The Simplex Method

Linear Programming

In order to solve a linear programming problem using the Simplex Method we require that the problem is stated in *Standard Form.* For this

I The right side of a constraint cannot be negative;

II All constraints must be stated as equations;

III All variables are restricted to non-negative values.

Standard Form

A linear programming problem which does not satisfy II can be changed to Standard Form by introducing new variables. A "less than or equal to" can be changed to an equality by adding a *slack variable* s.

Example

The constraint $7x_1 + 4x_2 \leq 61$ is rewritten as $7x_1 + 4x_2 + s_1 = 61$, with $s_1 \geq 0$.

Standard Form

A "greater than or equal to" can be changed to an equality by subtracting a *surplus variable* e and adding an *artificial variable* a.

Example

The constraint $2x_1 + x_2 \geq 17$ is rewritten as
$2x_1 + x_2 - e_2 + a_2 = 17$, with $e_2 \geq 0$, $a_2 \geq 0$.

Standard Form

For “equal to” constraints an artificial variable a is added to the left-hand-side of the constraint.

Example

The constraint $4x_1 + 3x_2 = 81$ is rewritten as $4x_1 + 3x_2 + a_3 = 81$, with $a_3 \geq 0$.

Standard Form

Transform the following set of constraints into the standard form

$$
\begin{aligned}
x_1 + 3x_2 &\leq 110 \\
2x_1 + 5x_2 &\geq 35 \\
x_1 - 3x_2 &= 19 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Solution:

$$
\begin{aligned}
x_1 + 3x_2 + s_1 &= 110 \\
2x_1 + 5x_2 - e_2 + a_2 &= 35 \\
x_1 - 3x_2 + a_3 &= 19 \\
x_1, x_2, s_1, e_2.a_2, a_3 &\geq 0
\end{aligned}
$$

Introduced Variables

s slack variables,

e surplus variables,

a artificial variables.

Standard Form

If Requirement I is not satisfied we multiply the inequality by -1.

Example: The constraint $3x_1 - 5x_2 \leq -72$ becomes the constraint $-3x_1 + 5x_2 \geq 72$. (We then subtract a surplus variable and add an artificial variable.)

Standard Form

Requirement III states that ALL variables are non-negative. This includes slack, surplus and artificial variables.

Example

Example

Rewrite the following constraint set in standard form.

$$
\begin{aligned}
x_1 + 3x_2 + 6x_3 &\geq 8 \\
3x_1 - 5x_3 &\leq -1 \\
4x_1 - 2x_2 + 11x_3 &\leq 21 \\
x_1 + 2x_3 &= 5 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Solution

Solution

$$
\begin{aligned}
x_1 + 3x_2 + 6x_3 - e_1 + a_1 &= 8 \\
-3x_1 + 5x_3 - e_2 + a_2 &= 1 \\
4x_1 - 2x_2 + 11x_3 + s_3 &= 21 \\
x_1 + 2x_3 + a_4 &= 5 \\
x_1, x_2, x_3, e_1, a_1, e_2, a_2, s_3, a_4 &\geq 0
\end{aligned}
$$

Standard From

An LP problem has 5 (decision) variables, 10 ($\leq$) constraints, 4 ($\geq$) constraints and 3 ($=$) constraints.
If we transform the problem into standard form (assuming they already meet requirement I), how many variables will there be?

Recall: Gaussian reduction from linear algebra

To determine the inverse of an $(n \times n)$ matrix A, we have to

1. Augment the matrix A with an $(n \times n)$ identity matrix, to get
$$(A \mid I).$$
2. Perform row operations on the entire augmented matrix, to transform A into the $(n \times n)$ identity matrix. The resulting matrix will have the form
$$(I \mid A^{-1}).$$

Standard From

This method is called the Gaussian reduction procedure. It allows us to do only the elementary row operations stated below

1. swap rows
2. multiply or divide each element in a row of A by a non-zero constant
3. add a multiple of one row to another row

Recall: Gaussian reduction from linear algebra

Example

Find the inverse of the matrix

$$\begin{pmatrix} 0 & 3 & 1 \\ 1 & 1 & 0 \\ 2 & 3 & 3 \end{pmatrix}$$

The augmented matrix is

$$\overline{A} = \left(\begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\overline{A} \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 0 & 1 \end{array}\right)$$

