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IERG 3050 Assignment 2 **Due: 21 March 2025**

- Submit a single .pdf file containing all your answers to the Blackboard before the due date.
- Answer all questions.
- Type or write your work neatly.

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1. A service center consists of two servers, each working at an exponential rate of 2 services per hour. Assume the system capacity is at most 3 customers (i.e., the queue size is at most 1). Suppose the customers arrive at a Poisson rate of 3 per hour. If the queue is full, then the arriving customer would leave the service center directly. What is fraction of potential customers entering the system?

   a) Approximate the answer by a simulation.

   b) Calculate the answer by using a continuous-time Markov chain.

2. Let $\mathcal{LN}(\gamma, \mu, \sigma^2)$ denote the shifted (three-parameter) lognormal distribution, which has density

$$f(x) = \begin{cases} \frac{1}{(x-\gamma)\sqrt{2\pi\sigma^2}} \exp\left(\frac{-[\ln(x-\gamma)-\mu]^2}{2\sigma^2}\right) & \text{if } x > \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

   for $\sigma > 0$ and any real numbers γ and μ. That is, $\mathcal{LN}(0, \mu, \sigma^2)$ is the original $\mathcal{LN}(\mu, \sigma^2)$ distribution.

   a) Verify that $X \sim \mathcal{LN}(\gamma, \mu, \sigma^2)$ if and only if $X - \gamma \sim \mathcal{LN}(\mu, \sigma^2)$.

   b) Show that for a fixed known value of γ, the MLEs of μ and σ in the $\mathcal{LN}(\gamma, \mu, \sigma^2)$ distribution are

$$\hat{\mu} = \frac{\sum_{i=1}^{n} \ln(X_i - \gamma)}{n} \qquad \text{and} \qquad \hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} [\ln(X_i - \gamma) - \hat{\mu}]^2}{n}}.$$

   That is, we simply shift the data by an amount $-\gamma$ and then treat them as being unshifted lognormal data.

3. Let $X \sim m$-Erlang(β). Suppose we observe 10 samples:

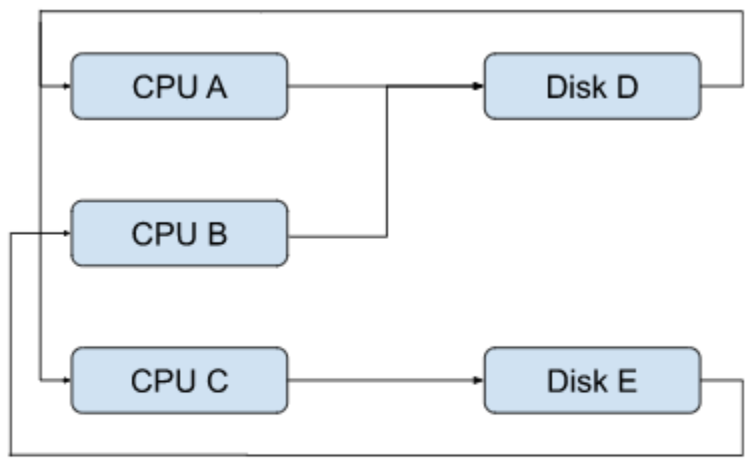
$$288, 302, 304, 308, 313, 314, 321, 347, 374, 413$$

   Deduce the value of m and β based on the above samples by

a) method of moments estimation.

b) maximum likelihood estimation. Apply the approximation of digamma function $\frac{d}{dx}\ln(\Gamma(x)) \approx \ln x - \frac{1}{2x}$ when necessary.

4. Consider a system with 3 CPUs and 2 Disks. CPUs A and B submit tasks to Disk D. CPU C submits tasks to Disk E. Disk D gives responses to CPUs A and C. Disk E gives responses to CPU B. This is illustrated in the following figure.



Suppose we know the visit ratios $V_A = 100$, $V_B = 80$ and the service rates $\mu_A = 0.05$, $\mu_B = 0.03$, $\mu_C = 0.04$, $\mu_D = 0.01$, $\mu_E = 0.06$. Which device is the bottleneck in the system?

