n}(\log n)^{2\xi+1}}{\eta^2 \delta} \left(\frac{1}{\sqrt{n}} + \frac{M - \alpha}{\|r - \alpha e\|_2} \right), \frac{(\sqrt{n} \log n)^{2/3}}{\eta^2 \delta^{2/3}} \right\}. \quad (4.4)$$

Then we have that $C(\cdot) \lesssim \frac{\delta}{\sqrt{n} \log n}$. Hence if additionally $p \gtrsim \max \left\{ \frac{M^2}{\eta^2 \|r - \alpha e\|_2^2}, \frac{\log n}{n} \right\}$ and $n = \Omega(1)$ hold, then Theorem 6 says that

$$\|\tilde{u}_2 - \beta u_2\|_\infty \lesssim \frac{\delta}{\sqrt{n} \log n}$$

holds w.h.p. Note that the condition on p is stricter than that in Theorem 5.

Example 2 Let us revisit the case where $r_i = i$ for all i . The condition (4.4) then simplifies to

$$p \gtrsim \max \left\{ \frac{1}{\eta^2 \delta^2} \left(\frac{(\log n)^3}{n} \right), \frac{1}{\eta^2 \delta} \left(\frac{(\log n)^{2\xi+1}}{n} \right), \frac{1}{\eta^2 \delta^{2/3}} \left(\frac{(\log n)^{2/3}}{n^{2/3}} \right) \right\}. \quad (4.5)$$

Since $\frac{M^2}{\|r - \alpha e\|_2^2} = \frac{1}{n}$, we conclude that for a fixed $\delta \in (0, 1)$, if $p \gtrsim \frac{1}{\eta^2 \delta^{2/3}} \left(\frac{(\log n)^{2/3}}{n^{2/3}} \right)$ and $n = \Omega(1)$, then $\|\tilde{u}_2 - \beta u_2\|_\infty \lesssim \frac{\delta}{\sqrt{n} \log n}$ holds with high probability.

Recovering ranks in the ℓ_∞ norm. Let us assume that \tilde{u}_2 is aligned with u_2 , i.e., $\|\tilde{u}_2 - u_2\|_\infty \leq \Upsilon(n, M, \eta, p, \varepsilon, r)$ where $\Upsilon(\cdot)$ is the bound in Theorem 6. For any $s \in \mathbb{R}^n$, we say that the permutation $\pi : [n] \rightarrow [n]$ is *consistent* with s if for all pairs (i, j) , $\pi(i) < \pi(j)$ implies $s_i \geq s_j$. For the purpose of recovering the rankings, we will consider the entries of r to be pairwise distinct, i.e. $r_i \neq r_j$ for all $i \neq j$. Hence there is a unique permutation π which is consistent with r . Let $\tilde{\pi}$ be a permutation that is consistent with \tilde{u}_2 , note that this is not unique since some entries of \tilde{u}_2 could have the same value. Our goal is to bound the maximum displacement error between π and $\tilde{\pi}$ defined as

$$\|\tilde{\pi} - \pi\|_\infty := \max_i \left(\sum_{\pi(j) > \pi(i)} \mathbf{1}_{\tilde{\pi}(j) < \tilde{\pi}(i)} + \sum_{\pi(j) < \pi(i)} \mathbf{1}_{\tilde{\pi}(j) > \tilde{\pi}(i)} \right). \quad (4.6)$$

To this end, we have the following theorem the proof of which is provided in Section 5.3. Notice that bounding $\|\pi - \tilde{\pi}\|_\infty$ requires a bound on $\|\tilde{u}_2 - u_2\|_\infty$ which is obtained from Theorem 6. The proof technique is essentially the same as that of (Fogel et al., 2016, Theorem 24).

Theorem 7 Assuming $r_i \neq r_j$ for all $i \neq j$, let π denote the (unique) ranking consistent with r and also define $\rho := \min_{i \neq j} |r_i - r_j|$. Assuming $\beta = 1$, denote $\Upsilon(n, M, \eta, p, \varepsilon, r)$ to be the bound on $\|\tilde{u}_2 - u_2\|_\infty$ in Theorem 6. Then under the notation and assumptions in Theorem 6, we have with probability at least $1 - 2n \exp\left(-\frac{20pn}{3c_\varepsilon}\right) - \frac{4}{n} - 2n^{1-\frac{1}{4}(\log_\mu n)^{\xi-1}(\log_\mu e)^{-\xi}}$ that

$$\|\tilde{\pi} - \pi\|_\infty \leq \frac{4\|r - \alpha e\|_2}{\rho} \Upsilon(n, M, \eta, p, \varepsilon, r) \quad (4.7)$$

holds for all rankings $\tilde{\pi}$ which are consistent with \tilde{u}_2 .

Example 3 Let us examine the bound in (4.7) for the case where $r_i = i$ for $i = 1, \dots, n$. We have seen that $\|r - \alpha e\|_2 = \Theta(n^{3/2})$. Theorem 6 tells us that for a fixed $\delta \in (0, 1)$, if $p \gtrsim \frac{1}{\eta^2 \delta^{2/3}} \left(\frac{(\log n)^{2/3}}{n^{2/3}} \right)$ and $n = \Omega(1)$, then $\|\tilde{u}_2 - u_2\|_\infty \leq \Upsilon(\cdot) \lesssim \frac{\delta}{\sqrt{n} \log n}$ w.h.p. Since $\rho = 1$, we obtain from (4.7) that

$$\|\tilde{\pi} - \pi\|_\infty \lesssim \delta \frac{n}{\log n}.$$

Recovering scores in the ℓ_2 and ℓ_∞ norms. We now bound the error between the score vector estimate \tilde{r} as in (3.6), and r up to a global shift and sign. This is shown in the following theorem for the ℓ_2 and ℓ_∞ norms. The proof is outlined in Section 5.4. Note that both \tilde{r} and $r - \alpha e$ are centered vectors.

Theorem 8 Recall $\tilde{r} \in \mathbb{R}^n$ as defined in (3.6).

1. Under the notation and assumptions of Theorem 5, there exists $\beta \in \{-1, 1\}$ such that

$$\|\tilde{r} - \beta(r - \alpha e)\|_2 \leq \frac{8M}{\eta} \sqrt{\frac{5}{3p}(2 + \varepsilon)} + \sqrt{\frac{120M(2 + \varepsilon)\|r - \alpha e\|_2}{\eta p^{1/2}}} \left(\frac{5}{3}\right)^{1/4} \quad (4.8)$$

with probability at least $1 - 2n \exp\left(-\frac{20pn}{3c_\varepsilon}\right)$.

2. Under the notation and assumptions of Theorem 6, there exists $\beta \in \{-1, 1\}$ such that

$$\begin{aligned} \|\tilde{r} - \beta(r - \alpha e)\|_\infty &\leq \frac{16}{3} \|r - \alpha e\|_2 [(2 + \sqrt{2})C(n, M, \eta, p, \varepsilon, r) + \sqrt{n}C^2(n, M, \eta, p, \varepsilon, r)] \\ &\quad + 8\sqrt{\frac{5}{3}(2 + \varepsilon)} \frac{M(M - \alpha)}{\eta \sqrt{p} \|r - \alpha e\|_2} \end{aligned} \quad (4.9)$$

with probability at least

$$1 - 2n \exp\left(-\frac{20pn}{3c_\varepsilon}\right) - \frac{4}{n} - 2n^{1-\frac{1}{4}(\log_\mu n)^{\xi-1}(\log_\mu e)^{-\xi}}.$$

Example 4 When $r_i = i$ for all i , the bound in (4.8) is of the form

$$\|\tilde{r} - \beta(r - \alpha e)\|_2 \lesssim \frac{n}{\eta \sqrt{p}} + \frac{n^{5/4}}{\sqrt{\eta p^{1/4}}}. \quad (4.10)$$

Recall from Theorem 5 that p is required to satisfy $p \gtrsim \frac{\log n}{\eta^2 \delta^2 n}$ for $\delta \in (0, 1)$. Then, (4.10) simplifies to

$$\|\tilde{r} - \beta(r - \alpha e)\|_2 \lesssim \frac{n^{3/2}}{(\log n)^{1/4}} \sqrt{\delta}.$$

Turning our attention to (4.9), recall from Theorem 6 that we require $p \gtrsim \frac{1}{\eta^2 \delta^{2/3}} \left(\frac{(\log n)^{2/3}}{n^{2/3}} \right)$ and $n = \Omega(1)$. This leads to

$$\|\tilde{r} - \beta(r - \alpha e)\|_\infty \lesssim \frac{n}{\log n} \delta + \frac{\sqrt{n}}{\eta \sqrt{p}} \lesssim \frac{n}{\log n} \delta + \frac{n^{5/6}}{(\log n)^{1/3}} \delta^{1/3} \lesssim \frac{n}{\log n} \delta.$$

4.2 Analysis of Algorithm 2 (SVD-NRS)

We now analyze Algorithm 2, namely SVD-NRS, under the ERO model. The random matrix $\bar{D}^{-1/2} H \bar{D}^{-1/2}$ will concentrate around $(\mathbb{E}[\bar{D}])^{-1/2} \mathbb{E}[H] (\mathbb{E}[\bar{D}])^{-1/2}$. Recall from (4.1) that $\mathbb{E}[H] = \eta p(re^T - er^T)$, hence $(\mathbb{E}[\bar{D}])^{-1/2} \mathbb{E}[H] (\mathbb{E}[\bar{D}])^{-1/2}$ is a rank-2 matrix. It is easy to verify that

$$\mathbb{E}[\bar{D}_{ii}] = \eta p \sum_{j=1}^n |r_i - r_j| + (1 - \eta)p \frac{M}{2}.$$

It will be useful to denote

$$\max_i \mathbb{E}[\bar{D}_{ii}] = p \left[\eta \max_i \left(\sum_{j=1}^n |r_i - r_j| \right) + (1 - \eta) \frac{M}{2} \right] = p \lambda_{\max}, \quad (4.11)$$

$$\min_i \mathbb{E}[\bar{D}_{ii}] = p \left[\eta \min_i \left(\sum_{j=1}^n |r_i - r_j| \right) + (1 - \eta) \frac{M}{2} \right] = p \lambda_{\min}. \quad (4.12)$$

Recall $\tilde{u}_2 \in \mathbb{R}^n$ obtained in Step 6 of SVD-NRS, and denote $u_2 = \frac{(\mathbb{E}[\bar{D}])^{-1/2}(r - \alpha e)}{\|(\mathbb{E}[\bar{D}])^{-1/2}(r - \alpha e)\|_2}$. The following theorem bounds the ℓ_2 error between \tilde{u}_2 and u_2 up to a global sign, its proof is given in Section 5.5.

Theorem 9 (ℓ_2 recovery, ERO model) Denote

$$\alpha = \frac{r^T (\mathbb{E}[\bar{D}])^{-1} e}{e^T (\mathbb{E}[\bar{D}])^{-1} e}, \quad A(\eta, M) = \eta M^2 + (1 - \eta) \frac{M}{2} \quad \text{and} \quad C_1(\eta, M) = 4A^{1/4}(\eta, M).$$

For $\varepsilon > 0$, denote

$$\begin{aligned} \sigma_{\min} &= \frac{\eta \|r - \alpha e\|_2 \sqrt{n}}{\lambda_{\max}}, \quad \sigma_{\max} = \frac{\eta \|r - \alpha e\|_2 \sqrt{n}}{\lambda_{\min}}, \\ \tilde{\Delta} &= 16M \sqrt{\frac{5}{3}pn} \frac{(2 + \varepsilon)}{p\lambda_{\min}} + \frac{C_1(\eta, M)(np \log n)^{1/4} \sigma_{\max}}{p^{3/2} \lambda_{\min}^{3/2}} \left(\frac{C_1(\eta, M)(np \log n)^{1/4}}{\sqrt{p\lambda_{\min}}} + 2\sqrt{2} \right). \end{aligned}$$

Let $\tilde{u}_2 \in \mathbb{R}^n$ be the vector obtained in Step 6 of SVD-NRS (with $\|\tilde{u}_2\|_2 = 1$) and denote $u_2 = \frac{(\mathbb{E}[\bar{D}])^{-1/2}(r - \alpha e)}{\|(\mathbb{E}[\bar{D}])^{-1/2}(r - \alpha e)\|_2}$. If

$$p \geq \frac{M^2}{9A(\eta, M)} \frac{\log n}{n}, \quad p \geq \frac{16(\sqrt{2} + 1)^2 A(\eta, M) n \log n}{\lambda_{\min}^2} \quad \text{and} \quad \tilde{\Delta} \leq \frac{\sigma_{\min}}{3} \quad (4.13)$$

hold, then there exists $\beta \in \{-1, 1\}$ and a constant $c_\varepsilon > 0$ depending only on ε such that

$$\|\tilde{u}_2 - \beta u_2\|_2^2 \leq \frac{15\tilde{\Delta}}{\sigma_{\min}}$$

holds with probability at least $1 - \frac{2}{n} - 2n \exp\left(-\frac{20pn}{3c_\varepsilon}\right)$.

The quantities $\sigma_{\min}, \sigma_{\max}$ are lower and upper bounds on the non-zero singular values of the rank 2 matrix $(\mathbb{E}[\bar{D}])^{-1/2}\mathbb{E}[H](\mathbb{E}[\bar{D}])^{-1/2}$, while $\tilde{\Delta}$ is an upper bound on the quantity $\|\bar{D}^{-1/2}H\bar{D}^{-1/2} - (\mathbb{E}[D])^{-1/2}\mathbb{E}[H](\mathbb{E}[D])^{-1/2}\|_2$. The theorem essentially states that if p is sufficiently large, then \tilde{u}_2 is close to u_2 up to a sign flip. In order to get a sense of the precise scaling of the quantities involved, we consider the special case $r_i = i$.

Example 5 Consider $r_i = i$ for all i with $\eta \in (0, 1)$ fixed, hence $M = n$. Denoting $S_i = \sum_{j=1}^n |r_i - r_j|$, one can verify that $S_i = i^2 - i(n+1) + \frac{n^2+n}{2}$. This implies that $\lambda_{\max}, \lambda_{\min} = \Theta(\eta n^2) = A(\eta, M)$, and so, $C(\eta, M) = \Theta(\eta^{1/4}n^{1/2})$. Now let us find the scaling of α . Since $\mathbb{E}[\bar{D}_{ii}] = \Theta(p\eta n^2)$, we obtain

$$\alpha = \frac{\sum_{i=1}^n i(\mathbb{E}[\bar{D}_{ii}])^{-1}}{\sum_{i=1}^n (\mathbb{E}[\bar{D}_{ii}])^{-1}} = \Theta\left(\frac{\sum_{i=1}^n i}{n}\right) = \Theta(n).$$

This in particular implies that $\|r - \alpha e\|_2^2 = \sum_{j=1}^n (j - \alpha)^2 = \Theta(n^3)$. We then obtain $\sigma_{\min}, \sigma_{\max} = \Theta(1)$; plugging the aforementioned scalings in the expression for $\tilde{\Delta}$ leads to

$$\tilde{\Delta} = O\left(\frac{1}{\eta\sqrt{np}} + \frac{(\log n)^{1/4}}{(\eta p)^{5/4}n^{9/4}} \left(\frac{(\log n)^{1/4}}{(\eta np)^{1/4}} + 2\sqrt{2}\right)\right).$$

The first two conditions on p in (4.13) translate to $p \gtrsim \frac{1}{\eta} \frac{\log n}{n}$. Since $\sigma_{\min} = \Theta(1)$, hence the third condition in (4.13) holds provided $\tilde{\Delta} \lesssim 1$. In fact, for any $\delta \in (0, 1)$, we have $\tilde{\Delta} \lesssim \frac{\delta}{\sqrt{\log n}}$ if p additionally satisfies

$$\frac{1}{\eta\sqrt{np}} \lesssim \frac{\delta}{\sqrt{\log n}} \Leftrightarrow p \gtrsim \frac{\log n}{\eta^2\delta^2 n} \quad \text{and} \quad \frac{(\log n)^{1/4}}{(\eta p)^{5/4}n^{9/4}} \lesssim \frac{\delta}{\sqrt{\log n}} \Leftrightarrow p \gtrsim \frac{(\log n)^{3/5}}{\eta n^{9/5}\delta^{4/5}}.$$

Clearly, the first condition on p dominates since $\frac{\log n}{\eta^2\delta^2 n} \gtrsim \frac{(\log n)^{3/5}}{\eta n^{9/5}\delta^{4/5}}$. To summarize, Theorem 9 states that if p satisfies

$$p \gtrsim \frac{\log n}{\eta^2\delta^2 n}, \tag{4.14}$$

then $\exists \beta \in \{-1, 1\}$ such that $\|\tilde{u}_2 - \beta u_2\|_2^2 \lesssim \delta$.

Recovering scores in the ℓ_2 norm. The following theorem bounds the ℓ_2 error between the score vector estimate \tilde{r} defined in (3.7), and r up to a global sign and shift. Its proof is outlined in Section 5.6. Note that both \tilde{r} and $r - (\alpha + \alpha')e$ are centered vectors.

Theorem 10 Recall $\tilde{r} \in \mathbb{R}^n$ as defined in (3.7). Under the notation and assumptions of Theorem 9, and denoting $\alpha' = \frac{e^T(r-\alpha e)}{n}$, there exists $\beta \in \{-1, 1\}$ such that

$$\begin{aligned} \|\tilde{r} - \beta(r - (\alpha + \alpha')e)\|_2 &\leq \frac{2}{\eta p} \left[\sqrt{\frac{3}{n}} \left(\sigma_{\max} + \tilde{\Delta} \right) (2\sqrt{2} + 1)^{1/2} (A(\eta, M)np \log n)^{1/4} \left(\sqrt{p\lambda_{\max}} + \frac{\lambda_{\max}}{\lambda_{\min}} \right) \right. \\ &\quad \left. + \frac{\tilde{\Delta}p\lambda_{\max}}{\sqrt{n}} + \sqrt{\frac{15\tilde{\Delta}}{\sigma_{\min}n} \sigma_{\max}p\lambda_{\max}} \right] \end{aligned}$$

holds with probability at least $1 - \frac{2}{n} - 2n \exp\left(-\frac{20pn}{3c_\varepsilon}\right)$.

Example 6 Consider the scenario where $r_i = i$ for each i . Then as discussed above for Theorem 9, if p satisfies (4.14), we obtain the bound

$$\begin{aligned} \|\tilde{r} - \beta(r - (\alpha + \alpha')e)\|_2 &\lesssim \frac{1}{\eta p} \left[\frac{(\eta m^3 p \log n)^{1/4}}{\sqrt{n}} (\sqrt{p\eta n^2} + 1) + \frac{p\eta n^2 \delta}{\sqrt{n \log n}} + \sqrt{\frac{\delta}{n \sqrt{\log n}}} p\eta n^2 \right] \\ &\lesssim \frac{1}{\eta p} \left[\eta^{3/4} n^{5/4} (\log n)^{1/4} p^{3/4} + \eta p \delta^{1/2} \frac{n^{3/2}}{(\log n)^{1/4}} \right] \\ &= \frac{n^{5/4} (\log n)^{1/4}}{(\eta p)^{1/4}} + \delta^{1/2} \frac{n^{3/2}}{(\log n)^{1/4}} \\ &\lesssim n^{3/2} \eta^{1/4} \delta^{1/2} + \delta^{1/2} \frac{n^{3/2}}{(\log n)^{1/4}} \quad (\text{using (4.14)}). \end{aligned}$$

5. Proofs

5.1 Proof of Theorem 5

We now outline the proof of Theorem 5 by highlighting the steps involved, along with the intermediate Lemmas (the proofs of which are in Appendix C). The proof is broken into the following steps.

Step 1: Singular values and singular vectors of C . We begin by finding the singular values and singular vectors of C .

