Problem 2.1

To walk "downhill" on the loss function (equation 2.5), we measure its gradient with respect to the parameters ϕ_0 and ϕ_1. Calculate expressions for the slopes $\frac{\partial L}{\partial \phi_0}$ and $\frac{\partial L}{\partial \phi_1}$.

Solution

Given the loss function (equation 2.5):

$$L[\phi] = \sum_{i=1}^{I} (f[x_i, \phi] - y_i)^2 = \sum_{i=1}^{I} (\phi_0 + \phi_1 x_i - y_i)^2$$

Differentiating with respect to ϕ_0:

$$\frac{\partial L}{\partial \phi_0} = \frac{\partial}{\partial \phi_0} \left(\sum_{i=1}^{I} (\phi_0 + \phi_1 x_i - y_i)^2 \right)$$

$$\frac{\partial L}{\partial \phi_0} = \sum_{i=1}^{I} 2(\phi_0 + \phi_1 x_i - y_i) \cdot \frac{\partial}{\partial \phi_0} (\phi_0 + \phi_1 x_i - y_i)$$

$$\frac{\partial L}{\partial \phi_0} = \sum_{i=1}^{I} 2(\phi_0 + \phi_1 x_i - y_i) \cdot 1$$

$$\frac{\partial L}{\partial \phi_0} = 2 \sum_{i=1}^{I} (\phi_0 + \phi_1 x_i - y_i)$$

Differentiating with respect to ϕ_1:

$$\frac{\partial L}{\partial \phi_1} = \frac{\partial}{\partial \phi_1} \left(\sum_{i=1}^{I} (\phi_0 + \phi_1 x_i - y_i)^2 \right)$$

$$\frac{\partial L}{\partial \phi_1} = \sum_{i=1}^{I} 2(\phi_0 + \phi_1 x_i - y_i) \cdot \frac{\partial}{\partial \phi_1} (\phi_0 + \phi_1 x_i - y_i)$$

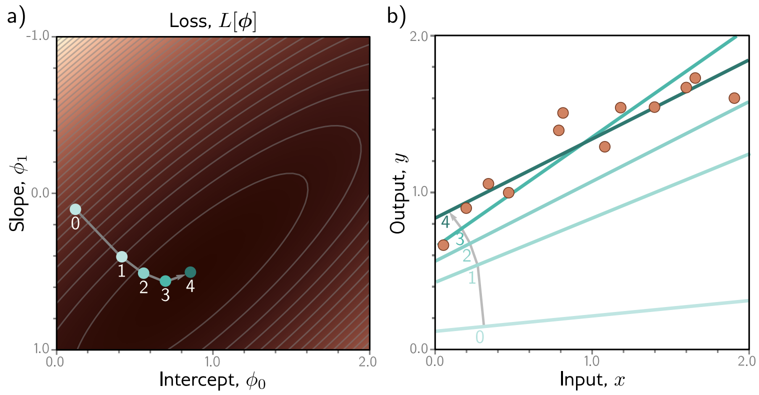
$$\frac{\partial L}{\partial \phi_1} = \sum_{i=1}^{I} 2(\phi_0 + \phi_1 x_i - y_i) \cdot x_i$$

$$\frac{\partial L}{\partial \phi_1} = 2 \sum_{i=1}^{I} (\phi_0 + \phi_1 x_i - y_i) x_i$$

The expressions for the slopes are:

$$\frac{\partial L}{\partial \phi_0} = 2 \sum_{i=1}^{I} (\phi_0 + \phi_1 x_i - y_i)$$

$$\frac{\partial L}{\partial \phi_1} = 2 \sum_{i=1}^{I} (\phi_0 + \phi_1 x_i - y_i) x_i$$



Problem 2.2

Show that we can find the minimum of the loss function in closed form by setting the expression for the derivatives from problem 2.1 to zero and solving for ϕ_0 and ϕ_1. Note that this works for linear regression but not for more complex models; this is why we use iterative model fitting methods like gradient descent (figure 2.4).

