6.1 Continuous Uniform Distribution

Example 1: Let X be the random number selected from $[a, b]$. Then X is continuous random variable. Intuitively, if each number in $[a, b]$ is equally likely to be selected, then for $a \leqslant x \leqslant b$,

$$F(x) = P(X \leqslant x) = \frac{x-a}{b-a}$$

$$P(X \leqslant x) = 0, \quad x < a; \quad P(X \leqslant x) = 1, \quad x > b$$

The c.d.f. of X is

$$F(x) = P(X \leqslant x) = \begin{cases} 0, & x < a; \\ \frac{x-a}{b-a}, & a \leqslant x \leqslant b; \\ 1, & x > b. \end{cases}$$

The p.d.f. of X is

$$f(x) = F'(x) = \begin{cases} \frac{1}{b-a}, & a \leqslant x \leqslant b; \\ 0, & \text{elsewhere.} \end{cases}$$

Definition If random variable X has density

$$f(x) = F'(x) = \begin{cases} \frac{1}{b-a}, & a \leqslant x \leqslant b; \\ 0, & \text{elsewhere.} \end{cases}$$

then X is said to have a (continuous) uniform distribution on $[a, b]$ and denoted as

Example 2: Customers arrive randomly at a bank teller's window. Given that a customer arrived during a particular 10-minute period. Let X equal the time within the 10 minutes that the customer arrived. If $X \sim U(0, 10)$, Find

(a) The pdf of X;

(b) $P(X \geqslant 8)$;

(c) $P(2 \leqslant X < 8)$;

(d) $E(X)$;

(e) $\mathrm{Var}(X)$.

Example 3: If $X \sim U(a, b)$, then show that the mean and variance of X are, respectively,

$$\mu = E(X) = \frac{a+b}{2}, \quad \sigma^2 = \mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

Example 4: If $U \sim U(0, 1)$, then $X = a + (b-a)U \sim U(a, b)$.

Pseudorandom Numbers (PRN's)

Linear Congruential Generator (LCG) is used by many software to generate PRN's:

$$x_{n+1} = ax_n + c \ mod \ m$$

where a, c and m are integers satisfy certain conditions.
For example, $a = 1664525$, $c = 1013904223$, $m = 2^{32}$ (*Numerical Recipes in C*).
x_0 is called the random seed.
$U_n = \frac{x_n}{m}$, $n = 1, 2, \ldots$, is a sequence of PRN's from $U(0, 1)$.
$Y_n = a + (b-a)U_n$, $n = 1, 2, \ldots$, is a sequence of PRN's from $U(a, b)$.

PRN's from Discrete Uniform Distribution:
If U_n are PRN's from $U(0, 1)$, then $X_n = \lfloor kU_n + 1 \rfloor$ a sequence of PRN's from discrete uniform distribution over $\{1, 2, 3, \ldots, k\}$,
where $\lfloor x \rfloor$ is the floor of x.

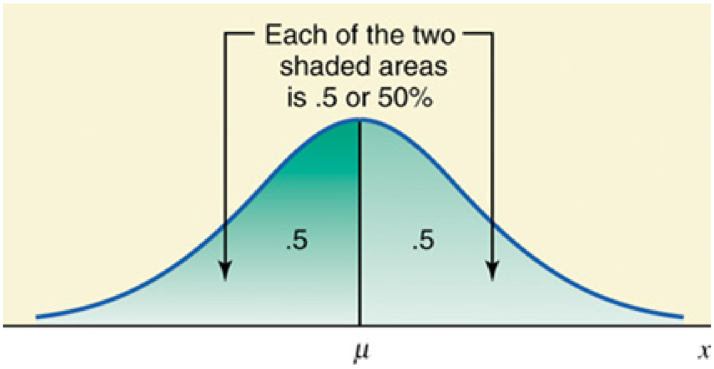
6.2 Normal Distribution

Normal/Gaussian Distribution:
A r.v. X is said to have a normal or Gaussian distribution with mean μ and variance σ^2, denoted by $X \sim N(\mu, \sigma^2)$, if its density is

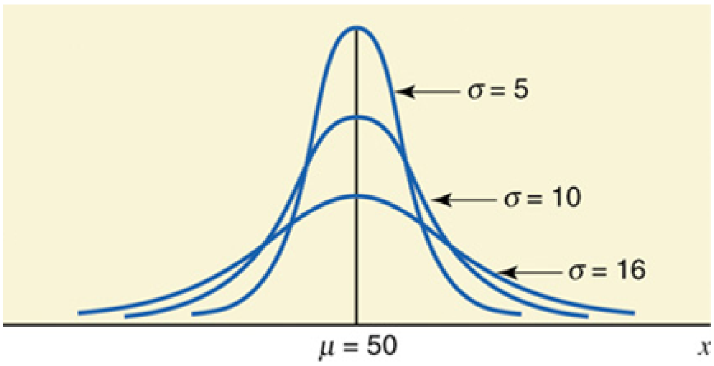
$$n(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$$

where $e = 2.71828\ldots$.

Normal curve is symmetric about the mean

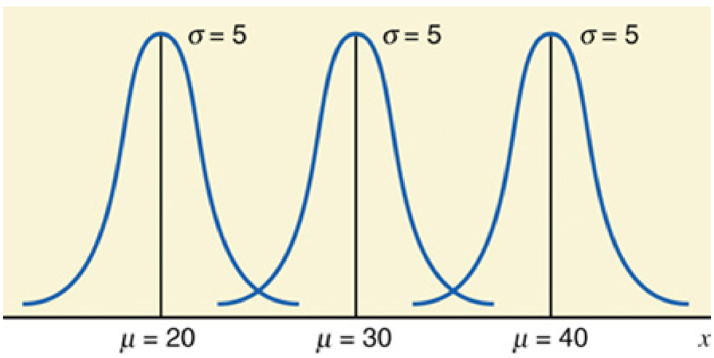


Shape of normal curve is determine by σ



The standard deviation determines the spread, but total area must be 100%. So σ determines the shape of normal curve.

