10.1 Statistical Hypotheses: General Concepts

Example 1.

- Let p be the proportion of printed circuits that fail. Current procedure has $p_0 = 0.06$.
- A new method is proposed. To see if the new method results in an improvement.
- We want to test **statistical hypotheses**

$$H_0 : p = p_0 = 0.06 \quad \text{vs} \quad H_1 : p < p_0 = 0.06.$$

- We decided to select $n = 200$ circuits to test. let Y be the number of circuits that fail.
- **Decision rule**: accept the improvement hypothesis H_1 if $Y \leqslant k$ (reject H_0), otherwise reject H_1 (accept H_0).
- $k =?$

Two Types of Errors

- For example, we choose $k = 7$ so that if $Y \leqslant 7$ or $\hat{p} \leqslant 0.035$, then we accept the improvement hypothesis.
- However, whatever k is, the decision made based on a sample could be wrong. There are two types of errors.

	H_0 true	H_0 false
Reject H_0	Type I error	Correct
Accept H_0	Correct	Type II error

- If we accept H_0, then we will keep the current procedure. This is called no change hypothesis. It is also called **null hypothesis**.
- H_1 is called **alternative hypothesis**.
- $y \leqslant k$ is called **rejection** or **critical region** of H_0.
- y or $\hat{p}$ is called the test statistic.

Chances to Make Mistakes: Type I Error

- The Type I error of the above rule is that $Y \leqslant 7$ but the truth is $p = 0.06$.
- The probability of this error is

$$P(Y \leqslant 7|p = 0.06) = \sum_{y=0}^{7} \binom{200}{y} (0.06)^y (0.94)^{200-y}$$
$$= 0.0829 = 8.29\%.$$

Chances to Make Mistakes: Type II Error

- Type II error: $Y > 7$ but the truth is $p < 0.06$.
- The probability of Type II error is, for $p < 0.06$,

$$\beta(p) = P(Y > 7|p) = \sum_{y=8}^{200} \binom{200}{y} p^y (1-p)^{200-y}$$

- For example,

$$\beta(0.03) = 1 - binomialcdf(7, n = 200, p = 0.03) = 0.267$$
$$\beta(0.02) = 1 - binomialcdf(7, n = 200, p = 0.02) = 0.038$$
$$\beta(0.01) = 1 - binomialcdf(7, n = 200, p = 0.01) = 4.64 \times 10^{-5}$$

Significance Level and Power of the Test

- Generally, we want to control the Type I error first: for a given **significance level** α, choose k so that
$$P(\text{Type I error}) \leqslant \alpha$$
- The **power** of the test is,
$$\text{Power}(p) = 1 - \beta(p).$$
- In the above example, for $p < 0.06$,
$$\text{Power}(p) = P(Y \leqslant 7|p) = 1 - P(Y > 7|p).$$
$$\begin{aligned}\text{Power}(0.03) &= 0.833\\ \text{Power}(0.02) &= 0.962,\\ \text{Power}(0.01) &\approx 1.\end{aligned}$$

Find Critical Value k If H_0 is true, approximately,
$$Z = \frac{Y - np_0}{\sqrt{np_0(1-p_0)}} \sim N(0,1).$$
Thus
$$\alpha = P(Y \leqslant k|p = p_0) \approx \Phi\left(\frac{k - np_0}{\sqrt{np_0(1-p_0)}}\right)$$
So
$$\frac{k - np_0}{\sqrt{np_0(1-p_0)}} = -z_\alpha.$$
and
$$k = np_0 - z_\alpha\sqrt{np_0(1-p_0)}.$$

Rejection Region of H_0

- The **rejection region** of H_0 is $R = \{Y : Y \leqslant k\}$
- or $R = \left\{Z = \frac{Y - np_0}{\sqrt{np_0(1-p_0)}} \leqslant -z_\alpha\right\}$.
- Here k and $-z_\alpha$ are called the critical values of the **test statistics** Y and Z respectively.
- In the above example, $n = 200$, $p_0 = 0.06$. Choose $\alpha = 0.05$. $z_\alpha = z_{0.05} = 1.645$.
$$k = np_0 - z_\alpha\sqrt{np_0(1-p_0)} = 6.42$$
- The rejection region of H_0 is
$$R = \{Y : Y \leqslant 6.4167\}$$
or
$$R = \left\{Z = \frac{Y - 12}{3.358571} \leqslant -1.645\right\}.$$

10.2 Testing a Statistical Hypothesis

Continuous Random Variable Examples

Example 2: Let X be the Brinell hardness measurement of ductile iron substantially annealed. Assume that the distribution of X is $N(\mu, 10^2)$. We shall test the null hypothesis $H_0 : \mu = 170$ against the alternative hypothesis $H_1 : \mu > 170$ based on a sample of size n.
(a) Define the test statistic;
(b) Define a critical region with a significance level $\alpha = 0.05$.
(c) A random sample of $n = 25$ observations of X yielded the following measurements.
170 167 174 179 179 156 163 156 187 156 183 179 174 179 170 156 187 179 183 174 187 167 159 170 179
Calculate the value of the test statistic.
(d) State your conclusion.

Solution of Example 2

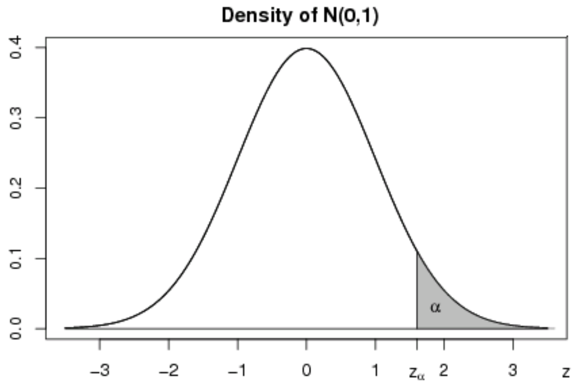
(a) Since $\bar{X}$ is an estimator of μ, we use $\bar{X}$ as the test statistic.

(b) The rejection region of $H_0 : \mu = 170$ in favor of $H_1 : \mu > 170$:

$$\bar{X} > c$$

for some pre-determined critical value c according to α.

$$0.05 = \alpha = Prob(\text{Type I}) =?$$
$$P(\text{Type I}) = P(\text{Rejec } H_0 \,|\, H_0 \text{ is true}) = P(\bar{X} > c \,|\, \mu = 170)$$
$$0.05 = \alpha = P\left(Z = \frac{\bar{X} - 170}{10/\sqrt{n}} > \frac{c - 170}{10/\sqrt{n}}\right)$$



So

$$\frac{c - 170}{10/\sqrt{n}} = z_\alpha \quad \to \quad c = 170 + \frac{10 z_\alpha}{\sqrt{n}}$$

The rejection region is

$$\bar{X} > 170 + \frac{10 z_\alpha}{\sqrt{n}}, \quad \text{or} \quad Z = \frac{\bar{X} - 170}{10/\sqrt{n}} > z_\alpha$$

If $\alpha = 0.05$ then $z_\alpha = 1.645$.

(c) $n = 25$, $\bar{x} = 172.52$, $\alpha = 0.05$, $z_\alpha = 1.645$.

$$z = \frac{\bar{x} - 170}{10/\sqrt{n}} = \frac{172.52 - 170}{10/\sqrt{25}} = 1.26$$

(d) Conclusion: Since $z = 1.26 < z_\alpha = 1.645$, don't reject H_0.

10.3 Using p-value

The p-value Approach

- One important statistic which is usually reported in statistical analysis is the **p-value**:
- the probability of rejecting true null H_0 using the observed value of test statistic as the critical value.
- For example, for the above hypotheses $H_0 : p = p_0$ vs $H_1 : p < p_0$, if the observed value of Y is y, then the observed value of Z is

$$z = \frac{y - np_0}{\sqrt{np_0(1-p_0)}}$$

- the p-value is

$$p_{value} = P(Z \leqslant z | p = p_0) = P(Y \leqslant y | p = p_0)$$
$$\approx \Phi\left(\frac{y - np_0}{\sqrt{np_0(1-p_0)}}\right).$$

Decision Based on p-value

- If p-value $< \alpha$, then we reject H_0 at the significance level α. Otherwise accept H_0.
- If in Example 1, the observed value of Y is $y = 4$, then the p-value is
$$p_{value} \approx \Phi\left(\frac{4-12}{3.358571}\right) = 0.0086.$$
- So we can reject H_0 at the significance level $\alpha = 0.05$. We can also reject H_0 at the significance level $\alpha = 0.01$.

P-value–the formal definition Definition If W is a test statistic, the *p-value*, or *attained significance level*, is the smallest level of significance α for which the observed data indicate that the null hypothesis should be rejected. Let RR_α be the rejection region of significance level α:
$$P(W \in RR_\alpha | H_0) = \alpha.$$
If the data resulted in an observed value w of W, then the
$$p\text{-value} = \min\{\alpha : w \in RR_\alpha\}.$$

Rejection Region: p-value $< \alpha$.

Example A health club claims that its members lose an average of 10 pounds or more within the 1st month after joining the club. A consumer agency wanted to test this claim.
A sample 36 members was selected and an average weight lost of 9.2 pounds was obtained. Assume the population standard deviation is known to be 2.4 pounds. What is the p-value of this test?
Solution:

Step 1. Hypotheses:
$$H_0 : \mu \geqslant 10 \quad H_A : \mu < 10$$

Step 2. Since $\sigma = 2.4$ known, $n = 36 > 30$, use normal distribution and z test statistic $Z = (\bar{X} - \mu_0)/(\sigma/\sqrt{n})$ and rejection region $RR_\alpha = \{Z < -z_\alpha\}$.

Step 3. $\bar{x} = 9.2$, the observed value of Z is $z = (\bar{x} - \mu_0)/(\sigma/\sqrt{n}) = -2.00$.
The p-value is
$$p\text{-value} = \min\{\alpha : -2.00 < -z_\alpha\} = P(Z < -2.00) = 0.0228$$
If $\alpha = 0.01$, then since p-value > 0.01, we do not reject H_0 at the significance level 0.01.
But p-value < 0.05, we reject H_0 at the significance level 0.05.

Finding p-value Let the observed value of the test statistic be w. If the rejection region is $\{W < k\}$, then the p-value is
$$p - \text{value} = P(W < w | H_0).$$
Given a significance level α, if $p -$ value is smaller than α, we reject the null hypothesis H_0. This is equivalent to the critical value approach.
Click here for tables of the commonly used tests.