Lemma 11 For $C = re^T - er^T$, it holds true that $C = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$, with $\sigma_1 = \sigma_2 = \|r - \alpha e\|_2 \sqrt{n}$, and

$$u_1 = v_2 = \frac{e}{\sqrt{n}}; \quad v_1 = -\frac{r - \alpha e}{\|r - \alpha e\|_2}; \quad u_2 = \frac{r - \alpha e}{\|r - \alpha e\|_2},$$

where $\alpha = \frac{r^T e}{n}$.

Step 2: Bounding the spectral norm of Z . Recall that $(Z_{ij})_{i \leq j}$ are independent zero-mean random variables with $Z_{ii} = 0$ and moreover, $Z_{ji} = -Z_{ij}$. Denote $Z = S - S^T$, where

S is an upper-triangular matrix with $S_{ij} = Z_{ij}$ for $i \leq j$. Note that $\|Z\|_2 \leq \|S\|_2 + \underbrace{\|S^T\|_2}_{= \|S\|_2} \leq 2\|S\|_2$. We define a symmetric matrix

$$\tilde{S} := \begin{bmatrix} 0 & S \\ S^T & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}.$$

It is easy to check that $\|\tilde{S}\|_2 = \|S\|_2$, and thus $\|Z\|_2 \leq 2\|\tilde{S}\|_2$. Since \tilde{S} is a random symmetric matrix with the entries $(\tilde{S}_{ij})_{i \leq j}$ being centered, independent and bounded random variables, we can bound $\|\tilde{S}\|_2$ using a recent result of (Bandeira and van Handel, 2016, Corollary 3.12, Remark 3.13) that gives non-asymptotic bounds on the spectral norm of such random symmetric matrices. This altogether leads to the following Lemma.

Lemma 12 *For $Z \in \mathbb{R}^{n \times n}$ as defined in (4.2), we have that*

$$\|Z\|_2 \leq 8M \sqrt{\frac{5}{3}pn(2 + \varepsilon)} \quad (5.1)$$

holds with probability at least $1 - 2n \exp\left(-\frac{20pn}{3c_\varepsilon}\right)$.

Step 3: Using Wedin's bound. Let us denote $U = [u_1 \ u_2] \in \mathbb{R}^{n \times 2}$ (recall u_1, u_2 from Lemma 11). Denote $\hat{U} = [\hat{u}_1 \ \hat{u}_2] \in \mathbb{R}^{n \times 2}$ where \hat{u}_1, \hat{u}_2 are the left singular vectors corresponding to the top two singular values of H . The following Lemma states that $\mathcal{R}(U)$ is close to $\mathcal{R}(\hat{U})$ if $\|Z\|_2$ is small. The proof uses Wedin's bound (Wedin, 1972) for perturbation of singular subspaces.

Lemma 13 *Given $H = \eta p C + Z$, where $\|Z\|_2 \leq \Delta$, if $\Delta < \eta p \|r - \alpha e\|_2 \sqrt{n}$, then*

$$\|(I - \hat{U}\hat{U}^T)U\|_2 \leq \frac{\Delta}{\eta p \|r - \alpha e\|_2 \sqrt{n} - \Delta} (=: \delta). \quad (5.2)$$

Step 4: Analyzing the projection step. Next, we project u_1 on $\text{span}(\hat{U})$ to obtain \bar{u}_1 , find a unit vector orthogonal to \bar{u}_1 lying in $\mathcal{R}(\hat{U})$ (denote this by \tilde{u}_2), and show that \tilde{u}_2 is close to u_2 up to a sign flip.

Lemma 14 *With δ as defined in Lemma 13 let $\delta \leq 1/2$ hold. Then, there exists $\beta \in \{-1, 1\}$ such that*

$$\|\tilde{u}_2 - \beta u_2\|_2^2 \leq 10\delta.$$

Step 5: Putting it together. We now use the above Lemma's to readily obtain Theorem 5 (see appendix for details). Using $\|Z\|_2 \leq \Delta = 8M \sqrt{\frac{5}{3}pn(2 + \varepsilon)}$ in the expression for δ , one can verify that

$$\delta \leq 1/2 \Leftrightarrow 3\Delta \leq \eta p \|r - \alpha e\|_2 \sqrt{n} \quad (5.3)$$

holds if $\|r - \alpha e\|_2 \geq \frac{24M}{\eta} \sqrt{\frac{5}{3p}}(2 + \varepsilon)$. By using the bound on Δ in (5.3) in the expression for δ , we finally obtain

$$\delta \leq \frac{12M}{\eta} \sqrt{\frac{5}{3p}} \frac{(2 + \varepsilon)}{\|r - \alpha e\|_2}.$$

Plugging this in Lemma 14 yields the stated ℓ_2 bound and completes the proof.

5.2 Proof of Theorem 6

Recall that $U = [u_1 \ u_2], \widehat{U} = [\widehat{u}_1 \ \widehat{u}_2] \in \mathbb{R}^{n \times 2}$ correspond to the two largest singular vectors of $\mathbb{E}[H]$, and H respectively. The proof is broken down into three main steps. The first step involves bounding $\|\widetilde{u}_2 - \beta u_2\|_\infty$ in terms of $\|\widehat{U} - UO^*\|_{\max}$ – where O^* is a 2×2 orthogonal matrix that “aligns” U with \widehat{U} – provided $\|Z\|_2$ is small. The second step involves bounding $\|\widehat{U} - UO^*\|_{\max}$, and in the third step we combine the results from the previous steps. The proofs of all lemmas are provided in Appendix D.

Step 1: Bounding $\|\widetilde{u}_2 - \beta u_2\|_\infty$ in terms of $\|\widehat{U} - UO^*\|_{\max}$. The following Lemma states this formally.

Lemma 15 *For Δ as defined in Lemma 13, let $\Delta \leq \frac{\eta p \|r - \alpha e\|_2 \sqrt{n}}{3}$ hold. Then for any orthogonal matrix $O \in \mathbb{R}^{2 \times 2}$, there exists $\beta \in \{-1, 1\}$ depending on O such that*

$$\|\widetilde{u}_2 - \beta u_2\|_\infty \leq 4\|\widehat{U} - UO\|_{\max} \left(2 + \frac{\sqrt{n}(M - \alpha)}{\|r - \alpha e\|_2} \right) + 4\sqrt{n}\|\widehat{U} - UO\|_{\max}^2. \quad (5.4)$$

Moreover, there exists an orthogonal matrix $O^* \in \mathbb{R}^{2 \times 2}$ such that

$$\|\widehat{U} - UO^*\|_2 \leq \frac{3\Delta}{\eta p \sqrt{n} \|r - \alpha e\|_2}. \quad (5.5)$$

Step 2: Bounding $\|\widehat{U} - UO^*\|_{\max}$. Since $H^T = -H$, therefore $-H^2 = HH^T$. Using the SVD $H = \widehat{U}\widehat{\Sigma}\widehat{V}$, we arrive at

$$\begin{aligned} HH^T\widehat{U} &= -H^2\widehat{U} = \widehat{U}\widehat{\Sigma}^2 \\ \Rightarrow \widehat{U} &= -H^2\widehat{U}\widehat{\Sigma}^{-2} = -(\mathbb{E}[H]^2 + \mathbb{E}[H]Z + Z\mathbb{E}[H] + Z^2)\widehat{U}\widehat{\Sigma}^{-2}. \end{aligned} \quad (5.6)$$

For convenience of notation, denote $\sigma = \sigma_i(\mathbb{E}[H]) = \eta p \sqrt{n} \|r - \alpha e\|_2$; $i = 1, 2$. Since $\mathbb{E}[H]^T = -\mathbb{E}[H]$, we get

$$\begin{aligned} -\mathbb{E}[H]^2 &= \mathbb{E}[H]\mathbb{E}[H]^T = U\Sigma^2U^T = \sigma^2(UO^*)(UO^*)^T \\ UO^* &= -\frac{1}{\sigma^2}(\mathbb{E}[H])^2(UO^*). \end{aligned} \quad (5.7)$$

Subtracting (5.7) from (5.6) leads to

$$\begin{aligned} \widehat{U} - UO^* &= \underbrace{-\mathbb{E}[H]^2(\widehat{U} - UO^*)\widehat{\Sigma}^{-2}}_{E_1} + \underbrace{\mathbb{E}[H]^2UO^*(\sigma^{-2}I - \widehat{\Sigma}^{-2})}_{E_2} - \underbrace{\mathbb{E}[H]Z\widehat{U}\widehat{\Sigma}^{-2}}_{E_3} \\ &\quad - \underbrace{Z\mathbb{E}[H]\widehat{U}\widehat{\Sigma}^{-2}}_{E_4} - \underbrace{Z^2\widehat{U}\widehat{\Sigma}^{-2}}_{E_5} \\ &= E_1 + E_2 - E_3 - E_4 - E_5 \\ \Rightarrow \|\widehat{U} - UO^*\|_{\max} &\leq \sum_{i=1}^5 \|E_i\|_{\max}. \end{aligned} \quad (5.8)$$

We now proceed to bound each term in the RHS of (5.8). This is stated precisely in the following Lemma.

Lemma 16 If $\Delta \leq \frac{\sigma}{3} = \frac{\eta p \sqrt{n} \|r - \alpha e\|_2}{3}$, then the following holds true.

1. $\|E_1\|_{\max} \leq \frac{27\Delta}{4\sigma} \left(\frac{1}{\sqrt{n}} + \frac{M-\alpha}{\|r - \alpha e\|_2} \right).$
2. $\|E_2\|_{\max} \leq \frac{21\Delta}{4\sigma} \left(\frac{1}{\sqrt{n}} + \frac{M-\alpha}{\|r - \alpha e\|_2} \right).$
3. $\|E_3\|_{\max} \leq \frac{9\Delta}{4\sigma} \left(\frac{1}{\sqrt{n}} + \frac{M-\alpha}{\|r - \alpha e\|_2} \right).$
4. $\|E_4\|_{\max} \leq \frac{9}{4\sigma} (\|Zu_1\|_{\infty} + \|Zu_2\|_{\infty}).$
5. $\|E_5\|_{\max} \leq \frac{27\Delta^3}{4\sigma^3} + \frac{9}{4\sigma^2} (\|Z^2u_1\|_{\infty} + \|Z^2u_2\|_{\infty}).$

Next, we will bound $\|Zu_i\|_{\infty}$ and $\|Z^2u_i\|_{\infty}$ for $i = 1, 2$. Bounding $\|Zu_i\|_{\infty}$ is a relatively straightforward consequence of Bernstein's inequality (Boucheron et al., 2013, Corollary 2.11), we state this in the following Lemma.

Lemma 17 If $p \geq \frac{2 \log n}{15n}$ then the following holds.

1. $\mathbb{P}(\|Zu_1\|_{\infty} \geq \frac{2\sqrt{2+4}}{3} M \sqrt{15p \log n}) \leq \frac{2}{n}.$
2. $\mathbb{P}(\|Zu_2\|_{\infty} \geq \frac{2\sqrt{2+4}}{3} M \sqrt{15p B n \log n}) \leq \frac{2}{n}$ where $B = \frac{(M-\alpha)^2}{\|r - \alpha e\|_2^2}.$

Bounding $\|Z^2u_i\|_{\infty}$ is a considerably more challenging task since the entries within a row of Z^2 will not be independent now. However, leveraging a recent entry-wise concentration result for the product of a random matrix (raised to a power) and a fixed vector (Eldridge et al., 2018, Theorem 15) (see Theorem 25), we are able to do so and arrive at the following Lemma.

Lemma 18 Let $p \geq \frac{1}{2n}$ hold. Choose $\xi > 1, 0 < \kappa < 1$ and define $\mu = \frac{2}{\kappa+1}$. Then if $\frac{16}{\kappa} \leq (\log n)^{\xi}$, the following holds true.

1. $\mathbb{P}(\|Z^2u_1\|_{\infty} \geq 8\sqrt{np}M^2(\log n)^{2\xi}) \leq n^{1-\frac{1}{4}(\log_{\mu} n)^{\xi-1}(\log_{\mu} e)^{-\xi}}.$
2. $\mathbb{P}(\|Z^2u_2\|_{\infty} \geq 8npM^2 \frac{M-\alpha}{\|r - \alpha e\|_2} (\log n)^{2\xi}) \leq n^{1-\frac{1}{4}(\log_{\mu} n)^{\xi-1}(\log_{\mu} e)^{-\xi}}.$

Step 3: Putting it together. We now combine the results of the preceding Lemmas to derive the final approximation bound. Recall from Lemma 12 that $\Delta = 8M\sqrt{\frac{5}{3}pn}(2 + \varepsilon)$. Also, $\sigma = \eta p \sqrt{n} \|r - \alpha e\|_2$ and so

$$\frac{\Delta}{\sigma} = \frac{8M}{\eta} \sqrt{\frac{5}{3p}} \frac{(2 + \varepsilon)}{\|r - \alpha e\|_2}. \quad (5.9)$$

Using Lemmas 16,17,18 and (5.9) in (5.8), we get

$$\begin{aligned}
 \|\widehat{U} - UO^*\|_{\max} &\leq \left(\frac{114M}{\eta} \sqrt{\frac{5}{3p}} \frac{(2+\varepsilon)}{\|r - \alpha e\|_2} \right) \left(\frac{1}{\sqrt{n}} + \frac{M-\alpha}{\|r - \alpha e\|_2} \right) \\
 &\quad + \left(\frac{3\sqrt{15}(2\sqrt{2}+4)}{4} \right) \left(\frac{M\sqrt{\log n}}{\eta\sqrt{p}\|r - \alpha e\|_2} \right) \left(\frac{1}{\sqrt{n}} + \frac{M-\alpha}{\|r - \alpha e\|_2} \right) \\
 &\quad + \frac{18M^2(\log n)^{2\xi}}{\eta^2 p \|r - \alpha e\|_2^2} \left(\frac{1}{\sqrt{n}} + \frac{M-\alpha}{\|r - \alpha e\|_2} \right) + \frac{27}{4} \left(\frac{8\sqrt{5}}{3} \right)^3 \frac{M^3}{\eta^3 p^{3/2}} \frac{(2+\varepsilon)^3}{\|r - \alpha e\|_2^3} \\
 &\leq C_\varepsilon \left[\left(\frac{M\sqrt{\log n}}{\eta\sqrt{p}\|r - \alpha e\|_2} + \frac{M^2(\log n)^{2\xi}}{\eta^2 p \|r - \alpha e\|_2^2} \right) \left(\frac{1}{\sqrt{n}} + \frac{M-\alpha}{\|r - \alpha e\|_2} \right) + \frac{M^3}{\eta^3 p^{3/2} \|r - \alpha e\|_2^3} \right] \\
 &= C(n, M, \eta, p, \varepsilon, r)
 \end{aligned}$$

where $C_\varepsilon > 0$ is a universal constant depending only on ε . Plugging this in (5.4), we arrive at the stated ℓ_∞ bound in the Theorem. The lower bound on the success probability follows readily via a union bound on the events stated in Lemmas 12, 17, 18.

5.3 Proof of Theorem 7

We assume w.l.o.g that π is identity so that $\pi(i) = i$. For any $i \in [n]$, we have for all rankings $\tilde{\pi}$ which are consistent with \tilde{u}_2 that

$$|i - \tilde{\pi}(i)| = \sum_{j>i} \mathbf{1}_{\{\tilde{\pi}(j) < \tilde{\pi}(i)\}} + \sum_{j< i} \mathbf{1}_{\{\tilde{\pi}(j) > \tilde{\pi}(i)\}} \leq \sum_{j>i} \mathbf{1}_{\{\tilde{u}_{2,j} \geq \tilde{u}_{2,i}\}} + \sum_{j< i} \mathbf{1}_{\{\tilde{u}_{2,j} \leq \tilde{u}_{2,i}\}},$$

leading to the bound

$$\|\pi - \tilde{\pi}\|_\infty = \max_i |i - \tilde{\pi}(i)| \leq \sum_{j>i} \mathbf{1}_{\{\tilde{u}_{2,j} \geq \tilde{u}_{2,i}\}} + \sum_{j< i} \mathbf{1}_{\{\tilde{u}_{2,j} \leq \tilde{u}_{2,i}\}}.$$

Now for a given $i' \in [n]$, we can decompose $\tilde{u}_{2,i'} - \tilde{u}_{2,j}$ as

$$\tilde{u}_{2,i'} - \tilde{u}_{2,j} = (\tilde{u}_{2,i'} - u_{2,i'}) + (u_{2,i'} - u_{2,j}) + (u_{2,j} - \tilde{u}_{2,j}),$$

which in turn implies $\tilde{u}_{2,i'} - \tilde{u}_{2,j} \leq (u_{2,i'} - u_{2,j}) + 2\|\tilde{u}_2 - u_2\|_\infty$. Hence if j is such that $i' > j$, then $\|\tilde{u}_2 - u_2\|_\infty \leq \frac{|u_{2,i'} - u_{2,j}|}{2}$ implies

$$\tilde{u}_{2,i'} - \tilde{u}_{2,j} \leq \underbrace{(u_{2,i'} - u_{2,j})}_{\leq 0} + |u_{2,i'} - u_{2,j}| = 0.$$

Using this, we obtain

$$\begin{aligned}
 \sum_{j < i'} \mathbf{1}_{\{\tilde{u}_{2,j} < \tilde{u}_{2,i'}\}} &\leq \sum_{j < i'} \mathbf{1}_{\left\{ \|\tilde{u}_2 - u_2\|_\infty > \frac{|u_{2,i'} - u_{2,j}|}{2} \right\}} \\
 &\leq \sum_{j < i'} \mathbf{1}_{\left\{ \Upsilon(n, M, \eta, p, \varepsilon, r) > \frac{|u_{2,i'} - u_{2,j}|}{2} \right\}} \quad (\text{since } \|\tilde{u}_2 - u_2\|_\infty \leq \Upsilon(n, M, \eta, p, \varepsilon, r)) \\
 &= \sum_{j < i'} \mathbf{1}_{\{2\Upsilon(n, M, \eta, p, \varepsilon, r)\|r - \alpha e\|_2 > |r_{i'} - r_j|\}} \\
 &= \left\lfloor \frac{2\Upsilon(n, M, \eta, p, \varepsilon, r)\|r - \alpha e\|_2}{\rho} \right\rfloor \leq \frac{2\Upsilon(n, M, \eta, p, \varepsilon, r)\|r - \alpha e\|_2}{\rho}.
 \end{aligned}$$

The same bound holds for $\sum_{j > i'} \mathbf{1}_{\{\tilde{u}_{2,j} > \tilde{u}_{2,i'}\}}$ and hence the statement of the Theorem follows.

5.4 Proof of Theorem 8

Recall that $\tilde{w} = \frac{\hat{\sigma}_1}{\eta p \sqrt{n}} \tilde{u}_2$. Let us denote

$$w = \frac{\sigma}{\eta p \sqrt{n}} \beta u_2 = \|r - \alpha e\|_2 \beta u_2 = (r - \alpha e) \beta.$$

For any norm $\|\cdot\|$ on \mathbb{R}^n , we have by triangle inequality that

$$\|\tilde{r} - w\| \leq \|\tilde{w} - w\| + \left| \frac{e^T \tilde{w}}{n} \right| \|e\|.$$

Since $e^T w = 0$, therefore we obtain via Cauchy-Schwarz that

$$\left| \frac{e^T \tilde{w}}{n} \right| = \left| \frac{e^T (\tilde{w} - w)}{n} \right| \leq \frac{\|\tilde{w} - w\|_2}{\sqrt{n}}.$$

Hence we have that

$$\|\tilde{r} - w\|_2 \leq 2\|\tilde{w} - w\|_2 \quad \text{and} \quad \|\tilde{r} - w\|_\infty \leq \|\tilde{w} - w\|_\infty + \frac{\|\tilde{w} - w\|_2}{\sqrt{n}} \leq 2\|\tilde{w} - w\|_\infty.$$

It remains to bound $\|\tilde{w} - w\|_2$ and $\|\tilde{w} - w\|_\infty$. Our starting point will be

$$\begin{aligned}
 \|\tilde{w} - w\| &= \frac{1}{\eta p \sqrt{n}} \|\hat{\sigma}_1 \tilde{u}_2 - \sigma_1 \beta u_2\| \\
 &= \frac{1}{\eta p \sqrt{n}} \|(\hat{\sigma}_1 - \sigma_1) \tilde{u}_2 + \sigma_1 (\tilde{u}_2 - \beta u_2)\| \\
 \Rightarrow \|\tilde{w} - w\| &\leq \frac{1}{\eta p \sqrt{n}} (|\hat{\sigma}_1 - \sigma_1| \|\tilde{u}_2\| + \sigma_1 \|\tilde{u}_2 - \beta u_2\|) \\
 &\leq \frac{1}{\eta p \sqrt{n}} (\Delta \|\tilde{u}_2\| + \eta p \|r - \alpha e\|_2 \sqrt{n} \|\tilde{u}_2 - \beta u_2\|). \tag{5.10}
 \end{aligned}$$

1. **Bounding $\|\tilde{w} - w\|_2$.** From (5.10), we have that

$$\|\tilde{w} - w\|_2 \leq \frac{1}{\eta p \sqrt{n}} (\Delta + \eta p \|r - \alpha e\|_2 \sqrt{n} \|\tilde{u}_2 - \beta u_2\|_2)$$

Plugging the expression for Δ from Lemma 12, and the bound on $\|\tilde{u}_2 - \beta u_2\|_2$ from Theorem 5, we readily obtain the bound in part (i) of the Theorem.

