Implementing the Polar Method

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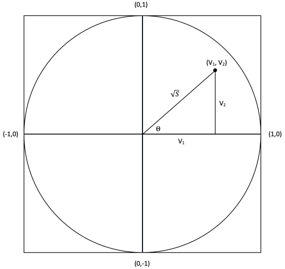
Inverse Transfrom Method

Let $F_X(x) = P(X \leq x)$ be the cumulative distribution function for the random variable X. If $U \sim \text{Unif}(0,1)$, then $F_X^{-1}(U)$ has the same culuative distribution function F_X. In other words, the generalized inverse function $F_X^{-1}(u) = \inf\{x : F(x) \geq u\}$ for all $u \in (0,1)$ has the CDF F_X.

Polar Method



where $S = V_1^2 + V_2^2$

Summary of Polar Method

1. Generate U_1 and U_2 from $U \sim U(0,1)$
2. Let $V_1 = 2U_1 - 1$ and $V_2 = 2U_2 - 1$
3. If $S = V_1^2 + V_2^2 > 1$ or $S = 0$, go back to Step 1.
4. Return the standard independent normal random variables $X = V_1\sqrt{\frac{-2\ln S}{S}}$ and $Y = V_2\sqrt{\frac{-2\ln S}{S}}$

Why does the Polar Method Work?

Let $U_1 \sim U(0,1)$ and $U_2 \sim U(0,1)$. Let $V_1 = 2U_1 - 1$ and $V_2 = 2U_2 - 1$. Then $V_1 \sim U(-1,1)$ and $V_2 \sim U(-1,1)$. Hence, the point (V_1, V_2) is uniformly distributed in the "unit square" (square with an area of 4 centered at the origin).

Let $S = V_1^2 + V_2^2$. Suppose we only accept points if $0 \leq S \leq 1$. Since (V_1, V_2) is uniformly distributed in the "unit square" then by rejecting points outside of the unit circle (where $S > 1$) we have that these accepted points are uniformly distributed in the unit circle.

In other words, this rejection gets us from the points being uniformly distributed in the square to the points being uniformly distributed in the unit circle.

CDF for S

Let's consider the CDF for $S = V_1^2 + V_2^2$. Our sample space for our accepted points then is the area of the unit circle, which is π. The area of a circle with radius $\sqrt{S}$ is πS. What is the probability for $S < t$?
If $t < 0$, then $P(S < t) = 0$. If $t > 1$, then $P(S < t) = 1$. If $t \in [0, 1]$, then $P(S < t) = \frac{\pi t}{\pi} = t$ (Same as $\frac{\pi(\sqrt{t})^2}{\pi} = t$). Hence, the CDF for S is

$$F(t) = P(S \leq t) = \begin{cases} 0, & \text{if } t < 0 \\ t, & \text{if } 0 \leq t \leq 1 \\ 1, & \text{if } t > 1 \end{cases}$$

Then the PDF of S is $f(t) = \frac{d}{dt}F(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$

This PDF is the same as the uniform distribution PDF for $U(0, 1)$, so we have that $S \sim U(0, 1)$.

PDF Transform

We can change from one set of coordinates to another using the following theorem.

(Theorem 7.3 - Ross) Let $f_{X_1,X_2,...,X_n}$ be the joint probability density function for the random variables $X_1, X_2, ..., X_n$ and $f_{Y_1,Y_2,...,Y_n}$ be the joint probability density function for the random variables Y_1, $Y_2, ...$, Y_n. If $Y_i = g_i(X_1, ..., X_n)$, g_i has continuous partial derivatives, the equations $y_i = g_i(x_1, ..., x_n)$ have the unique solutions $x_i = h_i(y_1, ..., y_n)$,and the Jacobian determinant is nonzero for all points $(x_1, ..., x_n)$ for each $1 \leq i \leq n$, then

$$f_{Y_1,...Y_n}(y_1, ..., y_n) = f_{X_1,...,X_n}(x_1, ..., x_n) \cdot |J(x_1, ..., x_n)|^{-1}$$

Derivation and Importance of Box-Muller Approach

Applying this theorem to the Polar Method, we have that
$r = g_1(x, y) = \sqrt{x^2 + y^2}$, $\theta = g_2(x, y) = \arctan(\frac{y}{x})$,
$x = h_1(r, \theta) = r\cos\theta$, and $y = h_2(r, \theta) = r\sin\theta$.

