Quantum Rational Preferences and Desirability

Alessio Benavoli[*]   ALESSIO@IDSIA.CH
Alessandro Facchini   ALESSANDRO.FACCHINI@IDSIA.CH
Marco Zaffalon   ZAFFALON@IDSIA.CH
Dalle Molle Institute for Artificial Intelligence (IDSIA), Switzerland

Editor: T.V. Guy, M. Kárný, D. Rios-Insua, D.H. Wolpert

Abstract

We develop a theory of quantum rational decision making in the tradition of Anscombe and Aumann's axiomatisation of preferences on horse lotteries. It is essentially the Bayesian decision theory generalised to the space of Hermitian matrices. Among other things, this leads us to give a representation theorem showing that quantum complete rational preferences are obtained by means of expected utility considerations.

Keywords: Quantum mechanics, Bayesian decision theory, Horse lotteries, Imprecise probability.

1. Introduction

The aim of this paper is simple. We have recently shown in Benavoli et al. (2016a) that Quantum Mechanics (QM) coincides with the Bayesian theory once this is formulated in the space of Hermitian matrices (so as to make possible to gamble on quantum experiments). Such an identification makes a number of things, at least in principle, straightforward: one of these is the extension of QM to make it handle non-linear utility. We do so by adapting the traditional axiomatisation of rational preferences by Anscombe and Aumann (1963) to the quantum setting.

After axiomatising quantum rational preferences in this way, we move on to give a representation theorem that shows that quantum rational preferences can be obtained by means of expected utility considerations. Our route to this results is based on the approach devised by Zaffalon and Miranda (2015) in the classical case: we show that the axiomatisation of quantum rational preferences is equivalent to the quantum theory of coherent desirable gambles—the very same theory at the heart of Benavoli et al.'s (2016a) formulation of QM—yet formulated so as to consider prizes other then events. Intuitively, this allows us to formally bring quantum rational preferences back to plain QM through an enlargement of the space of possibilities. We eventually show how this leads to quantum probabilities and utilities after enforcing axioms for state independence and completeness. All proofs can be found in the extended version (Benavoli et al., 2016c).

Before briefly reviewing related work and making some final comments, we illustrate one elegant consequence of the correspondence between preferences and desirability: namely, how to derive a coherent rule for updating preferences determining how should a subject rationally change her preferences in the prospect of obtaining new information in the form of an event.

Since the wording "quantum" is used nowadays with a number of acceptations in the literature, we would like to make precise what our framework actually addresses: that is, nothing else but

---

[*]. The authors are listed in alphabetic order.

gambling on quantum mechanics experiments; we are not, at this stage, endorsing any other interpretation of the quantum decision theory developed here. Moreover, we would like to remark that our framework is an actual generalisation of classical decision-theoretic approaches in the tradition of Anscombe and Aumann (1963): in fact, we can recover them by simply focusing on the subset of Hermitian matrices made by diagonal real-valued matrices; that is, by focusing on classical experiments.

2. Rational gambling on quantum experiments

We start by defining a gambling system about the results of a quantum experiment. To this end, we consider two subjects: the bookmaker and the gambler (Alice). The bookmaker prepares the quantum system in some quantum state. Alice has her personal knowledge (beliefs) about the experiment—possibly no knowledge at all.

1. The bookmaker announces that he will measure the quantum system along its n orthogonal directions and so the outcome of the measurement is an element of $\Omega = \{\omega_1, \ldots, \omega_n\}$, with ω_i denoting the elementary event "detection along i". Mathematically, it means that the quantum system is measured along its eigenvectors,[1] i.e., the projectors[2] $\Pi^* = \{\Pi_1^*, \ldots, \Pi_n^*\}$ and ω_i is the event "indicated" by the i-th projector. The bookmaker is fair, meaning that he will correctly perform the experiment and report the actual results to Alice.

2. Before the experiment, Alice declares the set of gambles she is willing to accept. Mathematically, a gamble G on this experiment is a Hermitian matrix in $\mathbb{C}_h^{n \times n}$, the space of all Hermitian $n \times n$ matrices being denoted by $\mathbb{C}_h^{n \times n}$. We will denote the set of gambles Alice is willing to accept by $\mathscr{K} \subseteq \mathbb{C}_h^{n \times n}$.

3. By accepting a gamble G, Alice commits herself to receive $\gamma_i \in \mathbb{R}$ utiles if the outcome of the experiment eventually happens to be ω_i. The value γ_i is defined from G and Π^* through $\Pi_i^* G \Pi_i^* = \gamma_i \Pi_i^*$ for $i = 1, \ldots, n$. It is a real number since G is Hermitian.