Recall: Gaussian reduction from linear algebra

$$\xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & -2 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & -2 & 1 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - 3R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -8 & 0 & -3 & -2 & 1 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \to -\frac{1}{8} R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3/8 & 1/4 & -1/8 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 3R_2]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/8 & 3/4 & 1/8 \\ 0 & 1 & 0 & 3/8 & 1/4 & -1/8 \\ 0 & 0 & 1 & -1/8 & -3/4 & 3/8 \end{array}\right)$$

Recall: Gaussian reduction from linear algebra

Hence, $A^{-1} = \begin{pmatrix} -3/8 & 3/4 & 1/8 \\ 3/8 & 1/4 & -1/8 \\ -1/8 & -3/4 & 3/8 \end{pmatrix}$

The Simplex Method

In a linear programming problem each pair of constraints determines a vertex of the problem. However, not all of these vertices may be in the feasible set.

Example

If we consider the set of constraints

$$\begin{aligned} x + 2y &\leq 6, \\ 2x + y &\leq 6, \\ x, y &\geq 0, \end{aligned}$$

then the points $(0, 0)$, $(3, 0)$, $(0, 3)$ and $(2, 2)$ correspond to vertices of the feasible set. The points $(0, 6)$ and $(6, 0)$ are intersections of lines bit are not in the feasible set.

The Simplex Method

One way of finding the extreme value of the objective function is to check each of the vertices of the feasible set. This can be a lot of points. The Simplex Method is a more efficient way to search for the optimal solution. We start at a feasible vertex and proceed through the feasible vertices until the extreme value is found. The basic tool which we use is Gaussian Elimination. Before we present the theory we consider and example.

Example

Consider the problem of maximizing $3x + 8y$ subject to

$$
\begin{aligned}
x + 3y &\leq 15, \\
4x + y &\leq 16, \\
x, y &\geq 0.
\end{aligned}
$$

First we bring the constraints into standard form:
we introduce two slack variables s_1 and s_2, to get

$$
\begin{array}{rcrcrcrl}
x & + & 3y & + & s_1 & & & = 15 \\
4x & + & y & & & + & s_2 & = 16
\end{array}
$$

with

$$x, y, s_1, s_2 \geq 0.$$

Example

We wish to optimize

$$z = 3x + 8y.$$

subject to the above constraints. We write this as solving the following equations

$$\begin{aligned} z - 3x - 8y \quad\quad\quad\quad &= 0 \\ x + 3y + s_1 \quad\quad &= 15 \\ 4x + y \quad\quad + s_2 &= 16 \end{aligned}$$

These are represented in tabular format (**tableau**).

z	x	y	s_1	s_2	c_i	Row number
1	-3	-8	0	0	0	(0)
0	1	3	1	0	15	(1)
0	4	1	0	1	16	(2)

Example

Note that there is one row for each equation and the table contains the coefficients of each variable in the equations. The c_i column contains the right-hand-side constants of the equations.

Example

Our starting point is obtained by putting x and y equal to 0. In this problem, the initial solution is $s_1 = 15$, $s_2 = 16$ and $z = 0$. This solution is feasible, because we made sure that all the entries in the c_i column are non-negative.
This solution, however, is not maximal: we can increase the value of y (or x) a bit and still have a feasible solution and a higher value of z. This leads to the following rule:

Rule (1)

The optimal solution has been found if all coefficients in row (0) are ≥ 0. *If any of the coefficients is negative, a better solution can be found by increasing the value of the corresponding variable.*

In the above example, the maximal solution has not yet been found.

Example

The best improvement can be obtained by choosing the column with the most negative coefficient in row (0).

Rule (2)

The **key column** *is the one having the most negative coefficient in row (0).*

In this example this is the column that correspond to the y variable.
Suppose the coefficients in the key column are called a_{ij} $(i = 1, \ldots, m)$. Pick the row i such that $a_{ij} > 0$ and c_i/a_{ij} is minimal.

Example

Rule (3)

For the key column, column j, we determine the row i associated with

$$\min_i \frac{c_i}{a_{ij}}$$

where $a_{ij} > 0$.

