Task 5.2 SPAB

$\mu = \mathbb{E}[X] \quad \sigma^2 := \mathbb{E}[|X-\mu|^2]$

$M := \frac{1}{n}(X_1 + \dots + X_n) = \frac{1}{n}\sum_{i=1}^n X_i$

$V := \frac{1}{n} \cdot \sum_{i=1}^n (X_i - M)^2$

① $\mathbb{E}[M] = E[\frac{1}{n}\sum_{i=1}^n X_i] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}X_i = \frac{1}{n}\sum_{i=1}^n p_i x_i$

$= \frac{1}{n}\sum_{i=1}^n \frac{1}{n} x_i = \frac{1}{n^2}\sum x_i$

② $Var[M] = Var[\frac{1}{n}\sum_{i=1}^n X_i] = \frac{1}{n^2} Var[\sum_{i=1}^n X_i]$

$= \frac{1}{n^2}\sum Var[X_i]$

$= \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}[X_i - \mathbb{E}[X_i]]^2$

$= \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}((X_i - \mathbb{E}X_i)^2]$

$= \frac{1}{n}\sum(\frac{1}{n}(x_i - \frac{1}{n}x_i)^2$

$= \frac{1}{n^2}\sum(x_i - \frac{1}{n}x_i)^2$

③ $\mathbb{E}[V] = \mathbb{E}[\frac{1}{n}\sum_{i=1}^n (X_i - M)^2]$

$= \frac{1}{n}\mathbb{E}[\sum_{i=1}^n (X_i - M)^2]$

$= \frac{1}{n}\sum_{i=1}^n \mathbb{E}(X_i - M)^2$

$= \frac{1}{n}\sum_{i=1}^n \mathbb{E}(X_i - M)\mathbb{E}(X_i - M)$

$= \frac{1}{n}\sum_{i=1}^n [\mathbb{E}(X_i) - \mathbb{E}(M)] \cdot [\mathbb{E}(X_i) - \mathbb{E}(M)]$

$= \frac{1}{n}\sum_{i=1}^n (\frac{1}{n}\sum X_i - \frac{1}{n^2}\sum X_i) \cdot (\frac{1}{n}\sum X_i - \frac{1}{n^2}\sum X_i)$

$= \frac{1}{n}\sum_{i=1}^n \frac{1}{n}(\sum X_i - \frac{1}{n}\sum X_i)\frac{1}{n}(\sum X_i - \frac{1}{n}\sum X_i)$

$= \frac{1}{n^3}\sum_{i=1}^n (\frac{n-1}{n}\sum X_i)^2 \quad \frac{n-1}{n}\sum X_i = \frac{(n-1)^2}{n^5}\sum X_i^2$

Task 5.3

Case a): Y is an "independent copy" of X. okay because of independence

1) obvious 2) $Var[X+Y] = Var[X] + [Y]$

$= 2\frac{(n-1)^2 - 1}{12} = \frac{(n-1)^2 - 1}{6}$

Case b) 1) come on, still obvious.

2) $Var(X+Y) = Var(X) + Var(X) + 2\,Cov(X,X)$

$= Var(X) + Var(X) + 2\,Var(X)$

$= 4\,Var(X) = \frac{4 \cdot ((n-1)^2 - 1)}{12} = \frac{4(n-1)^2 - 4}{12} = \frac{(n-1)^2 - 1}{3}$

BRUNNEN

case c)

$Y := n - X$

1) X is uniformly distributed in $\{0, \dots, n-1\}$
edge cases: $Y = n - 0 = n$
$Y = n - (n-1) = 1$
Since Y is just a linear combination of X it is also uniformly distributed on $\{1, \dots, n\}$
X and Y are not independent (when X changes it affects Y)

2) $$Var[X+Y] = Var[X] + Var[Y] + 2\,Cov[X, Y]$$
$$= \frac{(n-1)^2 - 1}{12} + \frac{n^2-1}{12} + 2\,E[(X - \tfrac{n}{2})(Y - \tfrac{n+1}{2})]$$
$$= \frac{2n^2 - 2n - 1}{12} + 2(E[XY] - \tfrac{n}{2} \cdot \tfrac{n+1}{2} - \tfrac{n}{2} \cdot \tfrac{n+1}{2} + \tfrac{n(n+1)}{4})$$
$$= \frac{2n^2 - 2n - 1}{12} + 2(E[X(n-X)] - \tfrac{n(n+1)}{4})$$
$$= \frac{2n^2 - 2n - 1}{12} + 2(E[X^2] - n\,E[X] - \tfrac{n(n+1)}{4})$$
$$= \frac{2n^2 - 2n - 1}{12} + 2\left(\frac{(n-1)^2}{3} - n\frac{n}{2} - \frac{n(n+1)}{4}\right)$$
$$= \frac{2n^2 - 2n - 1 + 8n^2 - 16n + 8 + 16n^2 - 6n^2 - 6n}{12}$$
$$= \frac{20n^2 - 24n + 7}{12}$$

case d)

$Y := (2X \bmod n)$

1) Since X is uniformly distributed in $\{0, \dots, n-1\}$ $2X$ lays in $\{0, 2, \dots, 2n-2\}$.
n is odd which means there is no $x_i \in X$ s.t. $2x_i \bmod n = 0$.
By multiplying x with 2 we create always an even number, so
$\nexists\, x_i \in X$ s.t. $2x_i \bmod n = n$.
$\rightarrow$ Y is uniformly distributed in $\{1, \dots, n-1\}$.
X and Y are not independent.

2) $$Var[X+Y] = Var[X] + Var[Y] + 2\,Cov[X, Y]$$
$$= \frac{2((n-1)^2 - 1)}{12} + 2\,E[(X - \tfrac{n}{2})(Y - \tfrac{n}{2})]$$
$$= \frac{(n-1)^2 - 1}{6} + 2\,E[X(2X \bmod n)] - \frac{2n^2}{4}$$

sry. too tired to continue