MATH20070/2020 - Optimization in Finance

2021/2022

Contents

Chapter 1

Unconstrained Optimization

1.1 Introduction

[1] We derive a Second Derivative Test for function of n-variables. In order to do this we will combine techniques from calculus of several variables and linear algebra.

1.2 Background and Notation

We let $\mathbb{R}$ denote the set of real numbers.

For $a, b \in \mathbb{R}$ with $a \leq b$, we let

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

be the closed interval with endpoints a and b and

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

be the open interval with endpoints a and b.

Let us recall how to find and classify extreme points of functions of one variable. Let $f\colon (a, b) \to \mathbb{R}$ be a differentiable function. We say that $c \in (a, b)$ is a critical point of f if $f'(c) = 0$. The follows Theorem tells us if we have a local maximum, a local minimum or a point of inflection.

Theorem 1.1 (Second Derivative Test) Let $f\colon (a, b) \to \mathbb{R}$ be a twice differentiable function and $c \in (a, b)$ be a critical point of f (that is, $f'(c) = 0$).

(i) If $f''(c) > 0$, then f has a relative minimum at c.

(ii) If $f''(c) < 0$, then f has a relative maximum at c.

(iii) If $f''(c) = 0$ AND f'' changes sign when passing through c then c is an inflection point for f

1.2.1 Partial Derivatives

Let us start by considering what happens for functions of two variables. Let $f(x, y)$ be a function of two variables x and y. Then $f(x, y)$ has two first order partial derivatives:

$$f_x = \frac{\partial f}{\partial x}$$

is the derivative with respect to x, obtained by differentiating f with respect to x treating y as a constant,

$$f_y = \frac{\partial f}{\partial y}$$

is the derivative with respect to y, obtained by differentiating f with respect to y treating x as a constant.

Formally we obtain $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ by the formulae

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and

$$\frac{\partial f}{\partial y}(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

We call f_x and f_y the first order partial derivatives of f with respect to x and y respectively.

Example 1.2 If $f(x, y) = xe^{x^2+y^2}$ then

$$\frac{\partial f}{\partial x}(x, y) = (2x^2 + 1)e^{x^2+y^2},$$

and

$$\frac{\partial f}{\partial y}(x, y) = 2xye^{x^2+y^2}.$$

For functions of n variables, $f(x_1, x_2, \ldots, x_n)$ we have n first order partial derivatives

$$f_{x_1} = \frac{\partial f}{\partial x_1}, \qquad f_{x_2} = \frac{\partial f}{\partial x_2}, \ldots, f_{x_n} = \frac{\partial f}{\partial x_n}.$$

The partial derivative f_{x_j} is obtained by differentiating f with respect to x_j treating the other $n - 1$ variables $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ as constants.

Example 1.3 If

$$f(x, y, z) = 5x^2 + 4xy + y^3 - 2y + z^5 + xe^z$$

then

$$f_x(x, y, z) = 10x + 4y + e^z, \qquad f_y(x, y, z) = 4x + 3y^2 - 2,$$

$$f_z(x, y, z) = 5y^4 + xe^z.$$

Exercise 1.4 Determine the first order partial derivatives of each of the following functions

(a) $f(x, y) = x^3 + x^2y^3 - 2y^2$;

(b) $f(x, y) = \dfrac{x}{(x+y)^2}$;

(c) $f(x, y, z) = xz - 5x^2y^3z^4$;

(d) $f(x, y, z) = ze^{xyz}$;

(e) $f(x, y, z) = \log\left(x + \sqrt{y^2 + z^2}\right)$.

1.2.2 Critical Points

As with functions of one variable an extremum (maximum or minimum) can only occur at a vector where all first order partial derivatives are 0.

We write this as

$$\nabla f(\mathbf{x}) = 0,$$

where

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right).$$

We call ∇f the gradient of f.

If $n = 3$ we have

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x}(\mathbf{x}), \frac{\partial f}{\partial y}(\mathbf{x}), \frac{\partial f}{\partial z}(\mathbf{x})\right)$$

Let us first consider a function of two variables.

Example 1.5 Let

$$f(x, y) = -2x^2 - y^2 + 8x + 10y - 5xy.$$

We have

$$\frac{\partial f}{\partial x}(x,y) = -4x + 8 - 5y, \qquad \frac{\partial f}{\partial y}(x,y) = -2y + 10 - 5x.$$

If (x,y) is a critical point of f then we have

$$\begin{aligned} 4x + 5y &= 8 \\ 5x + 2y &= 10 \end{aligned}$$

Multiply the first equation by 5 and the second by 4 to get

$$\begin{aligned} 20x + 25y &= 40 \\ 20x + 8y &= 40 \end{aligned}$$

Subtracting we get that $17y = 0$. Hence $y = 0$ and therefore $x = 2$. This means that the point $(2,0)$ is a critical point of f.

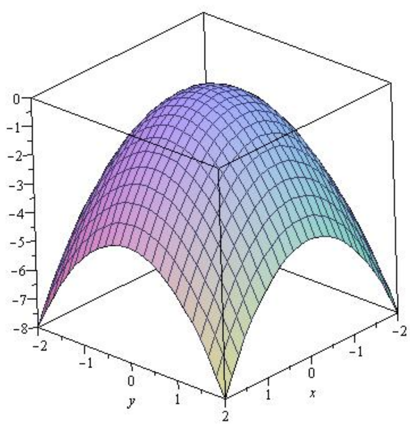
Exercise 1.6 Find the critical points of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$.

Exercise 1.7 Find the critical points of $f(x,y) = x^3 - 12xy + 8y^3$.

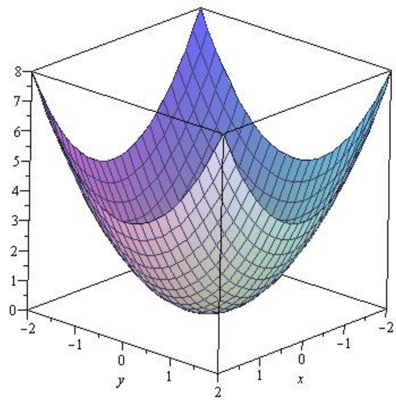
At a critical point, a function can have

- a (local) maximum,
- a (local) minimum,
- a saddle point.

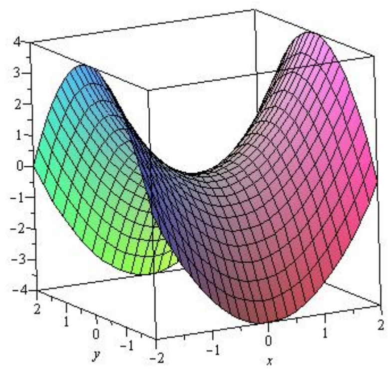
An example of a maximum point:



An example of a minimum point:



An example of a saddle point:



We know that extreme values will occur at critical points. But how do we know the nature of a critical point?

As with functions functions of one variable, to classify critical points of functions of several variables we need higher order partial derivatives.

For functions of two variables, there are four second order partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x},$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}.$$

We also use the notation $f_{xy}=\frac{\partial^2 f}{\partial y \partial x}$. Note the order of the indices!

For functions of n variables there are n^2 second order partial derivatives

$$f_{x_i x_j}=\frac{\partial^2 f}{\partial x_j \partial x_i}=\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right).$$

Note that in most of the examples we will consider that $f_{x_i x_j}=f_{x_j x_i}$.

Example 1.8 If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is given by

$$f(x,y,z)=5x^2+4xy+y^3-2y+z^5+xe^z$$

then

$$\begin{array}{ccc} f_{xx}=10 & f_{xy}=4 & f_{xz}=e^z \\ f_{yx}=4 & f_{yy}=6y & f_{yz}=0 \\ f_{xz}=e^z & f_{yz}=0 & f_{zz}=20z^3+xe^z. \end{array}$$

Exercise 1.9 Determine the second order partial derivatives of each of the following functions

(a) $f(x,y)=x^3+x^2y^3-2y^2$;

(b) $f(x,y)=\dfrac{x}{(x+y)^2}$;

(c) $f(x,y,z)=xz-5x^2y^3z^4$;

(d) $f(x,y,z)=ze^{xyz}$;

(e) $f(x,y,z)=\log\left(x+\sqrt{y^2+z^2}\right)$.

1.3 The Second Derivative Test

We need to introduce some linear algebra.

Otto Hesse 1811-74

1.3.1 The Hessian Matrix

The Hessian matrix H of the function f is the matrix of all second order partial derivatives of f.

$$H(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}x) & f_{x_1x_2}(\mathbf{x}) & f_{x_1x_3}(\mathbf{x}) & \cdots & f_{x_1x_n}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) & f_{x_2x_3}(\mathbf{x}) & \cdots & f_{x_2x_n}(\mathbf{x}) \\ f_{x_3x_1}(\mathbf{x}) & f_{x_3x_2}(\mathbf{x}) & f_{x_3x_3}(\mathbf{x}) & \cdots & f_{x_2x_n}(\mathbf{x}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\mathbf{x}) & f_{x_nx_2}(\mathbf{x}) & f_{x_nx_3}(\mathbf{x}) & \cdots & f_{x_nx_n}(\mathbf{x}) \end{pmatrix},$$

For most functions used in practical applications, we have

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i},$$

hence the Hessian matrix is symmetric.