Denote by $\mathscr{H}^+ = \{G \in \mathbb{C}_h^{n \times n} : G \gneq 0\}$ the subset of all positive semi-definite and non-zero (PSDNZ) matrices in $\mathbb{C}_h^{n \times n}$: we call them the set of *positive gambles*. The set of negative gambles is similarly given by $\mathscr{H}^- = \{G \in \mathbb{C}_h^{n \times n} : G \lneq 0\}$. Alice examines the gambles in $\mathbb{C}_h^{n \times n}$ and comes up with the subset $\mathscr{K}$ of the gambles that she finds desirable. Alice's rationality is characterised as follows:

1. Any gamble $G \in \mathbb{C}_h^{n \times n}$ such that $G \gneq 0$ must be desirable for Alice, given that it may increase Alice's utiles without ever decreasing them (**accepting partial gain**): $\mathscr{H}^+ \subseteq \mathscr{K}$.

2. Any gamble $G \in \mathbb{C}_h^{n \times n}$ such that $G \lneq 0$ is not desirable for Alice, given that it may only decrease Alice's utiles without ever increasing them (**avoiding partial loss**): $\mathscr{H}^- \cap \mathscr{K} = \emptyset$.

3. If Alice finds G desirable, that is $G \in \mathscr{K}$, then also νG must be desirable for her for any $0 < \nu \in \mathbb{R}$ (**positive homogeneity**).

4. If Alice finds G_1 and G_2 desirable, that is $G_1, G_2 \in \mathscr{K}$, then she must also accept $G_1 + G_2$, i.e., $G_1 + G_2 \in \mathscr{K}$ (**additivity**).

To understand these rationality criteria, originally presented in Benavoli et al. (2016a, Sec. III), we must remember that mathematically the payoff for any gamble G is computed as $\Pi_i^* G \Pi_i^*$ if the outcome of the experiment is the event indicated by Π_i^*. Then the first two rationality criteria above hold no matter the experiment Π^* that is eventually performed. In fact, from the properties of PSDNZ matrices, if $G \gneq 0$ then $\Pi_i^* G \Pi_i^* = \gamma_i^* \Pi_i^*$ with $\gamma_i^* \geq 0$ for any i and $\gamma_j^* > 0$ for some j.

---

1. We mean the eigenvectors of the density matrix of the quantum system.
2. A projector Π is a set of n positive semi-definite matrices in $\mathbb{C}_h^{n \times n}$ s.t. $\Pi_i \Pi_k = 0$, $(\Pi_i)^2 = \Pi_i = (\Pi_i)^\dagger$, $\sum_{i=1}^n \Pi_i = I$.

Therefore, by accepting $G \gneq 0$, Alice can only increase her utiles. Symmetrically, if $G \lneq 0$ then $\Pi_i^* G \Pi_i^* = \gamma_i \Pi_i^*$ with $\gamma_i \leq 0$ for any i. Alice must then avoid the gambles $G \lneq 0$ because they can only decrease her utiles. This justifies the first two rationality criteria. For the last two, observe that

$$\Pi_i^*(G_1 + G_2)\Pi_i^* = \Pi_i^* G_1 \Pi_i^* + \Pi_i^* G_2 \Pi_i^* = (\gamma_i + \vartheta_i)\Pi_i^*,$$

where we have exploited the fact that $\Pi_i^* G_1 \Pi_i^* = \gamma_i \Pi_i^*$ and $\Pi_i^* G_2 \Pi_i^* = \vartheta_i \Pi_i^*$. Hence, if Alice accepts G_1, G_2, she must also accept $G_1 + G_2$ because this will lead to a reward of $\gamma_i + \vartheta_i$. Similarly, if G is desirable for Alice, then also $\Pi_i^*(\nu G)\Pi_i^* = \nu \Pi_i^* G \Pi_i^*$ should be desirable for any $\nu > 0$.

In other words, as in the case of classical desirability (Williams, 1975; Walley, 1991), the four conditions above state only minimal requirements: that Alice would like to increase her wealth and not decrease it (conditions 1 and 2); and that Alice's utility scale is linear (conditions 3 and 4). The first two conditions should be plainly uncontroversial. The linearity of the utility scale is routinely assumed in the theories of personal, and in particular Bayesian, probability as a way to isolate considerations of uncertainty from those of value (wealth).

We can characterise $\mathscr{K}$ also from a geometric point of view. In fact, from the above properties, it follows that a coherent set of desirable gambles $\mathscr{K}$ is a convex cone in $\mathbb{C}_h^{n \times n}$ that includes all positive gambles (accepting partial gains) and excludes all negative gambles (avoiding partial losses). Without loss of generality we can also assume that $\mathscr{K}$ is not pointed, i.e., $0 \notin \mathscr{K}$: Alice does not accept the null gamble. A coherent set of desirable gambles is therefore a non-pointed convex cone.