Then $|J(x, y)| = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}}$
$\implies |J(x, y)|^{-1} = \sqrt{x^2 + y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$

Assuming that X and Y are independent standard normal random variables,

$$\begin{aligned}
f_{R,\Theta}(r,\theta) &= f_{X,Y}(x,y)\cdot|J(x,y)|^{-1} \\
&= f_{X,Y}(x,y)\cdot r \\
&= f_X(x)f_Y(y)\cdot r \\
&= \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\right)r \\
&= \frac{1}{2\pi}re^{-\frac{1}{2}(x^2+y^2)} \\
&= \frac{1}{2\pi}re^{-\frac{r^2}{2}} \\
&= \left(re^{-\frac{r^2}{2}}\right)\left(\frac{1}{2\pi}\right) \\
&= f_R(r)f_\Theta(\theta)
\end{aligned}$$

where R is Rayleigh distributed with $\sigma = 1$ and for $r > 0$ and Θ is uniformly distributed over $[0, 2\pi]$. So R and Θ are independent.

We have shown that $X = R\cos\Theta$ and $Y = R\sin\Theta$ are standard independent normal random variable when $R \sim \text{Rayleigh}(1)$ and $\Theta \sim U(0, 2\pi)$. Now we want to use the Inverse Transform Method to simulate draws from R and Θ in order to simulate a draws from X and Y.

Since $R \sim \text{Rayleigh}(1)$, then the CDF of R is $F_R(r) = 1 - e^{-\frac{r^2}{2}}$ for $r > 0$. Let $U \sim U(0, 1)$. Then using the Inverse Transform Method, we can let $U = F_R(r)$. So we have that $U = 1 - e^{-\frac{r^2}{2}}$ which implies that $r = \sqrt{-2\ln(1 - U)}$. Let $U_1 = (1 - U)$. Since $U \sim U(0, 1)$, then $U_1 \sim U(0, 1)$, so $r = \sqrt{-2\ln U_1}$ is a simulated draw from the Rayleigh(1) distribution.

Now we apply the Inverse Transform Method again to simulate values for Θ. Since $\Theta \sim U(0, 2\pi)$, then the CDF for Θ is

$$F_\Theta(\theta) = \begin{cases} 0, \text{ for } \theta < 0 \\ \frac{\theta}{2\pi}, \text{ for } 0 \leq \theta \leq 2\pi \\ 1, \text{ for } \theta > 2\pi \end{cases}$$

Let $U_2 \sim U(0, 1)$. Again, using the Inverse Transform Method $U_2 = F_\Theta(\theta)$. So we have that $U_2 = \frac{\theta}{2\pi}$ which implies that $\theta = 2\pi U_2$. So $\theta = 2\pi U_2$ is a simulated draw from $U(0, 2\pi)$.

Box-Muller Transform

Putting all of these ideas together and plugging in these simulated values for R and Θ, we have that

$$X = R\cos\Theta = \sqrt{-2\ln U_1}\cos(2\pi U_2)$$
$$Y = R\sin\Theta = \sqrt{-2\ln U_1}\sin(2\pi U_2)$$

are standard independent normal random variables.