2. **Bounding $\|\tilde{w} - w\|_\infty$.** From (5.10), we have that

$$\begin{aligned} \|\tilde{w} - w\|_\infty &\leq \frac{1}{\eta p \sqrt{n}} [\Delta \|\tilde{u}_2\|_\infty + \sigma_1 \|\tilde{u}_2 - \beta u_2\|_\infty] \\ &\leq \frac{1}{\eta p \sqrt{n}} [\Delta (\|\tilde{u}_2 - \beta u_2\|_\infty + \|u_2\|_\infty) + \sigma_1 \|\tilde{u}_2 - \beta u_2\|_\infty] \\ &= \frac{1}{\eta p \sqrt{n}} \left[(\sigma_1 + \Delta) \|\tilde{u}_2 - \beta u_2\|_\infty + \Delta \frac{(M - \alpha)}{\|r - \alpha e\|_2} \right]. \end{aligned} \quad (5.11)$$

Plugging the expression for σ_1 (from Lemma 11) and Δ (from Lemma 12), followed by some simplification, we arrive at the bound in part (ii) of the Theorem.

5.5 Proof of Theorem 9

The outline is similar to the proof of Theorem 5 with some technical changes. We first note that the SVD of $(\mathbb{E}[\bar{D}])^{-1/2} \mathbb{E}[H] (\mathbb{E}[\bar{D}])^{-1/2}$ is given by $\sigma(u_1 v_1^T + u_2 v_2^T)$ where

$$\begin{aligned} \sigma &= \eta p \|(\mathbb{E}[\bar{D}])^{-1/2}(r - \alpha e)\|_2 \|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2, \quad v_2 = u_1 = \frac{(\mathbb{E}[\bar{D}])^{-1/2}e}{\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2}, \\ v_1 &= -u_2 = -\frac{(\mathbb{E}[\bar{D}])^{-1/2}(r - \alpha e)}{\|(\mathbb{E}[\bar{D}])^{-1/2}(r - \alpha e)\|_2}, \end{aligned} \quad (5.12)$$

and $\alpha = \frac{r^T(\mathbb{E}[\bar{D}])^{-1}e}{e^T(\mathbb{E}[\bar{D}])^{-1}e}$. This is verified easily by proceeding as in the proof of Lemma 11. One can also readily see that $\sigma_{\min} \leq \sigma \leq \sigma_{\max}$. Now let us write

$$\bar{D}^{-1/2} H \bar{D}^{-1/2} = (\mathbb{E}[\bar{D}])^{-1/2} \mathbb{E}[H] (\mathbb{E}[\bar{D}])^{-1/2} + \tilde{Z}$$

where $\tilde{Z} = \bar{D}^{-1/2} (H - \mathbb{E}[H]) \bar{D}^{-1/2} + \bar{D}^{-1/2} \mathbb{E}[H] \bar{D}^{-1/2} - (\mathbb{E}[\bar{D}])^{-1/2} \mathbb{E}[H] (\mathbb{E}[\bar{D}])^{-1/2}$. In order to bound $\|\tilde{Z}\|_2$, we will first need to establish the concentration of \bar{D} around $\mathbb{E}[\bar{D}]$.

Lemma 19 Denote $A(\eta, M) = \eta M^2 + (1 - \eta) \frac{M}{2}$. If $p \geq \frac{M^2}{9A(\eta, M)} \frac{\log n}{n}$ then,

$$\mathbb{P}(\|\bar{D} - \mathbb{E}[\bar{D}]\|_2 \geq 2(\sqrt{2} + 1) \sqrt{A(\eta, M) np \log n}) \leq 2/n.$$

The proof is deferred to Appendix E. Conditioned on the event in Lemma 19, we have that

$$\bar{D}_{ii} \in [\mathbb{E}[\bar{D}_{ii}] \pm 2(\sqrt{2} + 1) \sqrt{A(\eta, M) np \log n}], \quad \forall i = 1, \dots, n.$$

In particular, if $2(\sqrt{2} + 1) \sqrt{A(\eta, M) np \log n} \leq \frac{1}{2} \min_i \mathbb{E}[\bar{D}_{ii}] = \frac{p \lambda_{\min}}{2}$, then

$$\min_i \bar{D}_{ii} \geq p \lambda_{\min}/2. \quad (5.13)$$

Using triangle inequality, we obtain the bound

$$\|\tilde{Z}\|_2 \leq \underbrace{\|\bar{D}^{-1/2}(H - \mathbb{E}[H])\bar{D}^{-1/2}\|_2}_{\tilde{Z}_1} + \underbrace{\|\bar{D}^{-1/2}\mathbb{E}[H]\bar{D}^{-1/2} - (\mathbb{E}[\bar{D}])^{-1/2}\mathbb{E}[H](\mathbb{E}[\bar{D}])^{-1/2}\|_2}_{\tilde{Z}_2}.$$

From sub-multiplicativity of the spectral norm and also (5.13), we have that

$$\|\tilde{Z}_1\|_2 \leq \frac{\|H - \mathbb{E}[H]\|_2}{\min_i \bar{D}_{ii}} \leq \frac{2\Delta}{p\lambda_{\min}}. \quad (5.14)$$

Here, we used the bound $\|H - \mathbb{E}[H]\|_2 \leq \Delta$ from Lemma 12. In order to bound $\|\tilde{Z}_2\|_2$, we add and subtract $\mathbb{E}[\bar{D}]$ from \bar{D} and apply triangle inequality. This yields

$$\begin{aligned} \|\tilde{Z}_2\|_2 &\leq \|(\bar{D}^{-1/2} - (\mathbb{E}[\bar{D}])^{-1/2})\mathbb{E}[H](\bar{D}^{-1/2} - (\mathbb{E}[\bar{D}])^{-1/2})\|_2 \\ &\quad + \|(\bar{D}^{-1/2} - (\mathbb{E}[\bar{D}])^{-1/2})\mathbb{E}[H]\bar{D}^{-1/2}\|_2 + \|\bar{D}^{-1/2}\mathbb{E}[H](\bar{D}^{-1/2} - (\mathbb{E}[\bar{D}])^{-1/2})\|_2 \\ &\leq \|\bar{D}^{-1/2} - (\mathbb{E}[\bar{D}])^{-1/2}\|_2^2 \|\mathbb{E}[H]\|_2 + 2\|\bar{D}^{-1/2} - (\mathbb{E}[\bar{D}])^{-1/2}\|_2 \|\mathbb{E}[H]\|_2 \|\bar{D}^{-1/2}\|_2. \end{aligned} \quad (5.15)$$

Note that

$$\begin{aligned} \|(\bar{D}^{-1/2} - (\mathbb{E}[\bar{D}])^{-1/2})\|_2 &= \|\bar{D}^{-1/2}(\bar{D}^{1/2} - (\mathbb{E}[\bar{D}])^{1/2})(\mathbb{E}[\bar{D}])^{-1/2}\|_2 \\ &\leq \|\bar{D}^{-1/2}\|_2 \|(\bar{D}^{1/2} - (\mathbb{E}[\bar{D}])^{1/2})\|_2 \|(\mathbb{E}[\bar{D}])^{-1/2}\|_2. \end{aligned}$$

Moreover, we can bound $\|(\bar{D}^{1/2} - (\mathbb{E}[\bar{D}])^{1/2})\|_2 \leq \|\bar{D} - \mathbb{E}[\bar{D}]\|_2^{1/2}$ since $\bar{D}, \mathbb{E}[\bar{D}] \succ 0$ and $(\cdot)^{1/2}$ is operator monotone (see (Bhatia, 1996, Theorem X.1.1)). Using $\|\bar{D}^{-1/2}\|_2 \leq \sqrt{\frac{2}{p\lambda_{\min}}}$, $\|(\mathbb{E}[\bar{D}])^{-1/2}\|_2 \leq \sqrt{\frac{1}{p\lambda_{\min}}}$ and the bound from Lemma 19 in (5.15), we get

$$\|\tilde{Z}_2\|_2 \leq C_1^2(\eta, M) \frac{(np \log n)^{1/4}}{p\lambda_{\min}} \|\mathbb{E}[H]\|_2 + \frac{2\sqrt{2}C_1(\eta, M)\|\mathbb{E}[H]\|_2(np \log n)^{1/4}}{(p\lambda_{\min})^{3/2}}, \quad (5.16)$$

where $C_1(\eta, M) = 4A^{1/4}(\eta, M)$. From (5.14), (5.16), we get

$$\begin{aligned} \|\tilde{Z}\|_2 &\leq \frac{2\Delta}{p\lambda_{\min}} + \frac{C_1(\eta, M)\|\mathbb{E}[H]\|_2(np \log n)^{1/4}}{(p\lambda_{\min})^{3/2}} \left[\frac{C_1(\eta, M)(np \log n)^{1/4}}{\sqrt{p\lambda_{\min}}} + 2\sqrt{2} \right] \\ &\leq 16M \sqrt{\frac{5}{3}pn} \frac{(2+\varepsilon)}{p\lambda_{\min}} + \frac{C_1(\eta, M)(np \log n)^{1/4}\sigma}{(p\lambda_{\min})^{3/2}} \left(\frac{C_1(\eta, M)(np \log n)^{1/4}}{\sqrt{p\lambda_{\min}}} + 2\sqrt{2} \right) \\ &= \tilde{\Delta} \end{aligned}$$

where the second inequality follows using Lemma 12 and (5.12). Let $\widehat{U} = [\widehat{u}_1 \ \widehat{u}_2]$ denote the top two left singular vectors of $\bar{D}^{-1/2}H\bar{D}^{-1/2}$. Then from Wedin's bound, we know that

$$\tilde{\Delta} < \sigma \Rightarrow \|(I - \widehat{U}\widehat{U}^T)U\|_2 \leq \frac{\tilde{\Delta}}{\sigma - \tilde{\Delta}} = \delta. \quad (5.17)$$

Finally, Lemma 14 can be used here unchanged. Hence if $\delta \leq 1/2 \Leftrightarrow \tilde{\Delta} \leq \sigma/3$, then there exists $\beta \in \{-1, 1\}$ such that $\|\tilde{u}_2 - \beta u_2\|_2^2 \leq 10\delta$. Since $\tilde{\Delta} \leq \sigma_{\min}/3 \Leftrightarrow \delta \leq \frac{3\tilde{\Delta}}{2\sigma_{\min}} \Rightarrow \tilde{\Delta} \leq \sigma/3$, we obtain the stated bound on $\|\tilde{u}_2 - \beta u_2\|_2^2$. The lower bound on the success probability follows by applying the union bound to the events in Lemmas 12, 19.

5.6 Proof of Theorem 10

Recall that $\tilde{w} = \frac{\widehat{\sigma}_1}{\eta p \|\bar{D}^{-1/2}e\|_2} \bar{D}^{1/2} \tilde{u}_2$ and $\tilde{r} = \tilde{w} - \frac{e^T \tilde{w}}{n} e$. Let us denote

$$w = \frac{\sigma}{\eta p \|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2} (\mathbb{E}[\bar{D}])^{1/2} \beta u_2 = (r - \alpha e) \beta,$$

$$w' = w - \frac{e^T w}{n} e = \beta(r - (\alpha + \alpha')e),$$

where $\alpha' = \frac{e^T(r - \alpha e)}{n}$. Then using triangle inequality, we can bound $\|\tilde{r} - w'\|_2$ as

$$\|\tilde{r} - w'\|_2 \leq \|w - \tilde{w}\|_2 + \left| \frac{e^T(\tilde{w} - w)}{n} \right| \|e\|_2 \leq 2\|w - \tilde{w}\|_2.$$

In order to bound $\|w - \tilde{w}\|_2$, note that we can write $\tilde{w} - w = \frac{1}{\eta p}(a_1 + a_2 + a_3 + a_4)$ where

$$a_1 = \frac{\widehat{\sigma}_1}{\|\bar{D}^{-1/2}e\|_2} \bar{D}^{1/2} \tilde{u}_2 - \frac{\widehat{\sigma}_1}{\|\bar{D}^{-1/2}e\|_2} (\mathbb{E}[\bar{D}])^{1/2} \tilde{u}_2,$$

$$a_2 = \frac{\widehat{\sigma}_1}{\|\bar{D}^{-1/2}e\|_2} (\mathbb{E}[\bar{D}])^{1/2} \tilde{u}_2 - \frac{\widehat{\sigma}_1}{\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2} (\mathbb{E}[\bar{D}])^{1/2} \tilde{u}_2,$$

$$a_3 = \frac{\widehat{\sigma}_1}{\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2} (\mathbb{E}[\bar{D}])^{1/2} \tilde{u}_2 - \frac{\sigma}{\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2} (\mathbb{E}[\bar{D}])^{1/2} \tilde{u}_2,$$

$$a_4 = \frac{\sigma}{\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2} (\mathbb{E}[\bar{D}])^{1/2} \tilde{u}_2 - \frac{\sigma}{\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2} (\mathbb{E}[\bar{D}])^{1/2} \beta u_2.$$

Since $\|\tilde{w} - w\|_2 \leq \frac{1}{\eta p} \sum_{i=1}^4 \|a_i\|_2$ we will now bound $\|a_i\|_2$ for each i . Before proceeding, recall from the proof of Theorem 5.5 that $\min_i \bar{D}_{ii} \geq \frac{p\lambda_{\min}}{2}$. We also have

$$\max_i \bar{D}_{ii} \leq p\lambda_{\max} + \frac{p\lambda_{\min}}{2} \leq \frac{3p\lambda_{\max}}{2},$$

and $\widehat{\sigma}_1 \leq \sigma + \widetilde{\Delta}$ where the latter is due to Weyl's inequality.

Bounding $\|a_1\|_2$. Since $\sigma \leq \sigma_{\max}$, we have that

$$\|a_1\|_2 \leq \frac{\widetilde{\Delta} + \sigma_{\max}}{\|\bar{D}^{-1/2}e\|_2} \|\bar{D}^{1/2} - (\mathbb{E}[\bar{D}])^{1/2}\|_2 \leq \frac{\widetilde{\Delta} + \sigma_{\max}}{\|\bar{D}^{-1/2}e\|_2} \|\bar{D} - \mathbb{E}[\bar{D}]\|_2^{1/2} \quad (5.18)$$

where the second inequality is due to $\bar{D}, \mathbb{E}[\bar{D}] \succ 0$ and since $(\cdot)^{1/2}$ is operator monotone (see (Bhatia, 1996, Theorem X.1.1)). Using the bound $\|\bar{D}^{-1/2}e\|_2 \geq \sqrt{\frac{2n}{3p\lambda_{\max}}}$, along with the bound for $\|\bar{D} - \mathbb{E}[\bar{D}]\|_2$ (from Lemma 19) in (5.18), we arrive at

$$\|a_1\|_2 \leq \sqrt{\frac{3}{2n} p\lambda_{\max} (\sigma_{\max} + \widetilde{\Delta}) (2\sqrt{2} + 1)^{1/2}} (A(\eta, M) np \log n)^{1/4}.$$

Bounding $\|a_2\|_2$. We begin by noting that

$$\|a_2\|_2 = \widehat{\sigma}_1 \left| \frac{1}{\|\bar{D}^{-1/2}e\|_2} - \frac{1}{\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2} \right| \|(\mathbb{E}[\bar{D}])^{1/2}\tilde{u}_2\|_2 \leq (\sigma_{\max} + \tilde{\Delta}) \left| \frac{\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2 - \|\bar{D}^{-1/2}e\|_2}{\|\bar{D}^{-1/2}e\|_2 \|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2} \right|.$$

Since $\|\bar{D}^{-1/2}e\|_2 \geq \sqrt{\frac{2n}{3p\lambda_{\max}}}$, $\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2 \geq \frac{\sqrt{n}}{\sqrt{p\lambda_{\max}}}$, and

$$\begin{aligned} \left| \|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2 - \|\bar{D}^{-1/2}e\|_2 \right| &\leq \|(\bar{D}^{-1/2} - (\mathbb{E}[\bar{D}])^{-1/2})e\|_2 \\ &\leq \|\bar{D}^{-1/2}\|_2 \|\bar{D}^{1/2} - (\mathbb{E}[\bar{D}])^{1/2}\|_2 \|(\mathbb{E}[\bar{D}])^{-1/2}\|_2 \sqrt{n} \\ &\leq \frac{\sqrt{2}}{p\lambda_{\min}} (2\sqrt{2} + 1)^{1/2} (A(\eta, M)np \log n)^{1/4}, \end{aligned}$$

we can bound $\|a_2\|_2$ as

$$\|a_2\|_2 \leq \sqrt{\frac{3}{n}} (\sigma_{\max} + \tilde{\Delta}) \frac{\lambda_{\max}}{\lambda_{\min}} (2\sqrt{2} + 1)^{1/2} (A(\eta, M)np \log n)^{1/4}.$$

Bounding $\|a_3\|_2$. This is easily achieved by noting that

$$\|a_3\|_2 = \frac{|\widehat{\sigma}_1 - \sigma|}{\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2} \|(\mathbb{E}[\bar{D}])^{1/2}\tilde{u}_2\|_2 \leq \frac{\tilde{\Delta}\sqrt{p\lambda_{\max}}}{\sqrt{n}/\sqrt{p\lambda_{\max}}} = \frac{\tilde{\Delta}p\lambda_{\max}}{\sqrt{n}}.$$

Bounding $\|a_4\|_2$. This is also easily achieved by noting that

$$\begin{aligned} \|a_4\|_2 &\leq \frac{\sigma}{\|(\mathbb{E}[\bar{D}])^{-1/2}e\|_2} \|(\mathbb{E}[\bar{D}])^{1/2}\|_2 \|\tilde{u}_2 - \beta u_2\|_2 \\ &\leq \frac{\sigma_{\max}\sqrt{p\lambda_{\max}}}{\sqrt{n}/\sqrt{p\lambda_{\max}}} \left(\sqrt{15} \sqrt{\frac{\tilde{\Delta}}{\sigma_{\min}}} \right) \\ &\leq \sqrt{\frac{15\tilde{\Delta}}{\sigma_{\min}n}} \sigma_{\max} p \lambda_{\max}. \end{aligned}$$

The stated bound now follows from $\|\tilde{r} - \beta(r - (\alpha + \alpha')e)\|_2 \leq \frac{2}{\eta p} \sum_{i=1}^4 \|a_i\|_2$.

6. Matrix completion as a preprocessing step for ranking

Low-rank matrix completion is the problem of recovering the missing entries of a low-rank matrix given a subset of its entries. This line of research started with the results in Candès and Recht (2009) and Candès and Tao (2010) which showed that given a rank- r matrix C of size $n_1 \times n_2$ (with $n = \max\{n_1, n_2\}$), one can recover it by observing only $O(nr \text{ polylog}(n))$ randomly selected entries (under some assumptions on C) via a simple convex optimization algorithm. This was partly inspired by similar approaches used previously in the compressed sensing literature (Candes et al., 2006; Candès and Tao, 2006). This problem has received tremendous attention in the last decade, due to a number of applications such as in phase

retrieval (Candès et al., 2013), computer vision (Tomasi and Kanade, 1992; Özyeşil et al., 2017) and sensor network localization (Cucuringu et al., 2012a) to name a few.

