Lectures Notes For
An Introduction to Conformal Field Theory
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Contents

1 Lecture 1: Introduction to Conformal Field Theory

Recommended references:

- *A Mathematical Introduction to Conformal Field Theory* by Schottenloher.
- *Applied Conformal Field Theory*, hep-th/9108028, by Ginsparg.
- *Conformal Field Theory* ("Yellow Book") by Francesco, Mathieu, Senechal.

Why study conformal field theory (CFT)?

- CFT provides a good description of systems at or near criticality.
- CFTs are the only true quantum field theories (QFTs), since they are cutoff-independent. One can think of QFTs as perturbations of CFTs. CFTs correspond to renormalization groups of fixed points, which dominate an effective theory at or near criticality.
- CFTs can be made, by and large, mathematically rigorous, at least in $(1+1)$-dimensional theories. There are three major competing mathematical descriptions for CFT, and advances are being made towards a single, unifying description.

Prerequisites for this material:

- Advanced quantum mechanics

  E.g, many-body theory and Fock spaces.
- Classical field theory

  E.g., symplectic geometry.
- Quantum field theory.
- Advanced quantum field theory.

What is CFT?

- A *conformal field theory* is a field theory, quantum or classical, that is invariant, or symmetric, under a group of transformations called the *conformal group G*.
- In a classical field theory, this means that the equations of motion are left invariant.
- In a quantum field theory, this means that, by Wigner's theorem, there is a projective unitary representation of the group G. In other words, symmetries, or transformations, that leave the transition amplitude invariant, are realized, up to a phase, by (anti)unitary operators.

Conformal Transformations in d Dimensions

Let $M = \mathbb{R}^{p,q}$ be a manifold $\mathbb{R}^d$, where $d = p + q$, and $p, q \in \mathbb{Z}_{\geq 0}$. To this manifold, assign the metric

$$g_{\mu\nu} \equiv \eta_{\mu\nu} = \mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1) \tag{1}$$

With the first p entries equal to one, and the last q entries equal to minus one. Note that this is not necessarily a Riemannian metric, since the signature can be negative. We have a few cases of interest for this metric

- $p = d$, Riemannian.
- $p = d - 1$, $q = 1$, Lorentz.
- $q > 1$, e.g., $q = 2$, AdS-CFT correspondence.

A conformal transformation leaves the metric invariant up to a scale factor.

Consider a smooth change of coordinates

$$x \to x' = x'(x) \text{ , with } x = (x^1, x^2, \ldots, x^p, x^{p+1}, \ldots, x^{p+q}) \tag{2}$$

Such that the metric, a type-$(2, 0)$ tensor, undergoes an *active coordinate* transformation as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') \equiv \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x') \tag{3}$$

And then impose the condition

$$g'_{\mu\nu}(x') = \Omega(x') g_{\mu\nu}(x'). \tag{4}$$

Where $\Omega(x) > 0$ is the (local) scale factor. Note that if the scale factor is zero, then we have a singularity, which we will discuss later. A transformation that obeys the last line is called *conformal*, and these transformations preserve angles

$$\angle\theta = \frac{g_{\mu\nu} u^\mu v^\nu}{\sqrt{(g_{\mu\nu} u^\mu v^\nu)^2}}. \tag{5}$$

The *conformal group* of a manifold M is denoted by $\mathrm{Conf}(M)$, and is the connected component of the group of all conformal transformations of M containing the identity, in a compact, open topology.

So, in a quantum conformal field theory, we are looking for a Hilbert space $\mathcal{H}$ and a projective unitary representation of the group G for *local* QFTs

$$G \to \pi(G). \tag{6}$$

This is unexpectedly nontrivial, and makes for a very rich field of study, since there is a tension between knowing the unitary representations of symmetries and demanding that the representation is locally implementable.

To classify the conformal group on our chosen manifold $G = \mathrm{Conf}(\mathbb{R}^{p,q})$, consider an infinitesimal conformal (active coordinate) transformation on the spacetime coordinates

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{7}$$

Which must leave the metric invariant up to the scale factor $\Omega(x)$. This places constraints on ϵ (**Exercise**)

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \mathcal{O}(\epsilon^2). \tag{8}$$

To satisfy the constraint placed by conformal invariance on the metric (c.f., $g_{\mu\nu} \to g'_{\mu\nu}(x') = \Omega(x')g_{\mu\nu}(x')$), as well as the constraint that the conformally transformed metric is still proptional to the diagonal flat spacetime metric $g'_{\mu\nu} \propto \eta_{\mu\nu}$, we must have that the second term is also diagonal, proportional to $\eta_{\mu\nu}$

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \propto \eta_{\mu\nu} \tag{9}$$

$$\implies (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \text{constant} \cdot \eta_{\mu\nu} \tag{10}$$

Take the trace of each side, set $\mu = \nu$, and solve for the constant

$$\text{constant} = \frac{2(\partial \cdot \epsilon)}{d} \tag{11}$$

So, the conformal transformation on the metric reads, tossing out higher order terms,

$$g'_{\mu\nu} = g_{\mu\nu} + \frac{2(\partial \cdot \epsilon)}{d} g_{\mu\nu}. \tag{12}$$

And substituting into the proportionality relation from above, we have

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}. \tag{13}$$

Combining this with the conformal transformation of the metric and comparing to the metric transformation law, we get that the scale factor $\Omega(x)$ for the conformal transformation of the spacetime metric is

$$\Omega(x) = 1 + \frac{2}{d}(\partial \cdot \epsilon). \tag{14}$$

Then it follows from $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}$, expanding and equating mixed partial derivatives to third order, and we get d^2 partial differential equations of the form (**Exercise**)

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu)(\partial \cdot \epsilon) = 0 \tag{15}$$

Where $\Box = \eta^{\mu\nu}\partial_\mu \partial_\nu$ is the d'Alembertian operator.

Classification of Infinitesimal Conformal Translations for $d > 2$

By examining the condition for ϵ and the d^2 equations

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{16}$$

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu)(\partial \cdot \epsilon) = 0 \tag{17}$$

We find that third order derivatives of $\epsilon(x)$ vanish and $\epsilon(x)$ is at most quadratic.

This leaves four types of infinitesimal transformations, defined via ϵ, allowable in a conformal transformation: one constant, two linear, and one quadratic in spacetime coordinates.

1. Spacetime translations

   $\epsilon = a^\mu$.

2. Rotations

   $\epsilon^\mu = \omega^\mu_{\ \nu} x^\nu$, ω antisymmetric.

3. Scale transformations

   $\epsilon^\mu = \lambda x^\mu$, $\lambda > 0$.

4. Special conformal transformations (SCT; inversion through a sphere)

   $\epsilon^\mu = b^\mu x^2 - 2x^\mu(b \cdot x)$.

Note that Lorentz and Poincaré transformations are always subgroups of the conformal group, leaving the metric invariant. since ω corresponds to boosts and Euclidean affine rotations complete the Poincaré group.

Theorem

Every conformal transformation that acts on an connected subset of Minkowski space, including the whole space itself, $\varphi : U \subset \mathbb{R}^{p,q}$, where $p + q > 2$, is a composition of

- a translation

  $x^\mu \to x^\mu + a^\mu$, where $a \in \mathbb{R}^d$,

- an orthogonal transformation (rotation)

  $x \to \Lambda x$, where $\Lambda \in O(p, q)$,

- a dilation (scale)

  $x^\mu \to \lambda x^\mu$, where $\lambda \in \mathbb{R}^+$,

- and an SCT

  $x \to \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}$, where $b \in \mathbb{R}^q$.

Note that it is possible to find a vector b such that the denominator is equal to zero, the SCT is not invertible, and this is no longer a group. Also note that if we don't compactify the space and include ∞ as a point available to the conformal transformation, the group becomes significantly smaller and more constrained.

Classification of Infinitesimal Conformal Translations for $d = 2$

If $d = 2$, the spacetime metric becomes the identity

$$g_{\mu\nu} = \delta_{\mu\nu} \tag{18}$$

And $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}$ becomes the Cauchy-Riemann equations and $\epsilon(x)$ is complex-valued, complex-differentiable, and analytic

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \text{ and } \partial_1 \epsilon_2 = -\partial_2 \epsilon_1. \tag{19}$$

Introduce the complex coordinates

$$z = x^1 + ix^2 \text{ and } \bar{z} = x^1 - ix^2. \tag{20}$$

Then we can complexify ϵ as

$$\epsilon(z) = \epsilon^1 + i\epsilon^2 \text{ and } \bar{\epsilon}(\bar{z}) = \epsilon^1 - i\epsilon^2. \tag{21}$$

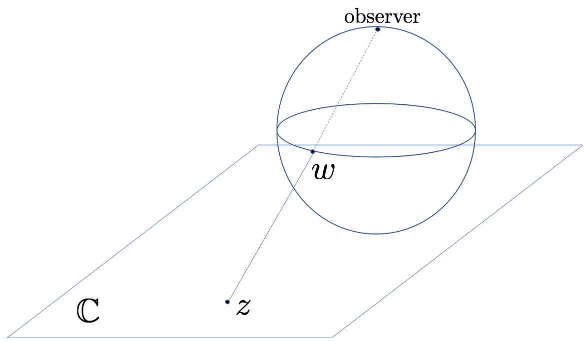
Two-dimensional global, gotten via exponentiation of an infinitesimal transformation, conformal transformations correspond to *entire* (no singularities, invertible everywhere), *holomorphic* functions $z \to f(z)$ with holomorphic inverses $f^{-1}(z)$. The only allowable form for a conformal transformation that corresponds to an entire, holomorphic function is linear in the complex coordinates

$$f(z) = \alpha z + \beta, \text{ where } \alpha, \beta \in \mathbb{C}. \tag{22}$$

We may expect a larger group of symmetries with entirety and holomorphism enforced, since the space seems less constrained, but this actually constrains the space more and the group becomes smaller. So, if we were to not compactify, and add infinity as a point, as we demonstrated, the conformal space becomes linear and boring: only rotations and scaling are allowed.

To include this complex representation of the spacetime coordinates, we *extend* our manifold to the complex numbers $\mathbb{C}$ and compactify complex space to a Riemann sphere $\mathbb{C} \cup \{\infty\}$ we get the proper space for the conformal transformations to act in

$$\mathbb{R}^{2,0} \to \mathbb{C} \to \mathbb{C} \cup \{\infty\} \to \text{Conf}(\mathbb{C} \cup \{\infty\}) \tag{23}$$



Where the conformal group is

$$\mathrm{Conf}(\mathbb{C} \cup \{\infty\}) = \left\{ f(z) = \frac{\alpha z + \beta}{\gamma z + \delta};\ \alpha, \beta, \gamma, \delta \in \mathbb{C}, \alpha\delta - \beta\gamma \neq 0 \right\}. \tag{24}$$

This is also called the group of Moebius transformations, and is a slightly larger group of conformal transformations (symmetires), since we can map to and from infinity as a point.

Summary

A conformal field theory is a local quantum field theory that is invariant under the conformal group, a set of transformations, a change in coordinates, that leave the metric invariant up to a scale factor. In different spacetime dimensions, the conformal group takes on significantly different forms.

The global conformal group in dimensions greater than two is comprised of translations, rotations, scaling, and special conformal transformations, as well as dimensions equal to two, as long as the space is compactified. If singularities are included, functions with poles are allowed, the symmetry gets larger.

2 Lecture 2: Local Conformal Transformations

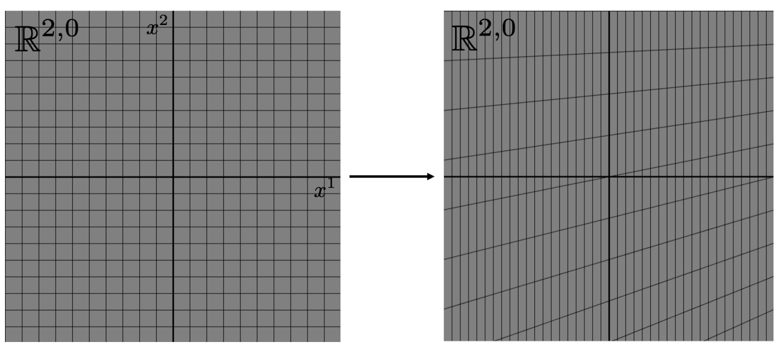
In the last lecture we introduced global conformal transformations/symmetries of some manifold M that form a (symmetry) group G which can be promoted to a symmetry group of some quanutm system, where the kinematics of the system are described by a Hilbert space $\mathcal{H}$. The quantum system is said to be globally conformally invariant if these is some unitary representation, operators U that act on the Hilbert space,

$$U: \ G \to U(\mathcal{H}). \tag{25}$$

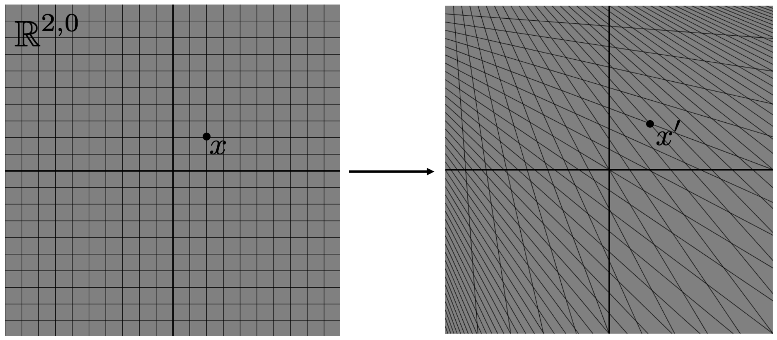
Recall that in the generalized Minkowski space $\mathbb{R}^{p,q}$, the structure of the group of global conformal transformations G consists of compositions of translations, dilations, rotations(boosts), and special conformal transformations (SCTs).

Here we now study the case where $d = 2$, which will expand our notion of what a symmetry is and will allow us to define local, infinitesimal conformal transformations.

For example, a global conformal transformation, a $1-1$ differentiable map from $\mathbb{R}^2 \to \mathbb{R}^2$, may consist of a dilation and a rotation and look like



For contrast, consider an infinitesimal conformal transformation $\mathrm{id} + \epsilon X$, where $X = X(x)$ is a vector field, the derivative of a diffeomorphism, that acts on the two-dimensional Minkowski space as



This transformation preserves all of the right angles in the untransformed Minkowski space, and the action is close to the identity, such that $|x - x'| \sim \mathcal{O}(\epsilon)$. The transformation $\mathrm{id} + \epsilon X$ is conformal to first order in ϵ, as is required by the definition of infinitesimal.

Although the vector field $X(x)$ is a generator of the infinitesimal conformal transformation, it does not necessarily define a global transformation via exponentiation, as it just may not be well defined globally.

To begin to make sense of this, consider in quantum mechanics, where we talk about quantum systems symmetric under a group G with HIlbert space

$$(\mathcal{H},\ U: G \to U(\mathcal{H})). \tag{26}$$

So, in quantum mechanics, we are reduced to finding these unitary representations of G. If G is finite, it does not make sense to speak infinitesimally (e.g., one one-hundredth of a reflection).

We assume G is a manifold, and then we may as well go as far to assume that G is a Lie group with an associated Lie algebra $\mathfrak{g}$, which consists of vector fields that exponentiate to the Lie group. Then the quantum system is symmetric under the Lie algebra if you get a representation

$$(\mathcal{H},\ \pi: \mathfrak{g} \to L(\mathcal{H})) \tag{27}$$

Where $L(\mathcal{H})$ is the set of (bounded and unbounded) linear operators, and π generates a unitary operator on the Hilbert space, such that $\pi(X) = e^{isX}$, $s \in \mathbb{R}$.

