Ch4. Probability

Now that you have learned to describe a data set, how can you use sample data to draw conclusions about the sampled populations? This involves a statistical tool called *probability*. To use this tool correctly, you must first understand how it works. This chapter will present the basic concepts of probability, along with some simple examples.

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What is Probability?

- In Chapters 2 and 3, we used graphs and numerical measures to describe data sets which were usually **samples**.
- We measured "how often" using

Relative Frequency = f/n

- As *n* gets larger,

Sample ⟶ **Population**

And "how often"
= Relative frequency ⟶ **Probability**

Basic Concepts

- An **experiment** is the process by which an observation (or measurement) is obtained.

Example)

- ✓ Recording a test grade
- ✓ Measuring daily rainfall
- ✓ Interviewing a householder to obtain his or her opinion
- ✓ Testing a product to determine whether it is a defective product or an acceptable product
- ✓ Tossing a coin and observing the face that appears

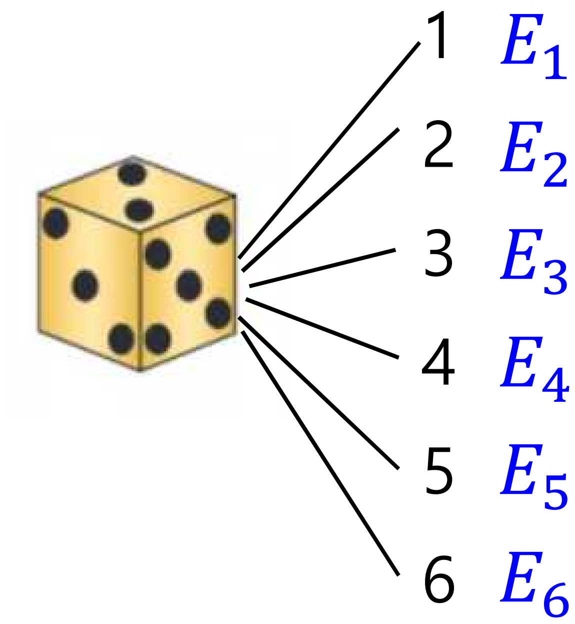
Basic Concepts

- A **simple event** is the outcome that is observed on a single repetition of the experiment.
  - The basic element to which probability is applied.
  - One and only one simple event can occur when the experiment is performed.
- A **simple event** is denoted by E with a subscript.
- Each simple event will be assigned a probability, measuring "how often" it occurs.
- The set of all simple events of an experiment is called the **sample space, S.**

Example

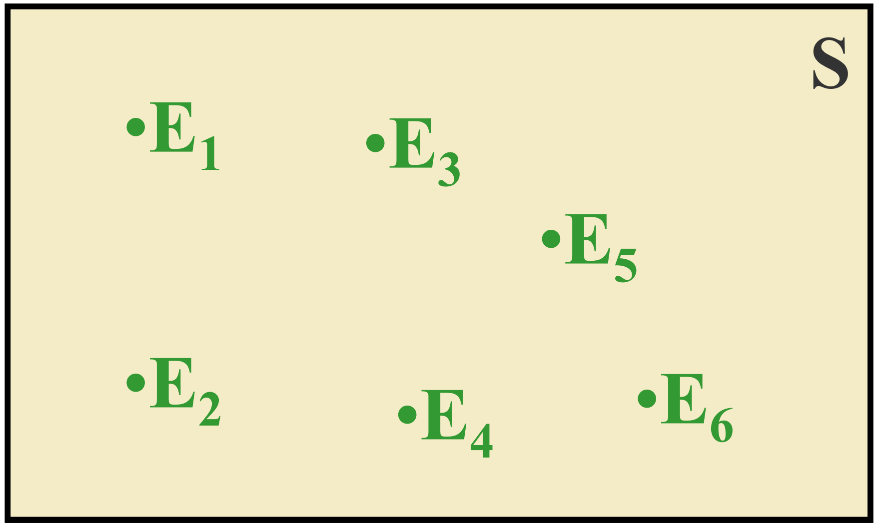
- Experiment: Toss a die and observe the number that appears on the upper face.