1. A service center consists of two servers, each working at an exponential rate of 2 services per hour. Assume the system capacity is at most 3 customers (i.e., the queue size is at most 1). Suppose the customers arrive at a Poisson rate of 3 per hour. If the queue is full, then the arriving customer would leave the service center directly. What is fraction of potential customers entering the system?

a) Approximate the answer by a simulation.

b) Calculate the answer by using a continuous-time Markov chain.

a) Arrival Rate : $\lambda = 3$ per hour

Service Rate : $\mu = 2$ per hour

System Capacity : 3 Max. (2 served , 1 in queue)

⇒ M/M/2/3 queue

⇒ Stimulation Setup :

- System State (0, 1, 2, 3) :
  - 0 : No customers
  - 1 : 1 server busy, 1 idle, queue = 0
  - 2 : 2 servers busy, queue = 0
  - 3 : 2 servers busy, queue = 1, system full

- Events :

  Arrival : At $\lambda = 3$, if state < 3 → state + 1 (customer enters)

  if state = 3 → customer blocked

  Departure : Occur at $\mu \times$ (No. of busy servers), where $\mu = 2$.

  Server finishes → State − 1 → queued customer moves to service (If have)

- Objective :

$$\text{Fraction Entering} = \frac{\text{Total Arrivals} - \text{Blocked Arrivals}}{\text{Total Arrivals}}$$

⇒ Stimulation Logic :

1. Maintain a clock + an event list.
2. Process the earliest event.
3. Update system state, track arrivals & blocked customers, schedule the next events.

- Arrival Times : Exponential Function with rate $\lambda = 3$.

  Mean interarrival time $= \frac{1}{\lambda} = \frac{1}{3}$ hour.

- Service Times : Exponential Function with rate $\mu = 2$.

  Mean service time $= \frac{1}{\mu} = \frac{1}{2}$ hour.

- Departure Rate : If 1 server is busy : Departure rate $= \mu = 2$

  If 2 servers are busy : Departure rate $= 2\mu = 4$

```python
import numpy as np

# Set random seed for reproducibility
np.random.seed(42)

# Parameters
lambda_arrival = 3  # Arrival rate (customers per hour)
mu_service = 2      # Service rate per server (customers per hour)
num_servers = 2     # Number of servers
system_capacity = 3 # Maximum number of customers in the system
num_arrivals = 100000  # Number of arrivals to simulate

# Initialize simulation variables
current_time = 0.0
system_state = 0  # Number of customers in the system
total_arrivals = 0  # Total number of arrival attempts
turned_away = 0   # Number of customers turned away
event_list = []   # List of events: (time, type, server_id if departure)

# Generate the first arrival
next_arrival_time = np.random.exponential(1/lambda_arrival)
event_list.append((next_arrival_time, "arrival", None))

# Lists to track departure times for each server
# -1 indicates the server is idle
server_busy_until = [-1, -1]  # Departure times for each server

# Run the simulation
while total_arrivals < num_arrivals:
    # Sort events by time
    event_list.sort()

    # Get the next event
    event_time, event_type, server_id = event_list.pop(0)
    current_time = event_time

    if event_type == "arrival":
        total_arrivals += 1

        # Schedule the next arrival
        next_arrival_time = current_time + np.random.exponential(1/lambda_arrival)
        event_list.append((next_arrival_time, "arrival", None))

        # Process the arrival
        if system_state < system_capacity:
            # Customer enters the system
            system_state += 1

            # Check for an idle server
            for i in range(num_servers):
                if server_busy_until[i] <= current_time:
                    # Server i is idle, assign the customer
                    service_time = np.random.exponential(1/mu_service)
                    departure_time = current_time + service_time
                    server_busy_until[i] = departure_time
                    event_list.append((departure_time, "departure", i))
                    break
        else:
            # System is full, customer is turned away
            turned_away += 1

    elif event_type == "departure":
        # Process the departure
        system_state -= 1
        server_busy_until[server_id] = -1  # Mark server as idle

        # If there are customers waiting (system_state >= num_servers after departure),
        # assign the next customer to the server
        if system_state >= num_servers:
            service_time = np.random.exponential(1/mu_service)
            departure_time = current_time + service_time
            server_busy_until[server_id] = departure_time
            event_list.append((departure_time, "departure", server_id))

# Calculate the fraction of customers who entered the system
fraction_entered = (total_arrivals - turned_away) / total_arrivals

print(f"Total arrivals: {total_arrivals}")
print(f"Customers turned away: {turned_away}")
print(f"Fraction of customers who entered the system: {fraction_entered:.4f}")
```

Stimulation Result :

```
lifehater@LifedeMacBook-Pro Desktop %  /usr/bin/env /usr/bin/python3
- /Users/lifehater/Desktop/Random.py
Total arrivals: 100000
Customers turned away: 18547
Fraction of customers who entered the system: 0.8145
```

b) • System State (0, 1, 2, 3):

