Lectures Notes For
An Introduction to Conformal Field Theory
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Lecture 1: Introduction to Conformal Field Theory

Recommended references:

- *A Mathematical Introduction to Conformal Field Theory* by Schottenloher.
- *Applied Conformal Field Theory*, hep-th/9108028, by Ginsparg.
- *Conformal Field Theory* ("Yellow Book") by Francesco, Mathieu, Senechal.

Why study conformal field theory (CFT)?

- CFT provides a good description of systems at or near criticality.
- CFTs are the only true quantum field theories (QFTs), since they are cutoff-independent. One can think of QFTs as perturbations of CFTs. CFTs correspond to renormalization groups of fixed points, which dominate an effective theory at or near criticality.
- CFTs can be made, by and large, mathematically rigorous, at least in $(1+1)$-dimensional theories. There are three major competing mathematical descriptions for CFT, and advances are being made towards a single, unifying description.

Prerequisites for this material:

- Advanced quantum mechanics

  E.g, many-body theory and Fock spaces.
- Classical field theory

  E.g., symplectic geometry.
- Quantum field theory.
- Advanced quantum field theory.

What is CFT?

- A *conformal field theory* is a field theory, quantum or classical, that is invariant, or symmetric, under a group of transformations called the *conformal group* G.
- In a classical field theory, this means that the equations of motion are left invariant.
- In a quantum field theory, this means that, by Wigner's theorem, there is a projective unitary representation of the group G. In other words, symmetries, or transformations, that leave the transition amplitude invariant, are realized, up to a phase, by (anti)unitary operators.

Conformal Transformations in d Dimensions

Let $M = \mathbb{R}^{p,q}$ be a manifold $\mathbb{R}^d$, where $d = p + q$, and $p, q \in \mathbb{Z}_{\geq 0}$. To this manifold, assign the metric

$$g_{\mu\nu} \equiv \eta_{\mu\nu} = \mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1) \tag{1}$$

With the first p entries equal to one, and the last q entries equal to minus one. Note that this is not necessarily a Riemannian metric, since the signature can be negative. We have a few cases of interest for this metric

- $p = d$, Riemannian.
- $p = d - 1$, $q = 1$, Lorentz.
- $q > 1$, e.g., $q = 2$, AdS-CFT correspondence.

A conformal transformation leaves the metric invariant up to a scale factor.

Consider a smooth change of coordinates

$$x \to x' = x'(x) \text{ , with } x = (x^1, x^2, \ldots, x^p, x^{p+1}, \ldots, x^{p+q}) \tag{2}$$

Such that the metric, a type-$(2, 0)$ tensor, undergoes an *active coordinate* transformation as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') \equiv \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x') \tag{3}$$

And then impose the condition

$$g'_{\mu\nu}(x') = \Omega(x') g_{\mu\nu}(x'). \tag{4}$$

Where $\Omega(x) > 0$ is the (local) scale factor. Note that if the scale factor is zero, then we have a singularity, which we will discuss later. A transformation that obeys the last line is called *conformal*, and these transformations preserve angles

$$\angle\theta = \frac{g_{\mu\nu} u^\mu v^\nu}{\sqrt{(g_{\mu\nu} u^\mu v^\nu)^2}}. \tag{5}$$

The *conformal group* of a manifold M is denoted by $\mathrm{Conf}(M)$, and is the connected component of the group of all conformal transformations of M containing the identity, in a compact, open topology.

So, in a quantum conformal field theory, we are looking for a Hilbert space $\mathcal{H}$ and a projective unitary representation of the group G for *local* QFTs

$$G \to \pi(G). \tag{6}$$

This is unexpectedly nontrivial, and makes for a very rich field of study, since there is a tension between knowing the unitary representations of symmetries and demanding that the representation is locally implementable.

To classify the conformal group on our chosen manifold $G = \mathrm{Conf}(\mathbb{R}^{p,q})$, consider an infinitesimal conformal (active coordinate) transformation on the spacetime coordinates

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{7}$$

Which must leave the metric invariant up to the scale factor $\Omega(x)$. This places constraints on ϵ (**Exercise**)

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \mathcal{O}(\epsilon^2). \tag{8}$$

To satisfy the constraint placed by conformal invariance on the metric (c.f., $g_{\mu\nu} \to g'_{\mu\nu}(x') = \Omega(x')g_{\mu\nu}(x')$), as well as the constraint that the conformally transformed metric is still proptional to the diagonal flat spacetime metric $g'_{\mu\nu} \propto \eta_{\mu\nu}$, we must have that the second term is also diagonal, proportional to $\eta_{\mu\nu}$

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \propto \eta_{\mu\nu} \tag{9}$$

$$\implies (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \text{constant} \cdot \eta_{\mu\nu} \tag{10}$$

Take the trace of each side, set $\mu = \nu$, and solve for the constant

$$\text{constant} = \frac{2(\partial \cdot \epsilon)}{d} \tag{11}$$

So, the conformal transformation on the metric reads, tossing out higher order terms,

$$g'_{\mu\nu} = g_{\mu\nu} + \frac{2(\partial \cdot \epsilon)}{d} g_{\mu\nu}. \tag{12}$$

And substituting into the proportionality relation from above, we have

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}. \tag{13}$$

Combining this with the conformal transformation of the metric and comparing to the metric transformation law, we get that the scale factor $\Omega(x)$ for the conformal transformation of the spacetime metric is

$$\Omega(x) = 1 + \frac{2}{d}(\partial \cdot \epsilon). \tag{14}$$

Then it follows from $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}$, expanding and equating mixed partial derivatives to third order, and we get d^2 partial differential equations of the form (**Exercise**)

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu)(\partial \cdot \epsilon) = 0 \tag{15}$$

Where $\Box = \eta^{\mu\nu}\partial_\mu \partial_\nu$ is the d'Alembertian operator.

Classification of Infinitesimal Conformal Translations for $d > 2$

By examining the condition for ϵ and the d^2 equations

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{16}$$

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu)(\partial \cdot \epsilon) = 0 \tag{17}$$

We find that third order derivatives of $\epsilon(x)$ vanish and $\epsilon(x)$ is at most quadratic.

This leaves four types of infinitesimal transformations, defined via ϵ, allowable in a conformal transformation: one constant, two linear, and one quadratic in spacetime coordinates.

1. Spacetime translations

   $\epsilon = a^\mu$.

2. Rotations

   $\epsilon^\mu = \omega^\mu_{\ \nu} x^\nu$, ω antisymmetric.

3. Scale transformations

   $\epsilon^\mu = \lambda x^\mu$, $\lambda > 0$.

4. Special conformal transformations (SCT; inversion through a sphere)

   $\epsilon^\mu = b^\mu x^2 - 2x^\mu (b \cdot x)$.

Note that Lorentz and Poincaré transformations are always subgroups of the conformal group, leaving the metric invariant. since ω corresponds to boosts and Euclidean affine rotations complete the Poincaré group.

Theorem

Every conformal transformation that acts on an connected subset of Minkowski space, including the whole space itself, $\varphi : U \subset \mathbb{R}^{p,q}$, where $p + q > 2$, is a composition of

- a translation

  $x^\mu \to x^\mu + a^\mu$, where $a \in \mathbb{R}^d$,

- an orthogonal transformation (rotation)

  $x \to \Lambda x$, where $\Lambda \in O(p,q)$,

- a dilation (scale)

  $x^\mu \to \lambda x^\mu$, where $\lambda \in \mathbb{R}^+$,

- and an SCT

  $x \to \frac{x^\mu - bx^2}{1 - 2b \cdot x + b^2 x^2}$, where $b \in \mathbb{R}^q$.

