Lagrange Multipliers

Lagrange Multipliers

Let us recall the method of Lagrange multipliers for functions of two variables and one constraint

Problem

Find the dimensions of the rectangle with maximum area, given that the perimeter is 10 m.

We can translate this into a mathematical problem:

Problem

For what (x, y) *is the function* $f(x, y) = xy$ *maximal, given the constraint* $g(x, y) = 2x + 2y = 10$.

The Lagrangian Method: two variables, one constraint

The Lagrangian method tells us, that if we want to find the maximum (minimum) of $f(x, y)$ subject to $g(x, y) = c$, We construct the Lagrangian Function

$$F(x, y, \lambda) = f(x, y) + \lambda[c - g(x, y)],$$

determine its critical points by setting

$$F_x(x, y, \lambda) = 0,$$

$$F_y(x, y, \lambda) = 0,$$

$$F_\lambda(x, y, \lambda) = 0$$

and solving.
The variable λ is known as the *Lagrange Multiplier*

Lagrange Multiplier

Find the maximum (minimum) of $f(x, y) = x^2 + y^2$ subject to $x^2 + xy + y^2 = 3$

Cobb-Douglas Model

Consider the Cobb-Douglas production function

$$q(k, l) = 25k^{1/3}l^{1/6}$$

where q, k and l denote quantity, capital and labour respectively in a certain industrial process.
Maximise production subject to $10k + 5l = 30$.

The Lagrangian Method: n variables, one constraint

If we have a function $f : \mathbb{R}^n \to \mathbb{R}$ and we want to find the maximum (minimum) of $f(x_1, \ldots, x_n)$ subject to $g(x_1, \ldots, x_n) = c$, we consider the Lagrangian function

$$F(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda[c - g(x_1, \ldots, x_n)].$$

We then solve the $n + 1$ equations

$$\begin{aligned} F_{x_1}(x_1, \ldots, x_n, \lambda) &= 0 \\ F_{x_2}(x_1, \ldots, x_n, \lambda) &= 0 \\ \vdots \qquad & \quad \vdots \\ F_{x_n}(x_1, \ldots, x_n, \lambda) &= 0 \\ F_{\lambda}(x_1, \ldots, x_n, \lambda) &= 0 \end{aligned}$$

Example 1:

Example

Find the maximum of xyz subject to $x + 2y + 4z = 12$.

Maximuming Utility

Suppose

$$U(x, y, z) = xyz$$

is the utility function of a person consuming x, y and z units of three commodities X, Y and Z.
Suppose that X costs €1 per unit, Y costs €4 per unit, and Z costs €2 per unit.

(a) If a person has a budget of €48, how much of each unit should he or she buy in order to maximise utility?

(b) What is maximum utility?

Maximuming Utility

The constraint is given by $x + 4y + 2z = 48$ since each unit of X costs €1, each unit of Y costs €4, and each unit of Z costs €2. Since the person has €48 to spend, we must have that $g(x, y, z) = x + 4y + 2z = 48$.

The Lagrangian Method: n variables, m constraints

Consider the following problem: Find the maximum(minimum) of $f(x_1, \ldots, x_n)$ subject to $g_1(x_1, \ldots, x_n) = c_1, g_2(x_1, \ldots, x_n) = c_2, \ldots, g_m(x_1, \ldots, x_n) = c_m$.
We consider the Lagrangian function

$$\begin{aligned} F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) =& f(x_1, \ldots, x_n) + \lambda_1[c_1 - g_1(x_1, \ldots, x_n)] \\ &+ \lambda_2[c_2 - g_2(x_1, \ldots, x_n)] + \ldots \\ &+ \lambda_m[c_m - g_m(x_1, \ldots, x_n)]. \end{aligned}$$

The Lagrangian Method: n variables, m constraints

To find the maximum(minimum) of $f(x_1, \ldots, x_n)$ subject to $g_1(x_1, \ldots, x_n) = c_1, g_2(x_1, \ldots, x_n) = c_2, \ldots, g_m(x_1, \ldots, x_n) = c_m$ we solve the $n + m$ equations

$$
\begin{aligned}
F_{x_1}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) &= 0 \\
F_{x_2}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) &= 0 \\
\vdots &\quad \vdots \\
F_{x_n}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) &= 0 \\
F_{\lambda_1}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) &= 0 \\
F_{\lambda_2}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) &= 0 \\
\vdots &\quad \vdots \\
F_{\lambda_m}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) &= 0
\end{aligned}
$$

Example

Find the maximum (minimum) of $f(x, y, z) = x^2 + y^2 + z^2$ subject to

$$\begin{aligned} x + 2y + z &= 1, \\ 2x - y - 3z &= 4. \end{aligned}$$

Example

Find the maximum (minimum) of $f(x, y, z) = x$ subject to

$$z = x + y,$$
$$x^2 + 2y^2 + 2z^2 = 8.$$

Economic interpretation of the Lagrange multiplier

Consider the problem of finding the extreme value of $f(x, y)$ subject to the condition that $g(x, y) = c$.
Suppose we solve the problem and find that x^* and y^* are the values of x and y that solve this problem. These values will depend on c, so

$$x^* = x^*(c),$$
$$y^* = y^*(c).$$

Thus the optimal value of f, f^* depends on c. So

$$f^*(c) = f^*(x^*(c), y^*(c)).$$

Economic interpretation of the Lagrange multiplier

If the optimal value of f is obtained using Lagrange Multipliers we have that the Lagrange multiplier, λ, is a function of c, $\lambda(c)$.
We *assume* that these are differentiable functions of c. Then it can be shown that that

$$\frac{d}{dc} f^*(c) = \lambda(c).$$

The rate of change of f^* is the Lagrange multiplier of the problem $\lambda = \lambda(c)$.

Economic interpretation of the Lagrange multiplier

We call λ the shadow price. If we increase c by one unit then we have that

$$\frac{f^*(c+1) - f^*(c)}{1} \approx \frac{df^*(c)}{dc} = \lambda(c)$$

So the change in f^* in going from the constraint $g(x, y) = c$ to the constraint $g(x, y) = c + 1$ is *approximately* equal to $\lambda(c)$.

Example

Consider the Cobb-Douglas production function

$$q(k, l) = 25k^{1/3}l^{1/6}$$

where q, k and l denote quantity, labour and capital respectively in a certain industrial process. This process is subject to cost constraint

$$10k + 5l = 30.$$

Estimate what happens if the cost constraint is changed to

$$10k + 5l = 31.$$

Summary

To find the maximum(minimum) of $f(x_1, \ldots, x_n)$ subject to $g_1(x_1, \ldots, x_n) = c_1, g_2(x_1, \ldots, x_n) = c_2, \ldots, g_m(x_1, \ldots, x_n) = c_m$ let

$$\begin{aligned} F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = & f(x_1, \ldots, x_n) + \lambda_1[c_1 - g_1(x_1, \ldots, x_n)] \\ & + \lambda_2[c_2 - g_2(x_1, \ldots, x_n)] + \ldots \\ & + \lambda_m[c_m - g_m(x_1, \ldots, x_n)]. \end{aligned}$$

Summary

solve the $n + m$ equations

$$
\begin{aligned}
F_{x_1}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) =& 0 \\
F_{x_2}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) =& 0 \\
\vdots \qquad\qquad & \vdots \\
F_{x_n}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) =& 0 \\
F_{\lambda_1}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) =& 0 \\
F_{\lambda_2}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) =& 0 \\
\vdots \qquad\qquad & \vdots \\
F_{\lambda_m}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) =& 0
\end{aligned}
$$

Use symmetry, the nature of the problem and reason.