An Example of One-Sided Test

Example 1. Let x be the Brinell hardness measurement of ductile iron subcritically annealed. Assume that x has a normal distribution with mean μ. Test $H_0 : \mu = 170$ against the alternative hypothesis that the mean hardness is greater than 170 based on the following 25 observations(Use significance level $\alpha = 0.05$.)

```
170, 167, 174, 179, 179, 156, 163, 156, 187
156, 183, 179, 174, 179, 170, 156, 187,
179, 183, 174, 187, 167, 159, 170, 179
```

Hypotheses:
$$H_0 : \mu = 170 \quad H_1 : \mu > 170$$

An Example of Two-Sided Test

Example 2. A company that manufactures brackets for an auto maker selected 15 brackets from the production line and performs a torque test. The goal is for mean torque to equal 125. Let the toque have a normal distribution. The 15 observations are

```
128, 149, 136, 114, 126, 142, 124,
122, 118, 122, 129, 118, 122, 129, 136
```

Hypotheses: $H_0 : \mu = 125 \quad H_1 : \mu \neq 125$

Another Example of Two-Sided Test

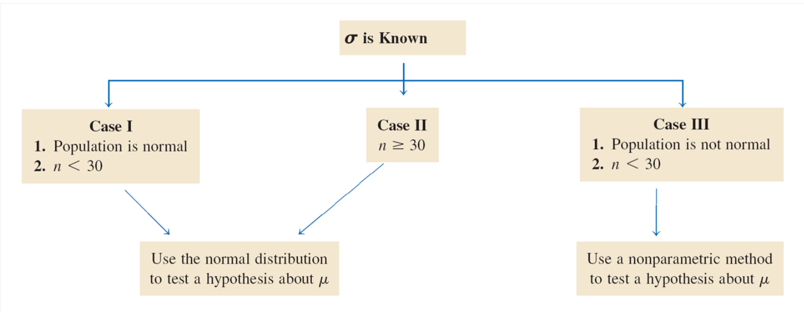
Example 3. A Casino want to test whether the probability p that a red number comes up on a Nevada roulette wheel equals 18/38 or not.

Hypotheses: $H_0 : p = 18/38 \quad H_1 : p \neq 18/38.$

10.4 Single Sample: Tests Concerning a Single Mean

Tests Concerning a Single Mean (Variance Known)

Three Cases



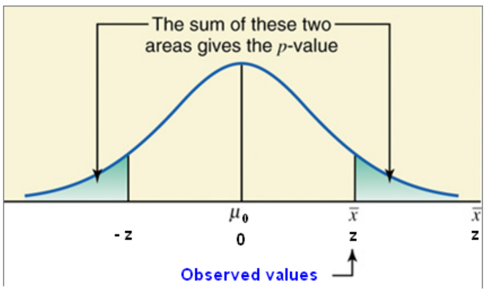
Cases I and II: The z Test

The Two-Tailed z-Test for $H_0 : \mu = \mu_0$

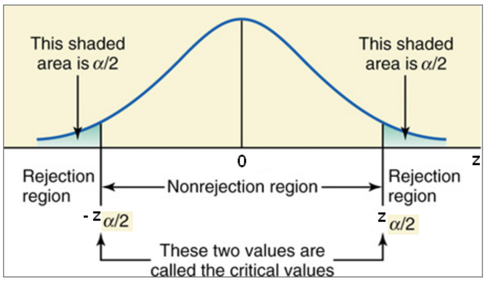
Step 1. State the hypotheses $H_0 : \mu = \mu_0, \quad H_1 : \mu \neq \mu_0$

Step 2. Choose the distribution: If σ is known and either population is normal or $n \geqslant 30$, then use normal distribution;

Step 3. Calculate p-value or find critical value. The test statistic is $z = \sqrt{n}(\bar{x} - \mu_0)/\sigma$
Find the p-value:



Step 3. Find p-value or critical value.
Find the critical value $z_{\alpha/2}$ for z:



Step 4. Make decision:
If p-value $< \alpha$, reject H_0, otherwise, do not reject H_0.
if $z > z_{\alpha/2}$ or $z < -z_{\alpha/2}$, reject H_0, otherwise, do not reject H_0.

Example 1. Assume that the the thickness of spearmint gum manufactured for vending machines has a normal distribution with mean μ and $\sigma = 0.1$. Test the null hypothesis $H_0 : \mu = 7.5$ hundredths of an inch against $H_1 : \mu \neq 7.5$ based on the 10 observations

```
7.65, 7.60, 7.65, 7.70, 7.55
7.55, 7.40, 7.40, 7.50, 7.50
```

Solution:

Step 1. Hypotheses: $H_0 : \mu = 7.5 \quad H_1 : \mu \neq 7.5$

Step 2. This is Case I: $\sigma = 0.1$, $n = 10 < 30$, population is normal.

Step 3.

$$\bar{x} = 7.55, \quad \alpha = 0.05,$$

$$z = (\bar{x} - \mu_0)/(\sigma/\sqrt{n}) = 1.581$$

The p-value Approach: The p-value is

$$p\text{-value} = 2P(\bar{x} > 7.55) = 2P(z > 1.581) = 0.1139$$

The Critical Value Approach: The critical value is $z_{\alpha/2} = 1.96$.

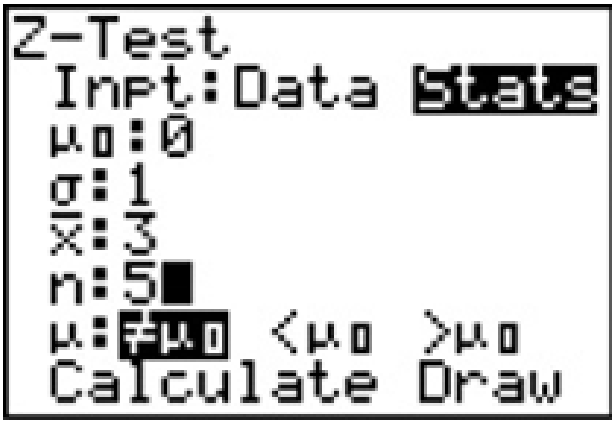
Step 4. Since the p-value $> \alpha$, we do not reject H_0.
Since $|z| < z_{\alpha/2}$, we do not reject H_0.

Using Excel

ZTEST =2*ZTEST(B5:F6,7.5,0.1)

	A	B	C	D	E	F	G
1	Two-Tailed Z-Test						
2	Example 1:		H0: mu = mu0 vs H1: mu != mu0				
3	mu0=	7.5	sigma=	0.1			
4	alpha=	0.05	n=	10			
5	Data:	7.65	7.6	7.65	7.7	7.55	
6		7.55	7.4	7.4	7.5	7.5	
7	x-bar=	7.55	z=	1.5811	C. V. =	1.95996	
8	Using Z-test		P-val=	=2*ZTEST(B5:F6,7.5,0.1)			
9				ZTEST(array, x, [sigma])			
10							

Using TI-84



The two-tailed z-test can be done by CI

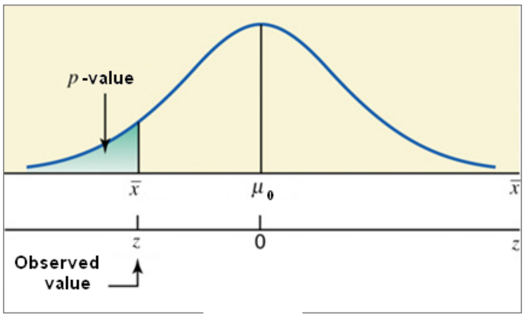
- Construct a $100(1-\alpha)\%$ CI.
- If the hypothesized μ_0 is not contained in the $100(1-\alpha)\%$ CI, we reject H_0 at the significant level α,
- otherwise we do not reject H_0 at the significant level α.

The Left-Tailed z-Test

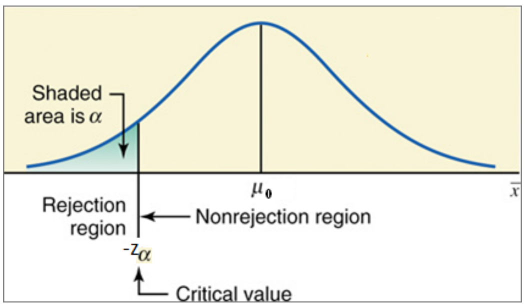
Step 1. State the null and alternative hypotheses $H_0 : \mu = \mu_0,$ or $H_0 : \mu \geqslant \mu_0, H_1 : \mu < \mu_0$

Step 2. Choose the distribution;

Step 3. Find p-value or critical value; $z = \sqrt{n}(\bar{x} - \mu_0)/\sigma$
Find the p-value:



Step 3. Find p-value or critical value;
The Critical Value Approach:
Find critical value $-z_\alpha$ for z:



Step 4. Make decision:
If p-value $< \alpha$, reject H_0, otherwise, do not reject H_0.
If $z < -z_\alpha$, reject H_0, otherwise, do not reject H_0.

Example 2. A health club claims that its members lose an average of 10 pounds or more within the 1st month after joining the club. A consumer agency wanted to test this claim.
A sample 36 members was selected and an average weight lost of 9.2 pounds was obtained. Assume the population standard deviation is known to be 2.4 pounds. What is your decision if $\alpha = 0.01$? What if $\alpha = 0.05$?
Solution:

Step 1. Hypotheses:

$$H_0 : \mu \geqslant 10 \quad H_1 : \mu < 10$$

Step 2. This is Case II. $\sigma = 2.4$, $n = 36$, use normal distribution.

Step 3. $\bar{x} = 9.2$, $z = (\bar{x} - \mu_0)/(\sigma/\sqrt{n}) = -2.00$ The p-value Approach: The p-value is

$$p\text{-value} = P(\bar{x} < 9.2) = 2P(z < -2.00) = 0.0228$$

The Critical Value Approach:
If $\alpha = 0.01$, then $-z_\alpha = -2.34$.
If $\alpha = 0.05$, then $-z_\alpha = -1.64$.

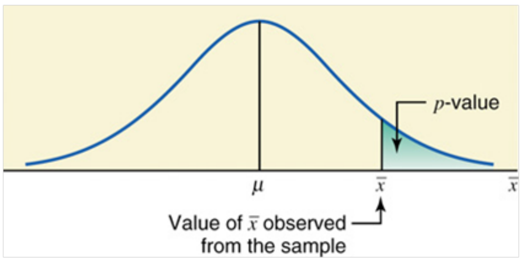
Step 4. p-value > 0.01, we do not reject H_0 at the significant level 0.01.
But p-value < 0.05, we reject H_0 at the significant level 0.05.
$z > -z_\alpha$, we do not reject H_0 at the significant level 0.01.
$z < -z_\alpha$, we reject H_0 at the significant level 0.05.

The Right-Tailed z-Test

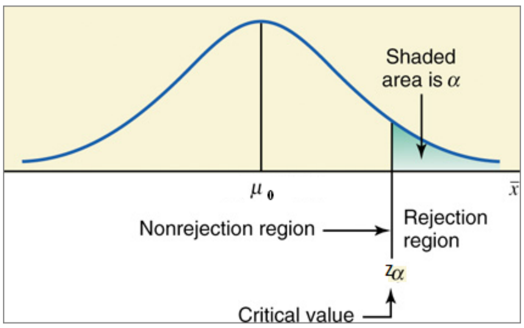
Step 1. State the null and alternative hypotheses $H_0 : \mu = \mu_0$, or $H_0 : \mu \leqslant \mu_0$, $H_1 : \mu > \mu_0$.