Simple Events



Sample Space

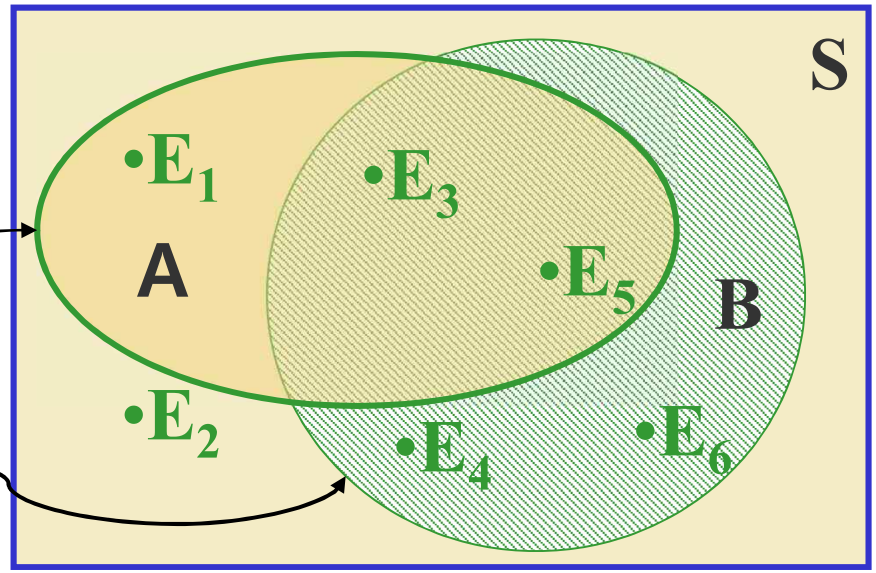
$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$



Basic Concepts

- A **event** is a collection of one or more **simple events.**
- The die toss:
  - A: an odd number
  - B: a number > 2

Venn diagram



A =$\{E_1, E_3, E_5\}$

B =$\{E_3, E_4, E_5, E_6\}$

Basic Concepts

- Two events are **mutually exclusive** if, when one event occurs, the other cannot, and vice versa.

- Experiment: Toss a die

  event A : observe an odd number

  event B : observe a number greater than 2

  event C : observe a 6

  event D : observe a 3

✓ event A and B are not mutually exclusive

✓ event C and D are mutually exclusive

✓ event A and C ?

✓ event B and D ?

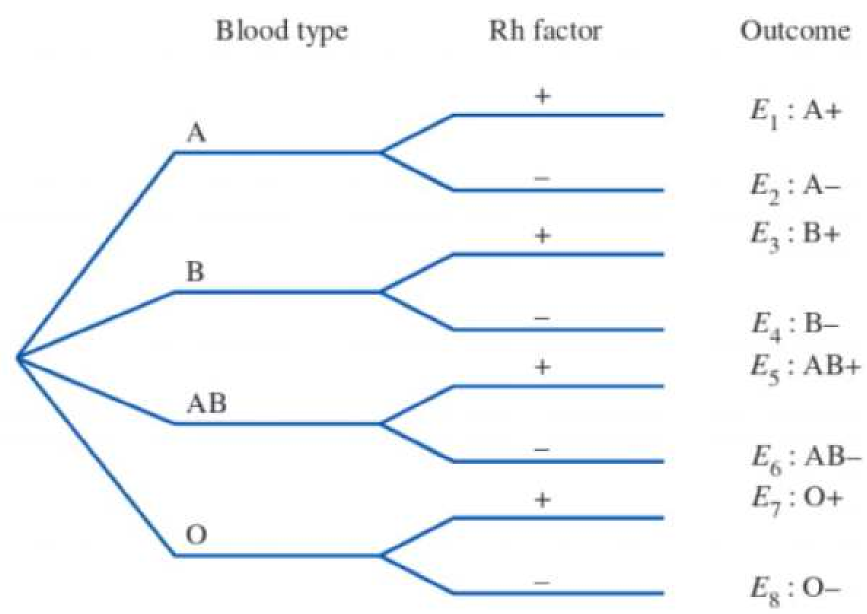
Basic Concepts

- The sample space can be displayed in a **tree diagram**.
- Each successive level of branching on the tree corresponds to a step required to generate the final outcome.