Note that for an infinite-dimensional group, (1) the operator e^{isX} may not be continuous, and (2) the Lie algebra may not exponeniate to a Lie group, which we will encounter in conformal field theory. In other words, in contrast to when we used infinitesimal quanities to build global representations, we find that infinitesimal conformal transformations don't necessarily exponentiate to a group.

Therefore, in the infinitesimal case, we abandon looking for (full, continuous) unitary representations of the Lie group, and instead focus in on finding Hermitian representations that generate the Lie algebra.

Local algebra of infinitesimal conformal transformations

Recall that for global conformal transformations, we have $z \to f(z)$, where f is holomorphic with inverse f^{-1}. For infinitesimal f, this transformation, including the complex conjugate, becomes

$$z \to z + \epsilon(z) \text{ and } \bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z}) \tag{28}$$

Where ϵ is a holomorphic function. A convenient choice of basis, which is infinite dimensional, is

$$\epsilon_n(z) = -\epsilon z^{n+1}, \text{ where } n \in \mathbb{Z}. \tag{29}$$

Given a diffeomorphism $[z \to z + \epsilon_n(z)] = e^{\epsilon \ell_n}$, the corresponding vector field tangent to every point in the manifold is defined by the operators

$$\ell_n \equiv -z^{n+1}\partial_z \text{ and } \bar{\ell}_n \equiv -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{30}$$

These differential operators form a basis, since they obey the commutation relations (**Exercise**)

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} \tag{31}$$
$$[\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n} \tag{32}$$
$$[\bar{\ell}_m, \ell_n] = 0. \tag{33}$$

They also as form a closed, infinite-dimensional Lie algebra $\forall m, n \in \mathbb{Z}$, called the Witt algebra WItt $= \mathcal{A} \oplus \bar{\mathcal{A}}$, where $\mathcal{A}$ is generated by $\{\ell\}$, and $\bar{\mathcal{A}}$ is generated by $\{\bar{\ell}\}$. Everything commutes in the basis, so the direct sum of the Witt algebra is justified.

The Witt algebra is generated infinitesimally, and could also be used to infinitiseimally generate a Lie group. This turns out to be true, but the Lie group is not the conformal group.

Which operators ℓ_n correspond to global transformations?

Consider a vector field

$$v(z) = -\sum_{n=-\infty}^{\infty} v_n \ell_n = \sum_{n=-\infty}^{\infty} v_n z^{n+1} \partial_z. \tag{34}$$

For this vector field to correspond to a global transformation, $v(z)$ must exponentiate to a holomorhpic map f, which is nonsingular in the limit as $z \to 0$. This places constraints on the coefficients of the vector field

$$v_n = 0,\ n < -1. \tag{35}$$

The inverse of the vector field must also exponentiate to a holomorphic map f^{-1}, which is nonsingular in the limit as $z \to \infty$ (e.g., exists on the Riemann sphere). This places the constraint on the coefficients of the vector field:

$$v_n = 0,\ n > 1. \tag{36}$$

Note that if we demand holomorphism on the full complex plane without compactifying, the only allowed global transformations will be linear transformation (**Exercise**). By compactifying $\pm\infty$ as a point onto the Riemann sphere, we have more freedom in allowed global transformations.

With these constraints in place, we are left with three (six with complex conjugates) generators of infinitesimal global conformal transformations

$$\{\ell_{-1}, \ell_0, \ell_1\} \cup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}. \tag{37}$$

The generators close to form a subalgebra under the commutator bracket $[,]$ defined above (**Exercise**), and generate thegroup of *linear fractional (Moebius) transformations*, also known as the projective special linear group $\mathrm{PSL}(2, \mathbb{C})$

$$z \to \frac{az+b}{cz+d},\ ad - bc = 1. \tag{38}$$

The set of global conformal transformations allowed in this basis are (**Exercise**), for $s \in \mathbb{R}$,

$$\text{Translation:} \qquad e^{s\ell_{-1}} \equiv \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \qquad \equiv z \to z - s \tag{39}$$

$$\text{Dilation:} \qquad e^{s(\ell_0 + \bar{\ell}_0)} \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \equiv z \to e^{-s} \tag{40}$$

$$\text{Rotation:} \qquad e^{is(\bar{\ell}_0 - \ell_0)} \equiv \begin{pmatrix} \exp(i\frac{\theta}{2}) & 0 \\ 0 & \exp(-i\frac{\theta}{2}) \end{pmatrix} \qquad \equiv z \to e^{is} \tag{41}$$

$$\text{Special Conformal:} \qquad e^{s\ell_1} \equiv \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \qquad \equiv z \to \frac{z}{1+cz}. \tag{42}$$

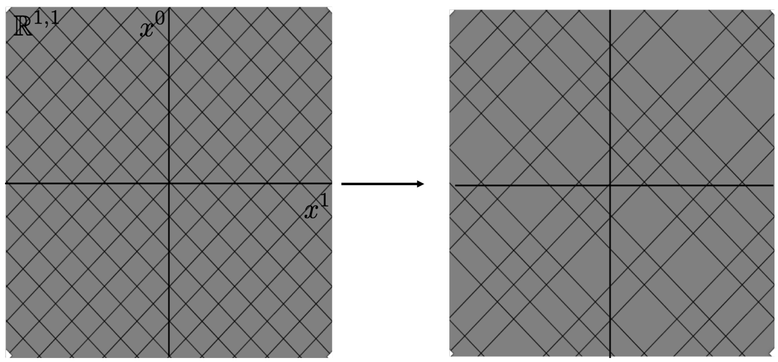
Note that for $\mathbb{R}^{d,0}$, $d > 2$, the local transformations are also global! Also note that in one-dimensional spacetime, $(1, 0)$ or $(0, 1)$, conformal transformations are all monotonic increasing functions $\mathbb{R} \to \mathbb{R}$.

$d = 2$ Minkowski space, $\mathbb{R}^{1,1}$

The conformal group of $\mathbb{R}^{1,1}$ is *special*.

Theorem:

A smooth map $\varphi = (u, v) : M \to \mathbb{R}^{1,1}$ from a connected subset of $M \subset \mathbb{R}^{1,1}$ is conformal (pulls back metric to a scalar multiple of the diagonal metric), iff $u_x^2 > v_x^2$ and $u_x = v_y$, $u_y = v_x$ or $u_x = -v_y$, $u_y = -v_x$.



Theorem:

Consider an infinitely differentiable function on the real line $f \in C^\infty(\mathbb{R})$, and let $f_\pm \in C^\infty(\mathbb{R}^2, \mathbb{R})$, the infinitely differentiable functions from the real line to the real plane, be defined by $f_\pm(x, y) = f(x \pm y)$. Then the map

$$\Phi : C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^2, \mathbb{R}^2) \tag{43}$$

$$(f, g) \to \frac{1}{2}(f_+ + g_-, f_+ - g_-) \tag{44}$$

Has the following properties

- image$(\Phi) = \{(u, v) : u_x = v_y, u_y, v_x\}$
- $\Phi(f, g)$ is conformal iff $f' > 0$ and $g' > 0$ or $f' < 0$ and $g' < 0$
- Φ is bijective iff f and g are bijective
- $\Phi(f \circ h, g \circ k) = \Phi(f, g) \circ \Phi(h, k), \forall f, g, h, k \in C^\infty(\mathbb{R}) \equiv \Phi$ is a homomorphism.

The group of orientation-preserving transformations of $M = \mathbb{R}^{1,1}$ is isomorphic to

$$(\mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})) \cup (\mathrm{Diff}_-(\mathbb{R}) \times \mathrm{Diff}_-(\mathbb{R})) \tag{45}$$

Which consists of the infinitely-differentiable orientation-preserving maps of $\mathbb{R}$, diffeomorphisms of $\mathbb{R}$.

It is convenient to compactify $\mathbb{R}^{1,1} \to S^{1,1} \subset \mathbb{R}^{2,0} \times \mathbb{R}^{0,2}$. Then the group of orientation-preserving transformations of $M = S^{1,1}$ is isomorphic to

$$\mathrm{Conf}(\mathbb{R}^{1,1}) \equiv (\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)) \cup (\mathrm{Diff}_-(S^1) \times \mathrm{Diff}_-(S^1)). \tag{46}$$

This is the definition of the conformal group of Minkowski space. Typically, we throw away the second part of the union, the $''-''$ reversing part, since it is the same as preserving with $z \to -z$, and focus on the infinite-dimensional subgroup $\mathrm{Diff}_+(S^1)$, which we call the *chiral half* of the conformal group. This is admissable, since the symmetries of a quantum system can be understood by the symmetries of $\mathrm{Diff}_+(S^1)$, and the rest is easily gotten by tensor products to include the other light-cone axes.

In the next lecture, we will focus on which quantum systems are invariant under this infinite-dimensional group $\mathrm{Diff}_+(S^1)$ by going to the Lie algebra, which turns out to be isomorphic to the Witt algebra, in the Euclidean case. The unitary representations, gotten via infinitesimal generators, of $\mathrm{Diff}_+(S^1)$ will not be bounded below and are unstable. Therefore, *projective* unitary representaitons will be required, and are classified by the *central charge*.

3 Lecture 3: Classical Conformal Field Theory

We continue our discussion of systems that exhibit conformal symmetries. These symmetries are contained in the conformal group called Conf($\mathbb{R}^{p,q}$), which is the connected component containing the identity of all conformal diffeomorphisms of the pseudo-Riemannian manifold $\mathbb{R}^{p,q}$.

We discussed the infinitesimal conformal transformation, which led to a Lie algebra, the Witt algebra, in $(1+1)$ and $(2,0)$ dimensions. For $d = p + q = 2$, there is a bigger symmetry group (less constrained), yielding more conserved quantiteis, more degrees of freedom of the system. If $d \neq 2$, the symmetry group is too constrained to be that interesting.

A *conformal theory* is a theory with a representation of the group $G = \text{Conf}(\mathbb{R}^{p,q})$. This group contains transformations corresonding to temporal translations, spatial translations, boosts, dilations, and special conformal transformations (inversion about the origin, translation, and a second inversion about the origin). So, a conformal theory has a Hamiltonian H built in, since it is the generator of time translations.

Note that in a nonrelativistic theory, we demand that the Hamiltonian H commutes with everything, which introduces symmetries of the system, but the inclusion of boosts requires a relativistically invariant theory. This constrains the theory further to allow only certain symmetries and exhibit the desired properties.

Note that the Lorentz boost mixes energy and momentum through conjugation of spatial translations to temporal translations. This conjugation requires that all types of possible transformations in a nonrelativistic theory must be represented all at once, and they are not independent of each other.

Another property we need for our theory is *locality.*

So, we have a collection of observables $\phi_a(x)$, where $x \in \mathbb{R}^{p,q}$ and $a \in I$, an index set (labels by particle types, vector quantities, etc.), which can be classical (functions on phase space), quantum (self-adjoint operators), or even probabilistic (element of ordered unit vector space).

A representation of a group of symmetries is a map π that can be

$$\text{finite } \pi : G \to M_n(\mathbb{C}) \quad \text{, the } n \times n \text{ matrices over the complex numbers} \tag{47}$$

$$\text{infinite } \pi : G \to \mathcal{B}(\mathcal{H}) \quad \text{, the bounded operators on a Hilbert space, for example.} \tag{48}$$

The concern is that a given representation does not necessarily yield a set of observables $\phi_a(x)$. In the event that it does, it is likely that a representation which furnishes a collection of (local) observables is *reducible*, and can be decomposed into a direct sum of *irreducible* representations, or *irreps*. This makes for an infinite number of ways to build reducible representations.

So, although we can write down an irreducible representation of G and attempt to enforce locality, we prefer to take the stance, and shall from this point on, that the locality of the theory is the most important property, and find irreducible representations from there.

Classical field representations of conformal symmetries

The concept of the field easily puts forth the idea of locality, but what constraints does conformal symmetry place on a classical field?

Recall for symmetries in a classical field theory start with the action

$$S = \int d^d x \, \mathcal{L}(\phi, \partial_\mu \phi), \text{ where } \phi = \{\phi_a(x)\}. \tag{49}$$

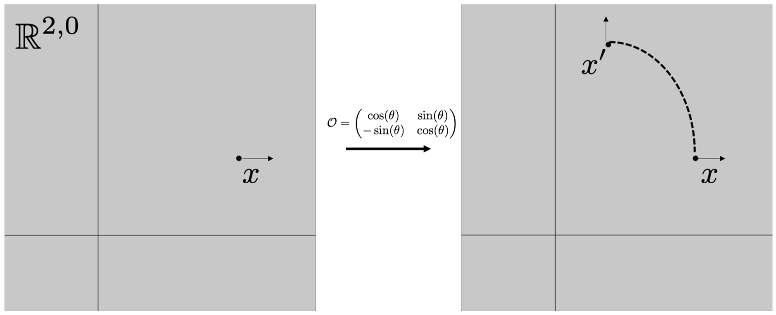
By writing down the action, we have assumed (1) that the equations of motion are represented by an action and (2) that the Lagrangian density depends only on the field and its first derivatives. We have effectively thrown out all non-local theories.

So, a symmetry transformation takes a spacetime location and maps it to its image under that transformation: $x \to x'$. If the transformation is *active*, the then fields transform as well

$$\phi(x) \to \phi'(x') \equiv \mathcal{F}(\phi(x)) \tag{50}$$

Where we note that $\mathcal{F}(\phi(x))$ depends on the previous field configuration.

For example, in an active rotation of a vector field $\mathbb{R}^{2,0}$, a nontrivial representation rotates the spacetime coordinate as well as the vector at each spacetime coordinate, the field (A trivial representation will not rotate the vector.)



After the rotation, the new field configuration at x is

$$\phi'_a(x) = \sum_b \pi(\mathcal{O})_{ab}\phi_b(\mathcal{O}^{-1}x). \tag{51}$$

The trivial representation of the field component b would simply be the identity $\pi(\mathcal{O})_{ab} = \delta_{ab}$, and the fundamental, nontrivial representation is written

$$\pi(\mathcal{O})_{ab} = [\mathcal{O}]_{ab}. \tag{52}$$

How does the action S transform under a symmetry transformation?

$$S' = \int d^d x \left|\det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)\right| \mathcal{L}\left(\mathcal{F}\left(\phi(x)\right), \frac{\partial x^{\nu}}{\partial x'^{\mu}}\partial_\nu \mathcal{F}\left(\phi(x)\right)\right) \tag{53}$$

We also know that our theory is a conformal field theory, conformally invariant, conformally symemtric if the equations of motion are invariant. This is equivalent to the Lagrangian density transforming up to a total derivative

$$\mathcal{L}' = \mathcal{L} + \text{total derivative}. \tag{54}$$

Now, let's study the infinitesimal generators of the conformal group $\mathrm{Conf}(\mathbb{R}^{p,q})$.

- Translation

  $P_\mu = -i\partial_\mu$

- Dilation

  $D = -ix^\mu \partial_\mu$

- Rotation (Boost)

  $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

- Special Conformal

  $K_\mu = -i(2x_\mu x'^\nu \partial_\nu - x^2 \partial_\mu)$

Work out the commutation relations to form a Lie algebra (**Exercise**).

$$
\begin{aligned}
[D, P_\mu] &= iP_\mu && (55)\\
[D, K_\mu] &= -iK_\mu && (56)\\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) && (57)\\
[K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) && (58)\\
[P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) && (59)\\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) && (60)
\end{aligned}
$$

And the rest commute.