Step 2. Choose the distribution;

Step 3. Find p-value or critical value; $z = \sqrt{n}(\bar{x} - \mu_0)/\sigma$
Find the p-value:



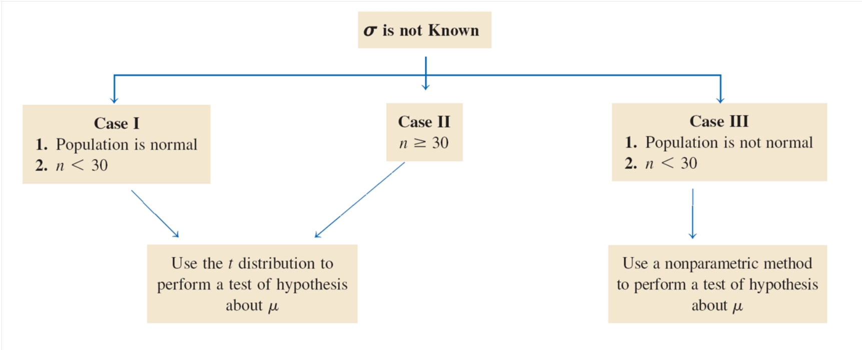
Step 3. Find p-value or critical value.
Find critical value z_α for z:



Step 4. Make decision:
If p-value $< \alpha$, reject H_0, otherwise, do not reject H_0.
If $z > z_\alpha$, reject H_0, otherwise, do not reject H_0.

Tests Concerning a Single Mean (Variance Unknown)

Three Cases



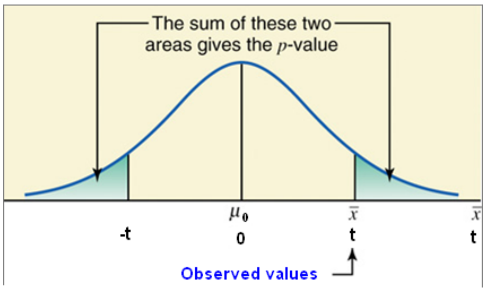
Cases I and II: The t Test

The Two-Tailed t-Test for $H_0 : \mu = \mu_0$

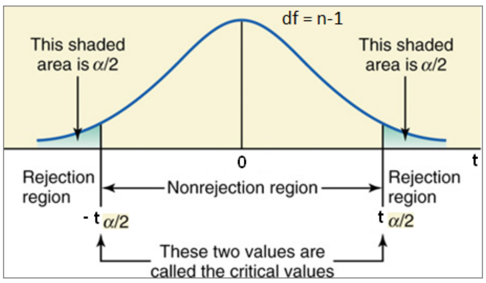
Step 1. State the hypotheses $H_0 : \mu = \mu_0, \quad H_1 : \mu \neq \mu_0$

Step 2. Choose the distribution: If σ is unknown and either population is normal or $n \geqslant 30$, then use t distribution;

Step 3. Calculate p-value or find critical value. The test statistic is $t = \sqrt{n}(\bar{x} - \mu_0)/s$, $df = n - 1$.
Find the p-value:



Step 3. Find p-value or critical value.
Find the critical value $t_{\alpha/2}$ for t:



Step 4. Make decision:
If p-value $< \alpha$, reject H_0, otherwise, do not reject H_0.
if $t > t_{\alpha/2}$ or $t < -t_{\alpha/2}$, reject H_0, otherwise, do not reject H_0.

Example 1. A company that manufactures brackets for an auto maker selected 15 brackets from the production line and performs a torque test. The goal is for mean torque to equal 125. Let the toque have a normal distribution. The 15 observations are

```
128, 149, 136, 114, 126, 142, 124,
122, 118. 122, 129, 118, 122, 129, 136
```

Test $H_0 : \mu = 125$ against a two-tailed alternative hypothesis ($\alpha = 0.05$).
Solution:

Step 1. Hypotheses: $H_0 : \mu = 125 \quad H_1 : \mu \neq 125$

Step 2. This is Case I: $n = 15 < 30$, population is normal, use t distribution.

Step 3.

$$t = (\bar{x} - \mu_0)/(s/\sqrt{n}) = 1.076$$

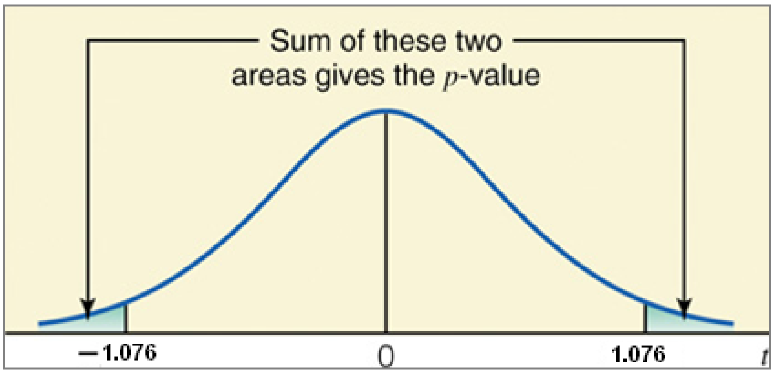
The p-value Approach: The p-value is

$$p\text{-value} = 2P(t > 1.076) = 0.3001$$

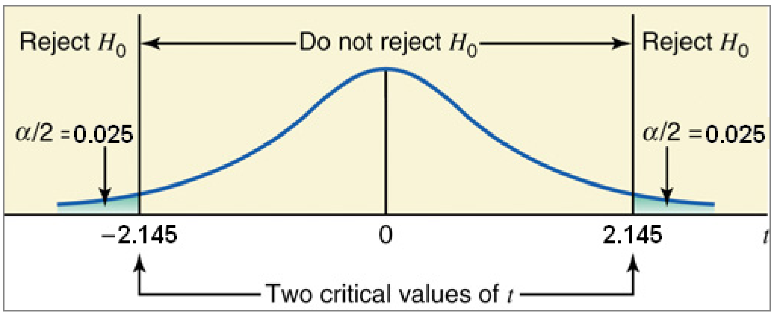
The Critical Value Approach: $\alpha = 0.05$, The critical value is $t_{\alpha/2} = 2.145$.

Step 4. Since the p-value $> \alpha$, we do not reject H_0.
Since $|t| < t_{\alpha/2}$, we do not reject H_0.

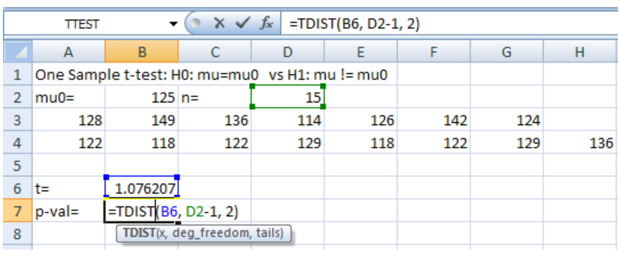
P-value Value of t–Two-Tailed



Critical Value of t–Two-Tailed

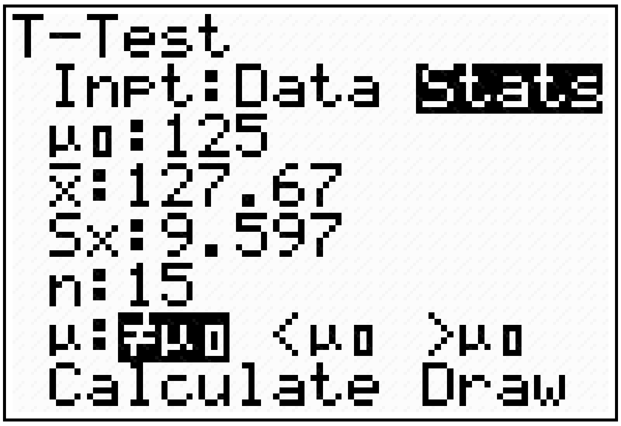


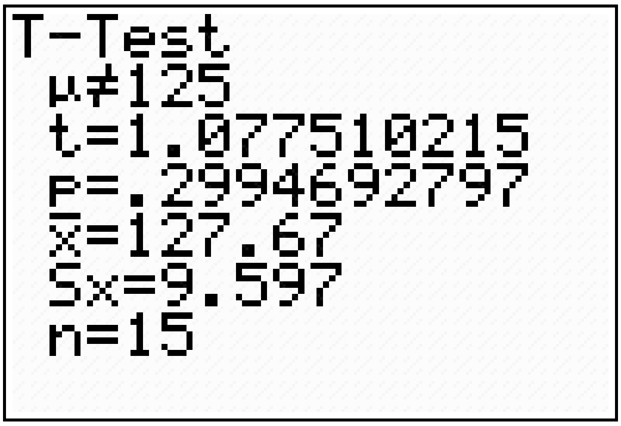
Using Excel


TTEST =TDIST(B6, D2-1, 2)

	A	B	C	D	E	F	G	H
1	One Sample t-test: H0: mu=mu0 vs H1: mu != mu0							
2	mu0=	125	n=	15				
3	128	149	136	114	126	142	124	
4	122	118	122	129	118	122	129	136
5								
6	t=	1.076207						
7	p-val=	=TDIST(B6, D2-1, 2)						
8		TDIST(x, deg_freedom, tails)						

Using TI-84





The two-tailed t-test can be done by CI

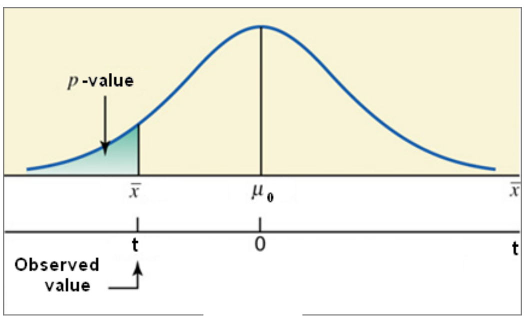
- Construct a $100(1-\alpha)\%$ CI using t distribution.
- If the hypothesized μ_0 is not contained in the $100(1-\alpha)\%$ CI, we reject H_0 at the significant level α,
- otherwise we do not reject H_0 at the significant level α.

The Left-Tailed t-Test

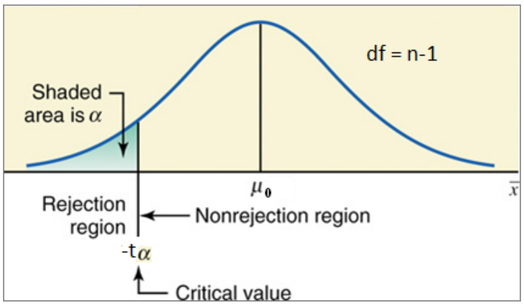
Step 1. State the null and alternative hypotheses $H_0 : \mu = \mu_0$, or $H_0 : \mu \geqslant \mu_0, H_1 : \mu < \mu_0$

Step 2. Choose the distribution;

Step 3. Find p-value or critical value; $t = \sqrt{n}(\bar{x} - \mu_0)/s$, $df = n - 1$,
Find the p-value:



Step 3. Find p-value or critical value; The Critical Value Approach:
Find critical value $-t_\alpha$ for t:



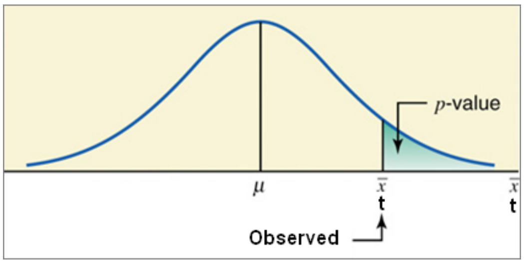
Step 4. Make decision:
If p-value $< \alpha$, reject H_0, otherwise, do not reject H_0.
If $t < -t_\alpha$, reject H_0, otherwise, do not reject H_0.