Example : a person's blood type and Rh factor

$$S = \{A+, A-, B+, B-, AB+, AB-, O+, O-\}$$

Tree diagram

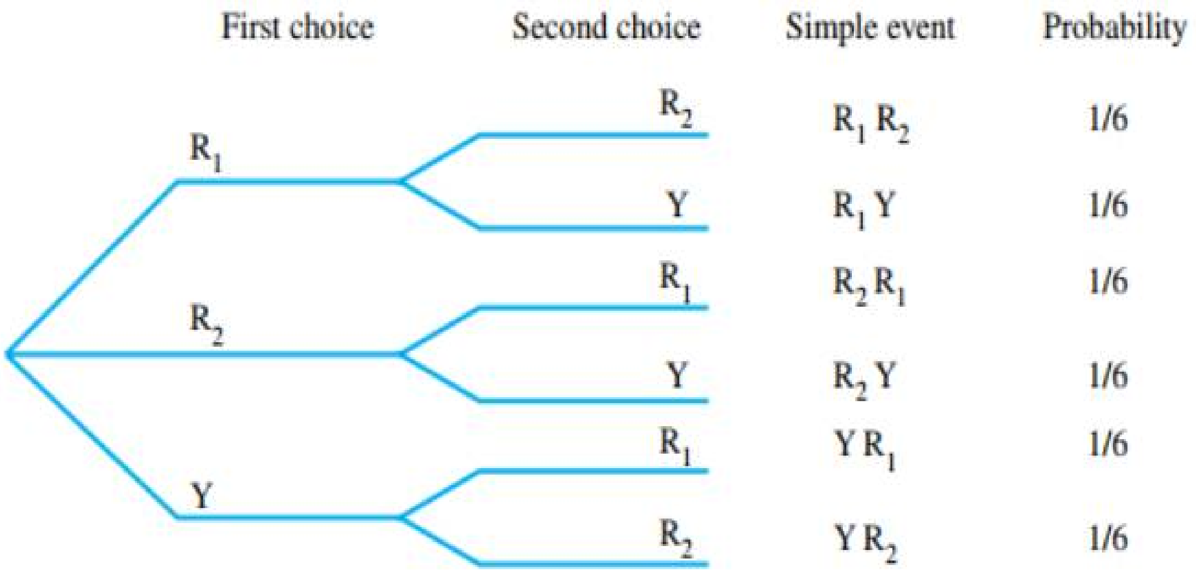


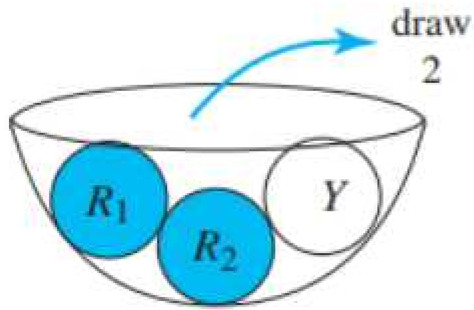
Example

A candy dish contains one yellow and two red candies. You close your eyes, choose two candies one at a time from the dish, and record their colors. What is the probability that both candies are red?

Sol) $A = \{ R_1R_2, R_2R_1\}$

$$P(A) = P(R_1R_2) + P(R_2R_1) = 1/6 + 1/6 = 1/3$$





The Probability of an Event

- The probability of an event A measures "how often" we think A will occur. We write **P(A)**.
- Suppose that an experiment is performed *n* times.

  The relative frequency for an event A is

$$\frac{\text{Number of times A occurs}}{n} = \frac{f}{n}$$

- If we let *n* get infinitely large,

$$P(A) = \lim_{n \to \infty} \frac{f}{n}$$

The Probability of an Event

- P(A) must be between 0 and 1.
  - If event A can never occur, P(A) = 0.
  - If event A always occurs when the experiment is performed, P(A) =1.
- The sum of the probabilities for all simple events in S equals 1.
- The **probability of an event A** is found by adding the probabilities of all the simple events contained in A.

Finding Probability

- Probabilities can be found using
  - Estimates from empirical studies
  - Common sense estimates based on equally likely events.

Examples

- Toss a fair coin. $\Rightarrow$ $\mathrm{P(Head)} = 1/2$

- 10% of the U.S. population has red hair.
  Select a person at random. $\Rightarrow$ $\mathrm{P(Red\ hair)} = 0.1$

