# Notes On Introduction to Conformal Field Theory
# A Course Given By Dr. Tobias Osborne

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# Lecture 1: Introduction to Conformal Field Theory

Recommended references:

- *A Mathematical Introduction to Conformal Field Theory* by Schottenloher.
- *Applied Conformal Field Theory*, hep-th/9108028, by Ginsparg.
- *Conformal Field Theory* ("Yellow Book") by Francesco, Mathieu, Senechal.

Why study conformal field theory (CFT)?

- CFT provides a good description of systems at or near criticality.
- CFTs are the only true quantum field theories (QFTs), since they are cutoff-independent. One can think of QFTs as perturbations of CFTs. CFTs correspond to renormalization groups of fixed points, which dominate an effective theory at or near criticality.
- CFTs can be made, by and large, mathematically rigorous, at least in $(1+1)$-dimensional theories. There are three major competing mathematical descriptions for CFT, and advances are being made towards a single, unifying description.

Prerequisites for this material:

- Advanced quantum mechanics

  E.g, many-body theory and Fock spaces.
- Classical field theory

  E.g., symplectic geometry.
- Quantum field theory.
- Advanced quantum field theory.

What is CFT?

- A *conformal field theory* is a field theory, quantum or classical, that is invariant, or symmetric, under a group of transformations called the *conformal group* $G$.
- In a classical field theory, this means that the equations of motion are left invariant.
- In a quantum field theory, this means that, by Wigner's theorem, there is a projective unitary representation of the group $G$. In other words, symmetries, or transformations, that leave the transition amplitude invariant, are realized, up to a phase, by (anti)unitary operators.

## Conformal Transformations in $d$ Dimensions

Let $M = \mathbb{R}^{p,q}$ be a manifold $\mathbb{R}^d$, where $d = p + q$, and $p, q \in \mathbb{Z}_{\geq 0}$. To this manifold, assign the metric

$$g_{\mu\nu} \equiv \eta_{\mu\nu} = \text{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1) \tag{1}$$

With the first $p$ entries equal to one, and the last $q$ entries equal to minus one. Note that this is not necessarily a Riemannian metric, since the signature can be negative. We have a few cases of interest for this metric

- $p = d$ , Riemannian.
- $p = d - 1$, $q = 1$, Lorentz.
- $q > 1$, e.g., $q = 2$, AdS-CFT correspondence.

A conformal transformation leaves the metric invariant up to a scale factor. Consider a smooth change of coordinates

$$x \to x' = x'(x) \text{ , with } x = (x^1, x^2, \ldots, x^p, x^{p+1}, \ldots, x^{p+q}) \tag{2}$$

Such that for the metric, as a type-$(2, 0)$ tensor,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \equiv \Omega(x) g_{\mu\nu}(x). \tag{3}$$

Where $\Omega(x) > 0$ is the (local) scale factor. Note that if the scale factor is zero, then we have a singularity, which we will discuss later. Such a transformation is called *conformal*, and these transformations preserve angles

$$\angle\theta = \frac{g_{\mu\nu} u^\mu v^\nu}{\sqrt{(g_{\mu\nu} u^\mu v^\nu)^2}}. \tag{4}$$

The *conformal group* of a manifold M is denoted by $\text{Conf}(M)$, and is the connected component of the group of all conformal transformations of $M$ containing the identity, in a compact, open topology.

So, in a quantum conformal field theory, we are looking for a Hilbert space $\mathcal{H}$ and a projective unitary representation of the group $G$ for *local* QFTs

$$G \to \pi(G). \tag{5}$$

This is unexpectedly difficult, and makes for a very rich field of study, since there is a tension between knowing the unitary representations of symmetries and demanding that the representation is locally implementable.

To classify the conformal group on our chosen manifold $G = \text{Conf}(\mathbb{R}^{p,q})$, consider an infinitesimal conformal transformation on the spacetime coordinates

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{6}$$

Which acts on the metric, placing constraints on $\epsilon$, as (**Exercise**)

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \mathcal{O}(\epsilon^2). \tag{7}$$

To satisfy the condition $g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)$, we must have that the second term be diagonal, proportional to $\eta_{\mu\nu}$

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \propto \eta_{\mu\nu}. \tag{8}$$

Calculate the proportionality factor by tracing both sides with $\eta_{\mu\nu}$ to get

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}. \tag{9}$$

Comparing this with the metric transformation law, we get that the scale factor $\Omega(x)$ is

$$\Omega(x) = 1 + \frac{2}{d}(\partial \cdot \epsilon). \tag{10}$$

From equality of mixed partial derivatives (to third order), it follows that (**Exercise**)