The Right-Tailed t-Test

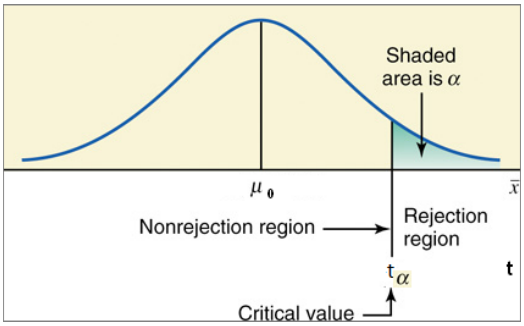
Step 1. State the null and alternative hypotheses $H_0 : \mu = \mu_0$, or $H_0 : \mu \leqslant \mu_0$, $H_1 : \mu > \mu_0$

Step 2. Choose the distribution;

Step 3. Find p-value or critical value; $t = \sqrt{n}(\bar{x} - \mu_0)/s$, $df = n - 1$,
Find the p-value:



Step 3. Find p-value or critical value.
Find critical value t_α for t:



Step 4. Make decision:
If p-value $< \alpha$, reject H_0, otherwise, do not reject H_0.
If $t > t_\alpha$, reject H_0, otherwise, do not reject H_0.

Example 2. Let x be the Brinell hardness measurement of ductile iron subcritically annealed. Assume that x has a normal distribution with mean μ. Test $H_0 : \mu = 170$ against the alternative hypothesis that the mean hardness is greater than 170 based on the following 25 observations(Use significance level $\alpha = 0.05$.)

```
170, 167, 174, 179, 179, 156, 163, 156, 187
156, 183, 179, 174, 179, 170, 156, 187,
179, 183, 174, 187, 167, 159, 170, 179
```

Solution:

Step 1. Hypotheses:

$$H_0 : \mu = 170 \quad H_1 : \mu > 170$$

Step 2. This is Case I. $s = 10.31$, $n = 25$, σ is not known, use t distribution.

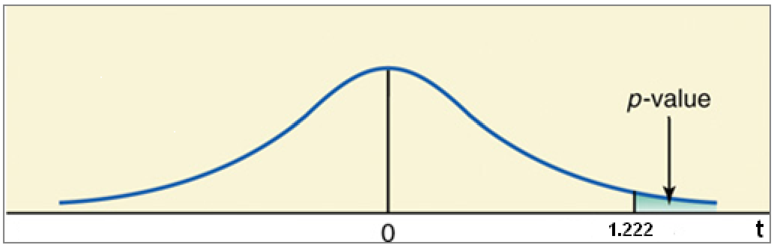
Step 3. $\bar{x} = 172.52$, $s = 10.31$, $t = (\bar{x} - \mu_0)/(s/\sqrt{n}) = 1.2218$, $df = 24$.
The p-value Approach: The p-value is

$$p\text{-value} = P(t > 1.2218) = 0.1168$$

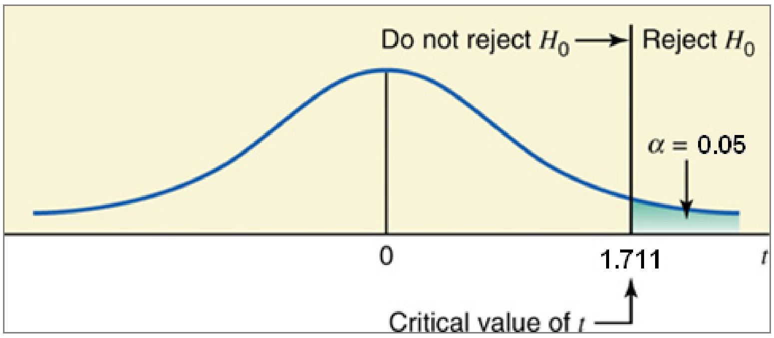
The Critical Value Approach:
$\alpha = 0.05$, $t_\alpha = 1.7109$.

Step 4. p-value > 0.05, we do not reject H_0 at the significant level 0.05.
$t < t_\alpha$, we do not reject H_0 at the significant level 0.05.

P-value of t–Right-Tailed



Critical value of t–Right-Tailed

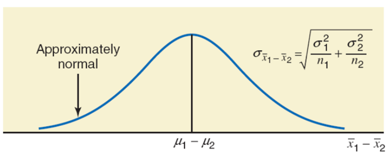


10.5 Two Samples: Tests on Two Means

σ_1 and σ_2 known

1. The two samples are independent.
2. The two population standard deviations σ_1 and σ_2 are known
3. At least one of the following conditions is fulfilled:
   i. Both samples are large (i.e., $n_1 \geqslant 30$ and $n_2 \geqslant 30$)
   ii. Both populations from which the samples are drawn are normally distributed

Then the sampling distribution of $\bar{x}_1 - \bar{x}_2$ is (approximately) normal with mean $\mu_{\bar{x}_1 - \bar{x}_2} = \mu_1 - \mu_2$ and standard deviation $\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$.

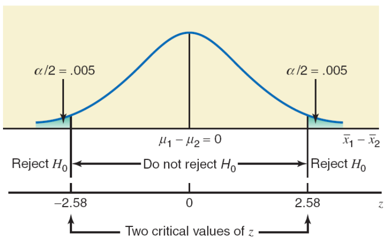


Two-Tailed Test $H_0 : \mu_1 - \mu_2 = d_0$ (The difference is d_0.)
$H_1 : \mu_1 - \mu_2 \neq d_0$ (The difference is not d_0.)

The z-test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2 - d_0}{\sigma_{\bar{x}_1 - \bar{x}_2}}$$

Example 1
According to Kaiser Family Foundation survey in 2011 and 2010, the average premium for employer-sponsored health insurance for family coverage was $15,073 in 2011 and $13,770 in 2010 (USA TODAY, September 29, 2011). Suppose that these averages were based on random samples of 250 and 200 employees who had such employer-sponsored health insurance plans for 2011 and 2010, respectively. Further assume that the population standard deviations for 2011 and 2010 were $2160 and $1990, respectively.
Let μ_1 and μ_2 be the respective population means for such annual premiums for the years 2011 and 2010, respectively. Test at the 1% significance level whether the population means for the two years are different.
Solution Step 1.
$H_0 : \mu_1 - \mu_2 = 0$ (The two population means are not different.)
$H_1 : \mu_1 - \mu_2 \neq 0$ (The two population means are different.)
Step 2.
Population standard deviations, σ_1 and σ_2, are known Both samples are large; $n_1 \geqslant 30$ and $n_2 \geqslant 30$ Therefore, we use the normal distribution to perform the hypothesis test.
Step 3.
$\alpha = .01$.
The $\neq$ sign in the alternative hypothesis indicates that the test is two-tailed
Area in each tail = $\alpha/2 = .01/2 = .005$ The critical values of z are 2.58 and -2.58.



Step 4.

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{(2160)^2}{250} + \frac{(1990)^2}{200}} = \$196.12$$

The z-test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2 - d_0}{\sigma_{\bar{x}_1 - \bar{x}_2}} = \frac{\$15,073 - \$13,770 - 0}{\$196.12} = 6.64$$

Step 5. Because the value of the test statistic $z = 6.64$ falls in the rejection region, we reject the null hypothesis H_0. Therefore, we conclude that the average annual premiums for employer-sponsored health insurance for family coverage were different for 2011 and 2010.

$\sigma_1 = \sigma_2$ but unknown

If

1. the two samples are independent
2. σ_1 and σ_2 are equal but unknown
3. both populations are normally distributed

then the pooled estimator of $\sigma_1 = \sigma_2$ is

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

and the sampling distribution of

$$t = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{s_{\bar{x}_1 - \bar{x}_2}}$$

is t-distribution with $df = n_1 + n_2 - 2$. The estimator of the standard deviation of $\bar{x}_1 - \bar{x}_2$ is

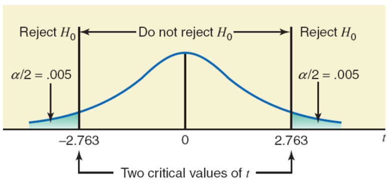
$$s_{\bar{x}_1 - \bar{x}_2} = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Two-Tailed Test $H_0 : \mu_1 - \mu_2 = d_0$ (The difference is d_0.)
$H_1 : \mu_1 - \mu_2 \neq d_0$ (The difference is not d_0.)

The t-test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2 - d_0}{s_{\bar{x}_1 - \bar{x}_2}}$$

Example 2
A sample of 14 cans of Brand I diet soda gave the mean number of calories of 23 per can with a standard deviation of 3 calories. Another sample of 16 cans of Brand II diet soda gave the mean number of calories of 25 per can with a standard deviation of 4 calories.
At the 1% significance level, can you conclude that the mean number of calories per can are different for these two brands of diet soda? Assume that the calories per can of diet soda are normally distributed for each of the two brands and that the standard deviations for the two populations are equal.
Solution Step 1.
$H_0 : \mu_1 - \mu_2 = 0$ (The mean numbers of calories are not different.)
$H_1 : \mu_1 - \mu_2 \neq 0$ (The mean numbers of calories are different.)
Step 2.
The two samples are independent.
Population standard deviations, σ_1 and σ_2, are unknown but equal. Both samples are small but both populations are normal. Therefore, we use the t distribution to perform the hypothesis test.
Step 3.
$\alpha = .01$.
The $\neq$ sign in the alternative hypothesis indicates that the test is two-tailed.
Area in each tail = $\alpha/2 = .01/2 = .005$. $df = n_1 + n_2 - 2 = 14 + 16 - 2 = 28$. The critical values of t are -2.763 and 2.763.



Step 4.

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{(14 - 1)(3)^2 + (16 - 1)(4)^2}{14 + 16 - 2}} = 3.5707$$

$$s_{\bar{x}_1 - \bar{x}_2} = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 3.5707\sqrt{\frac{1}{14} + \frac{1}{16}} = 1.3067$$

The t-test statistic

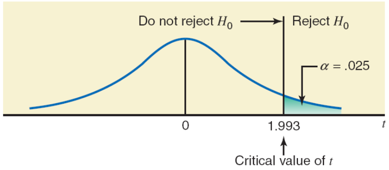
$$t = \frac{\bar{x}_1 - \bar{x}_2 - d_0}{s_{\bar{x}_1 - \bar{x}_2}} = \frac{23 - 25 - 0}{1.3067} = -1.531$$

Step 5. Because the value of the test statistic $t = -1.531$ falls in the nonrejection region, we fail to reject the null hypothesis H_0.

