Quandoquidem inter nos sanctissima divitiarum maiestas

Since the majesty of wealth is most sacred with us

Juvenal (55–120), Sat. 1, 113

20

Path Integrals and Financial Markets

An important field of applications for path integrals are financial markets. The prices of assets fluctuate as a function of time and, if the number of participants in the market is large, the fluctuations are pretty much random. Then the time dependence of prices can be modeled by fluctuating paths.

20.1 Fluctuation Properties of Financial Assets

Let $S(t)$ denote the price of a stock or another financial asset. Over long time spans, i.e., if data recording frequency is low, the average over many stock prices has a time behavior that can be approximated by pieces of exponentials. This is why they are usually plotted on a logarithmic scale. This is best illustrated by a plot of the Dow-Jones industrial index over 60 years in Fig. 20.1. The fluctuations of the index have a certain average width called the *volatility* of the market. Over

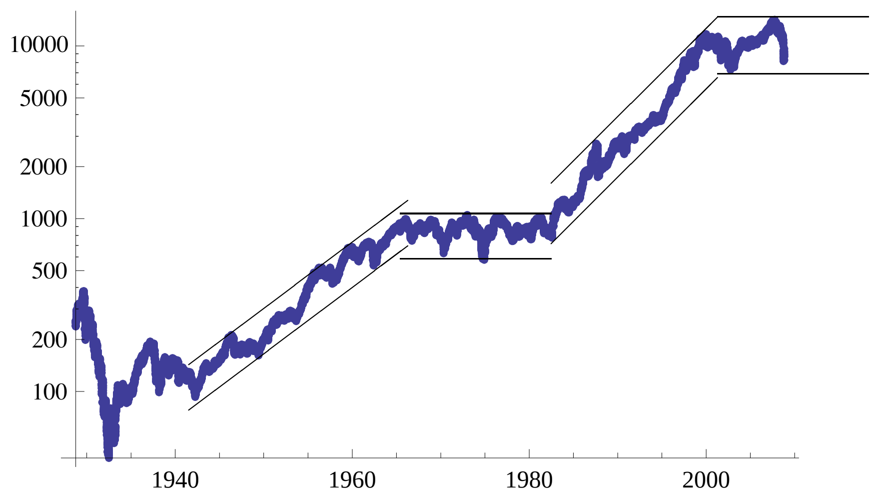

Figure 20.1 Logarithmic plot of Dow Jones industrial index over 80 years. There are four roughly linear regimes, two of exponential growth, two of stagnation [1].

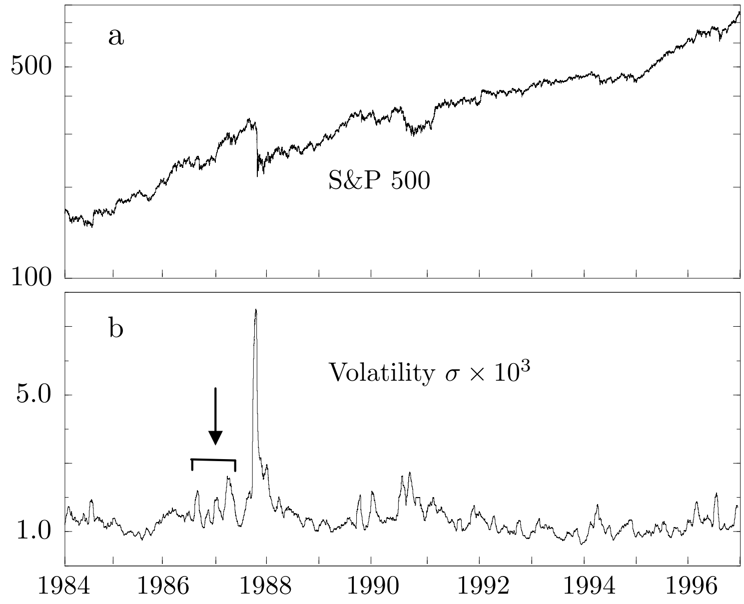

Figure 20.2 (a) Index S&P 500 for 13-year period Jan. 1, 1984 — Dec. 14, 1996, recorded every minute, and (b) volatility in time intervals 30 min (from Ref. [2]).

long times, the volatility is not constant but changes stochastically, as illustrated by the data of the S&P 500 index over the years 1984-1997, as shown in Fig. 20.2 [3]. In particular, there are strong increases shortly before a market crash.

The theory to be developed will at first ignore these fluctuations and assume a constant volatility. Attempts to include them have been made in the literature [3]–[79] and a promising version will be described in Section 20.4.

The volatilities follow approximately a Gamma distribution, as illustrated in Fig. 20.3.

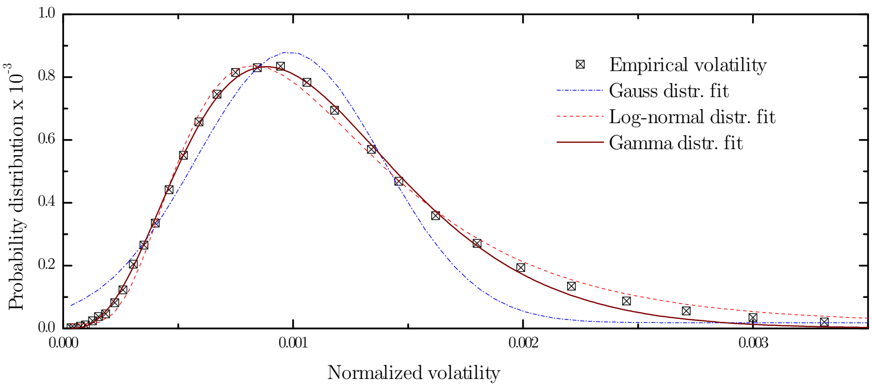

Figure 20.3 Comparison of best Gaussian, log-normal, and Gamma distribution fits to volatilities over 300 min (from Ref. [80]). The normalized log-normal distribution has the form $D^{\log-\mathrm{normal}}(z) = (2\pi\sigma^2 z^2)^{-1/2} e^{-(\log z-\mu)^2/2\sigma^2}$. The Gamma distribution will be discussed further in Subsection 20.1.5.

An individual stock will in general be more volatile than an averge market index, especially when the associated company is small and only few shares are traded per day.

20.1.1 Harmonic Approximation to Fluctuations

To lowest approximation, the stock price $S(t)$ satisfies a stochastic differential equation for exponential growth

$$\frac{\dot{S}(t)}{S(t)} = r_S + \eta(t), \tag{20.1}$$

where r_S is the growth rate, and $\eta(t)$ is a white noise variable defined by the correlation functions

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \sigma^2 \delta(t - t'). \tag{20.2}$$

The standard deviation σ is a precise measure for the volatility of the stock price. The squared volatility $v \equiv \sigma^2$ is called the *variance*.

The quantity $dS(t)/S(t)$ is called the *return* of the asset. From financial data, the return is usually extracted for finite time intervals Δt rather than the infinitesimal dt since prices $S(t)$ are listed for certain discrete times $t_n = t_0 + n\Delta t$. There are, for instance, abundant tables of daily closing prices of the market $S(t_n)$, from which one obtains the *daily returns* $\Delta S(t_n)/S(t_n) = [S(t_{n+1}) - S(t_n)]/S(t_n)$. The set of available $S(t_n)$ is called the *time series* of prices.

For a suitable choice of the time scales to be studied, the assumption of a white noise is fulfilled quite well by actual fluctuations of asset prices, as illustrated in Fig. 20.4.

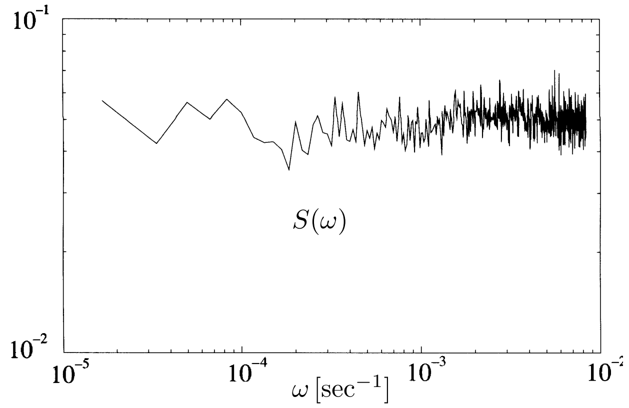

Figure 20.4 Fluctuation spectrum of exchange rate DM/US$ as function of frequency in units 1/sec, showing that the noise driving the stochastic differential equation (20.1) is approximately white (from [13]).

For the logarithm of the stock or asset price[1]

$$x(t) \equiv \log S(t) \tag{20.3}$$

this implies a stochastic differential equation for linear growth [14, 15, 16, 17]

$$\dot{x}(t) = \frac{\dot{S}}{S} - \frac{1}{2}\sigma^2 = r_x + \eta(t), \tag{20.4}$$

where

$$r_x \equiv r_S - \frac{1}{2}\sigma^2 \tag{20.5}$$

is the drift of the process [compare (18.405)]. A typical set of solutions of (20.4) is shown in Fig. 20.5.

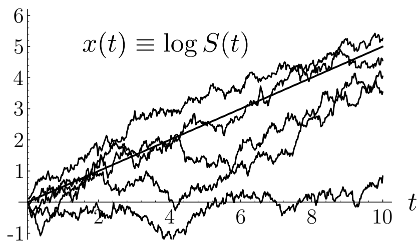

Figure 20.5 Behavior of logarithmic stock price following the stochastic differential equation (20.3).

The finite differences $\Delta x(t_n) = x(t_{n+1}) - x(t_n)$ and the corresponding differentials dx are called *log-returns*.

The extra term $\sigma^2/2$ in (20.5) is due to Itō's Lemma (18.413) for functions of a stochastic variable $x(t)$. Recall that the formal expansion in powers of dt:

$$\begin{aligned} dx(t) &= \frac{dx}{dS}dS(t) + \frac{1}{2}\frac{d^2x}{dS^2}dS^2(t) + \ldots \\ &= \frac{\dot{S}(t)}{S(t)}dt - \frac{1}{2}\left[\frac{\dot{S}(t)}{S(t)}\right]^2 dt^2 + \ldots \end{aligned} \tag{20.6}$$

may be treated in the same way as the expansion (18.426) using the mnemonic rule (18.429), according to which we may substitute $\dot{x}^2 dt \to \langle \dot{x}^2 \rangle dt = \sigma^2$, and thus

$$\left[\frac{\dot{S}(t)}{S(t)}\right]^2 dt \to \dot{x}^2(t)dt = \sigma^2. \tag{20.7}$$

The higher powers in dt do not contribute for Gaussian fluctuations since they carry higher powers of dt. For the same reason the constant rates r_S and r_x in $\dot{S}(t)/S(t)$ and $\dot{x}(t)$ do not show up in $[\dot{S}(t)/S(t)]^2\, dt = \dot{x}^2(t)dt$.

[1]To form the logarithm, the stock or asset price $S(t)$ is assumed to be dimensionless, i.e. the numeric value of the price in the relevant currency.

In charts of stock prices, relation (20.5) implies that if we fit a straight line through a plot of the logarithms of the prices with slope r_x, the stock price itself grows on the average like

$$\langle S(t)\rangle = S(0)\, e^{r s t} = S(0)\langle e^{r_x t+\int_0^t dt'\,\eta(t')}\rangle = S(0)\, e^{(r_x+\sigma^2/2)t}. \tag{20.8}$$

This result is, of course, a direct consequence of Eq. (18.425).

The description of the logarithms of the stock prices by Gaussian fluctuations around a linear trend is only a rough approximation to the real stock prices. The volatilities depend on time. If observed at small time intervals, for instance every minute or hour, they have distributions in which frequent events have an exponential distribution [see Subsection 20.1.6]. Rare events, on the other hand, have a much higher probability than in Gaussian distributions. The observed probability distributions possess *heavy tails* in comparison with the extremely light tails of Gaussian distributions. This was first noted by Pareto in the 19th century [18], reemphasized by Mandelbrot in the 1960s [19], and investigated recently by several authors [20, 22]. The theory needs therefore considerable refinement. As an intermediate generalization we shall introduce, beside the heavy power-like tails, also the so-called *semi-heavy tails*, which drop off faster than any power, such as $e^{-x^a}\, x^b$ with arbitrarily small $a > 0$ and any $b > 0$. We shall see later in Section 20.4 that semi-heavy tails of financial distribution may be viewed as a consequence of Gaussian fluctuations with fluctuating volatilities. Before we come to these we may fit the data phenomenologically with various non-Gaussian distributions and explore the consequences.

20.1.2 Lévy Distributions

Following Pareto and Mandelbrot we may attempt to approximately fit the distributions of the price changes $\Delta S_n = S(t_{n+1}) - S(t_n)$, the returns $\Delta S_n/S(t_n)$, and the log-returns $\Delta x_n = x(t_{n+1}) - x(t_n)$ for a certain time difference $\Delta t = t_{n+1} - t_n$ with the help of Lévy distributions [19, 22, 24, 23]. For brevity we shall, from now on, use the generic variable z to denote any of the above differences. The Lévy distributions are defined by the Fourier transform

$$\tilde{L}^{\lambda}_{\sigma^2}(z) \equiv \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{ipz}\, L^{\lambda}_{\sigma^2}(p), \tag{20.9}$$

with

$$L^{\lambda}_{\sigma^2}(p) \equiv \exp\left[-(\sigma^2 p^2)^{\lambda/2}/2\right]. \tag{20.10}$$

For an arbitrary distribution $\tilde{D}(z)$, we shall write the Fourier decomposition as

$$\tilde{D}(z) = \int \frac{dp}{2\pi}\, e^{ipz} D(p) \tag{20.11}$$

and the Fourier components $D(p)$ as an exponential

$$D(p) \equiv e^{-H(p)}, \tag{20.12}$$

where $H(p)$ plays a similar role as the Hamiltonian in quantum statistical path integrals. By analogy with this we shall also define $\tilde{H}(z)$ so that

$$\tilde{D}(z) = e^{-\tilde{H}(z)}. \tag{20.13}$$

An equivalent definition of the Hamiltonian is

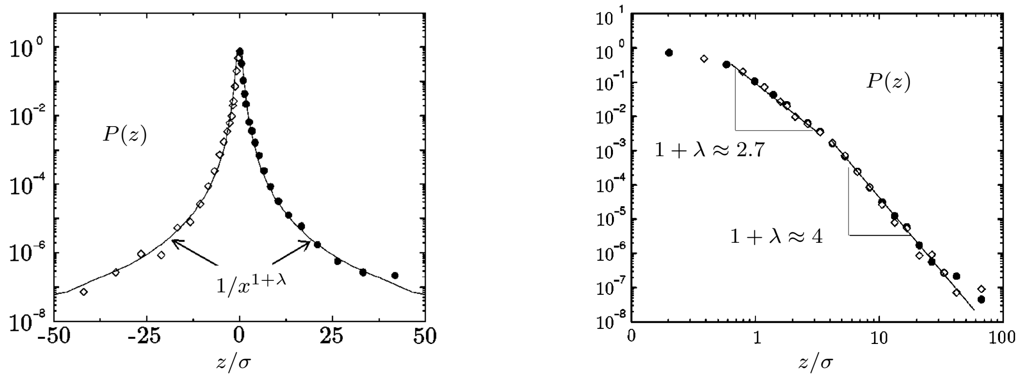

Figure 20.6 Left: Lévy tails of the S&P 500 index (1 minute log-returns) plotted against z/δ. Right: Double-logarithmic plot exhibiting power-like tail regions of the S&P 500 index (1 minute log-returns) (after Ref. [23])

$$e^{-H(p)} \equiv \langle e^{-ipz} \rangle. \tag{20.14}$$

For the Lévy distributions (20.9), the Hamiltonian is

$$H(p) = \frac{1}{2}(\sigma^2 p^2)^{\lambda/2}. \tag{20.15}$$

The Gaussian distribution is recovered in the limit $\lambda \to 2$ where the Hamiltonian simply becomes $\sigma^2 p^2/2$.

For large z, the Lévy distribution (20.9) falls off with the characteristic power-law

$$\tilde{L}^{\lambda}_{\sigma^2}(z) \to A^{\lambda}_{\sigma^2} \frac{\lambda}{|z|^{1+\lambda}}. \tag{20.16}$$

This power falloff is the heavy tail of the distribution discussed above. It is also called *power tail*, *Paretian tail*, or *Lévy tail*). The size of the tails is found by approximating the integral (20.9) for large z, where only small momenta contribute, as follows:

$$\tilde{L}^{\lambda}_{\sigma^2}(z) \approx \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{ipz} \left[1 - \frac{1}{2}(\sigma^2 p^2)^{\lambda/2}\right] \underset{z\to\infty}{\to} A^{\lambda}_{\sigma^2} \frac{\lambda}{|z|^{1+\lambda}}, \tag{20.17}$$

with

$$A^{\lambda}_{\sigma^2} = -\frac{\sigma^\lambda}{2\lambda}\int_0^\infty \frac{dp'}{\pi}\, p'^\lambda \cos p' = \frac{\sigma^\lambda}{2\pi\lambda}\sin(\pi\lambda/2)\,\Gamma(1+\lambda). \tag{20.18}$$

The stock market data are fitted best with λ between 1.2 and 1.5 [13], and we shall use $\lambda = 3/2$ most of the time, for simplicity, where one has

$$A^{3/2}_{\sigma^2} = \frac{1}{4}\frac{\sigma^{3/2}}{\sqrt{2\pi}}. \tag{20.19}$$

The full Taylor expansion of the Fourier transform (20.10) yields the asymptotic series

$$\tilde{L}^{\lambda}_{\sigma^2}(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\infty \frac{dp}{\pi}\frac{\sigma^{\lambda n}p^{\lambda n}}{2^n}\cos pz = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{\sigma^{\lambda n}}{2^n\pi}\Gamma(1+n\lambda)\frac{\sin\frac{\pi\lambda}{2}}{|z|^{1+\lambda}}. \tag{20.20}$$

This series is not useful for practical calculations since it diverges. In particular, it is unable to reproduce the pure Gaussian distribution in the limit $\lambda \to 2$.