0 : No customers

1 : 1 server busy, 1 idle, queue = 0

2 : 2 servers busy, queue = 0

3 : 2 servers busy, queue = 1, system full

• Transition Rates:

$0 \to 1$: Arrival Rate $\lambda = 3$ (1 server)

$1 \to 2$: $\lambda = 3$ (2 servers)

$2 \to 3$: $\lambda = 3$ (queue fills)

$1 \to 0$: Service Rate $\mu = 2$ (1 server)

$2 \to 1$: $2\mu = 4$ (2 servers)

$3 \to 2$: $2\mu = 4$ (1 of 2 servers finishes, queue customer moves to service)

From 3 upwards : Blocked

• Balance Equations:

State 0 : $3\pi_0 = 2\pi_1 \to \pi_1 = \frac{3}{2}\pi_0$

State 1 : $(3+2)\pi_1 = 3\pi_0 + 4\pi_2 \to 5\pi_1 = 3\pi_0 + 4\pi_2$

State 2 : $(3+4)\pi_2 = 3\pi_1 + 4\pi_3 \to 7\pi_2 = 3\pi_1 + 4\pi_3$

State 3 : $4\pi_3 = 3\pi_2$

• Solve:

From 0 : $\pi_1 = \frac{3}{2}\pi_0 = 1.5\pi_0$

From 3 : $\pi_3 = \frac{3}{4}\pi_2 = 0.75\pi_2$

State 1 : $5(1.5\pi_0) = 3\pi_0 + 4\pi_2 \to 7.5\pi_0 = 3\pi_0 + 4\pi_2 \to 4.5\pi_0 = 4\pi_2 \to \pi_2 = \frac{4.5}{4}\pi_0 = 1.125\pi_0$

State 3 : $\pi_3 = 0.75 \times 1.125\pi_0 = 0.84375\pi_0$

Normalize : $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

- $\pi_0 + 1.5\pi_0 + 1.125\pi_0 + 0.84375\pi_0 = (1 + 1.5 + 1.125 + 0.84375)\pi_0 = 4.46875\pi_0 = 1$

  $\pi_0 = \frac{1}{4.46875} \approx 0.2238$

$\Rightarrow \pi_1 = \frac{3}{2}\pi_0 \approx 0.3357$

$\pi_2 = 1.125\pi_0 \approx 0.2517$

$\pi_3 = 0.84375\pi_0 \approx 0.1888$

$\therefore$ P(block) ≈ 0.1888

Fraction entering = 1 − P(block) = 0.8112 (or about 81.12%).

2. Let $\mathcal{LN}(\gamma,\mu,\sigma^2)$ denote the shifted (three-parameter) lognormal distribution, which has density

$$f(x)=\begin{cases}\frac{1}{(x-\gamma)\sqrt{2\pi\sigma^2}}\exp\left(\frac{-[\ln(x-\gamma)-\mu]^2}{2\sigma^2}\right) & \text{if } x>\gamma,\\ 0 & \text{otherwise,}\end{cases}$$

for $\sigma>0$ and any real numbers γ and μ. That is, $\mathcal{LN}(0,\mu,\sigma^2)$ is the original $\mathcal{LN}(\mu,\sigma^2)$ distribution.

a) Verify that $X\sim\mathcal{LN}(\gamma,\mu,\sigma^2)$ if and only if $X-\gamma\sim\mathcal{LN}(\mu,\sigma^2)$.

b) Show that for a fixed known value of γ, the MLEs of μ and σ in the $\mathcal{LN}(\gamma,\mu,\sigma^2)$ distribution are

$$\hat{\mu}=\frac{\sum_{i=1}^n\ln(X_i-\gamma)}{n}\qquad\text{and}\qquad\hat{\sigma}=\sqrt{\frac{\sum_{i=1}^n[\ln(X_i-\gamma)-\hat{\mu}]^2}{n}}.$$

That is, we simply shift the data by an amount $-\gamma$ and then treat them as being unshifted lognormal data.

a) $X-\gamma\sim\mathcal{LN}(\mu,\sigma^2)$ if $X\sim\mathcal{LN}(\gamma,\mu,\sigma^2)$

$X\sim\mathcal{LN}(\gamma,\mu,\sigma^2)$

$\ln(x-\gamma)\sim N(\mu,\sigma^2)$

Let $Y=x-\gamma$, $Y>0$,

$\ln(Y)=\ln(x-\gamma)\sim N(\mu,\sigma^2)$

$Y=x-\gamma\sim\mathcal{LN}(\mu,\sigma^2)$

$X\sim\mathcal{LN}(\gamma,\mu,\sigma^2)$ if $X-\gamma\sim\mathcal{LN}(\mu,\sigma^2)$