Consequently, we conclude that there is no difference in the mean numbers of calories per can for the two brands of diet soda.

Example 3
A sample of 40 children from New York State showed that the mean time they spend watching television is 28.50 hours per week with a standard deviation of 4 hours. Another sample of 35 children from California showed that the mean time spent by them watching television is 23.25 hours per week with a standard deviation of 5 hours.
Using a 2.5% significance level, can you conclude that the mean time spent watching television by children in New York State is greater than that for children in California? Assume that the standard deviations for the two populations are equal.
Solution Step 1.
$H_0 : \mu_1 - \mu_2 = 0$
$H_1 : \mu_1 - \mu_2 > 0$
Step 2.

The two samples are independent.
Population standard deviations, σ_1 and σ_2, are unknown but equal. Both samples are large. Therefore, we use the t distribution to perform the hypothesis test.
Step 3.
$\alpha = .025$.
The $>$ sign in the alternative hypothesis indicates that the test is right-tailed.
Area in the right tail = $\alpha = .025$. $df = n_1 + n_2 - 2 = 40 + 35 - 2 = 73$. The critical values of t is 1.993.



Step 4.

$$s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}} = \sqrt{\frac{(40-1)(4)^2 + (35-1)(5)^2}{40+35-2}} = 4.4935$$

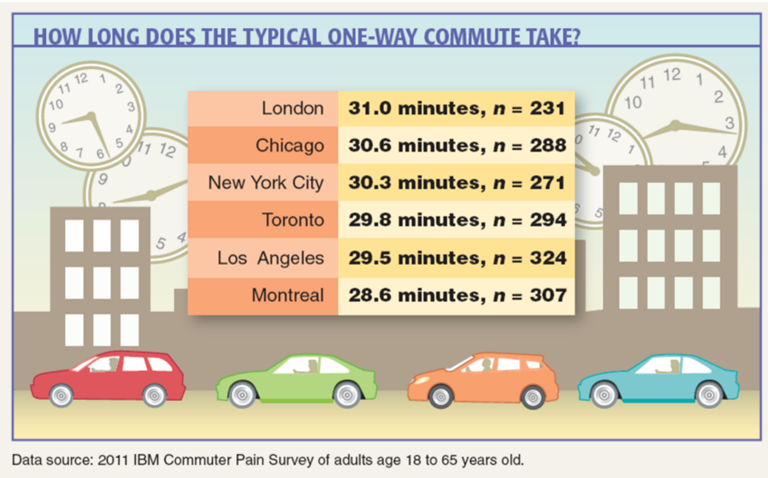
$$s_{\bar{x}_1-\bar{x}_2} = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 4.4935\sqrt{\frac{1}{40} + \frac{1}{35}} = 1.040$$

The t-test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2 - d_0}{s_{\bar{x}_1-\bar{x}_2}} = \frac{28.5 - 23.25 - 0}{1.040} = 5.048$$

Step 5. Because the value of the test statistic $t = 5.048$ falls in the rejection region, we reject the null hypothesis H_0.
Hence, we conclude that children in New York State spend more time, on average, watching TV than children in California.

Case Study 10-1 How Long Does the Typical One-Way Commute Take?



σ_1 and σ_2 unknown

If

1. The two samples are independent
2. The standard deviations σ_1 and σ_2 of the two populations are unknown and unequal
3. At least one of the following conditions is fulfilled:
   i. Both samples are large (i.e., $n_1 \geqslant 30$ and $n_2 \geqslant 30$)
   ii. Both populations from which the samples are drawn are normally distributed

then the sampling distribution of

$$t = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{s_{\bar{x}_1-\bar{x}_2}}$$

is approximately t-distribution with $df = \lfloor r \rfloor$, "floor" of r, which is the integer part of r and

$$r = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2}, \quad s_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Two-Tailed Test $H_0 : \mu_1 - \mu_2 = d_0$ (The difference is d_0.)
$H_1 : \mu_1 - \mu_2 \neq d_0$ (The difference is not d_0.)

The t-test statistic

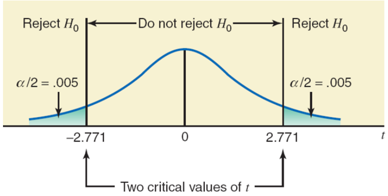
$$t = \frac{\bar{x}_1 - \bar{x}_2 - d_0}{s_{\bar{x}_1-\bar{x}_2}}$$

Example 4(Example 2 revisited)
A sample of 14 cans of Brand I diet soda gave the mean number of calories of 23 per can with a standard deviation of 3 calories. Another sample of 16 cans of Brand II diet soda gave the mean number of calories of 25 per can with a standard deviation of 4 calories.
At the 1% significance level, can you conclude that the mean number of calories per can are different for these two brands of diet soda? Assume that the calories per can of diet soda are normally distributed for each of the two brands and that the standard deviations for the two populations are unequal.
Solution Step 1.
$H_0 : \mu_1 - \mu_2 = 0$ (The mean numbers of calories are not different.)
$H_1 : \mu_1 - \mu_2 \neq 0$ (The mean numbers of calories are different.)
Step 2.
The two samples are independent.
Population standard deviations, σ_1 and σ_2, are unknown and unequal. Both samples are small but both populations are normal. Therefore, we use the t distribution to perform the hypothesis test.
Step 3. $\alpha = .01$.
The $\neq$ sign in the alternative hypothesis indicates that the test is two-tailed. Area in each tail = $\alpha/2 = .01/2 = .005$.

$$s_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{3^2}{14} + \frac{4^2}{16}} = 1.2817$$

$$r = \frac{s^4_{\bar{x}_1-\bar{x}_2}}{\frac{s^4_{\bar{x}_1}}{n_1-1} + \frac{s^4_{\bar{x}_2}}{n_2-1}} = \frac{1.2817^4}{\frac{\left(\frac{3^2}{14}\right)^2}{14-1} + \frac{\left(\frac{4^2}{16}\right)^2}{16-1}} = 27.41$$

$df = \lfloor 27.41 \rfloor = 27$. The critical values of t are -2.771 and 2.771.



Step 4. The t-test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2 - d_0}{s_{\bar{x}_1-\bar{x}_2}} = \frac{23 - 25 - 0}{1.2817} = -1.560$$

Step 5. Because the value of the test statistic $t = -1.560$ falls in the nonrejection region, we fail to reject the null hypothesis H_0.

Consequently, we conclude that there is no difference in the mean numbers of calories per can for the two brands of diet soda.

Using Graphing Calculator

TI 83/84 have 2-sampleTTest for $d_0 = 0$.

If $d_0 \neq 0$ then adjust the first sample as $x_{11} - d_0, \ldots, x_{1n_1} - d_0$ so that it has mean $\bar{x}_1 - d_0$.

When entering STATS, replace $\bar{x}_1$ by $\bar{x}_1 - d_0$ and keep others unchanged.

10.6 Choice of Sample Size for Testing Means

Example 0.1. *A random sample of 500 measurements on the length of stay in hospitals had sample mean 5.4 days and sample standard deviation 3.1 days. A federal regulatory agency hypothesizes that the average length of stay is in excess of 5 days.*

a. Do the data support this hypothesis? Use $\alpha = .05$.

b. Suppose that the agency wants to be able to detect a difference equal to half day in the average length of stay. That is he wishes to test $H_0 : \mu = 5$ against $H_1 : \mu > 5$. Find the $\beta = \beta(5.5)$.

c. If the agency wants the $\beta = \beta(5.5) = .01$, how many measurements are needed?

Solution:

a. $\alpha = .05$, $z_\alpha = 1.644854$.

$$z = (\bar{x} - \mu_0)/[\sigma/\sqrt{n}] \approx (5.4 - 5)/(3.1/\sqrt{500}) = 2.885249 > z_\alpha.$$

So the observed z falls in the rejection region. We reject H_0 in favor of the hypothesis of the agency.

b.
$$\beta = \beta(\mu_1) = P\left[\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leqslant z_\alpha \Big| \mu = \mu_1\right]$$

Under the alternative $H_1 : \mu > 5$, $\bar{X}$ is normal with mean μ and standard deviation $\sigma \approx 3.1$.

$$\beta = \beta(5.5) \approx P\left[\frac{\bar{Y} - 5}{3.1/\sqrt{500}} \leqslant z_\alpha \Big| \mu = 5.5\right]$$
$$= P\left[Z = \frac{\bar{X} - 5.5}{3.1/\sqrt{500}} \leqslant z_\alpha - \frac{5.5 - 5}{3.1/\sqrt{500}}\right]$$
$$= P\left[Z \leqslant z_\alpha - \frac{.5}{3.1/\sqrt{500}}\right] = P(Z \leqslant -1.961708) = 0.0249$$

c. Generally, Under the alternative $H_1 : \mu > \mu_0$, $\bar{Y}$ is normal with mean μ and standard deviation σ.

$$\beta = \beta(\mu_1) = P\left[\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leqslant z_\alpha \Big| \mu = \mu_1\right]$$
$$= P\left[Z = \frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} \leqslant z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}\right]$$
$$= P\left(Z \leqslant z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}\right)$$

Solving for n, we have

$$z_\alpha - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}} = -z_\beta \quad \to \quad n = \frac{(z_\alpha + z_\beta)^2\sigma^2}{(\mu_1 - \mu_0)^2}.$$

$\beta(\mu = 5.5) = .01$, $z_\beta = z_{0.101} = 2.326348$,

$$n \approx \frac{(z_\alpha + z_\beta)^2 s^2}{(\mu_1 - \mu_0)^2} = \frac{(1.644854 + 2.326348)^2(3.1^2)}{(5.5 - 5)^2} = 607$$

Left-Tailed Test

Similarly for testing $H_0 : \mu = \mu_0$ vs. $H_1 : \mu < \mu_0$, if we require $\beta(\mu_1) \leqslant \beta$ for some fixed $\mu_1 < \mu_0$, then $n = \frac{(z_\alpha+z_\beta)^2\sigma^2}{(\mu_0-\mu_1)^2}$.

Two-Tailed z Test

For testing $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$, if we require $\beta(\mu_1) \leqslant \beta$ for some fixed $\mu_1 \neq \mu_0$, then $n =$?

$$\beta(\mu_1) = P\left[-z_{\alpha/2} \leqslant \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leqslant z_{\alpha/2} \Big| \mu = \mu_1\right]$$
$$= P\left[-z_{\alpha/2} - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}} \leqslant Z \leqslant z_{\alpha/2} - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}\right]$$

If $\mu_1 > \mu_0$ then $\beta(\mu_1) < P\left(Z \leqslant z_{\alpha/2} - \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}}\right)$. We can choose n so that $z_{\alpha/2} - \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} = -z_\beta \quad \to \quad n = \frac{(z_{\alpha/2}+z_\beta)^2\sigma^2}{(\mu_1-\mu_0)^2}$.