Example 1.10 The Hessian matrix of $f(x,y,z) = 5x^2 + 4xy + y^3 - 2y + z^5 + xe^z$ is

$$H(\mathbf{x}) = \begin{pmatrix} 10 & 4 & e^z \\ 4 & 6y & 0 \\ e^z & 0 & 20z^3 + xe^z \end{pmatrix}.$$

Exercise 1.11 Determine the Hessian matrix of each of the following functions

(a) $f(x,y) = x^3 + x^2y^3 - 2y^2$;

(b) $f(x,y) = \dfrac{x}{(x+y)^2}$;

(c) $f(x,y,z) = xz - 5x^2y^3z^4$;

(d) $f(x,y,z) = ze^{xyz}$;

(e) $f(x,y,z) = \log\left(x + \sqrt{y^2+z^2}\right)$;

(f) $f(x,y,z) = xe^y + ye^z + ze^x$.

For an $n \times n$ Hessian matrix we identify n submatrices. The first of these, denoted by $H_1(\mathbf{x})$, is the 1×1 submatrix consisting of the element located in row 1 and column 1,

$$H_1(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) \end{pmatrix}.$$

The second submatrix is the 2×2 matrix

$$H_2(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & f_{x_1x_2}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) \end{pmatrix}.$$

The third submatrix is the 3×3 matrix

$$H_3(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & f_{x_1x_2}(\mathbf{x}) & f_{x_1x_3}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) & f_{x_2x_3}(\mathbf{x}) \\ f_{x_3x_1}(\mathbf{x}) & f_{x_3x_2}(\mathbf{x}) & f_{x_3x_3}(\mathbf{x}) \end{pmatrix},$$

The j^{th} submatrix is the $j \times j$ matrix

$$H_j(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & \cdots & f_{x_1x_j}(\mathbf{x}) \\ \vdots & & \vdots \\ f_{x_jx_1}(\mathbf{x}) & \cdots & f_{x_jx_j}(\mathbf{x}) \end{pmatrix}$$

and $H_n(\mathbf{x}) = H(\mathbf{x})$.

Example 1.12 If $f \colon \mathbb{R}^3 \to \mathbb{R}$ is given by

$$f(x,y,z) = 5x^2 + 4xy + y^3 - 2y + z^5 + xe^z$$

then

$$H(\mathbf{x}) = \begin{pmatrix} 10 & 4 & e^z \\ 4 & 6y & 0 \\ e^z & 0 & 20z^3 + xe^z \end{pmatrix}.$$

So

$$H_1(\mathbf{x}) = 10,$$

$$H_2(\mathbf{x}) = \begin{pmatrix} 10 & 4 \\ 4 & 6y \end{pmatrix}$$

and

$$H_3(\mathbf{x}) = \begin{pmatrix} 10 & 4 & e^z \\ 4 & 6y & 0 \\ e^z & 0 & 20z^3 + xe^z \end{pmatrix}.$$

The determinants of these submatrices are called principal minors and are denoted by $\Delta_k(\mathbf{x})$

$$\Delta_k(\mathbf{x}) = \det(H_k(\mathbf{x})) \qquad k = 1, \ldots, n.$$

Example 1.13 The principal minors of $f(x,y,z) = 5x^2 + 4xy + y^3 - 2y + z^5 + xe^z$ are

$$\Delta_1(\mathbf{x}) = H_1(\mathbf{x}) = 10$$

$$\Delta_2(\mathbf{x}) = \det H_2(\mathbf{x}) = 60y - 16$$

$$\Delta_3(\mathbf{x}) = \det H_3(\mathbf{x}) = e^z(-6ye^z) + (20z^3 + xe^z)(60y - 16)$$

Exercise 1.14 Determine the principal minors of each of the following functions

(a) $f(x,y,z) = xz - 5x^2y^3z^4$;

(b) $f(x,y,z) = ze^{xyz}$;

(c) $f(x,y,z) = \log(x + \sqrt{y^2 + z^2})$.

1.3.2 Second Derivative Test

Theorem 1.15 (Second Derivative Test) Let $\mathbf{p}$ be a critical point of a function $f\colon \mathbb{R}^n \to \mathbb{R}$.

- If $(-1)^k \Delta_k(\mathbf{p}) > 0$ for all $k = 1, \ldots, n$, then f has a local maximum at $\mathbf{p}$.
- If $\Delta_k(\mathbf{p}) > 0$ for all $k = 1, \ldots, n$, then f has a local minimum at $\mathbf{p}$.
- If $\Delta_n(\mathbf{p}) = \det(H(\mathbf{p})) \neq 0$, but none of the previous conditions holds, then f has a saddle point at $\mathbf{p}$.
- If $\Delta_n(\mathbf{p}) = \det(H(\mathbf{p})) = 0$, the test is inconclusive.

Exercise 1.16 Find and classify all critical points of the function

$$f(x, y, z) = 2x^2 + xy + 4y^2 + xz + z^2 + 2.$$

We have

$$f_x(x, y, z) = 4x + y + z, \qquad f_y(x, y, z) = x + 8y, \qquad f_z(x, y, z) = x + 2z.$$

Let p be a critical point of f. Then we have

$$\begin{aligned} f_x(x, y, z) = 0 &\Rightarrow 4x + y + z = 0 \\ f_y(x, y, z) = 0 &\Rightarrow x + 8y = 0 \\ f_z(x, y, z) = 0 &\Rightarrow x + 2z = 0 \end{aligned}$$

So $4x - \frac{1}{8}x - \frac{1}{2}x = 0$ which gives $3\frac{3}{8}x = 0$ and thus $x = 0$. This in turn gives $y = 0$ and $z = 0$. Therefore, the only critical point of f is $(0, 0, 0)$.

The Hessian matrix of f is given by

$$H_f(\mathbf{x}) = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

We have that the principal minors of f are

$$\Delta_1(\mathbf{x}) = 4, \qquad \Delta_2(\mathbf{x}) = 31, \qquad \Delta_3(\mathbf{x}) = 1(-8) + 2(31) = 54.$$

As $\Delta_1(\mathbf{x}) > 0$, $\Delta_2(\mathbf{x}) > 0$ and $\Delta_3(\mathbf{x}) > 0$, f has a local minimum at $(0, 0, 0)$.

The following illustrates out procedure in finding and classifying critical points of functions of several variables.

$$
\begin{tikzcd}
& \boxed{\text{Function } f \text{ of } n \text{ variables}} \arrow[d] & \\
& \boxed{\text{Determine first order derivatives of } f} \arrow[dl] \arrow[dr] & \\
\boxed{\text{Set } \nabla f = 0 \text{ and solve}} \arrow[dr] & & \boxed{\text{Find second order partial derivatives } f} \arrow[dl] \\
& \boxed{\text{Determine Hessian Matrix } H_f} \arrow[d] & \\
& \boxed{\text{Evaluate } H_f \text{ at critical points}} \arrow[d] & \\
& \boxed{\text{Evaluate Principal Minors at critical points}} \arrow[d] & \\
& \boxed{\text{Use Second Derivative Test to classify critical points}} &
\end{tikzcd}
$$

Example 1.17 Find and classify the critical points of

$$f(x, y, z) = x^2 + y^3 + z^2 - 2xy - 2yz + 1.$$

Finding Critical Points:

We have

$$f_x(x, y, z) = 2x - 2y, \qquad f_y(x, y, z) = 3y^2 - 2x - 2z,$$
$$f_z(x, y, z) = 2z - 2y.$$

Setting $f_x(x, y, z) = 0$ we get $x = y$, $f_z(x, y, z) = 0$ we get $z = y$. So $x = y = z$. Setting $f_z(x, y, z) = 0$ we get $3y^2 = 4y$ so $y = 0$ or $y = \frac{4}{3}$. Therefore we have two critical points, $(0, 0, 0)$ and $(\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$.

Classifying the Critical Points:

We have

$$
\begin{aligned}
f_{xx}(x, y, z) &= 2, & f_{xy}(x, y, z) &= -2, & f_{xz}(x, y, z) &= 0,\\
f_{yx}(x, y, z) &= -2, & f_{yy}(x, y, z) &= 6y, & f_{yz}(x, y, z) &= -2,\\
f_{zx}(x, y, z) &= 0, & f_{zy}(x, y, z) &= -2, & f_{zz}(x, y, z) &= 2.
\end{aligned}
$$

This gives a Hessian matrix of

$$H_f(x,y,z) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 6y & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

Therefore

$$H_f(0,0,0) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

giving $\Delta_1(0,0,0) = 2$, $\Delta_2(0,0,0) = -4$, $\Delta_3(0,0,0) = -16$. So by the Second Derivative Test f has a saddle point at $(0,0,0)$.

Also,

$$H_f\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 8 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

giving $\Delta_1(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) = 2$, $\Delta_2(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) = 12$, $\Delta_3(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) = 16$. So by the Second Derivative Test f has a local minimum at $(\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$.

Example 1.18 Find and classify all critical points of the function

$$f(x,y,z) = x^2 + x^2y + y^2z + z^2 - 4z.$$

We have

$$f_x(x,y,z) = 2x + 2xy, \qquad f_y(x,y,z) = x^2 + 2yz, \qquad f_z(x,y,z) = y^2 + 2z - 4.$$

Let p be a critical point of f. Then we have

$$\begin{aligned} f_x(x,y,z) = 0 &\Rightarrow 2x + 2xy = 0 \\ f_y(x,y,z) = 0 &\Rightarrow x^2 + 2yz = 0 \\ f_z(x,y,z) = 0 &\Rightarrow y^2 + 2z - 4 = 0. \end{aligned}$$

Multiply the first equation by z, the second equation by x and the third by xy to get

$$\begin{aligned} 2xz + 2xyz &= 0 \\ x^3 + 2xyz &= 0 \\ xy^3 + 2xyz - 4xy &= 0. \end{aligned}$$

Now subtract the second equation from the first, the third from the second and the third from the first to get

$$\begin{aligned} 2xz - x^3 &= 0 \\ x^3 - xy^3 + 4xy &= 0 \\ 2xz - xy^3 + 4xy &= 0. \end{aligned}$$

Therefore we have $x(x^2-2z)=0$ so either $x=0$ or $z=\frac{x^2}{2}$.