Solution

We will show that we can find the minimum of the loss function in closed form by setting the expressions for the derivatives to zero and solving for ϕ_0 and ϕ_1.

From Problem 2.1, we have:

$$\frac{\partial L}{\partial \phi_0} = 2\sum_{i=1}^{I}(\phi_0 + \phi_1 x_i - y_i)$$

$$\frac{\partial L}{\partial \phi_1} = 2\sum_{i=1}^{I}(\phi_0 + \phi_1 x_i - y_i)x_i$$

For $\frac{\partial L}{\partial \phi_0} = 0$:

$$2\sum_{i=1}^{I}(\phi_0 + \phi_1 x_i - y_i) = 0 \tag{1}$$

For $\frac{\partial L}{\partial \phi_1} = 0$:

$$2\sum_{i=1}^{I}(\phi_0 + \phi_1 x_i - y_i)x_i = 0 \tag{2}$$

From equation (1):

$$\sum_{i=1}^{I}(\phi_0 + \phi_1 x_i - y_i) = 0 \tag{3}$$

$$I\phi_0 + \phi_1 \sum_{i=1}^{I} x_i - \sum_{i=1}^{I} y_i = 0 \tag{3}$$

From equation (2):

$$\sum_{i=1}^{I}(\phi_0 + \phi_1 x_i - y_i)x_i = 0 \tag{4}$$

$$\phi_0 \sum_{i=1}^{I} x_i + \phi_1 \sum_{i=1}^{I} x_i^2 - \sum_{i=1}^{I} x_i y_i = 0 \tag{4}$$

From equation (3):

$$\phi_0 = \frac{\sum_{i=1}^{I} y_i - \phi_1 \sum_{i=1}^{I} x_i}{I} \tag{5}$$

Substitute this into equation (4):

$$\left(\frac{\sum_{i=1}^{I} y_i - \phi_1 \sum_{i=1}^{I} x_i}{I}\right) \sum_{i=1}^{I} x_i + \phi_1 \sum_{i=1}^{I} x_i^2 - \sum_{i=1}^{I} x_i y_i = 0$$

Simplify and solve for ϕ_1:

$$(\sum_{i=1}^{I} y_i)(\sum_{i=1}^{I} x_i) - \phi_1(\sum_{i=1}^{I} x_i)^2 + I\phi_1 \sum_{i=1}^{I} x_i^2 - I \sum_{i=1}^{I} x_i y_i = 0$$

$$\phi_1[I \sum_{i=1}^{I} x_i^2 - (\sum_{i=1}^{I} x_i)^2] = I \sum_{i=1}^{I} x_i y_i - (\sum_{i=1}^{I} x_i)(\sum_{i=1}^{I} y_i)$$

Therefore:

$$\phi_1 = \frac{I \sum_{i=1}^{I} x_i y_i - (\sum_{i=1}^{I} x_i)(\sum_{i=1}^{I} y_i)}{I \sum_{i=1}^{I} x_i^2 - (\sum_{i=1}^{I} x_i)^2} \tag{6}$$

Now substitute this value of ϕ_1 back into equation (5) to find ϕ_0:

$$\phi_0 = \frac{\sum_{i=1}^{I} y_i - \phi_1 \sum_{i=1}^{I} x_i}{I} \tag{7}$$

Equations (6) and (7) provide the closed-form solutions for ϕ_1 and ϕ_0 respectively, which minimize the loss function for linear regression.

Problem 3

Consider reformulating linear regression as a generative model, so we have $x = g[y, \phi] = \phi_0 + \phi_1 y$. What is the new loss function? Find an expression for the inverse function $y = g^{-1}[x, \phi]$ that we would use to perform inference. Will this model make the same predictions as the discriminative version for a given training dataset $\{x_i, y_i\}$? One way to establish this is to write code that fits a line to three data points using both methods and see if the result is the same.

Solution

Click here to see the answer of this question provided by the author.