Linear Programming

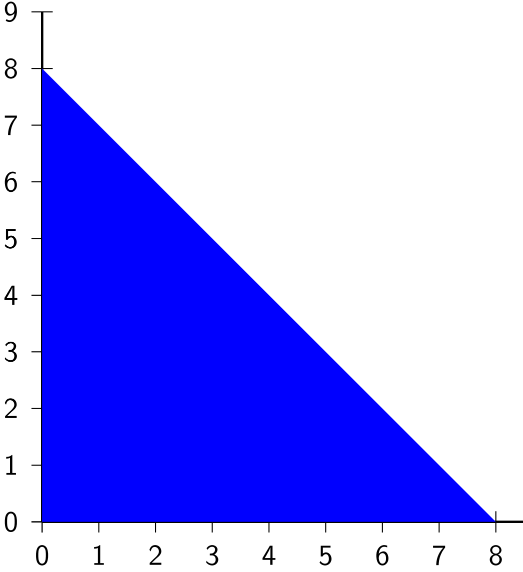
Linear Programming

In *Linear Programming* we consider the optimisation of a function over a region of $\mathbb{R}^n$ which is determined by a number of linear equations.
For example, the inequalities

$$\begin{aligned} x + y &\leq 8 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

is the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 8)$ and $(8, 0)$.

Linear Programming



Sketching Regions in $\mathbb{R}^{\not=}$

In $\mathbb{R}^2$ to sketch the region determined by the inequality $ax + by \leq c$ we first plot the line $ax + by = c$ and then see which side the inequality lies by picking a point and seeing if the inequality is satisfied or not.

Linear Programming (General Problem)

The general problem is to find the extreme values of a *linear objective function* subject to *linear inequality* constraints. That is,

$$\max_{x_1,\ldots,x_n} f_1 x_1 + f_2 x_2 + \ldots + f_n x_n$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq c_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq c_m \end{aligned}$$

and

$$x_1 \geq 0,\ x_2 \geq 0,\ \ldots,\ x_n \geq 0.$$

Linear inequalities

The equation

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = c_i$$

represents a *hyperplane* in $\mathbb{R}^n$.
For example

$$a_{i1}x_1 + a_{i2}x_2 = c_i$$

is the equation of a line in $\mathbb{R}^2$.

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 = c_i.$$

is the equation of a plane in $\mathbb{R}^3$.

Linear inequalities

The inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq c_i$$

defines all points lying on one side of that hyperplane.
The inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq c_i$$

defines all points lying on the other side of that hyperplane.

Linear inequalities (continued)

We represent a general linear inequality in $\mathbb{R}^n$ using the scalar or dot product as

$$\mathbf{v} \cdot \mathbf{x} \leq c.$$

This inequality is called *slack* at $\mathbf{x}$ if $<$ holds.
It is called *binding* at $\mathbf{x}$ if $=$ holds.
It is *unsatisfied* at $\mathbf{x}$ it it is not valid at $\mathbf{x}$.

The optimisation problem

In vector notation, the maximization problem is

$$\max_{\mathbf{x}} (\mathbf{f} \cdot \mathbf{x}),$$

subject to

$$\mathbf{A}\mathbf{x} \le \mathbf{c} \quad \text{and} \quad \mathbf{x} \ge 0.$$

Here $\mathbf{A}$ is the $m \times n$ matrix (a_{ij}) and $\mathbf{f}$ is the vector $(f_1, \ldots, f_n)$.
The set of variables which satisfy all the constraints

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \le \mathbf{c}, \mathbf{x} \ge 0\}$$

is called the *feasible* set.
The feasible set may be empty.

The optimization problem

We shall use two methods to solve a linear programming problem the *graphical method* and the *simplex method*.

The Graphical Method

Example: Maximise x subject to

$$\begin{aligned} x + y &\leq 10 \\ 3x + 2y &\leq 25 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

The Graphical Method

For the Graphical Method it is important that we can visualise the feasible set. This is possible in $\mathbb{R}^2$. It is not so easy in $\mathbb{R}^3$. In $\mathbb{R}^4$ it is almost impossible.

The Graphical Method

The Graphical Method consists of the following steps:

- Make a diagram for the feasible set.
- Plot the level sets of the (cost) function.
- Determine the highest (or lowest) level set that still has a point in common with the feasible set.

Corner points or *vertices* of the feasible set are those points on the boundary of the feasible set, that are the intersection points between two constraint lines.
The common direct of the level sets of the objective function is called the *preference direction*.
Note that the optimal solution to a linear programming problem can always be found at the vertices of the feasible set.

Example

The region A consists of all (x_1, x_2) satisfying $-3x_1 + x_2 \leq 3$, $x_1 + x_2 \leq 7$, $x_1 \geq 0$ and $x_2 \geq 0$.
Solve the following problems with A as the feasible region:

1. $\max\ 3x_1 + 2x_2$,
2. $\max\ x_1 - x_2$.

No Solution to Linear Programming Problem

An LP problem may not give us a maximum or minimum. This can be for two reasons:

1. The feasible set is empty;
2. The feasible set is unbounded.

Example: There are no points which satisfy the following constraints

$$\begin{aligned} x + y &\leq 1 \\ 2x + 3y &\geq 6 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

No Solution to Linear Programming Problem

Assume there are two food items, milk and bread, which cost €0.60 and €1 per unit. Assume that the nutrient content of milk and bread is:

Nutrient	unit milk	unit bread	daily requirement
calcium	10 mg	4 mg	20 mg
protein	5 g	5 g	20 g
vitamin B	2 mg	6 mg	12 mg

Minimize the cost of a diet consisting of milk and bread, while ensuring that it provides adequate amounts of the listed nutrients.

No Solution to Linear Programming Problem

Let x be the number of units of milk consumed, y the number of units of bread. We can rewrite the above problem as
Minimise $C(x, y) = 0.6x + y$,

$$\begin{aligned} 10x + 4y &\geq 20, \\ 5x + 5y &\geq 20, \\ 2x + 6y &\geq 12, \\ x, y &\geq 0. \end{aligned}$$

Because the feasible set in unbounded, C does not have a maximum.