Note that it is possible to find a vector b such that the denominator is equal to zero, the SCT is not invertible, and this is no longer a group. Also note that if we don't compactify the space and include ∞ as a point available to the conformal transformation, the group becomes significantly smaller and more constrained.

Classification of Infinitesimal Conformal Translations for $d = 2$

If $d = 2$, the spacetime metric becomes the identity

$$g_{\mu\nu} = \delta_{\mu\nu} \tag{18}$$

And $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}$ becomes the Cauchy-Riemann equations and $\epsilon(x)$ is complex-valued, complex-differentiable, and analytic

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \text{ and } \partial_1 \epsilon_2 = -\partial_2 \epsilon_1. \tag{19}$$

Introduce the complex coordinates

$$z = x^1 + ix^2 \text{ and } \bar{z} = x^1 - ix^2. \tag{20}$$

Then we can complexify ϵ as

$$\epsilon(z) = \epsilon^1 + i\epsilon^2 \text{ and } \bar{\epsilon}(\bar{z}) = \epsilon^1 - i\epsilon^2. \tag{21}$$

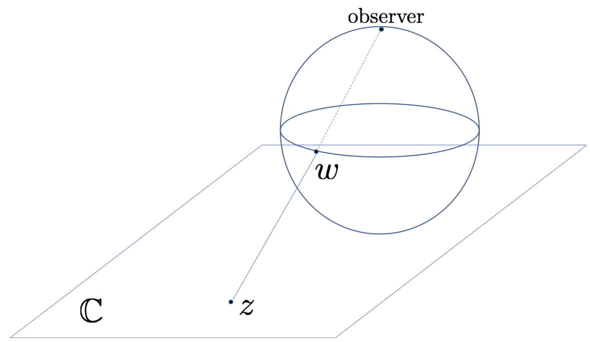
Two-dimensional global, gotten via exponentiation of an infinitesimal transformation, conformal transformations correspond to *entire* (no singularities, invertible everywhere), *holomorphic* functions $z \to f(z)$ with holomorphic inverses $f^{-1}(z)$. The only allowable form for a conformal transformation that corresponds to an entire, holomorphic function is linear in the complex coordinates

$$f(z) = \alpha z + \beta, \text{ where } \alpha, \beta \in \mathbb{C}. \tag{22}$$

We may expect a larger group of symmetries with entirety and holomorphism enforced, since the space seems less constrained, but this actually constrains the space more and the group becomes smaller. So, if we were to not compactify, and add infinity as a point, as we demonstrated, the conformal space becomes linear and boring: only rotations and scaling are allowed.

To include this complex representation of the spacetime coordinates, we *extend* our manifold to the complex numbers $\mathbb{C}$ and compactify complex space to a Riemann sphere $\mathbb{C} \cup \{\infty\}$ we get the proper space for the conformal transformations to act in

$$\mathbb{R}^{2,0} \to \mathbb{C} \to \mathbb{C} \cup \{\infty\} \to \text{Conf}(\mathbb{C} \cup \{\infty\}) \tag{23}$$



Where the conformal group is

$$\mathrm{Conf}(\mathbb{C} \cup \{\infty\}) = \left\{ f(z) = \frac{\alpha z + \beta}{\gamma z + \delta};\ \alpha, \beta, \gamma, \delta \in \mathbb{C}, \alpha\delta - \beta\gamma \neq 0 \right\}. \tag{24}$$

This is also called the group of Moebius transformations, and is a slightly larger group of conformal transformations (symmetires), since we can map to and from infinity as a point.

Summary

A conformal field theory is a local quantum field theory that is invariant under the conformal group, a set of transformations, a change in coordinates, that leave the metric invariant up to a scale factor. In different spacetime dimensions, the conformal group takes on significantly different forms.

The global conformal group in dimensions greater than two is comprised of translations, rotations, scaling, and special conformal transformations, as well as dimensions equal to two, as long as the space is compactified. If singularities are included, functions with poles are allowed, the symmetry gets larger.

Lecture 2: Local Conformal Transformations

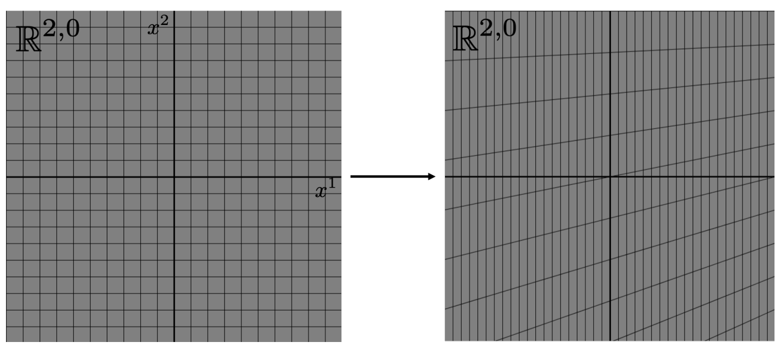
In the last lecture we introduced global conformal transformations/symmetries of some manifold M that form a (symmetry) group G which can be promoted to a symmetry group of some quanutm system, where the kinematics of the system are described by a Hilbert space $\mathcal{H}$. The quantum system is said to be globally conformally invariant if these is some unitary representation, operators U that act on the Hilbert space,

$$U: \; G \to U(\mathcal{H}). \tag{25}$$

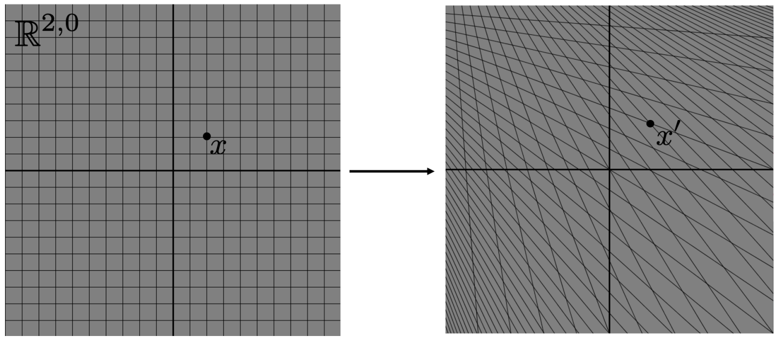
Recall that in the generalized Minkowski space $\mathbb{R}^{p,q}$, the structure of the group of global conformal transformations G consists of compositions of translations, dilations, rotations(boosts), and special conformal transformations (SCTs).

Here we now study the case where $d = 2$, which will expand our notion of what a symmetry is and will allow us to define local, infinitesimal conformal transformations.

For example, a global conformal transformation, a $1-1$ differentiable map from $\mathbb{R}^2 \to \mathbb{R}^2$, may consist of a dilation and a rotation and look like



For contrast, consider an infinitesimal conformal transformation $\mathrm{id} + \epsilon X$, where $X = X(x)$ is a vector field, the derivative of a diffeomorphism, that acts on the two-dimensional Minkowski space as



This transformation preserves all of the right angles in the untransformed Minkowski space, and the action is close to the identity, such that $|x - x'| \sim \mathcal{O}(\epsilon)$. The transformation $\mathrm{id} + \epsilon X$ is conformal to first order in ϵ, as is required by the definition of infinitesimal.

Although the vector field $X(x)$ is a generator of the infinitesimal conformal transformation, it does not necessarily define a global transformation via exponentiation, as it just may not be well defined globally.