Definition 1 *We say that $\mathscr{K} \in \mathbb{C}_h^{n \times n}$ is a* **coherent quantum set of desirable gambles** *(DG) if*
(S1) *$\mathscr{K}$ is a non-pointed convex-cone (positive homogeneity and additivity);*
(S2) *if $G \gneq 0$ then $G \in \mathscr{K}$ (accepting partial gain).*
If in addition a coherent set of desirable gambles satisfies the following property:
(S3) *if $G \in \mathscr{K}$ then either $G \gneq 0$ or $G - \varepsilon I \in \mathscr{K}$ for some strictly positive real number ε (openness),[3]*
then it is said to be a coherent quantum set of **strictly** *desirable gambles (SDGs).*

Note that the although the additional openness property (S3) is not necessary for rationality, it is technically convenient as it precisely isolates the kind of models we use in QM (as well as in classical probability), see Benavoli et al. (2016a). Property (S3) has a gambling interpretation too: it means that we will only consider gambles that are *strictly* desirable for Alice; these are the positive ones or those for which she is willing to pay a positive amount to have them. Note that assumptions (S1) and (S2) imply that SDGs also avoids partial loss: if $G \lneq 0$, then $G \notin \mathscr{K}$ (Benavoli et al., 2016a, Remark III.2).

In Benavoli et al. (2016a), we have shown that maximal set of SDGs, that is SDG sets that are not included in any larger set of SDG, and density matrices are one-to-one. The mapping between them is obtained through the standard inner product in $\mathbb{C}_h^{n \times n}$, i.e., $G \cdot R = Tr(G^\dagger R)$ with $G, R \in \mathbb{C}_h^{n \times n}$. This follows by a representation result whose proof is a direct application of Hahn-Banach theorem:

Theorem 2 (Representation theorem from Benavoli et al. (2016a)) *The map that associates a maximal SDG the unique density matrix ρ such that $Tr(G^\dagger \rho) \geq 0 \ \forall G \in \mathscr{K}$ defines a bijective corre-*

---

3. In Benavoli et al. (2016a) we used another formulation of openness, namely (S3'): if $G \in \mathscr{K}$ then either $G \gneq 0$ or $G - \Delta \in \mathscr{K}$ for some $0 < \Delta \in \mathbb{C}_h^{n \times n}$. (S3) and (S3') are provably equivalent given (S1) and (S2).

spondence between maximal SDGs and density matrices. Its inverse is the map $(\cdot)^\circ$ *that associates each density matrix* ρ *the maximal SDG*[4] $(\rho)^\circ = \{G \in \mathbb{C}_h^{n \times n} \mid G \gneq 0\} \cup \{G \in \mathbb{C}_h^{n \times n} \mid Tr(G^\dagger \rho) > 0\}.$

This representation theorem has several consequences. First, it provides a gambling interpretation of the first axiom of QM on density operators. Second, it shows that density operators are coherent, since the set $(\rho)^\circ$ that they induce is a valid SDG. This also implies that QM is self-consistent— a gambler that uses QM to place bets on a quantum experiment cannot be made a partial (and, thus, sure) loser. Third, the first axiom of QM on $\mathbb{C}_h^{n \times n}$ is structurally and formally equivalent to Kolmogorov's first and second axioms about probabilities on $\mathbb{R}^n$ (Benavoli et al., 2016a, Sec. 2). In fact, they can be both derived via duality from a coherent set of desirable gambles on $\mathbb{C}_h^{n \times n}$ and, respectively, $\mathbb{R}^n$. In Benavoli et al. (2016a) we have also derived Born's rule and the other three axioms of QM from the illustrated setting and shown that measurement, partial tracing and tensor product are just generalised probabilistic notions of Bayes rule, marginalisation and independence. Finally, as shown in (Benavoli et al., 2016b), the representation theorem enables us to derive a Gleason-type theorem that holds for any dimension n of a quantum system, hence even for $n = 2$.

3. Quantum horse lotteries and preference relations

As seen in Sec. 2, QM is just the Bayesian theory of probability formulated over Hermitian matrices. Now we proceed to extend such a theory of probability to make it handle non-linear utility. To this end, we work in the tradition of Anscombe and Aumann (1963). Central to this tradition is the notion of a horse lottery.

Consider a set of prizes $X = \{x_1, \ldots, x_m\}$ with $m \geq 2$ (this last constraint will be clarified later). A *horse lottery* is a compound lottery such that if $\omega \in \Omega$ occurs, it returns a simple lottery, which can depend on ω, over the prizes in X. The idea is that at some later point the subject (Alice) will play the simple lottery thus earning one of the prizes. Anscombe and Aumann's setting axiomatises rational preferences over horse lotteries; from this, it follows that there are probabilities and utilities that represent those preferences via maximum expected utility.