Location of normal curve is determine by μ



For a fixed standard deviation, changing the value of μ will move the normal curve horizontally.

Linear Transformation of $X \sim N(\mu, \sigma^2)$
Theorem 1. If $X \sim N(\mu, \sigma^2)$, then
(a)

$$Y = aX + b \sim N[a\mu + b, (|a|\sigma)^2].$$

(b)

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

Example: If $X \sim N(1,9)$, find the distribution of $Y = -2X+1$.
Solution: The distribution of Y is $N(-1, [2(3)]^2)$, the normal distribution with mean -1 and variance 36. **Standard Normal Distribution** Definition: If r.v. Z is $N(0,1)$, then Z has standard normal distribution. Its density is

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Distribution function is

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

Clearly, $\Phi(0) = \frac{1}{2}$ and $1 - \Phi(z) = \Phi(-z)$

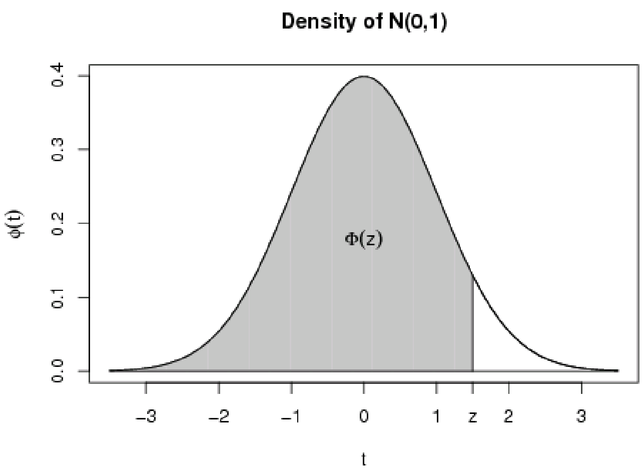
6.3 Area under the Normal Curve

Probability = Area Under Normal Curve

If r.v. X is $N(\mu, \sigma^2)$, then

$$\begin{aligned} P(x_1 < X < x_2) &= \text{area under normal curve between } x_1 \text{ and } x_2 \\ &= P\left(\frac{x_1-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{x_2-\mu}{\sigma}\right) \\ &= P\left(\frac{x_1-\mu}{\sigma} < Z < \frac{x_2-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{x_2-\mu}{\sigma}\right) - \Phi\left(\frac{x_1-\mu}{\sigma}\right) \end{aligned}$$

Standard Normal p.d.f. Curves $\Phi(z) = P(Z \leqslant z) =$ area of shaded region.



Example 1: Let X be a normal random variable with mean $\mu = 50$ and standard deviation $\sigma = 10$. Convert x values to z values and find

(a) $P(X \leqslant 55)$.

(b) $P(X < 35)$

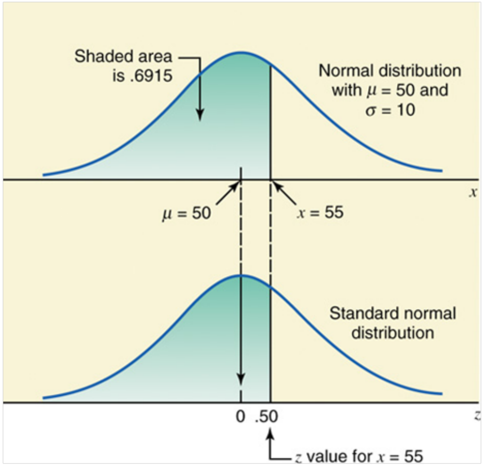
Solution of Example 1: $\mu = 50, \quad \sigma = 10$

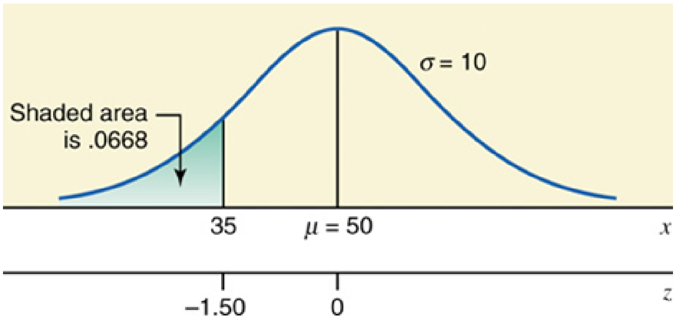
(a) $x = 55$.

$$z = \frac{x-\mu}{\sigma} = \frac{55-50}{10} = \frac{5}{10} = \frac{1}{2} = 0.5$$

(b) $x = 35$

$$z = \frac{x-\mu}{\sigma} = \frac{35-50}{10} = \frac{-15}{10} = -\frac{3}{2} = -1.5$$





Example 2: Let X be a normal random variable with mean $\mu = 25$ and standard deviation $\sigma = 4$. Convert x values to z values and find

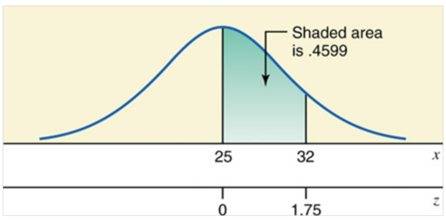
(a) $P(25 < X \leqslant 32)$.

(b) $P(18 \leqslant X \leqslant 34)$.

Solution of Example 2: $\mu = 25, \quad \sigma = 4$

(a) $x_1 = 25$ and $x_2 = 32$.