Similarly if $\mu_1 < \mu_0$ then $\beta(\mu_1) < P\left(Z \geqslant -z_{\alpha/2} - \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}}\right)$. We can choose n so that $-z_{\alpha/2} - \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} = z_\beta \quad \to \quad n = \frac{(z_{\alpha/2}+z_\beta)^2\sigma^2}{(\mu_1-\mu_0)^2}$.

In both cases if $n = \frac{(z_{\alpha/2}+z_\beta)^2\sigma^2}{(\mu_1-\mu_0)^2}$ then $\beta(\mu_1) < \beta$.

Two-Sample z-Test

Because increasing sample size always increases power, consider $n = n_1 = n_2$ for two-sample z-test.

Two-Tailed

$H_0 : \mu_1 - \mu_2 = d_0$ vs. $H_1 : \mu_1 - \mu_2 \neq d_0$.

If $\beta = \beta(\mu_1 - \mu_2 = d_1)$ for some $d_1 \neq d_0$,

then $n = \frac{(z_{\alpha/2}+z_\beta)^2(\sigma_1^2+\sigma_2^2)}{(d_1-d_0)^2}$.

Right-Tailed

$H_0 : \mu_1 - \mu_2 = d_0$ vs. $H_1 : \mu_1 - \mu_2 > d_0$.

If $\beta = \beta(\mu_1 - \mu_2 = d_1)$ for some $d_1 > d_0$,

then $n = \frac{(z_\alpha+z_\beta)^2(\sigma_1^2+\sigma_2^2)}{(d_1-d_0)^2}$.

Left-Tailed

$H_0 : \mu_1 - \mu_2 = d_0$ vs. $H_1 : \mu_1 - \mu_2 < d_0$.

If $\beta = \beta(\mu_1 - \mu_2 = d_1)$ for some $d_1 < d_0$,

then

$$n = \frac{(z_\alpha + z_\beta)^2\sigma^2}{(\mu_0 - \mu_1)^2}.$$

10.8 One Sample: Tests on a Single Proportion

The Two-Tailed z-Test for p

Step 1. State the hypotheses

$$H_0 : p = p_0, \quad vs \quad H_1 : p \neq p_0$$

Step 2. Choose the distribution: If

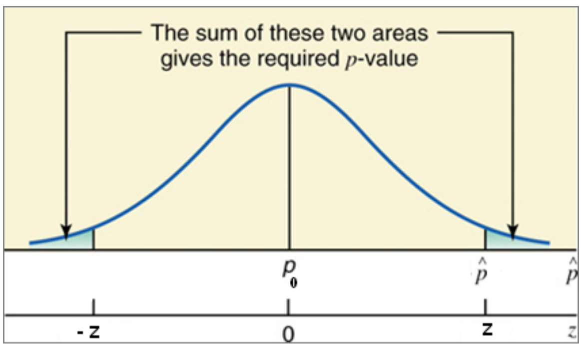
$$np_0 > 5 \quad \text{and} \quad nq_0 > 5 \quad q_0 = 1 - p_0$$

then use

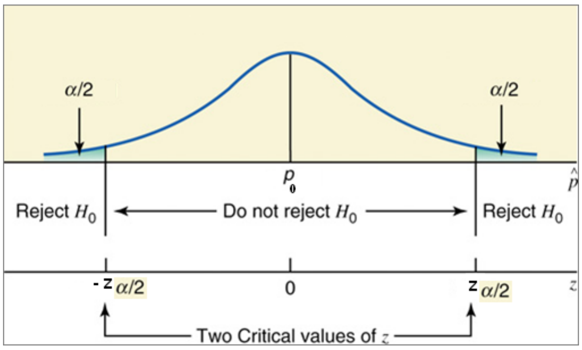
$$z = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0/n}}$$

as the test statistic and choose normal distribution.

Step 3. Calculate p-value or find critical value.
Find the p-value:



Step 3. Calculate p-value or find critical value.
Find the critical value $z_{\alpha/2}$ for z:



Step 4. Make decision:
If p-value $< \alpha$, reject H_0, otherwise, do not reject H_0.
if $z > z_{\alpha/2}$ or $z < -z_{\alpha/2}$, reject H_0, otherwise, do not reject H_0.

Example 1. In a *Time* magazine poll of adult Americans conducted by telephone March 15-17, 2005 by SRBI Public Affairs, 66% of the respondents said that there is too much violence on the television (*Time*, March 28, 2005). Assume that this result holds true for the 2005 population of all adults Americans. In a recent random sample of 1000 adult Americans, 70% said that there is too much violence on the television. Using the 2% significance level, can you conclude that the current percentage of adults Americans who think there is too much violence on the television is different from that for 2005?

Solution:

Step 1. Hypotheses: $H_0 : p = 0.66 \quad H_1 : p \neq 0.66$

Step 2. $n = 1000$,

$$np_0 = 660 > 5, \quad nq_0 = 340 > 5$$

Use z-test and normal distribution.

Step 3.

$$\hat{p} = \frac{104}{590} = 0.1763, \quad \alpha = 0.02,$$

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}} = \frac{0.7 - 0.66}{\sqrt{0.66(0.34)/1000}} = 2.67$$

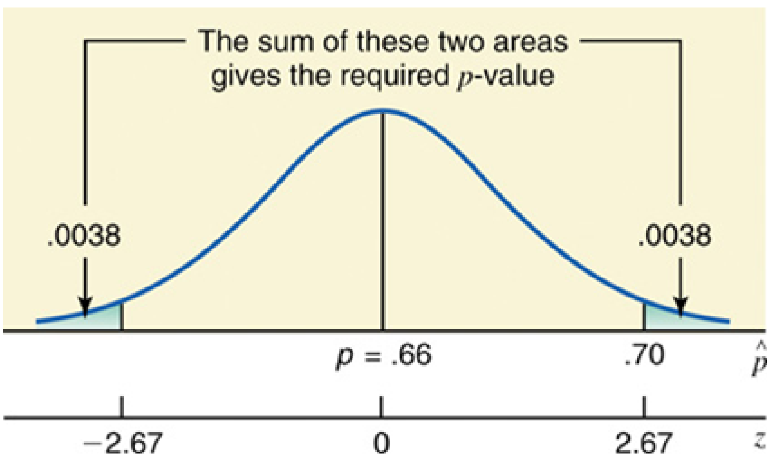
The p-value Approach: The p-value is

$$p\text{-value} = 2P(\hat{p} > 0.7) = 2P(z > 2.67) = 0.0076$$

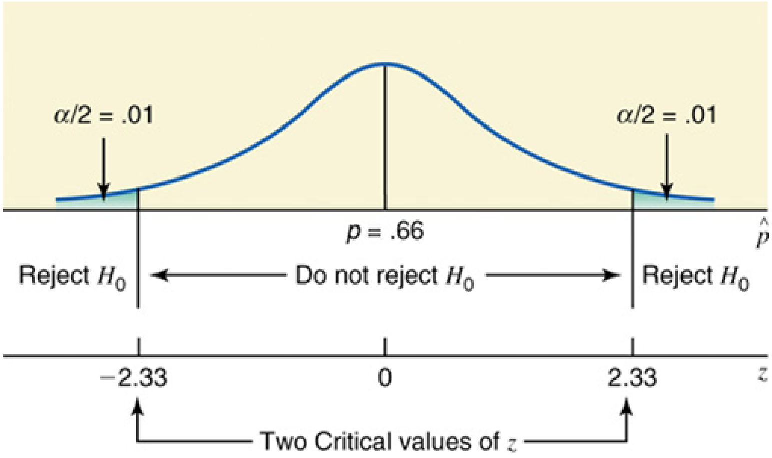
The Critical Value Approach: The critical value is $z_{\alpha/2} = 2.33$.

Step 4. Since the p-value $< \alpha$, we reject H_0.
Since $z > z_{\alpha/2}$, we reject H_0.

P-value of z–Two-Tailed



Critical Value of z–Two-Tailed



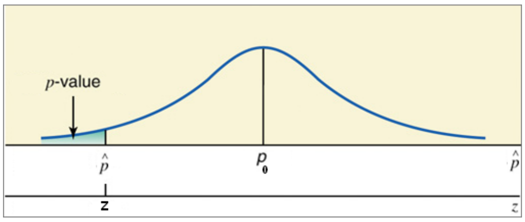
The Left-Tailed z-Test for p

Step 1. State the null and alternative hypotheses $H_0 : p = p_0,$ or $H_0 : p \geqslant p_0$, $H_1 : p < p_0$

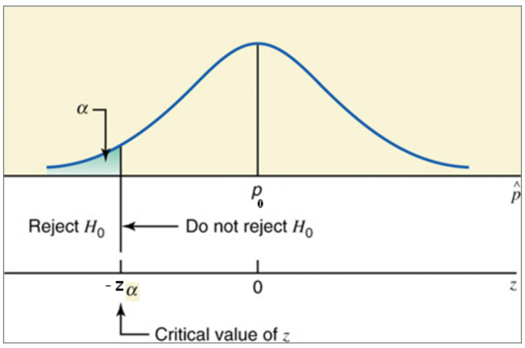
Step 2. Choose the distribution;

Step 3. Find p-value or critical value; $z = \frac{\hat{p}-p_0}{\sqrt{p_0 q_0/n}}$,

Find the p-value:



Step 3. Find p-value or critical value; The Critical Value Approach:
Find critical value z_α for z:



Step 4. Make decision:
If p-value $< \alpha$, reject H_0, otherwise, do not reject H_0.
If $z < z_\alpha$, reject H_0, otherwise, do not reject H_0.

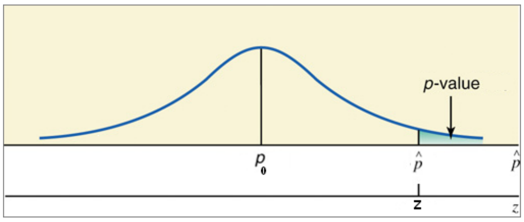
The Right-Tailed z-Test for p

Step 1. State the null and alternative hypotheses $H_0 : p = p_0,$ or $H_0 : p \leqslant p_0$, $H_1 : p > p_0$

Step 2. Choose the distribution;

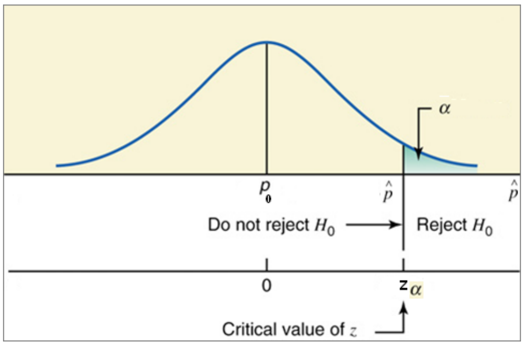
Step 3. Find p-value or critical value; $z = \frac{\hat{p}-p_0}{\sqrt{p_0 q_0/n}}$,

Find the p-value:



Step 3. Find p-value or critical value.