From the Box-Muller approach, we know that X and Y are standard independent normal random variables. From basic right triangle trigonometry, we have that

$$\cos\Theta = \frac{V_1}{\sqrt{V_1^2 + V_2^2}} = \frac{V_1}{\sqrt{S}}$$
$$\sin\Theta = \frac{V_2}{\sqrt{V_1^2 + V_2^2}} = \frac{V_2}{\sqrt{S}}$$

Polar Method

Using the fact that $S \sim U(0,1)$ and using these trig substitutions, we have that

$$X = R\cos\Theta = \sqrt{-2\ln S}\left(\frac{V_1}{\sqrt{S}}\right) = V_1\sqrt{\frac{-2\ln S}{S}}$$

$$Y = R\sin\Theta = \sqrt{-2\ln S}\left(\frac{V_2}{\sqrt{S}}\right) = V_2\sqrt{\frac{-2\ln S}{S}}$$

are standard independent normal random variables.

Chi-Square PDF

$$f(x;k) = \left\{ \begin{array}{ll} \frac{x^{\frac{k}{2}-1} e^{\frac{-x}{2}}}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})}, & \text{if } x > 0; \\ 0, & \textit{otherwise} \end{array} \right\}$$

This distribution is equal to that of a sum of squared independent standard normal variables, such as those obtained via marsaglia polar method.

We obtain $X^2 \sim \chi^2(k = 2)$ from the normal variable summation $X^2 + Y^2$

Given standard normal variable X we an random variable such that $Z = X^2$

CDF of Z is given by:

$$F_Z(z) = P(X^2 \leq z) = P(-z^{1/2} \leq X \leq z^{1/2})$$

$f_z(z)$ is given by evaluating the following, where $f_x(x)$ corresponds to the PDF of the normal distribution.

$$f_z(z) = \frac{d}{dz} F_z(z) = \frac{d}{dz} \int_{-z^{1/2}}^{z^{1/2}} f_x(x) dx$$

Relationship with Chi-Square Distribution

We arrive at, the following probability density function, which is equivalent to $X^2 \sim \chi^2(k = 1)$

$$f_z(z) = \frac{1}{\sqrt{2\pi}\sqrt{z}} \exp\left(\frac{-z}{2}\right)$$

From this it trivially follows that a summation of n squared normals represents a $X^2 \sim \chi^2(k = n)$.

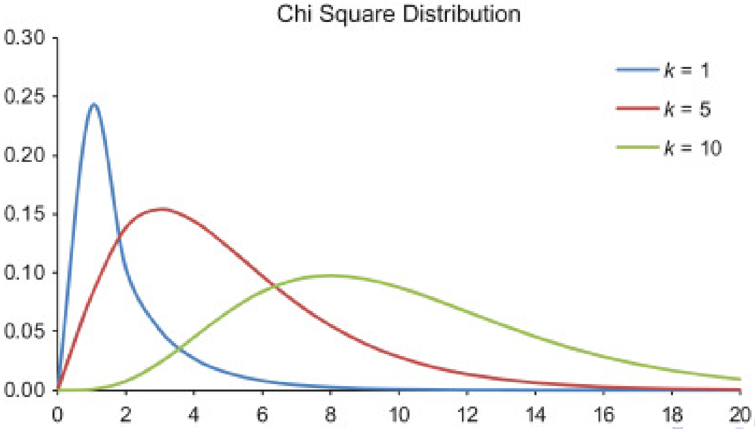
When $X^2 \sim \chi^2(k = 2)$, we obtain

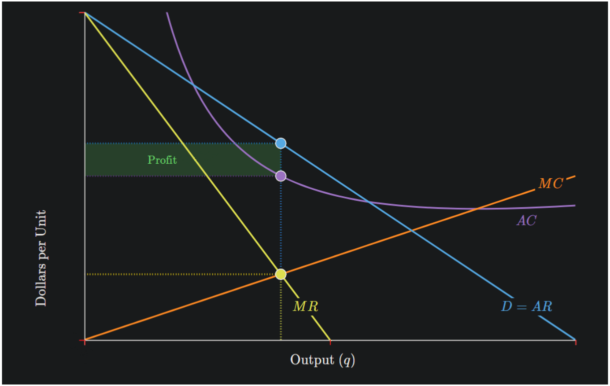
$$f(x; 2) = \frac{e^{-x/2}}{2}$$

Which is otherwise expressed as $\lambda = \frac{1}{2}$

Relationship with Chi-Square Distribution

Generating normal variables via marsaglia polar method allows for sampling from arbitrary chi-square distributions. Efficient pseudo-random generation from specific distributions has broad application in computing, including what