If $x=0$ then $2yz=0$ and hence $y=0$ or $z=0$. If $y=0$ then y^2+2z-4 gives $z=2$. If $z=0$ then y^2+2z-4 gives $y=\pm 2$. This gives us critical points of $(0,2,0)$, $(0,-2,0)$ and $(0,0,2)$.

If $z=\frac{x^2}{2}$ and $x\neq 0$ then $2x(1+y)=0$ gives $y=-1$. This now gives $(-1)^2+2z-4=0$ giving $z=\frac{3}{2}$. The equation $x^2=2z$ now gives $x=\pm\sqrt{3}$ giving two new critical points of $(\sqrt{3}.-1,\frac{3}{2})$ and $(-\sqrt{3}.-1,\frac{3}{2})$.

The second order partial derivatives of f are

$$\begin{array}{ccc} f_{xx}=2+2y & f_{xy}=2x & f_{xz}=0 \\ f_{yx}=2x & f_{yy}=6z & f_{yz}=2y \\ f_{zx}=0 & f_{zy}=2y & f_{zz}=2. \end{array}$$

and therefore the Hessian of f at (x,y,z) is given by

$$H_f(\mathbf{x})=\begin{pmatrix} 2+2y & 2x & 0 \\ 2x & 6z & 2y \\ 0 & 2y & 2 \end{pmatrix}.$$

We now check each of our critical points.

Let us start with $(0,2,0)$. We have

$$H_f((0,2,0))=\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 2 \end{pmatrix}.$$

Then $\Delta_1(0,2,0)=6$, $\Delta_2(0,2,0)=0$ and $\Delta_3(0,2,0)=-96$. Therefore f has a saddle point at $(0,2,0)$.

Next, we consider the point $(0,-2,0)$. We have

$$H_f((0,-2,0))=\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -4 & 2 \end{pmatrix}.$$

Then $\Delta_1(0,-2,0)=-2$, $\Delta_2(0,-2,0)=0$ and $\Delta_3(0,-2,0)=32$. Again f has a saddle point at $(0,-2,0)$.

Next, we consider the critical point $(0,0,2)$. We have

$$H_f((0,0,2))=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then $\Delta_1(0,0,2) = 2$, $\Delta_2(0,0,2) = 24$ and $\Delta_3(0,0,2) = 48$. Therefore f has a local minimum at $(0,0,2)$.

Next, we consider the critical point $(\sqrt{3}, -1, \frac{2}{2})$. We have

$$H_f((\sqrt{3}, -1, \tfrac{3}{2})) = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & 9 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

Then $\Delta_1(\sqrt{3}, -1, \frac{3}{2}) = 0$, $\Delta_2(\sqrt{3}, -1, \frac{3}{2}) = -12$ and $\Delta_3(\sqrt{3}, -1, \frac{3}{2}) = 24$. Therefore f has a saddle point at $(\sqrt{3}, -1, \frac{3}{2})$.

Finally, let us consider the critical point $(-\sqrt{3}, -1, \frac{2}{2})$. We have

$$H_f(-\sqrt{3}, -1, \tfrac{3}{2})) = \begin{pmatrix} 0 & -2\sqrt{3} & 0 \\ -2\sqrt{3} & 9 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

Then $\Delta_1(-\sqrt{3}, -1, \frac{3}{2}) = 0$, $\Delta_2(-\sqrt{3}, -1, \frac{3}{2}) = -12$ and $\Delta_3(-\sqrt{3}, -1, \frac{3}{2}) = 24$. And again f has a saddle point at $(-\sqrt{3}, -1, \frac{2}{2})$.

Example 1.19 For $a \in \mathbb{R}$ and consider the function

$$f(x,y,z) = x^2 + ay^2 + z^2 - 4xy.$$

Determine how the critical points of f and their nature depends on a.

Finding Critical Points:

We have

$$f_x(x,y,z) = 2x - 4y, \qquad f_y(x,y,z) = 2ay - 4x,$$
$$f_z(x,y,z) = 2z.$$

We see immediately from $f_z(x,y,z) = 0$ that $z = 0$.

Setting $f_x(x,y,z) = 0$ we get $x = 2y$. Setting $f_y(x,y,z) = 0$ we get $ay = 2x$. Therefore we have that $ay = 4y$ and hence either have $y = 0$ or $a = 4$. If $y = 0$ then $x = 0$ and we get the critical point $(0,0,0)$. If $a = 4$ we that any point of the form $(2y, y, 0)$ is a critical point of f.

Classifying the Critical Points:

We have

$$f_{xx}(x,y,z) = 2, \qquad f_{xy}(x,y,z) = -4, \qquad f_{xz}(x,y,z) = 0,$$
$$f_{yx}(x,y,z) = -4, \qquad f_{yy}(x,y,z) = 2a, \qquad f_{yz}(x,y,z) = 0,$$

$$f_{zx}(x,y,z) = 0, \qquad f_{zy}(x,y,z) = 0, \qquad f_{zz}(x,y,z) = 2.$$

This gives a Hessian matrix of

$$H_f(x,y,z) = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 2a & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Therefore

$$H_f(0,0,0) = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 2a & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

giving $\Delta_1(0,0,0) = 2$, $\Delta_2(0,0,0) = 4a - 16$, $\Delta_3(0,0,0) = 2(4a - 16)$. So by the Second Derivative Test f has a local minimum at $(0,0,0)$ if $a > 4$ and a saddle point $(0,0,0)$ if $a < 4$.

If $a = 4$ the Second Derivative Test is inconclusive.

When $a = 4$ we get critical points of the form $(2y, y, 0)$ for $y \in \mathbb{R}$. In this case we see that

$$H_f(2y,y,0) = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

giving $\Delta_1(2y,y,0) = 2$, $\Delta_2(2y,y,0) = 0$, $\Delta_3(2y,y,0) = 0$ which means that the Second Derivative Test is inconclusive.

When $a = 4$ we see that every points on the line $\{(2y,y,0) : y \in \mathbb{R}\}$ are all critical points of f. However, as we observe that $f(2y,y,0) = 0$ points of the form $(2y,y,0)$ are neither local maximum, local minimum of saddle points.

Exercise 1.20 Find and classify the critical points of the following functions

(a) $f(x,y,z) = x^2 + y^2 + 7z^2 - xy - 3yz$;

(b) $f(x,y,z) = x^4 + (x+y)^2 + (x+z)^2$;

(c) $f(x,y,z) = x^4 + x^2y + y^2 + z^2 + xy + 1$;

(d) $f(x,y,z) = x^3 - 2x^2 + y^2 + z^2 - 2xy + xz - yz + 3z$;

(e) $f(x,y,z,w) = -x^2 - y^2 - z^2 - w^2 + xy + wz$.

Keywords: Functions of Several Variables; Partial Derivative; Critical Point; Hessian Matrix; Second Derivatives Test.

Chapter 2

Lagrange Multipliers

2.1 Lagrange Multipliers with one constraint

[1]

Let us recall the method of Lagrange multiplies for functions of two variables and one constraint

> Problem 2.1 Find the dimensions of the rectangle with maximum area, given that the perimeter is 10 m.

We can translate this into a mathematical problem:

> Problem 2.2 For what (x, y) is the function $f(x, y) = xy$ maximal, given the constraint $g(x, y) = 2x + 2y = 10$.

The Lagrangian method tells us, that if we want to find the maximum (minimum) of $f(x, y)$ subject to $g(x, y) = c$,

We construct the Lagrangian Function

$$F(x, y, \lambda) = f(x, y) + \lambda[c - g(x, y)],$$

determine its critical points by setting

$$F_x(x, y, \lambda) = 0,$$
$$F_y(x, y, \lambda) = 0,$$
$$F_\lambda(x, y, \lambda) = 0$$

and solving.

The variable λ is known as the Lagrange Multiplier.

For the above problem we let $g(x, y) = 2x + 2y$ and set

$$\begin{aligned} F(x, y, \lambda) =& f(x, y) + \lambda[10 - g(x, y)] \\ =& xy + \lambda[10 - 2x - 2y] \end{aligned}$$

Then

$$\begin{aligned} F_x(x, y, \lambda) =& y + \lambda[-2], \\ F_y(x, y, \lambda) =& x + \lambda[-2], \\ F_\lambda(x, y, \lambda) =& 10 - 2x - 2 - y. \end{aligned}$$

Setting each of these equal to 0 we see that

$$\lambda = \frac{x}{2} = \frac{y}{2}$$

giving $x = y$. Putting this in the equation $2x + 2y = 10$ we get that $x = y = 2\frac{1}{2}$ and $\lambda = 1\frac{1}{4}$.

There are two 'challenges' with using the method of Lagrange Multipliers. The first is that there is no systematic way to solve the set of equations we obtain when we compute the partial derivatives. The second is that, unlike the Second Derivative Test, the method of Lagrange Multipliers does not tell us if our solution is a maximum, minimum or saddle point. We distinguish between them by looking at the relative values the objective function takes at the critical points.