Our task now is to find out which kinds of fields transform under the conformal group and give representations of the conformal group. We already know that a field transforming under $\mathrm{Conf}(\mathbb{R}^{p,q})$, a general conformal transformation, transforms under the subgroup Poincaré, which is generated by translation P_μ and rotation $L_{\mu\nu}$, as

$$\phi_a(x) \to_{\text{Poincaré}} \sum_b \pi(\Lambda)_{ab}\phi_b(\Lambda^{-1}x). \tag{61}$$

Let's focus on the subgroup of $\mathrm{Conf}(\mathbb{R}^{p,q})$ which leave the orgin fixed: rotations, dilations, and SCTs. The infinitesimal generators of this subgroup form a subalgebra by exponentiation

$$\Lambda = e^{i\omega^\alpha G_\alpha}, \text{ where } \omega^\alpha \text{ is infinitesimal.} \tag{62}$$

The group element G_α can be K_μ, D, or $L_{\mu\nu}$. At the origin, the Poincaré transformation of the field looks like

$$\phi_a(x=0) \to \sum_b \pi(e^{i\omega^\alpha G_\alpha})_{ab}\phi_b(\Lambda^{-1}x = 0) \tag{63}$$

Where the representation is, by Taylor expansion,

$$\pi(e^{i\omega^\alpha G_\alpha}) = \pi(\mathbb{I}) + i\omega^\alpha\pi(G_\alpha). \tag{64}$$

Rename the representations of the generators (group elements)

$$
\begin{aligned}
\pi(D) &= \tilde{\Delta}(\text{scaling dimension}) && (65)\\
\pi(K_\mu) &= \kappa_\mu && (66)\\
\pi(L_{\mu\nu}) &= S_{\mu\nu}\,(\text{spin}). && (67)
\end{aligned}
$$

The commutation of these representations are then

$$[\tilde{\Delta}, S_{\mu\nu}] = 0 \tag{68}$$
$$[\tilde{\Delta}, \kappa_\mu] = -i\kappa_\mu \tag{69}$$
$$[\kappa_\mu, \kappa_\nu] = 0. \tag{70}$$

Now, suppose that the generators $S_{\mu\nu}$ are irreducible representations, irreps, of the Lorentz group, the group that describes spin/helicity. By Schur's lemma (**Exercise**), we find that the scaling dimension is trivial, and, in turn, by the commuation relations, that all generators κ_μ are also trivial

$$\tilde{\Delta} \propto \mathbb{I} \implies -i\kappa_\mu = 0. \tag{71}$$

Now use this fact to show how dilations act on the fields. The coordinates transform as

$$x \to \lambda x \tag{72}$$
$$x \to \lambda^\epsilon \lambda^\epsilon \dots \lambda^\epsilon x \tag{73}$$

At the origin, the field transforms as

$$\begin{aligned} \phi_a(x=0) \to & (\mathbb{I} + i\epsilon\tilde{\Delta}) \dots (\mathbb{I} + i\epsilon\tilde{\Delta})\phi_a(0) && (74) \\ &= \lambda^{i\tilde{\Delta}_a}\phi_a(0) && (75) \\ &= \lambda^{-\Delta_a}\phi_a(0) && (76) \end{aligned}$$

Where we used the Taylor expansion of the infinitesimal ϵ, and the last line uses $\tilde{\Delta} = i\Delta\mathbb{I}$, since Schur's lemma tells us that the scaling deimension is trivial.

So, every conformal field has a behavior under dilations, defined by the scaling dimension Δ_a, with Jacobian

$$\left|\frac{\partial x'}{\partial x}\right| = \Lambda^{-\frac{d}{2}}, \text{ where } \Lambda = \lambda^{-2}. \tag{77}$$

And the metric transforms under dilations as

$$g'_{\mu\nu} = \lambda^{-2} g_{\mu\nu}. \tag{78}$$

Putting all this together, the field now transforms as

$$\phi_a(x) \to \phi'_a(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \phi_a(x). \tag{79}$$

Filling in the Jacobian, the new field in terms of the orignial field and the original spacetime location is

$$\phi'_a(x) = \sum_b \pi(\Lambda)_{ab}\phi_b(\Lambda^{-1}x) \tag{80}$$

$$= \sum_b [\lambda^{-\Delta}]_{ab}\phi_b(\Lambda^{-1}x) \tag{81}$$

$$= \lambda^{-\Delta}\phi_a(\Lambda^{-1}x) \tag{82}$$

$$= \lambda^{-\Delta}\phi_a(\lambda^{-1}x). \tag{83}$$

By the Baker-Campbell-Hausdorff (BCH) formula, we know how the new field looks at all spacetime lcoations, not just the origin (**Exercise**)

$$e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho} = D + x^\nu P_\nu \tag{84}$$

$$\implies D\phi_a(x) = (-ix^\nu \partial_\nu + \tilde{\Delta})\phi_a(x). \tag{85}$$

In the next lecture, we give this the quantum treatment, where we will look for unitary representations (self-adjoint operators), that are labelled by quantum numbers, such as spin, scaling dimension, and central charge. The central charge will require projective unitary representations.

4 Lecture 4: Constraints of Conformal Invariance on Quantum Field Theories

Recall that a scale transformation on a classical field is characterized per field, indexed by a, by the scaling dimension Δ_a in the mapping

$$\phi_a(x) \to \lambda^{-\Delta_a} \phi_a(x). \tag{86}$$

We now make a move to study quantum fields, using the classical limit to check our results. So, suppose we have done all of the hard work of quantization and have a quantum field theory.

Digression: In $(1+1)$ dimensions, some approaches to quantization include

- Vertex operator algebras.
- (Local) Algebraic QFT

  Top-down approach where one attaches an algebra of observables to each point in spacetime and then makes sense of it as a field theory.
- Functors from n-Categories to Hilbert spaces: $n\mathrm{Cat} \to \mathrm{Hilb}$ (Segal).

Digression: A main assumption of this course is that quantum field theories are a subset of quantum theories, although some schools argue that QFT is done via path integrals, which is not obviously a quantum theory.

The data of a quantum field theory includes

- Hilbert space of states $\mathcal{H}$
- Projective unitary representation of the conformal group $U(g)$, where $g \in \mathrm{Conf}(\mathbb{R}^{p,q})$
- Vacuum (reference) state $|0\rangle \in \mathcal{H}$

  Invariant, up to a phase, under global symmetries, but not necessarily local symmetries: $U(g)\,|0\rangle = e^{i\varphi(g)}\,|0\rangle$.

  If $p+q > 2$, then the global and local (full conformal) symmetries coincide.

  If $p = q = 1$, then the local conformal group of transformations (diffeomorphisms of the circle) is much larger than the global conformal group, making the global transformations subgroup of the local transformations.

- Observables.

For observables, we demand that we can measure some set of local observables from the set of self-adjoint linear operators on the Hilbert space

$$A_{j,x} \in L_{\text{self-adjoint}}(\mathcal{H});\ x \in \mathbb{R}^{p,q};\ j \in \text{index set (e.g., particle type)}. \tag{87}$$

Digression: We concern ourselves with *local* observables, and non-local observables are difficult to imagine. An example of a non-local observable that shows up in gauge theory is the Wilson loop or Wilson line, since they are spread over a nonzero dimensional submanifold of the gauge theory's manifold.

Note that locality is induced by causality, such that if $x - y$ is spacelike, then $A_{j,x}$ and $A_{k,y}$ are jointly observable, $\forall\, j, k$ and x, y.

Quasi-Primary Observables

The *quasi-primary* of a field is a subset of local observables $\{A_{j,x} : j \in J, x \in \mathbb{R}^{p,q}\}$ with the additional properties to satisfy the assumed constraints that enforce conformal invariance of the system, denoted here by $\{\hat{\phi}_k(x) : k \in K\}$ that transform as

$$U(g):\ \hat{\phi}_k(x) \to U^\dagger(g)\hat{\phi}_k(x)U(g) = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_k}{d}} \hat{\phi}_k(x') \tag{88}$$

Where $x' = gx$, and $g \in \mathrm{Conf}(\mathbb{R}^{p,q})$ is a conformal transformation.

Now we must demonstrate that these assumptions yield nontrivial examples of fields. Rest assured that there are such fields, such as free bosons and free fermions.

By these assumptions, the n-point correlation function, which is observable via scattering experiments, transforms under conformal transformations as

$$\langle 0|\, \hat{\phi}_{k_1}(x_1) \ldots \hat{\phi}_{k_n}(x_n)\, |0\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\frac{\Delta_{k_1}}{d}} \ldots \left|\frac{\partial x_n'}{\partial x_n}\right|^{\frac{\Delta_{k_n}}{d}} \langle 0|\, \hat{\phi}_{k_1}(x_1') \ldots \hat{\phi}_{k_n}(x_n')\, |0\rangle\,. \tag{89}$$

This equation constrains the structure of the n-point correlation functions. To understand these constraints, let's analyze how each type of transformation of the conformal group constrains the invariants.

Translations: For $x_j, x_k \in \mathbb{R}^{p,q}$, $j, k = 1, \ldots, n$, the difference $x_j - x_k$ is invariant, and there are $d(n-1)$ such quantities.

Rotations: For spinless objects (in large enough dimension d), the length $r_{jk} \equiv |x_j - x_k|$ is invariant, and there are $\binom{n}{2}$ such quantities.

Dilations: Under the scale transformations, the length r_{jk} is clearly not invariant, but the ratio $\frac{r_{jk}}{r_{lm}}$ can be invariant.

SCTs: Under special conformal transformations, invariant quantities must be cross ratios of the form $\frac{r_{jk}r_{lm}}{r_{jl}r_{km}}$, since the squared length under SCTs transforms as

$$|x_1' - x_2'|^2 = \frac{|x_1 - x_2|^2}{(1 + 2b \cdot x_1 + b^2 x_1^2)(1 + 2b \cdot x_2 + b^2 x_2^2)}. \tag{90}$$

Two-Point Correlation Functions

Consider the classical two-point correlation function, or Green's function, of the quasi-primary fields

$$G^{(2)}(x_1, x_2) = \langle 0 | \phi_1(x_1)\phi_2(x_2) | 0 \rangle = \left| \frac{\partial x_1'}{\partial x_1} \right|^{\frac{\Delta_1}{d}} \left| \frac{\partial x_2'}{\partial x_2} \right|^{\frac{\Delta_2}{d}} \langle 0 | \hat{\phi}_1(x_1')\hat{\phi}_2(x_2') | 0 \rangle . \tag{91}$$

For simplicity, let us denote

$$G^{(2)}(x_1, x_2) = f(x_1, x_2) \tag{92}$$

Exploiting the assumed conformal symmetries of the system, use the fact that the Jacobian for a translation, as well as for a rotation, is equal to one, such that $\left| \frac{\partial x'}{\partial x} \right| = 1$, and, therefore, the Green's function can only depend on the length

$$f(x_1, x_2) = f(Rx_1 + a, Rx_2 + a), \tag{93}$$

requiring that

$$G^{(2)}(x_1, x_2) = f(|x_1 - x_2|) = f(r_{12}). \tag{94}$$

For a dilation, the Jacobian obeys a scale factor, such that $\left| \frac{\partial x'}{\partial x} \right| = \lambda^{\Delta}$, and the Green's function has the form

$$G^{(2)}(x_1, x_2) = \lambda^{\Delta_1 + \Delta_2} f(\lambda r_{12}). \tag{95}$$

To calculate the function that obeys this constraint, expand $f(r_{12}) = \sum_a f_a r_{12}^a$ in a series, noting that a can be a continuous parameter, and compare to the above to get the condition that the coefficients f_a are all zero, except for $a = -\Delta_1 - \Delta_2$.

Therefore, under translations, rotation, and dilations, we find that the Green's function is constrained to the form

$$G^{(2)}(x_1, x_2) = \frac{f_{-\Delta_1 - \Delta_2}}{r_{12}^{\Delta_1 + \Delta_2}} = \frac{c_{12}}{r_{12}^{\Delta_1 + \Delta_2}} \tag{96}$$

Where c_{12} is a constant determined by the normalization condition.

So far, we have come to the conclusion that the correlation function must obey a power law, and we now apply the constraints of SCTs to find that the two-point correlation function of quasi-primary fields is zero unless the two fields have the same scaling dimension $\Delta_1 = \Delta_2 = \Delta$ (**Exercise**)

$$\langle 0| \phi_1(x_1)\phi_2(x_2) |0\rangle = \frac{c_{12}}{r_{12}^{2\Delta}}. \tag{97}$$

This means that if we solve a system and the calculate the two-point correlation function of two fields with different scaling dimensions, Δ_1 and Δ_2, respectively, where $\Delta_1 \neq \Delta_2$, and get a nonzero result, then our system is not conformally invariant under the full global conformal group, but under a subgroup consisting of translations, rotations, and dilations.

Example: Three-Point Correlation

Following similar procedure as in the two-point case, translations, rotations, and dilations lead to conclusion that the three-point correlation function must have the form

$$\langle 0| \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) |0\rangle = \sum_{a,b,c} \frac{c_{abc}}{r_{12}^a r_{23}^b r_{31}^c} \tag{98}$$

Where the summation is constrained by the field scaling dimensions to satisfy $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$.

Special conformal transformations further constrain the exponents in the power law to (**Exercise**)

$$a = \Delta_1 + \Delta_2 - \Delta_3, \quad b = -\Delta_1 + \Delta_2 + \Delta_3, \quad c = \Delta_1 - \Delta_2 + \Delta_3. \tag{99}$$

And the three-point correlation function has the form

$$\langle 0| \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) |0\rangle = \frac{c_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{23}^{-\Delta_1+\Delta_2+\Delta_3} r_{31}^{\Delta_1-\Delta_2+\Delta_3}}. \tag{100}$$

Example: Four-Point Correlation

The four-point correlation function has the form

$$\langle 0| \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) |0\rangle = F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{13}r_{24}}{r_{23}r_{14}}\right) \prod_{j<k} r_{jk}^{-\Delta_j - \Delta_k + \frac{\Delta}{3}} \tag{101}$$

Where $F(\cdot, \cdot)$ is an arbitrary function of the cross products and $\Delta = \sum_j \Delta_j$.

Conformal Theory in $(2+0)$ Dimensions

Thus far, we have been studying the structure of conformal theories in arbitrary spacetime dimension d. In $(2+0)$ dimensions, conformal theories will take on a new definition.

Recall that we defined the complex fields $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, and made the assumption that we can *analytically continue* to arbitrary, independent z and $\bar{z}$, such that

$$\Phi(x_1, x_2) = \Phi(z, \bar{z}) \tag{102}$$

And work out the consequences from there. We now extend our definition of quasi-primary fields, by analogy to the general, arbitrary dimension case. We will now refer to these fields as *primary fields* of type, or conformal weight, $(h, \bar{h})$ with the proposed form

$$\Phi(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})). \tag{103}$$

Note here that there are indeed enough examples of primary fields of conformal field theories to make these assumptions interesting to study.

Expand the primary field infinitesimally, as it is the more convenient method opposed to working with the full global transformation, by sending $z \to z + \epsilon(z)$, and complex conjugate, to order ϵ

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(z, \bar{z}) = ((h\partial_z \epsilon(z) + \epsilon(z)\partial_z) + (\bar{h}\partial_{\bar{z}}\bar{\epsilon}(\bar{z}) + \bar{\epsilon}(\bar{z})\partial_{\bar{z}}))\Phi(z, \bar{z}). \tag{104}$$

And write the two-point correlation function as

$$G^{(2)}(\underline{z}, \underline{\bar{z}}) = \langle 0| \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2) |0\rangle \tag{105}$$

Where $\underline{z} \equiv (z_1, z_2)$ and $\underline{\bar{z}} \equiv (\bar{z}_1, \bar{z}_2)$.