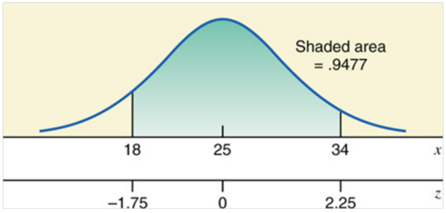
$$z_1 = \frac{x_1-\mu}{\sigma} = \frac{25-25}{4} = \frac{0}{4} = 0$$
$$z_2 = \frac{x_2-\mu}{\sigma} = \frac{32-25}{4} = \frac{7}{4} = 1.75$$



$\mu = 25, \quad \sigma = 4$

(b) $x_1 = 18$ and $x_2 = 34$.

$$z_1 = \frac{x_1-\mu}{\sigma} = \frac{18-25}{4} = \frac{-7}{4} = -1.75$$
$$z_2 = \frac{x_2-\mu}{\sigma} = \frac{34-25}{4} = \frac{9}{4} = 2.25$$



Excel Function for Normal Distribution:

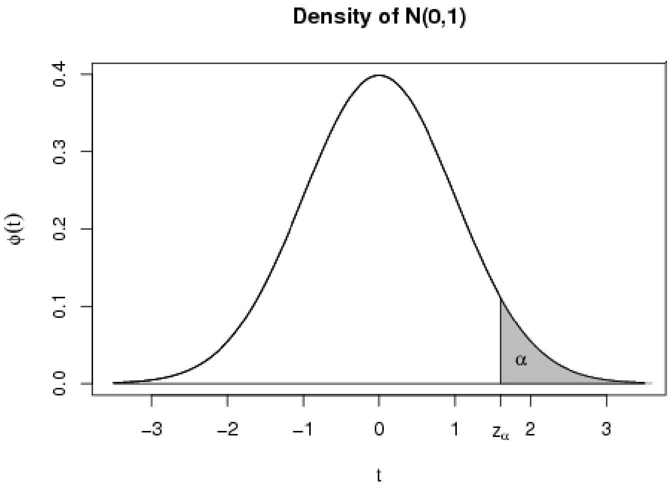
1) `NORMDIST`(x, μ, σ, `cumul`) returns $P(X \leqslant x)$ if `cumul=TRUE`, p.d.f. $n(x; \mu, \sigma)$ otherwise.
2) `NORMINV`(p, μ, σ) returns $\mu + z_\alpha \sigma$ if $p = 1 - \alpha$.
3) `NORMSDIST(z)` returns $\Phi(z)$ and
4) `NORMSINV`(p) returns z_α if $p = 1 - \alpha$.
TI-8x Use TI8x: "DISTR" $\to$ "normalcdf":
"normalcdf(x_1, x_2, μ, σ)" gives $P(x_1 < X < x_2)$;
"normalcdf($-E99$, x, μ, σ)" gives $P(X \leqslant x)$, where "E"=$\boxed{\text{2nd}}$ + $\boxed{\text{EE}}$.
The Upper 100α **Percent Point** The upper 100α percent point, z_α is $100(1-\alpha)$ percentile. That is

$$P(Z \geqslant z_\alpha) = \alpha$$

where $Z \sim N(0,1)$. Since

$$P(Z \leqslant -z_\alpha) = P(Z \geqslant z_\alpha) = \alpha$$
$$z_{1-\alpha} = -z_\alpha$$

Normal p.d.f. Curves



Example 3 If $X \sim N(6, 25)$, find

(a) $P(6 \leqslant X \leqslant 12)$;

(b) $P(X > 21)$;

(c) $P(|X - 6| < 10)$;

$$\begin{aligned}
(a)\quad & P(6 \leqslant X \leqslant 12) = P\big(\tfrac{6-6}{5} \leqslant \tfrac{X-6}{5} \leqslant \tfrac{12-6}{5}\big) = P(0 \leqslant Z \leqslant 1.2) \\
& \qquad = \Phi(1.2) - \Phi(0) = 0.8849 - 0.5 = 0.3849 \\
(b)\quad & P(X > 21) = 1 - P(X \leqslant 21) = 1 - P\big(\tfrac{X-6}{5} \leqslant \tfrac{21-6}{5}\big) = 1 - P(Z \leqslant 3) \\
& \qquad = 1 - \Phi(3) = 1 - 0.999 = 0.001 \\
(c)\quad & P(|X - 6| < 10) = P(-10 < X - 6 < 10) \\
& \qquad = P\big(-\tfrac{10}{5} \leqslant \tfrac{X-6}{5} \leqslant \tfrac{10}{5}\big) = P(-2 \leqslant Z \leqslant 2) = 2\Phi(2) - 1 = 0.9544
\end{aligned}$$

Example 4 If $X \sim N(\mu, \sigma^2)$, find the probability that X differs from the mean by more than $k\sigma$ where $k > 0$. Compare the result with one obtained by using Chebyshev Theorem.

6.4 Applications of the Normal Distribution

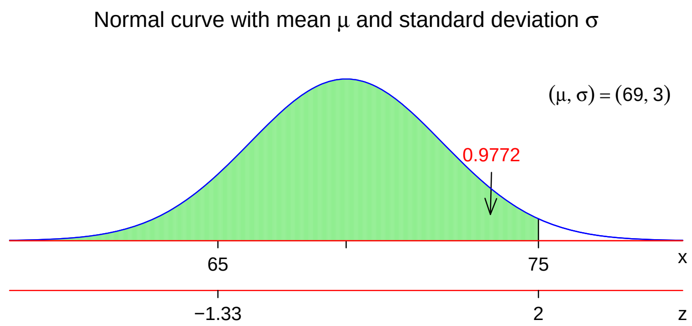
Example 4: Suppose that heights of adult males are normally distributed with average of 69 inches and a standard deviation of 3 inches.