3.1 Horse lotteries over complex numbers

As in the classical case, now we consider that when Π_i^* is observed, Alice receives a probability mass function p_i (pmf) over the prices X rather than the value γ_i as in Sec. 2. This framework is a composite system made of a quantum experiment and a classical experiment (on X). To describe it in a mathematically convenient way, we need to define gambles on this composite system. First, we define the form of the gambles. Since the experiment on X is classical, it can be described by the subspace of $\mathbb{C}_h^{m \times m}$ of diagonal matrices; we denote it as $\mathbb{D}^m$. Gambles on this composite system are therefore elements of $\mathbb{D}^m \otimes \mathbb{C}_h^{n \times n} \subset \mathbb{C}_h^{mn \times mn}$, where $\otimes$ denotes the tensor product. It can be observed that G is a block diagonal matrix with elements in $\mathbb{C}_h^{n \times n}$, i.e. $G = \text{Diag}(G_1, \ldots, G_m)$ with $G_k \in \mathbb{C}_h^{n \times n}$.

We are interested in the special case of gambles on $\mathbb{D}^m \otimes \mathbb{C}_h^{n \times n}$ that return a pmf p_i when the quantum system is measured along some projector.

Definition 3 *Let* $Q \in \mathbb{D}^m \otimes \mathbb{C}_h^{n \times n}$. *Q is said to be a* quantum horse lottery *(QH-lottery) if*

$$\forall \Pi, \forall \Pi_i \in \Pi, \exists p_i \in \mathbb{P}^m : (I_m \otimes \Pi_i) Q (I_m \otimes \Pi_i) = p_i \otimes \Pi_i, \tag{1}$$

---

4. Here the gambles $G \gneq 0$ are treated separately because they are always desirable and, thus, they are not informative on Alice's beliefs about the quantum system. Alice's knowledge is determined by the gambles that are not $G \gneq 0$.

where $\mathbb{P}^m \subset \mathbb{D}^m$ denotes the subset of trace one diagonal-matrices whose elements are non-negative.

The set $\mathbb{P}^m$ is isomorphic to the set of all probability mass functions (pmf) on $\mathbb{R}^m$. Therefore, with an abuse of terminology we improperly refer to the diagonal matrix p_i as a pmf. We denote the subspace of $\mathbb{D}^m \otimes \mathbb{C}_h^{n \times n}$ of QH-lotteries as QL. By Definition 3, a QH-lottery is therefore a gamble that returns to Alice the pmf p_i on X whenever a measurement Π is performed on the quantum system and the projector $\Pi_i \in \Pi$ is observed. In what follows, we determine some properties of QL.

Consider the matrix Q in $\mathbb{D}^m \otimes \mathbb{C}_h^{n \times n}$ defined as $Q = \sum_{j=1}^n q_j \otimes V_j$, with $q_j \in \mathbb{P}^m$ and $V = \{V_j\}_{j=1}^n$ is an orthogonal decomposition (OD) on $\mathbb{C}_h^{n \times n}$. It turns out that actually Q is a QH-lottery. We call it a *simple* QH-lottery. Note that a convex combination of simple QH-lotteries is a QH-lottery. However, such a combination need not be simple anymore. The next theorem isolates necessary and sufficient conditions for an element of the composite space $\mathbb{D}^m \otimes \mathbb{C}_h^{n \times n}$ to be a QH-lottery.

Theorem 4 *Let $Q \in \mathbb{D}^m \otimes \mathbb{C}_h^{n \times n}$ of the form $Q = Diag(Q_1, \ldots, Q_m)$. Then $Q \in$ QL if and only if $Q_j \geq 0$ for every $j = 1, \ldots, m$ and $\sum_{j=1}^m Q_j = I_n$.*

Remark 5 *It should be observed that $Q_1, \ldots, Q_m$ are Hermitian positive semi-definite matrices that sum up to the identity operator. This is the definition of positive-operator valued measure (POVM). Therefore the generalisation of horse lotteries to the quantum setting naturally leads to POVMs.*

Remark 6 *The classical definition of horse lotteries can be recovered by the quantum one just by considering, instead of the space $\mathbb{C}_h^{n \times n}$, the space of diagonal real-valued matrices. Obviously, the composite system under consideration is $\mathbb{D}^m \times \mathbb{D}^n$. This space is isomorphic to the space $\mathscr{L}(X \times \Omega)$ of real valued functions whose domain is $X \times \Omega$, where $\Omega = \{1, \ldots, n\}$ and $X = \{1, \ldots, m\}$. By applying Definition 3, we immediately obtain that an object $Q \in \mathscr{L}(X \times \Omega)$ satisfies Property 1 if and only if $Q(\cdot, \omega)$ is a pmf on X for each $\omega \in \Omega$, meaning that Q is a (classical) horse lottery.*

3.2 Coherent preference relations

Horse lotteries are given a behavioural interpretation through a notion of preference. The idea is that Alice, who aims at receiving a prize from X, will prefer some horse lotteries over some others, depending on her knowledge about the quantum experiment, as well as on her attitude towards the prizes. We consider the following well-known axioms of rational preferences, formulated here in the quantum setting.