Applying the infinitesimal equation for $\delta_{\epsilon,\bar{\epsilon}}\Phi(z, \bar{z})$ from above, and setting equal to zero by conformal invariance to order ϵ, we get

$$\delta_{\epsilon,\bar{\epsilon}}G^{(2)}(\underline{z}, \underline{\bar{z}}) = \langle 0| \delta_{\epsilon,\bar{\epsilon}}\Phi_1, \Phi_2 |0\rangle + \langle 0| \Phi_1, \delta_{\epsilon,\bar{\epsilon}}\Phi_2 |0\rangle = 0. \tag{106}$$

From here, as before in the arbitrary dimension case, we can work out the constraints of each type of conformal transformation on the correlation function.

Translation:

$$\epsilon(z) = \epsilon \implies G^{(2)}(\underline{z}, \underline{\bar{z}}) \propto z_{12} = z_1 - z_2 \text{ and } \bar{z}_{12} = \bar{z}_1 - \bar{z}_2. \tag{107}$$

Rotation & Dilation:

$$\epsilon(z) = z \implies G^{(2)}(\underline{z}, \underline{\bar{z}}) = \frac{c_{12}}{z_{12}^{h_1+h_2} \ \bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}. \tag{108}$$

SCT:

$$\epsilon(z) = z^2 \implies G^{(2)}(\underline{z}, \underline{\bar{z}}) = \frac{c_{12}}{z_{12}^{2h} \ \bar{z}_{12}^{2\bar{h}}}. \tag{109}$$

Now, for example, suppose we have a bosonic field which applies the constraint $h - \bar{h} = 0$ to the conformal weights. Setting $h + \bar{h} = \Delta$, the analytically continued two-point correlation function for the bosonic field becomes

$$G^{(2)}(\underline{z}, \underline{\bar{z}}) = \frac{c_{12}}{|z_{12}|^{2\Delta}}. \tag{110}$$

Example: Three-Point Correlation

Similarly, the three-point correlation function has the form

$$G^{(3)}(\underline{z}, \underline{\bar{z}}) = c_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{-h_1+h_2+h_3} z_{31}^{h_1-h_2+h_3}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{-\bar{h}_1+\bar{h}_2+\bar{h}_3} \bar{z}_{31}^{\bar{h}_1-\bar{h}_2+\bar{h}_3}} \tag{111}$$

In summary, we have introduced what we consider quantum field theories and have analyzed the consequences of conformal invariance of these quantum theories. We have also analyzed subclasses of theories that obey additional constraints (e.g., $\Delta = h + \bar{h}$ is real).

5 Lecture 5: Quantum CFT: Ward Identities & Radial Quantization

We have been on this journey from classical conformal symmetries to the implementation of the symmetries in a quantum setting. Thus far, we have found that the constraints of conformal symmetry in the context of quantum field theory, in $(2+0)$, $(1+1)$, and $(d+1)$ dimensions, the two-point correlation functions must decay polynomially as

$$\langle 0| \hat{\phi}(x)\hat{\phi}(y) |0\rangle \propto \frac{1}{|x-y|^{\alpha}}. \tag{112}$$

Radial "Quantization"

Note that using quotes around quantization indicates that this approach is an inspired guess, and is not derived via a functor.

To build a quantum field theory with conforml symmetry, invariant under the group $\mathrm{Conf}(\mathbb{R}^{2,0})$, we need four pieces of data.

(1) We need a Hilbert space $\mathcal{H}$ which is preferrably an infinite, separable, kinematic space of states isomorphic to the vector space of linear operators $L^2(\mathbb{R})$, (2) a vacuum state, or a reference vector, $|0\rangle$, a subset of linear operators in the Hilbert space $\mathcal{L}(\mathcal{H})$ called *observables* $\hat{\phi}(x,y)$, $(x,y) \in \mathbb{R}^2$, which are distribution-valued objects, but can be thought of, without harm, as self-adjoint operators.

The choice of set of observables, the subset of linear operators from the Hilbert space, define and distinguish a quantum field theory from others. E.g., the bosonic quantum field and the hydrogen atom share the same Hilbert space, but their observables are different and define what is allowed to be measured in each quantum field theory.

Our conformal quantum field theory yields (projective) unitary representations of $\mathrm{Conf}(\mathbb{R}^{2,0})$. We can use analytic continuation to map to a Minkowski theory where we have representations of $\mathrm{Conf}(\mathbb{R}^{1,1})$.

Introduce the construction of "imaginary time"

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ it \end{pmatrix};\ t \in \mathbb{R}. \tag{113}$$

If possible, the correlation functions analytically continue, such that

$$\langle 0| \hat{\phi}(x,y)\hat{\phi}(0,0) |0\rangle = f(x,y) \to f(x,it). \tag{114}$$

If all n-point correlation functions are analytically continued in this way, it *often* happens that we get a mapping

$$G^{(n)}((x_1, y_1), \dots) \to G^{(n)}((x_1, t_1), \dots) \tag{115}$$

Which are invariant, respectively, under symmetry transformations

$$\mathrm{Conf}(\mathbb{R}^{2,0}) \to \mathrm{Conf}(\mathbb{R}^{1,1}). \tag{116}$$

This is the analytic continuation to a *Minkowski theory*, and the criteria to ensure that this mapping exists is called *reflection positivity*. See the work of Glimm and Jaffe for a rigorous account of this machinery.

With a Minkowski quantum field theory, enter the Heisenberg picture to apply time evolution, and analytically continue the "time" variable to define "inverse temperature" $t \to -i\beta$, which essentially gives the correlation function for the thermal state of the Hamiltonian $\hat{H}$ with inverse temperature β

$$\langle 0| \hat{\phi}(x,y)\hat{\phi}(0,0) |0\rangle = \langle 0| e^{-it\hat{H}}\hat{\phi}(x,0)e^{it\hat{H}}\hat{\phi}(0,0) |0\rangle \tag{117}$$

$$= \langle 0| e^{-\beta\hat{H}}\hat{\phi}(x,0)e^{\beta\hat{H}}\hat{\phi}(0,0) |0\rangle . \tag{118}$$

Digression: What does "physical" mean for this course?

To define a physical theory, we require *kinematics* and *observables*, which make for a perfectly fine physical theory, even without *dynamics*.

The kinematics of a system introduces a Hilbert space $\mathcal{H}$ and a set of density operators $\rho(\mathcal{H})$ that describe the states of the system, but do not allow measurement of those states. The observables of a system $\mathcal{O} \subset \mathcal{L}(\mathcal{H})$ are the measurements of the states of the system, and are labelled by points in the underlying manifold $\mathcal{M}$ of the theory, such that $x, x^\dagger, \mathbb{I} \in \mathcal{O}$, for $x \in \mathcal{M}$.

Dynamics of a system are introduced via the group of isometries on the manifold $G = \mathrm{Isom}(\mathcal{M})$, and, for a quantum field theory, we search for projective, unitary representations of G on the Hilbert space. The construct of time only comes in as a choice of the manifold and its one-dimensional subgroups that act like time, but time is not necessarily an axiom of a quantum theory.

End Digression.

So far, we have labelled our observables in our Minkowski CFT by $(\sigma^0, \sigma^1) \in \mathbb{R}^{1,1}$. We now complexify these coordinates by "Euclideanizing" and sending the timelike coordinate to be imaginary, such that $\sigma^0 \to i\sigma^0$, and

$$z = \sigma^1 + i\sigma^0 \text{ and } \bar{z} = \sigma^1 - i\sigma^0 \tag{119}$$

Which correspsonds to an analytic continuation of the n-point correlation functions

$$G^{(n)}(\underline{\sigma}_1, \underline{\sigma}_2, \dots) \to G^{(n)}(\underline{z}_1, \underline{\bar{z}}_1; \underline{z}_2, \underline{\bar{z}}_2; \dots). \tag{120}$$

In this complexifed, Euclidean spacetime, with locations defined by coordinates z and $\bar{z}$, compactify space to a cylinder, such that space corresponds to the transverse direction $\sigma^1 \to \sigma^1 + 2\pi$, and time corresponds to the longitudinal direction, and our complex coordinates are now

$$z = e^{\sigma^1 + i\sigma^0} \text{ and } \bar{z} = e^{\sigma^1 - i\sigma^0}. \tag{121}$$

Circles in the complex plane correspond to constant time, the radial direction on the cylinder, and the real coordinates σ^0, σ^1 are mapped to the complex coordinates as

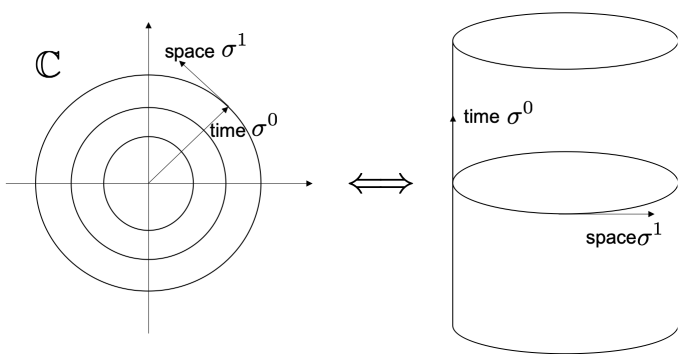
$$\sigma^0 = \infty \to z = \infty \text{ and } \sigma^0 = -\infty \to z = 0 \tag{122}$$

And transformations maps as

$$\text{Time reversal } [\sigma^0 \to -\sigma^0] \to \text{Inversion } [z \to \frac{1}{z}] \tag{123}$$

$$\text{Time translation } [\sigma^0 \to \sigma^0 + a] \to \text{Dilation } [z \to e^a z] \tag{124}$$

$$\text{Spatial translation } [\sigma^1 \to \sigma^1 + a] \to \text{Rotation } [z \to e^{ia} z] \tag{125}$$



Now we make a move to constructing a conformal field theory in complexified, Euclidean spacetime.

Ward Identities

Witha conformally invariant classical theory, we work towards getting a collection of observables obeying the correct symmetry for a conformally invariant quantum theory.

Recall from Noether's theorem that conserved charges correspond to classical symmetries Q obeying the anticommutation bracket

$$\{Q_j, Q_k\} = f_{jk}{}^{l} Q_l. \tag{126}$$

We may try to naively quantize by "puttings hats on" the conserved quantities that obey the commutation brackets

$$[\hat{Q}_j, \hat{Q}_k] = i f_{jk}{}^{l} \hat{Q}_l, \tag{127}$$

But without a Hilbert space, we can not write these operators as functions of the fields (e.g., $\hat{Q} = f(\hat{\phi})$.

From the classical symmetries, we now derive the quantum generators of the symmetries.

Suppose we have a classical field theory with action $S[\underline{\phi}]$, where $\underline{\phi}$ is a vector of classical fields. Assume that S is symmetric under the Lie group of infinitesimal transformations, such that

$$\underline{\phi}'(\underline{x}) = \underline{\phi}(\underline{x}) - i\omega_a(\underline{x})\mathbf{G}_a\underline{\phi}(\underline{x}) = e^{-i\omega_a(\underline{x})\mathbf{G}_a}\underline{\phi}(\underline{x}) \tag{128}$$

Where $\omega_a(\underline{x})$ is infinitesimal and $\mathbf{G}_a$ is a matrix acting on the vector labels of $\underline{\phi}$.

Use the path integral prescription to work out how this transformation affects the correlation functions

$$\langle 0| \hat{\underline{\phi}}(\underline{x}_1) \ldots \hat{\underline{\phi}}(\underline{x}_n) |0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(\underline{x}_1)\ldots\phi(\underline{x}_n) e^{iS[\underline{\phi}]}}{\int \mathcal{D}\phi\, e^{iS[\underline{\phi}]}}. \tag{129}$$

The first step is to change variables $\underline{\phi} \to \underline{\phi}'$ and assume that the measure is invariant $\mathcal{D}\phi' = \mathcal{D}\phi$. Define the operator $\hat{X}$ to be the time-ordered product of quantum fields

$$\hat{X} = \mathcal{T}[\hat{\underline{\phi}}(\underline{x}_1)\ldots\hat{\underline{\phi}}(\underline{x}_n)]. \tag{130}$$

Note that the time-ordering will be assumed from here on, and consider the expectation value of $\hat{X}$ in the path integral prescription

$$\langle\hat{X}\rangle = \frac{\int \mathcal{D}\phi' \left(\hat{\underline{\phi}}(\underline{x}_1)\ldots\hat{\underline{\phi}}(\underline{x}_n) + i\omega_a(\underline{x})\mathbf{G}_a(\hat{\underline{\phi}}(\underline{x}_1)\ldots\hat{\underline{\phi}}(\underline{x}_n) + \ldots\right) e^{i(S+\int dx\, \partial_\mu j^\mu_a \omega_a(\underline{x}))}}{\int \mathcal{D}\phi\, e^{iS}}. \tag{131}$$

To zeroth order in ω_a, this is exactly the correlation function prior to the change of variables. To first order in ω_a, we get the equality for quantum fields

$$\frac{\partial}{\partial x^\mu}\langle \hat{j}^\mu_a(\underline{x})\hat{\phi}(\underline{x}_1)\ldots\hat{\phi}(\underline{x}_n)\rangle = -i\sum_{j=1}^{n}\delta(\underline{x}-\underline{x}_j)\langle\hat{\phi}(\underline{x}_1)\ldots\mathbf{G}_a\hat{\phi}\left(\underline{x}_j\right)\ldots\hat{\phi}(\underline{x}_n)\Big)\rangle. \tag{132}$$

The generator of symmetry $\hat{j}^\mu_a(\underline{x})$ is handed over by the oath integral approach.

6 Lecture 6: Ward Identities

We are on our way to building a conformal quantum field theory (CQFT). So far, we have analyzed the consequences of conformal symmetries on the observables of a quantum field theory.

The observables of a quantum field theory, not necessarily self-adjoint as in quantum theory, are indirectly observable, as they are the correlation functions derivable from the direct observables of scattering amplitudes in scattering experiments.

We have found that the two-point correlation functions under the assumed constraints of a conformal field theory behave as

$$\langle \hat{\phi}_\alpha(x)\hat{\phi}_\beta(y)\rangle \sim \frac{1}{|x-y|^{d_\alpha+d_\beta}}. \tag{133}$$

To actually construct a conformal quantum field theory (CQFT), we use the path integral approach as an efficient tool to (hopefully) yield proper, according to the theory at hand, quantum observables. In the path integral sppraoch, we have a "box" into which we put classical data, namely the action $S[\phi]$, a functional of the fields, and get wuantum data as output in the form of time-ordered correlation functions

$$\langle \mathcal{T}[\hat{\phi}(x_1)\dots\hat{\phi}(x_n)]\rangle \equiv \frac{\int \mathcal{D}\phi\,\phi(x_1)\dots\phi(x_n)e^{iS[\phi]}}{\int \mathcal{D}\phi\, e^{iS[\phi]}}. \tag{134}$$

Goal: Our goal is to take a classical CFT and calculate the quantum generators of conformal symmetries corresponding to some CQFT.

Note on imaginary time:

We work with imaginary time $t \to -i\beta$ via analytic continuation, also called Wick rotation for two reasons.