Example 2.3 Find the maximum (minimum) of $f(x, y) = x^2 + y^2$ subject to $x^2 + xy + y^2 = 3$.

We let $g(x, y) = x^2 + xy + y^2$ and set

$$\begin{aligned} F(x, y, \lambda) =& f(x, y) + \lambda[3 - g(x, y)] \\ =& x^2 + y^2 + \lambda[3 - x^2 - xy - y^2] \end{aligned}$$

Then

$$\begin{aligned} F_x(x, y, \lambda) =& 2x + \lambda[-2x - y], \\ F_y(x, y, \lambda) =& 2y + \lambda[-x - 2y], \\ F_\lambda(x, y, \lambda) =& 3 - x^2 - xy - y^2. \end{aligned}$$

Setting each of these equal to 0 we obtain the equations

$$\begin{aligned} 2x + \lambda[-2x - y] =& 0, \\ 2y + \lambda[-x - 2y] =& 0, \\ 3 - x^2 - xy - y^2 =& 0. \end{aligned}$$

Note that $2x + y \neq 0$ as $2x + y = 0$ implies $x = 0$ and therefore $y = 0$. This is not possible as $x^2 + xy + y^2 = 3$.

Similary, $x + 2y \neq 0$.

From the first and second we get that

$$\lambda = \frac{2x}{2x + y} = \frac{2y}{2y + x}.$$

So

$$\begin{aligned} 2x(2y + x) =& 2y(2y + x) \\ \Rightarrow& 2x^2 = 2y^2 \\ \Rightarrow& x = \pm y \end{aligned}$$

If $x = y$ then $3x^2 = 3$ giving $x = \pm 1$ and $y = \pm 1$.

If $x = -y$ then $x^2 = 3$ giving $x = \pm\sqrt{3}$ and $y = \mp\sqrt{3}$.

This gives us four possible extreme points $(1, 1)$, $(-1, -1)$, $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$. We have $f(1, 1) = f(-1, -1) = 2$ while $f(\sqrt{3}, -\sqrt{3}) = f(-\sqrt{3}, \sqrt{3}) = 6$.

Hence f has a maximum at 6 which occurs at $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ and a minimum of 1 which occurs at $(1, 1)$ and $(-1, -1)$.

Example 2.4 Consider the Cobb-Douglas production function

$$q(k, l) = 25k^{1/3}l^{1/6}$$

where q, k and l denote quantity, labour and capital respectively in a certain industrial process. Maximise production subject to $10k+5l = 30$.

Let us set $f(k, l) = 25k^{1/3}l^{1/6}$, $g(k, l) = 10k + 5l$ and $c = 30$. Then our Lagrangian function is

$$F(k, l, \lambda) = 25k^{1/3}l^{1/6} + \lambda[30 - 10k - 5l].$$

We have

$$\begin{aligned} F_k(k, l, \lambda) =& \frac{25}{3}k^{-2/3}l^{1/6} - 10\lambda, \\ F_l(k, l, \lambda) =& \frac{25}{6}k^{1/3}l^{-5/6} - 5\lambda, \\ F_\lambda(k, l, \lambda) =& 30 - 10k - 5l. \end{aligned}$$

Setting each of these equal to 0 we get

$$\begin{aligned}\frac{25}{3}k^{-2/3}l^{1/6} - 10\lambda =& 0.\\ \frac{25}{6}k^{1/3}l^{-5/6} - 5\lambda =& 0.\\ 30 - 10k + 5\lambda =& 0.\end{aligned}$$

This gives that

$$\lambda = \frac{5}{6}k^{-2/3}l^{1/6} = \frac{5}{6}\frac{\sqrt[6]{l}}{\sqrt[3]{k^2}}$$

and

$$\lambda = \frac{5}{6}k^{1/3}l^{-5/6} = \frac{5}{6}\frac{\sqrt[3]{k}}{\sqrt[6]{l^5}}.$$

So

$$\lambda = \frac{5}{6}\frac{\sqrt[3]{k}}{\sqrt[6]{l^5}} = \frac{5}{6}\frac{\sqrt[6]{l}}{\sqrt[3]{k^2}}.$$

This gives $\sqrt[3]{k^3} = \sqrt[6]{l^6}$ or that $k = l$.

Putting this into $10k + 5l = 30$ we get $15k = 30$. So $k = l = 2$ and $\lambda = \frac{5}{6}\frac{\sqrt[3]{2}}{\sqrt[6]{2^5}} = \frac{5}{6}\frac{1}{\sqrt{2}}$ and $q = \sqrt{2}(25)$.

Exercise 2.5 Use the method of Lagrange Multipliers to find

(a) the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to $xy = 1$,

(b) the maximum and minimum values of $f(x, y) = 3x + y$ subject to $x^2 + y^2 = 10$,

(c) the maximum and minimum values of $f(x, y) = y^2 - x^2$ subject to $\frac{1}{4}x^2 + y^2 = 1$,

(d) the maximum and minimum values of $f(x, y) = e^{xy}$ subject to $x^2 + y^2 = 16$.

2.2 Lagrangian method with n variables and one constraint

Suppose we have a function $f\colon \mathbb{R}^n \to \mathbb{R}$ and we want to find the maximum (minimum) of $f(x_1, \ldots, x_n)$ subject to $g(x_1, \ldots, x_n) = c$. We consider the Lagrangian function

$$F(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda[c - g(x_1, \ldots, x_n)].$$

Joseph-Louis Lagrange 1736-1813

We then solve the $n+1$ equations

$$\begin{aligned}
F_{x_1}(x_1,\ldots,x_n,\lambda) =& 0\\
F_{x_2}(x_1,\ldots,x_n,\lambda) =& 0\\
\vdots \qquad & \vdots\\
F_{x_n}(x_1,\ldots,x_n,\lambda) =& 0\\
F_{\lambda}(x_1,\ldots,x_n,\lambda) =& 0
\end{aligned}$$

Example 2.6 Find the maximum of xyz subject to $x+2y+4z=12$.

Let us set $g(x,y,z)=x+2y+4z$ and $c=12$. Then our Lagrangian function is

$$F(x,y,z,\lambda)=xyz+\lambda[12-x-2y-4z].$$

We have

$$\begin{aligned}
F_x(x,y,z,\lambda) =& yz-\lambda,\\
F_y(x,y,z,\lambda) =& xz-2\lambda,\\
F_z(x,y,z,\lambda) =& xy-4\lambda,\\
F_{\lambda}(x,y,z,\lambda) =& 12-x-2y-4z.
\end{aligned}$$

Setting each of these equal to 0 we get

$$\begin{aligned} yz - \lambda =& 0, \\ xz - 2\lambda =& 0, \\ xy - 4\lambda =& 0 \\ 12 - x - 2y - 4z =& 0. \end{aligned}$$

Comparing values for λ we get

$$\lambda = yz = \frac{xz}{2} = \frac{xy}{4}.$$

For a maximum we must have $x \neq 0$, $y \neq 0$ and $z \neq 0$. So we can divide by y and z to get that

$$z = \frac{x}{4}, \qquad y = \frac{x}{2}.$$

As $x + 2y + 4z = 12$ we get that $3x = 12$ and therefore $x = 4$, $y = 2$, $z = 1$ and $\lambda = 1$. Therefore a maximum of 8 occur when $x = 4$, $y = 2$ and $z = 1$.

Example 2.7 Suppose

$$U(x, y, z) = xyz$$

is the utility function of a person consuming x, y and z units of three commodities X, Y and Z.

Suppose that X costs €1 per unit, Y costs €4 per unit and Z costs €2 unit.

(a) If a person has a budget of €48, how much of each unit should he or she buy in order to maximise utility?

(b) What is the maximum utility?

In mathematical terms the problem is to find the maximum of $U(x, y, z) = xyz$ subject to to $x + 4y + 2z = 48$.

Let us set $g(x, y, z) = x + 4y + 2z$ and $c = 48$. Then our Lagrangian function is

$$F(x, y, z, \lambda) = xyz + \lambda[48 - x - 4y - 2z].$$

We have

$$\begin{aligned} F_x(x, y, z, \lambda) =& yz - \lambda, \\ F_y(x, y, z, \lambda) =& xz - 4\lambda, \\ F_z(x, y, z, \lambda) =& xy - 2\lambda, \\ F_\lambda(x, y, z, \lambda) =& 48 - x - 4y - 2z. \end{aligned}$$

Setting each of these equal to 0 we get

$$\begin{aligned}
yz - \lambda =& 0, \\
xz - 4\lambda =& 0, \\
y - 2\lambda =& 0 \\
48 - x - 4y - 2z =& 0.
\end{aligned}$$

Comparing values for λ we get

$$\lambda = yz = \frac{xz}{4} = \frac{xy}{2}.$$

For a maximum we must have $x \neq 0$, $y \neq 0$ and $z \neq 0$. So we can divide by y and z to get that

$$z = \frac{x}{2}, \qquad y = \frac{x}{4}.$$

As $x + 4y + 2z = 48$ we get that $3x = 48$ and therefore $x = 16$, $y = 4$, $z = 8$ and $\lambda = 32$. Therefore a maximum of 512 occur when $x = 16$, $y = 4$ and $z = 8$.