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Let $Y=x-\gamma$,

$Y\sim\mathcal{LN}(\mu,\sigma^2)$

$\ln(Y)\sim N(\mu,\sigma^2)$

$Y=x-\gamma$

$\ln(x-\gamma)=\ln(Y)\sim N(\mu,\sigma^2)$

$x=\mathcal{LN}(\gamma,\mu,\sigma^2)$

$\therefore$ $X\sim\mathcal{LN}(\gamma,\mu,\sigma^2)$ if and only if $X-\gamma\sim\mathcal{LN}(\mu,\sigma^2)$.

b) For $\mathcal{LN}(\gamma,\mu,\sigma^2)$:

Joint Likelihood function: $L(\mu,\sigma^2)=\prod_{i=1}^n f_x(x_i)=\prod_{i=1}^n\frac{1}{(x_i-\gamma)\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln(x_i-\gamma)-\mu)^2}{2\sigma^2}\right)$

Likelihood function: $\ln L(\mu,\sigma^2)=\sum_{i=1}^n\ln(x_i-\gamma)-n\ln(\sigma\sqrt{2\pi})-\frac{1}{2\sigma^2}\sum_{i=1}^n(\ln(x_i-\gamma)-\mu)^2$

⇒ Then we find the partial d. with respect to μ:

$$\frac{\partial\ln L}{\partial\mu}=\frac{1}{\sigma^2}\sum_{i=1}^n(\ln(x_i-\gamma)-\mu)=0$$

$$\hat{\mu}=\frac{1}{n}\sum_{i=1}^n\ln(x_i-\gamma)$$

Partial d. with respect to σ^2:

$$\frac{\partial\ln L}{\partial\sigma^2}=-\frac{n}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{i=1}^n(\ln(x_i-\gamma)-\mu)^2=0$$

$$\hat{\sigma}=\sqrt{\frac{1}{n}\sum_{i=1}^n(\ln(x_i-\gamma)-\hat{\mu})^2}$$

3. Let $X \sim m$-Erlang(β). Suppose we observe 10 samples:

$$288, 302, 304, 308, 313, 314, 321, 347, 374, 413$$

Deduce the value of m and β based on the above samples by

a) method of moments estimation.

b) maximum likelihood estimation. Apply the approximation of digamma function $\frac{d}{dx}\ln(\Gamma(x)) \approx \ln x - \frac{1}{2x}$ when necessary.

a) PDF : $f(x) = \dfrac{\beta^m x^{m-1} e^{-\beta x}}{(m-1)!}$, $x > 0$.

$$E[X] = \frac{m}{\beta}$$

$$Var[X] = \frac{m}{\beta^2}$$

Sample Mean : $\bar{X} = \dfrac{288+302+304+308+313+314+321+347+374+413}{10} = 328.4$

Sample Variance (S^2) : $\sum(X_i - \bar{X})^2 = (288-328.4)^2 + (302-328.4)^2 + \cdots + (413-328.4)^2$

$$= (-40.4)^2 + (-26.4)^2 + (-24.4)^2 + (-20.4)^2 + (-15.4)^2 + (-14.4)^2 + (-7.4)^2 + (18.6)^2 + (45.6)^2 + (84.6)^2$$

$$= 13422.4$$

$$S^2 = \frac{\sum(X_i - \bar{X})^2}{n-1} = \frac{13422.4}{9} \approx 1491.38$$

$$S = \sqrt{1491.38} \approx 38.62$$

First moment (mean) : $E[X] = \frac{m}{\beta} = \bar{X} = 328.4$ —— ①

Second moment (variance) : $Var(X) = \frac{m}{\beta^2} \approx s^2 = 1491.38$ —— ②

From ① and ② : $\beta \approx \dfrac{328.4}{1491.38} \approx 0.2202$

Sub $\beta = 0.2202$ into ① : $m = 328.4 \times \beta \approx 72.31$

$m = 72$

$\therefore$ Using Method of Moments estimates : $m \approx 72$, $\beta \approx 0.2202$.

b) For $X \sim \text{Erlang}(m, \beta)$, PDF: $f(x) = \dfrac{\beta^m x^{m-1} e^{-\beta x}}{(m-1)!}$

With $n=10$, likelihood function: $L(m,\beta) = \prod_{i=1}^{10} \dfrac{\beta^m x_i^{m-1} e^{-\beta x_i}}{(m-1)!} = \dfrac{\beta^{10m} \left(\prod_{i=1}^{10} x_i\right)^{m-1} e^{-\beta \sum x_i}}{[(m-1)!]^{10}}$

Log-likelihood: $\ln L = 10m\ln\beta + (m-1)\sum \ln x_i - \beta \sum x_i - 10\ln[(m-1)!]$

$$\sum x_i = 3284$$

$$\begin{aligned}\sum \ln x_i &= \ln(288) + \ln(302) + \cdots \ln(413) \\ &\approx 5.66296 + 5.71043 + 5.71703 + 5.73010 + 5.74620 \\ &\quad + 5.74940 + 5.77144 + 5.84932 + 5.92426 + 6.02345 \\ &\approx 57.88458\end{aligned}$$

$\therefore \ln L = 10m\ln\beta + (m-1) \times 57.88458 - \beta \times 3284 - 10\ln[(m-1)!]$

$\Rightarrow \dfrac{\partial \ln L}{\partial \beta} = \dfrac{10m}{\beta} - 3284 = 0 \Rightarrow \beta = \dfrac{10m}{3284}$

$\Rightarrow$ MLE of $\beta\ (\hat{\beta}) = \dfrac{10m}{3284}$

Sub $\beta = \dfrac{10m}{3284}$ into log-likelihood:

$$\begin{aligned}\ln L(m) &= 10m\ln\left(\frac{10m}{3284}\right) + (m-1)\times 57.88458 - \left(\frac{10m}{3284}\right)\times 3284 - 10\ln[(m-1)!] \\ &= 10m\ln 10m - 10m\ln 3284 + (m-1)\times 57.88458 - 10m - 10\ln[(m-1)!] \\ &= 10m\ln 10m - 80.968m + 57.88458m - 57.88458 - 10m - 10\ln[(m-1)!] \\ &= 10m\ln 10m - 33.08342m - 57.88458 - 10\ln[(m-1)!]\end{aligned}$$

$$\boxed{\begin{aligned}\ln 3284 &\approx 8.0968 \\ 10\ln 3284 &\approx 80.968\end{aligned}}$$

Given approximation: $\dfrac{d}{dx}\ln(\Gamma(x)) \approx \ln x - \dfrac{1}{2x}$ with $\ln[(m-1)!] = \ln\Gamma(m)$, $\psi(x) = \dfrac{d}{dx}\ln\Gamma(x)$

$\Rightarrow \ln\Gamma(m) \approx (m-\tfrac{1}{2})\ln m - m + \tfrac{1}{2}\ln(2\pi)$

$\therefore 10\ln\Gamma(m) \approx 10(m-\tfrac{1}{2})\ln m - 10m + 5\ln(2\pi)$

Substitute $\ln L(m) \approx 10m\ln 10m - 33.08342m - 57.88458$
$\qquad - 10(m-\tfrac{1}{2})\ln m + 10m - 5\ln(2\pi)$
$\qquad = 10m\ln 10 - 33.08342m + 10\ln m - 57.88458 - 5\ln(2\pi)$

Take derivative: $\dfrac{d}{dm}\ln L \approx 10\ln 10 - 33.08342 + \dfrac{10}{m} = 0$

$$\begin{aligned}10\ln 10 + \frac{10}{m} &= 33.08342 \\ 23.02585 + \frac{10}{m} &= 33.08342 \\ \frac{10}{m} &= 33.08342 - 23.02585 \\ \frac{10}{m} &= 10.05757 \\ m &= \frac{10}{10.05757} \approx 0.99428\end{aligned}$$

$\Leftarrow$ Completely Wrong

As the previous method is not calculating the right output, I will evaluate $\ln L(m)$ for m :

$$\ln L(m) = 10m\ln\left(\frac{10m}{3284}\right) + (m-1)\times 57.88458 - 10m - 10\ln[(m-1)!]$$

$\Rightarrow$ I will try for $m = 71, 72, 73$:

$m = 71$: $\beta = \dfrac{10\times 71}{3284} \approx 0.2162$

$\ln\beta = \ln 0.2162 \approx -1.5316$

$10m\ln\beta = 710 \times -1.5316 \approx -1087.402$

$(m-1)\times 57.88458 = 70 \times 57.88458 \approx 4051.9206$

$\beta\sum x_i = 0.2162 \times 3284 \approx 710$

$$\boxed{\begin{array}{l}\ln 70! \approx 230.44 \\ 10\ln 70! \approx 2304.4\end{array}}$$

