Combinatorial Counting & Probability (3cr)
Chapter 6 Some Continuous Probability Distributions

Zhong Guan

Math, IUSB

Outline

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Memoryless Property

If $X \sim Exp(\beta)$, then

$$F(x) = P(X \leqslant x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - e^{-\frac{x}{\beta}}, & \text{if } x \geqslant 0. \end{cases}$$

For $s > 0$ and $t > 0$,

$$P(X > s + t | X > s) = P(X > t).$$

$$\begin{aligned} P(X > s + t | X > s) &= \frac{P[(X > s + t) \cap (X > s)]}{P(X > s)} \\ &= \frac{P(X > s + t)}{P(X > s)} = \frac{1 - F(s + t)}{1 - F(s)} \\ &= \frac{e^{-\frac{s+t}{\beta}}}{e^{-\frac{s}{\beta}}} = e^{-\frac{t}{\beta}} = P(X > t) \end{aligned}$$

Outline

Generating Exponential PRN's

(1) Generate PRN's U_i from $U(0, 1)$;

(2) $X_i = -\beta \ln(U_i)$;

(3) X_i are independent exponentials.

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Discrete events Method

Since the times between events in a Poisson process are independent exponentials, we can simulate Poisson process by simulating exponentials.

Definition of Chi-Square Distribution

Chi-square distribution

— an important **special case of gamma distribution.**
If $X \sim \Gamma(\alpha, \theta)$ with $\theta = 2$ and $\alpha = v/2$, $v > 0$ is an integer, then p.d.f. of X is

$$f(x) = \frac{x^{\frac{v}{2}-1}}{\Gamma(\frac{v}{2})2^{\frac{v}{2}}} e^{-\frac{x}{2}}, \quad 0 \leqslant x < \infty$$

We say X has a **chi-square distribution** with v degrees of freedom, denote $X \sim \chi^2(v)$.

Excel Function for χ^2 Distribution:

`CHIDIST(x,df)`
returns $P(X > x) = 1 - P(X \leqslant x)$ and
`CHIINV(prob,df)`
returns $\chi^2_\alpha(v)$ if `prob` = α and `df` = v.

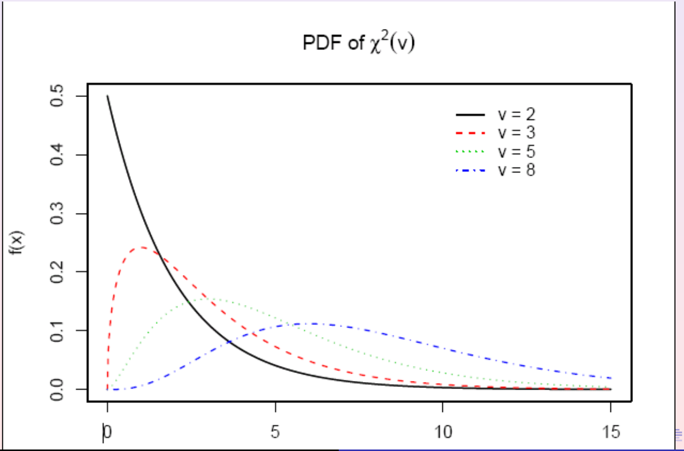
TI-8x χ^2 Distribution:

χ^2`cdf(`x_1`, `x_2`, df)`
returns $P(x_1 < X < x_2)$ and
χ^2`pdf(`x`, df)`
returns the density.

Mean and Variance of χ^2 Distribution:

The mean and variance of $\chi^2(v)$ are

$$\mu = \alpha\theta = \left(\frac{v}{2}\right)2 = v, \quad \sigma^2 = \alpha^2\theta = \left(\frac{v}{2}\right)2^2 = 2v.$$



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Algorithm

STEP 1: Generate $U_1, U_2 \sim U(0,1)$.
STEP 2: Set

$$V_1 = 2U_1 - 1, \quad V_2 = 2U_2 - 1, \quad S = V_1^2 + V_2^2.$$

STEP 3: If $S > 1$ return to STEP 1.
STEP 4:

$$X = \sqrt{\frac{-2\log S}{S}}\, V_1$$

$$Y = \sqrt{\frac{-2\log S}{S}}\, V_2$$

Then X and Y are independent $N(0,1)$ r.v.'s.

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