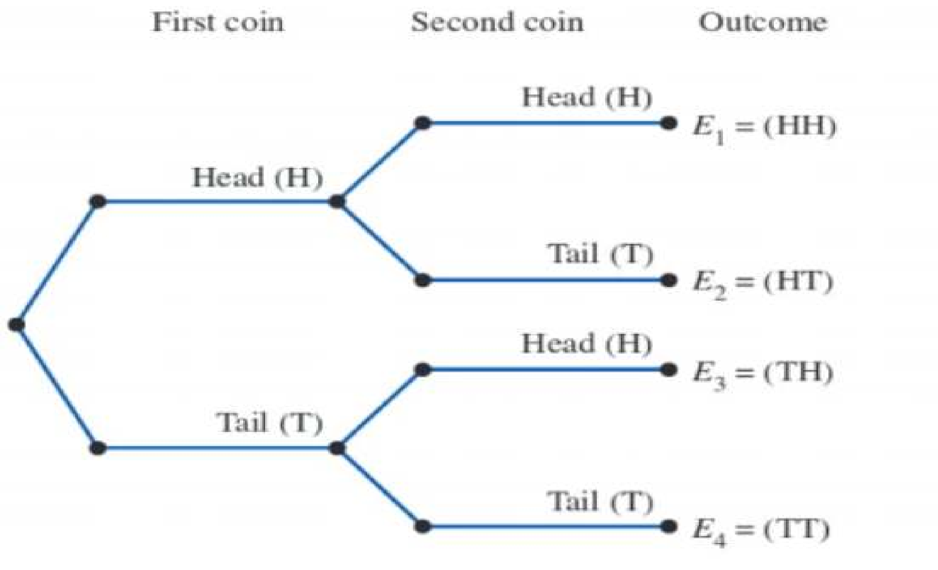
Example

- Toss a fair coin twice. What is the probability of observing exactly one head?

  - Event A : observe exactly one head

  - $P(A) = P(E_2) + P(E_3) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Tree diagram



Table

Event	First Coin	Second Coin	$P(E_i)$
E_1	H	H	1/4
E_2	H	T	1/4
E_3	T	H	1/4
E_4	T	T	1/4

Each simple event is equally likely probability

Example

Probability table of Blood type-Rh factor

	Blood Type			
Rh Factor	A	B	AB	O
Negative	.07	.02	.01	.08
Positive	.33	.09	.03	.37

Each simple event is not equally likely probability

- What is the probability that he or she will have either type O positive or type A?

$$P(\text{either}\quad O+\ or\ A) = P(O+) + P(A+) + P(A-)$$
$$= 0.37 + 0.33 + 0.07 = 0.77$$

Useful Counting Rules

- If the simple events in an experiment are **equally likely**, you can calculate

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

  - You can use **counting rules** to find n_A and N.

The *mn* Rule

- If an experiment is performed in two stages, with ***m*** ways to accomplish the first stage and ***n*** ways to accomplish the second stage, then there are ***mn*** ways to accomplish the experiment.
- This rule is easily extended to ***k*** stages, with the number of ways equal to $n_1 n_2 n_3 \ldots n_k$

Examples

Ex1. Toss two coins. The total number of simple events is:

$$2 \times 2 = 4$$

Ex2. Toss three coins. The total number of simple events is:

$$2 \times 2 \times 2 = 8$$

Ex3. Toss two dice. The total number of simple events is:

$$6 \times 6 = 36$$

Ex4. A candy dish contains one yellow and two red candies. Two candies are selected one at a time from the dish, and their colors are recorded. The total number of simple events is:

$$3 \times 2 = 6$$

Permutations

- The number of ways you can arrange ***n*** distinct objects, taking them ***r*** at a time is

$$P_r^n = \frac{n!}{(n-r)!}$$

$$\text{where } n! = n(n-1)(n-2)\ldots(2)(1) \text{ and } 0! \equiv 1.$$

Example: How many 3-digit lock combinations can we make from the numbers 1, 2, 3, and 4?

$$P_3^4 = \frac{4!}{1!} = 4(3)(2) = 24$$

✓ The order of the choice is important!

Arranging n items

The number of ways to arrange an entire set of n distinct items is $P_r^n = n!$

Example

A lock consists of five parts and can be assembled in any order.

A quality control engineer wants to test each order for efficiency of assembly. How many orders are there?

$$P_5^5 = \frac{5!}{0!} = 5(4)(3)(2)(1) = 120$$

✓ The order of the choice is important!

Combinations

- The number of distinct combinations of ***n*** distinct objects that can be formed, taking them ***r*** at a time is

$$C_r^n = \frac{n!}{r!\,(n-r)!}$$

Example: Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?

$$C_3^5 = \frac{5!}{3!\,(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$

✓ The order of the choice is not important!

Example

Five manufacturers produce a certain electronic device, whose quality varies from manufacturer to manufacturer. If you were to select three manufacturers at random, what is the chance that the selection would contain exactly two of the best three?