1. It is convenient since things will converge better.
2. It makes direct contact with statistical physics.

To the second point, Wick rotation of the time coordinate turns all formulae for a quantum system, eith unitary time evolution, into statements about partition functions and thermal systems at temperature β. In other words, Wick rotation maps the unitary generators of time translation to a non-unitary semigroup with fixed points being the ground state

$$U(t) = e^{-it\hat{H}} \to S(\beta) = e^{-\beta\hat{H}}. \tag{135}$$

From $S(\beta)$, we simply take the trace and get the partition function, for a system in thermal equilibrium with a reservoir at inverse temperature β

$$Z = \mathrm{tr}(S(\beta)) = \mathrm{tr}(e^{-\beta\hat{H}}). \tag{136}$$

This contact with statistical physics and thermal states of systems is also interesting because in studying phase transitions, at critical points, the symmetry group is enlarged, and the system exhibits, for example, dilation and scale invariance. Phase transitions are examples of conformal field theories, though with non-unitary representations of the symmetry group. See *Cardy, Scaling & Phase Transitions* for reference.

Since our goal is to construct unitary representations of the conformal group of Minkowski space, for the reasons above, we Wick rotate and work with imaginary time. After our calculations, we Wick rotate back into real time, as much as allowed by holomorphic functions.

The path integral, our tool for calculating quantum obervables, under Wick rotation $t \to -i\beta$ transforms as

$$\int \mathcal{D}\phi\, e^{iS[\phi]} \to \int \mathcal{D}\phi\, e^{iS[\phi]}. \tag{137}$$

Constructing a CQFT

Suppose we have a tuple of classical fields $\phi(x)$, and let S be the action of the fields with assumed invariance under infinitesimal conformal symmetry transformations (up to a total derivative).

$$\phi(x) \to \phi'(x) = \phi(x) - i\omega_a(x) G_a \phi(x) \tag{138}$$

Where G_a is a symmetry transformation matrix. For example,

$$\text{Translations: } G_a \simeq -i\partial_\nu \tag{139}$$

$$\text{Dilations: } G_a \simeq -ix^\nu \partial_\nu - i\Delta_a. \tag{140}$$

Note that time-ordering of quantum field operators is implicit whenever the LHS is a correlator and the RHS is a path integral, and define

$$\hat{X} \equiv \mathcal{T}[\hat{\phi}(x_1)\dots\hat{\phi}(x_n)]. \tag{141}$$

By the path integral prescription, the time-ordered quantum correlation function is given by classical (not time-ordered) data, after Wick rotation, as

$$\langle \hat{X} \rangle = \frac{1}{Z}\int \mathcal{D}\phi\, X e^{iS[\phi]} = \frac{1}{Z}\int \mathcal{D}\phi\, \phi(x_1)\dots\phi(x_n) e^{iS[\phi]} \tag{142}$$

Where we have written the partition function Z, since we have Wick-rotated into imaginary time and

$$\int \mathcal{D}\phi \, e^{iS} \to \mathrm{tr}(e^{-\beta H}) = Z. \tag{143}$$

Under the infinitesimal transformation above, *assume* that the measure is invariant, such that $\mathcal{D}\phi' = \mathcal{D}\phi$. Recall that the action is invariant up to a total derivative. Then the correlation function, including the product of time-ordered quantum field operators, becomes

$$\langle \hat{X} \rangle = \frac{1}{Z} \int \mathcal{D}\phi' \, (X + \delta X) e^{-S[\phi] - \int dx \, \partial_\mu j_a^\mu \omega_a(x)} \tag{144}$$

Where $X' = X + \delta X$, and to order ω_a

$$X' = (\phi(x_1) - i\omega_a G_a \phi(x_1))(\phi(x_2) - i\omega_a G_a \phi(x_2)) \ldots \tag{145}$$

Which makes the infinitesimal change in X to be

$$\delta X = -i \sum_{j=1}^{n} \left(\phi(x_1) \ldots G_a \phi(x_j) \ldots \phi(x_n)\right) \omega_a(x_j) \tag{146}$$

$$= -i \int dx \; \omega_a(x) \sum_{j=1}^{n} \left(\phi(x_1) \ldots G_a \phi(x_j) \ldots \phi(x_n)\right) \delta(x - x_j). \tag{147}$$

Insert this into the equation for $\langle \hat{X} \rangle$ and distribute to get

$$\langle \hat{X} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \left(X e^{-S} + \delta X e^{-S} + X e^{-S} \left(\int dx \partial_\mu j_a^\mu \omega_a(x) \right) \right) \tag{148}$$

$$\langle \hat{X} \rangle = \langle \hat{X} \rangle + \langle \delta \hat{X} \rangle - \int dx \, \partial_\mu \langle \hat{j}_a^\mu \hat{X} \rangle \omega_a(x) \tag{149}$$

$$0 = \langle \delta \hat{X} \rangle - \int dx \, \partial_\mu \langle \hat{j}_a^\mu \hat{X} \rangle \omega_a(x). \tag{150}$$

By the infintesimal transformation and the invariance of the action, we have inserted the conserved classical current into the path integral presecription and have gotten a quantum operator in return! We find that the quantization of the infinitesimal change δX is equal to the integral of the divergence of the quantization of the product $j_a^\mu X$.

From this equality, which holds for all $\omega_a(x)$, extract the local relation of how the quantum fields transform according to the transformation-generating conserved current. Pull off the integral by the insertion of the delta function into δX above. The local equality, called the *Ward identity* is then

$$\frac{\partial}{\partial x^\mu} \langle \hat{j}_a^\mu(x) \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \rangle = -i \sum_{j=1}^{n} \delta(x - x_j) \langle \hat{\phi}(x_1) \ldots G_a \hat{\phi}(x_j) \ldots \hat{\phi}(x_n) \rangle \tag{151}$$

The Ward identity is the principal tool by which we quantize symmetries and showing how the symmetries are implemented on quantum correlation functions.

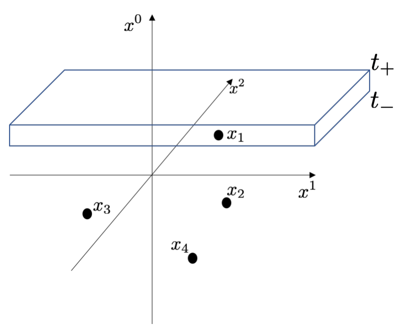
Integrate the Ward identity over all of spacetime (assuming good decay), and show that the quantum correlation function is invariant under the infinitesimal symmetry transformation

$$\delta_\omega \langle \hat{\phi}(x_1) \dots \hat{\phi}(x_n) \rangle \equiv -i\omega_a(x) \sum_{j=1}^{n} \langle \hat{\phi}(x_1) \dots G_a \hat{\phi}(x_j) \dots \hat{\phi}(x_n) \rangle = 0. \tag{152}$$

Recall that the conserved charge corresponding to the *generator* of some symmetry is gotten by integrating the time component of the conserved current

$$\hat{Q}_a = \int d^{d_a}x \; \hat{j}^0_a(x). \tag{153}$$

Integrate the Ward identity now over a thin slice of time $t_- < t < t_+$, instead of the entire spacetime, and suppose that there is one point of time $x^0_1 = t$ that is contained within the slice, and all others are sufficiently far away.



The LHS of the Ward identity becomes

$$\langle \hat{Q}_a(t_+) \hat{\phi}(x_1) \hat{Y} \rangle - \langle \hat{Q}_a(t_-) \hat{\phi}(x_1) \hat{Y} \rangle = -i \langle G_a \hat{\phi}(x_1) \hat{Y} \rangle \tag{154}$$

Where $\hat{Y} = \hat{\phi}(x_2) \dots \hat{\phi}(x_n)$.

Let the inverse temperature go to infinity $\beta \to \infty$, such that we have the vacuum correlation function $\langle \hat{X} \to \langle 0| \hat{X} |0\rangle$. Suppose that there is a point of time ahead of the slice-contained point $x^0_j > x^0_1$, and in the limit of an inifnitesimal time slice$t_- \to t_+$, the expression for the Ward identity above becomes

$$\langle 0| [\hat{Q}_a, \hat{\phi}(x_1)] \hat{Y} |0\rangle = -i \langle 0| G_a \hat{\phi}(x_1) |0\rangle . \tag{155}$$

This is true for all $\hat{Y}$, such that we have the commutator bracket, starting with classical data, for conserved charges and quantum field operators that shows that passing the quantum conserved charge operator over the field is tantamount to applying the symmetry transformation G_a

$$[\hat{Q}_a, \hat{\phi}] = -iG_a\hat{\phi}. \tag{156}$$

For example, translations transform the field as $\phi'(x) = \phi(x) - \epsilon^\mu \partial_\mu \phi(x)$ with symmetry transformation matrix $G_a = i\partial_\mu$, and the conserved charge operator is the total momentum operator $\hat{Q}_\mu \equiv \hat{P}_\mu$.

Ward Identities for Conformal Symmetries

Let $\hat{X}$ be the product of local quantum field operators as before, and consider the invariants of conformal symmetry transformations. The conserved current is the *energy-momentum tensor* $\hat{T}^\mu_\nu$, where we assume that T^ν_μ (classical) is traceless $T^\mu_\mu = 0$ and symmetric $T_{\mu\nu} = T_{\nu\mu}$, which does not necessarily hold quantumly.

Translation:

$$\partial_\mu \langle \hat{T}^\mu_\nu \hat{X}\rangle = -i\sum_j \delta(x - x_j)\partial_{x^\nu_j}\langle \hat{X}\rangle. \tag{157}$$

Rotation:

$$\langle (\hat{T}^{\rho\nu} - \hat{T}^{\nu\rho})\hat{X}\rangle = -i\sum_j \delta(x - x_j) s^{\nu\rho}_j \langle \hat{X}\rangle \tag{158}$$

Since $j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$ (**Exercise**). Note that $s^{\nu\rho}_j$ is the spin matrix representation of the j^{th} field vector, and is equal to one for spinless particles.

Dilation:

$$\langle \hat{T}^\mu_\mu \hat{X}\rangle = -\sum_j \delta(x - x_j)\Delta_j \langle \hat{X}\rangle. \tag{159}$$

Since $G \equiv D = -ix^\nu \partial_\nu - i\Delta$ (**Exercise**).

Next, we work out the two-dimensional case and rewrite the Ward identities in complex coordinates. We will then work out examples of energy-momentum tensors for free bosons, and see that the remaining degrees of freedom of the energy-momentum tensor for the *Virasoro algebra*.

7 Lecture 7: Ward Identities in $(2+0)d$ and Radial Quantization

Suppose we have a quantum theory with a set of observables in the linear operator space of a Hilbert space $\mathcal{O} \subset \mathcal{L}(\mathcal{H})$, and a field theory with a subset of field operators $\mathcal{F} \subset \mathcal{O}$, where the fields are labelled by position and spin, for example, $\hat{\phi}_{x,\alpha} \in \mathcal{F}$. This turns into a quantum field theory when we have symmetries that act on the field observables. For example, the symmetry of translation invariance in an expectation value has the form

$$\begin{aligned}\langle A| \hat{\phi}_{x+\epsilon,\alpha} |B\rangle &= f(x+\epsilon, \alpha; |A\rangle, |B\rangle) && (160)\\ &= f(x, \alpha; |A\rangle, |B\rangle) + \epsilon \frac{\partial f}{\partial x} + \mathcal{O}(\epsilon^2). && (161)\end{aligned}$$

Note that $\hat{\phi}_{x,\alpha}$ is an operator-valued distribution (like a delta function whose values are operators, not numbers) and is not mathematically well-defined outside of the expectation value. This equality should be true for all "nice" $|A\rangle$ and $|B\rangle$.

Notationally, we express the above as (omitting the expectation value)

$$\hat{\phi}_{x+\epsilon,\alpha} = \hat{\phi}_{x,\alpha} + \epsilon \partial_x \hat{\phi}_{x,\alpha}. \tag{162}$$

Translation invariance, acting on spacetime four-vectors, also includes time translation invariance. If $\epsilon = (\epsilon, 0, 0, 0)$, then

$$\hat{\phi}_{x+\epsilon,\alpha} = \hat{\phi}_{x,\alpha} + \epsilon \partial_t \hat{\phi}_{x,\alpha}. \tag{163}$$

As symmetries, the translation transformation vector acts on unitaries of the Hilbert space

$$U(\epsilon) = e^{-i\epsilon \cdot \hat{P}}, \text{ where } \hat{P} = (\hat{H}, \hat{P}_x, \hat{P}_y, \hat{P}_z). \tag{164}$$

In the Heisenberg picture, we can write the time translation transformation (time evolution) in another way with commutator brackets

$$\hat{\phi}_{x+\epsilon,\alpha} = U^\dagger(\epsilon) \hat{\phi}_{x,\alpha} U(\epsilon) = \hat{\phi}_{x,\alpha} + i[\epsilon^\mu \hat{P}_\mu, \hat{\phi}_{x,\alpha}]. \tag{165}$$

So, these are two ways to carry out the symmetry transformation, and the Ward identity gives us a way to connect them, inside correlation functions, show that they are equivalent, and then calculate the generators from the path integral prescription. Comparing the results of the last lecture, for time translation invariance

$$[\hat{Q}_a, \hat{\phi}] = -iG_a \hat{\phi} \equiv [\hat{H}, \hat{\phi}_{x,\alpha}] = -i\partial_t \hat{\phi}_{x,\alpha}. \tag{166}$$

Recall from the last lecture, that we derived the Ward identity for conformal transformations

Translation: $$\partial_\mu \langle \hat{T}^\mu_\nu \hat{X} \rangle = -i \sum_j \delta(x - x_j) \partial_{x_j^\nu} \langle \hat{X} \rangle \tag{167}$$

Rotation: $$\langle (\hat{T}^{\rho\nu} - \hat{T}^{\nu\rho}) \hat{X} \rangle = -i \sum_j \delta(x - x_j) s_j^{\nu\rho} \langle \hat{X} \rangle \tag{168}$$

Dilation: $$\langle \hat{T}^\mu_\mu \hat{X} \rangle = -\sum_j \delta(x - x_j) \Delta_j \langle \hat{X} \rangle. \tag{169}$$

We noted that although the stress-energy tensor is classically traceless $T^\mu_\mu = 0$, yielding a zero vacuum expectation value, it is not necessarily traceless quantumly $\hat{T}^\mu_\mu \neq 0$, but there is the fact that

$$\langle 0 | \hat{T}^\mu_\mu(x) \hat{T}^\mu_\mu(0) | 0 \rangle = 0. \tag{170}$$

Ward Identities in $(2+0)d$ and Complex Coordinates

Our task today, is to specialize to $(2+0)d$ in complex coordinates $ds^2 = dx^2 + dy^2 = dzd\bar{z}$, since

$$ds^2 = g_{zz} dzdz + g_{z\bar{z}} dzd\bar{z} + g_{\bar{z}z} d\bar{z}dz + g_{\bar{z}\bar{z}} d\bar{z}d\bar{z} \tag{171}$$
$$= 0 \cdot dzdz + \frac{1}{2} \cdot dzd\bar{z} + \frac{1}{2} \cdot d\bar{z}dz + 0 \cdot d\bar{z}d\bar{z} = dzd\bar{z}. \tag{172}$$

Vector quantities transform into complex coordinates as

$$F = F^x \partial_x + F^y \partial_y = F^z \partial_z + F^{\bar{z}} \partial_{\bar{z}} \tag{173}$$

Where $F^z = F^x + iF^y$ and $F^{\bar{z}} = F^x - iF^y$.