More formally, let us assume for simplicity that $r = O(1)$ and the SVD of $C = \sum_{i \in [r]} \sigma_i u_i v_i^T$ satisfies

$$\|u_i\|_\infty \leq \sqrt{\frac{\mu}{n_1}}, \quad \|v_i\|_\infty \leq \sqrt{\frac{\mu}{n_2}}; \quad \forall i, j, \quad (6.1)$$

for some $\mu \geq 1$. Here, μ is a coherence parameter which measures how spread out the entries of u_i, v_i are - the smaller the value of μ , the better. Say we observe m entries of C on a subset $\Omega \subset [n_1] \times [n_2]$, sampled uniformly at random. Denoting $P_\Omega : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}^{n_1 \times n_2}$ to be the projection operator on Ω , it was shown in Candes and Tao (2010) that the solution \widehat{C} of

$$\min \|X\|_* \quad \text{s.t.} \quad P_\Omega(X) = P_\Omega(C) \quad (6.2)$$

equals C with high probability, provided $m = \Omega(\mu^4 n \log^2 n)$. The nuclear norm minimization problem (6.2) is a SDP and hence can be solved in polynomial time using, for eg., interior point methods. In fact, the result holds under a different Bernoulli sampling model too, wherein each entry of M is observed independently with a certain probability (Candès and Recht, 2009, Section 4.1). Moreover, say that the observations are noisy, i.e, we observe

$$Y_{i,j} = C_{i,j} + Z_{i,j}; \quad (i, j) \in \Omega \quad \Leftrightarrow \quad P_\Omega(Y) = P_\Omega(C) + P_\Omega(Z),$$

where Z is the noise matrix. Say $\|P_\Omega(Z)\|_F \leq \delta$. Then, it was shown in Candes and Plan (2010) (under certain additional conditions) that the solution of

$$\min \|X\|_* \quad \text{s.t.} \quad \|P_\Omega(X) - P_\Omega(Y)\|_F \leq \delta, \quad (6.3)$$

is stable, i.e., the estimation error $\|\widehat{C} - C\|_F$ is bounded by a term proportional to δ .

In our setting, we observe (noisy versions of) a subset of the entries of the $n \times n$ matrix $C = re^T - er^T$ where each off-diagonal entry of C is revealed with probability p . Since C has rank 2, it is natural to consider estimating C via matrix completion as a preprocessing step, and then subsequently applying SVD-RS or SVD-NRS on the obtained estimate of C for recovering the underlying ranks and scores. In order to understand the sample complexity for successful matrix completion, we need to express μ defined in (6.1) in terms of the score vector r . To this end, we see from Lemma 11 that

$$\|u_2\|_\infty, \|v_1\|_\infty = \frac{\|r - \alpha e\|_\infty}{\|r - \alpha e\|_2} \leq \frac{M - \alpha}{\|r - \alpha e\|_2},$$

and $\|u_1\|_\infty, \|v_2\|_\infty = 1/\sqrt{n}$. Hence it follows that

$$\mu = \max \left\{ \frac{(M - \alpha)\sqrt{n}}{\|r - \alpha e\|_2}, 1 \right\}.$$

In order to get a sense of the scaling of μ , consider the setting $r_i = i$. As seen before, we then have $\|r - \alpha e\|_2 = \Theta(n^{3/2})$, $M = n$, and $\alpha = \Theta(n)$. This in turn implies that $\mu = \Theta(1)$ which is the ideal scenario for matrix completion. In our numerical experiments,

we rely on the TFOCS software library (Becker et al.), that allows for construction of first-order methods for a variety of convex optimization problems (Becker et al., 2011). In our implementation, we also set the diagonal entries of X to be equal to zero since the same is true for C . We do not enforce the skew-symmetry constraints $X_{ij} = -X_{ji}$ for $(i, j) \in \Omega$, so the solution \widehat{C} to C is not guaranteed to be skew-symmetric. Instead, we output $\frac{\widehat{C} - \widehat{C}^T}{2}$ as the final (skew-symmetric) estimate obtained from the preprocessing step. Note that since $C^T = -C$, we have

$$\left\| \frac{\widehat{C} - \widehat{C}^T}{2} - C \right\|_F = \left\| \frac{\widehat{C} - \widehat{C}^T - C + C^T}{2} \right\|_F \leq \|C - \widehat{C}\|_F.$$

Remark 20 Note that in the setting of matrix completion, we perform the scaling recovery procedure outlined in Section 3.2 as follows. We build the matrix Π from equation (3.2) by only consider the entries/edges $\{i, j\} \in E$ from the original measurement graph G , and do not include the entries filled in during the matrix completion step.

Finally, we end with a motivating discussion for the applicability of the matrix completion approach in the setting of ranking from pairwise comparisons. The recent work of Udell and Townsend (2019) provides a compelling argument on why big data matrices are approximately low-rank. The authors consider a simple generative model for matrices, assuming that each row or column is associated to a (possibly high dimensional) bounded latent variable, with the individual matrix entries being generated by applying a piecewise analytic function to these latent variables. While the initial resulting matrices are typically full rank, the authors show that one can approximate every entry of an $m \times n$ matrix drawn from the above model to within a fixed absolute error by a low-rank matrix whose rank grows as $O(\log(m+n))$. In other words, any sufficiently large matrix from such a latent variable model can be approximated by a low-rank matrix (up to a small entrywise error). The paradigm that “nice latent variables models are of log-rank” (Udell and Townsend, 2019) is also applicable in the ranking setting, where one assumes that the final ranking (or skill) of each player varies smoothly as a function of covariate information, which is typically available in many real word applications. For example, in sport or clinical data, covariates may provide additional information about both the ranked players and the rankers/judges, which can overall lead to better aggregated results.

7. Numerical Experiments

We compare the performance of Algorithm 1 SVD-RS (SVD in the figure legends for brevity) and Algorithm 2 SVD-NRS (SVD-N in the figure legends) with that of seven other algorithms from the literature, namely RowSUM RANKING (RSUM) (Gleich and Lim, 2011), LEAST-SQUARES-RANK (LS) as considered in Cucuringu (2016), SERIAL-RANK (SER) (Fogel et al., 2016), SPRING-RANK (SPR) (De Bacco et al., 2018), BRADLEY–TERRY (BTL) (Bradley and Terry, 1952), PAGE-RANK (PGR) (Page et al., 1998), and SYNC-RANK (SYNC) (Cucuringu, 2016). We refer the reader to Section 2.2 for a brief survey of the ranking literature, including the above algorithms. We compare the performance of all algorithms on synthetic data in Section 7.1, and on real data in Section 7.2. We

consider a variety of performance metrics summarized further below, altogether highlighting the competitiveness of our proposed SVD-based algorithms with that of state-of-the-art methods.

7.1 Synthetic data

This section details the outcomes of synthetic numerical experiments, under the ERO model, where the measurement graph is Erdős-Rényi and the noise comes in the form of outliers from a uniform distribution.

We consider two sets of experiments, a first one where the strengths r_i are uniformly distributed in $[0, 1]$, and a second one where they are Gamma distributed with shape parameter $a = 0.5$ and scale parameter $b = 1$. This choice of parameters for the Gamma distribution leads to a skewed distribution of player strengths, and subsequently a skewed distribution of node degrees, which is a setting realistic in practice but known to be challenging for spectral methods. We fix the number of nodes (n), and vary the edge density (p), and noise level (γ) as follows.

We broadly consider two main experimental settings.

- In Figure 2 (uniform scores) and Figure 3 (Gamma scores) we consider a synthetic model with $n = 1000$, with sparsity parameters $p = 0.05$ (column 1) and $p = 1$ (column 3). For the $p = 0.05$ scenario, we also show the results after running all algorithms on top of a reprocessing step that applies low-rank matrix completion (column 2).
- In Figure 4 (uniform scores) and Figure 5 (Gamma scores), we consider a synthetic model with $n = 3000$, and sparsity parameter $p \in \{0.01, 0.05, 0.1\}$, indexing the columns.

Across all the above experiments, we consider three different performance metrics, as we vary the noise level γ on the x-axis. Whenever ground truth is available, we plot in the top row of Figures 2, 3, 4, 5 the *Kendall Distance* between the recovered strength vector \hat{r} and the ground truth r , for different noise levels γ . The Kendall distance counts the number of pairs of candidates that are ranked in different order (flips), in the original ranking versus the recovered one. The middle row of each Figure plots the *Correlation Score*, computed as the Pearson correlation between the ground truth r and the recovered \hat{r} . Finally, the bottom row in each Figure plots the *RMSE* error, defined as $\sqrt{\frac{1}{n} \|r - \hat{r}\|_2^2}$, after having centered r and \hat{r} . Note that the low-rank matrix completion is illustrated in the middle column of Figures 2 and 3.

Performance comparison. For the remainder of this section, we compare and contrast the performance of our algorithms with that of the other methods, across four synthetic data sets, detailed below.

- In Figure 2 (uniform scores with $n = 1000$): in the setting $p = 0.05$, our methods perform better than SER but are in general outperformed by the other methods in terms of Kendall Distance (KD) and Correlation Score (CS) (except at very low levels of noise), and they perform on par with all the methods in terms of RMSE (in particular for $\gamma > 0.30$ our methods outperform SER, PGR, BTL, and SPR). After the matrix completion step, both SVD and SVD-N outperform all other methods except LS and RSUM in terms of

RMSE, to which they are comparable (visually indistinguishable) up until $\gamma < 0.50$, and only slightly outperformed for higher noise. In terms of KD and CS, our two SVD-based algorithms perform on par with most other methods, and clearly outperform both SER and PGR. Finally, for the complete graph case $p = 1$, our methods perform at the top of the rankings in terms of KD, CS, and RMSE, and are only matched in performance or slightly outperformed depending on the noise regime, by SYNC, LS, and RSUM.

- In Figure 3 (Gamma scores with $n = 1000$): for the very sparse regime SYNC clearly outperforms all other methods, with the SVD-based methods outperforming only SER. After the matrix completion step, our proposed algorithms outperform all other methods except LS and RSUM. For the complete graph case, SYNC is by far the winner in terms of KD, with the remaining methods performing comparably, except PGR. In terms of RMSE, our methods perform best along with LS and RSUM, and clearly outperform SYNC, SPR, PGR, BTL and SER.
- In Figure 4 (uniform scores with $n = 3000$): for $p = 0.01$ there is very clear ordering of the methods, mostly consistent across all three performance metrics. We observe that SYNC, SPR, RSUM, and LS, are the top three best performing methods, followed by BTL, PGR, SVD, SVD-N, and SER. At slightly higher edge density $p = 0.05$, our SVD methods perform better than SER, PGR, and BTL, for the first part of the noise spectrum, while in terms of RMSE, SVD and SVD-N clearly outperform SER, PGR, BTL, SPR, and SYNC, and are surpassed only by LS and RSUM. The relative performance is similar for $p = 0.10$.
- Finally, we comment on the results from Figure 5 (Gamma scores with $n = 3000$): for $p = 0.01$, SYNC is the clear winner in terms of KD, CS, and RMSE, while SER, SVD and SVD-N are at the bottom of the ranking. The relative ordering is roughly preserved for higher edge densities, with the comment that SER is clearly the worst performer, while SYNC is by far the most accurate method, especially in terms of KD, where it outperforms all other methods by a large margin.

7.2 Real data

We also apply all ranking algorithms to a variety of real world networks, and measure the performance of the outcome by the number of upsets in the final ranking, computed with respect to the given pairwise measurements. To this end, we consider two types of upsets, as detailed below. Let \hat{r}_i denote the estimated score of item i as computed by the method of choice. Note that higher values of \hat{r}_i correspond to higher ranks, meaning stronger players or more preferred items. Next, we construct the induced (possibly incomplete) matrix of estimated/denoised pairwise rank-offsets

$$\hat{R}_{ij} = \begin{cases} \hat{r}_i - \hat{r}_j; & \text{if } \{i, j\} \in E \\ 0; & \text{if } (i, j) \notin E, \end{cases}$$

with $\hat{R}_{ij} > 0$ denoting that the rank of player i is higher than the rank of player j . To measure the accuracy of a proposed reconstruction, we rely on the following two metrics.

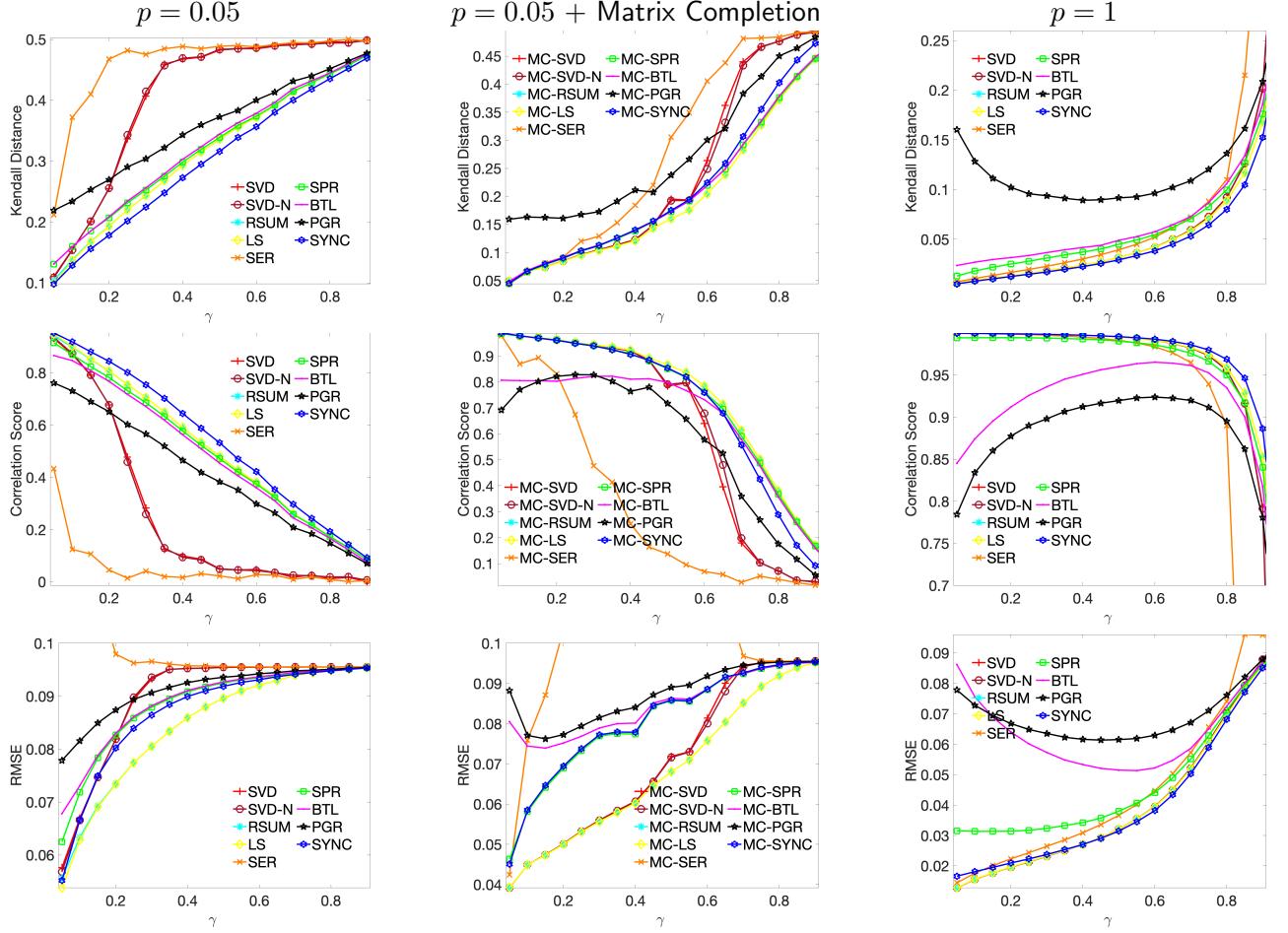


Figure 2: Performance statistics in terms of Kendall distance (top row; lower is better), correlation score (middle row; higher is better), and RMSE (bottom row; lower is better), for synthetic data with scores generated from a **Uniform** distribution with $n = 1000$. The first and third columns pertain to the case of sparsity $p = 0.05$, respectively $p = 1$, without the matrix completion step, while the middle column, for $p = 0.05$, employs a matrix-completion preprocessing step. Results are averaged over 20 runs.

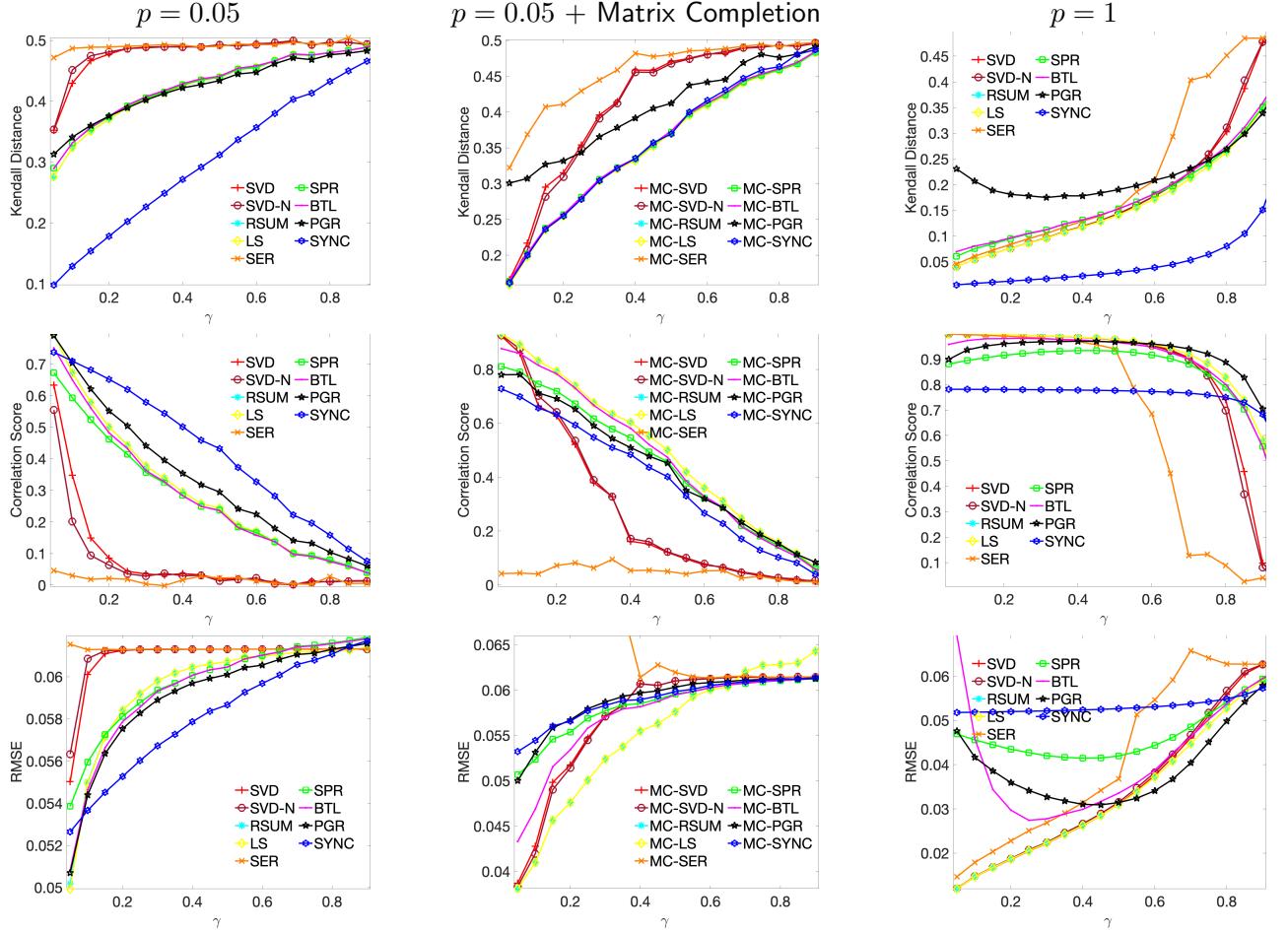


Figure 3: Performance statistics in terms of Kendall distance (top row), correlation score (middle row), and RMSE (bottom row), for synthetic data with scores generated from a **Gamma** distribution with $n = 1000$. The first and third columns pertain to the case of sparsity $p = 0.05$, respectively $p = 1$ without the matrix completion step, while the middle column, for $p = 0.05$, employs a matrix-completion preprocessing step. Results are averaged over 20 runs.