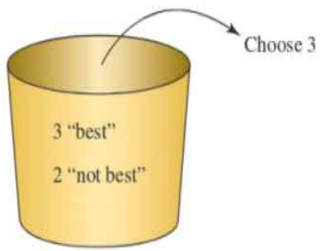
Sol) “best”: 3 manufacturers , “not best” : 2 manufacturers

- N = the number of ways to choose three of the five manufacturers

$$N = C_3^5 = \frac{5!}{3!\,(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$

- event A : select two of the best three and one of the two “not best”

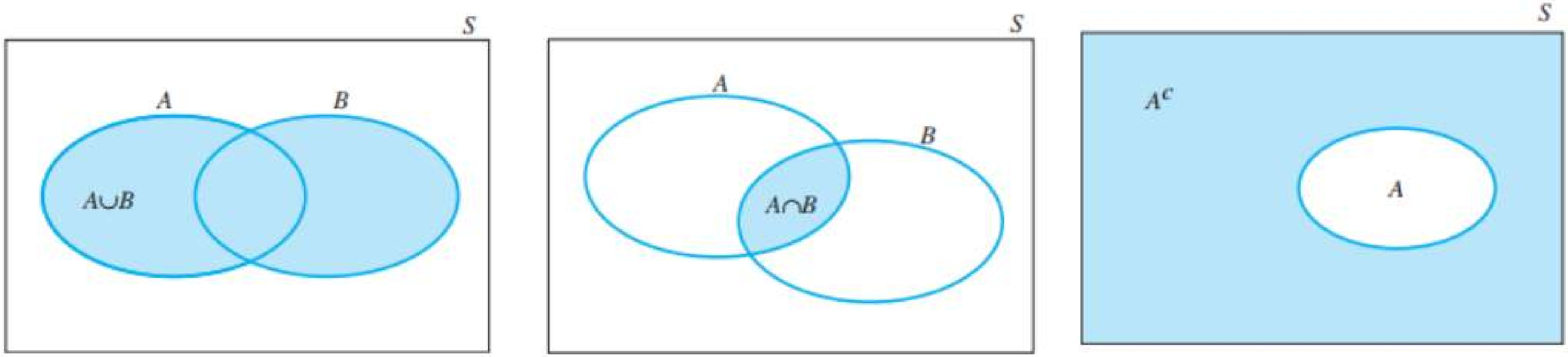
$$n_A = C_2^3 \times C_1^2 = \frac{3!}{2!1!} \times \frac{2!}{1!1!} = 3 \times 2 = 6$$



- $P(A) = \frac{n_A}{N} = \frac{6}{10}$

Rules for Calculating Probabilities

- **Union**
  - The union of events A and B, denoted by $A \cup B$, is the event that either A or B or both occur.
- **Intersection**
  - The intersection of events A and B, denoted by $A \cap B$, is the event that both A and B occur.
- **Complement**
  - The complement of an event A, denoted by A^C is the event that A does not occur.



Example : Two fair coins are tossed

- Event A: Observe at least one head
- Event B: Observe at least one tail

$$S = \{HH, HT, TH, TT\} = \{E_1, E_2, E_3, E_4\}$$

$$A = \{E_1, E_2, E_3\} \quad A^C = \{E_4\} \quad P(A) = \frac{3}{4}, \quad P(A^C) = 1/4$$

$$B = \{E_2, E_3, E_4\} \quad B^C = \{E_1\}$$

$$A \cup B = \{E_1, E_2, E_3, E_4\} \qquad P(A \cup B) = \frac{4}{4} = 1$$

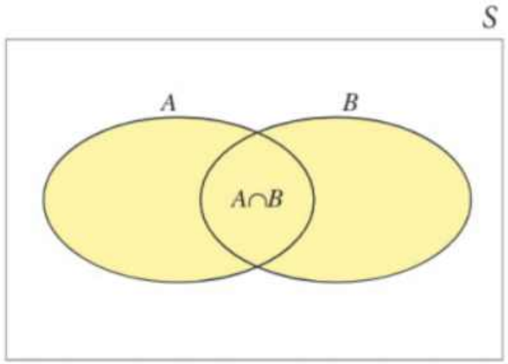
$$A \cap B = \{E_2, E_3\} \qquad P(A \cap B) = \frac{2}{4}$$

