Combinatorial Counting & Probability (3cr)

Chapter 6 Some Continuous Probability Distributions

Zhong Guan

Math, IUSB

Outline

Outline

Recall from Chapter 2

Binomial experiment is an experiment consists of n repeated independent trials, each trial has two outcomes: **success** and **failure** and the probability of success p remains constant. Each trial is called a **Bernoulli trial.**

Binomial distribution: Let X be the number of "successes" in a binomial experiment of n trials with probability p of success. The distribution of X is called a **binomial distribution** with p.m.f.

$$P(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

with mean $\mu = np$ and standard deviation $\sigma = \sqrt{np(1-p)}$.

Recall from Chapter 2

Binomial experiment is an experiment consists of n repeated independent trials, each trial has two outcomes: **success** and **failure** and the probability of success p remains constant. Each trial is called a **Bernoulli trial**.

Binomial distribution: Let X be the number of "successes" in a binomial experiment of n trials with probability p of success. The distribution of X is called a **binomial distribution** with p.m.f.

$$P(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

with mean $\mu = np$ and standard deviation $\sigma = \sqrt{np(1-p)}$.

Example of binomial close to normal

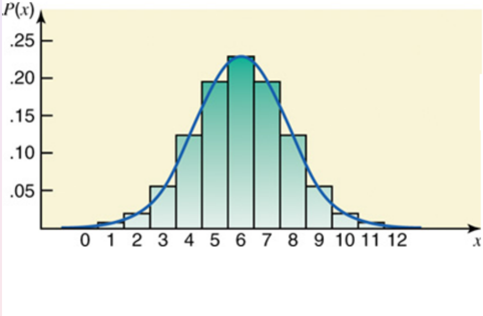
Table 6.5 The Binomial Probability Distribution for $n = 12$ and $p = .50$

x	$P(x)$
0	.0002
1	.0029
2	.0161
3	.0537
4	.1208
5	.1934
6	.2256
7	.1934
8	.1208
9	.0537
10	.0161
11	.0029
12	.0002

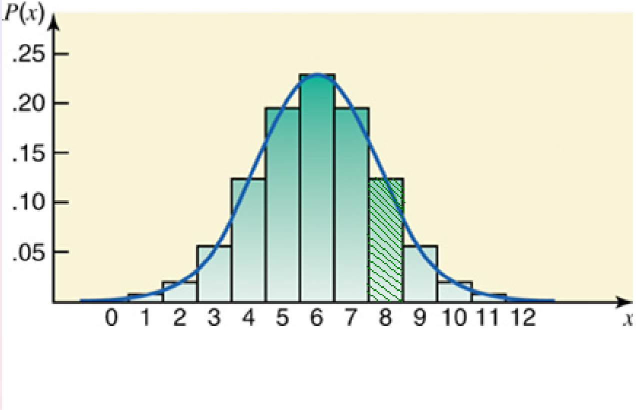
Example of binomial close to normal

Table 6.5 The Binomial Probability Distribution for $n = 12$ and $p = .50$

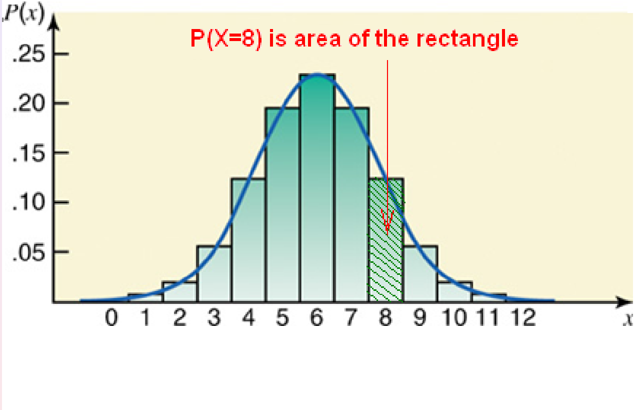
x	$P(x)$
0	.0002
1	.0029
2	.0161
3	.0537
4	.1208
5	.1934
6	.2256
7	.1934
8	.1208
9	.0537
10	.0161
11	.0029
12	.0002



