MATH20070/2020 - Optimization in Finance

2021/2022

Contents

Chapter 1

Unconstrained Optimization

1.1 Introduction

[1] We derive a Second Derivative Test for function of n-variables. In order to do this we will combine techniques from calculus of several variables and linear algebra.

1.2 Background and Notation

We let $\mathbb{R}$ denote the set of real numbers.

For $a, b \in \mathbb{R}$ with $a \leq b$, we let

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

be the closed interval with endpoints a and b and

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

be the open interval with endpoints a and b.

Let us recall how to find and classify extreme points of functions of one variable. Let $f: (a, b) \to \mathbb{R}$ be a differentiable function. We say that $c \in (a, b)$ is a critical point of f if $f'(c) = 0$. The follows Theorem tells us if we have a local maximum, a local minimum or a point of inflection.

> Theorem 1.1 (Second Derivative Test) Let $f: (a, b) \to \mathbb{R}$ be a twice differentiable function and $c \in (a, b)$ be a critical point of f (that is, $f'(c) = 0$).

(i) If $f''(c) > 0$, then f has a relative minimum at c.

(ii) If $f''(c) < 0$, then f has a relative maximum at c.

(iii) If $f''(c) = 0$ AND f'' changes sign when passing through c then c is an inflection point for f

1.2.1 Partial Derivatives

Let us start by considering what happens for functions of two variables. Let $f(x, y)$ be a function of two variables x and y. Then $f(x, y)$ has two first order partial derivatives:

$$f_x = \frac{\partial f}{\partial x}$$

is the derivative with respect to x, obtained by differentiating f with respect to x treating y as a constant,

$$f_y = \frac{\partial f}{\partial y}$$

is the derivative with respect to y, obtained by differentiating f with respect to y treating x as a constant.

Formally we obtain $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ by the formulae

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and

$$\frac{\partial f}{\partial y}(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

We call f_x and f_y the first order partial derivatives of f with respect to x and y respectively.

Example 1.2 If $f(x, y) = xe^{x^2+y^2}$ then

$$\frac{\partial f}{\partial x}(x, y) = (2x^2 + 1)e^{x^2+y^2},$$

and

$$\frac{\partial f}{\partial y}(x, y) = 2xye^{x^2+y^2}.$$

For functions of n variables, $f(x_1, x_2, \ldots, x_n)$ we have n first order partial derivatives

$$f_{x_1} = \frac{\partial f}{\partial x_1}, \qquad f_{x_2} = \frac{\partial f}{\partial x_2}, \ldots, f_{x_n} = \frac{\partial f}{\partial x_n}.$$

The partial derivative f_{x_j} is obtained by differentiating f with respect to x_j treating the other $n-1$ variables $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ as constants.

Example 1.3 If

$$f(x, y, z) = 5x^2 + 4xy + y^3 - 2y + z^5 + xe^z$$

then

$$f_x(x, y, z) = 10x + 4y + e^z, \qquad f_y(x, y, z) = 4x + 3y^2 - 2,$$

$$f_z(x, y, z) = 5y^4 + xe^z.$$

Exercise 1.4 Determine the first order partial derivatives of each of the following functions

(a) $f(x, y) = x^3 + x^2y^3 - 2y^2$;

(b) $f(x, y) = \dfrac{x}{(x + y)^2}$;

(c) $f(x, y, z) = xz - 5x^2y^3z^4$;

(d) $f(x, y, z) = ze^{xyz}$;

(e) $f(x, y, z) = \log\left(x + \sqrt{y^2 + z^2}\right)$.

1.2.2 Critical Points

As with functions of one variable an extremum (maximum or minimum) can only occur at a vector where all first order partial derivatives are 0.

We write this as

$$\nabla f(\mathbf{x}) = 0,$$

where

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right).$$

We call ∇f the gradient of f.

If $n = 3$ we have

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x}(\mathbf{x}), \frac{\partial f}{\partial y}(\mathbf{x}), \frac{\partial f}{\partial z}(\mathbf{x})\right)$$

Let us first consider a function of two variables.

Example 1.5 Let

$$f(x, y) = -2x^2 - y^2 + 8x + 10y - 5xy.$$

We have

$$\frac{\partial f}{\partial x}(x,y) = -4x + 8 - 5y, \qquad \frac{\partial f}{\partial y}(x,y) = -2y + 10 - 5x.$$

If (x, y) is a critical point of f then we have

$$\begin{aligned} 4x + 5y &= 8 \\ 5x + 2y &= 10 \end{aligned}$$

Multiply the first equation by 5 and the second by 4 to get

$$\begin{aligned} 20x + 25y &= 40 \\ 20x + 8y &= 40 \end{aligned}$$

Subtracting we get that $17y = 0$. Hence $y = 0$ and therefore $x = 2$. This means that the point $(2, 0)$ is a critical point of f.

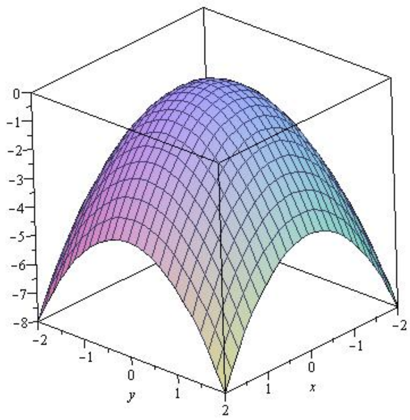
Exercise 1.6 Find the critical points of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$.

Exercise 1.7 Find the critical points of $f(x,y) = x^3 - 12xy + 8y^3$.

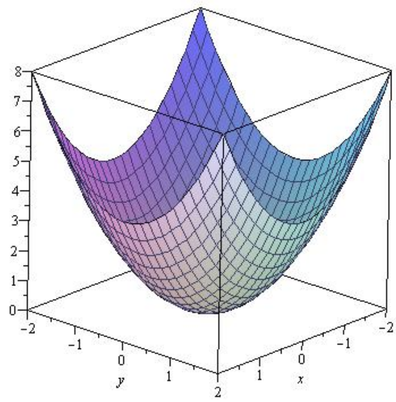
At a critical point, a function can have

- a (local) maximum,
- a (local) minimum,
- a saddle point.

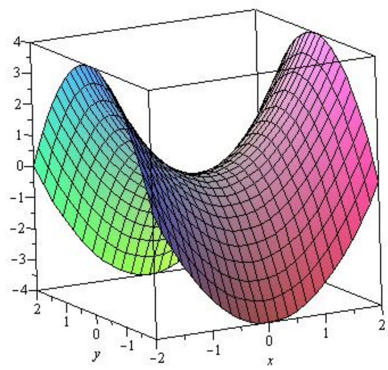
An example of a maximum point:



An example of a minimum point:



An example of a saddle point:



We know that extreme values will occur at critical points. But how do we know the nature of a critical point?

As with functions functions of one variable, to classify critical points of functions of several variables we need higher order partial derivatives.

For functions of two variables, there are four second order partial derivatives:

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial x^2}, & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial y \partial x}, \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial x \partial y}, & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial y^2}.
\end{aligned}
$$

We also use the notation $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$. Note the order of the indices!