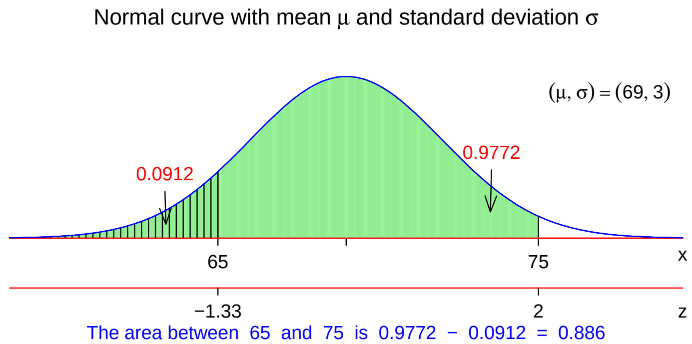
(a) What is the probability that a randomly selected adult male taller than 65 inches and shorter than 75 inches?

(b) What percentage of adult males are taller than 80 inches?

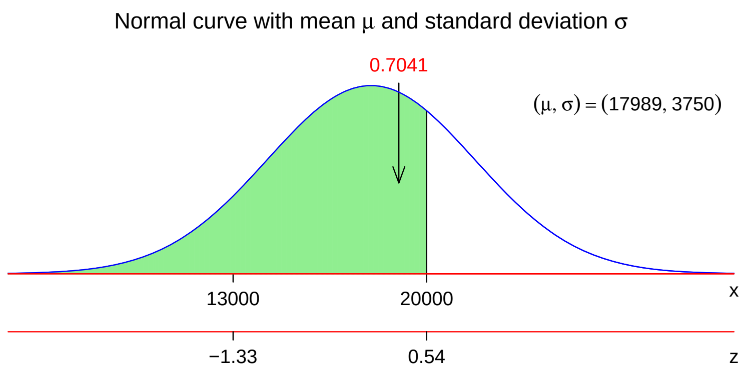
(c) How many adult males are taller than Shaquille O'Neal who is 85 inches?

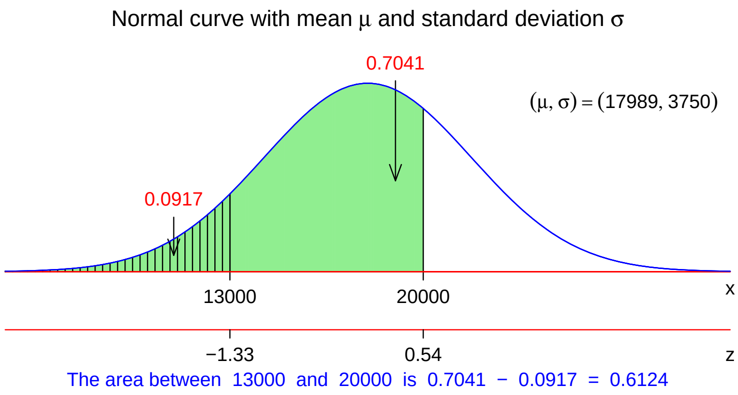
Solution of Example 4(a)





Example 5: Suppose that consumer debt (owed on cars, credit cards, and so forth) for all U.S. households have a normal distribution with mean of \$17,989 and a standard deviation of \$3750. Find the probability that such consumer debt of a randomly selected U.S. household is between \$13,000 and \$20,000.
Solution:





Log-normal Distribution

Definition 0.1. Random variable X is said to have a log-normal distribution $LN(\mu, \sigma^2)$ if its natural logarithm $\ln(X)$ has normal distribution $N(\mu, \sigma^2)$. That is X has pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}x} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\}, \quad x > 0.$$

Application in Stock price: Log-normal is commonly used to model stock returns: Let S_i be stock price at the end of the i-th day, $i = 0, 1, \ldots, n$, and $R_i = \frac{S_i}{S_{i-1}}$, $i = 1, \ldots, n$. Under certain conditions, we can assume that $R_1, \ldots, R_n$ are independent and identically distributed log-normal random variables.

Cumulative Distribution Function Since $Y = \ln(X)$ has normal distribution $N(\mu, \sigma^2)$, the cumulative distribution function of X is

$$\begin{aligned} F(x) &= P(X \leqslant x) \\ &= P[\ln(X) \leqslant \ln(x)] = P[Y \leqslant \ln(x)] \\ &= \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right], \quad x > 0. \end{aligned}$$

Mean: $E(X) = \exp\left(\mu + \sigma^2/2\right)$

$$\begin{aligned} E(X) &= \int_0^\infty x f(x)dx = \int_0^\infty \cancel{x} \frac{1}{\sigma\sqrt{2\pi}\cancel{x}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\} dx \\ &\underset{dx = e^y dy}{\overset{y=\ln(x),\, x=e^y}{=\!=\!=}} \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} e^y dy \\ &= \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu)^2 - 2\sigma^2 y}{2\sigma^2}\right\} dy \\ &= \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{[y-(\mu+\sigma^2)]^2}{2\sigma^2}\right\} \exp\left\{-\frac{(\mu+\sigma^2)^2 - \mu^2}{2\sigma^2}\right\} dy \\ &= \exp\left(\mu + \sigma^2/2\right) \end{aligned}$$

Variance

$$\begin{aligned} E(X^2) &= \int_0^\infty x^2 f(x)dx = \int_0^\infty x^{\cancel{2}} \frac{1}{\sigma\sqrt{2\pi}\cancel{x}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\} dx \\ &\underset{dx = e^y dy}{\overset{y=\ln(x),\, x=e^y}{=\!=\!=}} \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} e^{2y} dy \\ &= \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu)^2 - 4\sigma^2 y}{2\sigma^2}\right\} dy \end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{[y-(\mu+2\sigma^2)]^2}{2\sigma^2}\right\} \exp\left\{-\frac{(\mu+2\sigma^2)^2-\mu^2}{2\sigma^2}\right\} dy$$
$$= \exp\left\{2(\mu+\sigma^2)\right\}$$
$$V(X) = E(X^2) - [E(X)]^2 = e^{2(\mu+\sigma^2)} - e^{2(\mu+\sigma^2/2)} = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$$

6.5 Normal Approximation to Binomial and Poisson

Recall from Chapter 2

Binomial experiment is an experiment consists of n repeated independent trials, each trial has two outcomes: **success** and **failure** and the probability of success p remains constant. Each trial is called a **Bernoulli trial**.