There also exists an asymmetric Lévy distribution whose Hamiltonian is

$$H_{\lambda,\sigma,\beta}(p) = \frac{1}{2}|\sigma p|^\lambda\,[1 - i\beta\epsilon(p)F_{\lambda,\sigma}(p)], \tag{20.21}$$

where $\epsilon(p)$ is the step function (1.316), and

$$F_{\lambda,\sigma}(p) = \begin{cases} \tan(\pi\lambda/2) & \text{for } \lambda \neq 1, \\ -(1/\pi)\log p^2 & \text{for } \lambda = 1. \end{cases} \tag{20.22}$$

The large-$|z|$ behavior of this distribution is again given by (20.17), except that the prefactor (20.18) is multiplied by a factor $(1+\beta)$.

20.1.3 Truncated Lévy Distributions

Mathematically, an undesirable property of the Lévy distributions is that their fluctuation width diverges for $\lambda < 2$, since the second moment

$$\sigma^2 = \langle z^2\rangle \equiv \int_{-\infty}^{\infty} dz\, z^2\, \tilde{L}^{\lambda}_{\sigma^2}(z) = -\frac{d^2}{dp^2}L^{\lambda}_{\sigma^2}(p)\bigg|_{p=0} \tag{20.23}$$

is infinite. If one wants to describe data which show heavy tails for large log-returns but have finite widths one must make them fall off at least with semi-heavy tails at very large returns. Examples are the so-called *truncated Lévy distributions* [22]. They are defined by

$$\tilde{L}^{(\lambda,\alpha)}_{\sigma^2}(z) \equiv \int_{-\infty}^{\infty}\frac{dp}{2\pi}\, e^{ipz}\, L^{(\lambda,\alpha)}_{\sigma^2}(p) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\, e^{ipz - H(p)}\,, \tag{20.24}$$

with a Hamiltonian which generalizes the Lévy Hamiltonian (20.15) to

$$\begin{aligned} H(p) &\equiv \frac{\sigma^2}{2}\frac{\alpha^{2-\lambda}}{\lambda(1-\lambda)}\left[(\alpha+ip)^\lambda+(\alpha-ip)^\lambda-2\alpha^\lambda\right] \\ &= \sigma^2\frac{(\alpha^2+p^2)^{\lambda/2}\cos[\lambda\arctan(p/\alpha)]-\alpha^\lambda}{\alpha^{\lambda-2}\lambda(1-\lambda)}. \end{aligned} \tag{20.25}$$

The asymptotic behavior of the truncated Lévy distributions differs from the power behavior of the Lévy distribution in Eq. (20.17) by an exponential factor $e^{-\alpha z}$ which guarantees the finiteness of the width σ and of all higher moments. A rough estimate of the leading term is again obtained from the Fourier transform of the lowest expansion term of the exponential function $e^{-H(p)}$:

$$\begin{aligned} \tilde{L}^{(\lambda,\alpha)}_{\sigma^2}(z) &\approx e^{2s\alpha^\lambda}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,e^{ipz}\left\{1-s\left[(\alpha+ip)^\lambda+(\alpha-ip)^\lambda\right]\right\} \\ &\underset{z\to\infty}{\longrightarrow} e^{2s\alpha^\lambda}\Gamma(1+\lambda)\frac{\sin(\pi\lambda)}{\pi}s\frac{e^{-\alpha|z|}}{|z|^{1+\lambda}}, \end{aligned} \tag{20.26}$$

where

$$s\equiv\frac{\sigma^2}{2}\frac{\alpha^{2-\lambda}}{\lambda(1-\lambda)}. \tag{20.27}$$

The integral follows directly from the formulas [25]

$$\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,e^{ipz}\,(\alpha+ip)^\lambda=\frac{\Theta(z)}{\Gamma(-\lambda)}\frac{e^{-\alpha z}}{z^{1+\lambda}},\quad \int_{-\infty}^{\infty}\frac{dp}{2\pi}\,e^{ipz}\,(\alpha-ip)^\lambda=\frac{\Theta(-z)}{\Gamma(-\lambda)}\frac{e^{-\alpha|z|}}{|z|^{1+\lambda}}, \tag{20.28}$$

and the identity for Gamma functions[2]

$$\frac{1}{\Gamma(-z)}=-\Gamma(1+z)\sin(\pi z)/\pi. \tag{20.29}$$

The full expansion is integrated with the help of the formula [26]

$$\begin{aligned} &\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,e^{ipz}\,(\alpha+ip)^\lambda(\alpha-ip)^\nu \\ &= (2\alpha)^{\lambda/2+\nu/2}\frac{1}{|z|^{1+\lambda/2+\nu/2}}\left\{\begin{array}{ll} \dfrac{1}{\Gamma(-\lambda)}W_{(\nu-\lambda)/2,(1+\lambda+\nu)/2}(2\alpha z) & \\ & \text{for} \\ \dfrac{1}{\Gamma(-\nu)}W_{(\lambda-\nu)/2,(1+\lambda+\nu)/2}(2\alpha z) & \end{array}\right. \begin{array}{l} z>0, \\ \\ z<0, \end{array} \end{aligned} \tag{20.30}$$

where the Whittaker functions $W_{(\nu-\lambda)/2,(1+\lambda+\nu)/2}(2\alpha z)$ can be expressed in terms of Kummer's confluent hypergeometric function ${}_1F_1(a;b;x)$ of Eq. (9.45) as

$$\begin{aligned} W_{\delta,\kappa}(x) &= \frac{\Gamma(-2\kappa)}{\Gamma(1/2-\kappa-\delta)}x^{\kappa+1/2}e^{-x/2}{}_1F_1(1/2+\kappa-\delta;2\kappa+1;x) \\ &+ \frac{\Gamma(2\kappa)}{\Gamma(1/2+\kappa-\delta)}x^{-\kappa+1/2}e^{-x/2}{}_1F_1(1/2-\kappa-\delta;-2\kappa+1;x), \end{aligned} \tag{20.31}$$

[2] M. Abramowitz and I. Stegun, op. cit., Formula 6.1.17.

as can be seen from (9.39), (9.46) and Ref. [27]. For $\nu = 0$, only $z > 0$ gives a nonzero integral (20.30), which reduces, with $W_{-\lambda/2,1/2+\lambda/2}(z) = z^{-\lambda/2}e^{-z/2}$, to the left equation in (20.28). Setting $\lambda = \nu$ we find

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{ipz}\, (\alpha^2 + p^2)^\nu = (2\alpha)^{\nu/2} \frac{1}{|z|^{1+\nu}} \frac{1}{\Gamma(-\nu)} W_{0,1/2+\nu}(2\alpha|z|). \tag{20.32}$$

Inserting

$$W_{0,1/2+\nu}(x) = \sqrt{\frac{2z}{\pi}} K_{1/2+\nu}(x/2), \tag{20.33}$$

we may write

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{ipz}\, (\alpha^2 + p^2)^\nu = \left(\frac{2\alpha}{|z|}\right)^{1/2+\nu} \frac{1}{\sqrt{\pi}\Gamma(-\nu)} K_{1/2+\nu}(\alpha|z|). \tag{20.34}$$

For $\nu = -1$ where $K_{-1/2}(x) = K_{1/2}(x) = \sqrt{\pi/2x}e^{-x}$, this reduces to

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{ipz}\, \frac{1}{\alpha^2 + p^2} = \frac{1}{2\alpha} e^{-\alpha|z|}. \tag{20.35}$$

Summing up all terms in the expansion of the exponential function $e^{-H(p)}$:

$$\tilde{L}_{\sigma^2}^{(\lambda,\alpha)}(z) \;\approx\; e^{2s\alpha^\lambda} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-s)^n}{n!} \left[(\alpha + ip)^\lambda + (\alpha - ip)^\lambda \right]^n \right\} e^{ipz} \tag{20.36}$$

yields the true asymptotic behavior

$$\tilde{L}_{\sigma^2}^{(\lambda,\alpha)}(z) \;\underset{z\to\infty}{\to}\; e^{(2-2^\lambda)s\alpha^\lambda} \Gamma(1+\lambda) \frac{\sin(\pi\lambda)}{\pi} s \frac{e^{-\alpha|z|}}{|z|^{1+\lambda}}, \tag{20.37}$$

which differs from the estimate (20.26) by a constant factor (see Appendix 20A for details) [28]. Hence the tails are semi-heavy.

In contrast to Gaussian distributions which are characterized completely by their width σ, the truncated Lévy distributions contain three parameters σ, λ, and α. Best fits to two sets of fluctuating market prices are shown in Fig. 20.7. For the S&P 500 index we plot the cumulative distributions

$$P_<(z) = \int_{-\infty}^{z} dz'\, \tilde{L}_{\sigma^2}^{(\lambda,\alpha)}(z'), \qquad P_>(z) = \int_{z}^{\infty} dz'\, \tilde{L}_{\sigma^2}^{(\lambda,\alpha)}(z') = 1 - P_<(z), \tag{20.38}$$

for the price differences $z = \Delta S$ over $\Delta t = 15$ minutes. For the ratios of the changes of the currency rates DM/$ we plot the returns $z = \Delta S/S$ with the same Δt. The plot shows the negative and positive branches $P_<(-z)$, and $P_>(z)$ both plotted on the positive z axis. By definition:

$$\begin{aligned} &P_<(-\infty) = 0, \;\; P_<(0) = 1/2, \;\; P_<(\infty) = 1, \\ &P_>(-\infty) = 1, \;\; P_>(0) = 1/2, \;\; P_>(\infty) = 0. \end{aligned} \tag{20.39}$$

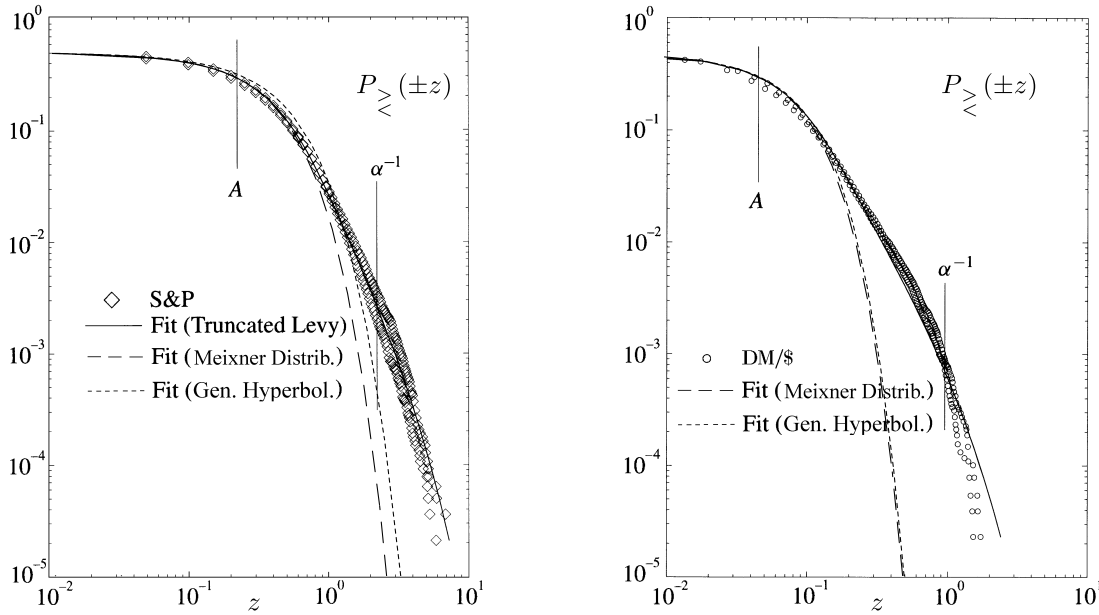

Figure 20.7 Best fit of cumulative versions (20.38) of truncated Lévy distribution to financial data. For the S&P 500 index, the fluctuating variable z is directly the index change ΔS every Δt=15 minutes (fit with $\sigma^2 = 0.280$ and $\kappa = 12.7$). For the DM/US$ exchange ratio, the variable z is equal to $100\Delta S/S$ every fifteen minutes (fit with $\sigma^2 = 0.0163$ and $\kappa = 20.5$). The negative fluctuations lie on a slightly higher curve than the positive ones. The difference is often neglected. The parameters A and α are the size and truncation parameters of the distribution. The best value of λ is 3/2 (from [13]). The dashed curves show the best fits of generalized hyperbolic functions (20.117) (l.h.s. $\lambda = 1.46$, $\alpha = 4.93$, $\beta = 0$, $\delta = 0.52$; r.h.s. $\lambda = 1.59$, $\alpha = 32.1$, $\beta = 0$, $\delta = 0.221$).

The fits are also compared with those by other distributions explained in the figure captions. A fit to most data sequences is possible with a rather universal parameter λ close to $\lambda = 3/2$. The remaining two parameters fix all expansion coefficients of Hamiltonian (20.25):

$$H(p) = \frac{1}{2} c_2 \, p^2 - \frac{1}{4!} c_4 \, p^4 + \frac{1}{6!} c_6 \, p^6 - \frac{1}{8!} c_8 \, p^8 + \dots . \tag{20.40}$$

The numbers $c_{2n} = -(-1)^n H^{(2n)}(0)$ are the *cumulants* of the truncated Lévy distribution [compare (3.587)], also denoted by $\langle z^n \rangle_c$. Here they are equal to

$$\begin{aligned}
\langle z^2 \rangle_c &= c_2 = \sigma^2, \\
\langle z^4 \rangle_c &= c_4 = \sigma^2 (2-\lambda)(3-\lambda)\alpha^{-2}, \\
\langle z^6 \rangle_c &= c_6 = \sigma^2 (2-\lambda)(3-\lambda)(4-\lambda)(5-\lambda)\alpha^{-4}, \\
&\vdots \\
\langle z^{2n} \rangle_c &= c_{2n} = \sigma^2 \frac{\Gamma(2n-\lambda)}{\Gamma(2-\lambda)} \alpha^{2-2n}.
\end{aligned} \tag{20.41}$$

The first cumulant c_2 determines the quadratic fluctuation width

$$\langle z^2 \rangle \equiv \int_{-\infty}^{\infty} dz\, z^2\, \tilde{L}_{\sigma^2}^{(\lambda,\alpha)}(z) = -\frac{d^2}{dp^2} e^{-H(p)}\bigg|_{p=0} = c_2 = \sigma^2, \tag{20.42}$$

the second the expectation of the fourth power of z

$$\langle z^4 \rangle \equiv \int_{-\infty}^{\infty} dz\, z^4\, \tilde{L}_{\sigma^2}^{(\lambda,\alpha)}(z) = \frac{d^4}{dp^4} e^{-H(p)}\bigg|_{p=0} = c_4 + 3c_2^2, \tag{20.43}$$

and so on:

$$\langle z^6 \rangle = c_6 + 15c_4c_2 + 15c_2^3, \quad \langle z^8 \rangle = c_8 + 28c_6c_2 + 35c_4^2 + 210c_4c_2^2 + 105c_2^4, \dots . \tag{20.44}$$

In a first analysis of the data, one usually determines the so-called *kurtosis*, which is the normalized fourth-order cumulant

$$\kappa \equiv \bar{c}_4 \equiv \frac{c_4}{c_2^2} = \frac{\langle z^4 \rangle_c}{\langle z^2 \rangle_c^2} = \frac{\langle z^4 \rangle_c}{\sigma^4}. \tag{20.45}$$

It depends on the parameters σ, λ, α as follows

$$\kappa = \frac{(2-\lambda)(3-\lambda)}{\sigma^2 \alpha^2}. \tag{20.46}$$

Given the volatility σ and the kurtosis κ, we extract the Lévy parameter α from the equation

$$\alpha = \frac{1}{\sigma} \sqrt{\frac{(2-\lambda)(3-\lambda)}{\kappa}}. \tag{20.47}$$

In terms of κ and λ, the normalized expansion coefficients are

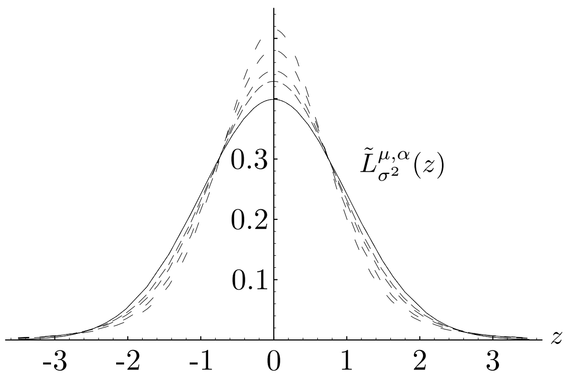

Figure 20.8 Change in shape of truncated Lévy distributions of width $\sigma = 1$ with increasing kurtoses $\kappa = 0$ (Gaussian, solid curve), 1, 2 ,5, 10.

$$\bar{c}_4 = \kappa, \quad \bar{c}_6 = \kappa^2 \frac{(5-\lambda)(4-\lambda)}{(3-\lambda)(2-\lambda)}, \quad \bar{c}_8 = \kappa^2 \frac{(7-\lambda)(6-\lambda)(5-\lambda)(4-\lambda)}{(3-\lambda)^2(2-\lambda)^2},$$
$$\vdots$$
$$\bar{c}_n = \kappa^{n/2-1} \frac{\Gamma(n-\lambda)/\Gamma(4-\lambda)}{(3-\lambda)^{n/2-2}(2-\lambda)^{n/2-2}}. \tag{20.48}$$

For $\lambda = 3/2$, the second equation in (20.47) becomes simply

$$\alpha = \frac{1}{2}\sqrt{\frac{3}{\sigma^2 \kappa}}, \tag{20.49}$$

and the coefficients (20.50):

$$\bar{c}_4 = \kappa, \quad \bar{c}_6 = \frac{5 \cdot 7}{3}\kappa^2, \quad \bar{c}_8 = 5 \cdot 7 \cdot 11\, \kappa^2,$$
$$\vdots$$
$$\bar{c}_n = \frac{\Gamma(n-3/2)/\Gamma(5/2)}{3^{n/2-2}/2^{n-4}} \kappa^{n/2-1}. \tag{20.50}$$

At zero kurtosis, the truncated Lévy distribution reduces to a Gaussian distribution of width σ. The change in shape for a fixed width and increasing kurtosis is shown in Fig. 20.8.