Calculating Probabilities for Unions and Complements

The Addition Rule

- Given two events, A and B, the probability of their union, A B, is equal to

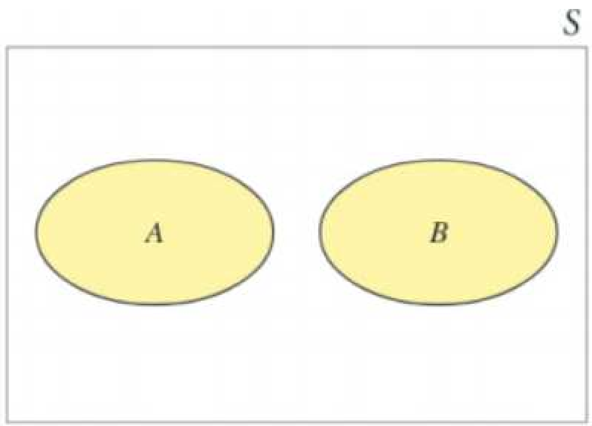
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



✓ When two events A and B are **mutually exclusive** or **disjoint**,

$$P(A \cap B) = 0$$

$$\Rightarrow P(A \cup B) = P(A) + P(B)$$

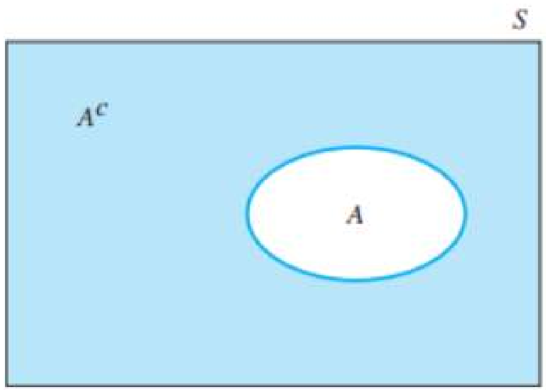


Rule for Complements

$$P\left(A \cap A^C\right) = 0 \text{ and } P\left(A \cup A^C\right) = 1$$

$$\Rightarrow P\left(A \cup A^C\right) = P(A) + P\left(A^C\right) = 1$$

$$\Rightarrow P\left(A^C\right) = 1 - P(A)$$



Example : Suppose that there were 120 students in the classroom, and that they could be classified as follows:

	Brown	Not Brown
Male	20	40
Female	30	30

Ex1)

A: brown hair, P(A) = 50/120

B: female, P(B) = 60/120

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

= 50/120 + 60/120 – 30/120 = 80/120

Ex2)

A: male with brown hair, P(A) = 20/120

B: female with brown hair, P(B)= 30/120

Since A and B are mutually exclusive

$$\boldsymbol{P(A \cup B) = P(A) + P(B)}$$

= 20/120 + 30/120 = 50/120

Ex3) A: male, B: female

Since A and B are complementary, P(B)=1-P(A)= 1-60/120=40/120

Calculating Probabilities for Intersections

- In the previous example, we found $P(A \cap B)$ directly from the table. Sometimes this is impractical or impossible.
- The rule for calculating $P(A \cap B)$ depends on the idea of **independent and dependent events.**

Two events, **A** and **B**, are said to be **independent** if and only if the probability that event **B** is not influenced or changed by the occurrence of event **A**, or vice versa.

Conditional Probabilities

The probability that A occurs, given that event B has occurred is called the **conditional probability** of A given B and is defined a

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) \neq 0}$$

"given"

Example : Toss a fair coin twice

- A: head on first toss
- B: head on second toss

$P(B|A) = ½$, $P(B|\text{not } A) = ½$

P(B) does not change, whether A happens or not…

A and B are independent!

Defining Independence

- We can redefine independence in terms of conditional probabilities:

Two events A and B are independent if and only if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Otherwise, they are dependent.

For any two events, **A** and **B**, the probability that both **A** and **B** occur is

$$\mathbf{P(A \cap B) = P(A)P(B|A)}$$
$$\mathbf{= P(A)\ P(B\ given\ that\ A\ occurred)}$$

If A and B are independent, $\mathbf{P(A \cap B) = P(A)\ P(B)}$

If A, B, and C are mutually independent, $\mathbf{P(A \cap B \cap C) = P(A)P(B)\ P(C)}$