In this example this is row (1).

Examples

We now use row operations to make the coefficient in the row we found with Rule (3) 1 and all the other coefficients in the key column 0:
We divide the key row by a suitable multiple to get 1 in the key column, and we add suitable multiples of the key row to the other rows, to get 0 in the key column.
In this example we get

z	x	y	s_1	s_2	c_i	Row number
1	-1/3	0	8/3	0	40	(0)
0	1/3	1	1/3	0	5	(1)
0	11/3	0	-1/3	1	11	(2)

On this new tableau we repeat rules (1)–(3).

The solution is yet not maximal as the negative coefficient for y variable in row (0) is negative. The key column is the one corresponding to x and the row in Rule (3) is row (2).
The new tableau becomes

z	x	y	s_1	s_2	c_i	Row number
1	0	0	29/11	1/11	41	(0)
0	0	1	4/11	-1/11	4	(1)
0	1	0	-1/11	3/11	3	(2)

Using Rule (1) we see that this solution is optimal, as all the coefficients in row (0) are ≥ 0.

Finding the optimal solution

Once we' have found the tableau that corresponds to the optimal solution, we read that solution off as follows:

1. all variables that have a non-zero coefficient in row (0) are set to 0;
2. the corresponding values for the other variables can be read off from the other rows.

In this example this gives $x = 3$, $y = 4$, $s_1 = 0$, $s_2 = 0$. The maximum value is 41

Summary of rules in simplex method

We have seen the following rules:

Rule (1)

The optimal solution has been found if all coefficients in row (0) are ≥ 0*. If any of the coefficients is negative, a better solution can be found by increasing the value of the corresponding variable.*

Rule (2)

The **key column** *is the one having the most negative coefficient in row (0).*

Rule (3)

For the key column, column j*, we determine the row* i *associated with*

$$\min_i \frac{c_i}{a_{ij}} \qquad \text{where } a_{ij} > 0.$$

Summary of the simplex method

First bring the constraints into standard form by adding slack variables to each constraint. Also rewrite the objective function. Place all coefficients and right-hand-side constants in a simplex tableau. Then:

1. Determine if the current solution is optimal (rule 1). If it is, stop!
2. Determine the key column (rule 2).
3. Determine the key row (rule 3).
4. Apply Gaussian elimination to generate a new tableau.
5. Go to step 1.

From the final tableau we can then read off the solution.

An example

Maximize $z = 5x_1 + 2x_2 + 8x_3$ subject to

$$
\begin{aligned}
2x_1 - 4x_2 + x_3 &\leq 42, \\
2x_1 + 3x_2 - x_3 &\leq 42, \\
6x_1 - x_2 + 3x_3 &\leq 42, \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

Adding slack variables the problem becomes maximize $z = 5x_1 + 2x_2 + 8x_3$ subject to

$$
\begin{aligned}
2x_1 - 4x_2 + x_3 + s_1 &= 42, \\
2x_1 + 3x_2 - x_3 + s_2 &= 42, \\
6x_1 - x_2 + 3x_3 + s_3 &= 42, \\
x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0.
\end{aligned}
$$

An example

The initial simplex tableau is

z	x_1	x_2	x_3	s_1	s_2	s_3	c_i	Row
1	-5	-2	-8	0	0	0	0	‘(0)
0	2	-4	1	1	0	0	42	(1)
0	2	3	-1	0	1	0	42	(2)
0	6	-1	3	0	0	1	42	(3)

This solution is not optimal. The key column is the x_3-column
We take row (3).
We perform the operations $R_3 \to \frac{1}{3}R_3,\ R_0 \to R_0 + 8R_3$, $R_1 \to R_1 - R_3$ and $R_2 \to R_2 + R_3$ to get the tableau

z	x_1	x_2	x_3	s_1	s_2	s_3	c_i	Row
1	11	$-\frac{14}{3}$	0	0	0	$\frac{8}{3}$	112	(0)
0	0	$-\frac{11}{3}$	0	1	0	$-\frac{1}{3}$	28	(1)
0	4	$\frac{8}{3}$	0	0	1	$\frac{1}{3}$	56	(2)
0	2	$-\frac{1}{3}$	1	0	0	$\frac{1}{3}$	14	(3)