$\ln L(71) \approx -1087.402 + 4051.9206 - 710 - 2304.4 \approx -49.8814$

$m = 72$: $\beta = \dfrac{10\times 72}{3284} \approx 0.2192$

$\ln\beta = \ln 0.2192 \approx -1.5176$

$10m\ln\beta = 720 \times -1.5176 \approx -1092.6477$

$(m-1)\times 57.88458 = 71 \times 57.88458 \approx 4109.8052$

$\beta\sum x_i = 0.2192 \times 3284 \approx 720$

$$\boxed{\begin{array}{l}\ln 71! \approx 234.70 \\ 10\ln 71! \approx 2347.0\end{array}}$$

$\ln L(72) \approx -1092.6477 + 4109.8052 - 720 - 2347.0 \approx -49.8425$

$m = 73$: $\beta = \dfrac{10\times 73}{3284} \approx 0.2223$

$\ln\beta = \ln 0.2223 \approx -1.5038$

$10m\ln\beta = 730 \times -1.5038 \approx -1097.7542$

$(m-1)\times 57.88458 = 72 \times 57.88458 \approx 4167.6898$

$\beta\sum x_i = 0.2223 \times 3284 \approx 730$

$$\boxed{\begin{array}{l}\ln 72! \approx 238.98 \\ 10\ln 72! \approx 2389.8\end{array}}$$

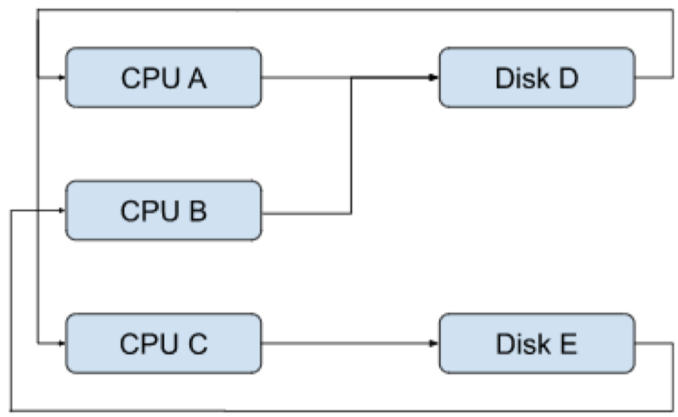
$\ln L(73) \approx -1097.7542 + 4167.6898 - 730 - 2389.8 \approx -49.8644$

$\because$ The log-likelihood is highest at $m = 72$, so the MLE for m is 72 :

$\hat{\beta} = \dfrac{10\times 72}{3284} \approx 0.2192$

$\therefore$ Using Method of Maximum Likelihood estimates : $m \approx 72$, $\beta \approx 0.2192$.

4. Consider a system with 3 CPUs and 2 Disks. CPUs A and B submit tasks to Disk D. CPU C submits tasks to Disk E. Disk D gives responses to CPUs A and C. Disk E gives responses to CPU B. This is illustrated in the following figure.



Suppose we know the visit ratios $V_A = 100$, $V_B = 80$ and the service rates $\mu_A = 0.05$, $\mu_B = 0.03$, $\mu_C = 0.04$, $\mu_D = 0.01$, $\mu_E = 0.06$. Which device is the bottleneck in the system?

Utilization $U = \frac{\text{Arrival Rate}}{\text{Service Rate}} = \frac{\lambda_i}{\mu_i}$

$$U_A = \frac{\lambda_A}{\mu_A} = \frac{100}{0.04} = 2000\lambda$$

$$U_B = \frac{\lambda_B}{\mu_B} = \frac{80}{0.03} = 2666.7\lambda$$

$$U_C = \frac{\lambda_C}{\mu_C} = \frac{1}{0.04} = 25\lambda$$

As Disk D receives tasks from CPU A and B, the arrival rate of Disk D is:

$$\lambda_D = \lambda_A + \lambda_B$$
$$= 100 + 180$$
$$= 180\lambda$$

$$U_D = \frac{\lambda_D}{\mu_D} = \frac{180}{0.01} = 18000\lambda$$

$$\lambda_E = \lambda_C$$

$$U_E = \frac{\lambda_E}{\mu_E} = \frac{1}{0.06} = 16.7\lambda$$

$\therefore$ U_D has the highest utilization, Disk D is the bottleneck.