Definition 7 *A preference relation over quantum horse lotteries is a subset $\succ \subseteq$ QL $\times$ QL. It is said to be* **coherent** *if it satisfies the following axioms:*

(A.1) $(\forall P, Q, R \in$ QL$)P \not\succ P$ *and* $P \succ Q \succ R \Rightarrow P \succ R$ *[strict partial order];*

(A.2) $(\forall P, Q \in$ QL$)P \succ Q \Leftrightarrow (\forall R \in$ QL$)(\forall \alpha \in (0,1]) \alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R$ *[mixture independence].*

Our approach is therefore a straightforward generalisation of the classical setting to the quantum case. In the classical axiomatisations of rational preferences, it is customary to assume that the preference relation has a best and a worst horse lottery. For us it is enough to assume that the worst

one exists and that it actually corresponds to a worst element in X.[5] Formally, we denote the last (m-th) element of X as z. By $p^z \in \mathbb{P}^m$ we denote the pmf that assigns all the mass to z. Finally, by Z we denote the QH-lottery $p_z \otimes I_n$. Notice that $Z = \sum_{i=n}^n p_z \otimes \Pi_i$, for every OD $\Pi = \{\Pi_i\}_{i=1}^n$, and therefore $(I_m \otimes \Pi_i)Z(I_m \otimes \Pi_i) = p_z \otimes \Pi_i$, for every $\Pi_i \in \Pi$.

Definition 8 *Let $\succ \subset \mathsf{QL} \times \mathsf{QL}$ be a preference relation. We say that $\succ$ has the worst outcome if there is $z \in X$ such that $P \succ Z$ for every $P \neq Z$.*

In what follows we assume that preference relations have such a worst outcome. The rationale is that the elements of $X \setminus \{z\}$ are actual prizes, whereas z represents the event that no prize in $X \setminus \{z\}$ is won (nothing is won). We have assumed that $m \geq 2$ precisely as a consequence of the assumption that there is the worst-outcome z among the elements of X.

The scaled differences of QL-lotteries is the set defined by

$$\mathscr{A} = \{\lambda (P - Q) \mid \lambda > 0, P, Q \in \mathsf{QL}\}, \tag{2}$$

where λ is a positive real. The set $\mathscr{A}$ constitutes a vector space.

Theorem 9 *The map $\succ \mapsto \mathscr{C} = \{\lambda (P - Q) \mid P, Q \in \mathsf{QL}, \ \lambda > 0, P \succ Q\}$ determines a bijection between coherent preference relations over QL and non-pointed convex cones in $\mathscr{A}$.*

Thus, it turns out that non-pointed cones and coherent quantum preference relations are just two ways of looking at the same thing.

4. Quantum desirability vs quantum preference: two faces of the same coin

In this section we follow the same strategy as in Zaffalon and Miranda (2015) to establish an equivalence between the theories of coherent quantum preference and coherent quantum desirability. To this end, we first define the projection operator that drops the z-components from an act.

Definition 10 *The projection operator is the functional $\mathsf{Proj} : \mathbb{D}^m \otimes \mathbb{C}_h^{n \times n} \to \mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$ that takes the QL-lottery (m-block diagonal matrix) Q and returns $\mathsf{Proj}(Q) = Diag(Q_1, \ldots, Q_{m-1})$.*

In this paper, we are going to use this operator to project QH-lotteries in $\mathbb{D}^m \otimes \mathbb{C}_h^{n \times n}$ into gambles on $\mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$. However, instead of working directly with the space QL, in what follows it will be more convenient to reason on the space $\mathscr{A}$ of scaled differences of QH-lotteries defined in (2). Note also that the restriction of Proj to $\mathscr{A}$ is injective.

Based on the correspondence between cones on $\mathscr{A}$ and preference relations, it is then an easy step to show (see (Benavoli et al., 2016c, Proposition 22)) that given a coherent preference relation $\succ$, one can define a coherent set $\mathscr{K}$ of desirable gambles on $\mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$ as $\mathscr{K} = \{\lambda \, \mathsf{Proj}(P - Q) : P \succ Q, \lambda > 0\}$ and with the property that

$$P \succ Q \Leftrightarrow \mathsf{Proj}(P - Q) \in \mathscr{K}. \tag{3}$$

One can actually verify that there is an exact correspondence between coherent sets of desirable gambles and coherent preference relations.

Theorem 11 *There is a one-to-one correspondence between coherent sets of desirable gambles over $\mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$ and coherent preference relations over $\mathsf{QL} \times \mathsf{QL}$.*

---

5. The two requirements—having a worst horse lottery and a worst element in X—have been shown equivalent in Zaffalon and Miranda (2015, Proposition 8).