From the S&P and DM/US$ data with time intervals $\Delta t = 15\,\mathrm{min}$ one extracts $\sigma^2 = 0.280$ and 0.0163, and the kurtoses $\kappa = 12.7$ and 20.5, respectively. This implies $\alpha \approx 0.46$ and $\alpha \approx 1.50$, respectively. The other normalized cumulants $(\bar{c}_6, \bar{c}_8, \ldots)$ are then all determined to be $(1881.72, 788627.46, \ldots)$ and $(-4902.92, 3.3168 \times 10^6, \ldots)$, respectively. The cumulants increase rapidly showing that the expansion needs resummation.

The higher normalized cumulants are given by the following ratios of expectation values

$$\bar{c}_6 = \frac{\langle z^6 \rangle}{\langle z^2 \rangle^3} - 15\frac{\langle z^4 \rangle}{\langle z^2 \rangle^2} + 30,$$
$$\bar{c}_8 = \frac{\langle z^8 \rangle}{\langle z^2 \rangle^4} - 28\frac{\langle z^6 \rangle}{\langle z^2 \rangle^3} - 35\frac{\langle z^4 \rangle^2}{\langle z^2 \rangle^4} + 420\frac{\langle z^4 \rangle}{\langle z^2 \rangle^2} - 630, \ldots\ . \tag{20.51}$$

In praxis, the high-order cumulants cannot be extracted from the data since they are sensitive to the extremely rare events for which the statistics is too low to fit a distribution function.

20.1.4 Asymmetric Truncated Lévy Distributions

We have seen in the data of Fig. 20.7 that the price fluctuations have a slight asymmetry: Price drops are slightly larger than rises. This is accounted for by an asymmetric truncated Lévy distribution. It has the general form [24]

$$L_{\sigma^2}^{(\lambda,\alpha,\beta)}(p) \equiv e^{-H(p)}, \tag{20.52}$$

with a Hamiltonian function

$$
\begin{aligned}
H(p) &\equiv \frac{\sigma^2}{2}\frac{\alpha^{2-\lambda}}{\lambda(1-\lambda)}\left[(\alpha+ip)^\lambda(1+\beta)+(\alpha-ip)^\lambda(1-\beta)-2\alpha^\lambda\right] \\
&= \sigma^2\frac{(\alpha^2+p^2)^{\lambda/2}\left\{\cos[\lambda\arctan(p/\alpha)]+i\beta\sin[\lambda\arctan(p/\alpha)]\right\}-\alpha^\lambda}{\alpha^{\lambda-2}\lambda(1-\lambda)}. \quad (20.53)
\end{aligned}
$$

This has a power series expansion

$$
H(p)=ic_1p+\frac{1}{2}c_2\,p^2-i\frac{1}{3!}c_3p^3-\frac{1}{4!}c_4\,p^4+i\frac{1}{5!}c_5p^5+\dots\ . \quad (20.54)
$$

There are now even and odd cumulants $c_n=-i^nH^{(n)}(0)$ with the values

$$
c_n=\sigma^2\frac{\Gamma(n-\lambda)}{\Gamma(2-\lambda)}\alpha^{2-n}\begin{cases}1\\ \beta\end{cases}\ \text{for}\ \begin{matrix}n=\text{even},\\ n=\text{odd}.\end{matrix} \quad (20.55)
$$

The even cumulants are the same as before in (20.41). Similarly, the even expectation values (20.42)–(20.44) are extended by the odd expectation values:

$$
\begin{aligned}
\langle z\rangle &\equiv \int_{-\infty}^{\infty}dz\,z\,\tilde L_{\sigma^2}^{(\lambda,\alpha,\beta)}(z) = i\frac{d}{dp}e^{-H(p)}\Big|_{p=0} = c_1,\\
\langle z^2\rangle &\equiv \int_{-\infty}^{\infty}dz\,z^2\,\tilde L_{\sigma^2}^{(\lambda,\alpha,\beta)}(z) = -\frac{d^2}{d^2p}e^{-H(p)}\Big|_{p=0} = c_2+c_1^2,\\
\langle z^3\rangle &\equiv \int_{-\infty}^{\infty}dz\,z^3\,\tilde L_{\sigma^2}^{(\lambda,\alpha,\beta)}(z) = -i\frac{d^3}{d^3p}e^{-H(p)}\Big|_{p=0} = c_3+3c_2c_1+c_1^3,\\
\langle z^4\rangle &\equiv \int_{-\infty}^{\infty}dz\,z^4\,\tilde L_{\sigma^2}^{(\lambda,\alpha,\beta)}(z) = \frac{d^4}{d^4p}e^{-H(p)}\Big|_{p=0} = c_4+4c_3c_1+3c_2^2+6c_2c_1^2+c_1^4,\\
&\ \vdots \quad . \qquad (20.56)
\end{aligned}
$$

The inverse relations are

$$
\begin{aligned}
c_1 &= \langle z\rangle_c = \langle z\rangle,\\
c_2 &= \langle z^2\rangle_c = \langle z^2\rangle-\langle z\rangle^2 = \langle(z-\langle z\rangle_c)\rangle^2,\\
c_3 &= \langle z^3\rangle_c = \langle z^3\rangle-3\langle z\rangle\langle z^2\rangle+2\langle z\rangle^3 = \langle(z-\langle z\rangle_c)\rangle^3,\\
c_4 &= \langle z^4\rangle_c = \langle z^4\rangle-3\langle z^2\rangle^2-4\langle z\rangle\langle z^3\rangle+12\langle z\rangle^2\langle z^2\rangle-6\langle z\rangle^4\\
&= \langle(z-\langle z\rangle_c)\rangle^4-3\langle z^2-\langle z\rangle_c^2\rangle^2 = \langle(z-\langle z\rangle_c)\rangle^4-3c_2^2. \qquad (20.57)
\end{aligned}
$$

These are, of course, just simple versions of the cumulant expansions (3.585) and (3.587).

The distribution is now centered around a nonzero average value:

$$
\mu\equiv\langle z\rangle=c_1. \quad (20.58)
$$

The fluctuation width is given by

$$\sigma^2 \equiv \langle z^2 \rangle - \langle z \rangle^2 = \langle (z - \langle z \rangle)^2 \rangle = c_2. \tag{20.59}$$

For large z, the asymmetric truncated Lévy distributions exhibit semi-heavy tails, obtained by a straightforward modification of (20.28):

$$\tilde{L}_{\sigma^2}^{(\lambda,\alpha)}(z) \approx \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{ipz} \left\{ 1 - \frac{\sigma^2}{2} \frac{\alpha^{2-\lambda}}{\lambda(1-\lambda)} \left[(\alpha + ip)^\lambda (1+\beta) + (\alpha - ip)^\lambda (1-\beta) - 2\alpha^\lambda \right] \right\}$$
$$\underset{z\to\infty}{\to} \sigma^2\, e^{2s\alpha^\lambda} \Gamma(1+\lambda) \frac{\sin(\pi\lambda)}{\pi} s \frac{e^{-\alpha|z|}}{|z|^{1+\lambda}} [1 + \beta\, \mathrm{sgn}(z)]. \tag{20.60}$$

In analyzing the data, one uses the *skewness*

$$s \equiv \frac{\langle (z - \langle z \rangle)^3 \rangle}{\sigma^3} = \bar{c}_3 = \frac{c_3}{c_2^{3/2}}. \tag{20.61}$$

It depends on the parameters σ, λ, β, and α or κ as follows

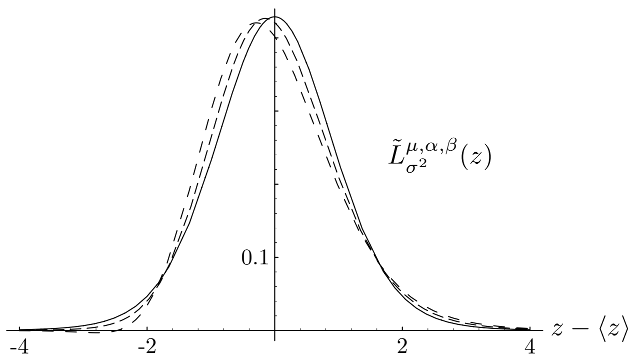

Figure 20.9 Change in shape of truncated Lévy distributions of width $\sigma = 1$ and kurtosis $\kappa = 1$ with increasing skewness $s = 0$ (solid curve), 0.4, 0.8. The curves are centered around $\langle z \rangle$.

$$s = \frac{(2-\lambda)\beta}{\sigma\alpha}. \tag{20.62}$$

The kurtosis can also be defined by [compare (20.45)][3]

$$\kappa \equiv \bar{c}_4 \equiv \frac{c_4}{c_2^2} = \frac{\langle z^4 \rangle_c}{\langle z^2 \rangle_c^2} = \frac{\langle z^4 \rangle_c}{\sigma^4} = \frac{\langle (z - \langle z \rangle)^4 \rangle}{\sigma^4} - 3. \tag{20.63}$$

From the data one extracts the three parameters volatility σ, skewness s, and kurtosis κ, which completely determine the asymmetric truncated Lévy distribution.

[3]Some authors call the ratio $\langle (z - \langle z \rangle) \rangle^4 / \sigma^4$ in (20.63) kurtosis and the quantity κ the *excess kurtosis*. Their kurtosis is equal to 3 for a Gaussian distribution, ours vanishes.

The data are then plotted against $z - \langle z \rangle = z - \mu$, so that they are centered at the average position. This centered distribution will be denoted by $\bar{L}_{\sigma^2}^{(\lambda,\alpha,\beta)}(z)$, i.e.

$$\bar{L}_{\sigma^2}^{(\lambda,\alpha,\beta)}(z) \equiv \tilde{L}_{\sigma^2}^{(\lambda,\alpha,\beta)}(z-\mu). \tag{20.64}$$

The Hamiltonian associated with this zero-average distribution is

$$\bar{H}(p) \equiv H(p) - H'(0)p, \tag{20.65}$$

and its expansion in power of the momenta starts out with p^2, i.e. the first term in (20.54) is subtracted.

In terms of σ, s, and κ, the normalized expansion coefficients are

$$\bar{c}_n = \kappa^{n/2-1} \frac{\Gamma(n-\lambda)/\Gamma(4-\lambda)}{(3-\lambda)^{n/2-2}(2-\lambda)^{n/2-2}} \begin{cases} 1 & \text{for} \quad n = \text{even}, \\ \sqrt{(3-\lambda)/(2-\lambda)\kappa}\, s & \phantom{\text{for}} \quad n = \text{odd}. \end{cases} \tag{20.66}$$

The change in shape of the distributions of a fixed width and kurtosis with increasing skewness is shown in Fig. 20.9. We have plotted the distributions centered around the average position $z = \langle c_1 \rangle$ which means that we have removed the linear term $ic_1 p$ from $H(p)$ in (20.52), (20.53), and (20.54). This subtracted Hamiltonian whose power series expansion begins with the term $c_2 p^2/2$ will be denoted by

$$\bar{H}(p) \equiv H(p) - H'(0)p = \frac{1}{2} c_2\, p^2 - i\frac{1}{3!} c_3 p^3 - \frac{1}{4!} c_4\, p^4 + i\frac{1}{5!} c_5 p^5 + \ldots\ . \tag{20.67}$$

20.1.5 Gamma Distribution

For a Hamiltonian

$$H_+(p) = \nu \log\left(1 - ip/\mu\right) \tag{20.68}$$

one obtains the normalized *Gamma distribution* of mathematical statistics:

$$\tilde{D}_{\mu,\nu}^{\text{Gamma}}(z) = \frac{1}{\Gamma(\nu)} \mu^\nu z^{\nu-1} e^{-\mu z}, \qquad \int_0^\infty dz\, \tilde{D}_{\mu,\nu}^{\text{Gamma}}(z) = 1, \tag{20.69}$$

which is restricted to positive variables z.

Expanding $H_+(p)$ in a power series $-\sum_{n=1}^\infty i^n \nu \mu^{-n} p^n/n = -\sum_{n=1}^\infty i^n c_n p^n/n!$, we identify the cumulants

$$c_n = (n-1)!\, \nu/\mu^n, \tag{20.70}$$

so that the lowest moments are

$$\bar{z} = \nu/\mu, \quad \sigma^2 = \nu/\mu^2, \quad s = 2/\sqrt{\nu}, \quad \kappa = 6/\nu. \tag{20.71}$$

The maximum of the distribution lies slightly below the average value $\bar{z}$ at

$$z_{\max} = (\nu - 1)/\mu = \bar{z} - 1/\mu. \tag{20.72}$$

The Gamma distribution was seen in Fig. 20.3 to yield an optimal fit to the fluctuating volatilities when is it is plotted as a distribution of the volatility $\sigma = \sqrt{z}$:

$$\tilde{D}^{\text{Chi}}_{\mu,\nu}(\sigma) \equiv 2\sigma \tilde{D}^{\text{Gamma}}_{\mu,\nu}(\sigma^2)\,, \qquad \int_0^\infty d\sigma\, \tilde{D}^{\text{Chi}}_{\mu,\nu}(\sigma) = 1. \tag{20.73}$$

As a normalized distribution of the volatility σ rather than the variance $v = \sigma^2$, one calls this function a *Chi distribution*.

In the limit $\nu \to \infty$ at fixed $\bar{z} = \nu/\mu$, the Gamma distribution becomes a Dirac δ-function:

$$\tilde{D}^{\text{Gamma}}_{\mu,\nu}(z) \to \delta\left(z - \frac{\nu}{\mu}\right)\,. \tag{20.74}$$

If we add to $H_+(p)$ the Hamiltonian

$$H_-(p) = \nu \log(1 + ip/\mu) \tag{20.75}$$

we obtain the distribution of negative z-values:

$$\tilde{D}^{\text{Gamma}}_{\mu,\nu}(z) = \frac{1}{\Gamma(\nu)} \mu^{-\nu} |z|^{\nu-1} e^{-\mu|z|}, \qquad z \leq 0. \tag{20.76}$$

By combining the two Hamiltonians with different parameters μ one can set up a skewed two-sided distribution.

20.1.6 Boltzmann Distribution

The highest-frequency returns ΔS of NASDAQ 100 and S&P 500 indices have a special property: they display a purely exponential behavior for positive as well as negative z, as long as the probability is rather large [32]. The data are fitted by the Boltzmann distribution

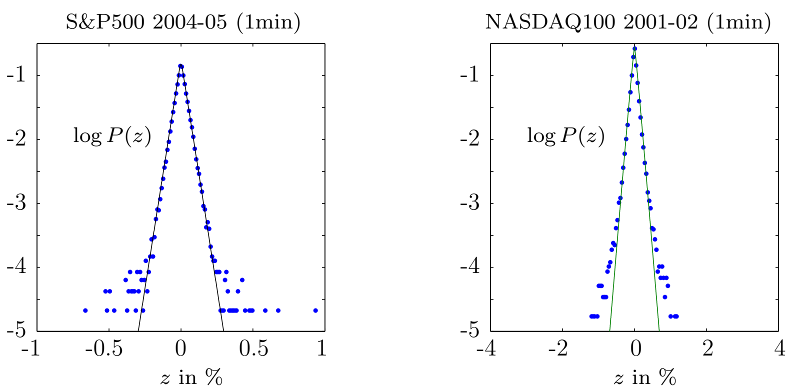

Figure 20.10 Boltzmann distribution of S&P 500 and NASDAQ 100 high-frequency log-returns recorded by the minute.

$$\tilde{B}(z) = \frac{1}{2T} e^{-|z|/T}. \tag{20.77}$$

We can see in Fig. 20.10, that only a very small set of rare events of large $|z|$ does not follow the Boltzmann law, but displays heavy tails. This allows us to assign a temperature to the stock markets [33]. The temperature depends on the volatility of the selected stocks and changes only very slowly with the general economic and political environment. Near a crash it reaches maximal values, as shown in Fig. 20.11.

It is interesting to observe the historic development of Dow Jones temperature over the last 78 years (1929-2006) in Fig. 20.12. Although the world went through a lot of turmoil and economic development in the 20th century, the temperature remained almost a constant except for short heat bursts. The hottest temperatures occurred in the 1930's, the time of the great depression. These temperatures were never reached again. An especially hot burst occurred during the crash year 1987. Thus, in the long run, the market temperature is not related to the value of the index but indicates the riskiness of the market. Only in bubbles, high temperatures go along with high stock prices. An extraordinary increase in market temperature before a crash may be useful to investors as a signal to go short.

