SPAB assignment 08

8.1

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad \forall x \in \mathbb{R}$$

1) to prove: Φ is a probability density, $\int_{-\infty}^{\infty} \Phi(x)\,dx = 1$

idea: $\int_{-\infty}^{\infty} \Phi(x)\,dx = I \overset{!}{=} 1 \Rightarrow I^2 = 1$

$$I^2 = \int_{-\infty}^{\infty} \Phi(x)\,dx \cdot \int_{-\infty}^{\infty} \Phi(y)\,dy$$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}}}_{\text{const}} \exp\left(-\frac{x^2}{2}\right) dx \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dx$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{(y^2+x^2)}{2}\right) dy\,dx$$

idea: use polar coordinates

$x = r\cos\varphi,\ y = r\sin\varphi$

$y^2 + x^2 = (r\sin\varphi)^2 + (r\cos\varphi)^2 = r^2\sin^2\varphi + r^2\cos^2\varphi = r^2 \underbrace{(\sin^2\varphi + \cos^2\varphi)}_{=1}$

$= r^2$

$dy\,dx = r\,d\varphi\,dr$

$$= \frac{1}{2\pi} \int_0^{\infty} \underbrace{\int_0^{2\pi} \exp\left(-\frac{r^2}{2}\right) r\,d\varphi}\,dr$$

$$= \left[\exp\left(-\frac{r^2}{2}\right) r\,dr\,\varphi\right]_0^{2\pi}$$

$$= \exp\left(-\frac{r^2}{2}\right) r\,dr\,2\pi - \underbrace{\exp\left(-\frac{r^2}{2}\right) r\,dr\,0}_{=0}$$

$$= \frac{1}{2\pi} \int_0^{\infty} \exp\left(-\frac{r^2}{2}\right) r\,dr\,2\pi$$

substitution $u = \frac{r^2}{2},\ du = r\,dr$

$$= \frac{1}{2\pi}\left[-2\pi\exp(-u)\right]_0^{\infty} = \frac{1}{2\pi}\left((-2\pi \underbrace{e^{-\infty}}_{=0}) - (-2\pi e^{-0})\right)$$

$$= \frac{1}{2\pi}(0 + 2\pi) = \frac{2\pi}{2\pi}$$

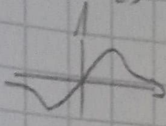
$$= 1$$

$$\Rightarrow I = \sqrt{I^2} = \sqrt{1} = 1$$

sources
- lecture notes SPAB, HMJ 1 - 4a
- e-pandu.com

2) $X \sim \mathcal{N}(0,1) \rightarrow E[X] = 0$, $\mu = \int_{-\infty}^{\infty} x \, \phi(x) \, dx = 0$

$x \exp(-\frac{x^2}{2})$



$$E[x] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) \, dx$$

when we look at the plot of $x \exp(-\frac{x^2}{2})$ it makes sense to split the integral at 0

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} x \exp(-\frac{x^2}{2}) dx + \int_{0}^{\infty} x \exp(-\frac{x^2}{2}) dx \right)$$

Wolfram Alpha

$$= \frac{1}{\sqrt{2\pi}} \left(\left[-\exp(-\frac{x^2}{2}) \right]_{-\infty}^{0} + \left[-\exp(-\frac{x^2}{2}) \right]_{0}^{\infty} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left((-1 - 0) + (0 - (-1)) \right)$$

$$= \frac{1}{\sqrt{2\pi}} (-1 + 1)$$

$$= 0 \qquad \blacksquare$$

3) $Var[x] = 1$, $Var[x] = E[x^2] - \underbrace{E[x]^2}_{= 0^2 = 0}$

$$Var[x] = E[x^2] - 0$$

$$= \int_{-\infty}^{\infty} x^2 \, \phi(x) \, dx$$

use tipp and split at 0

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} \underbrace{x}_{u} \underbrace{(x \exp(-\frac{x^2}{2}))}_{v'} dx + \int_{0}^{\infty} x (x \exp(-\frac{x^2}{2})) \, dx \right)$$

$$\int_a^b u(x) v'(x) \, dx = [u(x) v(x)]_a^b - \int_a^b u'(x) v(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[\underbrace{-x}_{u} \underbrace{\exp(-\frac{x^2}{2})}_{v} \right]_{-\infty}^{0} - \left(- \int_{-\infty}^{0} \underbrace{1}_{u'} \cdot \underbrace{\exp(-\frac{x^2}{2})}_{v} dx \right) + \left[-x \exp(-\frac{x^2}{2}) \right]_{0}^{\infty} + \int_{0}^{\infty} \exp(-\frac{x^2}{2}) dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left((0 - 0) + \int_{-\infty}^{0} \exp(-\frac{x^2}{2}) dx + (0 - 0) + \int_{0}^{\infty} \exp(-\frac{x^2}{2}) dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2}) \, dx$$

shown in task 8.1 1)

$$= 1 \qquad \blacksquare$$