Exercise 2.8 Use the method of Lagrange Multipliers to find

(a) the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x + y + z = 12$,

(b) the maximum and minimum values of $f(x, y, z) = 2x + 2y + z$ subject to $x^2 + y^2 + z^2 = 9$,

(c) the maximum and minimum values of $f(x, y, z) = x^2y^2z^2$ subject to $x^2 + y^2 + z^2 = 1$,

(d) the maximum and minimum values of $f(x, y, z) = x^4 + y^4 + z^4$ subject to $x^2 + y^2 + z^2 = 1$,

(e) the maximum and minimum values of $f(x, y, z, t) = x + y + z + t$ subject to $x^2 + y^2 + z^2 + t^2 = 1$.

2.3 The Lagrangian Method: n variables, m constraints

Consider the following problem: Find the maximum(minimum) of $f(x_1, \ldots, x_n)$ subject to $g_1(x_1, \ldots, x_n) = c_1, g_2(x_1, \ldots, x_n) = c_2, \ldots, g_m(x_1, \ldots, x_n) = c_m$.

We consider the Lagrangian function

$$\begin{aligned}
F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) =& f(x_1, \ldots, x_n) + \lambda_1[c_1 - g_1(x_1, \ldots, x_n)] \\
&+ \lambda_2[c_2 - g_2(x_1, \ldots, x_n)] + \ldots \\
&+ \lambda_m[c_m - g_m(x_1, \ldots, x_n)].
\end{aligned}$$

To find the maximum(minimum) of $f(x_1,\dots,x_n)$ subject to $g_1(x_1,\dots,x_n)=c_1, g_2(x_1,\dots,x_n)=c_2,\dots,g_m(x_1,\dots,x_n)=c_m$ we solve the $n+m$ equations

$$\begin{aligned}
F_{x_1}(x_1,\dots,x_n,\lambda_1,\dots,\lambda_m) =&0,\\
F_{x_2}(x_1,\dots,x_n,\lambda_1,\dots,\lambda_m) =&0,\\
\vdots\qquad\qquad & \vdots\\
F_{x_n}(x_1,\dots,x_n,\lambda_1,\dots,\lambda_m) =&0,\\
F_{\lambda_1}(x_1,\dots,x_n,\lambda_1,\dots,\lambda_m) =&0,\\
F_{\lambda_2}(x_1,\dots,x_n,\lambda_1,\dots,\lambda_m) =&0,\\
\vdots\qquad\qquad & \vdots\\
F_{\lambda_m}(x_1,\dots,x_n,\lambda_1,\dots,\lambda_m) =&0.
\end{aligned}$$

Example 2.9 Find the maximum (minimum) of $f(x,y,z)=x^2+y^2+z^2$ subject to

$$\begin{aligned}
x+2y+z&=1,\\
2x-y-3z&=4.
\end{aligned}$$

Our Lagrangian function is

$$F(x,y,z,\lambda_1,\lambda_2)=x^2+y^2+z^2+\lambda_1[1-x-2y-z]+\lambda_2[4-2x+y+3z].$$

Then we have

$$\begin{aligned}
F_x(x,y,z,\lambda_1,\lambda_2) =&2x-\lambda_1-2\lambda_2,\\
F_y(x,y,z,\lambda_1,\lambda_2) =&2y-2\lambda_1+\lambda_2,\\
F_z(x,y,z,\lambda_1,\lambda_2) =&2z-\lambda_1+3\lambda_2,\\
F_{\lambda_1}(x,y,z,\lambda_1,\lambda_2) =&1-x-2y-z,\\
F_{\lambda_2}(x,y,z,\lambda_1,\lambda_2) =&4-2x+y+3z.
\end{aligned}$$

Setting each of these equal to 0 we get

$$\begin{aligned}
2x&=\lambda_1+2\lambda_2,\\
2y&=2\lambda_1-\lambda_2,\\
2z&=\lambda_1-3\lambda_2,\\
x+2y+z&=1,\\
2x-y-3z&=4.
\end{aligned}$$

So, replacing x, y and z with λ_1 and λ_2 we get

$$\begin{aligned}
(\lambda_1+2\lambda_2)+2(2\lambda_1-\lambda_2)+(\lambda_1-3\lambda_2) =&2,\\
2(\lambda_1+2\lambda_2)-(2\lambda_1-\lambda_2)-3(\lambda_1-3\lambda_2) =&8.
\end{aligned}$$

or

$$\begin{aligned} 6\lambda_1 - 3\lambda_2 =& 2, \\ -3\lambda_1 + 14\lambda_2 =& 8. \end{aligned}$$

This gives that $25\lambda_2 = 18$ and hence we have that $\lambda_2 = 18/25$ and $\lambda_1 = 52/75$. Using the above equations we get that f has a minimum value of $134/15$ at

$$x = 16/15, \qquad y = 1/3, \quad \text{and} \quad z = -/15.$$

Example 2.10 Find the maximum (minimum) of $f(x, y, z) = x$ subject to .

$$\begin{aligned} z &= x + y, \\ x^2 + 2y^2 + 2z^2 &= 8. \end{aligned}$$

We let $g_1(x, y, z) = x + y - z$ and $g_2(x, y, z) = x^2 + 2y^2 + 2z^2$. Then we have to find the extreme values of $f(x, y, z) = x$ subject to $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 8$.

Our Lagrangian function is

$$F(x, y, z, \lambda_1, \lambda_2) = x + \lambda_1[z - x - y] + \lambda_2[8 - x^2 - 2y^2 - 2z^2].$$

Then we have

$$\begin{aligned} F_x(x, y, z, \lambda_1, \lambda_2) =& 1 - \lambda_1 - 2\lambda_2 x, \\ F_y(x, y, z, \lambda_1, \lambda_2) =& -\lambda_1 - 4\lambda_2 y, \\ F_z(x, y, z, \lambda_1, \lambda_2) =& \lambda_1 - 4\lambda_2 z, \\ F_{\lambda_1}(x, y, z, \lambda_1, \lambda_2) =& z - x - y, \\ F_{\lambda_2}(x, y, z, \lambda_1, \lambda_2) =& 8 - x^2 - 2y^2 - 2z^2. \end{aligned}$$

Setting each of these equal to 0 we get

$$\begin{aligned} \lambda_1 + 2\lambda_2 x &= 1, \\ \lambda_1 + 4\lambda_2 y &= 0, \\ \lambda_1 - 4\lambda_2 z &= 0, \\ z &= x + y, \\ x^2 + 2y^2 + 2z^2 &= 8. \end{aligned}$$

If $\lambda_2 = 0$ then we would get that $\lambda_1 = 1$ and $\lambda_1 = 0$ which is clearly impossible. Therefore $\lambda_2 \neq 0$ and as

$$\lambda_1 = -4\lambda_2 y = 4\lambda_2 z$$

we get that $y = -z$. Hence $x = z - y = 2z$ and $x^2 + 2y^2 + 2z^2 = 8$ gives $8z^2 = 8$ and therefore $z = \pm 1$.

When $z = 1$ we get $x = 2$, $y = -1$ and $f(2, -1, 1) = 2$.

When $z = -1$ we get $x = -2$, $y = 1$ and $f(-2, 1, -1) = -2$.

Therefore f has a minimum value of -2 at $(-2, 1, -1)$ and a maximum value of 2 at $(2, -1, 1)$.

Exercise 2.11 Use the method of Lagrange Multipliers to find

(a) the maximum and minimum values of $f(x, y, z) = x + 2y$ subject to $x + y + z = 1$ and $y^2 + z^2 = 4$,

(b) the maximum and minimum values of $f(x, y, z) = 3x - y - 3z$ subject to $x + y - z = 0$ and $x^2 + 2z^2 = 1$,

(c) the maximum and minimum values of $f(x, y, z) = yz + xy$ subject to $xy = 1$ and $y^2 + z^2 = 1$,

(d) the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x - y = 1$ and $y^2 - z^2 = 4$,

(e) the maximum and minimum values of $f(x, y, z) = x + 2y + 3z$ subject to $x - y + z = 1$ and $x^2 + y^2 = 1$.

2.4 Economic interpretation of Lagrange Multipliers

Consider the problem of finding the extreme values of $f(x, y)$ given $g(x, y) = c$. Suppose we solve the problem and that x^* and y^* are the values of x and y that give the extreme values. These values of x and y will depend on c, so

$$x^* = x^*(c),$$
$$y^* = y^*(c).$$

We assume that these are differentiable functions of c.

We also consider the function f^* of maximal values, called the (optimal) value function

$$f^*(c) = f(x^*(c), y^*(c)).$$

If we use the Lagrangian method, the Lagrange multiplier $\lambda(c)$ also depends on c.

One can show that

$$\frac{d}{dc} f^*(c) = \lambda(c).$$

We see that the rate at which the optimal value changes, if we change c, is the Lagrange multiplier $\lambda = \lambda(c)$.

In economic applications, c often is the available stock of some resource and f the profit.

The λ is then called a shadow price of the resource. If we increase c by one unit then we have that

$$\frac{f^*(c+1)-f^*(c)}{1} \approx \frac{df^*}{d\lambda}(c) = \lambda(c).$$

So the change in f^* is going from the constraint $g(x,y)=c$ to the constraint $g(x,y)=c+1$ is approximately equal to $\lambda(c)$.

