Combinatorial Counting & Probability (3cr)
Chapter 6 Some Continuous Probability Distributions

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Math, IUSB

Outline

Examples

Example 1: Let X be the random number selected from $[a, b]$. Then X is continuous random variable. Intuitively, if each number in $[a, b]$ is equally likely to be selected, then for $a \leqslant x \leqslant b$,

$$F(x) = P(X \leqslant x) = \frac{x-a}{b-a}$$

$$P(X \leqslant x) = 0, \quad x < a; \quad P(X \leqslant x) = 1, \quad x > b$$

The c.d.f. of X is

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Example 2: Customers arrive randomly at a bank teller's window. Given that a customer arrived during a particular 10-minute period. Let X equal the time within the 10 minutes that the customer arrived. If $X \sim U(0, 10)$, Find

(a) The pdf of X;

(b) $P(X \geqslant 8)$;

(c) $P(2 \leqslant X < 8)$;

(d) $E(X)$;

(e) $\mathrm{Var}(X)$.

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Example 3: If $X \sim U(a, b)$, then show that the mean and variance of X are, respectively,

$$\mu = E(X) = \frac{a+b}{2}, \quad \sigma^2 = \mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

Example 4: If $U \sim U(0,1)$, then $X = a + (b-a)U \sim U(a, b)$.

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1. 6.1 Continuous Uniform Distribution
   - Pseudorandom Numbers (PRN's)
2. 6.2 Normal Distribution
3. 6.3 Area under the Normal Curve
4. 6.4 Applications of the Normal Distribution
   - Log-normal Distribution

Linear Congruential Generator (LCG)

Linear Congruential Generator (LCG) is used by many software to generate PRN's:

$$x_{n+1} = ax_n + c \ mod \ m$$

where a, c and m are integers satisfy certain conditions.

For example, $a = 1664525$, $c = 1013904223$, $m = 2^{32}$ (*Numerical Recipes in C*).

x_0 is called the random seed.

$U_n = \frac{x_n}{m}$, $n = 1, 2, \ldots,$ is a sequence of PRN's from $U(0, 1)$.

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A r.v. X is said to have a normal or Gaussian distribution with mean μ and variance σ^2, denoted by $X \sim N(\mu, \sigma^2)$, if its density is

$$n(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$$

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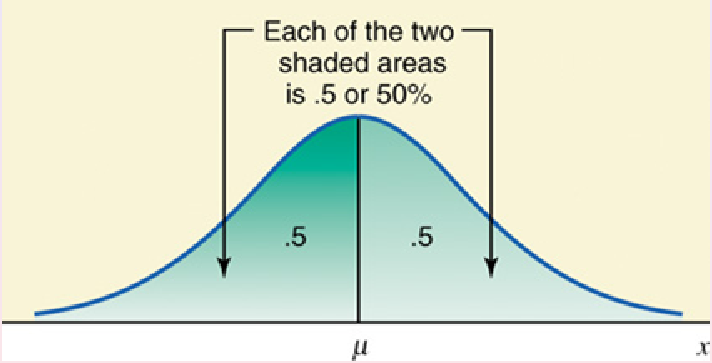
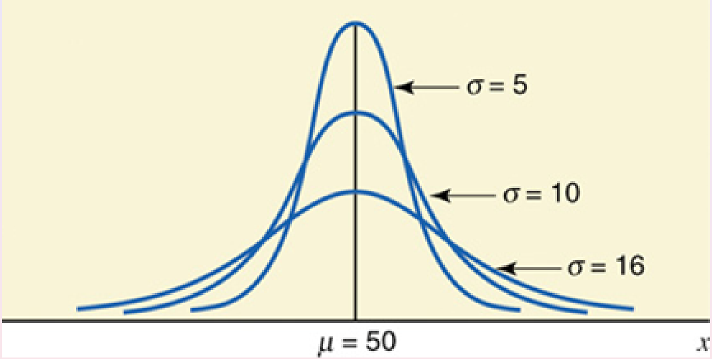
Normal curve is symmetric about the mean



