Task 8.3

$$\Phi_\mu := \left(\frac{1}{\sqrt{2\pi}}\right) \cdot e^{\frac{(x-\mu)^2}{2}} \quad \text{(density)}$$

$$f_\mu(x) = \left(\frac{1}{2}\right) \cdot [\Phi_{-\mu}(x) + \Phi_\mu(x)]$$

$$X_1 = \frac{-(x+\mu)^2}{2}$$

$$X_2 = \frac{-(x-\mu)^2}{2}$$

Curvature of $f_\mu''(x)$

$$\cancel{f_\mu'(x) = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\left[e^{\frac{-(x+\mu)^2}{2}}(-x-\mu) + e^{\frac{-(x-\mu)^2}{2}}(\mu-x)\right]}$$

$$f_\mu''(x) = \frac{1}{2\sqrt{2\pi}}\left[e^{X_1}(-x-\mu)(-x-\mu) + e^{X_1}(-1) + e^{X_2}(\mu-x)(\mu-x) + e^{X_2} \cdot 1\right]$$

$$= \frac{1}{2\sqrt{2\pi}}\left[-e^{X_2}(x-\mu) - e^{X_1}(x+\mu)\right]$$

I skip the side calculations and just provide the finished derivative

$$f_\mu''(x) = \frac{1}{2\sqrt{2\pi}} e^{-x^2-\mu^2}\left[e^{\left(\frac{x+\mu^2}{2}\right)}(x^2-2x\mu+\mu^2-1) + e^{\frac{(x-\mu)^2}{2}}(x^2+2x\mu+\mu^2-1)\right]$$

$$f_0''(0) = \frac{1}{2\sqrt{2\pi}} \cdot 1\left[1 \cdot (-1) + 1 \cdot (-1)\right] = \frac{1}{2\sqrt{2\pi}}$$

$$\boxed{f_1''(0) = \frac{1}{2\sqrt{2\pi}} \cdot 0{,}3678\left[1.6487 \cdot 0 + 1.6487 \cdot 0\right] = 0}$$

$$f_2''(0) = \frac{1}{2\sqrt{2\pi}} \cdot 0{,}0183\left[7.3890 \cdot 3 + 7.3890 \cdot 3\right] = 0{,}16183$$

This shows that T is exactly 1. Because curv. gets 0 at x_0 and curv of 0 at this point means unimodality.

z.z. $\Phi_\mu''(x) = ((x-\mu)^2-1) \cdot \Phi_\mu(x)$

$$\Phi_\mu'(x) = \left(\frac{1}{\sqrt{2\pi}}\right) \cdot e^{-\frac{(x-\mu)^2}{2}} \cdot \left[-\frac{1}{2} 2(x-\mu) \cdot 1\right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2}}(-(x-\mu))$$

$$\Phi_\mu''(x) = \frac{1}{\sqrt{2\pi}}\left[e^{X_2}(-(x-\mu))(-(x-\mu)) + e^{X_2} \cdot 1\right]$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} e^{X_2}}_{\Phi_\mu}\left[(x-\mu)^2 - 1\right]$$

$$\Phi_\mu''(x) = ((x-\mu)^2-1)\,\Phi_\mu(x) \quad \blacksquare$$

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