Binomial distribution: Let X be the number of "successes" in a binomial experiment of n trials with probability p of success. The distribution of X is called a **binomial distribution** with p.m.f.

$$P(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

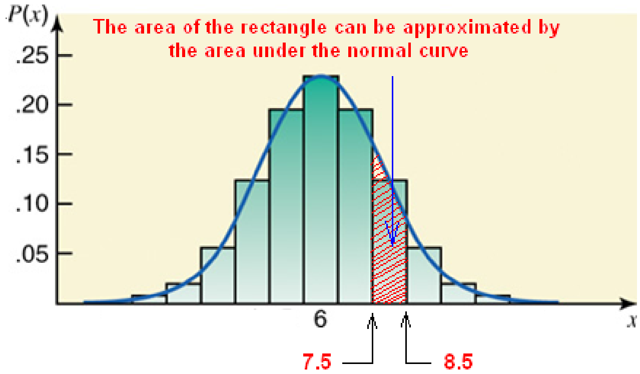
with mean $\mu = np$ and standard deviation $\sigma = \sqrt{np(1-p)}$.

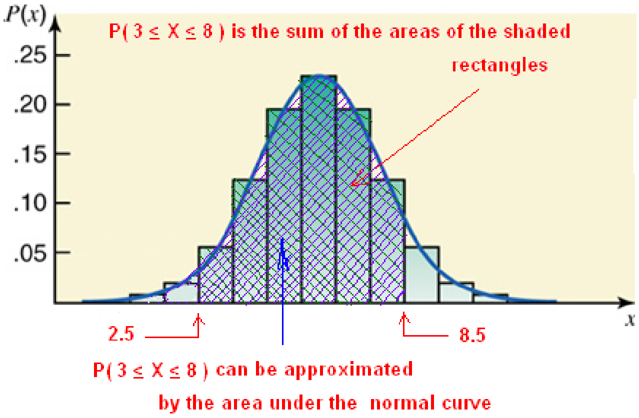
Example of binomial close to normal

Table 6.5 **The Binomial Probability Distribution for $n = 12$ and $p = .50$**

x	$P(x)$
0	.0002
1	.0029
2	.0161
3	.0537
4	.1208
5	.1934
6	.2256
7	.1934
8	.1208
9	.0537
10	.0161
11	.0029
12	.0002

Normal Approximation to Binomial





Normal Approximation to Binomial

If $Y \sim b(n, p)$, then for large n

$$\frac{Y}{n} \sim N(p, p(1-p)/n), \text{ approximately.}$$

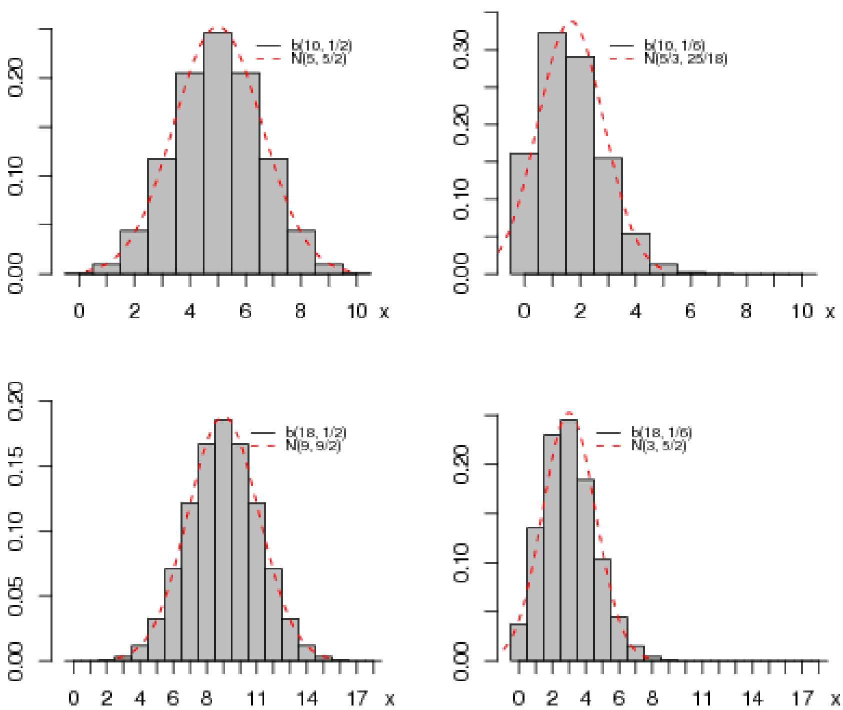
That is

$$Y \sim N[np, np(1-p)], \text{ approximately.}$$

So for large n, and integers a and b,

$$P(a \leqslant Y \leqslant b) = P(a - 0.5 \leqslant Y \leqslant b + 0.5) \approx \Phi\left[\frac{b+0.5-np}{\sqrt{np(1-p)}}\right] - \Phi\left[\frac{a-0.5-np}{\sqrt{np(1-p)}}\right]$$

Normal Approximation to Binomial



Example 1: Among the gifted 7th-graders who score very high on mathematics exam, approximately 20% are left-handed or ambidextrous. Let X equal the number of left-handed or ambidextrous students among a random sample of $n = 25$ gifted 7th-graders. Find $P(2 < X < 9)$, approximately.