The Fourier transform of the Boltzmann distribution is

$$B(p) = \int_{-\infty}^{\infty} dz\, e^{ipz} \frac{1}{2T} e^{-|z|/T} = \frac{1}{1+(Tp)^2} = e^{-H(p)}, \tag{20.78}$$

so that we identify the Hamiltonian as

$$H(p) = \log[1+(Tp)^2]. \tag{20.79}$$

This has only even cumulants:

$$c_{2n} = -(-1)^n H^{(2n)}(0) = 2(2n-1)!\, T^{2n}, \quad n = 1, 2, \dots\ . \tag{20.80}$$

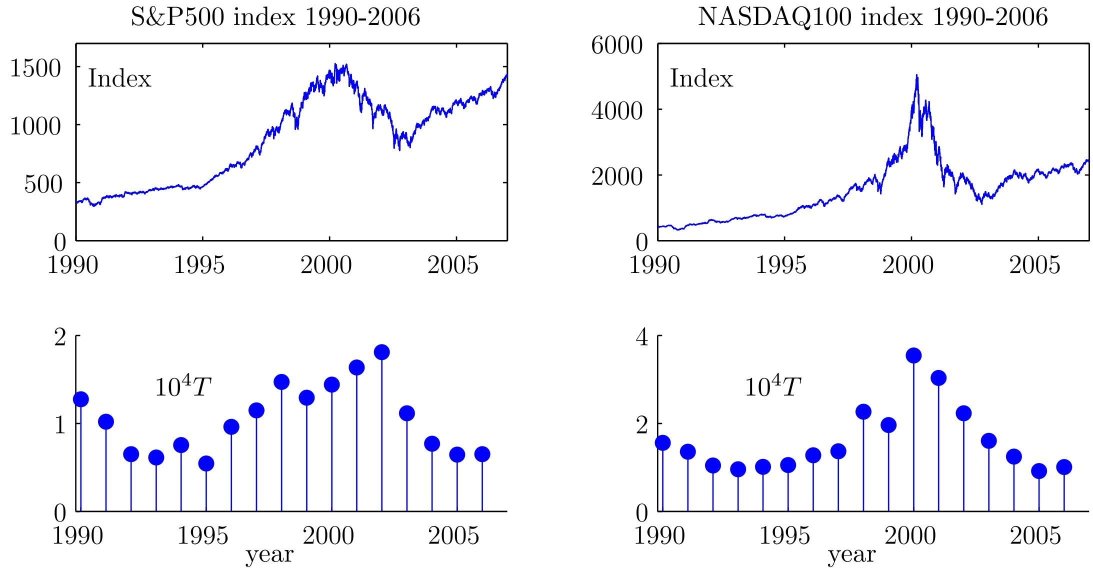

Figure 20.11 Market temperatures of S&P 500 and NASDAQ 100 indices from 1990 to 2006. The crash in the year 2000 occurred at the maximal temperatures $T_{\text{S\&P500}} \approx 2\times10^{-4}$ and $T_{\text{NASDAQ}} \approx 4 \times 10^{-4}$.

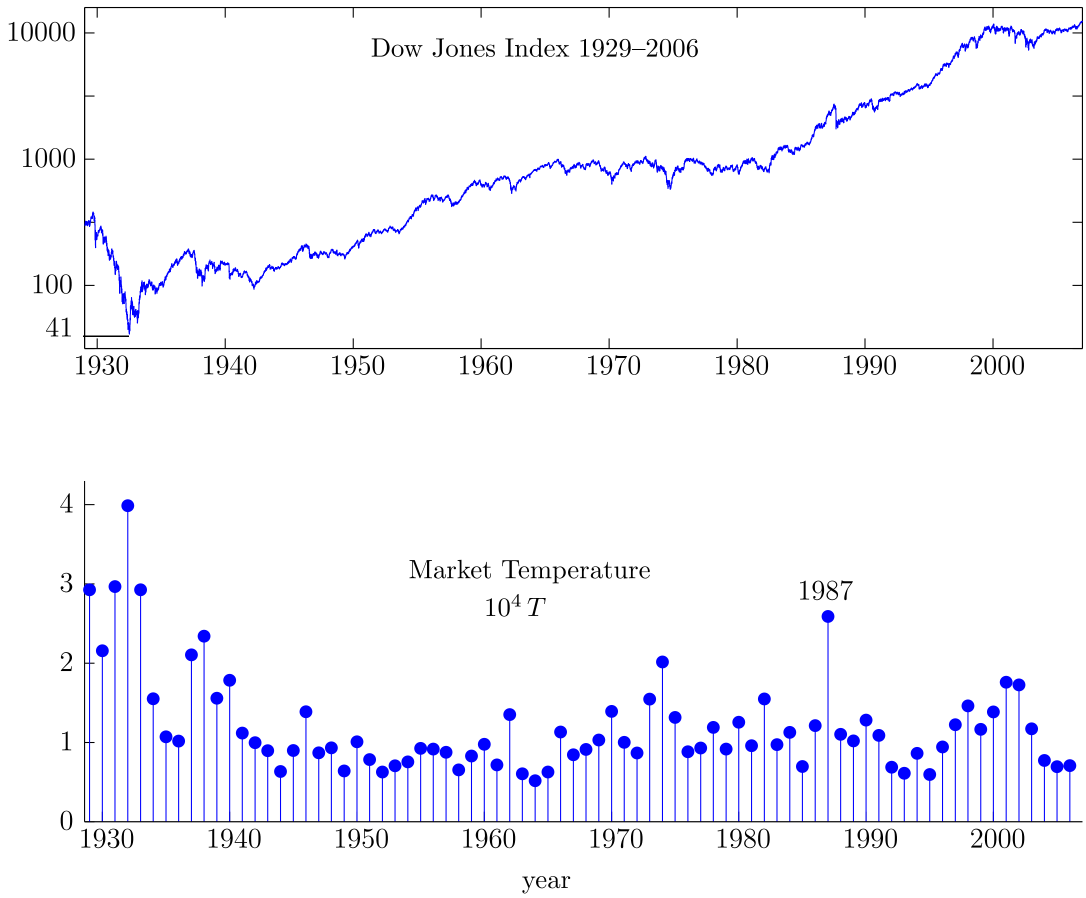

Figure 20.12 Dow Jones index over 78 years (1929-2006) and the annual market temperature, which is remarkably uniform, except in the 1930's, in the beginning of the great depression. An other high extreme temperature occurred in the crash year 1987.

The Boltzmann distribution is a special case of a two-sided Gamma distribution discussed in the previous subsection. It has semi-heavy tails.

We now observe that the Fourier transform can be rewritten as an integral [recall (2.499)]

$$B(p) = e^{-H(p)} = \frac{1}{1+(Tp)^2} = \int_0^\infty d\tau\, e^{-\tau(1+T^2p^2)}. \tag{20.81}$$

This implies that Boltzmann distribution can be obtained from a superposition of Gaussian distributions. Indeed, the prefactor of p^2 is equal to twice the variance $v=\sigma^2$ of the Gaussian distribution. Hence we change the variable of integration from τ to σ^2, substituting $\tau=\sigma^2/2T^2$, and obtain

$$B(p) = e^{-H(p)} = \frac{1}{2T^2}\int_0^\infty d\sigma^2\, e^{-\sigma^2/2T^2} e^{-\sigma^2p^2/2}. \tag{20.82}$$

The Fourier transform of this is the desired representation of the Boltzmann distribution as a volatility integral over Gaussian distributions of different widths:

$$\tilde{B}(z) = \frac{1}{2T^2}\int_0^\infty d\sigma^2\, e^{-\sigma^2/2T^2}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-z^2/2\sigma^2}. \tag{20.83}$$

The volatility distribution function is recognized to be a special case of the Gamma distribution (20.69) for $\mu = 1/2T^2$, $\nu = 1$.

20.1.7 Student or Tsallis Distribution

Recently, another non-Gaussian distribution has become quite popular through work of Tsallis [58, 59, 60]. It has been proposed as a good candidate for describing heavy tails a long time ago under the name Student distribution by Praetz [61] and by Blattberg and Gonedes [62]. This distribution can be written as

$$\tilde{D}_\delta(z) = N_\delta \frac{1}{\sqrt{2\pi\sigma_\delta^2}} e_\delta^{-z^2/2\sigma_\delta^2}, \tag{20.84}$$

where N_δ is a normalization factor

$$N_\delta = \frac{\sqrt{\delta}\Gamma(1/\delta)}{\Gamma(1/\delta - 1/2)}, \tag{20.85}$$

and $\sigma_K \equiv \sigma\sqrt{1-3\delta/2}$, where σ is the volatility of the distribution. The function $e_\delta(z)$ is an approximation of the exponential function called δ-exponential:

$$e_\delta^z \equiv (1-\delta z)^{-1/\delta}. \tag{20.86}$$

In the limit $\delta \to 0$, this reduces to the ordinary exponential function e^z.

Remarkably, the distribution of a fixed total amount of money W between N persons of equal earning talents follows such a distribution. The partition functions is given by

$$Z_N(W) = \left[\prod_{n=1}^N \int_0^W dw_n\right] \delta(w_1+\ldots+w_N - W). \tag{20.87}$$

After rewriting this as

$$Z_N(W) = \int_{-\infty}^\infty \frac{d\lambda}{2\pi}\left[\prod_{n=1}^N \int_0^W dw_n\right] e^{-i\lambda(w_1+\ldots+w_N)}e^{i\lambda W} = \int_{-\infty}^\infty \frac{d\lambda}{2\pi}\frac{(e^{-i\lambda W}-1)^N e^{i\lambda W}}{(-i\lambda+\epsilon)^N}, \tag{20.88}$$

where ϵ is an infinitesimal positive number, and a binomial expansion of $(e^{-i\lambda W} - 1)^N e^{i\lambda W}$, we can perform the integral over λ, using the formula[4]

$$\int_{-\infty}^\infty \frac{d\lambda}{2\pi}\frac{1}{(-i\lambda+\epsilon)^\nu}e^{-ipx} = \frac{p^{\nu-1}}{\Gamma(\nu)}e^{-\epsilon p}\,\Theta(p), \tag{20.89}$$

[4]See I.S. Gradshteyn and I.M. Ryzhik, *op. cit.*, Formula 3.382.7.

where $\Theta(z)$ is the Heaviside function (1.304), we obtain

$$Z_N(W) = \frac{W^{N-1}}{\Gamma(N)} \sum_{k=0}^{N-1} (-1)^k \binom{N}{N-k} (N-k-1)^{N-1} \tag{20.90}$$

The sum over binomial coefficients adds up to unity, due to a well-known identity for binomial coefficients[5], so that

$$Z_N(W) = \frac{W^{N-1}}{\Gamma(N)}. \tag{20.91}$$

If we replace W by $W - w_n$, this partition function gives us the unnormalized probability that an individual owns the part w_n of total wealth. The normalized probability is then

$$P_N(w_n) = Z_N^{-1} \frac{(W - w_n)^{N-2}}{\Gamma(N-1)} = \frac{N-1}{W}\left(1 - \frac{w_n}{E}\right)^{N-2}. \tag{20.92}$$

Defining $w \equiv W/(N-2)$, which for large N is the average wealth per person, $P_N(w_n)$ can be expressed in terms of the δ-exponential (20.86) as

$$P_N(w_n) = \frac{N-1}{W}\left[1 - \frac{w_n}{(N-2)w}\right]^{N-2} = \frac{N-1}{W} e_{-1/(N-2)}^{-w_n/w}. \tag{20.93}$$

In the limit of infinitely many individuals, this turns into the normalized Boltzmann distribution $w^{-1}e^{-w_n/w}$.

The *Student-Tsallis distribution* (20.84) is simply a normalized δ-exponential. Instead of the parameter δ one often uses the Tsallis parameter $q = \delta + 1$. A plot of these functions for different δ-values is shown on the left hand of Fig. 20.13. From Eq. (20.86) we see that the Student-Tsallis distribution with $\delta > 0$ has heavy tails with a power behavior $1/z^{1/\delta} = 1/z^{1/(q-1)}$. A fit of the log-returns of 10 NYSE top-volume stocks is shown in Fig. 20.13.

Remarkably, the δ-exponential in (20.84) can be written as a superposition of Gaussian distributions. With the help of the integral formula (2.499) we find

$$e_\delta^{-z^2/2\sigma_\delta^2} = \frac{1}{\Gamma(1/\delta)} \int_0^\infty \frac{ds}{s}\, s^{1/\delta} e^{-s} e^{-s\delta\, z^2/2\sigma_\delta^2}. \tag{20.94}$$

After a change of variables this can be rewritten as

$$e_\delta^{-z^2/2\sigma_\delta^2} = \frac{\mu^{1/\delta}}{\Gamma(1/\delta)} \int_0^\infty \frac{\mathrm{d}v}{v}\, v^{1/\delta} e^{-\mu v} e^{-vz^2/2}, \qquad \mu = \sigma_\delta^2/\delta. \tag{20.95}$$

This is a superposition of Gaussian functions $e^{-vz^2/2}$ whose *inverse variances* $v = 1/\sigma^2$ occur with a weight that is precisely the Gamma distribution of the variable v with $\nu = 1/\delta$:

$$\tilde{D}_{\mu,1/\delta}^{\rm Gamma}(v) = \frac{1}{\Gamma(1/\delta)} \mu^{1/\delta} v^\nu e^{-\mu v}. \tag{20.96}$$

[5] See I.S. Gradshteyn and I.M. Ryzhik, *op. cit.*, Formula 0.154.6.

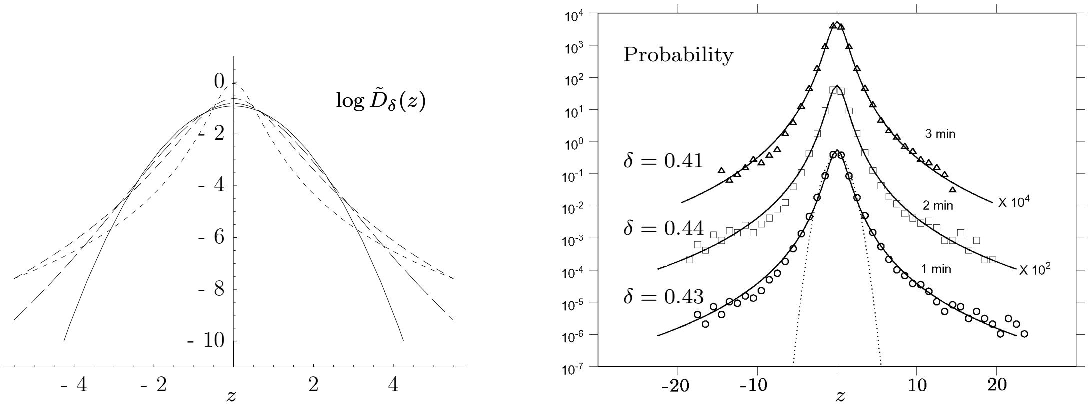

Figure 20.13 Left: Logarithmic plot of the normalized δ-exponential (20.84) (Student-Tsallis distribution) for $\delta = 0$ (Gaussian), 0.2, 0.4, 0.6, all for $\sigma = 1$. Right: Fit of the log-returns of 20 NYSE top-volume stocks over short time scales from 1 to 3 minutes by this distribution (after Ref. [60]). Dotted line is Gaussian distribution.

20.1.8 Tsallis Distribution in Momentum Space

When fitting the volatility distributions of the S&P 500 index in Fig. 20.3, we observed that the best fit was obtained by a Gamma distribution. Inspired by the discussion in the last section we observe that the δ-exponential (20.86) in momentum space

$$e^{-H_{\delta,\beta}(p)} \equiv e_\delta^{-\beta p^2/2} = \left(1 + \beta\delta\, p^2/2\right)^{-1/\delta}, \tag{20.97}$$

has precisely such a decomposition. We simply rewrite it by analogy with (20.94) as

$$e_\delta^{-\beta p^2/2} = \frac{1}{\Gamma(1/\delta)} \int_0^\infty \frac{ds}{s}\, s^{1/\delta} e^{-s} e^{-s\beta\delta\, p^2/2}, \qquad \beta \equiv \nu/\mu = 1/\mu\delta, \tag{20.98}$$

and change the variable of integration to obtain

$$e_\delta^{-\beta p^2/2} = \frac{\mu^\nu}{\Gamma(\nu)} \int_0^\infty \frac{\mathrm{d}v}{v}\, v^\nu e^{-\mu v} e^{-vp^2/2}, \qquad \nu = 1/\delta,\ \mu = \nu/\beta = 1/\beta\delta. \tag{20.99}$$

This is a superposition of Gaussian distributions whose variances $v = \sigma^2$ are weighted by the Gamma distribution (20.69):

$$e_\delta^{-\beta p^2/2} = \int_0^\infty dv\, D_{\mu,\nu}^{\mathrm{Gamma}}(v) e^{-vp^2/2}, \qquad \nu = 1/\delta,\ \mu = \nu/\beta = 1/\beta\delta. \tag{20.100}$$

The average of the Gamma distribution is $\bar{v} = \nu/\mu = \beta$ [recall (20.71)], so that the left-hand side can also be written as $e_\delta^{-\bar{v}p^2/2}$.

The lowest cumulants in the Hamiltonian $H_{\delta,\beta}(p) = \frac{1}{2!}c_2\, p^2 - \frac{1}{4!}c_4\, p^4 + \dots$ are

$$c_2 = \frac{\nu}{\mu},\ c_4 = 3\frac{\nu}{\mu^2},\ c_6 = \frac{5!!}{\mu^3}\left(2+3\nu-2\nu^2-\nu^3\right),\ c_8 = \frac{7!!}{\mu^3}\left(2+\nu-3\nu^2+\nu^3+nu^4\right). \tag{20.101}$$

For $\mu = 1/2T^2$ and $\nu = 1$, these reduce to those of the Boltzmann distribution in Eq. (20.80).

The superposition (20.99) can immediately be Fourier transformed to

$$\tilde{D}_{\delta,\beta}(z) = \frac{\mu^{1/\delta}}{\Gamma(1/\delta)} \int_0^\infty \frac{\mathrm{d}v}{v}\, v^{1/\delta} e^{-\mu v} \frac{1}{\sqrt{2\pi v}} e^{-z^2/2v}\,, \qquad \mu = 1/\beta\delta. \tag{20.102}$$

which is a superposition of Gaussians whose variances $v = \sigma^2$ follow a Gamma distribution (20.69) centered around $\bar{v} = 1/\delta\mu$ of width $\overline{(v-\bar{v})^2} = 1/\delta\mu^2$. Hence δ is given by the ratio $\delta = \overline{(v-\bar{v})^2}/\bar{v}^2$. Recalling Eq. (20.71) we see that δ determines the kurtosis of the distribution to be $\kappa = 6\delta$.