5. Archimedeanity and the representation theorem

Archimedeanity is an extra axiom adopted in traditional axiomatisations of rational preferences that tries to capture a form of continuity; it is such an axiom that makes it possible to have a representation of preferences in terms of expected utilities. Zaffalon and Miranda (2015, Prop. 11) (and in the quantum case, we, in (Benavoli et al., 2016c, Proposition 22)) have shown that the traditional Archimedean axiom has some drawbacks that can be fixed with a slight change in its definition. It is based on the notion of objective preference.

Definition 12 (Objective preference) *Let $P, Q \in \mathsf{QL}$. We say that P is objectively preferred to Q if* $\mathrm{Proj}(P - Q) \gneq 0$. *We denote objective preference by $P \rhd Q$.*

(Note that the definition neglects the outcome z, since it is not one any subject actually wants.)

Objective preference is a preference relation. Moreover, it is the least preference relation over $\mathsf{QL} \times \mathsf{QL}$ in the sense that it is included in any other preference relation (in this sense, we call it "objective"). Now we can directly rephrase Zaffalon and Miranda's Archimedean notion as follows for the quantum case:

(A.3) $(\forall P, Q \in \mathsf{QL})\, P \succ Q, P \not\rhd Q \Rightarrow (\exists \alpha \in (0,1))\, \alpha P + (1-\alpha)Z \succ Q$ [*Weak Archimedeanity*].

Analogously to their case, we obtain that it is equivalent to use coherent quantum sets of strictly desirable gambles in order to represent weakly Archimedean coherent preference relation on quantum horse lotteries. Recall also that a preference relation $\succ$ is said to be complete (or total) if either $P \succ Q$ or $Q \succ P$, for every $P, Q \in \mathsf{QL}$ with $P \neq Q$.

Theorem 13 *There is a one to one correspondence between coherent sets of SDG over $\mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$ and coherent preference relations over $\mathsf{QL} \times \mathsf{QL}$ that are weakly Archimedean. Moreover, such a correspondence induces a bijection between maximal coherent sets of SDG and complete weakly Archimedean coherent preference relations.*

Based on Theorem 13, we can then obtain a representation theorem for complete weakly Archimedean coherent preference relations as follows. First of all, Theorem 2 from Benavoli et al. (2016a) can be restated in the case of quantum horse lotteries as follows.

Theorem 14 *The map that associates a maximal SDG over $\mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$ the unique trace-one positive matrix $R \in \mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$ such that $Tr(G^\dagger R) \geq 0\, \forall G \in \mathcal{K}$ defines a bijective correspondence between maximal SDGs over $\mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$ and trace-one positive matrices over $\mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$. Its inverse is the map $(\cdot)^\circ$ that associates each trace-one positive matrix R the maximal SDG*

$$(R)^\circ = \{\mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n} \mid G \gneq 0\} \cup \{G \in \mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n} \mid Tr(G^\dagger R) > 0\}. \tag{4}$$

All trace-one positive matrices R are of the form $R = \mathrm{Diag}\,(p_1 \rho_1, \ldots, p_{m-1} \rho_{m-1})$ with $\rho_i \in \mathbb{C}_h^{n \times n}$ being density matrices and $\mathrm{Diag}(p_1, \ldots, p_{m-1}) \in \mathbb{P}^{m-1}$. Hence, applying Theorems 13 and 14 to Property 3 yields the following representation result for complete preference relations:

Corollary 15 *A relation $\succ$ over $\mathsf{QL} \times \mathsf{QL}$ is a complete weakly Archimedean coherent preference relation if and only if there is a unique trace-one positive matrix $R = \mathrm{Diag}\,(p_1 \rho_1, \ldots, p_{m-1} \rho_{m-1})$ such that*

$$P \succ Q \Leftrightarrow \left(either\ P \rhd Q\ or\ \sum_{i=1}^{m-1} p_i Tr(P_i^\dagger \rho_i) > \sum_{i=1}^{m-1} p_i Tr(Q_i^\dagger \rho_i)\right) \forall P, Q \in \mathsf{QL}. \tag{5}$$

Consistently with our generalisation of Gleason's theorem (Benavoli et al., 2016b), this result holds in any dimension (even $n = 2$), because we ask preference relations to be coherent.

6. Coherent updating and state independence

This section shows how to derive in a very simple, elegant, way a coherent rule for updating preferences. In particular our aim is to answer this question: how should Alice change her preferences in the prospect of obtaining new information in the form of an event?