Solution of Example 1: Since $X \sim b(n, 0.20)$, $n = 25$, $X \sim N(\mu, \sigma^2)$, approximately, with

$\mu = np = 25(0.2) = 5$, $\sigma^2 = 25(0.2)(0.8) = 4$.
Approximately,

$$\begin{aligned} P(2 < X < 9) &\approx \Phi(\tfrac{8.5-5}{\sqrt{4}}) - \Phi(\tfrac{2.5-5}{\sqrt{4}}) = \Phi(1.75) - \Phi(-1.25) \\ &= \Phi(1.75) - 1 + \Phi(1.25) = 0.854291 \end{aligned}$$

The exact value is

$$P(2 < X < 9) = P(X \leqslant 8) - P(X \leqslant 2) = 0.9532 - 0.0982 = 0.855.$$

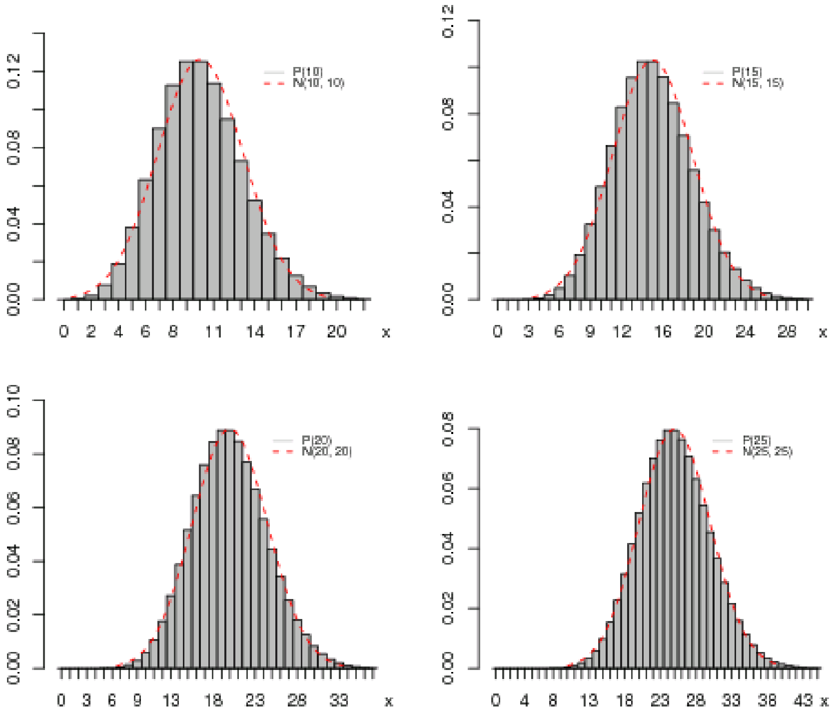
Normal Approximation to Poisson

If $Y \sim P(\mu)$ for large μ, then

$$W = \tfrac{Y-\mu}{\sqrt{\mu}} \sim N(0,1), \text{ approximately.}$$

So for large μ, and integers a and b,

$$P(a \leqslant Y \leqslant b) = P(a - 0.5 \leqslant Y \leqslant b + 0.5) \approx \Phi\left[\tfrac{b+0.5-\mu}{\sqrt{\mu}}\right] - \Phi\left[\tfrac{a-0.5-\mu}{\sqrt{\mu}}\right]$$



Example 2: Let X equal the number of alpha particles counted by a Geiger counter during 30 seconds. Assume that $X \sim P(\mu)$, with $\mu = \lambda t = 49$. Find $P(45 < X < 60)$ (a) exactly and (b) approximately.

Solution of Example 2: Since $X \sim P(49)$, $X \sim N(49, 49)$, approximately.
(a) Using Excel,

$$P(45 < X < 60) = P(X \leqslant 59) - P(X \leqslant 45) = 0.614817548.$$

(b) Approximately, using the normal approximation,

$$\begin{aligned} P(45 < X < 60) &\approx \Phi(\tfrac{59.5-49}{\sqrt{49}}) - \Phi(\tfrac{45.5-49}{\sqrt{49}}) = \Phi(1.5) - \Phi(-0.5) \\ &= \Phi(1.5) - 1 + \Phi(0.5) = 0.6246553 \end{aligned}$$

6.6 Gamma and Exponential Distributions

Waiting Time Distributions

Example 1. If the number of calls received per hour by a telephone answering service has a Poisson distribution with $\lambda = 6$.
(a) What is the distribution of the waiting time W for the first call?
(b) What is the distribution of the waiting time W for the 3rd call?

Gamma Distribution

Gamma Distribution is a Waiting Time Distribution
In a Poisson process with rate λ, mean number of events per unit "time", let W be the waiting time until the αth occurrence. Then

$$\begin{aligned} F(w) &= P(W \leqslant w) = 1 - P(W > w) \\ &= 1 - P(\text{fewer than } \alpha \text{ occurrences in } [0, w]) \\ &= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!} e^{-\lambda w} = 1 - e^{-\lambda w} \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!} \end{aligned}$$

If $w < 0$ then $F(w) = 0$.

The p.d.f. of Gamma Distribution
The p.d.f. is $f(w) = F'(x)$:
if $w \geqslant 0$ then

$$\begin{aligned} f(w) = F'(w) &= -\left[e^{-\lambda w} \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!}\right]' = \lambda e^{-\lambda w} \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!} - e^{\lambda w} \sum_{k=0}^{\alpha-1} \frac{k\lambda(\lambda w)^{k-1}}{k!} \\ &= \lambda e^{-\lambda w} \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!} - e^{\lambda w} \sum_{k=1}^{\alpha-1} \frac{\lambda(\lambda w)^{k-1}}{(k-1)!} = \lambda e^{-\lambda w} \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!} - \lambda e^{\lambda w} \sum_{j=0}^{\alpha-2} \frac{(\lambda w)^j}{j!} \\ &= \lambda e^{-\lambda w} \frac{(\lambda w)^{\alpha-1}}{(\alpha-1)!} = \frac{\lambda^\alpha w^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda w} \end{aligned}$$

Definition of Gamma Distribution The above density can be written as

$$f(w) = \frac{w^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{w}{\beta}}, \quad \lambda = \frac{1}{\beta}$$

Definition: Generally, if random variable X has p.d.f.

$$f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{x}{\beta}}, \qquad 0 < x < \infty,$$

where $\alpha > 0$ and $\beta > 0$, then X has a **gamma distribution**, and denote $X \sim \Gamma(\alpha, \beta)$.
Mean: $\mu = E(X) = \alpha\beta$.
Variance: $\sigma^2 = \mathrm{Var}(X) = \alpha\beta^2$.