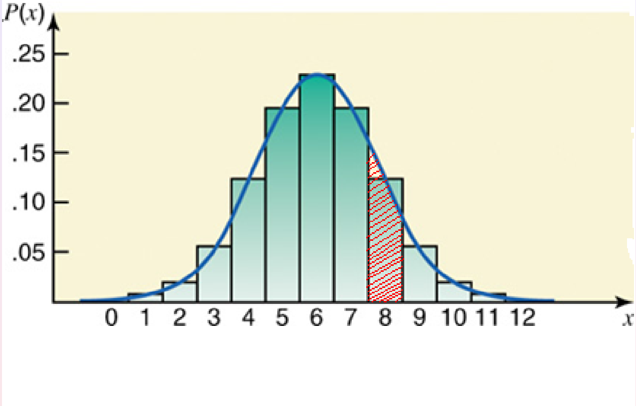
Normal Approximation to Binomial



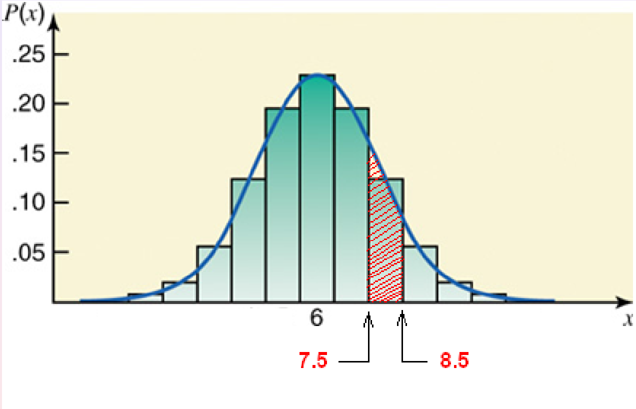
Normal Approximation to Binomial



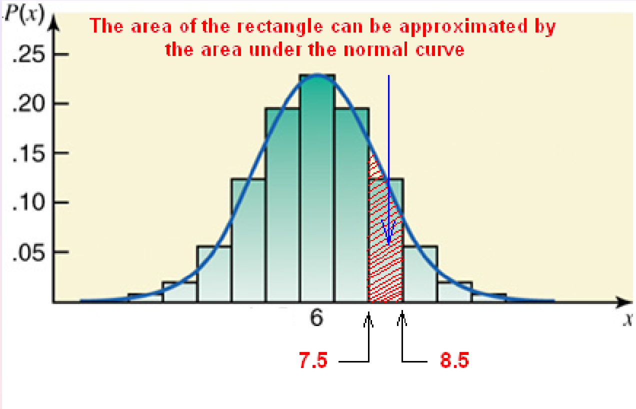
Normal Approximation to Binomial



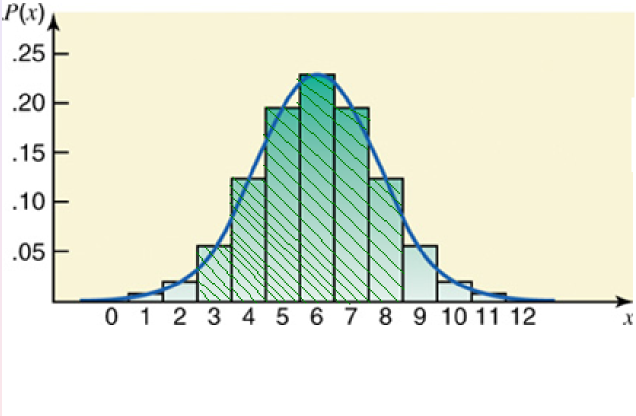
Normal Approximation to Binomial



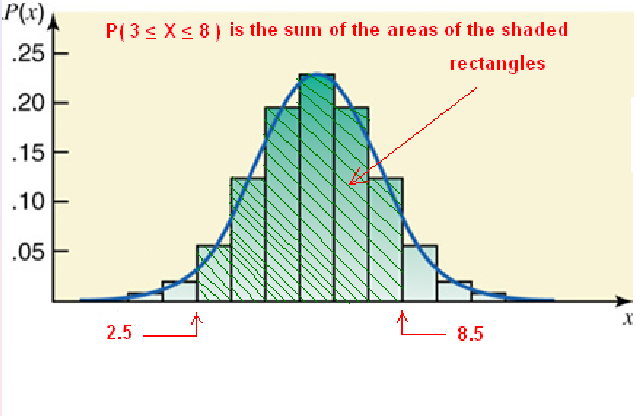
Normal Approximation to Binomial



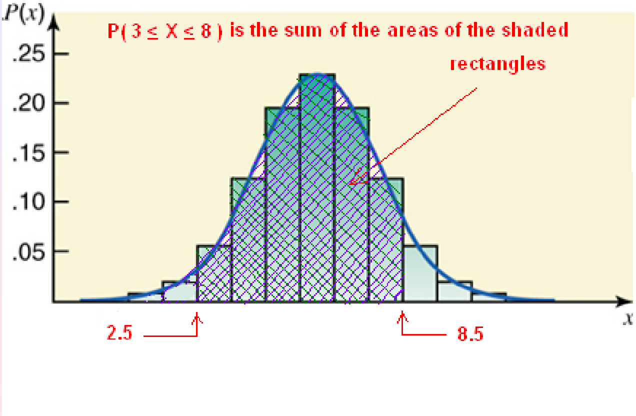
Normal Approximation to Binomial



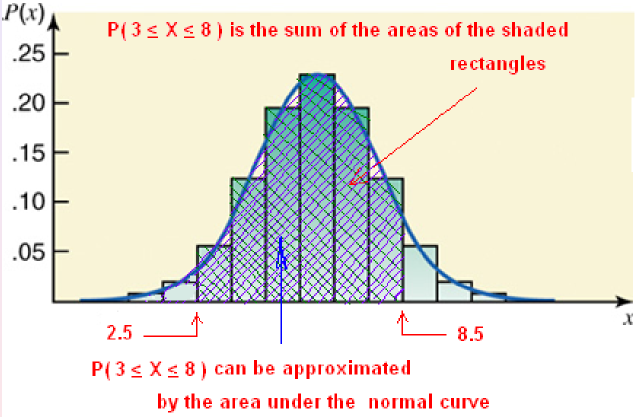
Normal Approximation to Binomial



Normal Approximation to Binomial



Normal Approximation to Binomial



Normal Approximation to Binomial

If $Y \sim b(n,p)$, then for large n

$$\frac{Y}{n} \sim N(p, p(1-p)/n), \text{ approximately.}$$

That is

$$Y \sim N[np, np(1-p)], \text{ approximately.}$$

So for large n, and integers a and b,

$$\begin{aligned} P(a \leqslant Y \leqslant b) &= P(a-0.5 \leqslant Y \leqslant b+0.5) \\ &\approx \Phi\left[\frac{b+0.5-np}{\sqrt{np(1-p)}}\right] - \Phi\left[\frac{a-0.5-np}{\sqrt{np(1-p)}}\right] \end{aligned}$$

Normal Approximation to Binomial

If $Y \sim b(n, p)$, then for large n

$$\frac{Y}{n} \sim N(p, p(1-p)/n), \text{ approximately.}$$

That is

$$Y \sim N[np, np(1-p)], \text{ approximately.}$$

So for large n, and integers a and b,

$$P(a \leqslant Y \leqslant b) = P(a - 0.5 \leqslant Y \leqslant b + 0.5)$$
$$\approx \Phi\left[\frac{b+0.5-np}{\sqrt{np(1-p)}}\right] - \Phi\left[\frac{a-0.5-np}{\sqrt{np(1-p)}}\right]$$