Example 2.12 Consider the Cobb-Douglas production function

$$q(k,l) = 25k^{1/3}l^{1/6}$$

where q, k and l denote quantity, labour and capital respectively in a certain industrial process. This process is subject to cost constraint

$$10k + 5l = 30.$$

Estimate what happens if the cost constraint is changed to

$$10k + 5l = 31.$$

The maximum value of $q(k,l) = 25k^{1/3}l^{1/6}$ subject to the constraint $10k+5l = 30$ is obtained when $k = 2$ and $l = 2$ and gives a maximum value of $q = 25\sqrt{2}$. Note that in this case $\lambda = \frac{5}{6\sqrt{2}}$. If we change the constraint to $10k + 5l = 31$ then the maximum value of q will be approximately $q = 25\sqrt{2} + \frac{5}{6\sqrt{2}}$.

Exercise 2.13 (a) We have previously found the maximum and minimum values of $f(x,y) = 3x + y$ subject to $x^2 + y^2 = 10$. What happens if the constraint is changed to $x^2 + y^2 = 11$? What happens if the constraint is changed to $x^2 + y^2 = 9$?

(b) We have previously found the maximum and minimum values of $f(x,y) = e^{xy}$ subject to $x^2 + y^2 = 16$. What happens if the constraint is changed to $x^2 + y^2 = 17$?

(c) We have previously found the maximum and minimum values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to $x + y + z = 12$. What happens if the constraint is changed to $x + y + z = 11$?

Keywords: Constraint; Lagrange multiplier; Lagrangian; Shadow price.

Chapter 3

Linear Programming: The Graphical Method

3.1 Introduction

[1] In Linear Programming we consider the optimisation of a function over a region of $\mathbb{R}^n$ which is determined by a number of linear equations.

For example, the inequalities

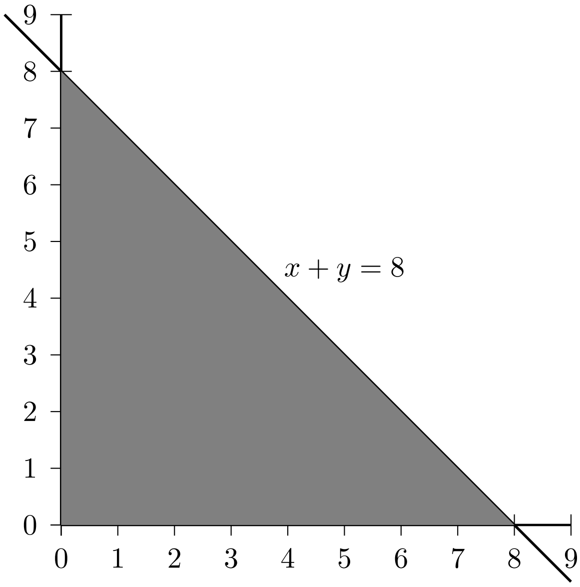
$$\begin{aligned} x + y &\leq 8 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

is the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 8)$ and $(8, 0)$.

In $\mathbb{R}^2$ to sketch the region determined by the inequality $ax + by \leq c$ we first plot the line $ax + by = c$ and then see which side the inequality lies by picking a point and seeing if the inequality is satisfied or not.

3.2 Linear Programming (General Problem)

The general problem is to find the extreme values of a linear objective function subject to linear inequality constraints.



That is,

$$\max_{x_1,\ldots,x_n} f_1x_1 + f_2x_2 + \ldots + f_nx_n$$

subject to

$$\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le c_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le c_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le c_m
\end{gathered}$$

and

$$x_1 \ge 0,\ x_2 \ge 0,\ \ldots,\ x_n \ge 0.$$

The equation

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = c_i$$

represents a hyperplane in $\mathbb{R}^n$.

For example

$$a_{i1}x_1 + a_{i2}x_2 = c_i$$

is the equation of a line in $\mathbb{R}^2$.

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 = c_i.$$

is the equation of a plane in $\mathbb{R}^3$.

The inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le c_i$$

defines all points lying on one side of that hyperplane.

The inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq c_i$$

defines all points lying on the other side of that hyperplane.

We represent a general linear inequality in $\mathbb{R}^n$ using the scalar or dot product as

$$\mathbf{v} \cdot \mathbf{x} \leq c.$$

This inequality is called slack at $\mathbf{x}$ if $<$ holds.

It is called binding at $\mathbf{x}$ if $=$ holds.

It is unsatisfied at $\mathbf{x}$ it it is not valid at $\mathbf{x}$.

In vector notation, the maximization problem is

$$\max_{\mathbf{x}} \, (\mathbf{f} \cdot \mathbf{x}),$$

subject to

$$\mathbf{A}\mathbf{x} \leq \mathbf{c} \quad \text{and} \quad \mathbf{x} \geq 0.$$

Here $\mathbf{A}$ is the $m \times n$ matrix (a_{ij}) and $\mathbf{f}$ is the vector $(f_1, \ldots, f_n)$.

The set of variables which satisfy all the constraints

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{c}, \mathbf{x} \geq 0\}$$

is called the feasible set.

The feasible set may be empty.

We shall use two methods to solve a linear programming problem the graphical method and the simplex method.

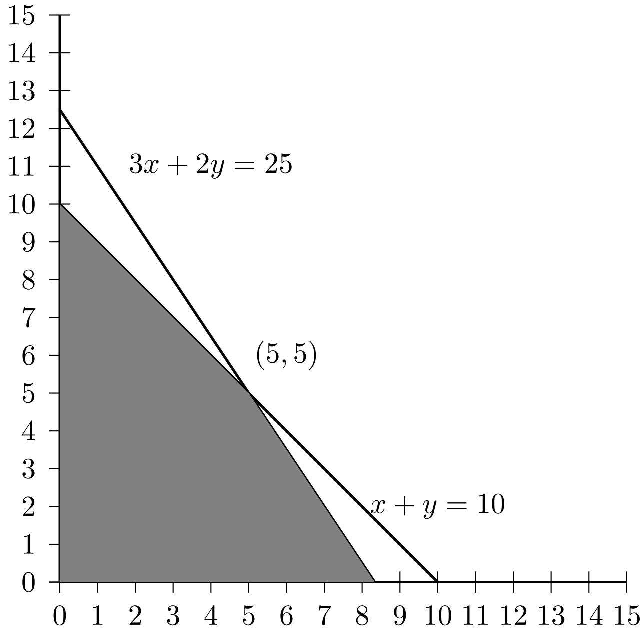


Figure 3.1: Feasible Region

3.3 The Graphical Method

Example 3.1 Maximise x subject to

$$x + y \leq 10$$
$$3x + 2y \leq 25 \; x \qquad\qquad \geq 0$$
$$y \geq 0$$

Let us first determine the feasible region.

The line $x + y = 10$ intersection the y-axis at $(0, 10)$ and the x-axis at $(10, 0)$ while the line $3x+2y = 25$ intersection the y-axis at $(0, 12.5)$ and the x-axis at $(8\frac{1}{3}, 0)$. The lines $x+y = 10$ and $3x + 2y = 25$ intersect at the point $(5, 5)$.

We know the optimal solution will occur at a vertex. As there are only finitely many vertices we can compare the relation values of f at the critical points.

(x,y)	f(x,y)
(0,0)	0
(0,10)	0
(5,5)	5
($8\frac{1}{3}$,0)	$8\frac{1}{3}$

So we see that the maximum value of f over the feasible region is $8\frac{1}{3}$ which occurs at $(8\frac{1}{3}, 0)$.

For the Graphical Method it is important that we can visualise the feasible set. This is

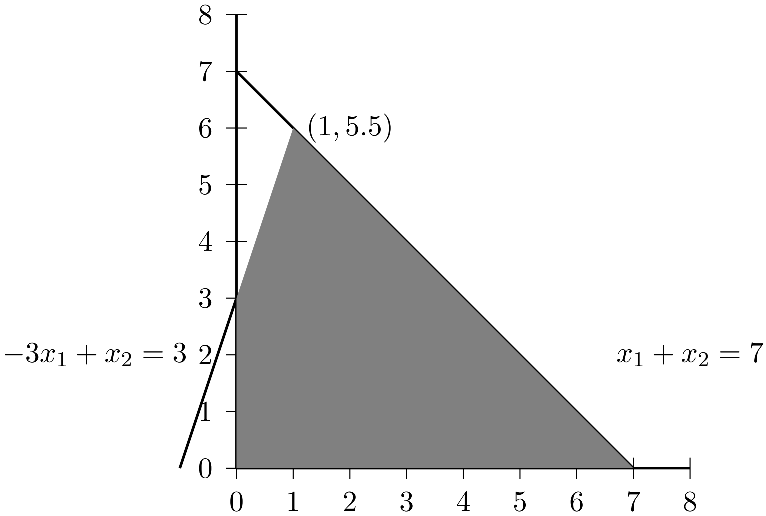


Figure 3.2: Feasible Region

possible in $\mathbb{R}^2$. It is not so easy in $\mathbb{R}^3$. In $\mathbb{R}^4$ it is almost impossible.

The Graphical Method consists of the following steps:

(a) Make a diagram for the feasible set.

(b) Plot the level sets of the (cost) function.

(c) Determine the highest (or lowest) level set that still has a point in common with the feasible set.

Corner points or vertices of the feasible set are those points on the boundary of the feasible set, that are the intersection points between two constraint lines.

The common direct of the level sets of the objective function is called the preference direction.

Note that the optimal solution to a linear programming problem can always be found at the vertices of the feasible set.