To begin to make sense of this, consider in quantum mechanics, where we talk about quantum systems symmetric under a group G with HIlbert space

$$(\mathcal{H},\ U : G \to U(\mathcal{H})). \tag{26}$$

So, in quantum mechanics, we are reduced to finding these unitary representations of G. If G is finite, it does not make sense to speak infinitesimally (e.g., one one-hundredth of a reflection).

We assume G is a manifold, and then we may as well go as far to assume that G is a Lie group with an associated Lie algebra $\mathfrak{g}$, which consists of vector fields that exponentiate to the Lie group. Then the quantum system is symmetric under the Lie algebra if you get a representation

$$(\mathcal{H},\ \pi : \mathfrak{g} \to L(\mathcal{H})) \tag{27}$$

Where $L(\mathcal{H})$ is the set of (bounded and unbounded) linear operators, and π generates a unitary operator on the Hilbert space, such that $\pi(X) = e^{isX}$, $s \in \mathbb{R}$.

Note that for an infinite-dimensional group, (1) the operator e^{isX} may not be continuous, and (2) the Lie algebra may not exponeniate to a Lie group, which we will encounter in conformal field theory. In other words, in contrast to when we used infinitesimal quanities to build global representations, we find that infinitesimal conformal transformations don't necessarily exponentiate to a group.

Therefore, in the infinitesimal case, we abandon looking for (full, continuous) unitary representations of the Lie group, and instead focus in on finding Hermitian representations that generate the Lie algebra.

Local algebra of infinitesimal conformal transformations

Recall that for global conformal transformations, we have $z \to f(z)$, where f is holomorphic with inverse f^{-1}. For infinitesimal f, this transformation, including the complex conjugate, becomes

$$z \to z + \epsilon(z) \text{ and } \bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z}) \tag{28}$$

Where ϵ is a holomorphic function. A convenient choice of basis, which is infinite dimensional, is

$$\epsilon_n(z) = -\epsilon z^{n+1}, \text{ where } n \in \mathbb{Z}. \tag{29}$$

Given a diffeomorphism $[z \to z + \epsilon_n(z)] = e^{\epsilon \ell_n}$, the corresponding vector field tangent to every point in the manifold is defined by the operators

$$\ell_n \equiv -z^{n+1}\partial_z \text{ and } \bar{\ell}_n \equiv -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{30}$$

These differential operators form a basis, since they obey the commutation relations (**Exercise**)

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} \tag{31}$$
$$[\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n} \tag{32}$$
$$[\bar{\ell}_m, \ell_n] = 0. \tag{33}$$

They also as form a closed, infinite-dimensional Lie algebra $\forall m, n \in \mathbb{Z}$, called the Witt algebra WItt $= \mathcal{A} \oplus \bar{\mathcal{A}}$, where $\mathcal{A}$ is generated by $\{\ell\}$, and $\bar{\mathcal{A}}$ is generated by $\{\bar{\ell}\}$. Everything commutes in the basis, so the direct sum of the Witt algebra is justified.

The Witt algebra is generated infinitesimally, and could also be used to infinitiseimally generate a Lie group. This turns out to be true, but the Lie group is not the conformal group.

Which operators ℓ_n correspond to global transformations?

Consider a vector field

$$v(z) = -\sum_{n=-\infty}^{\infty} v_n \ell_n = \sum_{n=-\infty}^{\infty} v_n z^{n+1}\partial_z. \tag{34}$$

For this vector field to correspond to a global transformation, $v(z)$ must exponentiate to a holomorhpic map f, which is nonsingular in the limit as $z \to 0$. This places constraints on the coefficients of the vector field

$$v_n = 0,\, n < -1. \tag{35}$$

The inverse of the vector field must also exponentiate to a holomorphic map f^{-1}, which is nonsingular in the limit as $z \to \infty$ (e.g., exists on the Riemann sphere). This places the constraint on the coefficients of the vector field:

$$v_n = 0,\, n > 1. \tag{36}$$

Note that if we demand holomorphism on the full complex plane without compactifying, the only allowed global transformations will be linear transformation (**Exercise**). By compactifying $\pm\infty$ as a point onto the Riemann sphere, we have more freedom in allowed global transformations.

With these constraints in place, we are left with three (six with complex conjugates) generators of infinitesimal global conformal transformations

$$\{\ell_{-1}, \ell_0, \ell_1\} \cup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}. \tag{37}$$

The generators close to form a subalgebra under the commutator bracket $[,]$ defined above (**Exercise**), and generate thegroup of *linear fractional (Moebius) transformations*, also known as the projective special linear group $\mathrm{PSL}(2,\mathbb{C})$

$$z \to \frac{az+b}{cz+d},\; ad - bc = 1. \tag{38}$$

The set of global conformal transformations allowed in this basis are (**Exercise**), for $s \in \mathbb{R}$,

$$\text{Translation:} \qquad e^{s\ell_{-1}} \equiv \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \qquad \equiv z \to z - s \tag{39}$$

$$\text{Dilation:} \qquad e^{s(\ell_0+\bar{\ell}_0)} \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \equiv z \to e^{-s} \tag{40}$$

$$\text{Rotation:} \qquad e^{is(\bar{\ell}_0-\ell_0)} \equiv \begin{pmatrix} \exp(i\frac{\theta}{2}) & 0 \\ 0 & \exp(-i\frac{\theta}{2}) \end{pmatrix} \qquad \equiv z \to e^{is} \tag{41}$$

$$\text{Special Conformal:} \qquad e^{s\ell_1} \equiv \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \qquad \equiv z \to \frac{z}{1+cz}. \tag{42}$$

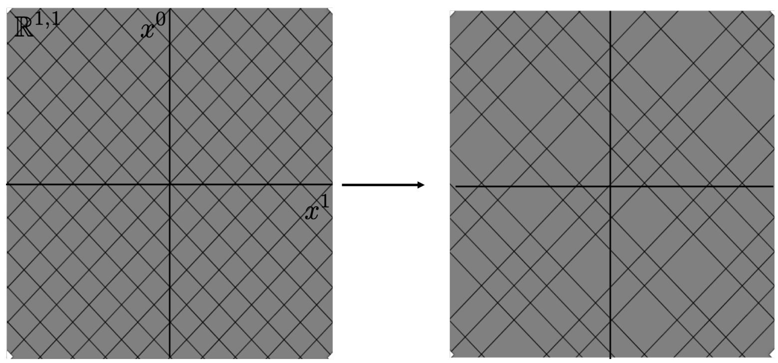
Note that for $\mathbb{R}^{d,0}$, $d > 2$, the local transformations are also global! Also note that in one-dimensional spacetime, $(1,0)$ or $(0,1)$, conformal transformations are all monotonic increasing functions $\mathbb{R} \to \mathbb{R}$.

$d = 2$ Minkowski space, $\mathbb{R}^{1,1}$

The conformal group of $\mathbb{R}^{1,1}$ is *special*.

Theorem:

A smooth map $\varphi = (u, v) : M \to \mathbb{R}^{1,1}$ from a connected subset of $M \subset \mathbb{R}^{1,1}$ is conformal (pulls back metric to a scalar multiple of the diagonal metric), iff $u_x^2 > v_x^2$ and $u_x = v_y$, $u_y = v_x$ or $u_x = -v_y$, $u_y = -v_x$.