Example

8 toys are place in a container. : two are red and six are green

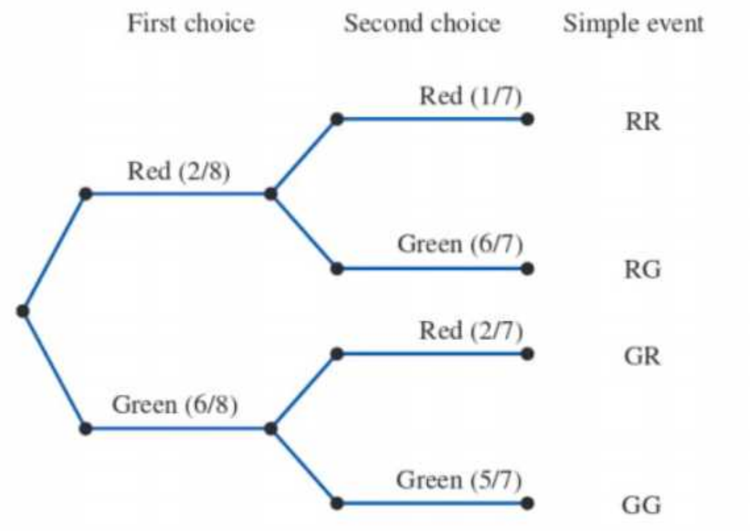
A child is asked to choose two toys at random. What is the probability that the child chooses the two red toys?

Sol) Let R: Red toy is chosen, G: Green toy is chosen

$P(A) = P(R \text{ on first choice} \cap R \text{ on second choice})$

$= P(R \text{ on first choice})\, P(R \text{ on second choice} \mid R \text{ on first})$

$= (2/8)*(1/7) = 2/56 = 1/28$



Example : Colorblindness

- Suppose that in the general population, there are 51% men and 49% women, and that the proportions of colorblind men and women are shown in the probability table below:

	Men(B)	Women (B^C)	Total
Colorblind (A)	.04	.002	.042
Not Colorblind (A^C)	.47	.488	.958
Total	.51	.49	1.00

- If a person is drawn at random from this population and is found to be a man (event B), what is the probability that the man is colorblind (event A)? $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.04}{.51} = .078$$
- What is the probability of being colorblind, given that the person is female? $$P(A|B^C) = \frac{P(A \cap B^C)}{P(B^C)} = \frac{.002}{.49} = .004$$

✓ *Probability of event A changed depending on event B occurred*
 ⇒ *These two events are dependent*

Example

In a telephone survey of 1000 adults, respondents were asked their opinion about the **cost of a college education**. The respondents were classified according to **whether they currently had a child in college** and whether they thought **the loan burden for most college students is too high, the right amount, or too little**.

Probability Table

	Too High (A)	Right Amount (B)	Too Little (C)	Total
Child in College (D)	.35	.08	.01	.44
No Child in College (E)	.25	.20	.11	.56

1. What is the probability that the respondent has a child in college?

   $P(D) = .35 + .08 + .01 = .44$

2. What is the probability that the respondent does not have a child in college?

   $P(E) = .25 + .20 + .11 = .56$ or $P(D^C)=1-P(D) = 1-.44 =.56$

3. What is the probability that the respondent has a child in college or thinks that the loan burden is too high or both?

$$P(A \cup D) = P(A) + P(D) - P(A \cap D)$$
$$= .6 + .44 - .35 = .69$$

4. Are events D and A independent?

a. $P(A \cap D) = .35 \quad P(A) = .60 \quad P(D) = .44$

$P(A) \times P(D) = (.60)(.44) = .264 \quad \neq \quad P(A \cap D) = .35$

$\Rightarrow$ *A and D are dependent*

b. $P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{.35}{.44} = .8 \quad \neq \quad P(A) = .60$

$\Rightarrow$ *A and D are dependent*

c. $P(D|A) = \frac{P(A \cap D)}{P(\mathrm{A})} = \frac{.35}{.60} = .58 \quad \neq \quad P(D) = .44$