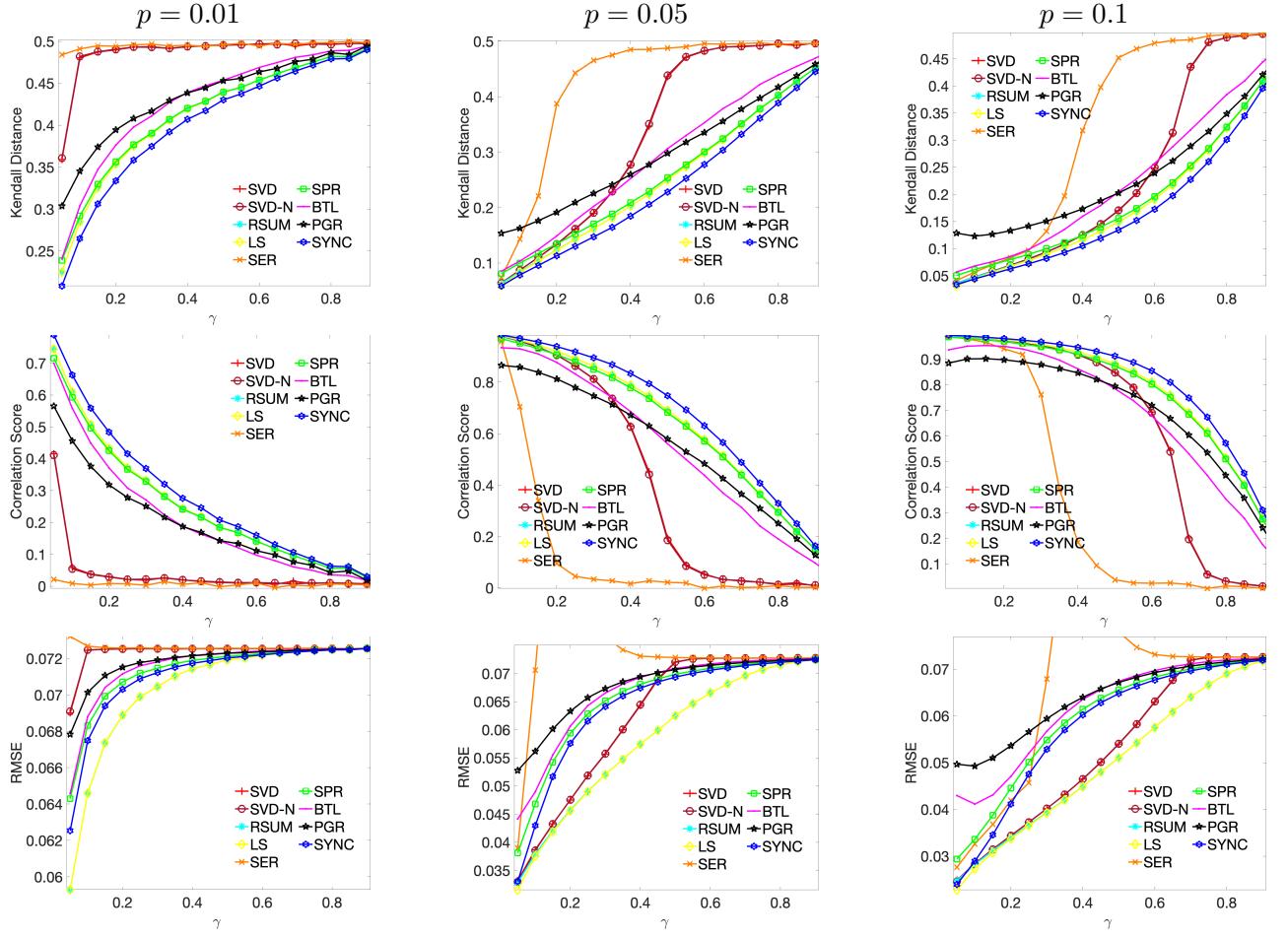


Figure 4: Performance statistics in terms of Kendall distance (top row), correlation score (middle row), and RMSE (bottom row), for synthetic data with scores generated by a **Uniform** distribution with $n = 3000$, and sparsity $p \in \{0.01, 0.05, 0.1\}$, without the matrix completion preprocessing step. Note that $\log(3000)/3000 = 0.00267$. Results are averaged over 20 runs.

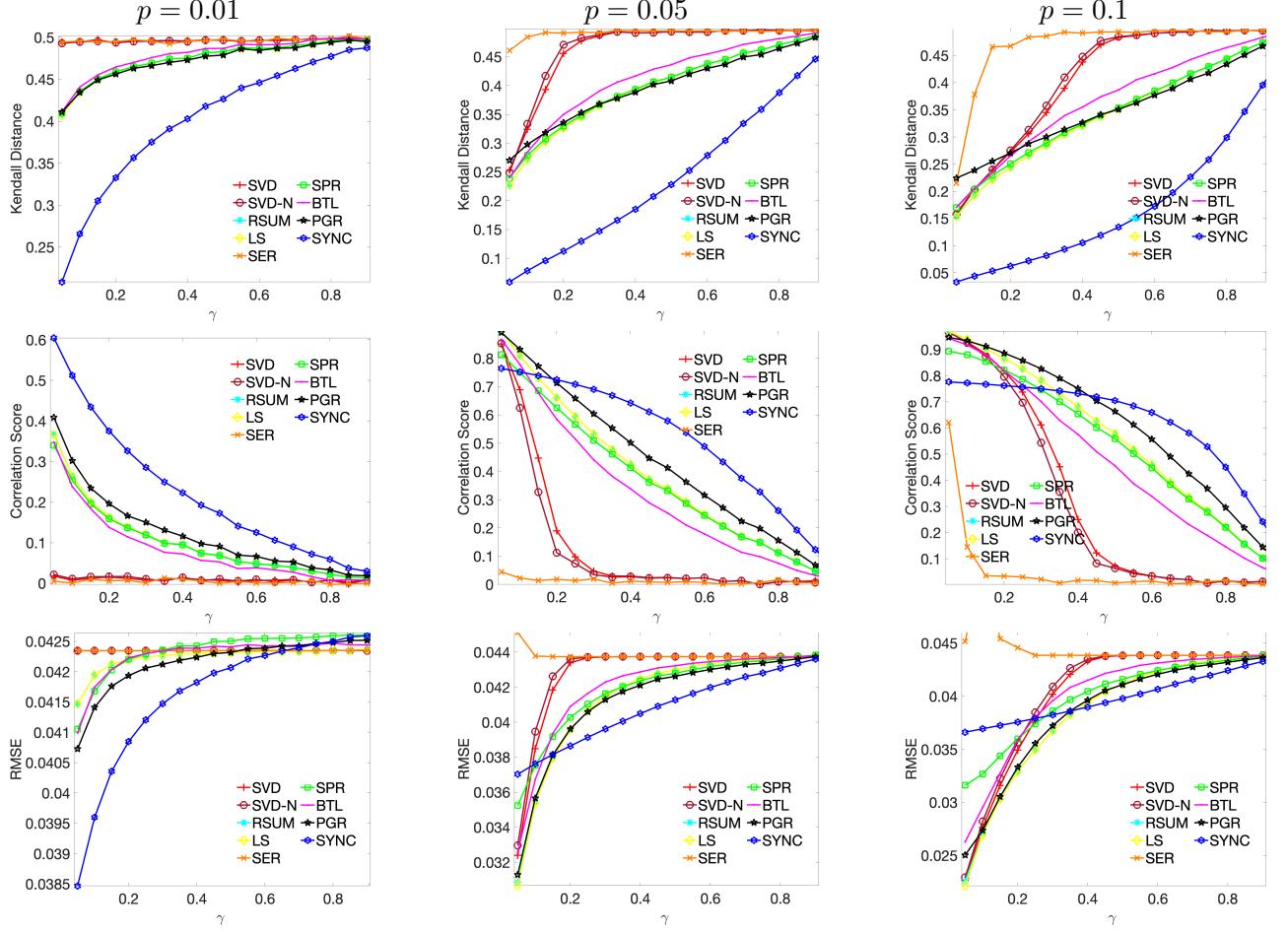


Figure 5: Performance statistics in terms of Kendall distance (top row), correlation score (middle row), and RMSE (bottom row), for synthetic data with scores generated by a **Gamma** distribution with $n = 3000$, and sparsity $p \in \{0.01, 0.05, 0.1\}$, without the matrix completion preprocessing step. Note that $\log(3000)/3000 = 0.00267$. Results are averaged over 20 runs.

First, we use the popular metric

$$\text{Number of upsets} = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \mathbf{1}_{\{\text{sign}(R_{ij}\hat{R}_{ij})=-1\}}, \quad (7.1)$$

which counts the number of *upsets* (lower is better), i.e., the number of disagreeing ordered comparisons. It contributes a $+1$ to the summation whenever the ordering in the provided data contradicts the ordering in the induced ranking. Next, we define a weighted version of the upset criterion (lower is better) between the given rank comparison data, and the one induced by the recovered solution

$$\text{Weighted upsets} = \sum_{i=1}^{n-1} \sum_{j=i+1}^n |R_{ij} - \hat{R}_{ij}|. \quad (7.2)$$

For ease of visualization and comparison, in all the barplot figures shown in this section, we color in red (resp. blue) the outcomes attained by SVD (resp. SVD-N). For brevity, for each real data set, we primarily comment on the performance of the two SVD-based methods relative to the other seven methods. We compare performance both before and after the matrix completion step. In each scenario, we compute the above two performance metrics, hence the four columns in the five Figures 6, 8, 9, 10, 11 corresponding to the five real data sets we consider.

NCAA College Basketball. Our first and most comprehensive real data set comes from sports, and contains the outcomes of NCAA College Basketball matches during the regular season, for the years 1985 - 2014. Each separate season provided a pairwise comparison matrix on which we evaluate all algorithms. This data set can essentially be construed as 30 separate problem instances. The input matrix corresponding to each season contains the point difference of the direct match between a pair of teams. If the same pair of teams played multiple matches against each other, the point differences are added up. The resulting pairwise comparison matrix is thus skew-symmetric by construction.

The experimental results shown in the left column of Figure 6 pertain to the setting without the matrix completion step, while the right column compares the outcome of all methods after the low-rank matrix completion pre-processing step. The first row shows performance in terms of the number of upsets across time for each season, while the barplot in the second row shows the average attained by all methods across all seasons. Before matrix completion, SVD-N and SVD are ranked 6th, respectively 8th, while after matrix completion, they are ranked 6th, respectively 7th, out of the 9 algorithms considered. The third and fourth row depict similar results, but the performance metric used is the weighted upsets, which is shown on a log scale for ease of visualization, primarily due to the poor performance of BTL, which would otherwise distort the y -axis. In this case, SVD-N and SVD are ranked 5th, respectively 6th, while after matrix completion, they are ranked 4th, respectively 5th, out of 9 methods, visibly outperforming the rest of the methods, and attaining a comparable performance to that of the top three methods.

The heatmaps shown in the top row of Figure 7 contain the pairwise correlations between the rankings attained by all methods over the 30 seasons. In other words, for any pair of methods, we compute the correlation between the respective rankings attained in each

season, which we then average over the entire set of 30 seasons. To the best of our knowledge, such a correlation study has not been performed in the literature, and provides insights on the potentially different latent rankings inherent in each data set, while still minimizing the number of upsets. SYNC, the best performing method in terms of the number of upsets, has a comparable performance to BTL, and a correlation of 86%. PGR and SVD-N attain a comparable number of upsets, but their Kendall score correlation is only 70%. The bottom barplot of the same Figure 7 show the average correlation of each of the methods, highlighting that SPR, LS and BTL yield the most correlated rankings with the other methods. A similar correlation analysis is performed on all rankings algorithms after the low-rank matrix completion step, which reveals a number of clusters. As expected, SVD and SVD-N produce very related rankings to each others, and similarly for RSUM and LS, and finally, SPR and BTL.

Animal dominance network. Our second real world example is a set of two networks of animal dominance among captive monk parakeets (Hobson and DeDeo, 2015). The input matrix H is skew-symmetric, with H_{ij} denoting the number of net aggressive wins of animal i toward animal j . The study covers two groups (of 21, respectively, 19 individuals), and spans across four quarters. We follow the same approach as in De Bacco et al. (2018), and only tested the algorithms on Quarters 3 and 4, based on guidance from the behavioral ecologist who collected and analyzed the data. Since the animals did not know each other well at the start of the experiment, the hierarchies were still in the transient process of forming. Figure 8 compares performance in terms of the number of upsets and weighted upsets, with and without the matrix completion step. Overall, we observe that SVD-N outperforms SVD, and the two methods typically rank halfway in the performance ranking, except in the leftmost column, where SVD performs worst.

Faculty hiring networks. Our next example covers three North American academic hiring networks, that track the flow of academics between universities (Clauset et al., 2015). The flow is captured in a directed graph with adjacency matrix A , such that A_{ij} is the number of faculty at university j who received their doctorate from university i . We then consider the skew symmetric matrix $H = A - A^T$, capturing the net flow of faculty between a pair of institutions. Figure 9 shows the number of upsets attained by each algorithm, for three different disciplines: (a) Computer Science ($n = 206$), (b) Business ($n = 113$) and (c) History ($n = 145$), before and after the matrix completion step, for both upsets criteria. Again, we observe that SVD-N outperforms SVD. Furthermore, SVD-N typically ranks in the top half of the rankings, and for the case of weighted upsets after the matrix completion step, it is the best performer for the Business and History fields, and second best in Computer Science.

Microsoft Halo 2 Tournament. Our third experiment was performed on a real data set of game outcomes collected during the Beta testing period for the Xbox game⁴ Halo 2. The graph has a total of $n = 606$ players, and 6227 head-to-head games. After removing the low-degree nodes, i.e., players who have played less than 3 games, we arrive at a graph with $n = 535$ nodes and 6109 edges, with an average degree of roughly 23. The skew-symmetric pairwise comparison matrix H captures the net number of wins of player i over

⁴Credits for using the Halo 2 Beta data set are given to Microsoft Research Ltd. and Bungie.

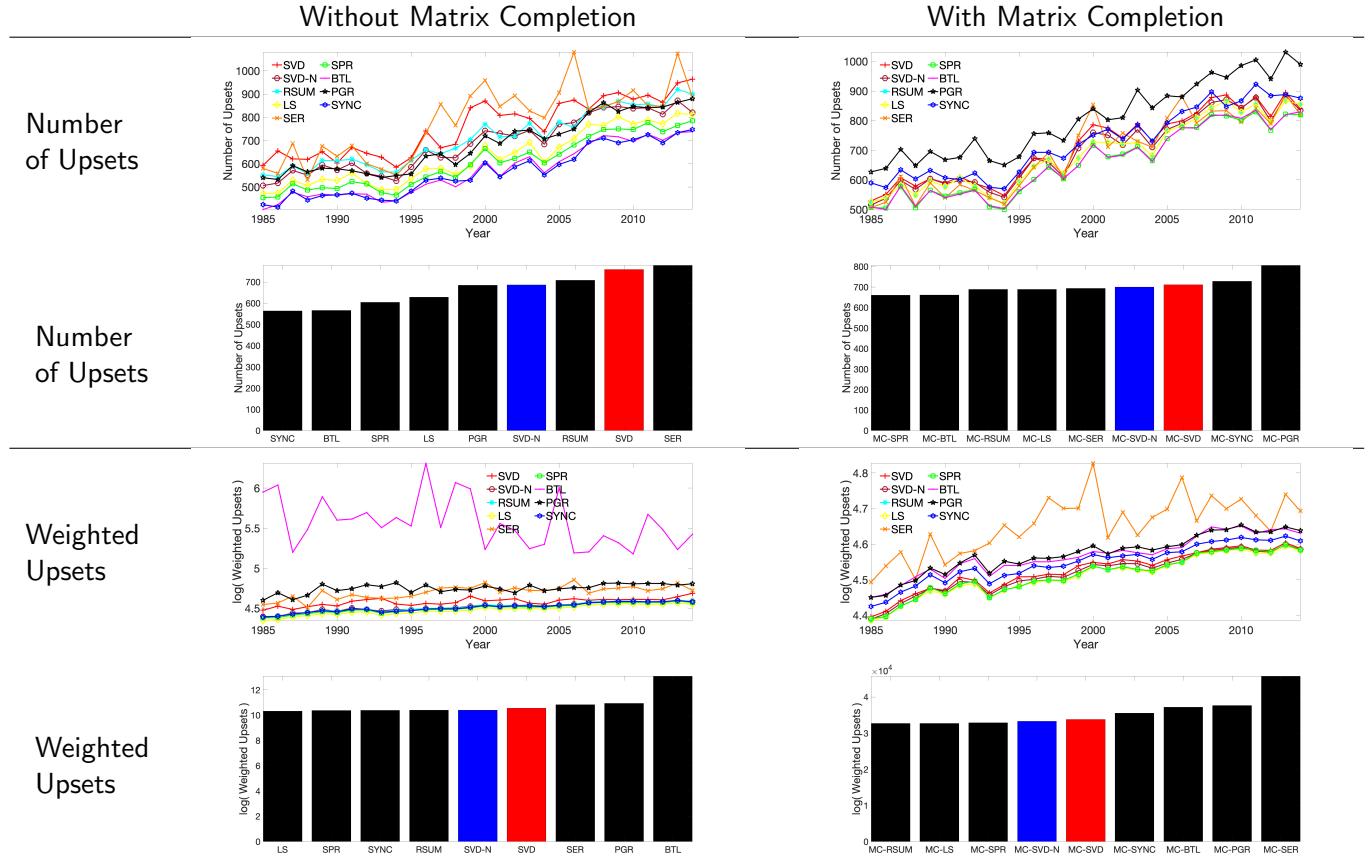


Figure 6: Comparison of the algorithms for the NCAA College Basketball data set (1984 - 2014), before (left column) and after (right column) the matrix completion step. We analyze 30 measurement graph instances, one for each season, and estimate rankings for each problem instance, for all the algorithms we considered. The first row shows the number of upsets for each season, and the second row displays the averages across all the seasons. Similar plots are shown in the third and fourth row, but this time we compare performance in terms of weighted upsets. The heatmaps in the fifth row show the average (across seasons) correlation between the rankings estimated by each pair of methods, with the average correlation degree depicted in the bottom row.