Theorem:

Consider an infinitely differentiable function on the real line $f \in C^\infty(\mathbb{R})$, and let $f_\pm \in C^\infty(\mathbb{R}^2, \mathbb{R})$, the infinitely differentiable functions from the real line to the real plane, be defined by $f_\pm(x, y) = f(x \pm y)$. Then the map

$$\Phi : C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^2, \mathbb{R}^2) \tag{43}$$

$$(f, g) \to \frac{1}{2}(f_+ + g_-, f_+ - g_-) \tag{44}$$

Has the following properties

- $\text{image}(\Phi) = \{(u, v) : u_x = v_y,\ u_y, v_x\}$
- $\Phi(f, g)$ is conformal iff $f' > 0$ and $g' > 0$ or $f' < 0$ and $g' < 0$
- Φ is bijective iff f and g are bijective
- $\Phi(f \circ h, g \circ k) = \Phi(f, g) \circ \Phi(h, k), \forall f, g, h, k \in C^\infty(\mathbb{R}) \equiv \Phi$ is a homomorphism.

The group of orientation-preserving transformations of $M = \mathbb{R}^{1,1}$ is isomorphic to

$$(\text{Diff}_+(\mathbb{R}) \times \text{Diff}_+(\mathbb{R})) \cup (\text{Diff}_-(\mathbb{R}) \times \text{Diff}_-(\mathbb{R})) \tag{45}$$

Which consists of the infinitely-differentiable orientation-preserving maps of $\mathbb{R}$, diffeomorphisms of $\mathbb{R}$.

It is convenient to compactify $\mathbb{R}^{1,1} \to S^{1,1} \subset \mathbb{R}^{2,0} \times \mathbb{R}^{0,2}$. Then the group of orientation-preserving transformations of $M = S^{1,1}$ is isomorphic to

$$\mathrm{Conf}(\mathbb{R}^{1,1}) \equiv (\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)) \cup (\mathrm{Diff}_-(S^1) \times \mathrm{Diff}_-(S^1)). \tag{46}$$

This is the definition of the conformal group of Minkowski space. Typically, we throw away the second part of the union, the $''-''$ reversing part, since it is the same as preserving with $z \to -z$, and focus on the infinite-dimensional subgroup $\mathrm{Diff}_+(S^1)$, which we call the *chiral half* of the conformal group. This is admissable, since the symmetries of a quantum system can be understood by the symmetries of $\mathrm{Diff}_+(S^1)$, and the rest is easily gotten by tensor products to include the other light-cone axes.

In the next lecture, we will focus on which quantum systems are invariant under this infinite-dimensional group $\mathrm{Diff}_+(S^1)$ by going to the Lie algebra, which turns out to be isomorphic to the Witt algebra, in the Euclidean case. The unitary representations, gotten via infinitesimal generators, of $\mathrm{Diff}_+(S^1)$ will not be bounded below and are unstable. Therefore, *projective* unitary representaitons will be required, and are classified by the *central charge*.

Lecture 3: Classical Conformal Field Theory

We continue our discussion of systems that exhibit conformal symmetries. These symmetries are contained in the conformal group called Conf($\mathbb{R}^{p,q}$), which is the connected component containing the identity of all conformal diffeomorphisms of the pseudo-Riemannian manifold $\mathbb{R}^{p,q}$.

We discussed the infinitesimal conformal transformation, which led to a Lie algebra, the Witt algebra, in $(1+1)$ and $(2,0)$ dimensions. For $d = p + q = 2$, there is a bigger symmetry group (less constrained), yielding more conserved quantiteis, more degrees of freedom of the system. If $d \neq 2$, the symmetry group is too constrained to be that interesting.

A *conformal theory* is a theory with a representation of the group $G = \mathrm{Conf}(\mathbb{R}^{p,q})$. This group contains transformations corresonding to temporal translations, spatial translations, boosts, dilations, and special conformal transformations (inversion about the origin, translation, and a second inversion about the origin). So, a conformal theory has a Hamiltonian H built in, since it is the generator of time translations.

Note that in a nonrelativistic theory, we demand that the Hamiltonian H commutes with everything, which introduces symmetries of the system, but the inclusion of boosts requires a relativistically invariant theory. This constrains the theory further to allow only certain symmetries and exhibit the desired properties.

Note that the Lorentz boost mixes energy and momentum through conjugation of spatial translations to temporal translations. This conjugation requires that all types of possible transformations in a nonrelativistic theory must be represented all at once, and they are not independent of each other.

Another property we need for our theory is *locality.*

So, we have a collection of observables $\phi_a(x)$, where $x \in \mathbb{R}^{p,q}$ and $a \in I$, an index set (labels by particle types, vector quantities, etc.), which can be classical (functions on phase space), quantum (self-adjoint operators), or even probabilistic (element of ordered unit vector space).

A representation of a group of symmetries is a map π that can be

$$\text{finite } \pi : G \to M_n(\mathbb{C}) \quad \text{, the } n \times n \text{ matrices over the complex numbers} \tag{47}$$

$$\text{infinite } \pi : G \to \mathcal{B}(\mathcal{H}) \quad \text{, the bounded operators on a Hilbert space, for example.} \tag{48}$$

The concern is that a given representation does not necessarily yield a set of observables $\phi_a(x)$. In the event that it does, it is likely that a representation which furnishes a collection of (local) observables is *reducible*, and can be decomposed into a direct sum of *irreducible* representations, or *irreps*. This makes for an infinite number of ways to build reducible representations.

So, although we can write down an irreducible representation of G and attempt to enforce locality, we prefer to take the stance, and shall from this point on, that the locality of the theory is the most important property, and find irreducible representations from there.

Classical field representations of conformal symmetries

The concept of the field easily puts forth the idea of locality, but what constraints does conformal symmetry place on a classical field?

Recall for symmetries in a classical field theory start with the action

$$S = \int d^d x \, \mathcal{L}(\phi, \partial_\mu \phi), \text{ where } \phi = \{\phi_a(x)\}. \tag{49}$$

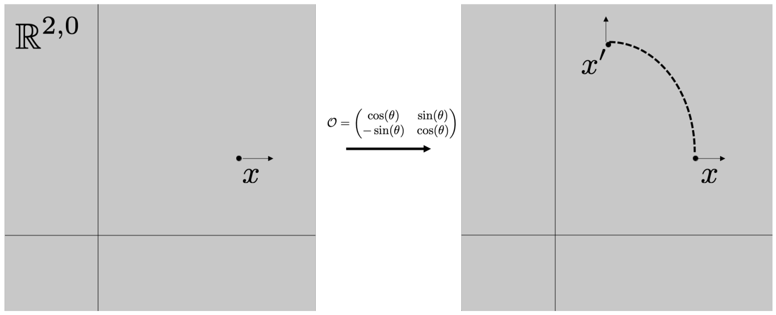
By writing down the action, we have assumed (1) that the equations of motion are represented by an action and (2) that the Lagrangian density depends only on the field and its first derivatives. We have effectively thrown out all non-local theories.

So, a symmetry transformation takes a spacetime location and maps it to its image under that transformation: $x \to x'$. If the transformation is *active*, the then fields transform as well

$$\phi(x) \to \phi'(x') \equiv \mathcal{F}(\phi(x)) \tag{50}$$

Where we note that $\mathcal{F}(\phi(x))$ depends on the previous field configuration.

For example, in an active rotation of a vector field $\mathbb{R}^{2,0}$, a nontrivial representation rotates the spacetime coordinate as well as the vector at each spacetime coordinate, the field (A trivial representation will not rotate the vector.)