$\Rightarrow$ *A and D are dependent*

Mutually Exclusive and Independent Events

- When two events are **mutually exclusive** or disjoint

$$P(A \cap B) = 0 \;\; and \;\; P(A \cup B) = P(A) + P(B)$$

$$P(A|B) = 0 \;\neq\; P(A) : dependent$$

- When two events are **independent**

$$P(A \cap B) = P(A)P(B)$$

$$and \;\; P(A \cup B) = P(A) + P(B) - P(A)P(B)$$

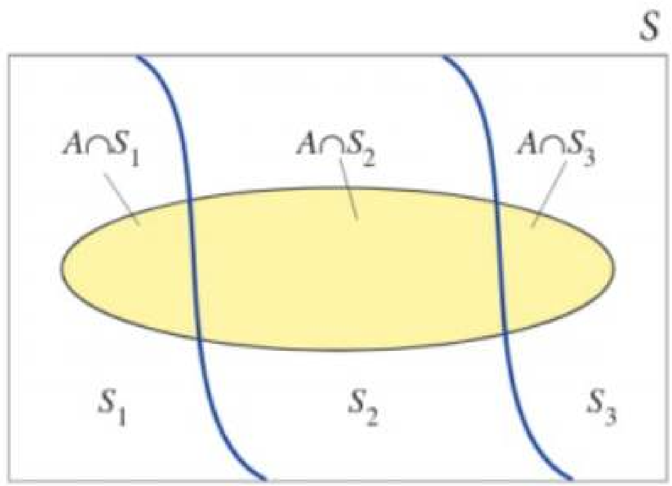
The law of Total Probability

- Let S_1 , S_2 , S_3 ,..., S_k be mutually exclusive and exhaustive events (that is, one and only one must happen). Then the probability of another event A can be written as

$$\mathbf{P(A) = P(A \cap S_1) + P(A \cap S_2) + \ldots + P(A \cap S_k)}$$

$$\mathbf{= P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + \ldots + P(S_k)P(A|S_k)}$$

$$= \sum P(S_i)P(A|S_i)$$



Bayes' Rule

- Let S_1 , S_2 , S_3 ,..., S_k be mutually exclusive and exhaustive events with prior probabilities $P(S_1)$, $P(S_2)$,…,$P(S_k)$. If an event A occurs, the posterior probability of S_i, given that A occurred is

$$\boldsymbol{P(S_i|A) = \frac{P(S_i)P(A|S_i)}{\sum P\,(S_i)P(A|S_i)}} \quad \textbf{for } \boldsymbol{i = 1, 2, \ldots k}$$

Example

Data: the fraction of U.S. adults 20 years of age and older who own five or more pairs of wearable sneakers, along with the fraction of the U.S. adult population 20 years or older in each of five age groups.

	Groups and Ages				
	G_1 20–24	G_2 25–34	G_3 35–49	G_4 50–64	G_5 ≥ 65
Fraction with ≥ 5 Pairs	.26	.20	.13	.18	.14
Fraction of U.S. Adults 20 and Older	.09	.18	.30	.25	.18

$P(A| G_i)$

- Let A ={ a person chosen at random from the U.S. adult population 20 years of age and older owns five or more pairs of wearable sneakers}
- Let G1, G2, . . . , G5 represent the event that the person selected belongs to each of the five age groups, respectively

$$P(A) = P(A \cap G_1) + P(A \cap G_2) + \ldots + P(A \cap G_5)$$

$$= P(G_1)P(A|G_1) + P(G_2)P(A|G_2) + \ldots + P(G_5)P(A|G_5)$$

$$= (.09)(.26) + (.18)(.20) + (.30)(.13) + (.25)(.18) + (.18)(.14)$$

$$= .0234 + .0360 + .0390 + .0450 + .0252 = .1686$$

- Find the probability that the person selected was 65 years of age or older, given that the person owned at least five pairs of wearable sneakers.

Sol) $$P(G_5|A) = \frac{P(G_5 \cap A)}{P(A)} = \frac{P(G_5)P(A\,|G_5\,)}{P(A)} = \frac{(.18).14)}{.1686} = .1495$$

Key Concepts

I. Experiments and the Sample Space

1. Experiments, events, mutually exclusive events, simple events
2. The sample space
3. Venn diagrams, tree diagrams, probability tables

II. Probabilities

1. Relative frequency definition of probability
2. Properties of probabilities
   a. Each probability lies between 0 and 1.
   b. Sum of all simple-event probabilities equals 1.
3. $P(A)$, the sum of the probabilities for all simple events in A

Key Concepts

III. Counting Rules

1. *mn* Rule; extended *mn* Rule
2. Permutations: $P_r^n = \frac{n!}{(n-r)!}$
3. Combinations: $C_r^n = \frac{n!}{r!(n-r)!}$

IV. Event Relations

1. Unions and intersections
2. Events
   a. Disjoint or mutually exclusive: $P(A \cap B) = 0$
   b. Complementary: $P(A) = 1 - P(A^C)$

Key Concepts

3. Conditional probability: $\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)}}$

4. Independent and dependent events

5. Additive Rule of Probability:

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

6. Multiplicative Rule of Probability:

$$\boldsymbol{P(A \cap B) = P(A)P(B|A)}$$

7. Law of Total Probability

8. Bayes' Rule