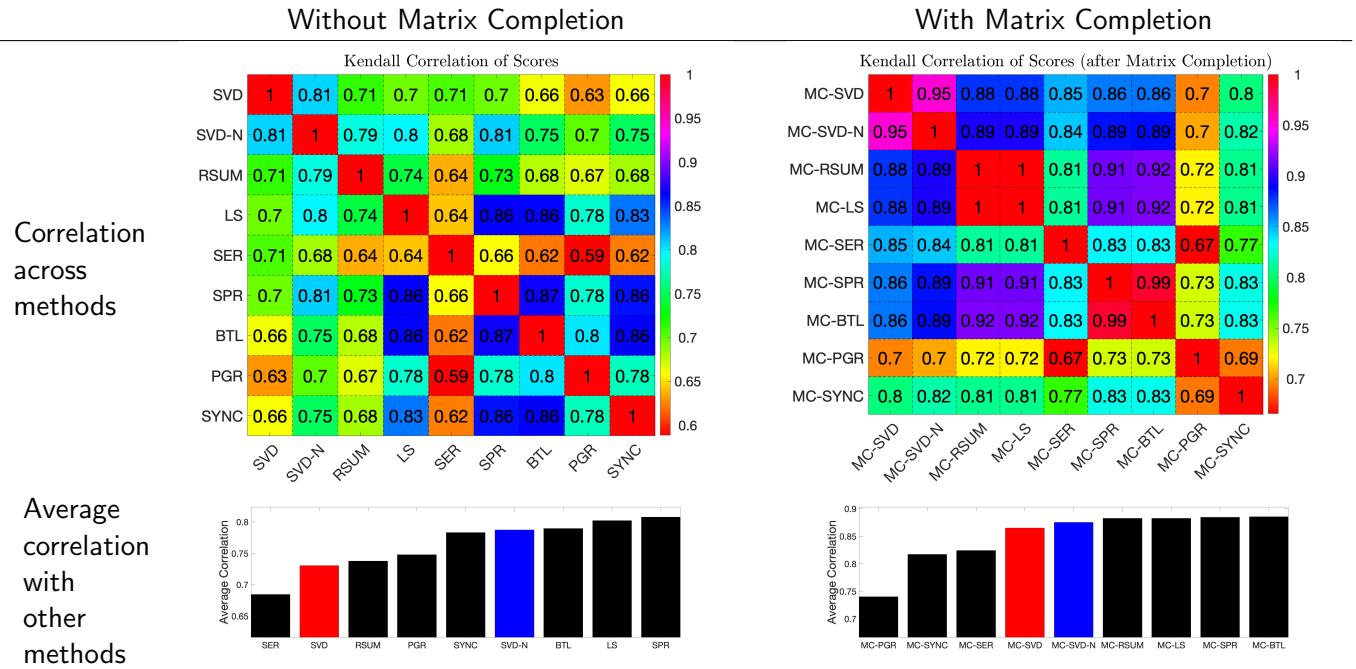


Figure 7: Comparison of the algorithms for the NCAA College Basketball data set (1984 - 2014), before (left column) and after (right column) the matrix completion step. The heatmaps in the first row show the average (across seasons) correlation between the rankings estimated by each pair of methods, with the average correlation degree depicted in the bottom row.

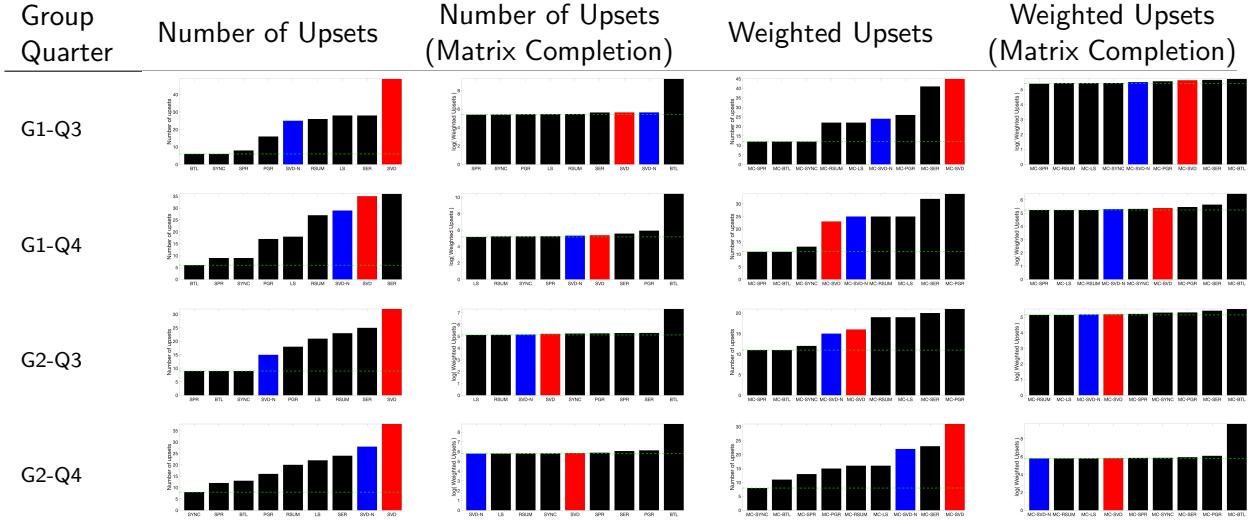


Figure 8: Performance comparison in the animal dominance networks, for two groups in Quarters 3-4, thus a total of four networks altogether. We compare both in terms of the number of upsets (first two columns) and weighted upsets (last two columns). The results without the matrix completion step pertain to the first and third columns, while columns two and four show results obtained after a low-rank matrix completion preprocessing step.

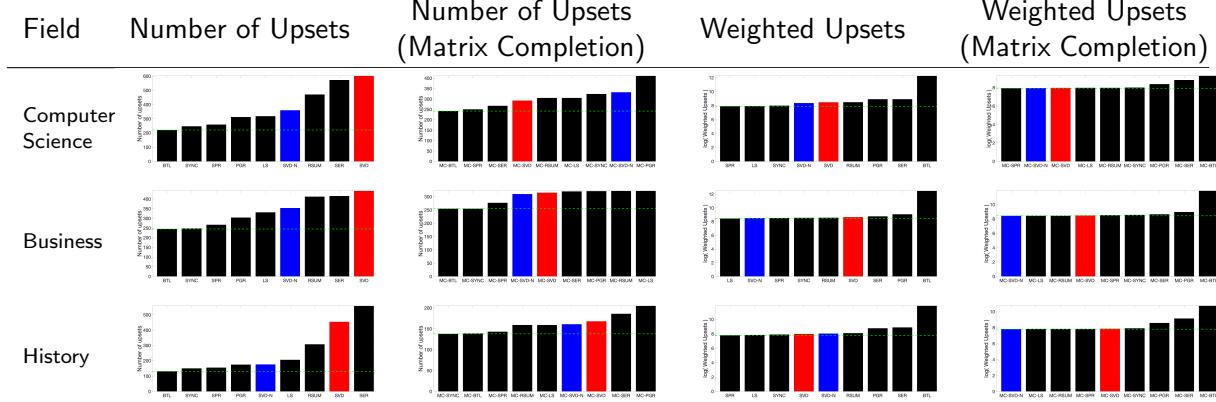


Figure 9: Performance comparison in the faculty hiring networks, for three networks corresponding to different fields. We compare both in terms of the number of upsets (first two columns) and weighted upsets (last two columns). The results without the matrix completion step pertain to the first and third columns, while columns two and four show results obtained after a low-rank matrix completion preprocessing step.

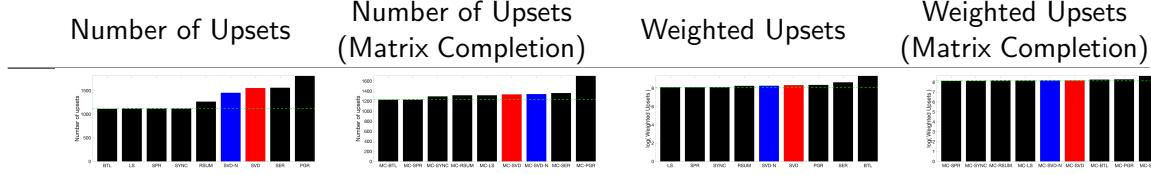


Figure 10: Recovery results for the Microsoft Halo Tournament data set, both in terms of the number of upsets (first two columns) and weighted upsets (last two columns). Columns one and three show results without the matrix completion step, while columns two and four show results obtained after a low-rank matrix completion preprocessing step.

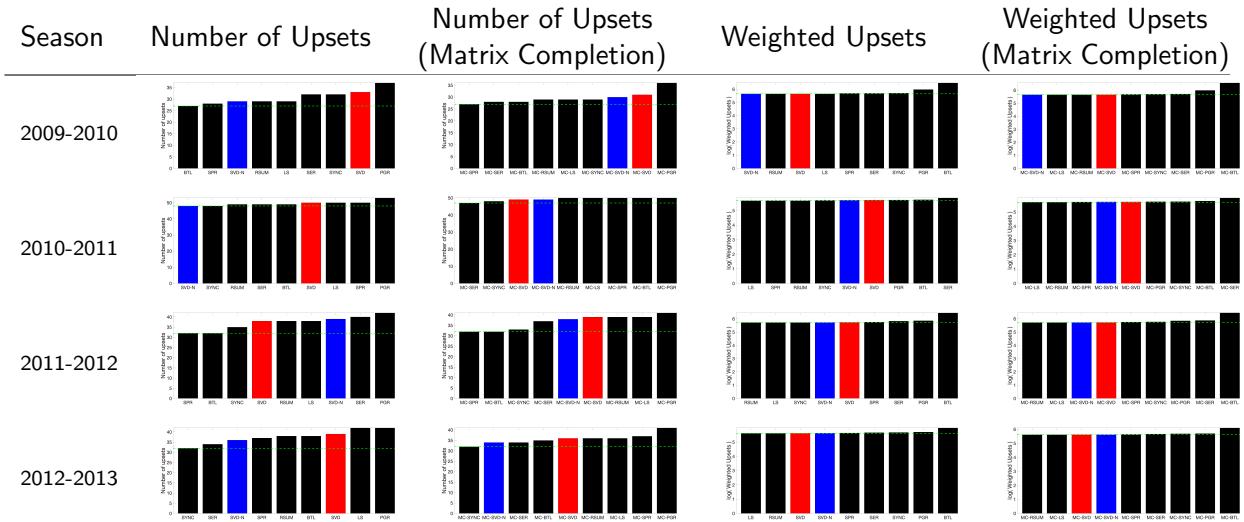


Figure 11: Performance comparison for the Premier League data set, for four seasons during 2009-2013. We compare both in terms of the number of upsets (first two columns) and weighted upsets (last two columns). Recovery performance after the matrix completion step is shown in columns two and four, while columns one and three do not rely on matrix completion.

player j . In this example illustrated in Figure 10, SVD-N outperforms SVD in three out of four instances, and the two methods are always ranked in the interval 5-7 out of the 9 algorithms considered.

Premier League. Our final real example is a Premier League data set, with four seasons (2009-2013), shown in Figure 11. The skew-symmetric pairwise comparison matrix H captures the net goal difference accrued in the two matches each pair of teams played against each other (home and away games). Across all the experimental setups considered and the different performance metrics, SVD-N outperforms SVD in 12 out of 16 rankings shown in Figure 11, and the two methods typically score in the top half best performing methods. Furthermore, SVD-N comes first in three instances. Note that Tsokos et al. (2019) also considers a subset of the same ranking methods in the context of football matches.

8. Conclusion and future directions

This paper considered the problems of ranking and time synchronization given a subset of noisy pairwise comparisons, and proposed an SVD-based algorithmic pipeline to solve both tasks.

We analyzed the robustness of SVD-RS in the form of ℓ_2 and ℓ_∞ recovery guarantees for the score vector r , against sampling sparsity and noise perturbation, using tools from matrix perturbation and random matrix theory. The ℓ_∞ analysis of SVD-RS leads to guarantees for rank recovery in terms of the maximum displacement error with respect to the ground truth. We also introduced SVD-NRS, a normalized version of SVD-RS, and provided ℓ_2 recovery guarantees for the score vector r .

We have augmented our theoretical analysis with a comprehensive set of numerical experiments on both synthetic and real data (five different real data sets, which altogether contained 42 distinct problem instances/comparison graphs), showcasing the competitiveness of our approach when compared to other seven algorithms from the literature. In particular, SVD-NRS was shown to perform particularly well on many of the real data sets, often ranking in the top three algorithms in terms of performance, out of the nine algorithms considered.

There are several avenues for future work. An interesting direction pertains to extending our analysis to the setting of very sparse graphs, with p on the order of $\frac{1}{n}$, by leveraging recent regularization techniques as introduced in Joseph and Yu (2016) and Le et al. (2015). Analysis of other, perhaps more realistic, noise models would also be of interest (such as the multiplicative noise models), as well as obtaining theoretical guarantees for the pipeline with low-rank matrix completion as a pre-processing step. Extending this work to the problem of ranking and synchronization in the heterogeneous setting, as recently explored in Cucuringu and Tyagi (2020) is another interesting direction, well motivated by real-world applications.

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Appendix A. Matrix perturbation analysis

Let $A, \tilde{A} \in \mathbb{C}^{m \times n}$ where we assume $m \geq n$ w.l.o.g. Let

$$\sigma_1(A) \geq \dots \geq \sigma_n(A), \quad \sigma_1(\tilde{A}) \geq \dots \geq \sigma_n(\tilde{A})$$

denote the singular values of A and \tilde{A} respectively, and denote $E = \tilde{A} - A$.

To begin with, we would like to quantify the perturbation of the singular values of \tilde{A} with respect to those of A . Weyl's inequality (Weyl, 1912) is a very useful result in this regard.

Theorem 21 (Weyl's inequality (Weyl, 1912)) *It holds that*

$$|\sigma_i(\tilde{A}) - \sigma_i(A)| \leq \|E\|_2, \quad i = 1, \dots, n.$$

Next, let us write the singular value decomposition of A as

$$A = [U_1 \ U_2 \ U_3] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \\ 0 & 0 \end{bmatrix} [V_1 \ V_2]^*,$$

and the same for \tilde{A} (conformal partition) so that

$$\tilde{A} = [\tilde{U}_1 \ \tilde{U}_2 \ \tilde{U}_3] \begin{bmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & \tilde{\Sigma}_2 \\ 0 & 0 \end{bmatrix} [\tilde{V}_1 \ \tilde{V}_2]^*.$$

We will like to quantify the perturbation of $\mathcal{R}(U_1), \mathcal{R}(V_1)$, this is given precisely by Wedin's theorem (Wedin, 1972). Before introducing the theorem, we need some definitions. Let $U, \tilde{U} \in \mathbb{C}^{n \times k}$ (for $k \leq n$) have orthonormal columns respectively and let $\sigma_1 \geq \dots \geq \sigma_k$ denote the singular values of $U^* \tilde{U}$. Then the k canonical angles between $\mathcal{R}(U), \mathcal{R}(\tilde{U})$ are defined as $\theta_i := \cos^{-1}(\sigma_i)$ for $1 \leq i \leq k$, with each $\theta_i \in [0, \pi/2]$. It is usual to define $k \times k$ diagonal matrices $\Theta(\mathcal{R}(U), \mathcal{R}(\tilde{U})) := \text{diag}(\theta_1, \dots, \theta_k)$ and $\sin \Theta(\mathcal{R}(U), \mathcal{R}(\tilde{U})) := \text{diag}(\sin \theta_1, \dots, \sin \theta_k)$. Denoting $\|\cdot\|$ to be any unitarily invariant norm (Frobenius, spectral, etc.), it is useful to know that the following relation holds (see for eg., (Li, 1994, Lemma 2.1), (Stewart and guang Sun, 1990, Corollary I.5.4)).

$$\|\sin \Theta(\mathcal{R}(U), \mathcal{R}(\tilde{U}))\| = \|(I - \tilde{U}\tilde{U}^*)U\|.$$

Let Φ be the matrix of canonical angles between $\mathcal{R}(U_1), \mathcal{R}(\tilde{U}_1)$ and let Θ be the matrix of canonical angles between $\mathcal{R}(V_1), \mathcal{R}(\tilde{V}_1)$.

Theorem 22 ((Wedin, 1972)) *Suppose that there are numbers $\alpha, \delta > 0$ such that*

$$\min \sigma(\tilde{\Sigma}_1) \geq \alpha + \delta \quad \text{and} \quad \max \sigma(\Sigma_2) \leq \alpha. \tag{A.1}$$

Then,

$$\max \{\|\sin \Theta\|_2, \|\sin \Phi\|_2\} \leq \frac{\|E\|_2}{\delta}.$$

Note that due to (A.1) the bounds are restricted to subspaces corresponding to the largest singular values. But one can derive bounds on $\max \{\|\sin \Theta\|_F, \|\sin \Phi\|_F\}$ without this restriction (see for eg. (Stewart and guang Sun, 1990, Theorem 4.1)).

Appendix B. Useful concentration inequalities

We outline here some useful concentration inequalities that are used in our proofs.

B.1 Spectral norm of random symmetric matrices

We will make use of the following result for bounding the spectral norm of random symmetric matrices with independent, centered and bounded entries.

Theorem 23 ((Bandeira and van Handel, 2016, Corollary 3.12, Remark 3.13)) *Let X be an $n \times n$ symmetric matrix whose entries X_{ij} ($i \leq j$) are independent, centered random variables. There there exists for any $0 < \varepsilon \leq 1/2$ a universal constant c_ε such that for every $t \geq 0$,*

$$\mathbb{P}(\|X\|_2 \geq (1 + \varepsilon)2\sqrt{2}\tilde{\sigma} + t) \leq n \exp\left(-\frac{t^2}{c_\varepsilon \tilde{\sigma}_*^2}\right) \quad (\text{B.1})$$

where

$$\tilde{\sigma} := \max_i \sqrt{\sum_j \mathbb{E}[X_{ij}^2]}, \quad \tilde{\sigma}_* := \max_{i,j} \|X_{ij}\|_\infty.$$

Note that it suffices to employ upper bound estimates on $\tilde{\sigma}, \tilde{\sigma}_*$ in (B.1).

B.2 Bernstein inequality

Theorem 24 ((Boucheron et al., 2013, Corollary 2.11)) *Let X_1, \dots, X_n be independent random variables with $|X_i| \leq b$ for all i , and $v = \sum_{i=1}^n \mathbb{E}[X_i^2]$. Then for any $t \geq 0$,*

$$\mathbb{P}\left(\left|\sum_{i=1}^n (X_i - \mathbb{E}[X_i])\right| \geq t\right) \leq 2 \exp\left(-\frac{t^2}{2(v + \frac{bt}{3})}\right).$$

B.3 Product of a random matrix (raised to a power) and a fixed vector

Given a random symmetric matrix X with independent entries on and above the diagonal, and a fixed vector u , (Eldridge et al., 2018, Theorem 15) provided an upper tail bound for $|(X^k u)_i|$. The proof technique therein followed that of Erdős et al. (2013). In fact, the proof goes through even if X is a random skew-symmetric matrix, the version stated below is for such matrices.

Theorem 25 ((Eldridge et al., 2018, Theorem 15)) *Let X be a $n \times n$ skew-symmetric and centered random matrix. Let $u \in \mathbb{R}^n$ be a fixed vector with $\|u\|_\infty = 1$. Choose $\xi > 1$, $0 < \kappa < 1$ and define $\mu = \frac{2}{\kappa+1}$. If $\mathbb{E}[|X_{ij}|^m] \leq 1/n$ for all $m \geq 2$, $1 \leq i, j \leq n$ and $k \leq \frac{\kappa}{8}(\log n)^{2\xi}$, then we have for any given $l \in [n]$ that*

$$\mathbb{P}(|(X^k u)_l| \geq (\log n)^{k\xi}) \leq n^{-\frac{1}{4}(\log_\mu n)^{\xi-1}(\log_\mu e)^{-\xi}}.$$

Appendix C. Proof of Theorem 5

C.1 Proof of Lemma 11

We begin by noting that for $\alpha = \frac{r^T e}{n}$, $\langle e, r - \alpha e \rangle = 0$ and so u_1, u_2 (similarly v_1, v_2) are orthogonal. Moreover, u_1, u_2, v_1, v_2 also have unit ℓ_2 norm. Finally, we note that

$$u_1 v_1^T + u_2 v_2^T = \frac{1}{\sqrt{n} \|r - \alpha e\|_2} [-e(r - \alpha e)^T + (r - \alpha e)e^T] = \frac{1}{\sqrt{n} \|r - \alpha e\|_2} \underbrace{[re^T - er^T]}_C,$$

which completes the proof.