Example 3.2 The region A consists of all (x_1, x_2) satisfying $-3x_1 + x_2 \leq 3$, $x_1 + x_2 \leq 7$, $x_1 \geq 0$ and $x_2 \geq 0$.
Solve the following problems with A as the feasible region:

(i) $\max\ 3x_1 + 2x_2$,

(ii) $\max\ x_1 - x_2$.

We first we determine the feasible region. The line $x_1 + x_2 = 7$ intersects the y-axis at $(0, 7)$ and the x-axis at $(7, 0)$ while the line $-3x_1 + x_2 = 3$ intersects the y-axis at $(0, 3)$ and the x-axis at $(-1, 0)$. The lines $x_1 + x_2 = 7$ and $-3x_1 + x - 2 = 3$ intersect at the point $(1, 6)$.

In this problem our vertices are $(0,0)$, $(0,3)$, $(1,5\frac{1}{2})$ and $(7,0)$. We can compare the relation values of the functions at the critical points.

(x_1, x_2)	$3x_1 + 2x_2$	$x_1 - x_2$
(0,0)	0	0
(0,3)	6	-3
$(1, 5\frac{1}{2})$	14	$-4\frac{1}{2}$
(7,0)	21	7

So we see that the maximum value of $3x_1 + 2x_2$ over the feasible region is 21 which occurs at $(7,0)$. The maximum value of $x_1 - x_2$ is 7 which also occurs at $(7,0)$.

Exercise 3.3

1. Use the Graphical Method to maximise $3x + 4y$ subject to

$$\begin{aligned} 2.5x + y &\leq 20 \\ 3x + 3y &\leq 30 \\ x + 2y &\leq 16 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

2. Use the Graphical Method to maximise $x + 1.5y$ subject to

$$\begin{aligned} 2x + 2y &\leq 16 \\ x + 2y &\leq 12 \\ 4x + 2y &\leq 28 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

3. Use the Graphical Method to maximise $50x_1 + 40x_2$ subject to

$$\begin{aligned} 4x_1 + 4x_2 &\leq 560 \\ 3x_1 + 2x - 2 &\leq 400 \\ 2x_1 + 4x_2 &\leq 400 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

4. Use the Graphical Method to maximise $24x + 8y$ subject to

$$\begin{aligned} 2x + 5y &\leq 40 \\ 4x + y &\leq 20 \\ 10x + 5y &\leq 60 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

5. Use the Graphical Method to maximise $50x + 120y$ subject to

$$\begin{aligned} x + y &\leq 110 \\ 100x + 200y &\leq 10,000 \\ 10x + 50y &\leq 1,200 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

An LP problem may not give us a maximum or minimum. This can be for two reasons:

(a) The feasible set is empty;

(b) The feasible set is unbounded.

Exercise 3.4 There are no points which satisfy the following constraints

$$\begin{aligned} x + y &\leq 1 \\ 2x + 3y &\geq 6 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

3.4 No Solution to Linear Programming Problem

Assume there are two food items, milk and bread, which cost €0.60 and €1 per unit. Assume that the nutrient content of milk and bread is:

Nutrient	unit milk	unit bread	daily requirement
calcium	10 mg	4 mg	20 mg
protein	5 g	5 g	20 g
vitamin B	2 mg	6 mg	12 mg

Minimize the cost of a diet consisting of milk and bread, while ensuring that it provides adequate amounts of the listed nutrients.

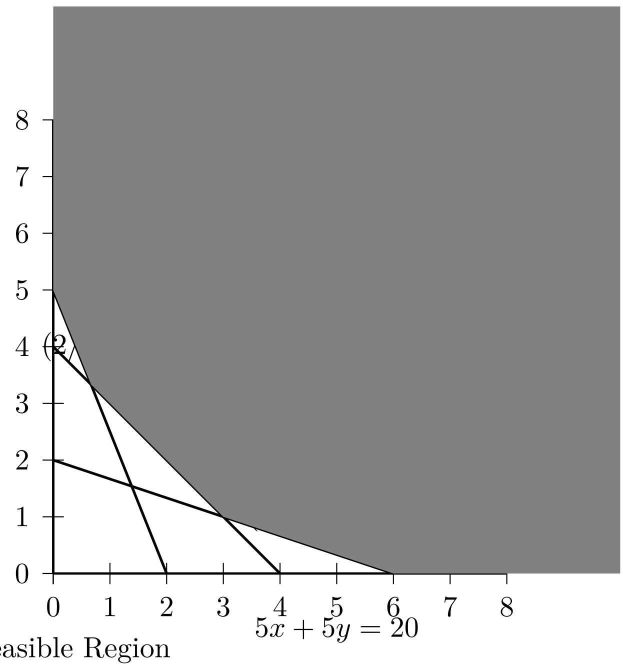


Figure 3.3: Feasible Region

Let x be the number of units of milk consumed, y the number of units of bread. We can rewrite the above problem as

Minimise $C(x, y) = 0.6x + y$,

$$\begin{aligned} 10x + 4y &\geq 20, \\ 5x + 5y &\geq 20, \\ 2x + 6y &\geq 12, \\ x, y &\geq 0. \end{aligned}$$

The line $10x + 4y = 20$ intersection the y-axis at $(0, 5)$ and the x-axis at $(2, 0)$, the line $5x+5y = 20$ intersection the y-axis at $(0, 4)$ and the x-axis at $(4, 0)$ while the line $2x+6y = 12$ intersection the y-axis at $(0, 2)$ and the x-axis at $(6, 0)$.

The lines $10x + 4y = 20$ and $5x + 5y = 20$ intersect at the point $(2/3, 10/3)$. The lines $10x + 4y = 20$ and $2x + 6y = 12$ intersect at the point $(18/13, 20/13)$. The lines $5x + 5y = 20$ and $2x + 6y = 12$ intersect at the point $(3, 1)$.

(x, y)	$0.6x + y$
(0,5)	5
(2/3,10/3)	3.7333
(3,1)	2.8
(6,0)	3.6

So the minimum is 2.8 which occurs at $(3, 1)$.

Because the feasible set in unbounded, C does not have a maximum.

Exercise 3.5 (a) A speciality steel manufacturer produces two types of steel, g_1 and g_2. Type 1 requires 2 hours of melting, 4 hours of rolling and 10 hours of cutting. Type 2 requires 5 hours of melting, 1 hour of rolling and 5 hours of cutting. Forty hours are available of melting, 20 for rolling and 60 for cutting. The profit margin for type 1 is 24; for type 2 it is 8. Reduce the data to the equations and inequalities necessary to determine the output mix that will maximise profits. Use the Graphical Method to determine the mix of type 1 and 2 that will maximise profit.

(b) Lisa makes sure that her chickens get at least 24 units of iron and 8 units of vitamins each day. Corn provides 2 units of iron and 5 units of vitamins. Bone meal provides 4 units of iron and 1 unit of vitamins. Millet provides 2 units of iron and 1 unit of vitamins. How should the feeds be mixed to provide the least-cost satisfaction of daily requirements if feeds are €40, €20 and €60.

Keywords: Inequalities; Feasible Region; Graphical Method.

Chapter 4

Linear Programming: The Simplex Method

4.1 Introduction

[1] In the previous chapter we used the Graphic Method to find the solution to a linear programming problem. This method required us to be able to visualise the feasible region. This is possible when we have two variables. With three variables it becomes more difficult. With four or more it is impossible. Because of this, when we want to solve a linear programming problem with multiple variable a new approach is required. In this chapter we will look at this method. It is know as the Simplex Method. The Simplex Method is an efficient way to search for the optimal solution. We start at a feasible vertex and only proceed to another vertex if this value is better than out current one. Eventually the optimal vertex is found.

4.2 Standard Form

n order to solve a linear programming problem using the Simplex Method we require that the problem is stated in Standard Form. For this

I The right side of a constraint cannot be negative;

II All constraints must be stated as equations;

III All variables are restricted to non-negative values.

Any linear programming problem which is not is Standard Form can be rewritten so that it appears in standard form.

A linear programming problem which does not satisfy II can be changed to Standard Form by introducing new variables. A "less than or equal to" can be changed to an equality by adding a slack variable s.

Example 4.1 The constraint $7x_1 + 4x_2 \leq 61$ is rewritten as $7x_1 + 4x_2 + s_1 = 61$, with $s_1 \geq 0$.

A "greater than or equal to" can be changed to an equality by subtracting a surplus variable e and adding an artificial variable a.

Example 4.2 The constraint $2x_1 + x_2 \geq 17$ is rewritten as $2x_1 + x_2 - e_2 + a_2 = 17$, with $e_2 \geq 0$, $a_2 \geq 0$.

For "equal to" constraints an artificial variable a is added to the left-hand-side of the constraint.

Example 4.3 The constraint $4x_1 + 3x_2 = 81$ is rewritten as $4x_1 + 3x_2 + a_3 = 81$, with $a_3 \geq 0$.

Example 4.4 Transform the following set of constraints into the standard form

$$\begin{aligned} x_1 + 3x_2 &\leq 110 \\ 2x_1 + 5x_2 &\geq 35 \\ x_1 - 3x_2 &= 19 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:

$$\begin{aligned} x_1 + 3x_2 + s_1 &= 110 \\ 2x_1 + 5x_2 - e_2 + a_2 &= 35 \\ x_1 - 3x_2 + a_3 &= 19 \\ x_1, x_2, s_1, e_2.a_2, a_3 &\geq 0 \end{aligned}$$

We have three types of variables s slack variables,

e surplus variables.

a artificial variables.

If Requirement I is not satisfied we multiply the inequality by -1.