So, elements of the stress-energy tensor transform as $T_{\mu\nu} dx^\mu dx^\nu \to T_{\mu\nu} dz^\mu d\bar{z}^\nu$, where

$$T_{zz} = \frac{1}{4}(T_{xx} - 2iT_{xy} - T_{yy}) \tag{174}$$
$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{xx} + 2iT_{xy} - T_{yy}) \tag{175}$$
$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{xx} + T_{yy}). \tag{176}$$

And the Ward identities in the complex coordinates are

$$\textbf{Translation, holomorphic:} \tag{177}$$

$$2\pi\partial_z\langle\hat{T}_{\bar{z}z}\hat{X}\rangle + 2\pi\partial_{\bar{z}}\langle\hat{T}_{zz}\hat{X}\rangle = -\sum_{j=1}^{n}\partial_{\bar{z}}\left(\frac{1}{z-w_j}\right)\partial_{w_j}\langle\hat{X}\rangle \tag{178}$$

$$\textbf{Translation, anti-holomorphic:} \tag{179}$$

$$2\pi\partial_z\langle\hat{T}_{\bar{z}\bar{z}}\hat{X}\rangle + 2\pi\partial_{\bar{z}}\langle\hat{T}_{z\bar{z}}\hat{X}\rangle = -\sum_{j=1}^{n}\partial_{z}\left(\frac{1}{\bar{z}-\bar{w}_j}\right)\partial_{\bar{w}_j}\langle\hat{X}\rangle \tag{180}$$

$$\textbf{Rotation:} \tag{181}$$

$$-2\pi\langle\hat{T}_{z\bar{z}}\hat{X}\rangle + 2\pi\langle\hat{T}_{\bar{z}z}\hat{X}\rangle = -\sum_{j=1}^{n}\partial_{\bar{z}}\left(\frac{1}{z-w_j}\right)s_j\langle\hat{X}\rangle \tag{182}$$

$$\textbf{Dilation:} \tag{183}$$

$$2\pi\langle\hat{T}_{z\bar{z}}\hat{X}\rangle + 2\pi\langle\hat{T}_{\bar{z}z}\hat{X}\rangle = -\sum_{j=1}^{n}\partial_{\bar{z}}\left(\frac{1}{z-w_j}\right)\Delta_j\langle\hat{X}\rangle \tag{184}$$

Where we used the identity $\delta(x) = \frac{1}{\pi}\partial_{\bar{z}}\left(\frac{1}{z}\right)$ and $w_j = x_j + iy_j$ is a complex number.

Digression: The delta function identity used above is a consequence of Gauss' theorem.

$$\int_{\mathcal{M}} d^2x\,\partial_\mu F^\mu = \int_{\partial\mathcal{M}} d\xi_\mu F^\mu = \frac{i}{2}\oint_{\partial\mathcal{M}}\left(-dzF^{\bar{z}} + d\bar{z}F^z\right) \tag{185}$$

Where the last equality is gotten by integration by parts, applying Guass' theorem, and the identity from residue calculus $\int_{\mathcal{M}} d^2x\,\delta(x)f(x) = f(0)$.

Add and subtract the Ward identities for **rotation** and **dilation** and substitute them into the **holomorphic** and **antiholomorphic translation** Ward identities to get a more compact form which only includes diagonal elements of the stress-energy tensor

$$\textbf{Holomorphic:} \tag{186}$$

$$\partial_{\bar{z}} \left(\langle \hat{T}(z,\bar{z})\hat{X} \rangle - \sum_j \left(\frac{1}{z-w_j} \partial_{w_j} \langle \hat{X} \rangle + \frac{h_j}{(z-w_j)^2} \langle \hat{X} \rangle \right) \right) = 0 \tag{187}$$

$$\textbf{Anti-holomorphic:} \tag{188}$$

$$\partial_{z} \left(\langle \hat{\bar{T}}(z,\bar{z})\hat{X} \rangle - \sum_j \left(\frac{1}{\bar{z}-\bar{w}_j} \partial_{\bar{w}_j} \langle \hat{X} \rangle + \frac{h_j}{(\bar{z}-\bar{w}_j)^2} \langle \hat{X} \rangle \right) \right) = 0 \tag{189}$$

$$\tag{190}$$

Where, notationally, we have written $\hat{T}(z,\bar{z}) = -2\pi \hat{T}_{zz}$ and $\hat{\bar{T}}(z,\bar{z}) = -2\pi \hat{T}_{\bar{z}\bar{z}}$. We've also used the definition of the conformal weights for the scaling dimension $\Delta_j = h_j + \bar{h}_j$ and the spin $s_j = h_j - \bar{h}_j$.

From these equalities, fropping $\hat{T}(z,\bar{z}) = \hat{T}(z)$, and similiarly for the anti-holomorphic equality, we derive that

$$\langle \hat{T}(z)\hat{X} \rangle = \sum_j \left(\frac{1}{z-w_j} \partial_{w_j} \langle \hat{X} \rangle + \frac{h_j}{(z-w_j)^2} \langle \hat{X} \rangle \right) + (\text{regular functions of } z) \tag{191}$$

$$\langle \hat{\bar{T}}(\bar{z})\hat{X} \rangle = \sum_j \left(\frac{1}{\bar{z}-\bar{w}_j} \partial_{\bar{w}_j} \langle \hat{X} \rangle + \frac{h_j}{(\bar{z}-\bar{w}_j)^2} \langle \hat{X} \rangle \right) + (\text{regular functions of } \bar{z})\,. \tag{192}$$

Note that the off-diagonal elements such as $\hat{T}_{z\bar{z}} = \hat{T}^x_{\ x} + \hat{T}^y_{\ y}$ and $\hat{T}_{\bar{z}z}$ are nonzero, but are not present in these equations.

Under an infinitesimal conformal transformation $x \to x + \epsilon$, all Ward identities are contained in the expression

$$\delta_{\epsilon,\bar{\epsilon}} \langle \hat{X} \rangle = \int_{\mathcal{M}} d^2x \, \partial_\mu \langle \hat{T}^{\mu\nu}(x) \epsilon_\nu(x) \hat{X} \rangle. \tag{193}$$

To see this, use the fact that for an infinitesimal conformal transformation, recall $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}$, which yields the two conditions

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \partial_\rho \epsilon^\rho \eta_{\mu\nu} \tag{194}$$

$$\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu = \epsilon^{\alpha\beta} \partial_\alpha \epsilon_\beta. \tag{195}$$

Then integrate $\partial_\mu(\epsilon_\nu T^{\mu\nu})$ and use Gauss' theorem to get the integrated for of the Ward identity

$$\delta_{\epsilon,\bar{\epsilon}}\langle\hat{X}\rangle = \frac{1}{2}i\oint_C \left(-dz\langle\hat{T}^{\bar{z}\bar{z}}\epsilon_{\bar{z}}\hat{X}\rangle + d\bar{z}\langle\hat{T}^{zz}\epsilon_z\hat{X}\rangle\right). \tag{196}$$

Where do we get expressions such as $\langle\hat{X}\rangle = \langle\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\rangle$?

We get them from path integrals in Euclidean space (imaginary time). A correlator in Euclidean space is given by

$$\langle\hat{A}_1(x_1, i\beta_1)\ldots\hat{A}_n(x_n, i\beta_n)\rangle = \langle\hat{A}_1(x_1, 0)e^{-\hat{H}(\beta_1-\beta_2)}\ldots e^{-\hat{H}(\beta_{n-1}-\beta_n)}\hat{A}_n(x_n, 0)\rangle \tag{197}$$

Where $\beta_j > \beta_{j-1}$ and the path integral obeys time ordering.

Substitute $\hat{\phi}_j = \hat{A}_j(x_j, i\beta_j)$, and then the path integral refelcts symmetries of Euclidean space, rather than Minkowski space, and the symmetries are implemented non-unitarily. E.g., for rotations with $t \to i\beta$,

$$\partial_\mu\phi\partial^\mu\phi = \partial_t\phi\partial_t\phi + \partial_x\phi\partial_x\phi = \partial_\beta\phi\partial_\beta\phi + \partial_x\phi\partial_x\phi. \tag{198}$$

8 Lecture 8: Radial Quantization and the OPE

The *operator prodtuct expansion* (OPE) for the stress-energy tensor of a conformal quantum field theory is derived from the general structure of radial quantization, and is the primary algebraic tool in conformal field theory, and will later be used to introduce the Virasoro algebra.

In the previous lecture(s), we have been building a conformal quantum field theory from classical theories using the path integral in Euclidean space with the n-point correlation functions defined by

$$\langle \mathcal{T}[\hat{\phi}(x_1)\dots\hat{\phi}(x_n)]\rangle = \lim_{\beta\to\infty} \frac{\int \mathcal{D}\phi\, \phi(x_1)\dots\phi(x_n)e^{-S[\phi]}}{\int \mathcal{D}\phi\, e^{-S[\phi]}}. \tag{199}$$

The choice of coordinate axis for "time" leads to a different time-ordering symbol, Ward identities, and Hilbert space of states. For example, in Cartesian coordinates $\beta \equiv y$, and our quantum system exists on an infinitely long one-dimensional line. In polar coordinates $\beta \equiv r$, and our quantum system exists on a circle.

There are several good reasons to quantize to a circle. One: IR divergences are avoided. Two: it is more physical, since a quantum system on a circle is preparable in the lab. Three: it is more mathematically interesting due to the diffeomorphisms of the circle S^1.

For these reasons, we work in polar coordinates

$$(x, y) \to (r, \theta) = \left(\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right). \tag{200}$$

In polar coordinates, spacetime forms a cylinder, rather then a plane or sheet, and "equal-time" slices are referred to as "equal-radius", and we will have equal-radius commutators for observables.

Introduce the *radial-ordering symbol*

$$\mathcal{R}[\hat{A}(z)\hat{B}(w)] = \hat{A}(z)\hat{B}(w),\ |z| > |w|\hat{B}(w)\hat{A}(z),\ |z| < |w|. \tag{201}$$

Note that we will often leave out the radial-ordering symbol, such that whenever we write the expectation value of a product of quantum field operators, unless noted otherwise, we mean the vacuum expectation value of the radial-ordering of the operators

$$\langle \hat{T}(z)\hat{A}(w)\rangle \equiv \langle 0|\, \mathcal{R}[\hat{T}(z)\hat{A}(w)]\, |0\rangle\,. \tag{202}$$

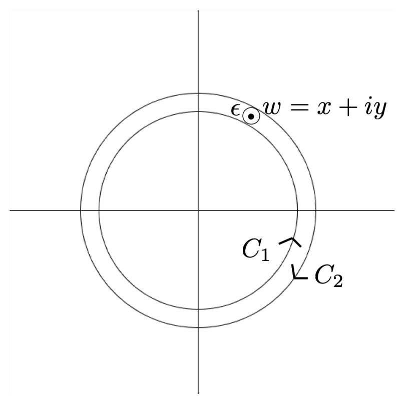
Now to make sense of the *equal-radius* commutators, in analogy to the equal-time integral $\int dx$, which integrates over the remaining three spatial degrees of freedom in the Ward identity, we replace this Cartesian integral with the polar, complex analog

$$\int dx \to \oint dz. \tag{203}$$

This contour integral will be over products of operators $\hat{a}(z)\hat{b}(w)\ldots$.

We want to build a "conserved charge" $\hat{A} = \oint dz\ \hat{a}(z)$ from these operators, and then use them to build equal-radius (equal-time) commutators like $[\hat{A}, \hat{b}(w)]$. and $\hat{a}$.

Consider the following set of contours C_1 and C_2 in the complex plane surrounding the point $w = x + iy$ by a distance ϵ, the radius of the enclosed contour W.



So, the commutator above can be written, assuming everything is holomorphic and nice,

$$[\hat{A}, \hat{b}(w)] = \oint_W dz\ \hat{a}(z)\hat{b}(w) \tag{204}$$

$$= \oint_{C_1} dz\ \hat{a}(z)\hat{b}(w) - \oint_{C_2} dz\ \hat{b}(w)\hat{a}(z). \tag{205}$$

Graphically, we can represent this as
Remember that expressions such as $\hat{a}(z)\hat{b}(w)\cdots = \hat{c}(z)\hat{d}(w)\ldots$ are valid only within evacuum xpectation values with radial ordering of the arguments.

Now, apply this to primary fields of a conformal field theory, and let $\hat{\phi}(w, \bar{w})$ be a primary field and vary the field with respect to an infinitesimal conformal transformation, which gives us the conformal Ward identity (which must obey the differential equation for the infinitesimal conformal transformation $(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu)(\partial \cdot \epsilon) = 0$).

$$\delta_{\epsilon,\bar{\epsilon}}\hat{\phi}(w,\bar{w}) = \frac{1}{2\pi i}\left(\oint dz\,\epsilon(z)\mathcal{R}[\hat{T}(z)\hat{\phi}(w,\bar{w})] + \oint d\bar{z}\,\bar{\epsilon}(\bar{z})\mathcal{R}[\hat{\bar{T}}(\bar{z})\hat{\phi}(w,\bar{w})]\right) \tag{206}$$

$$= \frac{1}{2\pi i}\left(\oint_{C_1} dz\,\epsilon(z)\mathcal{R}[\hat{T}(z)\hat{\phi}(w,\bar{w})] - \oint_{C_2} d\bar{z}\,\bar{\epsilon}(\bar{z})\mathcal{R}[\hat{\bar{T}}(\bar{z})\hat{\phi}(w,\bar{w})]\right) \tag{207}$$

$$= \frac{1}{2\pi i}\left(\oint dz\,\epsilon(z)[\hat{T}(z),\hat{\phi}(w,\bar{w})] + \oint d\bar{z}\,\bar{\epsilon}(\bar{z})[\hat{\bar{T}}(\bar{z}),\hat{\phi}(w,\bar{w})]\right). \tag{208}$$

Recall that in the previous lecture we derived the conformal Ward identity to be

$$\delta_{\epsilon,\bar{\epsilon}}\langle\hat{X}\rangle = \frac{i}{2}\oint_C \left(-dz\ \langle\hat{T}^{\bar{z}\bar{z}}\epsilon_{\bar{z}}\hat{X}\rangle + d\bar{z}\ \langle\hat{T}^{zz}\epsilon_z\hat{X}\rangle\right) \tag{209}$$

Where $\hat{T}(z) = -2\pi\hat{T}_{zz}$ and $\hat{\bar{T}}(\bar{z}) = -2\pi\hat{T}_{\bar{z}\bar{z}}$.

We now claim that the Ward identity derived above is the same as the one derived last lecture, when using the metric to raise and lower indices (e.g., $T_{zz} = T^{\bar{z}\bar{z}}$).

So, the conserved charge corresponding to an infinitesimal conformal symmetry

$$\hat{Q}_{\epsilon,\bar{\epsilon}} \equiv \frac{1}{2\pi i}\oint \left(dz\ \hat{T}(z)\epsilon(z) + d\bar{z}\ \hat{\bar{T}}(\bar{z})\bar{\epsilon}(\bar{z})\right). \tag{210}$$

For primary fields, by definition,

$$\hat{\phi}(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \hat{\phi}(f(z),\bar{f}(\bar{z})) \equiv \delta_{\epsilon,\bar{\epsilon}}\hat{\phi}(f(z),\bar{f}(\bar{z})) \tag{211}$$

Where $f(z) = z+\epsilon$ for an infinitesimal conformal transformation. Taylor expand $f(z)$ and substitute in the above to get, to order ϵ,

$$\delta_{\epsilon,\bar{\epsilon}}\hat{\phi}(z,\bar{z}) = h(\partial\epsilon(z))\hat{\phi}(z,\bar{z}) + \epsilon(z)(\partial\hat{\phi}(z,\bar{z})) + \bar{h}(\bar{\partial}\bar{\epsilon}(\bar{z}))\hat{\phi}(z,\bar{z}) + \bar{\epsilon}(\bar{z})(\bar{\partial}\hat{\phi}(z,\bar{z})). \tag{212}$$

For the conformal Ward identity and contour integrals to remain true under this constraint, this requires that

$$\mathcal{R}[\hat{T}(z)\hat{\phi}(w,\bar{w})] = \frac{h}{(z-w)^2}\hat{\phi}(w,\bar{w}) + \frac{1}{z-w}\partial_w\hat{\phi}(w,\bar{w}) + \text{holomorphic} \tag{213}$$

For the holomorphic parts. And, similarly for the anti-holomorphic parts

$$\mathcal{R}[\hat{\bar{T}}(\bar{z})\hat{\phi}(w,\bar{w})] = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\hat{\phi}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\hat{\phi}(w,\bar{w}) + \text{anti-holomorphic.} \tag{214}$$

These are the consequences of the conformal Ward identity for the stress-energy tensor and the primary fields. Note that $\hat{T}(z)$ may look like a primary field, but it is not.