Shape of normal curve is determine by σ



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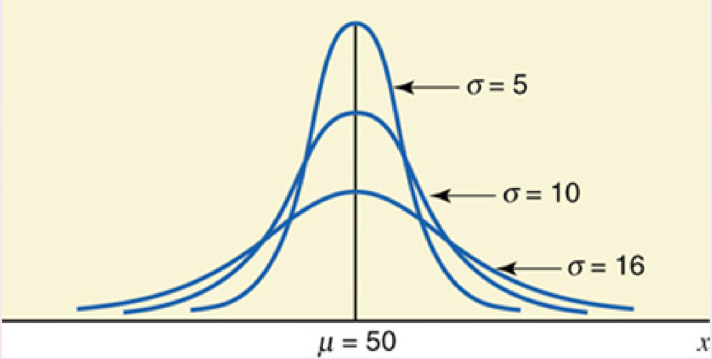

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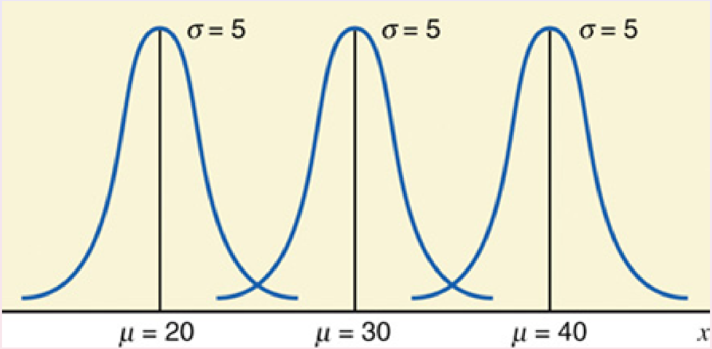

For a fixed standard deviation, changing the value of μ will move the normal curve horizontally.

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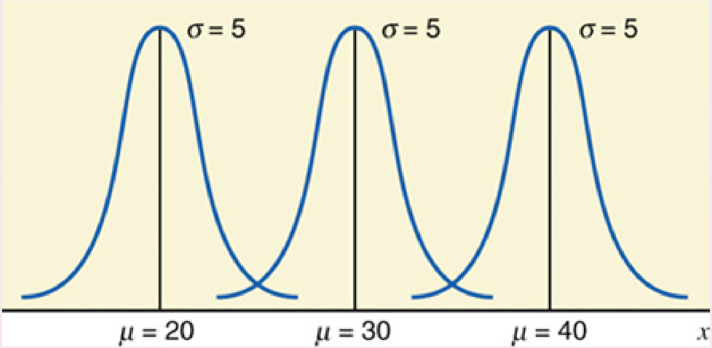

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Linear Transformation of Normal R. V.

Linear Transformation of $X \sim N(\mu, \sigma^2)$

Theorem 1. If $X \sim N(\mu, \sigma^2)$, then

(a)

$$Y = aX + b \sim N[a\mu + b, (|a|\sigma)^2].$$

(b)

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

Example: If $X \sim N(1, 9)$, find the distribution of $Y = -2X + 1$.

Solution: The distribution of Y is $N(-1, [2(3)]^2)$, the normal distribution with mean -1 and variance 36.

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Standard Normal Distribution

Definition: If r.v. Z is $N(0,1)$, then Z has standard normal distribution. Its density is

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Distribution function is

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

Clearly, $\Phi(0) = \frac{1}{2}$ and $1 - \Phi(z) = \Phi(-z)$

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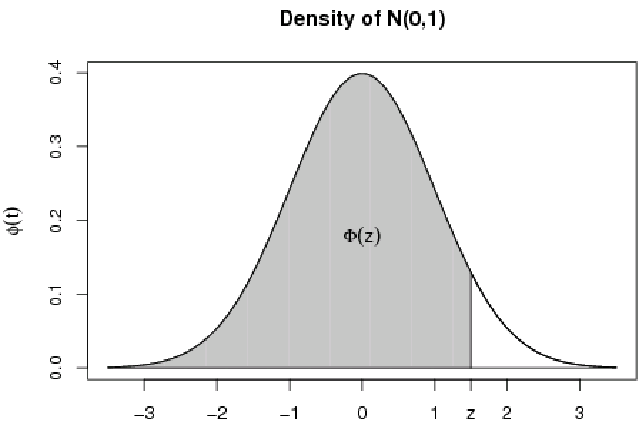
Probability = Area Under Normal Curve

If r.v. X is $N(\mu, \sigma^2)$, then

$$
\begin{aligned}
P(x_1 < X < x_2) &= \text{area under normal curve between } x_1 \text{ and } x_2 \\
&= P\left(\frac{x_1-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{x_2-\mu}{\sigma}\right) \\
&= P\left(\frac{x_1-\mu}{\sigma} < Z < \frac{x_2-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{x_2-\mu}{\sigma}\right) - \Phi\left(\frac{x_1-\mu}{\sigma}\right)
\end{aligned}
$$

Standard Normal p.d.f. Curves

$\Phi(z) = P(Z \leqslant z) =$ area of shaded region.



Example 1:

Let X be a normal random variable with mean $\mu = 50$ and standard deviation $\sigma = 10$. Convert x values to z values and find

(a) $P(X \leqslant 55)$.

(b) $P(X < 35)$

Solution of Example 1:

$\mu = 50, \quad \sigma = 10$

(a) $x = 55$.

$$z = \frac{x-\mu}{\sigma} = \frac{55-50}{10} = \frac{5}{10} = \frac{1}{2} = 0.5$$

(b) $x = 35$

$$z = \frac{x-\mu}{\sigma} = \frac{35-50}{10} = \frac{-15}{10} = -\frac{3}{2} = -1.5$$

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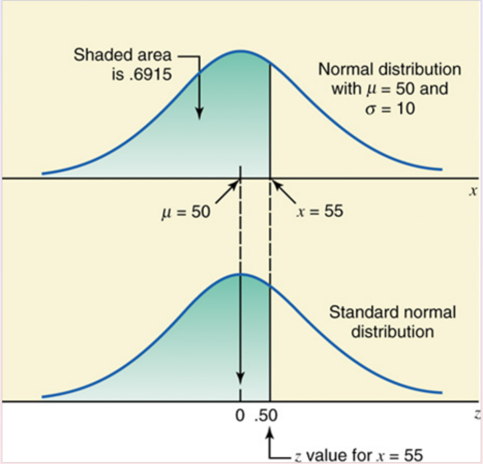
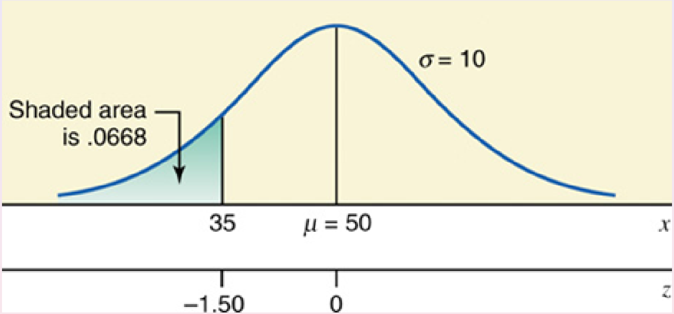

Shaded area is .6915
Normal distribution with $\mu = 50$ and $\sigma = 10$
x
$\mu = 50$
$x = 55$
Standard normal distribution
0 .50
z
z value for $x = 55$


σ = 10
Shaded area is .0668
35
μ = 50
x
−1.50
0
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Example 2:

Let X be a normal random variable with mean $\mu = 25$ and standard deviation $\sigma = 4$. Convert x values to z values and find

(a) $P(25 < X \leqslant 32)$.

(b) $P(18 \leqslant X \leqslant 34)$.

Solution of Example 2:

$\mu = 25, \quad \sigma = 4$

(a) $x_1 = 25$ and $x_2 = 32$.

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{25 - 25}{4} = \frac{0}{4} = 0$$

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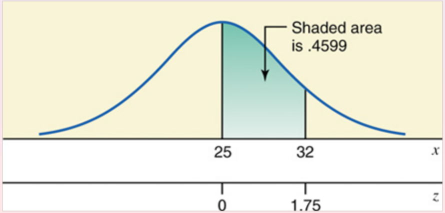

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$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{18 - 25}{4} = \frac{-7}{4} = -1.75$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{34 - 25}{4} = \frac{9}{4} = 2.25$$

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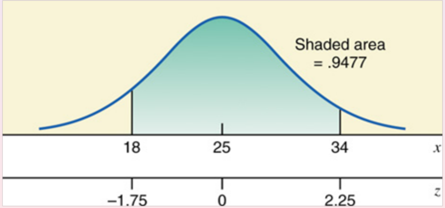

Excel Function for Normal Distribution:

1) `NORMDIST`(x, μ, σ, `cumul`) returns $P(X \leqslant x)$ if `cumul=TRUE`, p.d.f. $n(x; \mu, \sigma)$ otherwise.
2) `NORMINV`(p, μ, σ) returns $\mu + z_\alpha \sigma$ if $p = 1 - \alpha$.
3) `NORMSDIST(z)` returns $\Phi(z)$ and
4) `NORMSINV`(p) returns z_α if $p = 1 - \alpha$.

TI-8x

Use TI8x: “DISTR” $\rightarrow$ “normalcdf”:

“normalcdf(x_1, x_2, μ, σ)” gives $P(x_1 < X < x_2)$;

“normalcdf($-E99$, x, μ, σ)” gives $P(X \leqslant x)$, where

“E”= $\boxed{\text{2nd}}$ + $\boxed{\text{EE}}$.

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The Upper 100α Percent Point

The upper 100α percent point, z_α is $100(1-\alpha)$ percentile. That is

$$P(Z \geqslant z_\alpha) = \alpha$$

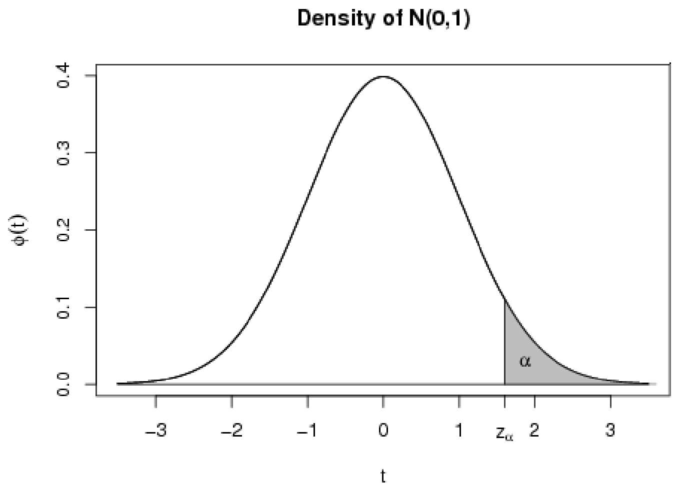
where $Z \sim N(0, 1)$. Since

$$P(Z \leqslant -z_\alpha) = P(Z \geqslant z_\alpha) = \alpha$$

$$z_{1-\alpha} = -z_\alpha$$

Use TI8x: “DISTR” $\rightarrow$ “invNorm(1-α)”;

Normal p.d.f. Curves



Examples

Example 3 If $X \sim N(6, 25)$, find

(a) $P(6 \leqslant X \leqslant 12)$;

(b) $P(X > 21)$;

(c) $P(|X - 6| < 10)$;

(a) $P(6 \leqslant X \leqslant 12) = P(\frac{6-6}{5} \leqslant \frac{X-6}{5} \leqslant \frac{12-6}{5}) = P(0 \leqslant Z \leqslant 1.2)$
$= \Phi(1.2) - \Phi(0) = 0.8849 - 0.5 = 0.3849$

(b) $P(X > 21) = 1 - P(X \leqslant 21)$
$= 1 - P(\frac{X-6}{5} \leqslant \frac{21-6}{5}) = 1 - P(Z \leqslant 3)$
$= 1 - \Phi(3) = 1 - 0.999 = 0.001$

(c) $P(|X - 6| < 10) = P(-10 < X - 6 < 10)$
$= P(-\frac{10}{5} \leqslant \frac{X-6}{5} \leqslant \frac{10}{5})$
$= P(-2 \leqslant Z \leqslant 2) = 2\Phi(2) - 1 = 0.9544$

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$$(c) \quad P(|X - 6| < 10) = P(-10 < X - 6 < 10)$$
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$$= P(-2 \leqslant Z \leqslant 2) = 2\Phi(2) - 1 = 0.9544$$

Examples

Example 4 If $X \sim N(\mu, \sigma^2)$, find the probability that X differs from the mean by more than $k\sigma$ where $k > 0$. Compare the result with one obtained by using Chebyshev Theorem.

Examples

Example 4 Suppose adult males average 69 inches tall with a standard deviation of 3 inches.

(a) What is the probability that a randomly selected adult male taller than 65 inches and shorter than 75 inches?

(b) What percentage of adult males are taller than 80 inches?

(c) How many adult males are taller than Shaquille O'Neal who is 85 inches?

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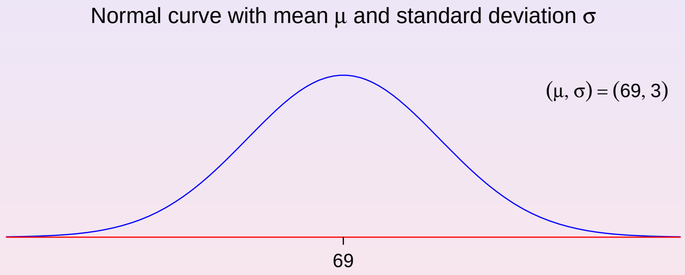
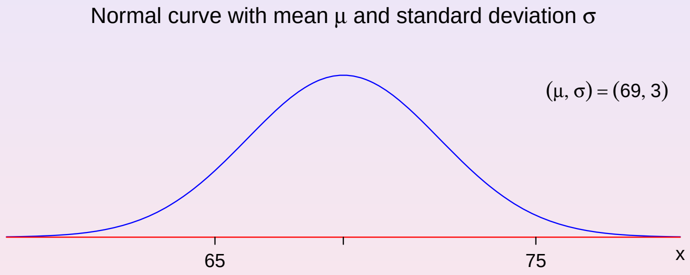
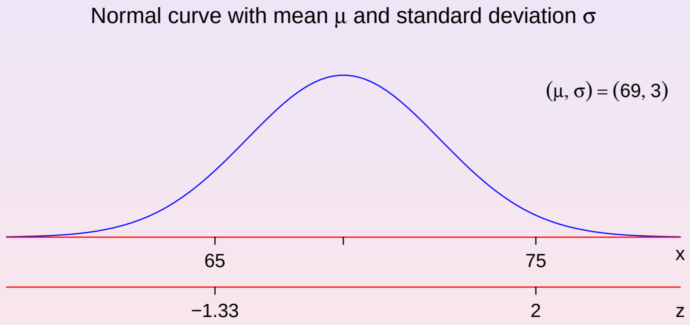
Solution of Example 4(a)



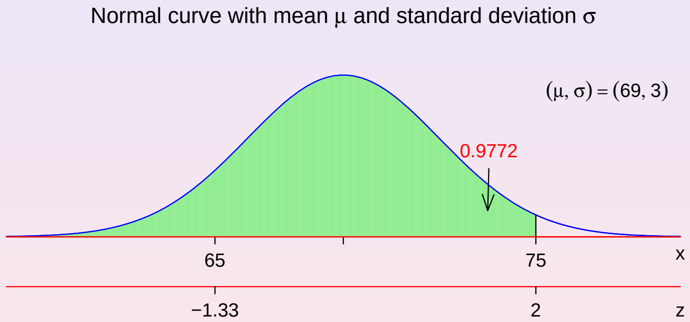
Solution of Example 4(a)



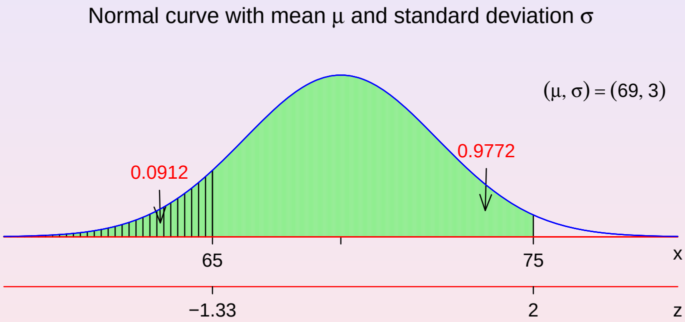
Solution of Example 4(a)



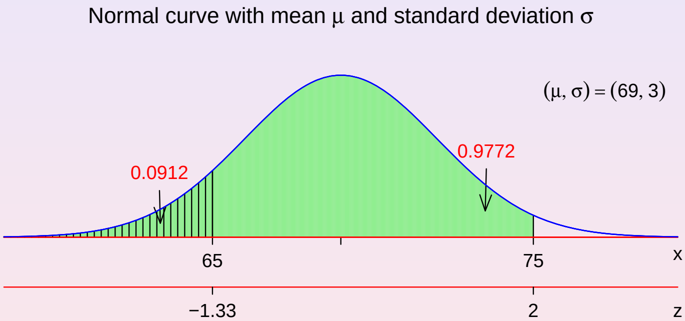
Solution of Example 4(a)



Solution of Example 4(a)

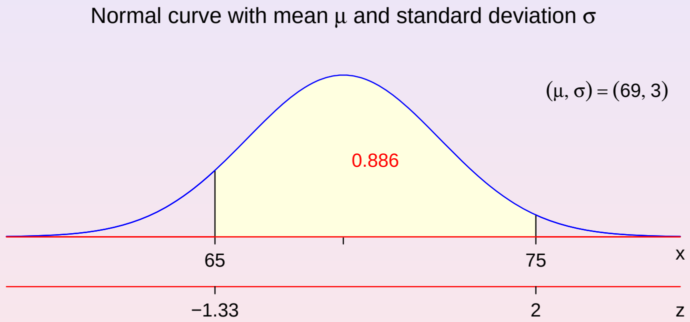


Solution of Example 4(a)



The area between 65 and 75 is 0.9772 − 0.0912 = 0.886

Solution of Example 4(a)



Examples

Example 5: Suppose that consumer debt (owed on cars, credit cards, and so forth) for all U.S. households have a normal distribution with mean of \$17,989 and a standard deviation of \$3750. Find the probability that such consumer debt of a randomly selected U.S. household is between \$13,000 and \$20,000.

Solution:

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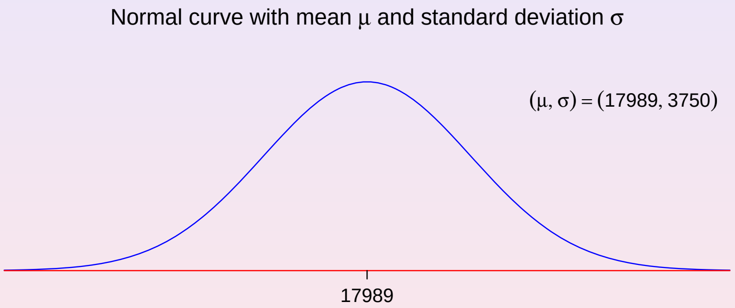
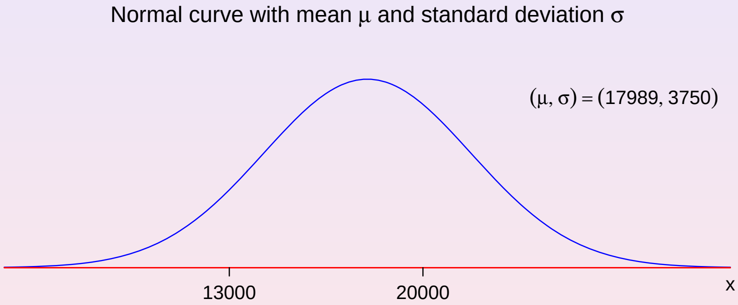


Solution of Example 5



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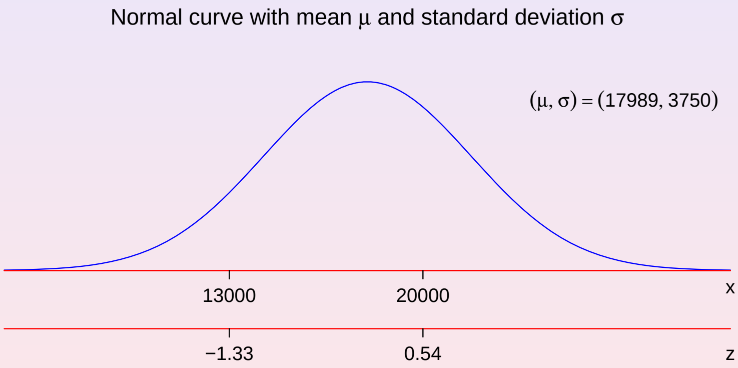

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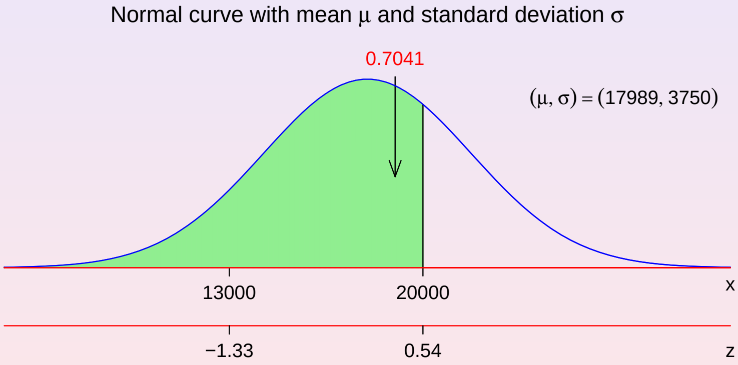

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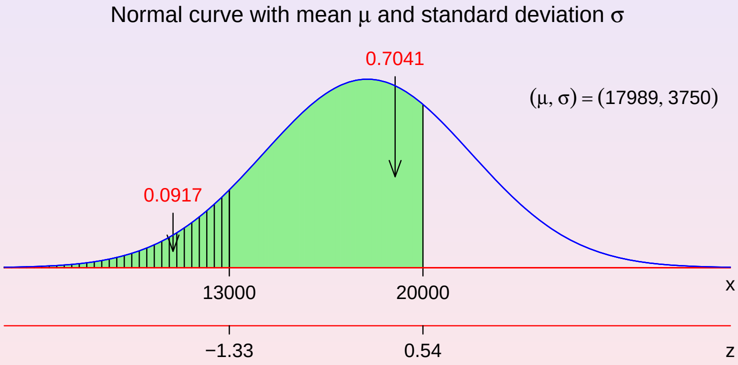

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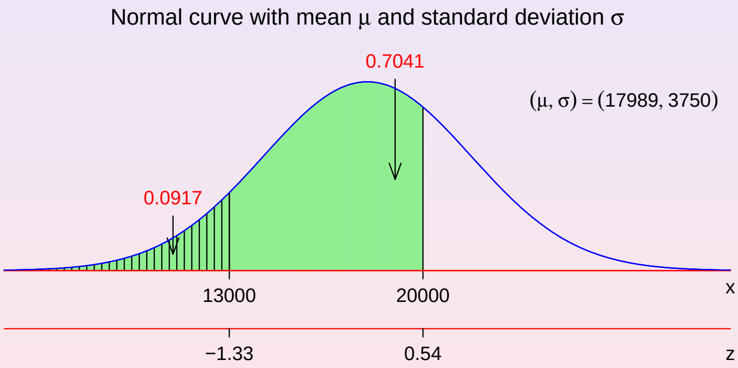
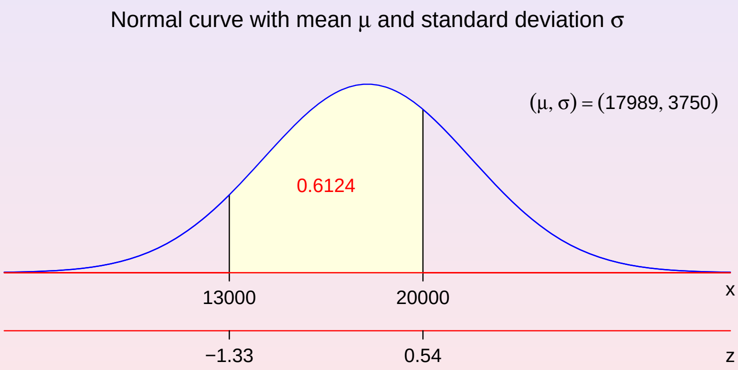


The area between 13000 and 20000 is 0.7041 − 0.0917 = 0.6124

Solution of Example 5



Outline

Log-normal Distribution

Definition

Random variable X is said to have a log-normal distribution $LN(\mu, \sigma^2)$ if its natural logarithm $\ln(X)$ has normal distribution $N(\mu, \sigma^2)$. That is X has pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}x} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\}, \quad x > 0.$$

Application in Stock price: Log-normal is commonly used to model stock returns: Let S_i be stock price at the end of the i-th day, $i = 0, 1, \ldots, n$, and $R_i = \frac{S_i}{S_{i-1}}$, $i = 1, \ldots, n$. Under certain conditions, we can assume that $R_1, \ldots, R_n$ are independent and identically distributed log-normal random variables.

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Cumulative Distribution Function

Since $Y = \ln(X)$ has normal distribution $N(\mu, \sigma^2)$, the cumulative distribution function of X is

$$F(x) = P(X \leqslant x)$$

$$= P[\ln(X) \leqslant \ln(x)] = P[Y \leqslant \ln(x)]$$

$$= \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right], \quad x > 0.$$

Mean

Mean: $E(X) = \exp\left(\mu + \sigma^2/2\right)$

$$E(X) = \int_0^{\infty} x f(x) dx = \int_0^{\infty} \cancel{x} \frac{1}{\sigma\sqrt{2\pi}\cancel{x}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\} dx$$

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Variance

Variance: $V(X) = E(X^2) - [E(X)]^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

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