We initially assume that Alice considers an event "indicated" by a certain projector $\Pi_i \in \mathbb{C}_h^{n \times n}$ in $\Pi = \{\Pi_i\}_{i=1}^n$. The information it represents is: an experiment Π is performed and the event indicated by Π_i happens.[6]

Now, assume that Alice's preferences are modelled by the coherent relation $\succ$ on QL. From Theorem 11 we can consider the coherent set $\mathscr{K}$ in $\mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$. Hence, we reason as in the derivation of the second axiom of QM in Benavoli et al. (2016a, Sec. V). Under the assumption that an experiment Π is performed and the event indicated by Π_i happens, Alice can focus on gambles that are contingent on $I_{m-1} \otimes \Pi_i$: these are the gambles such that "outside" $I_{m-1} \otimes \Pi_i$ no utile is received or due—status quo is maintained—; in other words, they represent gambles that are called off if the outcome of the experiment is not Π_i. Mathematically, these gambles are of the form

$$G = \begin{cases} H & \text{if } I_{m-1} \otimes \Pi_i \text{ occurs,} \\ 0 & \text{if } I_{m-1} \otimes \Pi_j \text{ occurs, with } j \neq i. \end{cases}$$

It is then clear that $H = \alpha I_{m-1} \otimes \Pi_i$ with $\alpha \in \mathbb{R}$ since $\Pi_i \Pi_j = 0$ for each $j \neq i$. In this light, we can define Alice's conditional preferences by moving to the equivalent view on gambles, restricting the attention to gambles of the form $I_{m-1} \otimes \Pi_i G I_{m-1} \otimes \Pi_i = \alpha I_{m-1} \otimes \Pi_i$ with $G \in \mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$, and finally updating the preferences by looking at the corresponding preference relation.

Definition 16 *Let $\succ$ be a preference relation. The relation obtained as $\succ_{\Pi_i} := \mathsf{Proj}_1^{-1}(\mathscr{K}_{\Pi_i})$, with*

$$\mathscr{K}_{\Pi_i} = \left\{ G \in \mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n} \mid G \gneq 0 \text{ or } (I_{m-1} \otimes \Pi_i) G (I_{m-1} \otimes \Pi_i) \in \mathscr{K} \right\} \qquad (6)$$

is called the **preference relation conditional** *on Π_i.*

By the same argument as in Benavoli et al. (2016a, Prop. A.6), one can prove that $\mathscr{K}_{\Pi_i}$ is a coherent set of (strictly) desirable gambles, whenever $\mathscr{K}$ is a coherent set of (strictly) desirable gambles. By Theorems 11 and 13 this yields that:

Theorem 17 *Let $\succ$ be a (weakly Archimedean) coherent preference relation. The relation $\succ_{\Pi_i}$ conditional on the event Π_i is also a (weakly Archimedean) coherent preference relation.*

Now, we rely on conditioning to introduce the concept of state-independent preferences. For this purpose, we use results in Benavoli et al. (2016a) to prove the fourth postulate of QM about composite systems. We first define the concept of epistemic irrelevance.

---

6. We assume that the quantum measurement device is a "perfect meter" (an ideal common assumption in QM), i.e., there are not observational errors—Alice can trust the received information.

Definition 18 *Let $\mathscr{R} \subset \mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$, and let us define*

$$
\begin{aligned}
\operatorname{marg}_{\mathbb{D}^{m-1}}(\mathscr{R}) &= \left\{ G \in \mathbb{D}^{m-1} \mid G \otimes I_n \in \mathscr{R} \right\}, \\
\mathscr{R}_{|\mathbb{C}_h^{n \times n}} &= \left\{ H \in \mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n} \mid H \gneq 0 \ or \ (I_{m-1} \otimes \Pi_i) H (I_{m-1} \otimes \Pi_i) \in \mathscr{R} \right\},
\end{aligned}
$$

An SDG $\mathscr{K}$ on $\mathbb{D}^{m-1} \otimes \mathbb{C}_h^{n \times n}$ is said to satisfy **epistemic irrelevance** *of the subsystems $\mathbb{D}^{m-1}$ to $\mathbb{C}_h^{n \times n}$ when $\operatorname{marg}_{\mathbb{D}^{m-1}}(\mathscr{K}) = \operatorname{marg}_{\mathbb{D}^{m-1}}(\mathscr{K}_{|\mathbb{C}_h^{n \times n}})$ for each projection measurement $\Pi = \{\Pi_i\}_{i=1}^n$.*

Let us briefly explain this definition. $\mathscr{K}$ is the SDG conditional on the event indicated by $I_{m-1} \otimes \Pi_i$, as it follows from its definition and (6). Thus, $\operatorname{marg}_{\mathbb{D}^{m-1}}(\mathscr{K}) = \operatorname{marg}_{\mathbb{D}^{m-1}}(\mathscr{K}_{\mathbb{C}_h^{n \times n}})$ means that Alice's marginal SDG $\operatorname{marg}_{\mathbb{D}^{m-1}}(\mathscr{K})$ on the subsystem $\mathbb{D}^{m-1}$ and the marginal on $\mathbb{D}^{m-1}$ of Alice's SDG updated with the information "the event indicated by Π_i has happened", which is $\operatorname{marg}_{\mathbb{D}^{m-1}}(\mathscr{K}_{\mathbb{C}_h^{n \times n}})$, coincide. If this holds for all possible Π_i's, then any information on $\mathbb{C}_h^{n \times n}$ does not change Alice's beliefs on $\mathbb{D}^{m-1}$: this is precisely the definition of epistemic irrelevance. In case $\mathscr{K}$ is maximal and satisfies epistemic irrelevance, we have shown in Benavoli et al. (2016a, Sec. VII.c) that the representation Theorem 14 applied to such $\mathscr{K}$ defines a matrix R that factorizes as $R = p \otimes \rho$. Therefore, as in the classical framework for decision theory, the "joint" density matrix R factorizes as the product of $p \in \mathbb{P}^m$ and the density matrix $\rho \in \mathbb{C}_h^{n \times n}$. Stated otherwise, $\mathscr{K}$ models independence between utility (p) and the "probabilistic" information (ρ) associated to the quantum system. Alice's preferences are state-independent in this case.