Gamma Function: The **gamma** function:

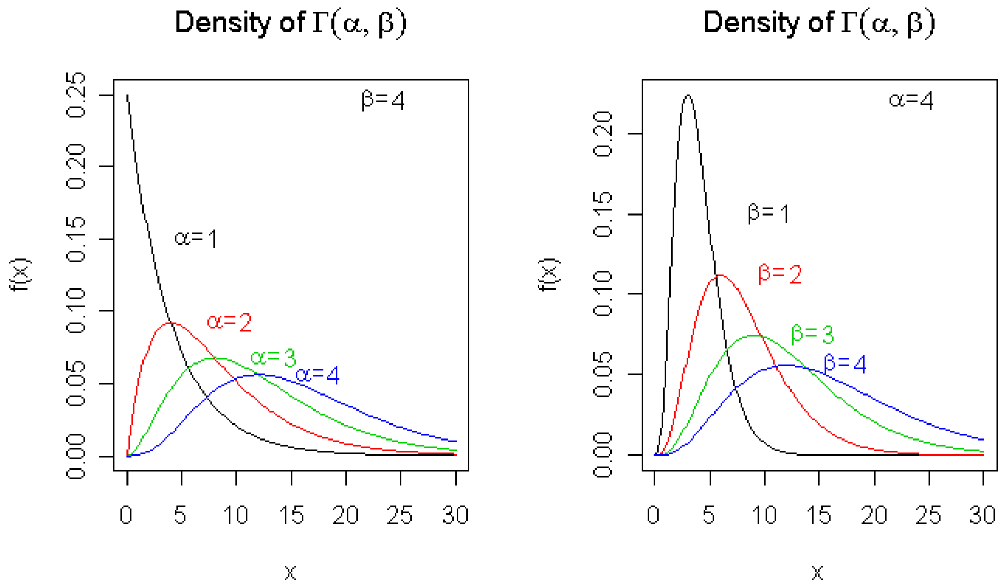
$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy, \qquad t > 0.$$

$$\Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(t) = (t-1)\Gamma(t-1)$$

For integer $n > 0$,

$$\Gamma(n) = (n-1)!$$

Gamma p.d.f. Curves



Excel Functions:
1) GAMMA.DIST(x, alpha, beta, cumulative) returns c.d.f. if cumulative = TRUE, p.d.f. otherwise. beta $= \theta$.
2) GAMMA.INV(probability,alpha,beta) returns the inverse (quantile) function of $\Gamma(\alpha, \beta)$.
3) GAMMALN(x) returns $\ln \Gamma(x)$
TI-8x:

- TI-83/84:
  - using interpolation or integral function `fnInt()`.
- Example: $\alpha = 2.2$, $\beta = 4$, $x = 20$
  - Excel: $P(X \leqslant 20) =$ GAMMA.DIST(20,2.2,4,1) $= 0.9473$
  - TI-8x: Using integral function on TI-8x: $P(X \leqslant x) = \int_0^x t^{\alpha-1}e^{-t/\beta}dt / \int_0^\infty t^{\alpha-1}e^{-t/\beta}dt$.

    ```
    fnInt(X^(2.2-1)*e^(-X/4),X,0,20)
    /fnInt(X^(2.2-1)*e^(-X/4),X,0,1000)
    ```

    =.94733

Using Table A.24: Incomplete Gamma Function
If X has a **gamma distribution** with parameters α and β, then

$$P(X \leqslant x) = F\left(\frac{x}{\beta}, \alpha\right)$$

because

$$P(X \leqslant x) = \int_0^x \tfrac{t^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{t}{\beta}} dt \overset{y=t/\beta}{=\!=\!=} \int_0^{x/\beta} \tfrac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy = F\left(\frac{x}{\beta}, \alpha\right)$$

For example, if $\alpha = 2$ and $\beta = 4$, then $P(X \geqslant 20) = 1 - P(X < 20) = 1 - F(20/4, 2) = 1 - F(5, 2) = 1 - 0.96 = 0.04$.

Exponential Distribution

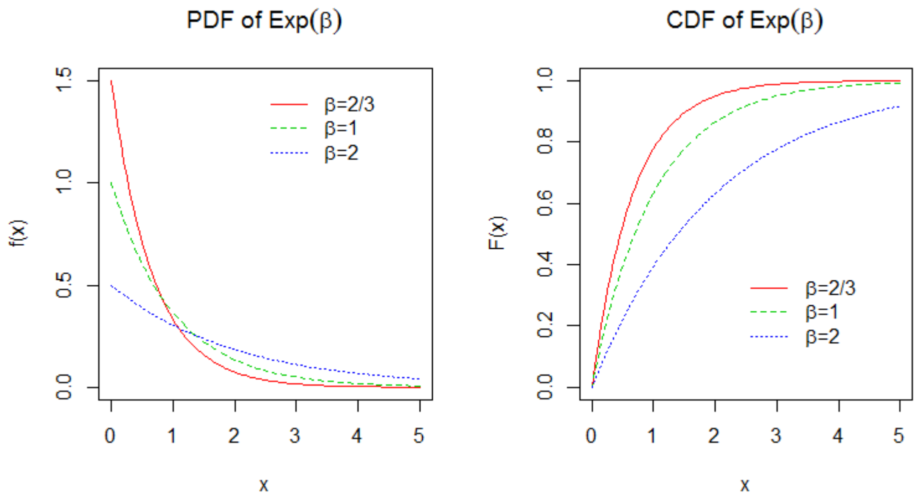
Exponential is a special gamma $(\alpha = 1)$ If X is the waiting time until the 1st occurrence, then X has an exponential distribution with p.d.f.

$$f(x) = \tfrac{1}{\beta} e^{-\frac{x}{\beta}}, \quad 0 \leqslant x < \infty$$

and denote $X \sim Exp(\beta)$.
Mean: $\mu = E(X) = \beta$.
Variance: $\sigma^2 = \text{Var}(X) = \beta^2$.
Cumulative Distribution Function :

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{\beta}}, & x \geqslant 0; \\ 0, & x < 0. \end{cases}$$

Exponential p.d.f. Curves



Examples of Exponentials

Example 2. The following three random variables are exponentially distributed.