After the rotation, the new field configuration at x is

$$\phi'_a(x) = \sum_b \pi(\mathcal{O})_{ab}\phi_b(\mathcal{O}^{-1}x). \tag{51}$$

The trivial representation of the field component b would simply be the identity $\pi(\mathcal{O})_{ab} = \delta_{ab}$, and the fundamental, nontrivial representation is written

$$\pi(\mathcal{O})_{ab} = [\mathcal{O}]_{ab}. \tag{52}$$

How does the action S transform under a symmetry transformation?

$$S' = \int d^d x \left|\det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)\right| \mathcal{L}\left(\mathcal{F}\left(\phi(x)\right), \frac{\partial x^{\nu}}{\partial x'^{\mu}}\partial_\nu \mathcal{F}\left(\phi(x)\right)\right) \tag{53}$$

We also know that our theory is a conformal field theory, conformally invariant, conformally symemtric if the equations of motion are invariant. This is equivalent to the Lagrangian density transforming up to a total derivative

$$\mathcal{L}' = \mathcal{L} + \text{total derivative}. \tag{54}$$

Now, let's study the infinitesimal generators of the conformal group $\mathrm{Conf}(\mathbb{R}^{p,q})$.

- Translation

  $P_\mu = -i\partial_\mu$

- Dilation

  $D = -ix^\mu \partial_\mu$

- Rotation (Boost)

  $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

- Special Conformal

  $K_\mu = -i(2x_\mu x'^\nu \partial_\nu - x^2 \partial_\mu)$

Work out the commutation relations to form a Lie algebra (**Exercise**).

$$[D, P_\mu] = iP_\mu \tag{55}$$
$$[D, K_\mu] = -iK_\mu \tag{56}$$
$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \tag{57}$$
$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \tag{58}$$
$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \tag{59}$$
$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \tag{60}$$

And the rest commute.

Our task now is to find out which kinds of fields transform under the conformal group and give representations of the conformal group. We already know that a field transforming under Conf($\mathbb{R}^{p,q}$), a general conformal transformation, transforms under the subgroup Poincaré, which is generated by translation P_μ and rotation $L_{\mu\nu}$, as

$$\phi_a(x) \to_{\text{Poincaré}} \sum_b \pi(\Lambda)_{ab}\phi_b(\Lambda^{-1}x). \tag{61}$$

Let's focus on the subgroup of Conf($\mathbb{R}^{p,q}$) which leave the orgin fixed: rotations, dilations, and SCTs. The infinitesimal generators of this subgroup form a subalgebra by exponentiation

$$\Lambda = e^{i\omega^\alpha G_\alpha}, \text{ where } \omega^\alpha \text{ is infinitesimal.} \tag{62}$$

The group element G_α can be K_μ, D, or $L_{\mu\nu}$. At the origin, the Poincaré transformation of the field looks like

$$\phi_a(x=0) \to \sum_b \pi(e^{i\omega^\alpha G_\alpha})_{ab}\phi_b(\Lambda^{-1}x=0) \tag{63}$$

Where the representation is, by Taylor expansion,

$$\pi(e^{i\omega^\alpha G_\alpha}) = \pi(\mathbb{I}) + i\omega^\alpha \pi(G_\alpha). \tag{64}$$

Rename the representations of the generators (group elements)

$$\pi(D) = \tilde{\Delta}(\text{scaling dimension}) \tag{65}$$
$$\pi(K_\mu) = \kappa_\mu \tag{66}$$
$$\pi(L_{\mu\nu}) = S_{\mu\nu}\,(\text{spin}). \tag{67}$$

The commutation of these representations are then

$$[\tilde{\Delta}, S_{\mu\nu}] = 0 \tag{68}$$
$$[\tilde{\Delta}, \kappa_\mu] = -i\kappa_\mu \tag{69}$$
$$[\kappa_\mu, \kappa_\nu] = 0. \tag{70}$$

Now, suppose that the generators $S_{\mu\nu}$ are irreducible representations, irreps, of the Lorentz group, the group that describes spin/helicity. By Schur's lemma (**Exercise**), we find that the scaling dimension is trivial, and, in turn, by the commuation relations, that all generators κ_μ are also trivial

$$\tilde{\Delta} \propto \mathbb{I} \implies -i\kappa_\mu = 0. \tag{71}$$

Now use this fact to show how dilations act on the fields. The coordinates transform as

$$x \to \lambda x \tag{72}$$
$$x \to \lambda^\epsilon \lambda^\epsilon \ldots \lambda^\epsilon x \tag{73}$$

At the origin, the field transforms as

$$\phi_a(x = 0) \to (\mathbb{I} + i\epsilon\tilde{\Delta}) \ldots (\mathbb{I} + i\epsilon\tilde{\Delta})\phi_a(0) \tag{74}$$
$$= \lambda^{i\tilde{\Delta}_a}\phi_a(0) \tag{75}$$
$$= \lambda^{-\Delta_a}\phi_a(0) \tag{76}$$

Where we used the Taylor expansion of the infinitesimal ϵ, and the last line uses $\tilde{\Delta} = i\Delta\mathbb{I}$, since Schur's lemma tells us that the scaling deimension is trivial.

So, every conformal field has a behavior under dilations, defined by the scaling dimension Δ_a, with Jacobian

$$\left|\frac{\partial x'}{\partial x}\right| = \Lambda^{-\frac{d}{2}}, \text{ where } \Lambda = \lambda^{-2}. \tag{77}$$

And the metric transforms under dilations as

$$g'_{\mu\nu} = \lambda^{-2} g_{\mu\nu}. \tag{78}$$

Putting all this together, the field now transforms as

$$\phi_a(x) \to \phi'_a(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \phi_a(x). \tag{79}$$

Filling in the Jacobian, the new field in terms of the orignial field and the original spacetime location is

$$\phi'_a(x) = \sum_b \pi(\Lambda)_{ab}\phi_b(\Lambda^{-1}x) \tag{80}$$
$$= \sum_b [\lambda^{-\Delta}]_{ab}\phi_b(\Lambda^{-1}x) \tag{81}$$
$$= \lambda^{-\Delta}\phi_a(\Lambda^{-1}x) \tag{82}$$
$$= \lambda^{-\Delta}\phi_a(\lambda^{-1}x). \tag{83}$$

By the Baker-Campbell-Hausdorff (BCH) formula, we know how the new field looks at all spacetime lcoations, not just the origin (**Exercise**)

$$e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho} = D + x^\nu P_\nu \tag{84}$$
$$\implies D\phi_a(x) = (-ix^\nu\partial_\nu + \tilde{\Delta})\phi_a(x). \tag{85}$$

In the next lecture, we give this the quantum treatment, where we will look for unitary representations (self-adjoint operators), that are labelled by quantum numbers, such as spin, scaling dimension, and central charge. The central charge will require projective unitary representations.

Lecture 4: Constraints of Conformal Invariance on Quantum Field Theories

Recall that a scale transformation on a classical field is characterized per field, indexed by a, by the scaling dimension Δ_a in the mapping

$$\phi_a(x) \to \lambda^{-\Delta_a} \phi_a(x). \tag{86}$$

We now make a move to study quantum fields, using the classical limit to check our results. So, suppose we have done all of the hard work of quantization and have a quantum field theory.

Digression: In $(1+1)$ dimensions, some approaches to quantization include

- Vertex operator algebras.
- (Local) Algebraic QFT

  Top-down approach where one attaches an algebra of observables to each point in spacetime and then makes sense of it as a field theory.
- Functors from n-Categories to Hilbert spaces: $n\text{Cat} \to \text{Hilb}$ (Segal).

Digression: A main assumption of this course is that quantum field theories are a subset of quantum theories, although some schools argue that QFT is done via path integrals, which is not obviously a quantum theory.

The data of a quantum field theory includes

- Hilbert space of states $\mathcal{H}$
- Projective unitary representation of the conformal group $U(g)$, where $g \in \text{Conf}(\mathbb{R}^{p,q})$
- Vacuum (reference) state $|0\rangle \in \mathcal{H}$

  Invariant, up to a phase, under global symmetries, but not necessarily local symmetries: $U(g)\,|0\rangle = e^{i\varphi(g)}\,|0\rangle$.