7. Related work

Axiomatic frameworks for the theory of subjective expected utility were originally given by Savage (1954) and by Anscombe and Aumann (1963). Karni (2013) provides a recent overview of several variations and extensions of these two models.

Busemeyer and Bruza's (2012) presents an overview and many references to quantum-like approaches to cognition and decision theory. Deutsch (1999), Khrennikov (2016) and Danilov et al. (2016) are examples of other works addressing similar issues. In particular the latter proposes an axiomatisation for quantum preferences directly in the space of Hermitian matrices similar to the one presented here. However, the authors only consider what we call *simple* quantum horse lotteries. In doing so, the traditional axiom of mixture independence is formulated relative to the particular orthogonal decomposition associated to a simple lottery; an additional axiom becomes then necessary to bind lotteries based on different orthogonal decompositions. Moreover their representation theorem crucially employs Gleason's theorem and therefore only works on spaces of dimension greater than two. Because of those characteristics, it is unclear to us whether or not that axiomatisation is coherent: e.g., whether it guarantees that a subject whose quantum preferences on a space of dimension two cannot be made a sure loser, that is, shown to be irrational. The case of dimension two is particularly critical as dispersion-free probabilities—which Benavoli et al. (2016b) have shown to be incoherent—could in principle be employed.

8. Concluding remarks

In this paper, we have axiomatised rational preferences over quantum horse lotteries. Such a development is a natural follow up of our recent work (Benavoli et al., 2016a), which has shown that Quantum Mechanics is the Bayesian theory of probability over Hermitian matrices. By bridging

those rational preferences with quantum desirability, we have given a representation theorem in terms of quantum probabilities and utilities.

There are many directions that can be explored starting from those in this paper. Two of them are particularly important in our view. The first regards the full extension of our setting to partial (i.e., incomplete) preferences; this would enable it to deal with sets of quantum probabilities and utilities. The second is the definition of horse lotteries in their full generality as compound quantum lotteries.

References

F. J. Anscombe and R. J. Aumann. A definition of subjective probability. *The Annals of Mathematical Statistics*, 34:199–2005, 1963.

A. Benavoli, A. Facchini, and M. Zaffalon. Quantum mechanics: The Bayesian theory generalized to the space of Hermitian matrices. *Physical Review A*, 94:042106, 2016a.

A. Benavoli, A. Facchini, and M. Zaffalon. A Gleason-type theorem for any dimension based on a gambling formulation of Quantum Mechanics. *ArXiv e-prints 1606.03615*, 2016b.

Alessio Benavoli, Alessandro Facchini, and Marco Zaffalon. Quantum rational preferences and desirability. *arXiv preprint arXiv:1610.06764*, 2016c.

J. R. Busemeyer and P. D. Bruza. *Quantum models of cognition and decision*. Cambridge University Press, 2012.

V. I. Danilov, A. Lambert-Mogiliansky, and V. Vergopoulos. Dynamic consistency of expected utility under non-classical(quantum) uncertainty. Preprint, May 2016. URL `https://halshs. archives-ouvertes.fr/halshs-01324046`.

D. Deutsch. Quantum theory of probability and decisions. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 455, pages 3129–3137. The Royal Society, 1999.

Edi Karni. Axiomatic foundations of expected utility and subjective probability. *Handbook of Economics of Risk and Uncertainty*, 1, 2013.

A. Khrennikov. Quantum Bayesianism as the basis of general theory of decision-making. *Philosophical Transactions of the Royal Society A*, 374(20150245), 2016.

Leonard J Savage. *The foundations of statistics*. John Wiley and Sons, New York, 1954.

P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, New York, 1991.

P. M. Williams. Notes on conditional previsions. Technical report, School of Mathematical and Physical Science, University of Sussex, UK, 1975. Reprinted in Williams (2007).

P. M. Williams. Notes on conditional previsions. *International Journal of Approximate Reasoning*, 44:366–383, 2007. Revised journal version of Williams (1975).

M. Zaffalon and E. Miranda. Desirability and the birth of incomplete preferences. *arXiv preprint arXiv:1506.00529*, 2015.