We will rewrite these identities using the symbol $\sim$, which means "equal up to expressions which are regular as $z \to w$ (holomorphic) and under radial ordering and expectation values"

$$\hat{T}(z)\hat{\phi}(w,\bar{w}) \sim \frac{h}{(z-w)^2}\hat{\phi}(w,\bar{w}) + \frac{1}{z-w}\partial_w\hat{\phi}(w,\bar{w}). \tag{215}$$

This expression tells us about short-distance behavior of the stress-energy tensor and the primary field as $z \to w$. It is called the operator product expansion, or OPE, of $\hat{T}(z)$ and $\hat{\phi}(w,\bar{w})$.

Expressions that tell us about how products of quantum fields getting close to each other at these positions were introduced by Wilson in the context of QCD, but it is also popular in CFT. The expression is a sum of terms, each a single field operators multiplied by a possibly-singular complex-valued function. Such an expression is called an *operator product expansion* (OPE). The OPE behaves like an algebra, as it "multiplies" two things and yields another thing back.

In general, an OPE has the form

$$\hat{A}(z)\hat{B}(w) \sim \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{216}$$

Where $\{AB\}_n(w)$ are composite fields nonsingular at w. In our example, one of the composite fields is

$$\{\hat{T}\hat{\phi}\}_1 = \partial_w\hat{\phi}(w,\bar{w}). \tag{217}$$

In summary, we have used the conformal Ward identity to derive a constraint on the behavior of fields the arguemnts tend towards each other, and we wrote an operator product expansion for the stress-energy tensor and the primary field. Next, we will apply these tools to the massless free boson.

9 Lecture 9: The Free Boson

We've spent all of this energy on understanding the consequences of conformal symmetry on classical and quantum systems. Now, we finally take an example. The free massless boson provides a good building-block example, as it and copies of it can be used to construct all kinds of conformal field theories.

The Lagrangian of the (massless, $m = 0$) free boson, from Klein-Gordon theory, where we have already solved the dynamics of this system, is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_\mu \phi)^2. \tag{218}$$

We set $m = 0$ and drop the massive term, since $m \neq 0$ induces a length scale and the theory is then not conformally invariant.

Specialize to the Euclidean picture in $(2 + 0)$ dimensions, and write the action in complex coordinates

$$S = \frac{g}{2}\int dx\ (\partial_{x^0} - i\partial_{x^1})\phi(x)(\partial_{x^0} + i\partial_{x^1})\phi(x) \tag{219}$$

Where $x = (x^0, x^1)$ and $dx = dx^0dx^1$, and we write the action in terms of a differential operator (making four integrals)

$$S = \frac{1}{2}\int dxdy\ \phi(x)A(x,y)\phi(y) \tag{220}$$

Where $A(x, y) = -g\delta(x - y)\partial_x^2$.

The Euclidean path integral prescription calculates n-point functions by the usual

$$\langle \hat{\phi}_1 \dots \hat{\phi}_n \rangle = \lim_{\beta\to\infty} \frac{\int \mathcal{D}\phi\ \phi_1 \dots \phi_n e^{-S}}{\int \mathcal{D}\phi\ e^{-S}}. \tag{221}$$

Let's instead use the generating functional, as it is more convenient when working with a theory that is quadratic in the fields since we get all n-point correlations functions via differentation with respect to an external current field $J(x)$ and setting $J = 0$. The generating functional is

$$Z[J] = Z[0]e^{\frac{1}{2}\int dxdy\ J(x)K(x,y)J(y)} \equiv \int \mathcal{D}\phi\ e^{-S+\int dx J(x)\phi(x)} \tag{222}$$

Where $K(x, y)$ is the Green's function for a differential operator

$$-g\partial_x^2 K(x,y) = \delta(x - y). \tag{223}$$

The solution to this Laplacian equation (by Fourier transform) is the same as the electrostatic potential of a point charge

$$K(x,y) = -\frac{1}{2\pi g}\log(\sqrt{(x-y)^2}). \tag{224}$$

In polar coordinates

$$K(r) = -\frac{1}{2\pi g}\log(r) = -\frac{1}{4\pi g}\log(r^2). \tag{225}$$

Note that correlation nonintuitively *grows* with separation.

The two-point correlation function is exactly the Green's function

$$\langle\hat{\phi}(x)\hat{\phi}(y)\rangle = \frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}Z[J]\Big|_{J=0} = K(x,y). \tag{226}$$

Now, because we can, analytically continue the two-point function and define the complex coordinates

$$z = x^0 + ix^1 \text{ and } w = y^0 + iy^1 \tag{227}$$

And separate out the holomorphic and anti-holomorphic parts

$$\langle\hat{\phi}(z)\hat{\phi}(w)\rangle = -\frac{1}{4\pi g}\log((\bar{z}-\bar{w})(z-w)) = -\frac{1}{4\pi g}(\log(z-w)+\log(\bar{z}-\bar{w})). \tag{228}$$

Recall that Wick's theorem will give all of the n-point functions in terms of the two-point functions, after some combinatorics, since we are working with a Gaussian theory, quadratic in the fields.

Now, with this form of the two-point functions, $\hat{\phi}(z)$ is *not* a primary field, but $\partial_z\hat{\phi}(z)$ is a primary field. To check this, we will need to compute $\hat{T}(z)$ in terms of the stress-energy tensor. The holomorphic and anti-holomorphic two-point functions for the alleged primary fields are

$$\langle\partial_z\hat{\phi}(z)\partial_w\hat{\phi}(w)\rangle = -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \tag{229}$$

$$\langle\partial_{\bar{z}}\hat{\phi}(\bar{z})\partial_{\bar{w}}\hat{\phi}(\bar{w})\rangle = -\frac{1}{4\pi g}\frac{1}{(\bar{z}-\bar{w})^2} \tag{230}$$

Which has the correct form to transform as a primary field under a conformal transformation. So, the operator product expansion (OPE) of the two primary fields is

$$\partial_z\hat{\phi}(z)\partial_w\hat{\phi}(w) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2}. \tag{231}$$

Now, we will compute OPEs that include the stress-energy tensor.

Intrepretations of the stress-energy tensor:

1. Momentum flux of the μ^{th} component through the ν^{th} coordinate constant plane.
2. Conserved chrage corresponding to a translation in the μ^{th}, or ν^{th}, since symmetric, direction.
3. Variation of the action with respect to the metric $\frac{\delta S}{\delta g_{\mu\nu}}$.

From Noether's theorem, for the Klein-Gordon field, the (traceless) stress-energy tensor is

$$T_{\mu\nu} = g(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho \phi \partial^\rho \phi). \tag{232}$$

Since this theory is quadratic in the fields, we quantize by "putting hats on" and applying normal-ordering

$$\hat{T}_{\mu\nu} = g\mathbf{:}(\partial_\mu \hat{\phi} \partial_\nu \hat{\phi} - \frac{1}{2}\eta_{\mu\nu}\partial_\rho \hat{\phi} \partial^\rho \hat{\phi})\mathbf{:}. \tag{233}$$

Apply the coordinate transformation to complex coordinates $(x^0, x^1) \to (z, \bar{z})$ (**Exercise**)

$$\hat{T}(z) = -2\pi \hat{T}_{zz} = 2\pi \frac{1}{4}\mathbf{:}(\hat{T}_{00} - 2i\hat{T}_{10} - \hat{T}_{11})\mathbf{:} \tag{234}$$

$$= -2\pi g \mathbf{:}\partial_z \hat{\phi}(z) \partial_z \hat{\phi}(z)\mathbf{:}. \tag{235}$$

Compute the OPE of the stress-energy tensor and the primary field, which is equivalent to "putting $\hat{T}(z)\partial_w \hat{\phi}(w)$ inside an n-point function, computing its singular behavior as $z \to w$, and dropping the regular part". Skipping several steps, by Wick's theorem and Taylor expansion about w

$$\hat{T}(z)\partial_w \hat{\phi}(w) = -2\pi g \left(\mathbf{:}\partial_z \overline{\hat{\phi}(z) \partial_z \hat{\phi}(z)\mathbf{:}\partial_w \hat{\phi}}(w) + \mathbf{:}\partial_z \hat{\phi}(z) \partial_z \overline{\hat{\phi}(z)\mathbf{:}\partial_w \hat{\phi}}(w) \right) \tag{236}$$

$$= -4\pi g \partial_z \hat{\phi}(z) \langle \partial_z \hat{\phi}(z) \partial_w \hat{\phi}(w) \rangle \tag{237}$$

$$= \frac{1}{(z-w)^2} \partial_z \hat{\phi}(z) \tag{238}$$

$$= \frac{1}{(z-w)^2} (\partial_w \hat{\phi}(w) + (z-w)\partial_w^2 \hat{\phi}(w) + \dots) \tag{239}$$

$$\sim \frac{1}{(z-w)^2} \partial_w \hat{\phi}(w) + \frac{1}{z-w} \partial_w^2 \hat{\phi}(w). \tag{240}$$

And we see that the OPE of $\hat{T}(z)\partial_w \hat{\phi}(w)$ is a primary field with $h = 1$.

Now, compute the OPE of the stress-energy tensor with itself to see what the product of two local conformal transformations looks like, which we hope is another local conformal transformation. Again, by Wick's theorem and Taylor expansion

$$\hat{T}(z)\hat{T}(w) = 4\pi^2 g^2 :\partial_z\hat{\phi}(z)\partial_z\hat{\phi}(z): :\partial_w\hat{\phi}(w)\partial_w\hat{\phi}(w): \tag{241}$$
$$= \frac{1}{2}\frac{1}{(z-w)^4} - \frac{4\pi g}{(z-w)^2} :\partial_z\hat{\phi}(z)\partial_w\hat{\phi}(w): \tag{242}$$
$$\sim \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial_w\hat{T}(w)}{z-w}. \tag{243}$$

And we see that the OPE of $\hat{T}(z)\hat{T}(w)$ is *almost* a primary field with $h = 2$, but it is spoiled bu the inverse quartic term. The quartic term arises, since the stress-energy tensor is of rank 2, and under dilations, behaves as $\hat{T}(z) \to \frac{1}{\lambda^2}\hat{T}(\lambda z)$.

In general, the OPE of the stress energy tensor with itself has the form

$$\hat{T}(z)\hat{T}(w) \sim \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial_w\hat{T}(w)}{z-w}. \tag{244}$$

We call c the *central charge*. It is the central extension of the Witt algebra, which comes from the fact that we allow $\mathrm{Conf}(\mathbb{R}^{1,1})$ to act projectively. Since the two-point function is positive, the central charge is also positive $c \geq 0$.

Now, relate the OPE to an infinite Lie algebra that is obeyed by the generators of local conformal transformations: the *Virasoro algebra*.

The Virasoro algebra

Make a mode expansion (Laurent series defined in an annular region) of the stress-energy tensor

$$\hat{T}(z) \equiv \sum_{n\in\mathbb{Z}} z^{-n-2}\hat{L}_n \tag{245}$$
$$\hat{\bar{T}}(\bar{z}) \equiv \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\hat{\bar{L}}_n. \tag{246}$$

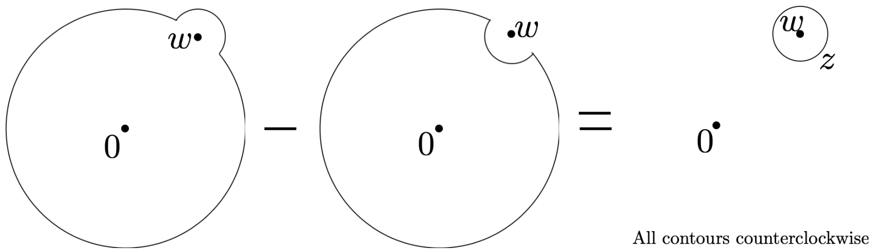
Invert these equations by multiplying by z^{n+1}, integrating around a circle, and applying the residue theorem

$$\hat{L}_n = \oint \frac{dz}{2\pi i}\, z^{n+1}\hat{T}(z) \tag{247}$$
$$\hat{\bar{L}}_n = \oint \frac{d\bar{z}}{2\pi i}\, \bar{z}^{n+1}\hat{\bar{T}}(\bar{z}). \tag{248}$$

Our goal is to work out the Lie algebra that quantumly generates conformal symmetry. To do this, we employ a commutator and contour integral trick that we've used before. Consider

$$[\hat{A},\hat{B}] = \oint_0 dw \oint_W dz\ \hat{a}(z)\hat{b}(w) \tag{249}$$

Where $\hat{A} = \oint dz\, \hat{a}(z)$ and $\hat{B} = \oint dz\, \hat{b}(z)$. Consider the contours



Where we fix w and then integrate around the circle around w to obtain the difference between the larger contours. Use OPE to do the integral and pick up singular behavior as $z \to w$, and the regular behavior naturally goes to zero.

Use this calculate the commutator of the $\hat{L}_n$ operators

$$[\hat{L}_n,\hat{L}_m] = \frac{1}{(2\pi i)^2}\oint_0 dw\, w^{m+1}\oint_W dz\, z^{n+1}\left(\frac{c}{2}\frac{1}{(z-w)^4}+\frac{2\hat{T}(w)}{(z-w)^2}+\frac{\partial_w\hat{T}(w)}{z-w}+\text{regular}\right) \tag{250}$$

$$= \frac{1}{2\pi i}\oint_0 dw\, w^{m+1}\left(\frac{c}{12}(n+1)n(n-1)w^{n-2}+2(n+1)w^n\hat{T}(w)+w^{n+1}\partial_w\hat{T}(w)\right) \tag{251}$$

$$= \frac{c}{12}n(n^2-1)\delta_{m+n,0}+2(n+1)\hat{L}_{m+n}-\frac{1}{2\pi i}\oint_0 dw\,(m+n+2)w^{m+n+1}\hat{T}(w) \tag{252}$$

$$[\hat{L}_n,\hat{L}_m] = \frac{c}{12}n(n^2-1)\delta_{m+n,0}+(-m+n)\hat{L}_{m+n}. \tag{253}$$

And similarly for the anti-holomorphic part $\hat{\bar{L}}_n$. And the "cross" commutators are zero

$$[\hat{L}_n,\hat{\bar{L}}_m] = 0. \tag{254}$$

The delta function piece of the commutator relation is a phase factor of the central extension of the Lie algebra, due to representing the Lie algebra projectively. This is the algebra of local conformal transformations, and is an infinite-dimensional Lie algebra. It contains the closed subalgebra formed by $\{\hat{L}_{-1},\hat{L}_0,\hat{L}_1\}$, which is isomorphic to the $SL_2(\mathbb{C})$ algebra. The global conformal transformations correspond to this three dimensional, or rank 1, subalgebra.