An example

z	x_1	x_2	x_3	s_1	s_2	s_3	c_i	Row
1	11	$-\frac{14}{3}$	0	0	0	$\frac{8}{3}$	112	(0)
0	0	$-\frac{11}{3}$	0	1	0	$-\frac{1}{3}$	28	(1)
0	4	$\frac{8}{3}$	0	0	1	$\frac{1}{3}$	56	(2)
0	2	$-\frac{1}{3}$	1	0	0	$\frac{1}{3}$	14	(3)

This solution is still not optimal. The key column is the one that belongs to x_2. We take row (2).
We perform the operations $R_2 \to \frac{3}{8}R_2$, $R_0 \to R_0 + \frac{14}{3}R_2$, $R_1 \to R_1 + \frac{11}{3}R_2$ and $R_3 \to R_3 + \frac{1}{3}R_2$ to get

z	x_1	x_2	x_3	s_1	s_2	s_3	c_i	Row
1	18	0	0	0	$\frac{7}{4}$	$\frac{13}{4}$	210	(0)
0	$\frac{11}{2}$	0	0	1	$\frac{11}{8}$	$\frac{1}{8}$	105	(1)
0	$\frac{3}{2}$	1	0	0	$\frac{3}{8}$	$\frac{1}{8}$	21	(2)
0	$\frac{5}{2}$	0	1	0	$\frac{1}{8}$	$\frac{3}{8}$	21	(3)

An example

z	x_1	x_2	x_3	s_1	s_2	s_3	c_i	Row
1	18	0	0	0	$\frac{7}{4}$	$\frac{13}{4}$	210	(0)
0	$\frac{11}{2}$	0	0	1	$\frac{11}{8}$	$\frac{1}{8}$	105	(1)
0	$\frac{3}{2}$	1	0	0	$\frac{3}{8}$	$\frac{1}{8}$	21	(2)
0	$\frac{5}{2}$	0	1	0	$\frac{1}{8}$	$\frac{3}{8}$	21	(3)

The tableau is now in optimal form. This optimal solution is: the maximum is 210 when $x_1 = 0$, $x_2 = 21$, $x_3 = 21$.($s_1 = 105$, $s_2 = 0$, $s_3 = 0$

Another example

Consider the problem of maximizing $z = 2x_1 + x_2$ subject to

$$\begin{aligned} 4x_1 + 3x_2 &\leq 12, \\ 4x_1 + x_2 &\leq 12, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Adding slack variables this becomes maximizing $z = 2x_1 + x_2$ subject to

$$\begin{aligned} 4x_1 + 3x_2 + s_1 &= 12, \\ 4x_1 + x_2 + s_2 &= 12, \\ x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

Example

Our initial tableau is

z	x_1	x_2	s_1	s_2	c_i	
1	-2	-1	0	0	0	(0)
0	4	3	1	0	12	(1)
0	4	1	0	1	12	(2)

The key column is the one that belongs to x_1.
When choosing our row, rows (1) and (2) are equally suitable.

Example

Suppose we pick row (1). Our tableau becomes

z	x_1	x_2	s_1	s_2	c_i	
1	0	1/2	1/2	0	6	(0)
0	1	3/4	1/4	0	3	(1)
0	0	-2	-1	1	0	(2)

We see that the maximum is obtained in $(3,0)$ ($s_1 = 0$, $s_2 = 0$). The maximum is 6.

Example

Alternatively, we could have picked row (2). Our tableau becomes

z	x_1	x_2	s_1	s_2	c_i	
1	0	-1/2	0	1/2	6	(0)
0	0	2	1	-1	0	(1)
0	1	1/4	0	1/4	3	(2)

This solution is still not optimal.
The key column is the one that belongs to x_2. We have to take row (1). Our tableau becomes

z	x_1	x_2	s_1	s_2	c_i	
1	0	0	1/4	1/4	6	(0)
0	0	1	1/2	-1/2	0	(1)
0	1	0	-1/8	3/8	3	(2)

From this we get that the maximum is 6, which is obtained in $(3, 0)$.

Example

Remark

Note that the first method is faster than the second.