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IERG 3050 Assignment 1 **Due: 7 March 2025**

- Submit a single .pdf file containing all your answers to the Blackboard before the due date.
- Answer all questions.
- Type or write your work neatly.

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1. A manufacturing process is supposed to produce ball bearings with a diameter of 0.5 inch. The company examines 50 ball bearings and finds that the sample mean is 0.45 inch and the sample variance is 0.06 inch^2. Test the null hypothesis $H_0\colon \mu = 0.5$ against the alternative hypothesis $H_1\colon \mu \neq 0.5$ at level $\alpha = 0.05$. Also, construct a 95% confidence interval for μ.

2. Suppose we flip an unfair coin n times and it comes up head x times. Let p be the probability of coming up head per flip. Assume $\Pr(p \mid n)$ is uniform. What is the probability density function of p?

3. When m is significantly large, the m-Erlang distribution has a bell-shape appearance. Explain this phenomenon.

4. Let X be a random variable with mean μ and variance σ^2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with the same distribution as X.

   a) Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Show that $E[\bar{X}] = \mu$ and $E[S^2] = \sigma^2$. That is, we are going to show that the sample variance is actually a bias-free estimator of σ^2 when the true value of mean is unknown.

   b) Let $\tilde{S}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$. Show that $E[\tilde{S}^2] = \sigma^2$. That is, when the true value of mean is known, we should we $\frac{1}{n}$ instead of $\frac{1}{n-1}$ as a bias-free estimator of σ^2.

1. A manufacturing process is supposed to produce ball bearings with a diameter of 0.5 inch. The company examines 50 ball bearings and finds that the sample mean is 0.45 inch and the sample variance is 0.06 inch^2. Test the null hypothesis $H_0\colon \mu = 0.5$ against the alternative hypothesis $H_1\colon \mu \neq 0.5$ at level $\alpha = 0.05$. Also, construct a 95% confidence interval for μ.

$\mu = 0.5$, $n = 50$, $\bar{x} = 0.45$, $s^2 = 0.06$, $\alpha = 0.05$

$s = \sqrt{s^2} = \sqrt{0.06} = 0.2449$

$H_0: \mu = 0.5$

$H_a: \mu \neq 0.5$

$D_f = n-1 = 49$

Test statistic: $t_c = \dfrac{\bar{x} - \mu}{\sqrt{\frac{s^2}{n}}} = \dfrac{0.45 - 0.5}{\sqrt{\frac{0.06}{50}}} = -\dfrac{0.05}{0.03464} = -1.4434$

Degrees of freedom (D_f) = $n-1 = 49$

Significance level (a) = $1 - CL = 0.05$

Probability (P) = $1 - \frac{a}{2} = 0.975$

$\Rightarrow$ T-score:

$T\,0.975\,(49) = 2.0096$

$\bar{x} \pm T\,0.975\,(49) \times \dfrac{0.245}{\sqrt{50}}$

$0.45 \pm 2.0096 \times 0.03464$

0.45 ± 0.06961

$\therefore$ Upper bound $= 0.5196$, Lower bound $= 0.3804$.

Mean confidence interval: $[0.3804, 0.5196]$.

2. Suppose we flip an unfair coin n times and it comes up head x times. Let p be the probability of coming up head per flip. Assume $\Pr(p \mid n)$ is uniform. What is the probability density function of p?

- Likelihood Function:

$$Pr(x|p,n) = \binom{n}{x} p^x (1-p)^{n-x}$$

- Prior Distribution:

$$Pr(p) = 1 \text{ for } p \in [0,1]$$

- Posterior Distribution:

$$Pr(p|x,n) \propto Pr(x|p,n) \cdot Pr(P)$$

$$Pr(p|x,n) \propto \binom{n}{x} p^x (1-p)^{n-x} \cdot 1$$

$$Pr(p|x,n) \propto p^x (1-p)^{n-x}$$

$\Rightarrow$ As the posterior distribution is a Beta distribution, it has the form:

$$Pr(p|x,n) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)} \quad \text{where} \quad \begin{matrix} \alpha = x+1 \\ \beta = n-x+1 \end{matrix}$$

$\therefore$ The posterior PDF of p is:

$$Pr(p|x,n) = \frac{p^x(1-p)^{n-x}}{B(x+1,\ n-x+1)}$$

3. When m is significantly large, the m-Erlang distribution has a bell-shape appearance. Explain this phenomenon.

PDF for m-Erlang: $f(x; m, \lambda) = \dfrac{\lambda^m x^{m-1} e^{-\lambda x}}{(m-1)!}, \quad x \geq 0$

$\Rightarrow$ The bell shape can be explained by the Central Limit Theorem and the properties of the Gamma distribution:

CLT: The distribution is the sum of m, each with mean $\frac{1}{\lambda}$ and variance $\frac{1}{\lambda^2}$.
As m becomes large, the sum of these exponential random variables tends towards a normal distribution by the CLT.

Gamma: Mean: $\mu = \frac{m}{\lambda}$
Variance: $\sigma^2 = \frac{m}{\lambda^2}$

So when m increases, the mean (μ) increases linearly with m.
$\Rightarrow$ The variance (σ^2) also increases linearly with m, but the SD (σ) increases only with $\sqrt{m}$.

$\therefore$ As the distribution becomes more symmetric and concentrated around the mean, it leads to a bell-shaped appearance.

4. Let X be a random variable with mean μ and variance σ^2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with the same distribution as X.

a) Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Show that $E[\bar{X}] = \mu$ and $E[S^2] = \sigma^2$. That is, we are going to show that the sample variance is actually a bias-free estimator of σ^2 when the true value of mean is unknown.

b) Let $\tilde{S}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$. Show that $E[\tilde{S}^2] = \sigma^2$. That is, when the true value of mean is known, we should we $\frac{1}{n}$ instead of $\frac{1}{n-1}$ as a bias-free estimator of σ^2.

(a) Showing $E[\bar{X}] = \mu$:

$$E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu \quad //$$

Showing $E[S^2] = \sigma^2$:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$\Rightarrow$ Using the fact that $\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$

$$E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = E\left[\sum_{i=1}^{n} X_i^2\right] - nE[\bar{X}^2]$$

$\Rightarrow$ Since $E[X_i^2] = \sigma^2 + \mu^2$, $E[\bar{X}^2] = \frac{\sigma^2}{n} + \mu^2$:

$$E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right) = (n-1)\sigma^2$$

$$\therefore E[S^2] = \frac{1}{n-1}E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \frac{1}{n-1}(n-1)\sigma^2 = \sigma^2 \quad //$$

(b) Expectation of $\hat{S}^2$:

$$\hat{S}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

Taking expection :

$$E[\hat{S}^2] = \frac{1}{n}\sum_{i=1}^{n} E[(X_i - \mu)^2] = \frac{1}{n}\sum_{i=1}^{n}\sigma^2 = \sigma^2$$

$\therefore$ Since each $(X_i - \mu)^2$ has expectation σ^2, the expectation of $\hat{S}^2$ is σ^2.