STATISTICAL INFERENCES (2cr)
Chapter 10 One- and Two-Sample Tests of Hypotheses

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Outline

1. 10.1 Statistical Hypotheses: General Concepts
2. 10.2 Testing a Statistical Hypothesis
3. 10.3 Using p-value

Example 1.

- Let p be the proportion of printed circuits that fail. Current procedure has $p_0 = 0.06$.
- A new method is proposed. To see if the new method results in an improvement.
- We want to test **statistical hypotheses**

$$H_0 : p = p_0 = 0.06 \quad \text{vs} \quad H_1 : p < p_0 = 0.06.$$

- We decided to select $n = 200$ circuits to test. Let Y be the number of circuits that fail.
- **Decision rule**: accept the improvement hypothesis H_1 if $Y \leqslant k$ (reject H_0), otherwise reject H_1 (accept H_0).
- $k = ?$

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Two Types of Errors

- For example, we choose $k = 7$ so that if $Y \leqslant 7$ or $\hat{p} \leqslant 0.035$, then we accept the improvement hypothesis.
- However, whatever k is, the decision made based on a sample could be wrong. There are two types of errors.

	H_0 true	H_0 false
Reject H_0	Type I error	Correct
Accept H_0	Correct	Type II error

- If we accept H_0, then we will keep the current procedure. This is called no change hypothesis. It is also called **null hypothesis**.
- H_1 is called **alternative hypothesis**.
- $y \leqslant k$ is called **rejection** or **critical region** of H_0.
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Chances to Make Mistakes: Type I Error

- The Type I error of the above rule is that $Y \leqslant 7$ but the truth is $p = 0.06$.
- The probability of this error is

$$P(Y \leqslant 7 | p = 0.06) = \sum_{y=0}^{7} \binom{200}{y} (0.06)^y (0.94)^{200-y}$$

$$= 0.0829 = 8.29\%.$$

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- Type II error: $Y > 7$ but the truth is $p < 0.06$.
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$$\beta(p) = P(Y > 7|p) = \sum_{y=8}^{200} \binom{200}{y} p^y (1-p)^{200-y}$$

- For example,

$$\beta(0.03) = 1 - binomialcdf(7, n = 200, p = 0.03) = 0.267$$

$$\beta(0.02) = 1 - binomialcdf(7, n = 200, p = 0.02) = 0.038$$

$$\beta(0.01) = 1 - binomialcdf(7, n = 200, p = 0.01) = 4.64 \times 10^{-5}$$

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Significance Level and Power of the Test

- Generally, we want to control the Type I error first: for a given **significance level** α, choose k so that

$$P(\text{Type I error}) \leqslant \alpha$$

- The **power** of the test is,

$$\text{Power}(p) = 1 - \beta(p).$$

- In the above example, for $p < 0.06$,

$$\text{Power}(p) = P(Y \leqslant 7|p) = 1 - P(Y > 7|p).$$

$$\text{Power}(0.03) = 0.833$$
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Find Critical Value k

If H_0 is true, approximately,

$$Z = \frac{Y - np_0}{\sqrt{np_0(1-p_0)}} \sim N(0,1).$$

Thus

$$\alpha = P(Y \leqslant k | p = p_0) \approx \Phi\left(\frac{k - np_0}{\sqrt{np_0(1-p_0)}}\right)$$

So

$$\frac{k - np_0}{\sqrt{np_0(1-p_0)}} = -z_\alpha.$$

and

$$k = np_0 - z_\alpha \sqrt{np_0(1-p_0)}.$$

Rejection Region of H_0

- The **rejection region** of H_0 is $R = \{Y : Y \leqslant k\}$
- or $R = \{Z = \frac{Y - np_0}{\sqrt{np_0(1-p_0)}} \leqslant -z_\alpha\}$.
- Here k and $-z_\alpha$ are called the critical values of the **test statistics** Y and Z respectively.
- In the above example, $n = 200$, $p_0 = 0.06$. Choose $\alpha = 0.05$. $z_\alpha = z_{0.05} = 1.645$.

$$k = np_0 - z_\alpha\sqrt{np_0(1 - p_0)} = 6.42$$

- The rejection region of H_0 is

$$R = \{Y : Y \leqslant 6.4167\}$$

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A Continuous Random Variable Example

Example 2: Let X be the Brinell hardness measurement of ductile iron substantially annealed. Assume that the distribution of X is $N(\mu, 10^2)$. We shall test the null hypothesis $H_0 : \mu = 170$ against the alternative hypothesis $H_1 : \mu > 170$ based on a sample of size n.

(a) Define the test statistic;

(b) Define a critical region with a significance level $\alpha = 0.05$.

(c) A random sample of $n = 25$ observations of X yielded the following measurements.

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Solution of Example 2

(a) Since $\bar{X}$ is an estimator of μ, we use $\bar{X}$ as the test statistic.

(b) The rejection region of $H_0 : \mu = 170$ in favor of $H_1 : \mu > 170$:

$$\bar{X} > c$$

for some pre-determined critical value c according to α.

$$0.05 = \alpha = Prob(\text{Type I}) = ?$$

$$P(\text{Type I}) = P(\text{Rejec } H_0 \,|\, H_0 \text{ is true}) = P(\bar{X} > c \,|\, \mu = 170)$$

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(b) The rejection region of $H_0 : \mu = 170$ in favor of $H_1 : \mu > 170$:

$$\bar{X} > c$$

for some pre-determined critical value c according to α.

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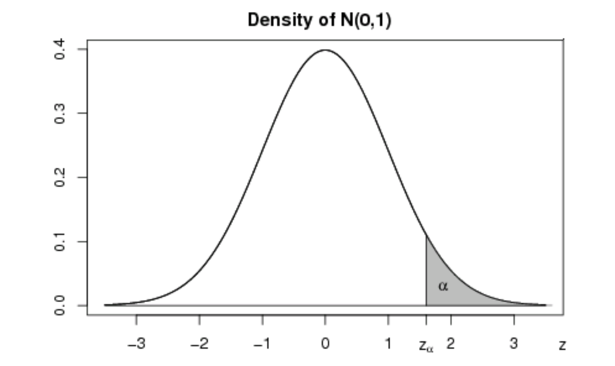

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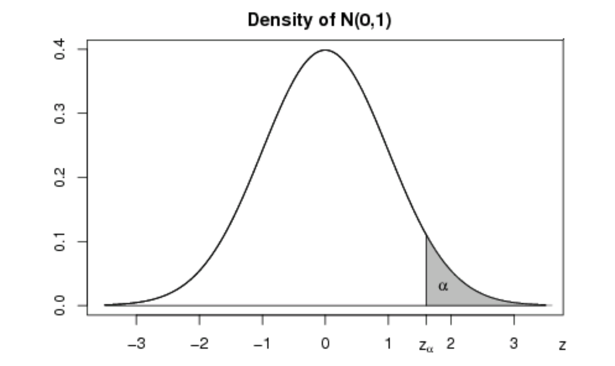

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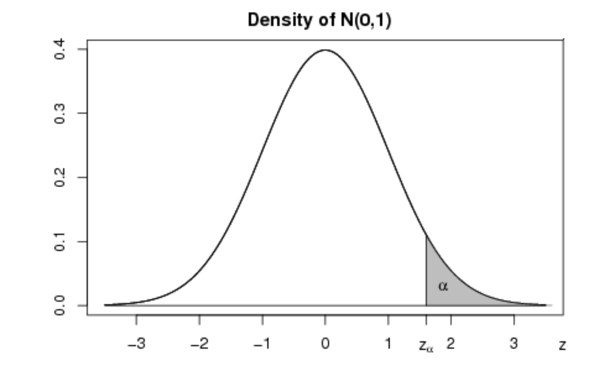

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If $\alpha = 0.05$ then $z_\alpha = 1.645$.

(c) $n = 25$, $\bar{x} = 172.52$, $\alpha = 0.05$, $z_\alpha = 1.645$.

$$z = \frac{\bar{x} - 170}{10/\sqrt{n}} = \frac{172.52 - 170}{10/\sqrt{25}} = 1.26$$

(d) Conclusion: Since $z = 1.26 < z_\alpha = 1.645$, don't reject H_0.

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The p-value Approach

- One important statistic which is usually reported in statistical analysis is the **p-value**:
- the probability of rejecting true null H_0 using the observed value of test statistic as the critical value.
- For example, for the above hypotheses $H_0 : p = p_0 \quad \text{vs} \quad H_1 : p < p_0$, if the observed value of Y is y, then the observed value of Z is

$$z = \frac{y - np_0}{\sqrt{np_0(1-p_0)}}$$

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$$p_{value} = P(Z \leqslant z | p = p_0) = P(Y \leqslant y | p = p_0) \approx \Phi\left(\frac{y - np_0}{\sqrt{np_0(1-p_0)}}\right).$$

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- If p-value $< \alpha$, then we reject H_0 at the significance level α. Otherwise accept H_0.
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P-value–the formal definition

Definition

If W is a test statistic, the *p-value*, or *attained significance level*, is the smallest level of significance α for which the observed data indicate that the null hypothesis should be rejected.

Let RR_α be the rejection region of significance level α:

$$P(W \in RR_\alpha | H_0) = \alpha.$$

If the data resulted in an observed value w of W, then the

$$p\text{-value} = \min\{\alpha : w \in RR_\alpha\}.$$

Rejection Region: p-value $< \alpha$.

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Example

A health club claims that its members lose an average of 10 pounds or more within the 1st month after joining the club. A consumer agency wanted to test this claim.

A sample 36 members was selected and an average weight lost of 9.2 pounds was obtained. Assume the population standard deviation is known to be 2.4 pounds. What is the p-value of this test?

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Step 1. Hypotheses:

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Step 2. Since $\sigma = 2.4$ known, $n = 36 > 30$, use normal distribution and z test statistic $Z = (\bar{X} - \mu_0)/(\sigma/\sqrt{n})$ and rejection region $RR_\alpha = \{Z < -z_\alpha\}$.

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$$p\text{-value} = \min\{\alpha : -2.00 < -z_\alpha\} = P(Z < -2.00) = 0.0228$$

If $\alpha = 0.01$, then since p-value > 0.01, we do not reject H_0 at the significance level 0.01.
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An Example of One-Sided Test

Example 1. Let x be the Brinell hardness measurement of ductile iron subcritically annealed. Assume that x has a normal distribution with mean μ. Test $H_0 : \mu = 170$ against the alternative hypothesis that the mean hardness is greater than 170 based on the following 25 observations(Use significance level $\alpha = 0.05$.)

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170, 167, 174, 179, 179, 156, 163, 156, 187
156, 183, 179, 174, 179, 170, 156, 187,
179, 183, 174, 187, 167, 159, 170, 179
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Another Example of Two-Sided Test

Example 3. A Casino want to test whether the probability p that a red number comes up on a Nevada roulette wheel equals 18/38 or not.

Hypotheses: $H_0 : p = 18/38 \quad H_1 : p \neq 18/38.$

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