  If $p+q > 2$, then the global and local (full conformal) symmetries coincide.

  If $p = q = 1$, then the local conformal group of transformations (diffeomorphisms of the circle) is much larger than the global conformal group, making the global transformations subgroup of the local transformations.

- Observables.

For observables, we demand that we can measure some set of local observables from the set of self-adjoint linear operators on the Hilbert space

$$A_{j,x} \in L_{\text{self-adjoint}}(\mathcal{H});\; x \in \mathbb{R}^{p,q};\; j \in \text{index set (e.g., particle type).} \tag{87}$$

Digression: We concern ourselves with *local* observables, and non-local observables are difficult to imagine. An example of a non-local observable that shows up in gauge theory is the Wilson loop or Wilson line, since they are spread over a nonzero dimensional submanifold of the gauge theory's manifold.

Note that locality is induced by causality, such that if $x - y$ is spacelike, then $A_{j,x}$ and $A_{k,y}$ are jointly observable, $\forall\, j, k$ and x, y.

Quasi-Primary Observables

The *quasi-primary* of a field is a subset of local observables $\{A_{j,x} : j \in J, x \in \mathbb{R}^{p,q}\}$ with the additional properties to satisfy the assumed constraints that enforce conformal invariance of the system, denoted here by $\{\hat{\phi}_k(x) : k \in K\}$ that transform as

$$U(g):\; \hat{\phi}_k(x) \to U^\dagger(g)\hat{\phi}_k(x)U(g) = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_k}{d}} \hat{\phi}_k(x') \tag{88}$$

Where $x' = gx$, and $g \in \mathrm{Conf}(\mathbb{R}^{p,q})$ is a conformal transformation.

Now we must demonstrate that these assumptions yield nontrivial examples of fields. Rest assured that there are such fields, such as free bosons and free fermions.

By these assumptions, the n-point correlation function, which is observable via scattering experiments, transforms under conformal transformations as

$$\langle 0|\, \hat{\phi}_{k_1}(x_1) \ldots \hat{\phi}_{k_n}(x_n)\, |0\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\frac{\Delta_{k_1}}{d}} \ldots \left|\frac{\partial x_n'}{\partial x_n}\right|^{\frac{\Delta_{k_n}}{d}} \langle 0|\, \hat{\phi}_{k_1}(x_1') \ldots \hat{\phi}_{k_n}(x_n')\, |0\rangle\,. \tag{89}$$

This equation constrains the structure of the n-point correlation functions. To understand these constraints, let's analyze how each type of transformation of the conformal group constrains the invariants.

Translations: For $x_j, x_k \in \mathbb{R}^{p,q},\; j, k = 1, \ldots, n$, the difference $x_j - x_k$ is invariant, and there are $d(n-1)$ such quantities.

Rotations: For spinless objects (in large enough dimension d), the length $r_{jk} \equiv |x_j - x_k|$ is invariant, and there are $\binom{n}{2}$ such quantities.

Dilations: Under the scale transformations, the length r_{jk} is clearly not invariant, but the ratio $\frac{r_{jk}}{r_{lm}}$ can be invariant.

SCTs: Under special conformal transformations, invariant quantities must be cross ratios of the form $\frac{r_{jk}r_{lm}}{r_{jl}r_{km}}$, since the squared length under SCTs transforms as

$$|x_1' - x_2'|^2 = \frac{|x_1 - x_2|^2}{(1 + 2b \cdot x_1 + b^2 x_1^2)(1 + 2b \cdot x_2 + b^2 x_2^2)}. \tag{90}$$

Two-Point Correlation Functions

Consider the classical two-point correlation function, or Green's function, of the quasi-primary fields

$$G^{(2)}(x_1, x_2) = \langle 0| \phi_1(x_1)\phi_2(x_2) |0\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\frac{\Delta_{k_1}}{d}} \left|\frac{\partial x_2'}{\partial x_2}\right|^{\frac{\Delta_{k_2}}{d}} \langle 0| \hat{\phi}_{k_1}(x_1')\hat{\phi}_{k_2}(x_2') |0\rangle . \tag{91}$$

Exploiting the assumed conformal symmetries of the system, use the fact that the Jacobian for a translation, as well as for a rotation, is equal to one, such that $\left|\frac{\partial x'}{\partial x}\right| = 1$, and, therefore, the Green's function can only depend on the length

$$G^{(2)}(x_1, x_2) = f(|x_1 - x_2|) = f(r_{12}). \tag{92}$$

For a dilation, the Jacobian obeys a scale factor, such that $\left|\frac{\partial x'}{\partial x}\right| = \lambda^{\Delta}$, and the Green's function has the form

$$G^{(2)}(x_1, x_2) = \lambda^{\Delta_1 + \Delta_2} f(\lambda r_{12}). \tag{93}$$

To calculate the function that obeys this constraint, expand $f(r_{12}) = \sum_a f_a r_{12}^a$ in a series, noting that a can be a continuous parameter, and compare to the above to get the condition that the coefficients f_a are all zero, except for $a = -\Delta_1 - \Delta_2$.

Therefore, under translations, rotation, and dilations, we find that the Green's function is constrained to the form

$$G^{(2)}(x_1, x_2) = \frac{f_{-\Delta_1 - \Delta_2}}{r_{12}^{\Delta_1 + \Delta_2}} = \frac{c_{12}}{r_{12}^{\Delta_1 + \Delta_2}} \tag{94}$$

Where c_{12} is a constant determined by the normalization condition.

So far, we have come to the conclusion that the correlation function must obey a power law, and we now apply the constraints of SCTs to find that the two-point correlation function of quasi-primary fields is zero unless the two fields have the same scaling dimension $\Delta_1 = \Delta_2 = \Delta$ (**Exercise**)

$$\langle 0| \phi_1(x_1)\phi_2(x_2) |0\rangle = \frac{c_{12}}{r_{12}^{2\Delta}}. \tag{95}$$

This means that if we solve a system and the calculate the two-point correlation function of two fields with different scaling dimensions, Δ_1 and Δ_2, respectively, where $\Delta_1 \neq \Delta_2$, and get a nonzero result, then our system is not conformally invariant under the full global conformal group, but under a subgroup consisting of translations, rotations, and dilations.

Example: Three-Point Correlation

Following similar procedure as in the two-point case, translations, rotations, and dilations lead to conclusion that the three-point correlation function must have the form

$$\langle 0| \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) |0\rangle = \sum_{a,b,c} \frac{c_{abc}}{r_{12}^a r_{23}^b r_{31}^c} \tag{96}$$

Where the summation is constrained by the field scaling dimensions to satisfy $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$.

Special conformal transformations further constrain the exponents in the power law to (**Exercise**)

$$a = \Delta_1 + \Delta_2 - \Delta_3, \quad b = -\Delta_1 + \Delta_2 + \Delta_3, \quad c = \Delta_1 - \Delta_2 + \Delta_3. \tag{97}$$

And the three-point correlation function has the form

$$\langle 0| \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) |0\rangle = \frac{c_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{23}^{-\Delta_1+\Delta_2+\Delta_3} r_{31}^{\Delta_1-\Delta_2+\Delta_3}}. \tag{98}$$

Example: Four-Point Correlation

The four-point correlation function has the form

$$\langle 0| \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) |0\rangle = F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{13}r_{24}}{r_{23}r_{14}}\right) \prod_{j<k} r_{jk}^{-\Delta_j - \Delta_k + \frac{\Delta}{3}} \tag{99}$$

Where $F(\cdot,\cdot)$ is an arbitrary function of the cross products and $\Delta = \sum_j \Delta_j$.