The integral over v in (20.102) can be done using Formula (2.559), yielding

$$\tilde{D}_{\delta,\beta}(z) = \frac{\sqrt{\mu}}{\Gamma(1/\delta)} \frac{1}{\sqrt{2\pi}} \left(\frac{\mu z^2}{2}\right)^{1/2\delta-1/4} 2K_{1/\delta-1/2}(\sqrt{2\mu}z), \qquad \mu = 1/\beta\delta. \tag{20.103}$$

For $\delta = 1$ and $\mu = 1/2T^2$, we use (1.349) and recover the Boltzmann distribution (20.83)

The small-z behavior of $K_\nu(z)$ is $(1/2)\Gamma(\nu)\,(z/2)^{-\nu}$ for $\mathrm{Re}\,\nu > 0$ [recall Eq. (1.351)]. If we assume $\delta < 2$, we obtain from (20.103) the value of the distribution function at the origin:

$$\tilde{D}_{\delta,\beta}(0) = \frac{\sqrt{\mu}\,\Gamma(1/\delta - 1/2)}{\Gamma(1/\delta)}. \tag{20.104}$$

The same result can, of course, be found directly from Eq. (20.102). This value diverges at $\delta = 2/(1-2n)$ ($n = 0, 1, 2, \ldots$).

20.1.9 Relativistic Particle Boltzmann Distribution

A physically important distribution in momentum space is the Boltzmann distribution $e^{-\beta E(p)}$ of relativistic particle energies $E(p) = \sqrt{p^2 + M^2}$, where β is the inverse temperature $1/k_BT$. The exponential can be expressed as a superposition of Gaussian distributions

$$e^{-\beta\sqrt{p^2+M^2}} = \int_0^\infty dv\, \omega_\beta(v) e^{-\beta v(p^2+M^2)/2} \tag{20.105}$$

with a weight function

$$\omega_\beta(v) \equiv \sqrt{\frac{\beta}{2\pi v^3}} e^{-\beta/2v}. \tag{20.106}$$

This is a special case of a *Weibull distribution*

$$\tilde{D}_{\mathrm{W}}(x) = \frac{b}{\Gamma(a/b)} x^{a-1} e^{-x^b}, \tag{20.107}$$

which has the simple moments

$$\langle x^n \rangle = \Gamma((a+n)/b)/\Gamma(a/b). \tag{20.108}$$

20.1.10 Meixner Distributions

Quite reasonable fits to financial data are provided by the *Meixner distributions* [34, 35] which read in configuration and momentum space:

$$\tilde{M}(z) = \frac{[2\cos(b/2)]^{2d}}{2a\pi\Gamma(2d)}|\Gamma(d+iz/a)|^2 \exp[bz/a], \tag{20.109}$$

$$M(p) = \left\{\frac{\cos(b/2)}{\cosh[(ap-ib)/2]}\right\}^{2d}. \tag{20.110}$$

They have the same semi-heavy tail behavior as the truncated Lévy distributions

$$\tilde{M}(z) \to C_\pm |z|^\rho e^{-\sigma_\pm |z|} \quad \text{for} \quad z \to \pm\infty, \tag{20.111}$$

with

$$C_\pm = \frac{[2\cos(b/2)]^{2d}}{2a\pi\Gamma(2d)}\frac{2\pi}{a^{2d-1}}e^{\pm 2\pi d\tan(b/2)},\ \rho = 2d-1,\ \sigma_\pm \equiv (\pi \pm b)/a. \tag{20.112}$$

The moments are

$$\mu = ad\tan(b/2), \quad \sigma^2 = a^2 d/2\cos^2(b/2), \quad s = \sqrt{2}\sin(b/2)/\sqrt{d}, \quad \kappa = [2-\cos b]/d, \tag{20.113}$$

such that we can calculate the parameters from the moments as follows:

$$a^2 = \sigma^2\left(2\kappa - 3s^2\right), \quad d = \frac{1}{\kappa - s^2}, \quad b = 2\arcsin\left(s\sqrt{d/2}\right). \tag{20.114}$$

As an example for the parameters of the distribution, a good fit of the daily Nikkei-225 index is possible with

$$a = 0.029828,\ b = 0.12716,\ d = 0.57295,\ \langle z\rangle = -0.0011243. \tag{20.115}$$

The curve has to be shifted in z by Δz to make $\Delta z + \mu$ equal to $\langle z\rangle$. Such a Meixner distribution has been used for option pricing in Ref. [35].

The Meixner distributions can be fitted quite well to the truncated Lévy distribution in the regime of large probability. In doing so we observe that the variance σ^2 and the kurtosis κ are not the best parameters to match the two distributions. The large-probability regime of the distributions can be matched perfectly by choosing, in the symmetric case, the value and the curvature at the origin to be the same in both curves. This is seen in Fig. 20.14.

In the asymmetric case we have to match also the first and third derivatives. The derivatives of the Meixner distribution are:

$$\begin{aligned}
\tilde{M}(0) &= \frac{2^{2d-1}\Gamma^2(d)}{\pi a\Gamma(2d)},\\
\tilde{M}'(0) &= b\frac{2^{2d-1}\Gamma^2(d)\,[1-d\psi(d)]}{\pi a^2\Gamma(2d)},\\
\tilde{M}''(0) &= -\frac{2^{2d}\Gamma^2(d)\psi(d)}{\pi a^3\Gamma(2d)},\\
\tilde{M}^{(3)}(0) &= -\frac{b}{2}\frac{2^{2d}\Gamma^2(d)\left[6\psi(d)-6d\psi^2(d)-d\psi^{(3)}(d)\right]}{\pi a^4\Gamma(2d)},\\
\tilde{M}^{(4)}(0) &= \frac{2^{2d}\Gamma^2(d)\left[6\psi^2(d)+\psi^{(3)}(d)\right]}{\pi a^5\Gamma(2d)},
\end{aligned} \tag{20.116}$$

where $\psi^{(n)}(z) \equiv d^{n+1}\log\Gamma(z)/dz^{n+1}$ are the Polygamma functions.

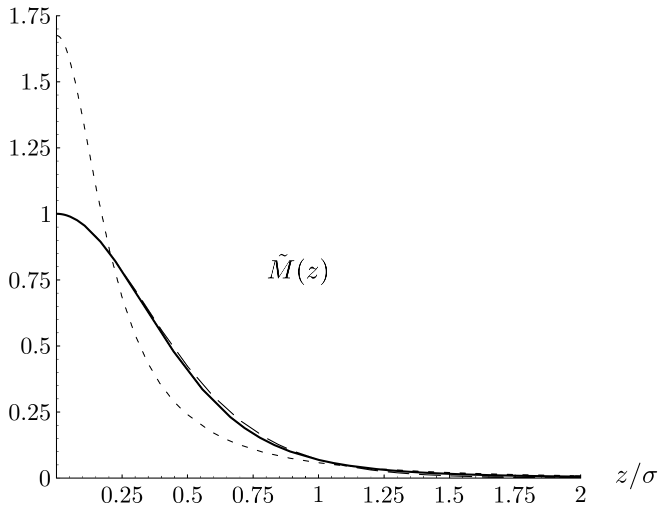

Figure 20.14 Comparison of best fit of Meixner distribution to truncated Lévy distributions. One of them (short dashed) has the same volatility σ and kurtosis κ. The other (long-dashed) has the same size and curvature at the origin. The parameters are $\sigma^2 = 0.280$ and $\kappa = 12.7$ as in the left-hand cumulative distribution in Fig. 20.7. The Meixner distribution with the same σ^2 and κ has parameters $a = 2.666$, $d = 0.079$, $b = 0$, the distribution with the same value and curvature at the origin has $a = 0.6145$, $d = 1.059$, $b = 0$. The very large σ-regime, however, is not fitted well as can be seen in the cumulative distributions which reach out to z of the order of 10σ in Fig. 20.7.

20.1.11 Generalized Hyperbolic Distributions

Another non-Gaussian distributions proposed in the literature is the so-called *generalized hyperbolic distributions*..[6] As the truncated Lévy and Meixner distributions, these have simple analytic forms both in z- and p-space:

$$\tilde{H}_G(z) = \frac{\left(\gamma^2 - \beta^2\right)^{\lambda/2} e^{\beta z}}{\gamma^{\lambda-1/2}\delta^\lambda \sqrt{2\pi}} \left[\delta^2 + z^2\right]^{\lambda/2-1/4} \frac{K_{\lambda-1/2}\left(\gamma\sqrt{\delta^2 + z^2}\right)}{K_\lambda\left(\delta\sqrt{\gamma^2 - \beta^2}\right)} \tag{20.117}$$

and

$$G(p) = \frac{\left(\delta\sqrt{\gamma^2 - \beta^2}\right)^\lambda}{K_\lambda\left(\delta\sqrt{\gamma^2 - \beta^2}\right)} \frac{K_\lambda\left(\delta\sqrt{\gamma^2 - (\beta + ip)^2}\right)}{\left[\delta\sqrt{\gamma^2 - (\beta + ip)^2}\right]^\lambda}, \tag{20.118}$$

the latter defining another Hamiltonian

$$H_G(p) \equiv -\log G(p). \tag{20.119}$$

But in contrast to the former, this set of functions is not closed under time evolution. The distributions at a later time t are obtained from the Fourier transforms $e^{-H(p)t}$. For truncated Lévy distributions and the Meixner distributions the factor t can simply be absorbed in the parameters of the functions ($\sigma^2 \to t\sigma^2$ in the first and $d \to td$ in the second case). For the generalized

[6]See the publications [36]–[74].

hyperbolic distributions, this is no longer true since $e^{-H_G(p)t} = [G(p)]^t$ involves higher powers of Bessel functions whose analytic Fourier transform cannot be found. In order to describe a complete temporal evolution we must therefore close the set of functions by adding all Fourier transforms of $e^{-H_G(p)t}$. In praxis, this is not a serious problem — it merely leads to a slowdown of numeric calculations which always involve a numeric Fourier transformation.

The asymptotic behavior of the generalized hyperbolic distributions has semi-heavy tails. From the large-z behavior of the Bessel function $K_\nu(z) \to \sqrt{\pi/2z}e^{-z}$ we find

$$\tilde{H}_G(z) \to \sqrt{\frac{\pi}{2\gamma}} \frac{\left(\gamma^2 - \beta^2\right)^{\lambda/2} e^{\beta z}}{\gamma^{\lambda-1/2}\delta^\lambda \sqrt{2\pi}} \frac{1}{K_\lambda\left(\delta\sqrt{\gamma^2-\beta^2}\right)} z^{\lambda-1} e^{-\gamma z}. \tag{20.120}$$

Introducing the variable $\zeta \equiv \delta\sqrt{\gamma^2-\beta^2}$, this can again be expanded in powers of p as in Eq. (20.54), yielding the first two cumulants:

$$c_1 = \beta\frac{\delta^2}{\zeta}\frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)}; \tag{20.121}$$

$$c_2 = \frac{\delta^2}{\zeta}\frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)} + \frac{\beta^2\delta^4}{\zeta^2}\left\{\frac{K_{2+\lambda}(\zeta)}{K_\lambda(\zeta)} - \left[\frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)}\right]^2\right\}. \tag{20.122}$$

Using the identity [50]

$$K_{\nu+1}(z) - K_{\nu-1}(z) = \frac{2\nu}{z}K_\nu(z), \tag{20.123}$$

the latter equation can be expressed entirely in terms of

$$\rho = \rho(\zeta) = \frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)} \tag{20.124}$$

as

$$c_2 = \frac{\delta^2}{\zeta}\rho + \frac{\beta^2\delta^4}{\zeta^3}\left[\zeta + 2\,(1+\lambda)\,\rho - \zeta\rho^2\right]. \tag{20.125}$$

Usually, the asymmetry of the distribution is small implying that c_1 is small, implying a small β. It is then useful to introduce the symmetric variance

$$\sigma_s^2 \equiv \delta^2\rho/\zeta, \tag{20.126}$$

and write

$$c_1 = \beta\sigma_s, \quad c_2 = \sigma^2 = \sigma_s^2 + \beta^2\left[\frac{\delta^4}{\zeta^2} + 2\,(1+\lambda)\,\frac{\delta^2}{\zeta^2}\sigma_s - \sigma_s^2\right]. \tag{20.127}$$

The cumulants c_3 and c_4 are most compactly written as

$$\begin{aligned} c_3 = \beta&\left[\frac{3\delta^4}{\zeta^2} + 6\,(1+\lambda)\,\frac{\delta^2}{\zeta^2}\sigma_s^2 - 3\sigma_s^4\right] \\ &+ \beta^3\left\{2\,(2+\lambda)\,\frac{\delta^6}{\zeta^4} + \left[4\,(1+\lambda)\,(2+\lambda) - 2\zeta^2\right]\frac{\delta^4}{\zeta^4}\sigma_s^2 - 6\,(1+\lambda)\,\frac{\delta^2}{\zeta^2}\,\sigma_s^4 + 2\sigma_s^6\right\} \end{aligned} \tag{20.128}$$

and

$$\begin{aligned} c_4 = \kappa\sigma^4 = &\frac{3\delta^4}{\zeta^2} + \frac{6\delta^2}{\zeta^2}\,(1+\lambda)\,\sigma_s^2 - 3\sigma_s^4 \\ &+ 6\beta^2\left\{2\,(2+\lambda)\,\frac{\delta^6}{\zeta^4} + \left[4\,(1+\lambda)\,(2+\lambda) - 2\zeta^2\right]\frac{\delta^4}{\zeta^4}\sigma_s^2 - 6\,(1+\lambda)\,\frac{\delta^2}{\zeta^2}\,\sigma_s^4 + 2\sigma_s^6\right\} \\ &+ \beta^4\left\{\left[4\,(2+\lambda)\,(3+\lambda) - \zeta^2\right]\frac{\delta^8}{\zeta^6} + \left[4\,(1+\lambda)\,(2+\lambda)\,(3+\lambda) - 2\,(5+4\lambda)\,\zeta^2\right]\frac{\delta^6}{\zeta^6}\sigma_s^2\right. \\ &\left. - 2\left[(1+\lambda)\,(11+7\lambda) - 2\zeta^2\right]\frac{\delta^4}{\zeta^4}\sigma_s^4 + 12\,(1+\lambda)\,\frac{\delta^2}{\zeta^2}\sigma_s^6 - 3\sigma_s^8\right\}. \end{aligned} \tag{20.129}$$

The first term in c_4 is equal to σ_s^4 times the kurtosis of the symmetric distribution

$$\kappa_s \equiv \frac{3\delta^4}{\zeta^2\sigma_s^4} + \frac{6\delta^2}{\zeta^2\sigma_s^2}(1+\lambda) - 3. \tag{20.130}$$

Inserting here σ_s^2 from (20.126), we find

$$\kappa_s \equiv \frac{3}{r^2(\zeta)} + (1+\lambda)\frac{6}{\zeta\, r(\zeta)} - 3. \tag{20.131}$$

Since all Bessel functions $K_\nu(z)$ have the same large-z behavior $K_\nu(z) \to \sqrt{\pi/2z}e^{-z}$ and the small-z behavior $K_\nu(z) \to \Gamma(\nu)/2(z/2)^\nu$, the kurtosis starts out at $3/\lambda$ for $\zeta = 0$ and decreases monotonously to 0 for $\zeta \to \infty$. Thus a high kurtosis can be reached only with a small parameter λ.

The first term in c_3 is $\beta\kappa_s\sigma_s^4$, and the first two terms in c_4 are $\kappa_s\sigma_s^4 + 6\left(c_3/\beta - \kappa_s\sigma_s^4\right)$. For a symmetric distribution with certain variance σ_s^2 and kurtosis κ_s we select some parameter $\lambda < 3/\kappa_s$, and solve the Eq. (20.131) to find ζ. This is inserted into Eq. (20.126) to determine

$$\delta^2 \quad = \quad \frac{\sigma_s^2\,\zeta}{\rho(\zeta)}. \tag{20.132}$$

If the kurtosis is larger, it is not an optimal parameter to determine generalized hyperbolic distributions. A better fit to the data is reached by reproducing correctly the size and shape of the distribution near the peak and allow for some deviations in the tails of the distribution, on which the kurtosis depends quite sensitively.

For distributions which are only slightly asymmetric, which is usually the case, it is sufficient to solve the above symmetric equations and determine the small parameter β approximately by the skew $s = c_3/\sigma^3$ from the first line in (20.128) as

$$\beta \approx \frac{s}{\kappa_s\sigma_s}. \tag{20.133}$$

This approximation can be improved iteratively by reinserting β into the second equation in (20.127) to determine, from the variance σ^2 of the data, an improved value of σ_s^2. Then Eq. (20.129) is used to determine from the kurtosis κ of the data an improved value of κ_s, and so on.

For the best fit near the origin where probabilities are large we use the derivatives

$$G(0) \quad = \quad \left(\frac{\delta}{\gamma}\right)^{\lambda-1/2}\zeta^{-\lambda}k_-, \tag{20.134}$$

$$G'(0) \quad = \quad \beta G(0), \tag{20.135}$$

$$G''(0) \quad = \quad -\left(\frac{\delta}{\gamma}\right)^{\lambda-3/2}\zeta^{-\lambda}k_+ + \left(\frac{\delta}{\gamma}\right)^{\lambda-1/2}\delta^{-2}\zeta^{-\lambda}\left(1-2\lambda-\beta^2\delta^2\right)k_-,$$

$$G^{(3)}(0) \quad = \quad -\frac{\beta}{2}\left[3\left(\frac{\delta}{\gamma}\right)^{\lambda-3/2}\zeta^{-\lambda}k_+ + \left(\frac{\delta}{\gamma}\right)^{\lambda-1/2}\delta^{-2}\zeta^{-\lambda}\left(3-6\lambda-\beta^2\delta^2\right)\right]k_-, \tag{20.136}$$

where we have abbreviated

$$k_\pm \equiv \frac{1}{\sqrt{2\pi}}\frac{K_{\lambda\pm1/2}(\delta\gamma)}{K_\lambda(\delta\gamma)}. \tag{20.137}$$

The generalized hyperbolic distributions with $\lambda = 1$ are called *hyperbolic distributions*. The option prices following from these can be calculated by inserting the appropriate parameters into an interactive internet page (see Ref. [51]). Another special case used frequently in the literature is $\lambda = -1/2$ in which case one speaks of a *normal inverse Gaussian distributions* also abbreviated as NIGs.