- The length of time between emergency arrivals at a hospital.
- The length of time between catastrophic events (floods, earthquakes etc.).
- The distance traveled by wildlife ecologist between sightings of an endangered species.

Example 3: Let T be the time in years to failure of a certain type of electronic component. The random variable T is modeled nicely by the exponential distribution with mean time to failure $\beta = 5$.

(a) What is the probability that a randomly selected component are still functioning at the end of 3 years? 5 years? 8 years?

(b) If 5 of these components are installed, what is the probability that at least 2 are still functioning at the end of 5 years?

6.7 Properties of Exponential Distribution

Memoryless Property

If $X \sim Exp(\beta)$, then

$$F(x) = P(X \leqslant x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - e^{-\frac{x}{\beta}}, & \text{if } x \geqslant 0. \end{cases}$$

For $s > 0$ and $t > 0$,

$$P(X > s + t | X > s) = P(X > t).$$

$$\begin{aligned} P(X > s + t | X > s) &= \frac{P[(X > s + t) \cap (X > s)]}{P(X > s)} \\ &= \frac{P(X > s + t)}{P(X > s)} = \frac{1 - F(s + t)}{1 - F(s)} \\ &= \frac{e^{-\frac{s+t}{\beta}}}{e^{-\frac{s}{\beta}}} = e^{-\frac{t}{\beta}} = P(X > t) \end{aligned}$$

Generating Exponential PRN's

(1) Generate PRN's U_i from $U(0,1)$;

(2) $X_i = -\beta \ln(U_i)$;

(3) X_i are independent exponentials.

Simulation of Poisson Process

Discrete events Method
Since the times between events in a Poisson process are independent exponentials, we can simulate Poisson process by simulating exponentials.

6.8 Chi-Square(kai-square) Distribution

Definition of Chi-Square Distribution

Chi-square distribution
— an important **special case of gamma distribution**.
If $X \sim \Gamma(\alpha, \theta)$ with $\theta = 2$ and $\alpha = v/2$, $v > 0$ is an integer, then p.d.f. of X is

$$f(x) = \frac{x^{\frac{v}{2}-1}}{\Gamma(\frac{v}{2})2^{\frac{v}{2}}} e^{-\frac{x}{2}}, \quad 0 \leqslant x < \infty$$

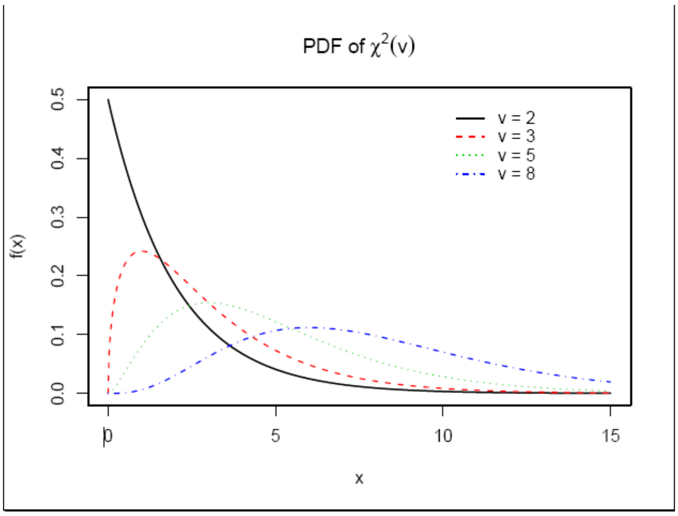
We say X has a **chi-square distribution** with v degrees of freedom, denote $X \sim \chi^2(v)$

Excel Function for χ^2 Distribution:
`CHIDIST(x,df)`
returns $P(X > x) = 1 - P(X \leqslant x)$ and
`CHIINV(prob,df)`
returns $\chi^2_\alpha(v)$ if `prob` $= \alpha$ and `df` $= v$.

TI-8x χ^2 Distribution: χ^2`cdf(`x_1`,` x_2`, df)`
returns $P(x_1 < X < x_2)$ and
χ^2`pdf(`x`, df)`
returns the density.

Mean and Variance of χ^2 Distribution:
The mean and variance of $\chi^2(v)$ are

$$\mu = \alpha\theta = \left(\frac{v}{2}\right)2 = v, \quad \sigma^2 = \alpha^2\theta = \left(\frac{v}{2}\right)2^2 = 2v.$$



Generating Normal PRN's

Polar Method

Algorithm

STEP 1: Generate $U_1, U_2 \sim U(0,1)$.
STEP 2: Set

$$V_1 = 2U_1 - 1, \quad V_2 = 2U_2 - 1, \quad S = V_1^2 + V_2^2.$$

STEP 3: If $S > 1$ return to STEP 1.
STEP 4:

$$\begin{aligned} X &= \sqrt{\tfrac{-2\log S}{S}} V_1 \\ Y &= \sqrt{\tfrac{-2\log S}{S}} V_2 \end{aligned}$$

Then X and Y are independent $N(0,1)$ r.v.'s.