Find critical value z_α for z:



Step 4. Make decision:

If p-value $< \alpha$, reject H_0, otherwise, do not reject H_0.

If $z > z_\alpha$, reject H_0, otherwise, do not reject H_0.

Example 2. Let p be the proportion of drivers who use a seat belt in a state that does not have a amendatory seat belt law. It was claimed that $p = 14\%$. An advertising campaign was conducted to increase the proportion. Two months after the campaign, 104 out of a random sample of 590 drivers were wearing their seat belts. Was the campaign successful?($\alpha = 0.01$).

Solution:

Step 1. Hypotheses: $H_0 : p = 0.14 \quad H_1 : p > 0.14$

Step 2. $n = 590$,

$$np_0 = 590(0.14) = 82.6 > 5, \quad nq_0 = 590(1 - 0.14) = 507.4 > 5$$

Use z-test and normal distribution.

Step 3.

$$\hat{p} = \frac{104}{590} = 0.1763, \quad \alpha = 0.01,$$

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0/n}} = \frac{0.1763 - 0.14}{\sqrt{0.14(0.86)/590}} = 2.541$$

The p-value Approach: The p-value is

$$p\text{-value} = P(\hat{p} > 0.1763) = P(z > 2.541) = 0.0055$$

The Critical Value Approach: The critical value is $z_\alpha = 2.33$.

Step 4. Since the p-value $< \alpha$, we reject H_0.

Since $z > z_\alpha$, we reject H_0.

10.9 Two Samples: Tests on Two Proportions

Objectives

- Hypothesis Testing About $p_1 - p_2$

Sampling distribution of $\hat{p}_1 - \hat{p}_2$ If

- the two samples are independent and
- the sample sizes are large, i.e., $n_1p_1, n_1(1-p_1), n_2p_2$, and $n_2(1-p_2)$ are greater than 5,

then the sampling distribution of $\hat{p}_1 - \hat{p}_2$ is approximately normal with mean $\mu_{\hat{p}_1-\hat{p}_2} = p_1 - p_2$ and standard deviation $\sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$.

Confidence Interval for $p_1 - p_2$ If $n_1\hat{p}_1, n_1(1-\hat{p}_1), n_2\hat{p}_2$, and $n_2(1-\hat{p}_2)$ are greater than 5, then the $(1-\alpha)100\%$ confidence interval for $p_1 - p_2$ is $\hat{p}_1 - \hat{p}_2 \mp z\ s_{\hat{p}_1-\hat{p}_2}$, where the value of z is read from the normal distribution table for the given confidence level, and $s_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$.

Hypothesis Testing About $p_1 - p_2$

For null hypothesis $H_0 : p_1 - p_2 = d_0$ (The difference is d_0.)
If $n_1\hat{p}_1, n_1(1-\hat{p}_1), n_2\hat{p}_2$, and $n_2(1-\hat{p}_2)$ are greater than 5, then the z-test statistic is

$$z = \frac{\hat{p}_1 - \hat{p}_2 - d_0}{s_{\hat{p}_1-\hat{p}_2}},$$

where if $d_0 \neq 0$ then $s_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$, otherwise if $d_0 = 0$, then

$$s_{\hat{p}_1-\hat{p}_2} = \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}, \quad \hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}$$

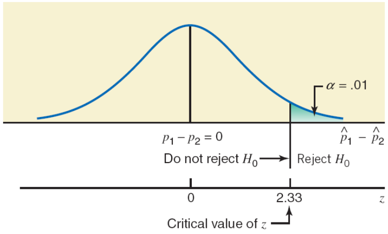
Example 1 A researcher wanted to estimate the difference between the percentages of users of two toothpastes who will never switch to another toothpaste. In a sample of 500 users of Toothpaste A taken by this researcher, 100 said that they will never switch to another toothpaste. In another sample of 400 users of Toothpaste B taken by the same researcher, 68 said that they will never switch to another toothpaste.

(a) Let p_1 and p_2 be the proportions of all users of Toothpastes A and B, respectively, who will never switch to another toothpaste. What is the point estimate of $p_1 - p_2$?

(b) Construct a 97% confidence interval for the difference between the proportions of all users of the two toothpastes who will never switch.

(c) At level $\alpha = 1\%$, can we conclude that the proportion of users of Toothpaste A who will never switch to another toothpaste is higher than the proportion of users of Toothpaste B who will never switch to another toothpaste?

Solution The two sample proportions are calculated as follows:
$\hat{p}_1 = 100/500 = 0.2$,
$\hat{p}_2 = 68/400 = 0.17$.
a. Point estimate of $p_1 - p_2$ is $\hat{p}_1 - \hat{p}_2 = .2 - .17 = .03$.
b. Both sample sizes are large because
$n_1\hat{p}_1 = 100, n_1(1-\hat{p}_1) = 400, n_2\hat{p}_2 = 68$, and $n_2(1-\hat{p}_2) = 332$ are greater than 5.
Consequently, we use the normal distribution to make a confidence interval for $p_1 - p_2$:
The estimated standard deviation of $\hat{p}_1 - \hat{p}_2$ is
$s_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$ =.02594, $z = 2.17$,
The 97% confidence interval for $p_1 - p_2$ is: $\hat{p}_1 - \hat{p}_2 \mp z s_{\hat{p}_1-\hat{p}_2} = .03 \mp 2.17(.02594) = .03 \mp .056 = [-.026, .086]$.
c.
Step 1.
$H_0 : p_1 - p_2$ =0 (p_1 is not greater than p_2)
$H_1 : p_1 - p_2$ ¿0 (p_1 is greater than p_2)
Step 2.
$\hat{p}_1 = 100/500 = 0.2, \hat{p}_2 = 68/400 = 0.17$.
Both sample sizes are large because
$n_1\hat{p}_1 = 100, n_1(1-\hat{p}_1) = 400, n_2\hat{p}_2 = 68$, and $n_2(1-\hat{p}_2) = 332$.We use the normal distribution to make the test.
Step 3. The $>$ sign in the alternative hypothesis indicates that the test is right-tailed. $\alpha = .01$. The critical value of z is 2.33.



Step 4.

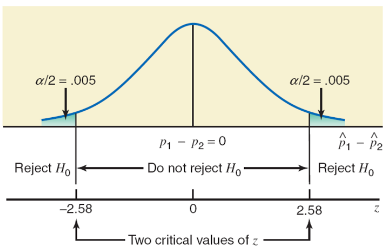
$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{100 + 68}{500 + 400} = .187$$

$$s_{\hat{p}_1 - \hat{p}_2} = \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sqrt{.187(1 - .187)\left(\frac{1}{500} + \frac{1}{400}\right)} = .026156$$

$$z = \frac{\hat{p}_1 - \hat{p}_2 - d_0}{s_{\hat{p}_1 - \hat{p}_2}} = \frac{.2 - .17 - 0}{.026156} = 1.15$$

Step 5. The value of the test statistic $z = 1.15$ It falls in the nonrejection region. We fail to reject the null hypothesis H_0.
Therefore, we conclude that the proportion of users of Toothpaste A who will never switch to another toothpaste is not greater that the proportion of users of Toothpaste B who will never switch to another toothpaste.

Example 2
According to a 2011 survey of college freshmen by UCLA's Cooperative Institutional Research Program, 39.5% of freshmen said that they had spent 6 or more hours a week studying or doing homework as high school seniors (USA TODAY, January 26, 2012). This percentage was 37.3% in the 2010 survey of freshmen by the same institution. The sample sizes for these surveys are usually very large, but for this example suppose the samples included 2000 freshmen in 2010 and 2200 freshmen in 2011. Test whether the proportions of 2010 and 2011 freshmen who spent 6 or more hours a week studying or doing homework as high school seniors are different. Use a 1% significance level.
Solution Step 1.
$H_0 : p_1 - p_2 = 0$ (p_1 and p_2 are not different.)
$H_1 : p_1 - p_2 \neq 0$ (p_1 and p_2 are different.)
Step 2. The samples are large and independent.
$n_1 = 2200$, $\hat{p}_1 = .395$, $n_2 = 2000$, $\hat{p}_2 = .373$.
Both sample sizes are large because $n_1\hat{p}_1, n_1(1-\hat{p}_1), n_2\hat{p}_2$, and $n_2(1-\hat{p}_2)$ are greater than 5.
We use the normal distribution to make the test.
Step 3. The $\neq$ sign in the alternative hypothesis indicates that the test is two-tailed. $\alpha = .01$. The critical values of z are ∓ 2.58.



$$\hat{p} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} = \frac{2200(.395) + 2000(.373)}{2200 + 2000} = .385$$

$$s_{\hat{p}_1 - \hat{p}_2} = \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sqrt{.385(1 - .385)\left(\frac{1}{2200} + \frac{1}{2000}\right)} = .01503$$

$$z = \frac{\hat{p}_1 - \hat{p}_2 - d_0}{s_{\hat{p}_1 - \hat{p}_2}} = \frac{.395 - .373 - 0}{.01503} = 1.46$$

Step 5. The value of the test statistic $z = 1.46$ It falls in the nonrejection region. We fail to reject the null hypothesis H_0.
Therefore, we conclude that the proportions of all freshmen in 2010 and 2011 who spent 6 or more hours a week studying or doing homework as high school seniors are not different.

10.10 One- and Two-Sample Tests Concerning Variances

Hypothesis Test for one Variance

The Chi-Square Distribution

The chi-square distribution has only one parameter called the degrees of freedom.

The shape of a chi-squared distribution curve is skewed to the right for small df and becomes symmetric for large df.

The entire chi-square distribution curve lies to the right of the vertical axis.

The chi-square distribution assumes nonnegative values only, and these are denoted by the symbol χ^2 (read as "chi-square").