20.1.12 Debye-Waller Factor for Non-Gaussian Fluctuations

At the end of Section 3.10 we calculated the expectation value of an exponential function $\langle e^{Pz}\rangle$ of a Gaussian variable which led to the Debye-Waller factor of Bragg scattering (3.311). This factor was introduced in solid state physics to describe the reduction of intensity of Bragg peaks due to the thermal fluctuations of the atomic positions. It is derivable from the Fourier representation

$$\langle e^{Pz}\rangle \equiv \int dz \, \frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/2\sigma^2} e^{Pz} = \int dz \int \frac{dp}{2\pi} \, e^{-\sigma^2 p^2/2} e^{ipz+Pz} = e^{\sigma^2 P^2/2}. \tag{20.138}$$

There exists a simple generalization of this relation to non-Gaussian distributions, which is

$$\langle e^{Pz}\rangle \equiv \int dz \int \frac{dp}{2\pi} \, e^{-H(p)} e^{ipz+Pz} = e^{-H(iP)}. \tag{20.139}$$

20.1.13 Path Integral for Non-Gaussian Distribution

Let us calculate the properties of the simplest process whose fluctuations are distributed according to any of the general non-Gaussian distributions. We consider the stochastic differential equation for the logarithms of the asset prices

$$\dot{x}(t) = r_x + \eta(t), \tag{20.140}$$

where the noise variable $\eta(t)$ is distributed according to an arbitrary distribution. The constant drift r_x in (20.140) is uniquely defined only if the average of the noise variable vanishes: $\langle \eta(t)\rangle = 0$. The general distributions discussed above can have a nonzero average $\langle x\rangle = c_1$ which has to be subtracted from $\eta(t)$ to identify r_x. The subsequent discussion will be simplest if we imagine r_x to have replaced c_1 in the above distributions, i.e. if the power series expansion of the Hamiltonian (20.54) is replaced as follows:

$$\begin{aligned} H(p) \to H_{r_x}(p) &\equiv H(p) - H'(0)p + ir_x p \equiv \bar{H}(p) + ir_x p \\ &\equiv ir_x \, p + \frac{1}{2} c_2 \, p^2 - i\frac{1}{3!} c_3 p^3 - \frac{1}{4!} c_4 \, p^4 + i\frac{1}{5!} c_5 p^5 + \ldots \, . \end{aligned} \tag{20.141}$$

Thus we may simply work with the original expansion (20.54) and replace at the end

$$c_1 \to r_x. \tag{20.142}$$

The stochastic differential equation (20.140) can be assumed to read simply

$$\dot{x}(t) = \eta(t). \tag{20.143}$$

With the ultimate replacement (20.142) in mind, the probability distribution of the endpoints $x_b = x(t_b)$ for the paths starting at a certain initial point $x_a = x(t_a)$ is given by a path integral of the form (18.342):

$$P(x_b t_b | x_a t_a) = \int \mathcal{D}\eta \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \exp\left[-\int_{t_a}^{t_b} dt \, \tilde{H}(\eta(t))\right] \delta[\dot{x} - \eta]. \tag{20.144}$$

The function $\tilde{H}(\eta)$ is the negative logarithm of the chosen distribution of log-returns [recall (20.13)]

$$\tilde{H}(\eta) = -\log \tilde{D}(\eta). \tag{20.145}$$

For example, $\tilde{H}(\eta)$ is given by $-\log \tilde{L}^{(\lambda,\alpha)}_{\sigma^2}(\eta)$ of Eq. (20.24) for the truncated Lévy distribution or by $-\log \tilde{B}(\eta)$ of Eq. (20.77) or the Boltzmann distributions.

In the mathematical literature, the measure in the path integral over the noise

$$\mathcal{D}\mu \equiv \mathcal{D}\eta\, P[\eta] = \mathcal{D}\eta\, e^{-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))} \tag{20.146}$$

of the probability distribution (20.144) is called the *measure of the process* $\dot{x}(t) = \eta(t)$. The path integral

$$\int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\, \delta[\dot{x}-\eta] \tag{20.147}$$

is called the *filter* which determines the distribution of x_b at time t_b for all paths $x(t)$ starting out at x_a at time t_a.

A measure differing from (20.146) only by the drift

$$\mathcal{D}\mu' = \mathcal{D}\eta\, P[\eta - r] = \mathcal{D}\eta\, e^{-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t)-r)} \tag{20.148}$$

is called *equivalent measure*. The ratio

$$\mathcal{D}\mu'/\mathcal{D}\mu = e^{-\int_{t_a}^{t_b}\left[\tilde{H}(\eta-r)-\tilde{H}(\eta)\right]} \tag{20.149}$$

is called the *Radon-Nikodym derivative*. For a Gaussian noise, this is simply

$$\mathcal{D}\mu'/\mathcal{D}\mu = e^{\int_{t_a}^{t_b}\left[r\eta(t)-r^2t/2\right]}. \tag{20.150}$$

If one wants to calculate the expectation value of any function $f(x(t))$, one has to split the filter into a product

$$\left[\int_{x(t)=x}^{x(t_b)=x_b} \mathcal{D}x\, \delta[\dot{x}-\eta]\right] \times \left[\int_{x(t_a)=x_a}^{x(t)=x} \mathcal{D}x\, \delta[\dot{x}-\eta]\right], \tag{20.151}$$

and perform an integral over $f(x)$ with this filter in the path integral (20.146). Going over to the probabilities $P(x_bt_b|x_at_a)$, we obtain the integral

$$\langle f(x(t))\rangle = \int dx\, P(x_bt_b|x\,t)f(x)P(x\,t|x_at_a). \tag{20.152}$$

The correlation functions of the noise variable $\eta(t)$ in the path integral (20.144) are given by a straightforward functional generalization of formulas (20.56). For this purpose, we express the noise distribution $P[\eta] \equiv \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right]$ in (20.144) as a Fourier path integral

$$P[\eta] = \int \frac{\mathcal{D}p}{2\pi} \exp\left\{\int_{t_a}^{t_b} dt\, [ip(t)\eta(t) - H(p(t))]\right\}, \tag{20.153}$$

and note that the correlation functions can be obtained from the functional derivatives

$$\langle\eta(t_1)\cdots\eta(t_n)\rangle = (-i)^n \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi}\left[\frac{\delta}{\delta p(t_1)}\cdots\frac{\delta}{\delta p(t_n)} e^{i\int_{t_a}^{t_b} dt\, p(t)\eta(t)}\right] e^{-\int_{t_a}^{t_b} dt\, H(p(t))}.$$

After n partial integrations, this becomes

$$\begin{aligned}\langle\eta(t_1)\cdots\eta(t_n)\rangle &= i^n \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi} e^{i\int_{t_a}^{t_b} dt\, p(t)\eta(t)}\, \frac{\delta}{\delta p(t_1)}\cdots\frac{\delta}{\delta p(t_n)} e^{-\int_{t_a}^{t_b} dt\, H(p(t))}\\ &= i^n \left[\frac{\delta}{\delta p(t_1)}\cdots\frac{\delta}{\delta p(t_n)} e^{-\int_{t_a}^{t_b} dt\, H(p(t))}\right]_{p(t)\equiv 0}. \end{aligned} \tag{20.154}$$

By expanding the exponential $e^{-\int_{t_a}^{t_b} dt\, H(p(t))}$ in a power series (20.54) we find immediately the lowest correlation functions

$$\langle\eta(t_1)\rangle \equiv Z^{-1}\int \mathcal{D}\eta\, \eta(t_1) \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] = 0, \tag{20.155}$$

$$\begin{aligned}\langle\eta(t_1)\eta(t_2)\rangle &\equiv Z^{-1}\int \mathcal{D}\eta\, \eta(t_1)\eta(t_2) \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right]\\ &= c_2\delta(t_1-t_2) + c_1^2, \end{aligned} \tag{20.156}$$

$$\begin{aligned}\langle\eta(t_1)\eta(t_2)\eta(t_3)\rangle &\equiv Z^{-1}\int \mathcal{D}\eta\, \eta(t_1)\eta(t_2)\eta(t_3) \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right]\\ &= c_3\delta(t_1-t_2)\delta(t_1-t_3)\\ &+ c_2c_1[\delta(t_1-t_2)+\delta(t_2-t_3)+\delta(t_1-t_3)] + c_1^3, \end{aligned} \tag{20.157}$$

$$\begin{aligned}\langle\eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4)\rangle &\equiv Z^{-1}\int \mathcal{D}\eta\, \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right]\\ &= c_4\delta(t_1-t_2)\delta(t_1-t_3)\delta(t_1-t_4)\\ &+ c_3c_1[\delta(t_1-t_2)\delta(t_1-t_3) + 3\,\text{cyclic perms}]\\ &+ c_2^2[\delta(t_1-t_2)\delta(t_3-t_4)+\delta(t_1-t_3)\delta(t_2-t_4)+\delta(t_1-t_4)\delta(t_2-t_3)]\\ &+ c_2c_1^2[\delta(t_1-t_2) + 5\,\text{pair terms}] + c_1^4, \end{aligned} \tag{20.158}$$

where

$$Z \equiv \int \mathcal{D}\eta\, \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right]. \tag{20.159}$$

The higher correlation functions are obvious generalizations of (20.44). The different contributions on the right-hand side of (20.156)–(20.262) are distinguishable by their connectedness structure.

Note that the term proportional to c_3 in the three-point correlation function (20.158) and the terms proportional to c_3 and to c_4 in the four-point correlation function (20.262) do not obey Wick's rule (3.305) since they contain contributions caused by the non-Gaussian terms $-ic_3p^3/3!$ $-c_4p^4/4!$ in the Hamiltonian (20.141).

20.1.14 Time Evolution of Distribution

The δ-functional in Eq. (20.144) may be represented by a Fourier integral leading to the path integral

$$P(x_bt_b|x_at_a)=\int\mathcal{D}\eta\int\mathcal{D}x\int\frac{\mathcal{D}p}{2\pi}\exp\left\{\int_{t_a}^{t_b}dt\left[ip(t)\dot{x}(t)-ip(t)\eta(t)-\tilde{H}(\eta(t))\right]\right\}. \tag{20.160}$$

Integrating out the noise variable $\eta(t)$ amounts to performing the inverse Fourier transform of the type (20.24) at each instant of time. Then we obtain

$$P(x_bt_b|x_at_a)=\int\mathcal{D}x\int\frac{\mathcal{D}p}{2\pi}\exp\left\{\int_{t_a}^{t_b}dt\left[ip(t)\dot{x}(t)-H(p(t))\right]\right\}. \tag{20.161}$$

Integrating over all $x(t)$ with fixed endpoints enforces a constant momentum along the path, and we remain with the single momentum integral

$$P(x_bt_b|x_at_a)=\int_{-\infty}^{\infty}\frac{dp}{2\pi}\exp\left[ip(x_b-x_a)-(t_b-t_a)H(p)\right]. \tag{20.162}$$

Given an arbitrary noise distribution $\tilde{D}(z)$ extracted from the data of a given frequency $1/\Delta t$, we simply identify the Hamiltonian $H(p)$ from the Fourier representation

$$\tilde{D}(z)=\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,e^{ikz-H(p)}, \tag{20.163}$$

and insert $H(p)$ into (20.170) to find the time dependence of the distribution. The time is then measured in units of the time interval Δt.

For a truncated Lévy distribution, the result is

$$P(x_bt_b|x_at_a)=\tilde{L}^{(\lambda,\alpha)}_{\sigma^2(t_b-t_a)}(x_b-x_a). \tag{20.164}$$

This is a truncated Lévy distribution of increasing width. The result for other distributions is analogous.

Since the distribution (20.170) depends only on $t=t_b-t_a$ and $x=x_b-x_a$, we shall write it shorter as

$$P(x,t)=\int_{-\infty}^{\infty}\frac{dp}{2\pi}\exp\left[ipx-tH(p)\right]. \tag{20.165}$$

20.1.15 Central Limit Theorem

For large t, the distribution $P(x,t)$ becomes more and more Gaussian (see Fig. 20.18 for the Boltzmann distribution). This is a manifestation of the *central limit theorem* of statistical mechanics which states that the convolution of infinitely many arbitrary distribution functions of finite width always approaches a Gaussian distribution. This is easily proved. We simply note that after an integer number t of convolutions, a probability distribution $\tilde{D}(z)$ with Hamiltonian $H(p)$ has the Fourier components $[D(p)]^t = e^{-tH(p)}$, so that it is given by the integral

$$\tilde{D}(z,t) = \int \frac{dp}{2\pi} e^{ipz} e^{-tH(p)}. \tag{20.166}$$

For large t, the integral can be evaluated in the saddle point approximation (4.51). Denoting the momentum at extremum of the exponent by p_z, which is determined implicitly by $tH'(p_z) = iz$, and setting $\sigma^2 = H''(p_z)$, we obtain the Gaussian distribution

$$\begin{aligned}\tilde{D}(z,t) \underset{t \text{ large}}{\to} & \; e^{ip_z z - tH(p_z)} \int \frac{dp}{2\pi} e^{i(p-p_z)z - t\sigma^2 (p-p_z)^2/2} = \frac{e^{ip_z z - tH(p_z)}}{\sqrt{2\pi\sigma^2}} e^{-z^2/2t\sigma^2} \\ = & \; \frac{e^{\sigma^2 p_z^2/2 - tH(p_z)}}{\sqrt{2\pi\sigma^2}} e^{-(z - t\sigma^2 p_z)^2/2t\sigma^2}. \end{aligned} \tag{20.167}$$

The same procedure can be applied to the integral (20.165).

The transition from Boltzmann to Gaussian distributions is shown for the S&P 500 index in Fig. 20.15.

If a distribution has a heavy power tail $\propto |z|^{-\lambda-1}$ with $\lambda < 2$, so that the volatility σ is infinite, the Hamiltonian starts out for small p like $|p|^\lambda$, and the saddle point approximation is governed by this term rather than the quadratic term p^2. For an asymmetric distribution, the leading terms in the Hamiltonian are those of the asymmetric Lévy distribution (20.21) plus a possible linear term irp accounting for a drift, and the asymptotic z-behavior will be given by Eq. (20.17) with an extra factor $(1+\beta)$ [21].

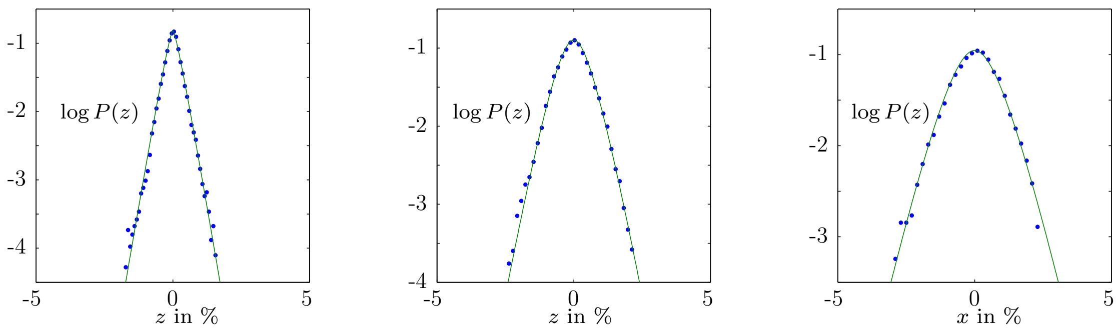

Figure 20.15 Fits of Gaussian distribution to S&P 500 log-returns recorded in intervals of 60 min, 240 min, and 1 day.

If a distribution has no second moments, the central limit theorem does not apply. Such distributions tend to another limit, the so-called *Pareto-Lévy-stable distribution*. Their Hamiltonian has precisely the generic form (20.21), except for a possible additional linear drift term [21]:

$$H(p) = -irp + H_{\lambda,\sigma,\beta}(p). \tag{20.168}$$

In this context, the Lévy parameter λ is also called the *stability parameter*. The parameter α has to be in the interval $(0, 2]$. The parameter β is called *skewness parameter*. For plots of $\tilde{D}(z, 1)$ see Ref. [75].

The convergence of variable with heavy-tail distributions is the content of the *generalized central limit theorem*.

20.1.16 Additivity Property of Noises and Hamiltonians

At this point we make the useful observation that each term in the Hamiltonian (20.141) can be attributed to an independent noise term in the stochastic differential equation (20.140). Indeed, if this differential equation has two noise terms

$$\dot{x}(t) = r_x + \eta_1(t) + \eta_2(t) \tag{20.169}$$

whose fluctuations are governed by two different Hamiltonians, the probability distribution (20.144) is replaced by

$$P(x_b t_b | x_a t_a) = \int \mathcal{D}\eta_1 \int \mathcal{D}\eta_2 \int \mathcal{D}x \exp\left[-\int_{t_a}^{t_b} dt [\tilde{H}_1(\eta_1(t)) + \tilde{H}_2(\eta_2(t))\right] \delta[\dot{x} - r_x - \eta_1 - \eta_2].$$

After a Fourier decomposition of the δ-functional, this becomes

$$\begin{aligned} P(x_b t_b | x_a t_a) = & \int \mathcal{D}p_1 \int \mathcal{D}p_2 \int \mathcal{D}\eta_1 \int \mathcal{D}\eta_2 \int \mathcal{D}x \int \mathcal{D}p e^{-\int_{t_a}^{t_b} dt\, [ip(\dot{x} - r_x - \eta_1 - \eta_2)]} \\ & \times \exp\left\{-\int_{t_a}^{t_b} dt\ [H_1(p_1) - ip_1\eta_1(t) + H_2(p_2) - ip_2\eta_2(t)]\right\}, \end{aligned}$$

after which the path integrals over $\eta_1(t)$, $\eta_2(t)$ lead to

$$P(x_b t_b | x_a t_a) = \int \mathcal{D}p \int \mathcal{D}x \exp\left\{\int_{t_a}^{t_b} dt\ [ip\dot{x} - ir_x p - H_1(p) - H_2(p)]\right\}. \tag{20.170}$$

This is the integral representation Eq. (20.161) with the combined Hamiltonian $H(p) = ir_x p + H_1(p) + H_2(p)$, which can be rewritten as a pure integral (20.170). By rewriting this integral trivially as

$$P(x_b t_b | x_a t_a) = \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} e^{ip_1(x_b - x_c) - (t_b - t_a)[ir_x p + H_1(p_1)]} e^{ip_2(x_c - x_a) - (t_b - t_a) H_2(p_2)}, \tag{20.171}$$

we see that the probability distribution associated with a sum of two Hamiltonians is the convolution of the individual probability distributions:

$$P(x_b t_b | x_a t_a) = \int_{-\infty}^{\infty} dx dx_c P_1(x_b t_b | x_c t_a) P_2(x_c t_b | x_a t_a). \tag{20.172}$$

Thus we may calculate the probability distributions associated with the noises $\eta_1(p)$ and $\eta_2(p)$ separately, and combine the results at the end by a convolution.