Conformal Theory in $(2+0)$ Dimensions

Thus far, we have been studying the structure of conformal theories in arbitrary spacetime dimension d. In $(2+0)$ dimensions, conformal theories will take on a new definition.

Recall that we defined the complex fields $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, and made the assumption that we can *analytically continue* to arbitrary, independent z and $\bar{z}$, such that

$$\Phi(x_1, x_2) = \Phi(z, \bar{z}) \tag{100}$$

And work out the consequences from there. We now extend our definition of quasi-primary fields, by analogy to the general, arbitrary dimension case. We will now refer to these fields as *primary fields* of type, or conformal weight, $(h, \bar{h})$ with the proposed form

$$\Phi(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})). \tag{101}$$

Note here that there are indeed enough examples of primary fields of conformal field theories to make these assumptions interesting to study.

Expand the primary field infinitesimally, as it is the more convenient method opposed to working with the full global transformation, by sending $z \to z + \epsilon(z)$, and complex conjugate, to order ϵ

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(z, \bar{z}) = ((h\partial_z \epsilon(z) + \epsilon(z)\partial_z) + (\bar{h}\partial_{\bar{z}}\bar{\epsilon}(\bar{z}) + \bar{\epsilon}(\bar{z})\partial_{\bar{z}}))\Phi(z, \bar{z}). \tag{102}$$

And write the two-point correlation function as

$$G^{(2)}(\underline{z}, \underline{\bar{z}}) = \langle 0| \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2) |0\rangle \tag{103}$$

Where $\underline{z} \equiv (z_1, z_2)$ and $\underline{\bar{z}} \equiv (\bar{z}_1, \bar{z}_2)$.

Applying the infinitesimal equation for $\delta_{\epsilon,\bar{\epsilon}}\Phi(z, \bar{z})$ from above, and setting equal to zero by conformal invariance to order ϵ, we get

$$\delta_{\epsilon,\bar{\epsilon}} G^{(2)}(\underline{z}, \underline{\bar{z}}) = \langle 0| \delta_{\epsilon,\bar{\epsilon}}\Phi_1\Phi_2 |0\rangle + \langle 0| \Phi_1\delta_{\epsilon,\bar{\epsilon}}\Phi_2 |0\rangle = 0. \tag{104}$$

From here, as before in the arbitrary dimension case, we can work out the constraints of each type of conformal transformation on the correlation function.

Translation:

$$\epsilon(z) = \epsilon \implies G^{(2)}(\underline{z}, \underline{\bar{z}}) \propto z_{12} = z_1 - z_2 \text{ and } \bar{z_{12}} = \bar{z}_1 - \bar{z}_2. \tag{105}$$

Rotation & Dilation:

$$\epsilon(z) = \epsilon z \implies G^{(2)}(\underline{z}, \underline{\bar{z}}) = \frac{c_{12}}{z_{12}^{h_1+h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}. \tag{106}$$

SCT:

$$\epsilon(z) = z^2 \implies G^{(2)}(\underline{z}, \underline{\bar{z}}) = \frac{c_{12}}{z_{12}^{2h} \bar{z}_{12}^{2\bar{h}}}. \tag{107}$$

Now, for example, suppose we have a bosonic field which applies the constraint $h - \bar{h} = 0$ to the conformal weights. Setting $h + \bar{h} = \Delta$, the analytically continued two-point correlation function for the bosonic field becomes

$$G^{(2)}(\underline{z}, \underline{\bar{z}}) = \frac{c_{12}}{|z_{12}|^{2\Delta}}. \tag{108}$$

Example: Three-Point Correlation

Similarly, the three-point correlation function has the form

$$G^{(3)}(\underline{z}, \underline{\bar{z}}) = c_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{-h_1+h_2+h_3} z_{31}^{h_1-h_2+h_3}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{-\bar{h}_1+\bar{h}_2+\bar{h}_3} \bar{z}_{31}^{\bar{h}_1-\bar{h}_2+\bar{h}_3}} \tag{109}$$

In summary, we have introduced what we consider quantum field theories and have analyzed the consequences of conformal invariance of these quantum theories. We have also analyzed subclasses of theories that obey additional constraints (e.g., $\Delta = h + \bar{h}$ is real).

Lecture 5:
Quantum CFT: Ward Identities & Radial Quantization

We have been on this journey from classical conformal symmetries to the implementation of the symmetries in a quantum setting. Thus far, we have found that the constraints of conformal symmetry in the context of quantum field theory, in $(2+0)$, $(1+1)$, and $(d+1)$ dimensions, the two-point correlation functions must decay polynomially as

$$\langle 0|\,\hat{\phi}(x)\hat{\phi}(y)\,|0\rangle \propto \frac{1}{|x-y|^{\alpha}}. \tag{110}$$

Radial "Quantization"

Note that using quotes around quantization indicates that this approach is an inspired guess, and is not derived via a functor.

To build a quantum field theory with conforml symmetry, invariant under the group $\mathrm{Conf}(\mathbb{R}^{2,0})$, we need four pieces of data.

(1) We need a Hilbert space $\mathcal{H}$ which is preferrably an infinite, separable, kinematic space of states isomorphic to the vector space of linear operators $L^2(\mathbb{R})$, (2) a vacuum state, or a reference vector, $|0\rangle$, a subset of linear operators in the Hilbert space $\mathcal{L}(\mathcal{H})$ called *observables* $\hat{\phi}(x,y)$, $(x,y) \in \mathbb{R}^2$, which are distribution-valued objects, but can be thought of, without harm, as self-adjoint operators.

The choice of set of observables, the subset of linear operators from the Hilbert space, define and distinguish a quantum field theory from others. E.g., the bosonic quantum field and the hydrogen atom share the same Hilbert space, but their observables are different and define what is allowed to be measured in each quantum field theory.

Our conformal quantum field theory yields (projective) unitary representations of $\mathrm{Conf}(\mathbb{R}^{2,0})$. We can use analytic continuation to map to a Minkowski theory where we have representations of $\mathrm{Conf}(\mathbb{R}^{1,1})$.

Introduce the construction of "imaginary time"

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ it \end{pmatrix};\ t \in \mathbb{R}. \tag{111}$$

If possible, the correlation functions analytically continue, such that

$$\langle 0|\,\hat{\phi}(x,y)\hat{\phi}(0,0)\,|0\rangle = f(x,y) \to f(x,it). \tag{112}$$

If all n-point correlation functions are analytically continued in this way, it *often* happens that we get a mapping

$$G^{(n)}((x_1, y_1), \dots) \to G^{(n)}((x_1, t_1), \dots) \tag{113}$$

Which are invariant, respectively, under symmetry transformations

$$\mathrm{Conf}(\mathbb{R}^{2,0}) \to \mathrm{Conf}(\mathbb{R}^{1,1}). \tag{114}$$

This is the analytic continuation to a *Minkowski theory*, and the criteria to ensure that this mapping exists is called *reflection positivity*. See the work of Glimm and Jaffe for a rigorous account of this machinery.

With a Minkowski quantum field theory, enter the Heisenberg picture to apply time evolution, and analytically continue the "time" variable to define "inverse temperature" $t \to -i\beta$, which essentially gives the correlation function for the thermal state of the Hamiltonian $\hat{H}$ with inverse temperature β

$$\langle 0| \hat{\phi}(x, y)\hat{\phi}(0, 0) |0\rangle = \langle 0| e^{-it\hat{H}} \hat{\phi}(x, 0)e^{it\hat{H}} \hat{\phi}(0, 0) |0\rangle \tag{115}$$

$$= \langle 0| e^{-\beta\hat{H}} \hat{\phi}(x, 0)e^{\beta\hat{H}} \hat{\phi}(0, 0) |0\rangle . \tag{116}$$

Digression: What does "physical" mean for this course?