Example 4.5 The constraint $3x_1 - 5x_2 \leq -72$ becomes the constraint $-3x_1 + 5x_2 \geq 72$.

(We then subtract a surplus variable and add an artificial variable.)

Requirement III states that ALL variables are non-negative. This includes slack, surplus and artificial variables.

Example 4.6 Rewrite the following constraint set in standard form.

$$\begin{aligned}
x_1 + 3x_2 + 6x_3 &\geq 8 \\
3x_1 - 5x_3 &\leq -1 \\
4x_1 - 2x_2 + 11x_3 &\leq 21 \\
x_1 + 2x_3 &= 5 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

Solution

$$\begin{aligned}
x_1 + 3x_2 + 6x_3 - e_1 + a_1 &= 8 \\
-3x_1 + 5x_3 - e_2 + a_2 &= 1 \\
4x_1 - 2x_2 + 11x_3 + s_3 &= 21 \\
x_1 + 2x_3 + a_4 &= 5 \\
x_1, x_2, x_3, e_1, a_1, e_2, a_2, s_3, a_4 &\geq 0
\end{aligned}$$

Exercise 4.7 (a) Transform the following set of constraints into the standard form

$$\begin{aligned}
x_1 - 3x_2 &\leq 210 \\
4x_1 - 5x_2 &\geq -23 \\
2x_1 - 3x_2 &= 41 \\
x_1, x_2 &\geq 0
\end{aligned}$$

(b) Transform the following set of constraints into the standard form

$$\begin{aligned}
5x_1 - 6x_2 + 8x_3 &\leq -45 \\
-x_1 + 12x_2 - 7x_3 &= -12 \\
8x_1 - 10x_2 + 181 &= 14 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

(c) Transform the following set of constraints into the standard form

$$\begin{aligned}
12x_1 + 15x_2 - 2x_3 &\geq 45 \\
11x_1 - 10x_2 + 70x_3 &\geq 2 \\
9x_1 + x_2 - 12 &\geq 0 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

(d) Transform the following set of constraints into the standard form

$$\begin{aligned} x_1 - 6x_2 - 3x_3 &\geq -15 \\ x_1 - 2x_2 + 4x_3 &\geq 0 \\ 3x_1 + 13x_2 - 5 &\leq 0 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Exercise 4.8 (a) A Linear Programming problem has 5 (decision) variables, 10 ($\leq$) constraints,
4 ($\geq$) constraints and 3 ($=$) constraints.

If we transform the problem into standard form (assuming they already meet requirement I), how many variables will there be?

(b) A Linear Programming problem has 35 (decision) variables, 210 ($\leq$) constraints,
26 ($\geq$) constraints and 23 ($=$) constraints.

If we transform the problem into standard form (assuming they already meet requirement I), how many variables will there be?

4.3 Gaussian reduction from linear algebra (Recall)

To determine the inverse of an $(n \times n)$ matrix A, we have to

(a) Augment the matrix A with an $(n \times n)$ identity matrix, to get

$$(A \mid I).$$

(b) Perform row operations on the entire augmented matrix, to transform A into the $(n \times n)$ identity matrix. The resulting matrix will have the form

$$(I \mid A^{-1}).$$

This method is called the Gaussian reduction procedure. It allows us to do only the elementary row operations stated below

(a) swap rows

(b) multiply or divide each element in a a row by a non-zero constant

(c) add a multiple of one row to another row

Example 4.9 Find the inverse of the matrix

$$\begin{pmatrix} 0 & 3 & 1 \\ 1 & 1 & 0 \\ 2 & 3 & 3 \end{pmatrix}$$

The augmented matrix is

$$\overline{A} = \left(\begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\overline{A} \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & \text{-}2 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & \text{-}2 & 1 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - 3R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & \text{-}8 & 0 & \text{-}3 & \text{-}2 & 1 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \to -\frac{1}{8} R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3/8 & 1/4 & \text{-}1/8 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 3R_2]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \text{-}3/8 & 3/4 & 1/8 \\ 0 & 1 & 0 & 3/8 & 1/4 & \text{-}1/8 \\ 0 & 0 & 1 & \text{-}1/8 & \text{-}3/4 & 3/8 \end{array}\right)$$

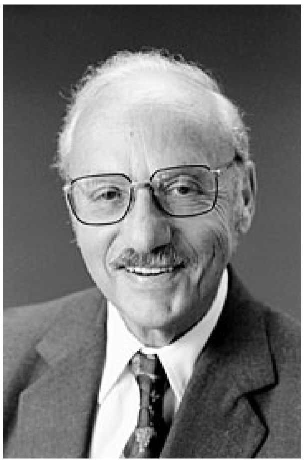
$$\text{Hence,} \quad A^{-1} = \begin{pmatrix} \text{-}3/8 & 3/4 & 1/8 \\ 3/8 & 1/4 & \text{-}1/8 \\ \text{-}1/8 & \text{-}3/4 & 3/8 \end{pmatrix}$$

Exercise 4.10 Use Gaussian elimination to determine the inverses of the following matrices

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ 1 & 7 & -2 \end{pmatrix}.$$

4.4 The Simplex Method

In a linear programming problem each pair of constraints determines a vertex of the problem. However, not all of these vertices may be in the feasible set.



George Dantzig 1914-2005

[2]

If we consider the set of constraints

$$\begin{aligned} x + 2y &\leq 6, \\ 2x + y &\leq 6, \\ x, y &\geq 0, \end{aligned}$$

then the points $(0, 0)$, $(3, 0)$, $(0, 3)$ and $(2, 2)$ correspond to vertices of the feasible set. The points $(0, 6)$ and $(6, 0)$ are intersections of lines bit are not in the feasible set.

The optimal solution to a linear programming problem will always occurs at a vertex. One way of finding the extreme value of the objective function is to check each of the vertices of the feasible set. This can be a lot of points.

The Simplex Method is a more efficient way to search for the optimal solution. We start at a feasible vertex and proceed through the feasible vertices until the extreme value is found.

[2]George Dantzig developed the Simplex Method in 1947 while working on problems for the US Air Force

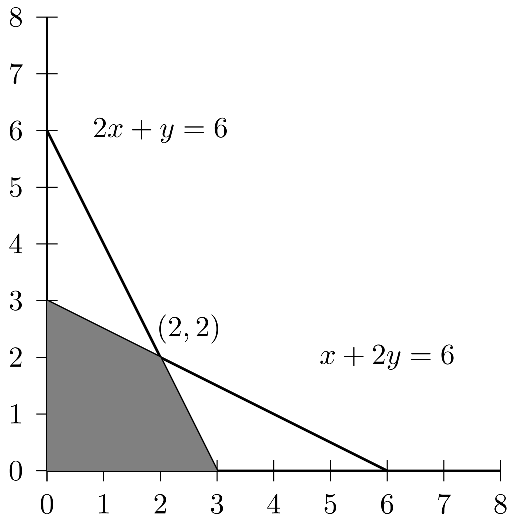


Figure 4.2: Feasible Region

The Simplex Method will only proceed to a new vertex if the solution at the new vertex is better than that which is offered at the present one.

The basic tool which we use is Gaussian Elimination. Before we present the theory we consider and example.

Example 4.11 Consider the problem of maximizing $3x + 2y$ subject to

$$\begin{aligned} 2x + y &\le 6, \\ x + 2y &\le 8, \\ x, y &\ge 0. \end{aligned}$$

First we bring the constraints into standard form:
we introduce two slack variables s_1 and s_2, to get

$$\begin{array}{rcrcrcrl} 2x &+& y &+& s_1 & & &= 6 \\ x &+& 2y & & &+& s_2 &= 8 \end{array}$$

with

$$x, y, s_1, s_2 \ge 0.$$

We wish to optimize

$$z = 3x + 2y.$$

subject to

$$\begin{array}{rcrcrcrcrl} z &-& 3x &-& 2y & & & & &= 0 \\ & & 2x &+& y &+& s_1 & & &= 6 \\ & & x &+& 2y & & &+& s_2 &= 8 \end{array}$$

These are represented in tabular format (tableau).

z	x	y	s_1	s_2	c_i	Row number
1	-3	-2	0	0	0	(0)
0	2	1	1	0	6	(1)
0	1	2	0	1	8	(2)

Note that there is one row for each equation and the table contains the coefficients of each variable in the equations. The c_i column contains the right-hand-side constants of the equations.

Our starting point is obtained by putting x and y equal to 0. In this problem, the initial solution is $s_1 = 6$, $s_2 = 8$ and $z = 0$. This solution is feasible, because we made sure that all the entries in the c_i column are non-negative.

This solution, however, is not maximal: we can increase the value of y (or x) a bit and still have a feasible solution and a higher value of z. This leads to the following rule:

> Rule 1 The optimal solution has been found if all coefficients in row (0) are ≥ 0. If any of the coefficients is negative, a better solution can be found by increasing the value of the corresponding variable.

In the above example, the maximal solution has not yet been found. he best improvement can be obtained by choosing the column with the most negative coefficient in row (0).

> Rule 2 The key column is the one having the most negative coefficient in row (0).

In this example this is the column that correspond to the x variable.

Suppose the coefficients in the key column are called a_{ij} $(i = 1, \ldots, m)$. Pick the row i such that $a_{ij} > 0$ and c_i/a_{ij} is minimal.

> Rule 3 For the key column, column j, we determine the row i associated with
> $$\min_i \frac{c_i}{a_{ij}}$$
> where $a_{ij} > 0$.

In this example this is row (1).