20.1.17 Lévy-Khintchine Formula

It is sometimes useful to represent the Hamiltonian in the form of a Fourier integral

$$H(p) = \int dz\, e^{ipz} F(z). \tag{20.173}$$

Due to the special significance of the linear term in $H(p)$ governing the drift, this is usually subtracted out of the integral by rewriting (20.173) as

$$H_r(p) = irp + \int dz \left(e^{ipz} - 1 - ipz\right) F(z). \tag{20.174}$$

The first subtraction ensures the property $H_r(0) = 0$ which guarantees the unit normalization of the distribution. This subtracted representation is known as the *Lévy-Khintchine formula*, and the function $F(z)$ is the so-called *Lévy weight* of the distribution. Some people also subtract out the quadratic term and write

$$H_r(p) = irp + \frac{\sigma^2}{2}p^2 + \int dz \left(e^{ipz} - 1\right) \bar{F}(z). \tag{20.175}$$

They employ a weight $\bar{F}(z)$ which has no first and second moment, i.e., $\int dz\, \bar{F}(z) z = 0$, $\int dz\, \bar{F}(z) z^2 = 0$, to avoid redundancy in the representation.

Note that according to the central limit theorem (20.167), the large-time probability distribution of x will become Gaussian for large t. Thus, in the Lévy-Khintchine decomposition (20.175) of the Hamiltonian $H(p)$, only the first two terms contribute for large t leading to the distribution:

$$P(x,t) \underset{t \text{ large}}{\to} \frac{e^{-(x-tr)^2/2t\sigma^2}}{\sqrt{2\pi\sigma^2}}. \tag{20.176}$$

The Lévy measure has been calculated explicitly for many non-Gaussian distributions. As an example, the generalized hyperbolic case has for $\lambda \geq 0$ a Lévy measure [76]

$$F(z) = \frac{e^{\beta z}}{|z|} \left\{ \frac{1}{\pi^2} \int_0^\infty \frac{dy}{y} \frac{e^{-\sqrt{y+\gamma^2}|z|}}{J_\lambda^2(\delta\sqrt{y}) + Y_\lambda^2(\delta\sqrt{y})} + \lambda e^{-\gamma|z|} \right\}, \tag{20.177}$$

where $J_\lambda(z)$ and $Y_\lambda(z)$ are standard Bessel functions.

The decomposition of the Hamiltonian according the Lévy-Khintchine formula and the additivity of the associated noises form the basis of the Lv́y-Itō theorem which states that an arbitrary stochastic differential equation with Hamiltonian (20.175) can be decomposed in the form

$$\dot{x} = r_x t + \eta_{\rm G} + \eta_{\leq 1} + \eta_{>1}, \tag{20.178}$$

where $\eta_{\rm G}$ is a Gaussian noise

$$\eta_{\leq 1} = \int_{|x|\leq 1} dz \left(e^{ipz} - 1\right) \bar{F}(z) \tag{20.179}$$

a superposition of discrete noises called Poisson point process with jumps smaller than or equal to unity, and

$$\eta_{>1} = \int_{|x|\leq 1} dz \left(e^{ipz} - 1\right) \bar{F}(z) \tag{20.180}$$

a noise with jumps larger than unity.

Consider the simplest noise of the type $\eta_{\leq 1}$ arising from a Lévy weight $\bar{F}_Z(z) = \tau_Z \delta(z + Z)$ with $0 < Z \leq 1$ in (20.179) so that the Hamiltonian is $H_Z(p) = \tau_Z(e^{-iZp} - 1)$. The associated distribution function $\tilde{D}_Z(z)$ is, according to (20.11),

$$\tilde{D}_Z(z) = \int \frac{dp}{2\pi}\, e^{ipz} e^{\tau_Z(e^{-ipZ}-1)}. \tag{20.181}$$

Expanding the last exponential in powers of e^{ipZ}, we obtain

$$\tilde{D}_Z(z) = \int \frac{dp}{2\pi}\, e^{ipz} \sum_{n=0}^{\infty} e^{-\tau_Z} \frac{\tau_Z^n}{n!} \tau_Z^n e^{-ipnZ} = \sum_{n=0}^{\infty} e^{-\tau_Z} \frac{\tau_Z^n}{n!} \delta(z - nZ). \tag{20.182}$$

This function describes jumps by nZ ($n = 0, 1, 2, 3, \dots$) whose probability follows a *Poisson distribution*

$$P(n, \tau_Z) = e^{-\tau_Z} \frac{\tau_Z^n}{n!}, \tag{20.183}$$

which is properly normalized to unity: $\sum_{n=0}^{\infty} P(n, \tau_Z) = 1$. The expectation values of powers of the jump number are

$$\langle n^k \rangle = \sum_{n=0}^{\infty} n^k P(n, \tau_Z) = e^{-\tau_Z} (\tau_Z \partial_{\tau_z})^k e^{\tau_Z} \sum_{n=0}^{\infty} P(n, \tau_Z) = e^{-\tau_Z} (\tau_Z \partial_{\tau_z})^k e^{\tau_Z} = \frac{\Gamma(\lambda_Z + k)}{\Gamma(\lambda_Z)}. \tag{20.184}$$

Thus $\langle n \rangle = \lambda_Z$, $\langle n^2 \rangle = \lambda_Z(\lambda_Z + 1)$, $\langle n^3 \rangle = \lambda_Z(\lambda_Z + 1)(\lambda_Z + 2)$, $\langle n^4 \rangle = \lambda_Z(\lambda_Z + 1)(\lambda_Z + 2)(\lambda_Z + 3)$, so that $\sigma^2 = \lambda_Z$, $s = 2/\sqrt{\lambda_Z}$, and $\kappa = 6/\lambda_Z$.

As typical noise curve is displayed in Fig. 20.16. An arbitrary Lévy weight $\bar{F}(z)$ in $\eta_{\leq 1}$ may

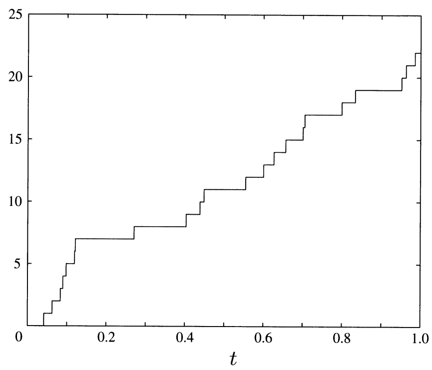

Figure 20.16 Typical Noise of Poisson Process.

always be viewed as a superposition $\bar{F}(z) = \int_{-1}^{1} dZ\, F(Z) \delta(z - Z)$ so that the distribution function becomes a superposition of Poisson noises:

$$\tilde{D}(z) = \int_{-1}^{1} dZ\, F(Z) \sum_{n=0}^{\infty} e^{-\tau_Z} \frac{\tau_Z^n}{n!} \delta(z - nZ). \tag{20.185}$$

20.1.18 Semigroup Property of Asset Distributions

An important property of the probability (20.144) is that it satisfies the semigroup equation

$$P(x_c t_c | x_a t_a) = \int_{-\infty}^{\infty} dx_b\, P(x_c t_c | x_b t_b) P(x_b t_b | x_a t_a). \tag{20.186}$$

In the stochastic context this property is referred to as *Chapman-Kolmogorov equation* or *Smoluchowski equation*. It is a general property of processes which a without memory. Such processes are referred to a *Markovian* [12]. Note that the semigroup property (20.186) implies the initial condition

$$P(x_c t_a | x_a t_a) = \delta(x_b - x_a). \qquad (20.187)$$

In Fig. 20.17 we show that the property (20.186) is satisfied reasonably well by experimental asset distributions, except for small deviations in the low-probability tails.

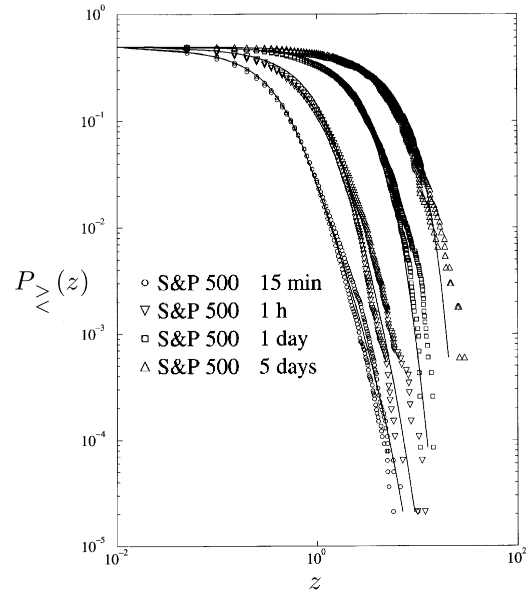

Figure 20.17 Cumulative distributions obtained from repeated convolution integrals of the 15-min distribution (from Ref. [13]). Apart from the far ends of the tails, the semigroup property (20.186) is reasonably well satisfied.

One may verify the semigroup property also using the high-frequency distributions of S&P 500 and NASDAQ 100 indices in Fig. 20.10 recorded by the minute, which display the Boltzmann distribution (20.77). If we convolute the minute distribution an integer number t times, the result aggress very well with the distribution of t-minute data. Only at the heavy tails of rare events do not follow this pattern

Note that due to the semigroup property (20.186) of the probabilities one has the trivial identity

$$\begin{aligned} \langle f(x(t_b), t_b) \rangle &= \int dx_b \, f(x_b, t_b) P(x_b t_b | x_a t_a) \\ &= \int dx \Big[\int dx_b \, f(x_b, t_b) P(x_b t_b | x\, t) \Big] P(x\, t | x_a t_a). \end{aligned} \qquad (20.188)$$

In the mathematical literature, the expectation value on the left-hand side of Eq. (20.188) is written as

$$\mathbb{E}[f(x(t_b), t_b) | x_a t_a] \equiv \int dx_b \, f(x_b, t_b) P(x_b t_b | x_a t_a). \qquad (20.189)$$

With this notation, the second line can be re-expressed in the form

$$\int dx\, \mathbb{E}[f(x_b,t_b)|x\,t]P(x\,t|x_at_a) = \mathbb{E}[\mathbb{E}[f(x_b,t_b)|x\,t]|x_at_a], \tag{20.190}$$

so that we obtain the property of expectation values

$$\mathbb{E}[f(x(t_b,t_b))|x_at_a] = \mathbb{E}[\mathbb{E}[f(x_b,t_b)|x\,t]|x_at_a]. \tag{20.191}$$

In mathematical finance, this complicated-looking but simple property is called the *towering property* of expectation values.

Note that since $P(x_bt_b|x_at_a)$ is equal to $\delta(x_b-x_a)$ for $t_a=t_b$, the expectation value (20.189) has the obvious property

$$\mathbb{E}[f(x(t_a),t_a)|x_at_a] = f(x_a,t_a) \tag{20.192}$$

20.1.19 Time Evolution of Moments of Distribution

From the time-dependent distribution (20.165) it is easy to calculate the time dependence of the moments:

$$\langle x^n\rangle(t) \equiv \int_{-\infty}^{\infty} dx\, x^n\, P(x,t) \tag{20.193}$$

Inserting (20.165), we obtain

$$\langle x^n\rangle(t) \;=\; \int_{\infty}^{\infty}\frac{dp}{2\pi}e^{-tH(p)}\int_{-\infty}^{\infty}dx\,x^ne^{ipx} = \int_{\infty}^{\infty}dpe^{-tH(p)}(-i\partial_p)^n\delta(p). \tag{20.194}$$

After n partial integrations, this becomes

$$\langle x^n\rangle(t) \;=\; (i\partial_p)^n e^{-tH(p)}\Big|_{p=0}. \tag{20.195}$$

All expansion coefficients c_n of $H(p)$ in Eq. (20.141) receive the same factor t, so that the cumulants of the moments all grow linearly in time:

$$\langle x^n\rangle_c(t) = -tH^{(n)}(0) = t\langle x^n\rangle_c(1) = tc_n. \tag{20.196}$$

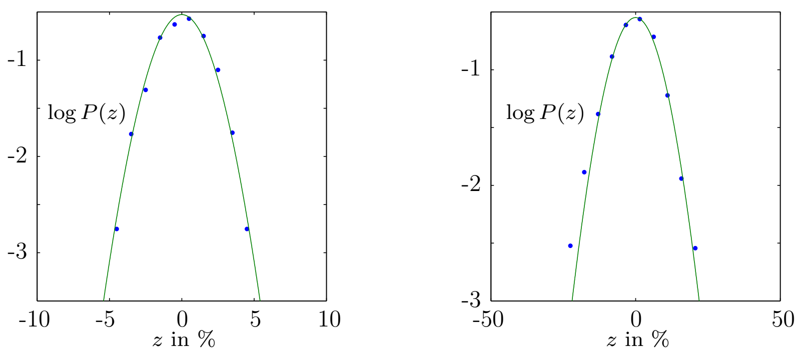

Figure 20.18 Gaussian distributions of S&P 500 and NASDAQ 100 weekly log-returns.

20.1.20 Boltzmann Distribution

As a an example, consider the Boltzmann distribution (20.77) of the minute data. The subsequent time-dependent variance and kurtosis are found by inserting the cumulants Eq. (20.80) into (20.196), yielding

$$\langle x^2 \rangle_c(t) = t\, 2T^2, \qquad \kappa(t) = \frac{c_4}{t c_2^2} = \frac{3}{t}. \tag{20.197}$$

The first increases linearly in time, the second decreases like one over time, which makes the distribution more and more Gaussian, as required by the central limit theorem (20.167). The two quantities are plotted in Figs. 20.19 and 20.20. The agreement with the data is seen to be excellent.

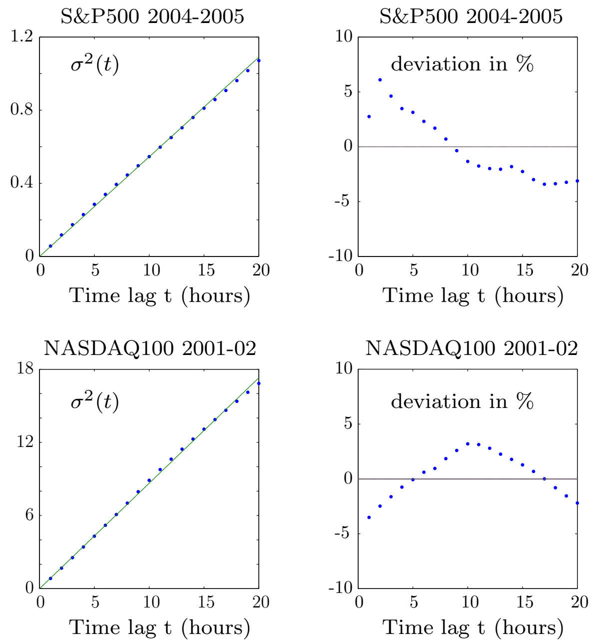

Figure 20.19 Variance of S&P 500 and NASDAQ 100 indices as a function of time. The slopes $\langle x^2 \rangle_c(t)/t$ are roughly $1.1/20 \times 1/60\,\mathrm{min}$ and $18.1/20 \times 1/60\,\mathrm{min}$, respectively, so that Eq. (20.198) yields the temperatures $T_{\mathrm{SP500}} \approx 0.075$ and $T_{\mathrm{NASDAQ100}} \approx 0.15$. The right-hand side shows the relative deviation from the linear shape in percent.

Note that if the minute data are not available, but only data taken with a frequency $1/t_0$ (in units 1/min) and an expectation $\langle x^2 \rangle_c(t_0)$, then we can find the temperature of the minute distribution from the formula

$$T = \sqrt{\langle x^2 \rangle_c(t_0)/2t_0}. \tag{20.198}$$

Since κ goes to zero like $1/t$, the distribution becomes increasingly Gaussian as the time grows, this being a manifestation of the central limit theorem (20.167) of statistical mechanics according to which the convolution of infinitely many arbitrary distribution functions of finite width always approaches a Gaussian distribution.

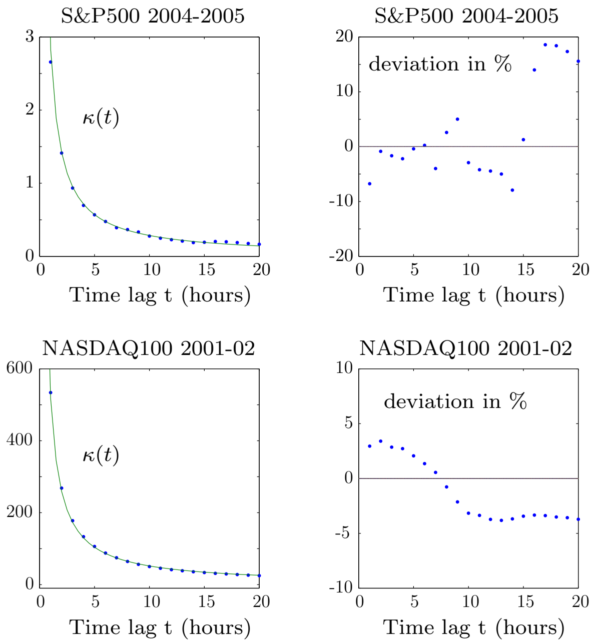

Figure 20.20 Kurtosis of S&P 500 and NASDAQ 100 indices as a function of time. The right-hand side shows the relative deviation from the $1/t$ behavior in percent.