Normal Approximation to Binomial

If $Y \sim b(n, p)$, then for large n

$$\frac{Y}{n} \sim N(p, p(1-p)/n), \text{ approximately.}$$

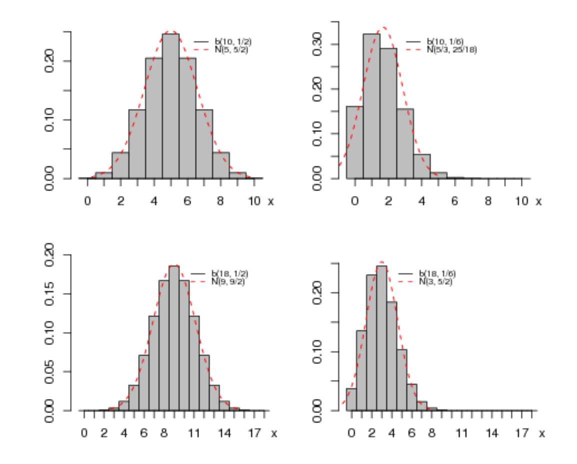
That is

$$Y \sim N[np, np(1-p)], \text{ approximately.}$$

So for large n, and integers a and b,

$$\begin{aligned} P(a \leqslant Y \leqslant b) &= P(a - 0.5 \leqslant Y \leqslant b + 0.5) \\ &\approx \Phi\left[\frac{b+0.5-np}{\sqrt{np(1-p)}}\right] - \Phi\left[\frac{a-0.5-np}{\sqrt{np(1-p)}}\right] \end{aligned}$$

Normal Approximation to Binomial



Examples

Example 1: Among the gifted 7th-graders who score very high on mathematics exam, approximately 20% are left-handed or ambidextrous. Let X equal the number of left-handed or ambidextrous students among a random sample of $n = 25$ gifted 7th-graders. Find $P(2 < X < 9)$, approximately.

Examples

Solution of Example 1: Since $X \sim b(n, 0.20)$, $n = 25$, $X \sim N(\mu, \sigma^2)$, approximately, with $\mu = np = 25(0.2) = 5$, $\sigma^2 = 25(0.2)(0.8) = 4$.
Approximately,

$$P(2 < X < 9) \approx \Phi(\tfrac{8.5-5}{\sqrt{4}}) - \Phi(\tfrac{2.5-5}{\sqrt{4}}) = \Phi(1.75) - \Phi(-1.25)$$
$$= \Phi(1.75) - 1 + \Phi(1.25) = 0.854291$$

The exact value is

$$P(2 < X < 9) = P(X \leqslant 8) - P(X \leqslant 2) = 0.9532 - 0.0982 = 0.855.$$

Outline

Normal Approximation to Poisson

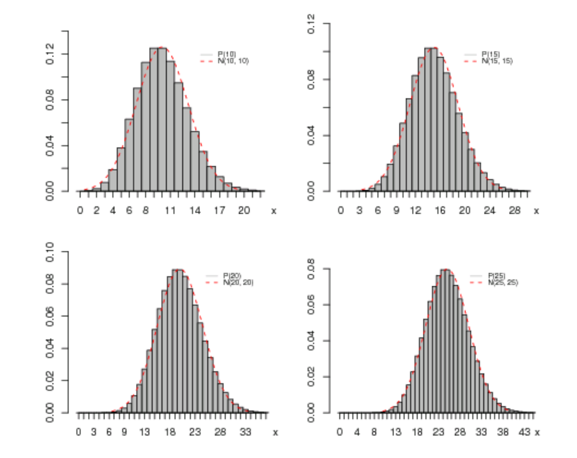
If $Y \sim P(\mu)$ for large μ, then

$$W = \frac{Y-\mu}{\sqrt{\mu}} \sim N(0,1), \text{ approximately.}$$

So for large μ, and integers a and b,

$$\begin{aligned} P(a \leqslant Y \leqslant b) &= P(a-0.5 \leqslant Y \leqslant b+0.5) \\ &\approx \Phi\left[\frac{b+0.5-\mu}{\sqrt{\mu}}\right] - \Phi\left[\frac{a-0.5-\mu}{\sqrt{\mu}}\right] \end{aligned}$$

Normal Approximation to Poisson



Examples

Example 2: Let X equal the number of alpha particles counted by a Geiger counter during 30 seconds. Assume that $X \sim P(\mu)$, with $\mu = \lambda t = 49$. Find $P(45 < X < 60)$ (a) exactly and (b) approximately.

Examples

Solution of Example 2: Since $X \sim P(49)$, $X \sim N(49, 49)$, approximately.
(a) Using Excel,

$$P(45 < X < 60) = P(X \leqslant 59) - P(X \leqslant 45) = 0.614817548.$$

(b) Approximately, using the normal approximation,

$$P(45 < X < 60) \approx \Phi(\tfrac{59.5-49}{\sqrt{49}}) - \Phi(\tfrac{45.5-49}{\sqrt{49}}) = \Phi(1.5) - \Phi(-0.5)$$
$$= \Phi(1.5) - 1 + \Phi(0.5) = 0.6246553$$