10 Lecture 10: The Hilbert Space of CFT

A Hilbert space is a vector space $\mathcal{H}$ with an inner product $\langle | \rangle$ which induces a topology such that $\mathcal{H}$ is complete with respect to $\langle | \rangle$.

How do we attach a Hilbert space to a conformal field theory?

In other words: "What are the states?", "What do the states correspond to?", "What are the observables and what do they correspond to?", "How do symmetries act on the Hilbert space (presumably unitarily)?".

We can use Wightman functions to build $\mathcal{H}$ for "good" quantum field theories. By "good", we mean that if we take data, namely n-point correlation functions, from a QFT and use them to build a (linear) vector space with an inner product, then we would eventually find out that we actually have a Hilbert space.

The *Wightman functions* are defined by

$$w_n \equiv \langle 0| \hat{\phi}_1(x_1) \dots \hat{\phi}(x_n) |0\rangle \,. \tag{255}$$

And define the ket vector by fields operating on the vacuum state

$$|x_1 \dots x_n\rangle \equiv \hat{\phi}(x_1) \dots \hat{\phi}(x_n) |0\rangle \tag{256}$$

Such that the inner product, assuming the fields are self-adjoint distribution-valued objects, is

$$\langle x_1 \dots x_n | y_1 \dots y_m\rangle = \langle 0| \hat{\phi}(x_1) \dots \hat{\phi}(x_n) \hat{\phi}(y_1) \dots \hat{\phi}(y_m) |0\rangle \,. \tag{257}$$

So, we actually use the Wightman functions $w_n(x_1, \dots, x_n; y_1, \dots, y_m)$ as "smeared out" versions of the operator products $\hat{\phi}(x_1) \dots \hat{\phi}(x_n) \hat{\phi}(y_1) \dots \hat{\phi}(y_m)$, and define f and g as *smearing functions*, with the inner product

$$\langle f_1 \dots f_n | g_1 \dots g_m\rangle = \int dx_1 \dots dx_n dy_1 \dots dy_m \; \bar{f}(x_1) \dots \bar{f}(x_n) g(y_1) \dots g(y_m) w(x_1, \dots, x_n; y_1, \dots, y_m). \tag{258}$$

This is roughly what is happening in today's argument.

So, we first denote the vacuum state of a conformal field theory by $|0\rangle$, and then associate higher states with operators.

Digression: Quantum Information Theory

Suppose we have the d-dimensional complex numbers as a Hilbert space $\mathcal{H} = \mathbb{C}^d$ and $d \times d$ matrices as operators on the space $A \in \mathcal{B}(\mathcal{H})$. Then we can associate a quantum state in $\mathcal{H} \otimes \mathcal{H}$ to A via the maximally entangled state

$$|\Phi^{+}\rangle = \sum_{j=1}^{d} \frac{1}{\sqrt{d}} |jj\rangle \text{ and } |A\rangle \equiv (A \otimes \mathbb{I}) |\Phi^{+}\rangle . \tag{259}$$

End Digression.

Let $\hat{A}(z, \bar{z})$ be a field associated to the "in" state

$$|A_{in}\rangle \equiv \lim_{\sigma^0 \to -\infty} \hat{A}(\sigma^0, \sigma^1) |0\rangle \tag{260}$$

$$= \lim_{\sigma^0 \to -\infty} e^{\sigma^0 \hat{H}} \hat{A}(0, \sigma^1) |0\rangle \tag{261}$$

$$= \lim_{z, \bar{z} \to 0} \hat{A}(z, \bar{z}) |0\rangle . \tag{262}$$

Since we compactified space (to the Riemann sphere), all points at infinity are the same point, and a single state coming from $-\infty$ can be associated to the operator $\hat{A}$.

Now identify an "out" state, as dual to the "in" state, as a bra vector

$$\langle A_{out}| \equiv \lim_{w, \bar{w} \to 0} \langle 0| \tilde{\hat{A}}(w, \bar{w}) \tag{263}$$

Where $w = \frac{1}{z}$ is a conformal transformation of the complex coordinates ($z = 0 \to w = \infty$), and the tilde operator is the conformally transformed operator with some multiplicative factors $\tilde{\hat{A}}(w, \bar{w}) = \ldots \hat{A}(z, \bar{z})$.

For a primary field $\hat{A}(z, \bar{z}) = \hat{\phi}(z, \bar{z})$, recall that we have the transformation law

$$f: \quad \hat{\phi}(w, \bar{w}) \to \hat{\phi}(f(w), \bar{f}(\bar{w}))(\partial_w f(w))^h (\partial_{\bar{w}} \bar{f}(\bar{w}))^{\bar{h}}. \tag{264}$$

So, the "out" state for a primary field looks like

$$\langle \phi_{out}| = \lim_{a, \bar{z} \to 0} \langle 0| \hat{\phi}\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) \frac{1}{z^{2h}} \frac{1}{\bar{z}^{2\bar{h}}} \tag{265}$$

$$\equiv \lim_{a, \bar{z} \to 0} \langle 0| \hat{\phi}(\bar{z}, z)^{\dagger} \tag{266}$$

$$= \left(\lim_{a, \bar{z} \to 0} \hat{\phi}(\bar{z}, z) |0\rangle \right)^{\dagger} \tag{267}$$

$$\langle \phi|_{out} = (|\phi_{in}\rangle)^{\dagger} \tag{268}$$

Where we defined the adjoint in the second line with the factors, and define the dual states.

Using these definitions, we get the inner product

$$\langle\phi_{out}|\phi_{in}\rangle = \lim_{z,\bar{z},w,\bar{w}\to 0} \langle 0|\,\hat{\phi}(z,\bar{z})^{\dagger}\hat{\phi}(w,\bar{w})\,|0\rangle \tag{269}$$

$$= \lim_{z,\bar{z},w,\bar{w}\to 0} \bar{z}^{-2h}z^{-2\bar{h}}\,\langle 0|\,\hat{\phi}\left(\frac{1}{\bar{z}},\frac{1}{z}\right)\hat{\phi}(w,\bar{w})\,|0\rangle \tag{270}$$

$$= \lim_{\xi,\bar{\xi}\to 0} \bar{\xi}^{2h}\xi^{2\bar{h}}\,\langle 0|\,\hat{\phi}(\bar{\xi},\xi)\hat{\phi}(0,0)\,|0\rangle \tag{271}$$

Where we have sent $w,\bar{w}\to 0$ and set $\xi = \frac{1}{z}$ and $\bar{\xi} = \frac{1}{\bar{z}}$. This is because we have a formula for this correlation function, and it is independent of spacetime location ξ.

Morally, we can think of the vacuum state $|0\rangle$ as being like an entangled state, a bipartite state of an operator and its adjoint $|\Phi^{+}\rangle_{A\bar{A}}$, between the degrees of freedom at $-\infty$ and ∞. Then the "in" and "out" states can be thought of as

$$|A_{in}\rangle \to \hat{A}\otimes\mathbb{I}\,|\Phi^{+}\rangle \tag{272}$$

$$|A_{out}\rangle \to \mathbb{I}\otimes\hat{\bar{A}}\,|\Phi^{+}\rangle\,. \tag{273}$$

Hilbert Space of CFT

Now, let's build out the states to occupy the Hilbert space of our conformal field theory.

We have worked out the operators that correspond to global conformal transformations and form a $SL(2,\mathbb{C})$ subalgebra : $\{\hat{L}_{-1},\hat{L}_0,\hat{L}_1\}$. Note that the stress-energy, or energy-momentum, tensor generates spacetime deformations and these three operators are the global conformal transformations in infinitesimal form.

First, we must ensure that the vacuum state is invariant under $SL(2,\mathbb{C})$ by demanding that $\hat{T}(z)\,|0\rangle$ and $\hat{\bar{T}}(\bar{z})\,|0\rangle$ are regular as $z,\bar{z}\to 0$. This requires that

$$\hat{T}(z)\,|0\rangle = \sum_{n\in\mathbb{Z}} \hat{L}_n z^{-n-2}\,|0\rangle \implies \hat{L}_n\,|0\rangle = 0,\ \forall\ n\geq -1 \tag{274}$$

$$\text{and } \hat{\bar{L}}_n\,|0\rangle = 0,\ \forall\ n\geq -1. \tag{275}$$

Otherwise, the stress-energy tensor acting on the vacuum blows up as $z\to 0$. So, we have found that our subalgebra annihilates the vacuum. In other words, the vacuum state is invariant under global conformation transformations, the Virasoro generators, such that

$$\hat{L}_{-1}\,|0\rangle = \hat{L}_0\,|0\rangle = \hat{L}_1\,|0\rangle = 0. \tag{276}$$

Now we wil use the OPE to show that the primary fields, our "in" states, are eigenstates of the Hamiltonian $\hat{H} = \hat{L}_0 + \hat{\bar{L}}_0$. Consider the commutator

$$[\hat{L}_n, \hat{\phi}(w,\bar{w})] = \frac{1}{2\pi i}\oint_W dz\ z^{n+1}\hat{T}(z)\hat{\phi}(w,\bar{w}) \tag{277}$$

$$= \frac{1}{2\pi i}\oint_W dz\ z^{n+1}\left(\frac{h}{(z-w)^2}\hat{\phi}(w,\bar{w}) + \frac{1}{z-w}\partial_w\hat{\phi}(w,\bar{w}) + \text{regular}\right) \tag{278}$$

$$[\hat{L}_n, \hat{\phi}(w,\bar{w})] = h(n+1)w^n\hat{\phi}(w,\bar{w}) + w^{n+1}\partial_w\hat{\phi}(w,\bar{w}). \tag{279}$$

Where the contour W is a tight circle around the point w, and, since $z \to w$, we need only to analyze the singular behaviour of this expression, and the regular part is tossed out.

Define the asymptotic "in" state by the conformal weights h and $\bar{h}$

$$|h,\bar{h}\rangle \equiv \hat{\phi}(0,\bar{0})\,|0\rangle\,. \tag{280}$$

Then we can work out the eigenvalues of the Virasoro generator by applying the above commutation relation to the defined "in" state

$$\hat{L}_0\,|h,\bar{h}\rangle = \hat{L}_0\hat{\phi}(0,\bar{0})\,|0\rangle = [\hat{L}_0, \hat{\phi}(0,\bar{0})]\,|0\rangle = h\,|h,\bar{h}\rangle\,. \tag{281}$$

Similarly for the dual Virasoro generator

$$\hat{\bar{L}}_0\,|h,\bar{h}\rangle = \bar{h}\,|h,\bar{h}\rangle\,. \tag{282}$$

And

$$\bar{L}_n\,|h,\bar{h}\rangle = \hat{\bar{L}}_n\,|h,\bar{h}\rangle = 0, \forall\ n > 0. \tag{283}$$

Excited states are created by expanding in Fourier modes of the primary field

$$\hat{\phi}(z,\bar{z}) = \sum_{m,n\in\mathbb{Z}} z^{-m+h}\bar{z}^{-n+\bar{h}}\hat{\phi}_{m,n}. \tag{284}$$

Inverting this to get

$$\hat{\phi}_{m,n} = \left(\frac{1}{2\pi i}\oint dz\, z^{m+h+1}\right)\left(\frac{1}{2\pi i}\oint d\bar{z}\ \bar{z}^{n+\bar{h}+1}\right)\hat{\phi}(z,\bar{z}). \tag{285}$$

Note that on a real surface, $z = \bar{z}$, and $\hat{\phi}^\dagger_{m,n} = \hat{\phi}_{-m,-n}$.

For the "in" and "out" states to be well-defined, it is required that the modes annihilate the ground state for

$$\hat{\phi}_{m,n} = 0, \forall\ m > -h \text{ and } n > -\bar{h}. \tag{286}$$

Note that we can drop the dependence on $\bar{z}$, since everything separates and we can reconstruct the $\bar{z}$-dependence from the z-dependence, such that

$$\hat{\phi}(z,\bar{z}) \to \hat{\phi}(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\hat{\phi}_n. \tag{287}$$

And define the holomorphic modes by

$$\hat{\phi}_n \equiv \frac{1}{2\pi i}\oint dz\, z^{n+h-1}\hat{\phi}(z). \tag{288}$$

Putting this together with the commutator above (of $\hat{L}_n$ and $\hat{\phi}(w,\bar{w})$) we get the commutation relation

$$[\hat{L}_n, \hat{\phi}_m] = (n(h-1)-m)\hat{\phi}_{m+n}. \tag{289}$$

And we see that these operators act like ladder (raising/lowering) operators. For example, $\hat{\phi}_{-m}$ increases the conformal dimension by m. It is worth noting that $\hat{L}_{-m}$ also raises the conformal dimension

$$[\hat{L}_0, \hat{L}_{-m}] = m\hat{L}_{-m}. \tag{290}$$

Now, we can apply these modes to the vacuum and build a big, intricate Hilbert space with vectors for each mode and inner products, but we'll just focus on the states constructed by applying $\hat{L}_n$ to the primary fields ("in" state), and build a subspace that is invariant under global conformal transformations.

Let $|h,\bar{h}\rangle = \hat{\phi}(0,\bar{0})\,|0\rangle$ be an asymptotic "in" state, and define for negative indices the *descendant fields*

$$\hat{L}_{-k_1}\dots\hat{L}_{-k_n}\,|h\rangle\,, \text{where } 1 \leq k_1 \leq \dots \leq k_n. \tag{291}$$

Enacting $\hat{L}_0$ on the descendant field yields the eigenvalue $h' = h+k_1+\dots+k_n \equiv h+N$, where we call $N = \sum_{j=1}^{n} k_j$ the *level*.

The subspace generated by $|h\rangle$ and the descendant fields forms an irreducible representation, a module, of the Virasoro algebra called a *Verma module*, or *conformal family*, of the primary field. The state $|h\rangle$ behaves as the highest state vector of the Virasoro algebra.

We can work out some of the relationship between the central charge c and the conformal weight h by using a few facts about the Virasoro algebra

$$\hat{L}_n^\dagger = \hat{L}_{-n}, \tag{292}$$

$$\hat{L}_0\,|h\rangle = h\,|h\rangle\,, \tag{293}$$

$$\hat{L}_n\,|h\rangle = 0, \forall\ n > 0, \tag{294}$$

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{m+n} + \frac{c}{12}(n^3-n)\delta_{m+n,0}, \tag{295}$$

And considering the commutator

$$\langle h| [\hat{L}_{-n}^{\dagger}, \hat{L}_{-n}] |h\rangle = \langle h| [\hat{L}_n, \hat{L}_{-n}] |h\rangle \tag{296}$$

$$= 2n \langle h| \hat{L}_0 |h\rangle + \frac{c}{12}(n^3 - n) \langle h|h\rangle \tag{297}$$

$$= \Big(2nh + \frac{c}{12}(n^3 - n)\Big) \langle h|h\rangle \,. \tag{298}$$

From this, we observe that as $n \to \infty$ the only nontrivial representations of the Virasoro algebra are given by $c > 0$.

For $n = 1$, the only nontrivial representations are given by $h \geq 0$.

If $c = 0$ and $h = 0$, we get a trivial representation.

For arbitrary h, we look at the Gram matrix of $\hat{L}_{-2n} |h\rangle$ and $\hat{L}_{-n} |h\rangle$ and take the determinant

$$\det(\text{Gram}) = 4n^3 h^2 (4h - 5n). \tag{299}$$

As $n \to \infty$, the determinant becomes negative, and yields a trivial representation.