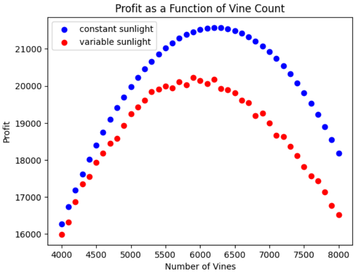


$$D = AR = n \cdot p(n)$$

$$MR = n \cdot \frac{dp}{dn} + p$$

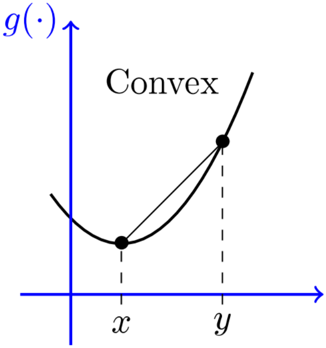
Profit as Function of Constant Sunlight and Variable Sunlight

- When comparing the polar method to other simulations of sunlight, the max profit came out to be $\sim$ \$20,143 from 5,990 vines. The polar method was a simulation of 10,000 scenarios (100 days with a range of 4,000-8,000 vines).
- The reason for the polar method is to eliminate the computational expense of sin and cos.
- However, the expected max profit overestimates due to variable sunlight. In the next slide, the solution to this will be explained.



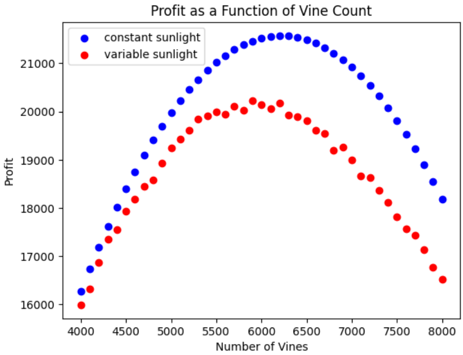
Definition: Jensen's Inequality

For any convex function f, $E[f(X)] \geq f(E[X])$. Conversely, for any concave function f, $E[f(X)] \leq f(E[X])$.

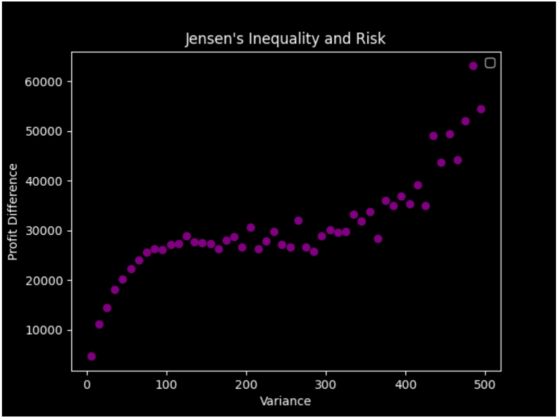


- Let X be a random variable which denotes the number of hours of sunlight an orchard receives per day, and suppose that $E[X] = k$.

- A vineyard with constant sunlight hours k yields a greater expected profit than that of a vineyard for which sunlight follows $f_X(x)$. Market actors have to account for variance when pricing products.



Jensen's Inequality and Risk Aversion



The payout from wine harvests is defined by a concave loss function, and necessarily $E[f(X)] \leq f(E[X])$. Actors will reduce production, and may even exit the market altogether should uncertainty increase beyond what their risk tolerance would allow for.

Future Work

- The vineyard was profitable and we were able to find the number of vines (5,990) that would give the max profit.
- Due to climate change, there will be an increase in volatile hours of sun throughout the vineyard which will lead to more restricted production (i.e. higher prices, and lower profits). This will amount to neither the vineyard owners nor the customers benefiting from the sales within the vineyard.

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