C.2 Proof of Lemma 12

We begin by considering the second moment of S_{ij} for $i < j$. Note that $\mathbb{E}[N_{ij}] = 0$ and $\mathbb{E}[N_{ij}^2] = M^2/3$. We thus obtain

$$\begin{aligned} \mathbb{E}[(S)_{ij}^2] &= \eta p(1 - \eta p)^2(r_i - r_j)^2 + (1 - \eta)p\mathbb{E}[N_{ij} - \eta p(r_i - r_j)]^2 + (1 - p)(r_i - r_j)^2\eta^2 p^2 \\ &= \eta p(1 - \eta p)^2(r_i - r_j)^2 + (1 - \eta)p \left[\frac{M^2}{3} + \eta^2 p^2(r_i - r_j)^2 \right] + (1 - p)(r_i - r_j)^2\eta^2 p^2 \end{aligned}$$

which together with $(r_i - r_j)^2 \leq M^2$ yields

$$\begin{aligned} \mathbb{E}[(S)_{ij}^2] &\leq \eta p(1 - \eta p)^2 M^2 + (1 - \eta)p \left[\frac{M^2}{3} + \eta^2 p^2 M^2 \right] + (1 - p)\eta^2 p^2 M^2 \\ &= [\eta p(1 - \eta p)^2 + (1 - \eta)p \left(\frac{1}{3} + \eta^2 p^2 \right) + (1 - p)\eta^2 p^2] M^2 \\ &\leq \left(\eta p + \frac{4}{3}p + \eta^2 p^2 \leq \frac{10}{3}p \right) M^2 = \frac{10p}{3} M^2. \end{aligned} \tag{C.1}$$

We thus have that $\mathbb{E}[(S)_{ij}^2] \leq \frac{10p}{3} M^2$ for $i < j$; the same bound holds for $i > j$ as well. Hence we can bound the quantities $\tilde{\sigma}, \tilde{\sigma}_*$ defined in Theorem 23 as follows.

$$\begin{aligned} \tilde{\sigma} &:= \max_i \sqrt{\sum_j \mathbb{E}[(\tilde{S})_{ij}^2]} \leq \sqrt{\frac{10p}{3} n M^2} = M \sqrt{\frac{10p}{3} n}, \\ \tilde{\sigma}_* &:= \max_{i,j} \|\tilde{S}_{ij}\|_\infty \leq 2M. \end{aligned}$$

Then by invoking Theorem 23, we obtain for any given $t \geq 0$, $0 < \varepsilon \leq 1/2$ that

$$\begin{aligned} \mathbb{P}(\|\tilde{S}\|_2 \geq (1 + \varepsilon)2\sqrt{2}M\sqrt{\frac{10p}{3}n} + t) &\leq \mathbb{P}(\|\tilde{S}\|_2 \geq (1 + \varepsilon)2\sqrt{2}\tilde{\sigma} + t) \\ &\leq 2n \exp\left(\frac{-t^2}{c_\varepsilon \sigma_*^2}\right) \leq 2n \exp\left(\frac{-t^2}{4c_\varepsilon M^2}\right), \end{aligned}$$

where $c_\varepsilon > 0$ depends only on ε . Plugging $t = 2\sqrt{2}M\sqrt{\frac{10p}{3}n}$ we obtain the stated bound.

C.3 Proof of Lemma 13

Note that Weyl's inequality (Weyl, 1912) (see Theorem 21) for singular value perturbation readily yields

$$\sigma_i(H) \in [\sigma_i(\eta p C) \pm \Delta], \quad \forall i = 1, \dots, n.$$

In particular, using the expressions for $\sigma_1(C), \sigma_2(C)$ from Lemma 11, we have that

$$\begin{aligned} \sigma_1(H), \sigma_2(H) &\geq \eta p \sqrt{n} \|r - \alpha e\|_2 - \Delta, \\ \sigma_3(H), \dots, \sigma_n(H) &\leq \Delta. \end{aligned}$$

Then using Wedin's bound for perturbation of singular subspaces (Wedin, 1972) (see Theorem 22), we obtain

$$\|\sin \Theta(\mathcal{R}(\hat{U}), \mathcal{R}(U))\|_2 = \|(I - \hat{U}\hat{U}^T)U\|_2 \leq \frac{\Delta}{\eta p \sqrt{n} \|r - \alpha e\|_2 - \Delta} (=: \delta),$$

provided $\Delta < \eta p \sqrt{n} \|r - \alpha e\|_2$ holds.

C.4 Proof of Lemma 14

To begin with, note that $\bar{u}_1 = \hat{U}\hat{U}^T u_1$ is the orthogonal projection of u_1 on $\mathcal{R}(\hat{U})$. A unit vector orthogonal to \bar{u}_1 and lying in $\mathcal{R}(\hat{U})$ is given by

$$\tilde{u}_2 = \frac{1}{\|\bar{u}_1\|_2} \hat{U} \begin{bmatrix} \mp \langle \hat{u}_2, \bar{u}_1 \rangle \\ \pm \langle \hat{u}_1, \bar{u}_1 \rangle \end{bmatrix}.$$

Denoting \bar{u}_2 to be the orthogonal projection of u_2 on $\mathcal{R}(\hat{U})$, we can decompose $\langle \tilde{u}_2, u_2 \rangle$ as

$$\langle \tilde{u}_2, u_2 \rangle = \langle \tilde{u}_2, u_2 - \frac{\bar{u}_2}{\|\bar{u}_2\|_2} \rangle + \langle \tilde{u}_2, \frac{\bar{u}_2}{\|\bar{u}_2\|_2} \rangle,$$

which leads to

$$|\langle \tilde{u}_2, u_2 \rangle| \geq |\langle \tilde{u}_2, \frac{\bar{u}_2}{\|\bar{u}_2\|_2} \rangle| - |\langle \tilde{u}_2, u_2 - \frac{\bar{u}_2}{\|\bar{u}_2\|_2} \rangle|. \quad (\text{C.2})$$

We will now bound the two terms in the RHS of (C.2) starting with the first term.

Lower bounding the first term in RHS of (C.2). Since $\langle \hat{u}_2, \bar{u}_1 \rangle = \langle \hat{u}_2, u_1 \rangle, \langle \hat{u}_1, \bar{u}_1 \rangle = \langle \hat{u}_1, u_1 \rangle$, we obtain

$$\begin{aligned} \langle \tilde{u}_2, \frac{\bar{u}_2}{\|\bar{u}_2\|_2} \rangle &= \frac{u_2^T}{\|\bar{u}_1\|_2 \|\bar{u}_2\|_2} \hat{U} \underbrace{\hat{U}^T \hat{U}}_I \begin{bmatrix} \mp \langle \hat{u}_2, \bar{u}_1 \rangle \\ \pm \langle \hat{u}_1, \bar{u}_1 \rangle \end{bmatrix} \\ &= \frac{1}{\|\bar{u}_1\|_2 \|\bar{u}_2\|_2} [\mp \langle u_2, \hat{u}_1 \rangle \langle \hat{u}_2, u_1 \rangle \pm \langle u_2, \hat{u}_2 \rangle \langle \hat{u}_1, u_1 \rangle] \\ &= \frac{1}{\|\bar{u}_1\|_2 \|\bar{u}_2\|_2} [\pm \det(U^T \hat{U})]. \end{aligned}$$

Recall the definition of δ in (5.2). The following simple claims will be useful for us.

Claim 1 *With δ as defined in (5.2) we have $|\det(U^T \hat{U})| \geq 1 - \delta^2$.*

Proof Since $U^T \hat{U}$ is a 2×2 matrix, we have that

$$|\det(U^T \hat{U})| \geq \sigma_{\min}^2(U^T \hat{U}) = 1 - \|\sin \Theta(\mathcal{R}(\hat{U}), \mathcal{R}(U))\|_2^2 \geq 1 - \delta^2.$$

■

Claim 2 It holds that $\|\bar{u}_1 - u_1\|_2, \|\bar{u}_2 - u_2\|_2 \leq \delta$.

Proof We will only show the bound for $\|\bar{u}_1 - u_1\|_2$ as an identical argument holds for bounding $\|\bar{u}_2 - u_2\|_2$. Using the facts $\bar{u}_1 = \hat{U} \hat{U}^T u_1$ and $u_1 = U U^T u_1$, we obtain

$$\|u_1 - \bar{u}_1\|_2 = \|(I - \hat{U} \hat{U}^T) u_1\|_2 = \|(I - \hat{U} \hat{U}^T) U U^T u_1\|_2 \leq \|(I - \hat{U} \hat{U}^T) U U^T\|_2 = \|(I - \hat{U} \hat{U}^T) U\|_2 \leq \delta$$

where the last inequality follows from (5.2). ■

Using Claims 1, 2 we can lower bound the first term in RHS of (C.2) as follows.

$$\left| \langle \tilde{u}_2, \frac{\bar{u}_2}{\|\bar{u}_2\|_2} \rangle \right| \geq \frac{1 - \delta^2}{(1 + \delta)^2} = \frac{1 - \delta}{1 + \delta} = 1 - \frac{2\delta}{1 + \delta} \geq 1 - 2\delta. \quad (\text{C.3})$$

Upper bounding the second term in RHS of (C.2). We can do this as follows.

$$\begin{aligned} \left| \langle \tilde{u}_2, u_2 - \frac{\bar{u}_2}{\|\bar{u}_2\|_2} \rangle \right| &\leq \|u_2 - \frac{\bar{u}_2}{\|\bar{u}_2\|_2}\|_2 \quad (\text{Cauchy-Schwartz and since } \|\tilde{u}_2\|_2 = 1) \\ &= \|u_2 - \bar{u}_2 + \bar{u}_2 - \frac{\bar{u}_2}{\|\bar{u}_2\|_2}\|_2 \\ &\leq \|u_2 - \bar{u}_2\|_2 + \|\bar{u}_2 - \frac{\bar{u}_2}{\|\bar{u}_2\|_2}\|_2 \\ &\leq \delta + \left| 1 - \frac{1}{\|\bar{u}_2\|_2} \right| \quad (\text{Using Claim 2}) \\ &= \delta + \frac{|\|\bar{u}_2\|_2 - 1|}{\|\bar{u}_2\|_2} \\ &\leq \delta + \frac{\delta}{1 - \delta} \quad (\text{since } \|\bar{u}_2\|_2 \in [1 - \delta, 1 + \delta] \text{ from Claim 2}) \\ &\leq 3\delta \quad (\text{whenever } \delta \leq 1/2). \end{aligned} \quad (\text{C.4})$$

Altogether, we conclude that if $\delta \leq 1/2$ then using (C.3), (C.4) in (C.2), we obtain

$$|\langle \tilde{u}_2, u_2 \rangle| \geq 1 - 5\delta,$$

and consequently, there exists $\beta \in \{-1, 1\}$ such that

$$\|\tilde{u}_2 - \beta u_2\|_2^2 = 2 - 2\langle \tilde{u}_2, \beta u_2 \rangle \leq 10\delta. \quad (\text{C.5})$$

Appendix D. Proof of Lemmas from Section 5.2

D.1 Proof of Lemma 15

Proof Recall from the proof of Lemma 14 that

$$\tilde{u}_2 = \frac{1}{\|\bar{u}_1\|_2} \hat{U} \begin{bmatrix} \mp \langle \hat{u}_2, \bar{u}_1 \rangle \\ \pm \langle \hat{u}_1, \bar{u}_1 \rangle \end{bmatrix} = \frac{1}{\|\bar{u}_1\|_2} \hat{U} \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix} \hat{U}^T \bar{u}_1 = \frac{1}{\|\bar{u}_1\|_2} \hat{U} \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix} \hat{U}^T u_1, \quad (\text{D.1})$$

where the last equality follows from the fact $\langle \hat{u}_2, \bar{u}_1 \rangle = \langle \hat{u}_2, u_1 \rangle$, $\langle \hat{u}_1, \bar{u}_1 \rangle = \langle \hat{u}_1, u_1 \rangle$. Let O be any 2×2 orthogonal matrix, clearly, UO also corresponds to the two largest left singular vectors of $\mathbb{E}[H]$ (since $\sigma_1(\mathbb{E}[H]) = \sigma_2(\mathbb{E}[H])$). Denote $P = \hat{U} - UO$. Plugging in (D.1), we obtain

$$\begin{aligned} \tilde{u}_2 &= \frac{1}{\|\bar{u}_1\|_2} \hat{U} \underbrace{\begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix}}_D \hat{U}^T \bar{u}_1 [O^T U^T + P] u_1 \\ &= \pm \det(O) u_2 + \underbrace{\frac{1}{\|\bar{u}_1\|_2} U O D P^T u_1}_{w_1} + \underbrace{\frac{1}{\|\bar{u}_1\|_2} P D O^T U^T u_1}_{w_2} + \underbrace{\frac{1}{\|\bar{u}_1\|_2} P D P^T u_1}_{w_3} \\ \Rightarrow \|\tilde{u}_2 \mp \det(O) u_2\|_\infty &\leq \|w_1\|_\infty + \|w_2\|_\infty + \|w_3\|_\infty. \end{aligned} \quad (\text{D.2})$$

Denoting

$$P = [p_1 \ p_2], \quad O = \begin{bmatrix} o_{11} & o_{12} \\ o_{21} & o_{22} \end{bmatrix}$$

we will now bound $\|w_1\|_\infty, \|w_2\|_\infty, \|w_3\|_\infty$.

To begin with,

$$\begin{aligned} w_1 &= \frac{1}{\|\bar{u}_1\|_2} [u_1 \ u_2] O \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix} \begin{bmatrix} \langle p_1, u_1 \rangle \\ \langle p_2, u_1 \rangle \end{bmatrix} \\ &= \frac{1}{\|\bar{u}_1\|_2} [u_1 \ u_2] O \begin{bmatrix} \mp \langle p_2, u_1 \rangle \\ \pm \langle p_1, u_1 \rangle \end{bmatrix} \\ &= \frac{1}{\|\bar{u}_1\|_2} [u_1 \ u_2] \begin{bmatrix} \mp o_{11} \langle p_2, u_1 \rangle \pm o_{12} \langle p_1, u_1 \rangle \\ \mp o_{21} \langle p_2, u_1 \rangle \pm o_{22} \langle p_1, u_1 \rangle \end{bmatrix} \\ \Rightarrow \|w_1\|_\infty &\leq \frac{1}{\|\bar{u}_1\|_2} [(|\langle p_2, u_1 \rangle| + |\langle p_1, u_1 \rangle|)(\|u_1\|_\infty + \|u_2\|_\infty)]. \end{aligned} \quad (\text{D.3})$$

Clearly, $\|u_1\|_\infty = \frac{1}{\sqrt{n}}$ and $\|u_2\|_\infty \leq \frac{M-\alpha}{\|r-\alpha e\|_2}$. Using Hölder's inequality,

$$|\langle p_i, u_1 \rangle| \leq \|P\|_{\max} \|u\|_1 = \|P\|_{\max} \sqrt{n}; \quad i = 1, 2.$$

Moreover, from Claim 2, we know that $\|\bar{u}_1\|_2 \geq 1 - \delta$ where we recall δ defined in (5.2). We saw in (5.3) that $\Delta \leq \frac{\eta p \|r-\alpha e\|_2 \sqrt{n}}{3} \Leftrightarrow \delta \leq 1/2$, which implies $\|\bar{u}_1\|_2 \geq 1/2$. Plugging these bounds in (D.3), we obtain

$$\|w_1\|_\infty \leq 4\|P\|_{\max} \left(1 + \frac{\sqrt{n}(M-\alpha)}{\|r-\alpha e\|_2} \right). \quad (\text{D.4})$$

Next, we can bound $\|w_2\|_\infty$ as follows.

$$\begin{aligned} w_2 &= \frac{1}{\|\bar{u}_1\|_2} [p_1 \ p_2] \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix} O^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\|\bar{u}_1\|_2} [p_1 \ p_2] \begin{bmatrix} \mp o_{12} \\ \pm o_{11} \end{bmatrix} \\ \Rightarrow \|w_2\|_\infty &\leq \frac{2\|P\|_{\max}}{\|\bar{u}_1\|_2} \leq 4\|P\|_{\max} \end{aligned} \quad (\text{D.5})$$

since $\Delta \leq \frac{\eta p \|r - \alpha e\|_2 \sqrt{n}}{3}$.

We now bound $\|w_3\|_\infty$.

$$\begin{aligned} w_3 &= \frac{1}{\|\bar{u}_1\|_2} [p_1 \ p_2] \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix} \begin{bmatrix} \langle p_1, u_1 \rangle \\ \langle p_2, u_1 \rangle \end{bmatrix} \\ &= \frac{1}{\|\bar{u}_1\|_2} [p_1 \ p_2] \begin{bmatrix} \mp \langle p_2, u_1 \rangle \\ \pm \langle p_1, u_1 \rangle \end{bmatrix} \\ \Rightarrow \|w_3\|_\infty &\leq \frac{1}{\|\bar{u}_1\|_2} ((|\langle p_2, u_1 \rangle| + |\langle p_1, u_1 \rangle|) \|P\|_{\max}) \\ &\leq 4\|P\|_{\max}^2 \|u_1\|_1 = 4\sqrt{n} \|P\|_{\max}^2 \quad (\text{since } \Delta \leq \frac{\eta p \|r - \alpha e\|_2 \sqrt{n}}{3} \text{ and using Hölder's inequality}). \end{aligned} \quad (\text{D.6})$$

Plugging (D.4), (D.5), (D.6) in (D.2), we obtain

$$\begin{aligned} \|\tilde{u}_2 \mp \det(O)u_2\|_\infty &\leq 4\|P\|_{\max} \left(2 + \frac{\sqrt{n}(M - \alpha)}{\|r - \alpha e\|_2} \right) + 4\sqrt{n} \|P\|_{\max}^2 \\ &= 4\|\widehat{U} - UO\|_{\max} \left(2 + \frac{\sqrt{n}(M - \alpha)}{\|r - \alpha e\|_2} \right) + 4\sqrt{n} \|\widehat{U} - UO\|_{\max}^2. \end{aligned}$$

The above bound is true for any orthogonal matrix O , it is not difficult to see that there exists an orthogonal matrix $O = O^*$ such that \widehat{U} is “aligned” with UO^* . Indeed, for any orthogonal O , we first obtain via triangle inequality that

$$\|\widehat{U} - UO\|_2 \leq \|(I - UU^T)\widehat{U}\|_2 + \|UU^T\widehat{U} - UO\|_2 = \|\sin \Theta(\mathcal{R}(U), \mathcal{R}(\widehat{U}))\|_2 + \|U^T\widehat{U} - O\|_2. \quad (\text{D.7})$$

Denoting $\tilde{U}\tilde{D}\tilde{V}^T$ to be the SVD of $U^T\widehat{U}$, we choose $O = O^* = \tilde{U}\tilde{V}^T$ (so O^* is orthogonal). Denoting θ_p to be the principal angle between $\mathcal{R}(U)$, $\mathcal{R}(\widehat{U})$, we obtain

$$\|U^T\widehat{U} - O^*\|_2 = \|I - \tilde{D}\|_2 = 1 - \cos \theta_p \leq \sin \theta_p = \|\sin \Theta(\mathcal{R}(U), \mathcal{R}(\widehat{U}))\|_2.$$

Plugging this in (D.7) leads to

$$\begin{aligned} \|\widehat{U} - UO^*\|_2 &\leq 2\|\sin \Theta(\mathcal{R}(U), \mathcal{R}(\widehat{U}))\|_2 \\ &\leq \frac{2\Delta}{\eta p \sqrt{n} \|r - \alpha e\|_2 - \Delta} \quad (\text{see proof of Lemma 13}) \\ &\leq \frac{3\Delta}{\eta p \sqrt{n} \|r - \alpha e\|_2}. \quad (\text{since } \Delta \leq \frac{\eta p \|r - \alpha e\|_2 \sqrt{n}}{3}) \end{aligned}$$

