Computing $Q[t = 9, W = 100.2, a = 0.5]$

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1 Introduction

This document outlines the computation of the optimal Q-value $Q_9^*(W = 100.2, a = 0.5)$ in a portfolio optimization problem under a Constant Absolute Risk Aversion (CARA) utility framework. The analysis is divided into two main sections: the first section defines the key parameters and assumptions, while the second section provides the mathematical derivation of the Q-value.

2 Parameter Settings

The following parameters are defined for the computation:

- $T = 10$: The total number of time steps.
- $R_u = 0.10$: The upward return of the risky asset (with probability $P_u = 0.6$).
- $R_d = -0.05$: The downward return of the risky asset (with probability $P_d = 0.4$).
- $r_f = 0.03$: The risk-free rate.
- $\gamma = 0.95$: The discount factor.
- $aversion_rate = 0.04$: The risk aversion coefficient used in the CARA utility function.
- Wealth transition: The wealth at the next time step is given by

$$W_{t+1} = W_t(1 + r_f) + aW_t(R - r_f),$$

  where a is the proportion of wealth invested in the risky asset, and R represents the random return of the risky asset.
- Initial state: At $t = 9$, the wealth $W_9 = 100.2$, and the action $a = 0.5$ is considered for evaluation.

3 Mathematical Derivation

This section derives the Q-value $Q_9^*(W_9 = 100.2, a = 0.5)$ using dynamic programming.

3.1 Value Function Form

The value function is assumed to take the form:

$$V_t^*(W_t) = -b_t \exp(-c_t W_t),$$

with the terminal condition at $t = 10$:

$$V_{10}^*(W_{10}) = -\frac{1}{aversion_rate} e^{-aversion_rate \cdot W_{10}} = -25e^{-0.04W_{10}},$$

where $b_{10} = 25$ and $c_{10} = 0.04$.

3.2 Bellman Equation ($t = 9$)

The Bellman optimality equation at $t = 9$ is:

$$Q_9^*(W_9, a) = \gamma \mathbb{E}\left[V_{10}^*(W_{10}) \mid W_9, a\right],$$

where the next-period wealth is:

$$W_{10} = W_9(1 + r_f) + aW_9(R - r_f).$$

Substituting the value function:

$$V_{10}^*(W_{10}) = -25 \exp\left(-0.04\left[W_9(1 + r_f) + aW_9(R - r_f)\right]\right),$$

thus:

$$Q_9^*(W_9, a) = \gamma \cdot (-25)\mathbb{E}\left[\exp\left(-0.04\left[W_9(1 + r_f) + aW_9(R - r_f)\right]\right)\right].$$

Decomposing the expression:

$$Q_9^*(W_9, a) = -25\gamma \exp\left(-0.04(1 + r_f)W_9\right) \mathbb{E}\left[\exp\left(-0.04aW_9(R - r_f)\right)\right].$$

3.3 Compute the Expectation

The expectation is computed based on the two possible returns:

- $R - r_f = 0.07$ (probability 0.6),
- $R - r_f = -0.08$ (probability 0.4).

Thus:

$$\mathbb{E}\left[e^{-0.04aW_9(R - r_f)}\right] = 0.6e^{-0.04aW_9(0.07)} + 0.4e^{-0.04aW_9(-0.08)},$$

which simplifies to:

$$= 0.6e^{-0.0028aW_9} + 0.4e^{0.0032aW_9}.$$

Substituting into the Q-value equation:

$$Q_9^*(W_9, a) = -25 \cdot 0.95 \exp\left(-0.04 \cdot 1.03 \cdot W_9\right)\left[0.6e^{-0.0028aW_9} + 0.4e^{0.0032aW_9}\right].$$

3.4 Substitute Specific Values

For $W_9 = 100.2$ and $a = 0.5$:

- $0.04 \cdot 1.03 \cdot 100.2 \approx 0.0412 \cdot 100.2 \approx 4.12824$,
- $\exp(-4.12824) \approx 0.01626$,
- $0.0028 \cdot 0.5 \cdot 100.2 \approx 0.0014 \cdot 100.2 \approx 0.14028$,
- $e^{-0.14028} \approx 0.8691$,
- $0.0032 \cdot 0.5 \cdot 100.2 \approx 0.0016 \cdot 100.2 \approx 0.16032$,
- $e^{0.16032} \approx 1.1741$,
- $\Phi(0.5) = 0.6 \cdot 0.8691 + 0.4 \cdot 1.1741 \approx 0.52146 + 0.46964 = 0.9911$.

$$Q_9^*(100.2, 0.5) = -25 \cdot 0.95 \cdot 0.01626 \cdot 0.9911,$$
$$\approx -25 \cdot 0.95 \cdot 0.01611 \approx -0.3826.$$

3.5 Verify the Optimal a^*

To find the optimal a, compute the derivative of $\Phi(a)$:

$$\frac{d}{da}\Phi(a) = -0.6 \cdot 0.0028 W_9 e^{-0.0028 a W_9} + 0.4 \cdot 0.0032 W_9 e^{0.0032 a W_9} = 0,$$

$$\begin{aligned}
0.6 \cdot 0.0028 e^{-0.0028 a W_9} &= 0.4 \cdot 0.0032 e^{0.0032 a W_9}, \\
\frac{0.00168}{0.00128} e^{-0.0028 a W_9} &= e^{0.0032 a W_9}, \\
1.3125 e^{-0.0028 a W_9} &= e^{0.0032 a W_9}, \\
e^{0.0032 a W_9 + 0.0028 a W_9} &= 1.3125, \\
e^{0.006 a W_9} &= 1.3125, \\
0.006 a W_9 &= \ln(1.3125) \approx 0.272, \\
a &= \frac{0.272}{0.006 \cdot 100.2} \approx \frac{0.272}{0.6012} \approx 0.4524.
\end{aligned}$$

Recalculate $\Phi(a^*)$ with $a^* = 0.4524$:

- $0.0028 \cdot 0.4524 \cdot 100.2 \approx 0.1269$, $e^{-0.1269} \approx 0.8808$,
- $0.0032 \cdot 0.4524 \cdot 100.2 \approx 0.1451$, $e^{0.1451} \approx 1.1562$,
- $\Phi(0.4524) = 0.6 \cdot 0.8808 + 0.4 \cdot 1.1562 \approx 0.5285 + 0.4625 = 0.9910$.

The result does not change significantly, $Q_9^*(100.2, 0.4524) \approx -0.3826$ (close to $a = 0.5$).