The Shape of The Chi-Square Distribution

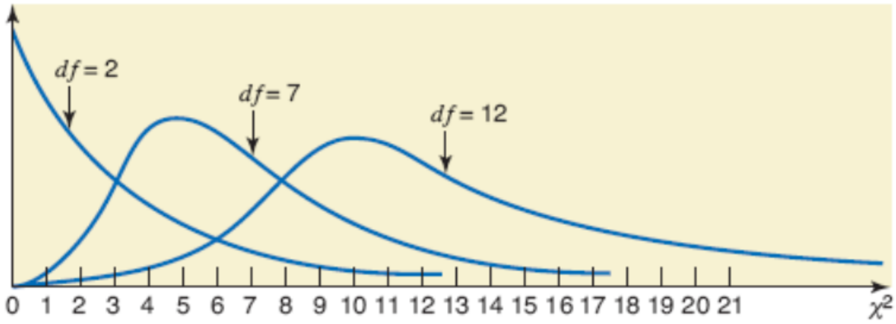


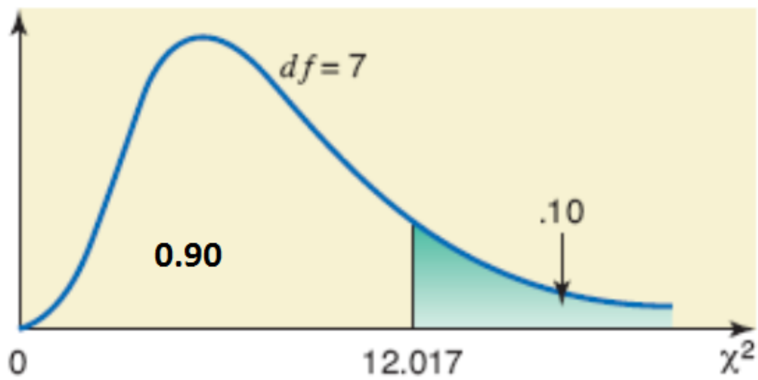
Table 5 Chi-Square Distribution

For df = 7 and .10 Area in the Right Tail

	Area in the Right Tail Under the Chi-Square Distribution Curve				
df	.995	...	.100	...	.005
1	0.000	...	2.706	...	7.879
2	0.010	...	4.605	...	10.597
.	...	...	...	...	...
.	...	...	...	...	...
.	...	...	...	...	...
7	0.989	...	12.017	...	20.278
.	...	...	...	...	...
.	...	...	...	...	...
.	...	...	...	...	...
100	67.328	...	118.498	...	140.169

Required value of χ^2

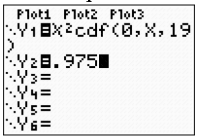
For df = 7 and .10 Area in the Right Tail



Use TI-83/84 to Find Critical χ^2 Value For example, $n = 20$, df $=n - 1$=19, to find $\chi^2_{.025}$:

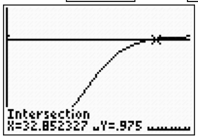
- Enter equation for χ^2 cdf : $\boxed{\text{Y=}}$ $\rightarrow$ Y1= χ^2cdf(0, X, 19)

- Enter equation: Y2=1-.025=0.975.



- Press $\boxed{\text{GRAPH}}$ and adjust $\boxed{\text{WINDOW}}$ if needed.

- Press $\boxed{\text{2ND}}$ and $\boxed{\text{CALC}}$ to find intersection of the two graphs. We can obtain $\chi^2_{.025} = 32.85$.



The Two-Tailed Test

Two–Tailed Test for σ^2 Let s^2 be the sample variance of $x_1, \ldots, x_n$ and σ^2 be the population variance.

Step 1. State the hypotheses

$$H_0 : \sigma^2 = \sigma_0^2, \quad vs \quad H_1 : \sigma^2 \neq \sigma_0^2$$

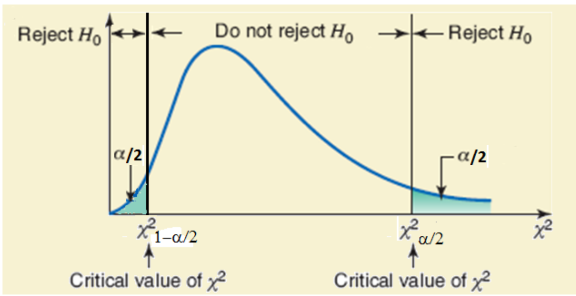
Step 2. Choose the distribution: If population is normal then use

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

which has χ^2-distribution with df = $n - 1$ if H_0 is true.

Two–Tailed Test for σ^2

Step 3. Calculate p-value or find critical value.
Find the p-value: Twice the smaller tail area at χ^2.
Using TI83/4, p-value is the smaller one of 2*χ^2cdf(0, χ^2, $n-1$) and 2*χ^2cdf(χ^2, E99, $n-1$). See example below.
The critical values for χ^2 are $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$:



Step 4. Make decision:
If p-value $< \alpha$, reject H_0, otherwise, do not reject H_0.
if $\chi^2 < \chi^2_{1-\alpha/2}$ or $\chi^2 > \chi^2_{\alpha/2}$, reject H_0, otherwise, do not reject H_0.

Example The variance of scores on a standardized mathematics test for all high school seniors was 150 in 2009. A sample of scores for 20 high school seniors who took this test this year gave a variance of 170. Test at the 5% significance level if the variance of current scores of all high school seniors on this test is different from 150. Assume that the scores of all high school seniors on this test are (approximately) normally distributed.
Solution:

Step 1. Hypotheses: $H_0 : \sigma^2 = 150 \quad H_A : \sigma^2 \neq 150$

Step 2. The population is (approximately) normal. We use the chi-square distribution to test a hypothesis about σ^2.

Step 3. $\alpha = 0.05$,

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(20-1)(170)}{150} = 21.533$$

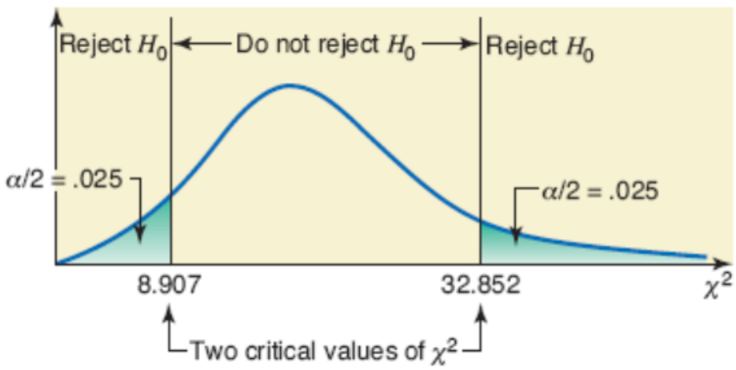
The p-value Approach: The p-value is

$$p\text{-value} = 2\min[\chi^2\text{cdf}(0, 21.533, 19), \chi^2\text{cdf}(21.533, \text{E}99, 19)]$$
$$= 0.6162207$$

The Critical Value Approach:
The critical values are $\chi^2_{.975} = 8.907$ and $\chi^2_{.025} = 32.852$.

Step 4. Since the p-value $> \alpha$, we do not reject H_0.
Since $\chi^2_{1-\alpha/2} < \chi^2 < \chi^2_{\alpha/2}$, we do not reject H_0.

Critical Value of Two-Tailed



Confidence Interval for σ^2

The $100(1-\alpha)\%$ confidence interval for σ^2 is

$$\left(\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{\alpha/2}}\right)$$

One-Tailed Test

$H_0 : \sigma^2 = \sigma_0^2$, normal population, $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$

Alternative Hypothesis	Rejection Region	p-value
$H_1 : \sigma^2 < \sigma_0^2$	$\chi^2 < \chi^2_{1-\alpha}(n-1)$	The area to the left of χ^2
$H_1 : \sigma^2 > \sigma_0^2$	$\chi^2 > \chi^2_{\alpha}(n-1)$	The area to the right of χ^2

Hypothesis Testing About σ_1^2/σ_2^2

Sampling distribution of s_1^2/s_2^2

- Let $x_{11}, \dots, x_{1n_1}$ be a sample from normal population with variance σ_1^2, and $x_{21}, \dots, x_{2n_2}$ be a sample from normal population with variance σ_2^2.
- Assume the two samples are independent and have sample variances s_1^2 and s_2^2, respectively.
- The distribution of $\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2}$ is F-distribution with numerator degrees of freedom $n_1 - 1$ and denominator degrees of freedom $n_2 - 1$.

The F Critical Value (Table 6)

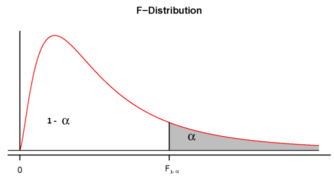
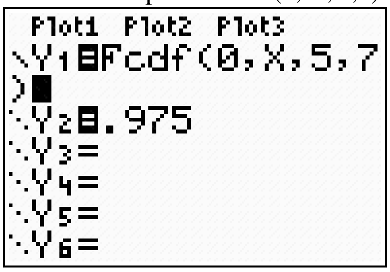
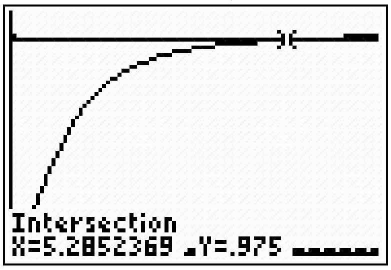


Table 7 gives F_α for $\alpha = .100, .050, .025, .010$, and $.005$.
For example, $\nu_1 = 5$, $\nu_2 = 7$, $F_{.025} = 5.29$ by Table 6.

The F Critical Value using Technology

- In Excel: $F_{.025}$=F.INV.RT(0.025,5,7)=5.285236852, or =F.INV(1-0.025,5,7)=5.285236852.
- In TI-8x: Graph Y1=Fcdf(0, X, 5,7) and Y2=1-.025=0.975, the x-coordinate of the intersection is $F_{.025}$=5.2852369.





- In R: $F_{.975}$= "qf(.975,5,7)=5.285237".

Hypothesis Testing About σ_1^2/σ_2^2 $H_0 : \sigma_1^2/\sigma_2^2 = 1$, normal populations, $F = \frac{s_1^2}{s_2^2}$.

Alternative Hypothesis	Rejection Region	p-value
$H_1 : \sigma_1^2/\sigma_2^2 \neq 1$	$F < F_{1-\alpha/2}(n_1 - 1, n_2 - 1)$ or $F > F_{\alpha/2}(n_1 - 1, n_2 - 1)$	Twice the smaller tail area at F
$H_1 : \sigma_1^2/\sigma_2^2 > 1$	$F > F_\alpha(n_1 - 1, n_2 - 1)$	The area to the left of F
$H_1 : \sigma_1^2/\sigma_2^2 < 1$	$F < F_{1-\alpha}(n_1 - 1, n_2 - 1)$	The area to the right of F

TI-83/84: $P(a < F(m, n) < b)$ =Fcdf(a, b, m, n).

If $H_0 : \sigma_1^2/\sigma_2^2 = 1$ is accepted, then we can assume equal variance in the two-sample t-test for comparing two population means.

Example

Pressures(mm Hg) under the Pelvis during Static Conditions for spinal cord injury(SCI) and healthy control groups.

CONTROL	131	115	124	131	122	117	88	114	150	169
SCI	60	150	130	180	163	130	121	119	130	148

Assume the populations are normal. Test $H_0 : \sigma_C^2 = \sigma_{SCI}^2$ vs $H_A : \sigma_C^2 \neq \sigma_{SCI}^2$ at level $\alpha = 0.05$.

Solution:

- $F = \frac{s_C^2}{s_{SCI}^2} = \frac{21.8^2}{32.2^2} = .458$, df.num =df.deno =10-1=9.
- p-value: tail areas at $F = .458$ are Fcdf(0, .458,9,9)=.1301 and Fcdf(.458, E99, 9,9)=.8699. Twice the smaller tail area = 2(.1301)=0.2602$>\alpha$.
- Do not reject $H_0 : \sigma_C^2 = \sigma_{SCI}^2$.
- We can assume that $\sigma_C^2 = \sigma_{SCI}^2$.