■

D.2 Proof of Lemma 16

Proof

1. **(Bounding $\|E_1\|_{\max}$)** Denoting $(\widehat{U} - UO^*)_i$ to be the i^{th} column of $\widehat{U} - UO^*$, we note that

$$\begin{aligned} E_1 &= -\mathbb{E}[H]^2(\widehat{U} - UO^*)\widehat{\Sigma}^{-2} \\ &= \sigma^2 UU^T(\widehat{U}^T - UO^*)\widehat{\Sigma}^{-2} \\ &= (u_1 u_1^T + u_2 u_2^T) \left[(\widehat{U} - UO^*)_1 \frac{\sigma^2}{\widehat{\sigma}_1^2} \quad (\widehat{U} - UO^*)_2 \frac{\sigma^2}{\widehat{\sigma}_2^2} \right] \\ &= \left[u_1 \langle u_1, (\widehat{U} - UO^*)_1 \rangle \frac{\sigma^2}{\widehat{\sigma}_1^2} + u_2 \langle u_2, (\widehat{U} - UO^*)_1 \rangle \frac{\sigma^2}{\widehat{\sigma}_1^2} \quad u_1 \langle u_1, (\widehat{U} - UO^*)_2 \rangle \frac{\sigma^2}{\widehat{\sigma}_2^2} + u_2 \langle u_2, (\widehat{U} - UO^*)_2 \rangle \frac{\sigma^2}{\widehat{\sigma}_2^2} \right]. \end{aligned}$$

For the i^{th} entry in the first column ($i = 1, \dots, n$), we have that

$$\begin{aligned} &\left| u_{1,i} \langle u_1, (\widehat{U} - UO^*)_1 \rangle \frac{\sigma^2}{\widehat{\sigma}_1^2} + u_{2,i} \langle u_2, (\widehat{U} - UO^*)_1 \rangle \frac{\sigma^2}{\widehat{\sigma}_1^2} \right| \\ &\leq \|u_1\|_\infty \underbrace{\|(\widehat{U} - UO^*)_1\|_2}_{\leq \|\widehat{U} - UO^*\|_2} \frac{\sigma^2}{\widehat{\sigma}_1^2} + \|u_2\|_\infty \underbrace{\|(\widehat{U} - UO^*)_1\|_2}_{\leq \|\widehat{U} - UO^*\|_2} \frac{\sigma^2}{\widehat{\sigma}_1^2}. \end{aligned}$$

Since $\|u_1\|_\infty = \frac{1}{\sqrt{n}}$, $\|u_2\|_\infty \leq \frac{M-\alpha}{\|r-\alpha e\|_2}$, and $\widehat{\sigma}_1 \geq \sigma - \Delta \geq \frac{2\sigma}{3}$, we obtain

$$\begin{aligned} &\left| u_{1,i} \langle u_1, (\widehat{U} - UO^*)_1 \rangle \frac{\sigma^2}{\widehat{\sigma}_1^2} + u_{2,i} \langle u_2, (\widehat{U} - UO^*)_1 \rangle \frac{\sigma^2}{\widehat{\sigma}_1^2} \right| \\ &\leq \frac{9}{4} \|\widehat{U} - UO^*\|_2 \left(\frac{1}{\sqrt{n}} + \frac{M-\alpha}{\|r-\alpha e\|_2} \right) \\ &\leq \frac{27\Delta}{4\sigma} \left(\frac{1}{\sqrt{n}} + \frac{M-\alpha}{\|r-\alpha e\|_2} \right) \end{aligned}$$

where in the last inequality, we used $\|\widehat{U} - UO^*\|_2 \leq \frac{3\Delta}{\sigma}$ (from Lemma 15). The same entry-wise bound holds for the second column, leading to the stated bound on $\|E_1\|_{\max}$.

2. **(Bounding $\|E_2\|_{\max}$)** Denoting the $(i, j)^{th}$ entry of O^* by o_{ij}^* , we have that

$$\begin{aligned} E_2 &= \mathbb{E}[H]^2 U O^* (\sigma^{-2} I - \widehat{\Sigma}^{-2}) \\ &= -UU^T \sigma^2 (U O^*) \begin{bmatrix} \sigma^{-2} - \widehat{\sigma}_1^{-2} & 0 \\ 0 & \sigma^{-2} - \widehat{\sigma}_2^{-2} \end{bmatrix} \\ &= -\sigma^2 [(\sigma^{-2} - \widehat{\sigma}_1^{-2})(u_1 o_{11}^* + u_2 o_{21}^*) \quad (\sigma^{-2} - \widehat{\sigma}_2^{-2})(u_1 o_{12}^* + u_2 o_{22}^*)]. \end{aligned}$$

Hence the i^{th} entry of the first column can be bounded as follows.

$$\begin{aligned} \sigma^2 |\sigma^{-2} - \hat{\sigma}_1^{-2}| |u_{1,i} o_{11}^* + u_{2,i} o_{12}^*| &\leq \frac{\Delta(2\sigma + \Delta)}{(\sigma - \Delta)^2} (\|u_1\|_\infty + \|u_2\|_\infty) \quad (\text{using } |\hat{\sigma}_1 - \sigma| \leq \Delta) \\ &\leq \frac{21\Delta}{4\sigma} \left(\frac{1}{\sqrt{n}} + \frac{M - \alpha}{\|r - \alpha e\|_2} \right) \quad (\text{using } \Delta \leq \frac{\sigma}{3}). \end{aligned}$$

The same entry-wise bound holds for the second column, leading to the stated bound on $\|E_2\|_{\max}$.

3. (**Bounding $\|E_3\|_{\max}$**) We have that

$$\begin{aligned} E_3 &= -\mathbb{E}[H]Z\hat{U}\hat{\Sigma}^{-2} \\ &= -\sigma(u_1 v_1^T + u_2 v_2^T)Z[\hat{\sigma}_1^{-2}\hat{u}_1 \quad \hat{\sigma}_2^{-2}\hat{u}_2] \\ &= -\sigma[u_1\langle v_1, Z\hat{u}_1 \rangle \hat{\sigma}_1^{-2} + u_2\langle v_2, Z\hat{u}_1 \rangle \hat{\sigma}_1^{-2} \quad u_1\langle v_1, Z\hat{u}_2 \rangle \hat{\sigma}_2^{-2} + u_2\langle v_2, Z\hat{u}_2 \rangle \hat{\sigma}_2^{-2}]. \end{aligned}$$

The i^{th} entry of the first column can be bounded as follows.

$$\begin{aligned} \sigma |u_{1,i}\langle v_1, Z\hat{u}_1 \rangle \hat{\sigma}_1^{-2} + u_{2,i}\langle v_2, Z\hat{u}_1 \rangle \hat{\sigma}_1^{-2}| &\leq \frac{\sigma}{\hat{\sigma}_1^2} (\|u_1\|_\infty |\langle v_1, Z\hat{u}_1 \rangle| + \|u_2\|_\infty |\langle v_2, Z\hat{u}_1 \rangle|) \\ &\leq \frac{\sigma}{\hat{\sigma}_1^2} \|Z\|_2 \left(\frac{1}{\sqrt{n}} + \frac{M - \alpha}{\|r - \alpha e\|_2} \right) \quad (\text{using Cauchy-Schwarz}) \\ &\leq \frac{9\Delta}{4\sigma} \left(\frac{1}{\sqrt{n}} + \frac{M - \alpha}{\|r - \alpha e\|_2} \right) \quad (\text{using } \hat{\sigma}_1 \geq 2\sigma/3). \end{aligned}$$

The same entry-wise bound holds for the second column, leading to the stated bound on $\|E_3\|_{\max}$.

4. (**Bounding $\|E_4\|_{\max}$**) We have that

$$\begin{aligned} E_4 &= Z\mathbb{E}[H]\hat{U}\hat{\Sigma}^{-2} = \sigma Z(u_1 v_1^T + u_2 v_2^T)[\hat{\sigma}_1^{-2}\hat{u}_1 \quad \hat{\sigma}_2^{-2}\hat{u}_2] \\ &= \sigma \left[Z u_1 \frac{v_1^T \hat{u}_1}{\hat{\sigma}_1^2} + Z u_2 \frac{v_2^T \hat{u}_1}{\hat{\sigma}_1^2} \quad Z u_1 \frac{v_1^T \hat{u}_2}{\hat{\sigma}_2^2} + Z u_2 \frac{v_2^T \hat{u}_2}{\hat{\sigma}_2^2} \right]. \end{aligned}$$

The magnitude of the i^{th} entry of the first column can be bounded as

$$\begin{aligned} \frac{\sigma}{\hat{\sigma}_1^2} |(v_1^T \hat{u}_1)(Z u_1)_i + (v_2^T \hat{u}_1)(Z u_2)_i| &\leq \frac{\sigma}{\hat{\sigma}_1^2} (\|Z u_1\|_\infty + \|Z u_2\|_\infty) \quad (\text{Using Cauchy-Schwarz}) \\ &\leq \frac{9}{4\sigma} (\|Z u_1\|_\infty + \|Z u_2\|_\infty) \quad (\text{using } \hat{\sigma}_1 \geq 2\sigma/3). \end{aligned}$$

The same entry-wise bound holds for the second column, leading to the stated bound on $\|E_4\|_{\max}$.

5. (**Bounding $\|E_5\|_{\max}$**) We can write

$$E_5 = Z^2 \hat{U} \hat{\Sigma}^{-2} = Z^2 (\hat{U} - U O^*) \hat{\Sigma}^{-2} + Z^2 (U O^*) \hat{\Sigma}^{-2}$$

which in turn implies $\|E_5\|_{\max} \leq \|Z^2(\widehat{U} - UO^*)\widehat{\Sigma}^{-2}\|_{\max} + \|Z^2(UO^*)\widehat{\Sigma}^{-2}\|_{\max}$. We will now bound these terms individually, below. To begin with,

$$\begin{aligned} \|Z^2(\widehat{U} - UO^*)\widehat{\Sigma}^{-2}\|_{\max} &\leq \|Z^2(\widehat{U} - UO^*)\widehat{\Sigma}^{-2}\|_2 \\ &\leq \Delta^2 \|\widehat{U} - UO^*\|_2 \|\widehat{\Sigma}^{-2}\|_2 \quad (\text{sub-multiplicativity of } \|\cdot\|_2) \\ &\leq \frac{3\Delta^3}{\sigma} \|\widehat{\Sigma}^{-2}\|_2 \quad (\text{from Lemma 15}) \\ &\leq \frac{27\Delta^3}{4\sigma^3} \quad (\text{using } \widehat{\sigma}_i \geq 2\sigma/3). \end{aligned}$$

Additionally, denoting $O^* = [o_1^* \ o_2^*]$, we can write

$$Z^2(UO^*)\widehat{\Sigma}^{-2} = \underbrace{[Z^2u_1 \ Z^2u_2]}_{\tilde{A}} [\widehat{\sigma}_1^{-2}o_1^* \ \widehat{\sigma}_2^{-2}o_2^*] = [\widehat{\sigma}_1^{-2}\tilde{A}o_1^* \ \widehat{\sigma}_2^{-2}\tilde{A}o_2^*]$$

which in turn implies

$$\begin{aligned} \|Z^2(UO^*)\widehat{\Sigma}^{-2}\|_{\max} &\leq \max \left\{ \frac{\|\tilde{A}o_1^*\|_{\infty}}{\widehat{\sigma}_1^2}, \frac{\|\tilde{A}o_2^*\|_{\infty}}{\widehat{\sigma}_2^2} \right\} \\ &\leq \frac{9}{4\sigma^2} \max \left\{ \|\tilde{A}o_1^*\|_{\infty}, \|\tilde{A}o_2^*\|_{\infty} \right\} \quad (\text{using } \widehat{\sigma}_i \geq 2\sigma/3) \\ &\leq \frac{9}{4\sigma^2} (\|Z^2u_1\|_{\infty} + \|Z^2u_2\|_{\infty}). \end{aligned}$$

■

D.3 Proof of Lemma 17

Proof

- For a fixed i , consider $\sum_{j=1}^n Z_{ij}u_{1j}$ which is a sum of independent random variables. From the definition of Z_{ij} , we can see that $|Z_{ij}u_{1j}| \leq \frac{2M}{\sqrt{n}}$ for each j . Moreover, $\mathbb{E}[(Z_{ij}u_{1j})^2] = \frac{1}{n}\mathbb{E}[Z_{ij}^2] \leq \frac{10pM^2}{3n}$ where the last inequality was shown in the proof of Lemma 12 (see (C.1)). Hence $\sum_{i=1}^n \mathbb{E}[(Z_{ij}u_{1j})^2] \leq \frac{10pM^2}{3}$ and it follows from Bernstein's inequality (see Theorem B.2) and the union bound that

$$\mathbb{P}(\|Zu_1\|_{\infty} \geq t) \leq 2n \exp \left(-\frac{3t^2}{20pM^2 + \frac{4tM}{\sqrt{n}}} \right). \quad (\text{D.8})$$

We want to choose t such that

$$3t^2 \geq 2 \log n \left(20pM^2 + \frac{4tM}{\sqrt{n}} \right) \Leftrightarrow 3t^2 - 40pM^2 \log n - \frac{8tM \log n}{\sqrt{n}} \geq 0. \quad (\text{D.9})$$

Since $t \geq 0$, the above inequality is achieved iff

$$t \geq \frac{\frac{8M \log n}{\sqrt{n}} + \sqrt{\frac{64M^2 \log^2 n}{n} + 480pM^2 \log n}}{6}. \quad (\text{D.10})$$

Hence if

$$\frac{64M^2 \log^2 n}{n} \leq 480pM^2 \log n \Leftrightarrow p \geq \frac{2 \log n}{15n}$$

holds, then the RHS of (D.10) is upper bounded by

$$\frac{\frac{8M \log n}{\sqrt{n}} + \sqrt{960pM^2 \log n}}{6} \leq \frac{\frac{8M \log n}{\sqrt{2} \log n} \sqrt{15p} + 8M \sqrt{15p \log n}}{6} = \frac{2\sqrt{2} + 4}{3} M \sqrt{15p \log n}.$$

Therefore if $t \geq \frac{2\sqrt{2}+4}{3} M \sqrt{15p \log n}$ is satisfied, then it implies that (D.9) holds. Hence plugging $t = \frac{2\sqrt{2}+4}{3} M \sqrt{15p \log n}$ in (D.8) leads to the statement of the Lemma.

2. For a fixed i , consider now $\sum_{j=1}^n Z_{ij} u_{2j}$ which is also a sum of independent random variables. One can verify that $|Z_{ij} u_{2j}| \leq 2M\sqrt{B}$ for each j , with B as defined in the Lemma. Moreover, $\mathbb{E}[(Z_{ij} u_{2j})^2] \leq B\mathbb{E}[Z_{ij}^2] \leq \frac{10pM^2B}{3}$ (see (C.1)) and hence, $\sum_{i=1}^n \mathbb{E}[(Z_{ij} u_{2j})^2] \leq \frac{10pM^2Bn}{3}$ and it follows from Bernstein's inequality (see Theorem B.2) and the union bound that

$$\mathbb{P}(\|Zu_2\|_\infty \geq t) \leq 2n \exp\left(-\frac{3t^2}{20pM^2Bn + 4t\sqrt{BM}}\right). \quad (\text{D.11})$$

Proceeding identically as before, the reader is invited to verify that if $p \geq \frac{2 \log n}{15n}$, then the stated bound on $\|Zu_2\|_\infty$ is obtained by plugging $t = \frac{2\sqrt{2}+4}{3} M \sqrt{15pBn \log n}$ in (D.11).

■

D.4 Proof of Lemma 18

Proof

1. For a given $l \in [n]$, we can write

$$\begin{aligned} |(Z^2 u_1)_l| &= \|u_1\|_\infty \left| \left(Z^2 \underbrace{\frac{u_1}{\|u_1\|_\infty}}_{\tilde{u}_1} \right)_l \right| \\ &= \frac{1}{\sqrt{n}} |(Z^2 \tilde{u}_1)_l| \quad (\text{with } \|\tilde{u}_1\|_\infty = 1) \\ &= \frac{\phi^2}{\sqrt{n}} |((Z/\phi)^2 \tilde{u}_1)_l| \end{aligned} \quad (\text{D.12})$$

for any $\phi > 0$. For any integer $m \geq 2$, and $1 \leq i, j \leq n$, we have that

$$\begin{aligned}\mathbb{E} \left[\left(\frac{|Z_{ij}|}{\phi} \right)^m \right] &= \frac{1}{\phi^m} [\eta p |r_i - r_j|^m (1 - \eta p)^m + (1 - \eta)p\mathbb{E}[|N_{ij} - \eta p(r_i - r_j)|^m] + (1 - p) |\eta p(r_i - r_j)|^m] \\ &\leq \frac{1}{\phi^m} [\eta p M^m + p(2M)^m + (\eta p)^m M^m] \\ &\leq \frac{1}{\phi^m} [2^{m+1} p M^m].\end{aligned}$$

Setting $\phi = 2^{3/2} \sqrt{pn} M$ leads to

$$\mathbb{E} \left[\left(\frac{|Z_{ij}|}{\phi} \right)^m \right] \leq \frac{2^{m+1} p M^m}{(2)^{3m/2} p^{m/2} M^m n^{m/2}} = \frac{1}{2^{\frac{m}{2}-1}} \frac{1}{p^{\frac{m}{2}-1}} \frac{1}{n^{\frac{m}{2}}}.$$

It is easy to check that $p \geq \frac{1}{2n}$ implies $\mathbb{E} \left[\left(\frac{|Z_{ij}|}{\phi} \right)^m \right] \leq 1/n$. Therefore invoking Theorem 25 for $k = 2$, we obtain for the stated choices of ξ, μ, κ that $|((Z/\phi)^2 \tilde{u}_1)_l| \leq (\log n)^{2\xi}$ holds with probability at least $1 - n^{-\frac{1}{4}(\log_\mu n)^{\xi-1}(\log_\mu e)^{-\xi}}$. Plugging this in (D.12) with the expression for ϕ , and using the union bound, we obtain the statement of the Lemma.

2. The steps are identical to those above, with the only difference being $\|u_2\|_\infty = \frac{M-\alpha}{\|r-\alpha e\|_2}$.

■

Appendix E. Proofs of Lemmas from Section 5.5

E.1 Proof of Lemma 19

Note that $\bar{D}_{ii} = \sum_{j=1}^n |H_{ij}|$ is the sum of independent random variables, where $|H_{ij}| \leq M \forall i, j$. Moreover,

$$\sum_{j=1}^n \mathbb{E}[|H_{ij}|^2] = \eta p \sum_{j=1}^n (r_i - r_j)^2 + (1 - \eta) \frac{npM}{2} \leq npnM^2 + (1 - \eta)np \frac{M}{2} = npA(\eta, M).$$

Invoking Bernstein's inequality, it follows for any given i and $t \geq 0$ that

$$\mathbb{P}(|\bar{D}_{ii} - \mathbb{E}[\bar{D}_{ii}]| \geq t) \leq 2 \exp \left(-\frac{t^2}{2(npA(\eta, M) + \frac{Mt}{3})} \right).$$

In order to get a high probability bound, we require $t^2 \geq 4 \log n (npA(\eta, M) + \frac{Mt}{3})$. Since $t \geq 0$, this is equivalent to saying that

$$t \geq \frac{\frac{4M}{3} \log n + \sqrt{(\frac{4M}{3} \log n)^2 + 16pn \log n A(\eta, M)}}{2}. \quad (\text{E.1})$$

If $(\frac{4M}{3} \log n)^2 \leq 6pn \log n A(\eta, M)$ or equivalently $p \geq \frac{M^2}{9A(\eta, M)} \frac{\log n}{n}$, then the RHS of (E.1) is bounded by

$$\sqrt{16pn \log n A(\eta, M)} \frac{\sqrt{2} + 1}{2} = 2(\sqrt{2} + 1) \sqrt{pn \log n} \sqrt{A(\eta, M)}.$$

Hence taking $t = 2(\sqrt{2} + 1) \sqrt{pn \log n} \sqrt{A(\eta, M)}$ and applying the union bound, we obtain the statement of the Lemma.