To define a physical theory, we require *kinematics* and *observables*, which make for a perfectly fine physical theory, even without *dynamics*.

The kinematics of a system introduces a Hilbert space $\mathcal{H}$ and a set of density operators $\rho(\mathcal{H})$ that describe the states of the system, but do not allow measurement of those states. The observables of a system $\mathcal{O} \subset \mathcal{L}(\mathcal{H})$ are the measurements of the states of the system, and are labelled by points in the underlying manifold $\mathcal{M}$ of the theory, such that $x, x^\dagger, \mathbb{I} \in \mathcal{O}$, for $x \in \mathcal{M}$.

Dynamics of a system are introduced via the group of isometries on the manifold $G = \mathrm{Isom}(\mathcal{M})$, and, for a quantum field theory, we search for projective, unitary representations of G on the Hilbert space. The construct of time only comes in as a choice of the manifold and its one-dimensional subgroups that act like time, but time is not necessarily an axiom of a quantum theory.

End Digression.

So far, we have labelled our observables in our Minkowski CFT by $(\sigma^0, \sigma^1) \in \mathbb{R}^{1,1}$. We now complexify these coordinates by "Euclideanizing" and sending the timelike coordinate to be imaginary, such that $\sigma^0 \to i\sigma^0$, and

$$z = \sigma^1 + i\sigma^0 \text{ and } \bar{z} = \sigma^1 - i\sigma^0 \tag{117}$$

Which correspsonds to an analytic continuation of the n-point correlation functions

$$G^{(n)}(\underline{\sigma}_1, \underline{\sigma}_2, \dots) \to G^{(n)}(\underline{z}_1, \underline{\bar{z}}_1; \underline{z}_2, \underline{\bar{z}}_2; \dots). \tag{118}$$

In this complexifed, Euclidean spacetime, with locations defined by coordinates z and $\bar{z}$, compactify space to a cylinder, such that space corresponds to the transverse direction $\sigma^1 \to \sigma^1 + 2\pi$, and time corresponds to the longitudinal direction, and our complex coordinates are now

$$z = e^{\sigma^1 + i\sigma^0} \text{ and } \bar{z} = e^{\sigma^1 - i\sigma^0}. \tag{119}$$

Circles in the complex plane correspond to constant time, the radial direction on the cylinder, and the real coordinates σ^0, σ^1 are mapped to the complex coordinates as

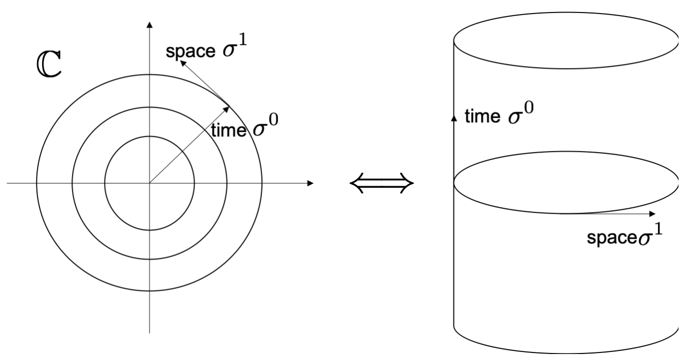
$$\sigma^0 = \infty \to z = \infty \text{ and } \sigma^0 = -\infty \to z = 0 \tag{120}$$

And transformations maps as

$$\text{Time reversal } [\sigma^0 \to -\sigma^0] \to \text{Inversion } [z \to \frac{1}{z}] \tag{121}$$

$$\text{Time translation } [\sigma^0 \to \sigma^0 + a] \to \text{Dilation } [z \to e^a z] \tag{122}$$

$$\text{Spatial translation } [\sigma^1 \to \sigma^1 + a] \to \text{Rotation } [z \to e^{ia} z] \tag{123}$$



Now we make a move to constructing a conformal field theory in complexified, Euclidean spacetime.

Ward Identities

Witha conformally invariant classical theory, we work towards getting a collection of observables obeying the correct symmetry for a conformally invariant quantum theory.

Recall from Noether's theorem that conserved charges correspond to classical symmetries Q obeying the anticommutation bracket

$$\{Q_j, Q_k\} = f_{jk}{}^{l} Q_l. \tag{124}$$

We may try to naively quantize by "puttings hats on" the conserved quantities that obey the commutation brackets

$$[\hat{Q}_j, \hat{Q}_k] = i f_{jk}{}^{l} \hat{Q}_l, \tag{125}$$

But without a Hilbert space, we can not write these operators as functions of the fields (e.g., $\hat{Q} = f(\hat{\phi})$.

From the classical symmetries, we now derive the quantum generators of the symmetries.

Suppose we have a classical field theory with action $S[\underline{\phi}]$, where $\underline{\phi}$ is a vector of classical fields. Assume that S is symmetric under the Lie group of infinitesimal transformations, such that

$$\underline{\phi}'(\underline{x}) = \underline{\phi}(\underline{x}) - i\omega_a(\underline{x})\mathbf{G}_a\underline{\phi}(\underline{x}) = e^{-i\omega_a(\underline{x})\mathbf{G}_a}\underline{\phi}(\underline{x}) \tag{126}$$

Where $\omega_a(\underline{x})$ is infinitesimal and $\mathbf{G}_a$ is a matrix acting on the vector labels of $\underline{\phi}$.

Use the path integral prescription to work out how this transformation affects the correlation functions

$$\langle 0| \underline{\hat{\phi}}(\underline{x}_1)\dots\underline{\hat{\phi}}(\underline{x}_n)|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(\underline{x}_1)\dots\phi(\underline{x}_n)e^{iS[\underline{\phi}]}}{\int \mathcal{D}\phi\, e^{iS[\underline{\phi}]}}. \tag{127}$$

The first step is to change variables $\underline{\phi} \to \underline{\phi}'$ and assume that the measure is invariant $\mathcal{D}\phi' = \mathcal{D}\phi$. Consider the expectation value of x

$$\langle \hat{x}\rangle = \frac{\int \mathcal{D}\phi' \left(\underline{\hat{\phi}}(\underline{x}_1)\dots\underline{\hat{\phi}}(\underline{x}_n) + i\omega_a(\underline{x})\mathbf{G}_a(\underline{\hat{\phi}}(\underline{x}_1)\dots\underline{\hat{\phi}}(\underline{x}_n) + \dots\right) e^{i(S+\int dx\, \partial_\mu j_a^\mu \omega_a(\underline{x}))}}{\int \mathcal{D}\phi\, e^{iS}}. \tag{128}$$

To zeroth order in ω_a, this is exactly the correlation function prior to the change of variables. To first order in ω_a, we get the equality for quantum fields

$$\frac{\partial}{\partial x^\mu}\langle \hat{j}_a^\mu(\underline{x})\hat{\phi}(\underline{x}_1)\dots\hat{\phi}(\underline{x}_n)\rangle = -i\sum_{j=1}^{n}\delta(\underline{x}-\underline{x}_j)\langle\hat{\phi}(\underline{x}_1)\dots\mathbf{G}_a\hat{\phi}(\underline{x}_j)\dots\hat{\phi}(\underline{x}_n)\rangle. \tag{129}$$

The generator of symmetry $\hat{j}_a^\mu(\underline{x})$ is handed over by the oath integral approach.

Lecture 6: