Combinatorial Counting & Probability (3cr)
Chapter 6 Some Continuous Probability Distributions

Zhong Guan

Math, IUSB

Outline

Waiting Time Distributions

Example 1. If the number of calls received per hour by a telephone answering service has a Poisson distribution with $\lambda = 6$.
(a) What is the distribution of the waiting time W for the first call?
(b) What is the distribution of the waiting time W for the 3rd call?

Outline

Gamma Distribution is a Waiting Time Distribution

In a Poisson process with rate λ, mean number of events per unit "time", let W be the waiting time until the αth occurrence. Then

$$\begin{aligned} F(w) = P(W \leqslant w) &= 1 - P(W > w) \\ &= 1 - P(\text{fewer than } \alpha \text{ occurrences in } [0, w]) \\ &= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!} e^{-\lambda w} \\ &= 1 - e^{-\lambda w} \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!} \end{aligned}$$

If $w < 0$ then $F(w) = 0$.

Gamma Distribution is a Waiting Time Distribution

In a Poisson process with rate λ, mean number of events per unit "time", let W be the waiting time until the αth occurrence. Then

$$\begin{aligned} F(w) = P(W \leqslant w) &= 1 - P(W > w) \\ &= 1 - P(\text{fewer than } \alpha \text{ occurrences in } [0, w]) \\ &= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!} e^{-\lambda w} \\ &= 1 - e^{-\lambda w} \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!} \end{aligned}$$

If $w < 0$ then $F(w) = 0$.

The p.d.f. of Gamma Distribution

The p.d.f. is $f(w) = F'(x)$: if $w \geqslant 0$ then

$$f(w) = F'(w) = -\left[e^{-\lambda w}\sum_{k=0}^{\alpha-1}\frac{(\lambda w)^k}{k!}\right]'$$

$$= \lambda e^{-\lambda w}\sum_{k=0}^{\alpha-1}\frac{(\lambda w)^k}{k!} - e^{\lambda w}\sum_{k=0}^{\alpha-1}\frac{k\lambda(\lambda w)^{k-1}}{k!}$$

$$= \lambda e^{-\lambda w}\sum_{k=0}^{\alpha-1}\frac{(\lambda w)^k}{k!} - e^{\lambda w}\sum_{k=1}^{\alpha-1}\frac{\lambda(\lambda w)^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda w}\sum_{k=0}^{\alpha-1}\frac{(\lambda w)^k}{k!} - \lambda e^{\lambda w}\sum_{j=0}^{\alpha-2}\frac{(\lambda w)^j}{j!}$$

$$= \lambda e^{-\lambda w}\frac{(\lambda w)^{\alpha-1}}{(\alpha-1)!} = \frac{\lambda^\alpha w^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda w}$$

Definition of Gamma Distribution

The above density can be written as

$$f(w) = \frac{w^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{w}{\beta}}, \quad \lambda = \frac{1}{\beta}$$

Definition: Generally, if random variable X has p.d.f.

$$f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{x}{\beta}}, \quad 0 < x < \infty,$$

where $\alpha > 0$ and $\beta > 0$, then X has a **gamma distribution**, and denote $X \sim \Gamma(\alpha, \beta)$.
Mean: $\mu = E(X) = \alpha\beta$.
Variance: $\sigma^2 = \mathrm{Var}(X) = \alpha\beta^2$.

Definition of Gamma Distribution

The above density can be written as

$$f(w) = \frac{w^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{w}{\beta}}, \quad \lambda = \frac{1}{\beta}$$

Definition: Generally, if random variable X has p.d.f.

$$f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{x}{\beta}}, \quad 0 < x < \infty,$$

where $\alpha > 0$ and $\beta > 0$, then X has a **gamma distribution**, and denote $X \sim \Gamma(\alpha, \beta)$.

Mean: $\mu = E(X) = \alpha\beta$.

Variance: $\sigma^2 = \mathrm{Var}(X) = \alpha\beta^2$.

Definition of Gamma Distribution

The above density can be written as

$$f(w) = \frac{w^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{w}{\beta}}, \quad \lambda = \frac{1}{\beta}$$

Definition: Generally, if random variable X has p.d.f.

$$f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{x}{\beta}}, \quad 0 < x < \infty,$$

where $\alpha > 0$ and $\beta > 0$, then X has a **gamma distribution**, and denote $X \sim \Gamma(\alpha, \beta)$.
Mean: $\mu = E(X) = \alpha\beta$.
Variance: $\sigma^2 = \mathrm{Var}(X) = \alpha\beta^2$.

Gamma Function

Gamma Function: The **gamma** function:

$$\Gamma(t) = \int_0^{\infty} y^{t-1} e^{-y} dy, \qquad t > 0.$$

$$\Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(t) = (t-1)\Gamma(t-1)$$

For integer $n > 0$,

$$\Gamma(n) = (n-1)!$$

Gamma Function

Gamma Function: The **gamma** function:

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy, \qquad t > 0.$$

$$\Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(t) = (t-1)\Gamma(t-1)$$

For integer $n > 0$,

$$\Gamma(n) = (n-1)!$$

Gamma Function

Gamma Function: The **gamma** function:

$$\Gamma(t) = \int_0^{\infty} y^{t-1} e^{-y} dy, \qquad t > 0.$$

$$\Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(t) = (t-1)\Gamma(t-1)$$

For integer $n > 0$,

$$\Gamma(n) = (n-1)!$$

Gamma Function

Gamma Function: The **gamma** function:

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy, \qquad t > 0.$$

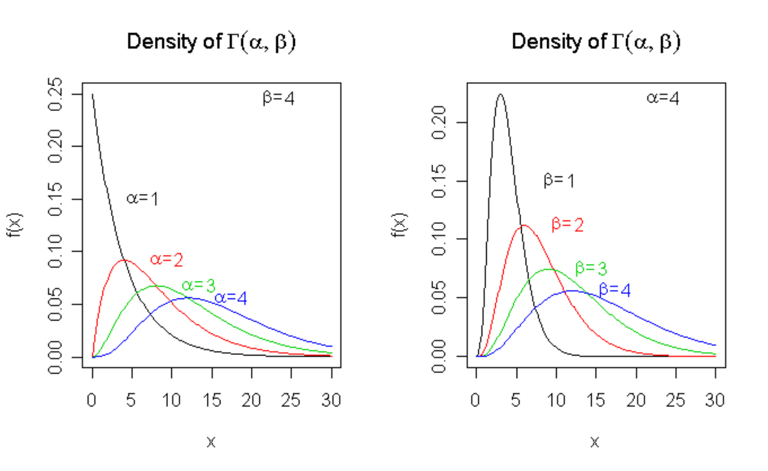
$$\Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(t) = (t-1)\Gamma(t-1)$$

For integer $n > 0$,

$$\Gamma(n) = (n-1)!$$

Gamma p.d.f. Curves



Excel:

Excel Functions:

1) GAMMA.DIST(x, alpha, beta, cumulative) returns c.d.f. if cumulative = TRUE, p.d.f. otherwise. beta = θ.

2) GAMMA.INV(probability, alpha, beta) returns the inverse (quantile) function of $\Gamma(\alpha, \beta)$.

3) GAMMALN(x) returns $\ln \Gamma(x)$

Excel:

Excel Functions:

1) GAMMA.DIST(x, alpha, beta, cumulative) returns c.d.f. if cumulative = TRUE, p.d.f. otherwise. beta = θ.

2) GAMMA.INV(probability, alpha, beta) returns the inverse (quantile) function of $\Gamma(\alpha, \beta)$.

3) GAMMALN(x) returns $\ln \Gamma(x)$

Excel:

Excel Functions:

1) GAMMA.DIST(x, alpha, beta, cumulative) returns c.d.f. if cumulative = TRUE, p.d.f. otherwise. beta = θ.
2) GAMMA.INV(probability, alpha, beta) returns the inverse (quantile) function of $\Gamma(\alpha, \beta)$.
3) GAMMALN(x) returns $\ln \Gamma(x)$

Using TI-8x:

- TI-83/84:
  - using interpolation or integral function `fnInt()`.
- Example: $\alpha = 2.2$, $\beta = 4$, $x = 20$.
  - Excel: $P(X \leqslant 20)$ =GAMMA.DIST(20,2.2,4,1) = 0.9473
  - TI-8x: Using integral function on TI-8x:
    $P(X \leqslant x) = \int_0^x t^{\alpha-1} e^{-t/\beta} dt / \int_0^\infty t^{\alpha-1} e^{-t/\beta} dt.$
    ```
    fnInt(X^(2.2-1)*e^(-X/4),X,0,20)
    /fnInt(X^(2.2-1)*e^(-X/4),X,0,1000)
    ```
    =.94733

Using TI-8x:

- TI-83/84:
  - using interpolation or integral function `fnInt()`.
- Example: $\alpha = 2.2$, $\beta = 4$, $x = 20$.
  - Excel: $P(X \leqslant 20)$ =GAMMA.DIST(20,2.2,4,1) = 0.9473
  - TI-8x: Using integral function on TI-8x:
    $P(X \leqslant x) = \int_0^x t^{\alpha-1} e^{-t/\beta} dt / \int_0^\infty t^{\alpha-1} e^{-t/\beta} dt.$

    ```
    fnInt(X^(2.2-1)*e^(-X/4),X,0,20)
    /fnInt(X^(2.2-1)*e^(-X/4),X,0,1000)
    ```
    =.94733

Using TI-8x:

- TI-83/84:
  - using interpolation or integral function `fnInt()`.
- Example: $\alpha = 2.2$, $\beta = 4$, $x = 20$.
  - Excel: $P(X \leqslant 20)$ =GAMMA.DIST(20,2.2,4,1) = 0.9473
  - TI-8x: Using integral function on TI-8x:
    $P(X \leqslant x) = \int_0^x t^{\alpha-1} e^{-t/\beta} dt / \int_0^\infty t^{\alpha-1} e^{-t/\beta} dt.$

    ```
    fnInt(X^(2.2-1)*e^(-X/4),X,0,20)
    /fnInt(X^(2.2-1)*e^(-X/4),X,0,1000)
    ```
    =.94733

Using TI-8x:

- TI-83/84:
  - using interpolation or integral function `fnInt()`.
- Example: $\alpha = 2.2$, $\beta = 4$, $x = 20$.
  - Excel: $P(X \leqslant 20) =$GAMMA.DIST(20,2.2,4,1) = 0.9473
  - TI-8x: Using integral function on TI-8x:
    $P(X \leqslant x) = \int_0^x t^{\alpha-1} e^{-t/\beta} dt / \int_0^\infty t^{\alpha-1} e^{-t/\beta} dt.$

    ```
    fnInt(X^(2.2-1)*e^(-X/4),X,0,20)
    /fnInt(X^(2.2-1)*e^(-X/4),X,0,1000)
    ```
    =.94733

Using TI-8x:

- TI-83/84:
  - using interpolation or integral function `fnInt()`.
- Example: $\alpha = 2.2$, $\beta = 4$, $x = 20$.
  - Excel: $P(X \leqslant 20)$ =GAMMA.DIST(20,2.2,4,1) = 0.9473
  - TI-8x: Using integral function on TI-8x:
    $P(X \leqslant x) = \int_0^x t^{\alpha-1} e^{-t/\beta} dt / \int_0^\infty t^{\alpha-1} e^{-t/\beta} dt.$
    ```
    fnInt(X^(2.2-1)*e^(-X/4),X,0,20)
    /fnInt(X^(2.2-1)*e^(-X/4),X,0,1000)
    ```
    =.94733

Using TI-8x:

- TI-83/84:
  - using interpolation or integral function `fnInt()`.
- Example: $\alpha = 2.2$, $\beta = 4$, $x = 20$.
  - Excel: $P(X \leqslant 20)$ =GAMMA.DIST(20,2.2,4,1) = 0.9473
  - TI-8x: Using integral function on TI-8x:
    $P(X \leqslant x) = \int_0^x t^{\alpha-1} e^{-t/\beta} dt / \int_0^\infty t^{\alpha-1} e^{-t/\beta} dt$.

```
fnInt(X^(2.2-1)*e^(-X/4),X,0,20)
/fnInt(X^(2.2-1)*e^(-X/4),X,0,1000)
```

=.94733

Using TI-8x:

- TI-83/84:
  - using interpolation or integral function `fnInt()`.
- Example: $\alpha = 2.2$, $\beta = 4$, $x = 20$.
  - Excel: $P(X \leqslant 20) =$GAMMA.DIST(20,2.2,4,1) = 0.9473
  - TI-8x: Using integral function on TI-8x: $P(X \leqslant x) = \int_0^x t^{\alpha-1}e^{-t/\beta}dt / \int_0^\infty t^{\alpha-1}e^{-t/\beta}dt.$

    ```
    fnInt(X^(2.2-1)*e^(-X/4),X,0,20)
    /fnInt(X^(2.2-1)*e^(-X/4),X,0,1000)
    ```

    =.94733

Using TI-8x:

- TI-83/84:
  - using interpolation or integral function `fnInt()`.
- Example: $\alpha = 2.2$, $\beta = 4$, $x = 20$.
  - Excel: $P(X \leqslant 20)$ =GAMMA.DIST(20,2.2,4,1) = 0.9473
  - TI-8x: Using integral function on TI-8x:
    $P(X \leqslant x) = \int_0^x t^{\alpha-1} e^{-t/\beta} dt / \int_0^\infty t^{\alpha-1} e^{-t/\beta} dt.$

    ```
    fnInt(X^(2.2-1)*e^(-X/4),X,0,20)
    /fnInt(X^(2.2-1)*e^(-X/4),X,0,1000)
    ```
    =.94733

Using Table A.24: Incomplete Gamma Function

If X has a **gamma distribution** with parameters α and β, then

$$P(X \leqslant x) = F\left(\frac{x}{\beta}, \alpha\right)$$

$$P(X \leqslant x) = \int_0^x \frac{t^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{t}{\beta}} dt \overset{y=t/\beta}{=\!=\!=} \int_0^{x/\beta} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy$$
$$= F\left(\frac{x}{\beta}, \alpha\right)$$

For example, if $\alpha = 2$ and $\beta = 4$, then $P(X \geqslant 20) = 1 - P(X < 20) = 1 - F(20/4, 2) = 1 - F(5, 2) = 1 - 0.96 = 0.04$.

Using Table A.24: Incomplete Gamma Function

If X has a **gamma distribution** with parameters α and β, then

$$P(X \leqslant x) = F\left(\frac{x}{\beta}, \alpha\right)$$

$$P(X \leqslant x) = \int_0^x \frac{t^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{t}{\beta}} dt \xlongequal{y=t/\beta} \int_0^{x/\beta} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy$$
$$= F\left(\frac{x}{\beta}, \alpha\right)$$

For example, if $\alpha = 2$ and $\beta = 4$, then $P(X \geqslant 20) = 1 - P(X < 20) = 1 - F(20/4, 2) = 1 - F(5, 2) = 1 - 0.96 = 0.04$.

Using Table A.24: Incomplete Gamma Function

If X has a **gamma distribution** with parameters α and β, then

$$P(X \leqslant x) = F\left(\frac{x}{\beta}, \alpha\right)$$

$$\begin{aligned} P(X \leqslant x) &= \int_0^x \frac{t^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{t}{\beta}} dt \xlongequal{y=t/\beta} \int_0^{x/\beta} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy \\ &= F\left(\frac{x}{\beta}, \alpha\right) \end{aligned}$$

For example, if $\alpha = 2$ and $\beta = 4$, then $P(X \geqslant 20) = 1 - P(X < 20) = 1 - F(20/4, 2) = 1 - F(5, 2) = 1 - 0.96 = 0.04$.

Outline

Exponential is a special gamma ($\alpha = 1$)

If X is the waiting time until the 1st occurrence, then X has an exponential distribution with p.d.f.

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad 0 \leqslant x < \infty$$

and denote $X \sim Exp(\beta)$.

Mean: $\mu = E(X) = \beta$.

Variance: $\sigma^2 = \mathrm{Var}(X) = \beta^2$.

Cumulative Distribution Function :

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{\beta}}, & x \geqslant 0; \\ 0, & x < 0. \end{cases}$$

Exponential is a special gamma ($\alpha = 1$)

If X is the waiting time until the 1st occurrence, then X has an exponential distribution with p.d.f.

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad 0 \leqslant x < \infty$$

and denote $X \sim Exp(\beta)$.

Mean: $\mu = E(X) = \beta$.

Variance: $\sigma^2 = \mathrm{Var}(X) = \beta^2$.

Cumulative Distribution Function :

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{\beta}}, & x \geqslant 0; \\ 0, & x < 0. \end{cases}$$

Exponential is a special gamma ($\alpha = 1$)

If X is the waiting time until the 1st occurrence, then X has an exponential distribution with p.d.f.

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad 0 \leqslant x < \infty$$

and denote $X \sim Exp(\beta)$.

Mean: $\mu = E(X) = \beta$.

Variance: $\sigma^2 = \mathrm{Var}(X) = \beta^2$.

Cumulative Distribution Function :

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{\beta}}, & x \geqslant 0; \\ 0, & x < 0. \end{cases}$$

Exponential is a special gamma ($\alpha = 1$)

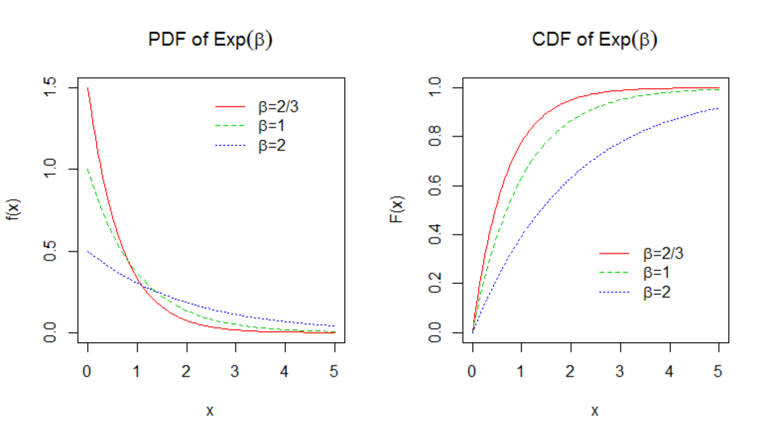
If X is the waiting time until the 1st occurrence, then X has an exponential distribution with p.d.f.

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad 0 \leqslant x < \infty$$

and denote $X \sim Exp(\beta)$.
Mean: $\mu = E(X) = \beta$.
Variance: $\sigma^2 = \mathrm{Var}(X) = \beta^2$.
Cumulative Distribution Function :

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{\beta}}, & x \geqslant 0; \\ 0, & x < 0. \end{cases}$$

Exponential p.d.f. Curves



Examples of Exponentials

Example 2. The following three random variables are exponentially distributed.

- The length of time between emergency arrivals at a hospital.
- The length of time between catastrophic events (floods, earthquakes etc.).
- The distance traveled by wildlife ecologist between sightings of an endangered species.

Examples of Exponentials

Example 2. The following three random variables are exponentially distributed.

- The length of time between emergency arrivals at a hospital.
- The length of time between catastrophic events (floods, earthquakes etc.).
- The distance traveled by wildlife ecologist between sightings of an endangered species.

Examples of Exponentials

Example 2. The following three random variables are exponentially distributed.

- The length of time between emergency arrivals at a hospital.
- The length of time between catastrophic events (floods, earthquakes etc.).
- The distance traveled by wildlife ecologist between sightings of an endangered species.

Examples

Example 3: Let T be the time in years to failure of a certain type of electronic component. The random variable T is modeled nicely by the exponential distribution with mean time to failure $\beta = 5$.

(a) What is the probability that a randomly selected component are still functioning at the end of 3 years? 5 years? 8 years?

(b) If 5 of these components are installed, what is the probability that at least 2 are still functioning at the end of 5 years?

Examples

Example 3: Let T be the time in years to failure of a certain type of electronic component. The random variable T is modeled nicely by the exponential distribution with mean time to failure $\beta = 5$.

(a) What is the probability that a randomly selected component are still functioning at the end of 3 years? 5 years? 8 years?

(b) If 5 of these components are installed, what is the probability that at least 2 are still functioning at the end of 5 years?