This result is in contrast to the pure Lévy distribution in Subsection 20.1.2 with $\lambda < 2$, which has no finite width and therefore maintains its power falloff at large distances.

If we omit the time argument in the cumulants $\langle x^n \rangle_c(1)$, for brevity, and insert these into the decompositions (20.56) we obtain the time dependence of the moments:

$$\begin{aligned}
\langle x \rangle(t) &= t\langle x \rangle_c, \\
\langle x^2 \rangle(t) &= t\langle x^2 \rangle_c + t^2\langle x \rangle_c^2, \\
\langle x^3 \rangle(t) &= t\langle x^3 \rangle_c + 3t^2\langle x \rangle_c\langle x^2 \rangle_c + t^3\langle x \rangle_c^3, \\
\langle x^4 \rangle(t) &= t\langle x^4 \rangle_c + 3t^2\langle x^2 \rangle_c^2 - 4t^2\langle x \rangle_c\langle x^3 \rangle_c + 6t^3\langle x \rangle_c^2\langle x^2 \rangle_c + t^4\langle x \rangle_c^4, \\
&\vdots \quad .
\end{aligned} \tag{20.199}$$

Let us now calculate the time evolution of the Boltzmann distribution (20.77) of the minute data. The Hamiltonian $H(p)$ was identified in Eq. (20.79), so that

$$e^{-tH(p)} = \frac{1}{[1+(Tp)^2]^t}. \tag{20.200}$$

The time-dependent distribution is therefore given by the Fourier integral

$$P(x,t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx - tH(p)} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{(1+T^2p^2)^t} e^{ipx}. \tag{20.201}$$

The calculation proceeds most easily rewriting (20.200), by analogy with (20.81) and using again formula (2.499), as an integral

$$e^{-tH(p)} = \frac{1}{[1+(Tp)^2]^t} = \frac{1}{\Gamma(t)}\int_0^\infty \frac{d\tau}{\tau}\,\tau^t e^{-\tau(1+T^2p^2)}. \qquad (20.202)$$

The prefactor of p^2 is equal to $t\sigma^2 = tv$ a Gaussian distribution in x-space. We therefore change the variable of integration from τ to v, substituting $\tau = tv/2T^2$, and obtain [compare (20.82)]

$$e^{-tH(p)} = \left(\frac{t}{2T^2}\right)^t \frac{1}{\Gamma(t)}\int_0^\infty \frac{dv}{v}\,v^t e^{-tv/2T^2} e^{-tvp^2/2}. \qquad (20.203)$$

The Fourier transform of this yields the time evolution of the Boltzmann distribution as a time-dependent superposition of Gaussian distributions of different widths [compare (20.83)]:

$$P(x,t) = \left(\frac{t}{2T^2}\right)^t \frac{1}{\Gamma(t)}\int_0^\infty \frac{dv}{v}\,v^t e^{-tv/2T^2}\frac{1}{\sqrt{2\pi tv}}e^{-x^2/2tv}. \qquad (20.204)$$

The integral can be performed using the integral formula (2.559), and we find the time-dependent distribution

$$P(x,t) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ipx-tH(p)} = \frac{1}{T\sqrt{\pi}\Gamma(t)}\left(\frac{|x|}{2T}\right)^{t-1/2} K_{t-1/2}(|x|/T), \qquad (20.205)$$

where t is measured in minutes.

For $t=1$, we reobtain the fundamental distribution (20.77), recalling the explicit form of $K_{1/2}(z)$ in Eq. (1.349).

In the limit of large t, the integral over v in (20.204) can be evaluated in the saddle-point approximation of Section 4.2. We expand the weight function in the integrand around the maximum at $v_m = 2T^2(1-1/t)$ as

$$v^{t-1}e^{-tv/2T^2} = \left[2T^2(1-1/t)\right]^{t-1} e^{-(t-1)}e^{-t\delta v^2/2[2T^2(1-1/t)]^2}\left[1+\mathcal{O}(\delta v^3)\right], (20.206)$$

where $\delta v = v - v_m$, and $\mathcal{O}(\delta v^3)$ is equal to $-t\delta v^3/3[2T^2(1-1/t)]^3$. Then we observe that for large t, the Gaussian can be expanded into derivatives of δ-functions as follows:

$$e^{-tz^2/2}\left[1+\frac{a}{3}z^3+\ldots\right] = \sqrt{2\pi/t}\left[\delta(z)+\frac{1}{2t}\delta''(z)+\frac{a}{t^2}\delta'(z)\ldots\right]. \qquad (20.207)$$

This can easily be verified by multiplying both sides with $f(z) = f(0) + zf'(0) + z^2 f''(0)/2 + \ldots$ and integrating over z. Thus we find to leading order

$$e^{-t\delta v^2/2[2T^2(1-1/t)]^2} \to \sqrt{\frac{2\pi}{t}}\,2T^2(1-1/t)\,\delta(\delta v). \qquad (20.208)$$

Using the large-t limit $(1 - 1/t)^t \to e^{-1}$, and including the first correction from (20.207), we obtain

$$v^{t-1}e^{-tv/2T^2} \to \sqrt{\frac{2\pi}{t}}(2T^2)^t e^{-t}\left\{\delta(\delta v) + \frac{[2T^2(1-1/t)]^2}{2t}\delta''(\delta v) + \dots\right\}. \quad (20.209)$$

If we now employ Stirling's formula (17.286) to approximate $t^t/\Gamma(t) \to \sqrt{t/2\pi}e^t$, we see that the integral (20.204) converges for large t against the Gaussian distribution $e^{-x^2/2tv_m}/\sqrt{2\pi t v_m}$ with the saddle point variance $v_m \to 2T^2$.

The behavior near the peak of the distribution can be calculated from the small-z expansion[7]

$$\left(\frac{z}{2}\right)^\nu K_\nu(z) = \frac{\pi}{2\sin\pi\nu\,\Gamma(1-\nu)}\left[1 + \frac{\Gamma(1-\nu)}{1!\Gamma(2-\nu)}\frac{z^2}{4} + \mathcal{O}(z^4, z^{2\nu})\right]. \quad (20.210)$$

For large ν and small z, this can be approximated by [recall (20.29)]

$$\left(\frac{z}{2}\right)^\nu K_\nu(z) \approx \frac{\Gamma(\nu)}{2}e^{-z^2/4(t-3/2)}. \quad (20.211)$$

We now use Stirling's formula (17.286) to find the large-t limit $\Gamma(t-1/2)/\Gamma(t) \to 1$, leading to the Gaussian behavior near the peak:

$$P(x,t) \underset{\text{small}\,|x|}{\approx} \frac{1}{\sqrt{2\pi\, 2T^2 t}}e^{-x^2/2\,2T^2(t-3/2)}. \quad (20.212)$$

This corresponds to the approximation of the Fourier transform (20.200) by $e^{-tT^2p^2}$. The Gaussian shape reaches out to $x \approx T\sqrt{t}$. For very large t, it holds everywhere, as it should by virtue of the central limit theorem (20.167).

20.1.21 Fourier-Transformed Tsallis Distribution

For the Hamiltonian defined in Eq. (20.97), the time evolution is given by the Fourier transform

$$e^{-tH_{\delta,\beta}(p)} = \left[1 - \beta\delta\, p^2/2\right]^{-t/\delta} = \frac{1}{\Gamma(t/\delta)}\int_0^\infty \frac{ds}{s}\; s^{t/\delta}e^{-s}e^{-s\beta\delta\, p^2/2},\ \beta \equiv \frac{\nu}{\mu} = \frac{1}{\mu\delta}, \quad (20.213)$$

which can be rewritten, by analogy with (20.99), as

$$e^{-tH_{\delta,\beta}(p)} = \frac{\mu^{t/\delta}}{\Gamma(t/\delta)}\int_0^\infty \frac{dv}{v}\; v^{t/\delta}e^{-\mu v}e^{-vp^2/2}, \quad \mu = 1/\beta\delta. \quad (20.214)$$

Taking the Fourier transform of this yields the time-dependent distribution function

$$P_{\delta,\beta}(x,t) = \frac{\mu^{t/\delta}}{\Gamma(t/\delta)}\int_0^\infty \frac{dv}{v}\; v^{t/\delta}e^{-\mu v}\frac{1}{\sqrt{2\pi v}}e^{-x^2/2v}, \quad \mu = 1/\beta\delta, \quad (20.215)$$

[7] M. Abramowitz and I. Stegun, op. cit., Formulas 9.6.2 and 9.6.10.

which becomes with the help of Formula (2.499):

$$P_{\delta,\beta}(x,t) = \frac{\mu^{t/\delta}}{\Gamma\,(t/\delta)} \frac{1}{\sqrt{2\pi}} \left(\frac{x^2}{2\mu}\right)^{t/2\delta-1/4} 2K_{t/\delta-1/2}(\sqrt{2\mu}x), \quad \mu = \frac{1}{\beta\delta}. \tag{20.216}$$

For $\delta = 1$ and $\mu = 1/2T^2$, this reduces to the time-dependent Boltzmann distribution (20.205). At the origin, its value is [compare (20.104)]

$$P_{\delta,\beta}(0,t) = \sqrt{\mu}\,\Gamma(t/\delta - 1/2)/\Gamma(t/\delta). \tag{20.217}$$

20.1.22 Superposition of Gaussian Distributions

All time-dependent distributions whose Fourier transforms have the form $e^{-tH(p)}$ possess a path integral representation (20.161) and which obeys the semigroup equation (20.186). In several examples we have seen that the Fourier transforms $e^{-H(p)}$ can be obtained from a superposition of Gaussian distributions of different variances $e^{-vp^2/2}$ with some weight function $w(v)$:

$$e^{-H(p)} = \int_0^\infty dv\, w(v) e^{-vp^2/2}. \tag{20.218}$$

This was true for the Boltzmann distribution in Eq. (20.81), for the Fourier-transformed Tsallis distribution in Eq. (20.100), and for the Boltzmann Distribution of a relativistic particle in Eq. (20.105). In all these cases it was possible to write a similar superposition for the time-dependent distributions:

$$e^{-tH(p)} = \int_0^\infty dv' w_t(v') e^{-v'p^2/2} = \int_0^\infty dv\, \omega_t(v) e^{-tvp^2/2}, \tag{20.219}$$

where we have introduced the time-dependent weight function

$$\omega_t(v) \equiv t\, w_t(vt). \tag{20.220}$$

See Eqs. (20.202), (20.214), and (20.105) for the above three cases.

The question arises as to general property of the time dependence of the weight function $w_t(v)$ which ensures that that the Fourier transform of the superposition (20.219) obtained as in Eq. (20.164) satisfies the semigroup condition (20.186) [52]. For this, the superposition has to depend on time as

$$e^{-(t_2+t_1)H(p)} = e^{-t_2 H(p)} e^{-t_1 H(p)},$$

or $e^{-tH(p)} = [e^{-H(p)}]^t$. This implies that the weight factor $w_t(v)$ has the property

$$\int dv_{12} w_{t_1+t_2}(v_{12}) e^{-v_{12}p^2/2} = \int dv_2 w_{t_2}(v_2) e^{-v_2 p^2/2} \int dv_1 w_{t_1}(v_1) e^{-v_1 p^2/2}. \tag{20.221}$$

For the Laplace transforms $\tilde{w}_t(v)$ of $w_t(v)$

$$\tilde{w}_t(p_v) \equiv \int dv\, e^{-p_v v} w_t(v) \tag{20.222}$$

this amounts to the factorization property

$$\tilde{w}_{t_1+t_2}(p_v) = \tilde{w}_{t_2}(p_v)\tilde{w}_{t_1}(p_v), \tag{20.223}$$

which is fulfilled by the exponential

$$\tilde{w}_t(p_v) = e^{-tH_v(p_v)}, \tag{20.224}$$

with some Hamiltonian $H_v(p_v)$. Since the integral over v in (20.222) runs only over the positive v-axis, the Hamiltonian $H_v(p_v)$ is analytic in the upper half-plane of p_v.

It is easy to relate $H_v(p_v)$ to $H(p)$. For this we form the inverse Laplace transformation, the so-called *Bromwich integral* [53]

$$w_t(v) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dp_v}{2\pi i}\, e^{p_v v - tH_v(p_v)}, \tag{20.225}$$

in which γ is some real number larger than the real parts of all singularities in $e^{-p_v v - tH_v(p_v)}$. Inserting this into (20.219) and performing the v-integral yields

$$e^{-tH(p)} = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dp_v}{2\pi i} \frac{1}{p^2/2 - p_v}\, e^{-tH_v(p_v)}. \tag{20.226}$$

Closing the integral over p_v in the complex p_v-plane and deforming the contour to an anticlockwise circle around $p_v = p^2/2$ we find the desired relation:

$$H_v(p^2/2) = H(p). \tag{20.227}$$

The time-dependent distribution associated with any Hamiltonian $H(p)$ via the Fourier integral (20.170) can be represented as a superposition of Gaussian distributions with the help of Eq. (20.219) if we merely choose $H_v(p_v) = H(\sqrt{2p_v})$.

Recalling the derivation of the Fourier representation (20.165) from the path integral (20.153) and the descendence of that path integral from the stochastic differential equation (20.140) with the noise Hamiltonian $H(p)$ we conclude that the Hamiltonian $H_v(ip_v)$ governs the noise in a stochastic differential equation for the volatility fluctuations.

Take, for example, the superposition of Gaussians (20.105). The Laplace transform of the weight function $\omega_\beta(v)$ is

$$\tilde{\omega}_\beta(p_v) = \int dv\, e^{-vp_v}\omega_\beta(v) = \int dv\, e^{-vp_v}\sqrt{\frac{\beta}{2\pi v^3}}e^{-\beta/2v} = e^{-\sqrt{2\beta p_v}}. \tag{20.228}$$

The Laplace transform of $w_\beta(v)$ is related to this by

$$\int dv' e^{-v' p_v} w_\beta(v') = \int dv\, e^{-\beta v p_v}\omega_\beta(v) = \tilde{\omega}_\beta(\beta p_v). \tag{20.229}$$

Hence it is given by $\tilde{w}_\beta(p_v) = e^{-\beta\sqrt{2p_v}} = e^{-\beta H_v(p_v)}$, which satisfies the relation (20.223). Moreover, $H_v(p^2/2)$ is equal to $H(p) = \sqrt{p^2}$, as required by Eq. (20.227) and in accordance with Eq. (20.105) for $M = 0$.

Note that instead of the *Bromwich integral* (20.226) it is sometimes more convenient to use *Post's Laplace inversion formula* [54]

$$\omega_t(v) = \lim_{k\to\infty} \frac{(-1)^k}{k!} x^{k+1} \left.\frac{\partial^k \tilde{\omega}_t(x)}{\partial x^k}\right|_{x=k/v}. \tag{20.230}$$

20.1.23 Fokker-Planck-Type Equation

From the Fourier representation (20.170) it is easy to prove that the probability satisfies a Fokker-Planck-type equation

$$\partial_t P(x_b t_b | x_a t_a) = -H(-i\partial_x) P(x_b t_b | x_a t_a). \tag{20.231}$$

Indeed, the general solution $\psi(x,t)$ of this differential equation with the initial condition $\psi(x,0)$ is given by the path integral generalizing (20.144)

$$\psi(x,t) = \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] \psi\left(x - \int_{t_a}^{t} dt' \eta(t')\right). \tag{20.232}$$

This satisfies the Fokker-Planck-type equation (20.231). To show this, we take $\psi(x,t)$ at a slightly later time $t+\epsilon$ and expand

$$\begin{aligned}
\psi(x,t+\epsilon) &= \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] \psi\left(x - \int_{t_a}^{t} dt'\eta(t') - \int_{t}^{t+\epsilon} dt'\eta(t')\right). \\
&= \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] \left\{\psi\left(x - \int_{t_a}^{t} dt'\eta(t')\right)\right. \\
&- \quad \psi'\ \left(x - \int_{t_a}^{t} dt'\eta(t')\right) \int_{t}^{t+\epsilon} dt'\eta(t') \\
&+ \frac{1}{2}\psi''\ \left(x - \int_{t_a}^{t} dt'\eta(t')\right) \int_{t}^{t+\epsilon} dt_1 dt_2\, \eta(t_1)\eta(t_2) \\
&- \frac{1}{3!}\psi'''\left(x - \int_{t_a}^{t} dt'\eta(t')\right) \int_{t}^{t+\epsilon} dt_1 dt_2 dt_3\, \eta(t_1)\eta(t_2)\eta(t_3) \\
&+ \left.\frac{1}{4!}\psi^{(4)}\left(x - \int_{t_a}^{t} dt'\eta(t')\right) \int_{t}^{t+\epsilon} dt_1 dt_2 dt_3 dt_4\, \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) + \dots\ \right\}.
\end{aligned} \tag{20.233}$$

Using the correlation functions (20.155)–(20.262) we obtain

$$\begin{aligned}
\psi(x,t+\epsilon) =& \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] \\
&\times \left[-\epsilon c_1 \partial_x + \left(\epsilon c_2 + \epsilon^2 c_1\right)\frac{1}{2}\partial_x^2 - \left(\epsilon c_3 + 3\epsilon^2 c_2 c_1\right)\frac{1}{3!}\partial_x^3\right. \\
&+ \left.\left(\epsilon c_4 + \epsilon^2 4 c_3 c_1 + \epsilon^2 3 c_2^2 + \epsilon^3 c_2 c_1^2 + \epsilon^4 c_1^2\right)\frac{1}{4!}\partial_x^4 + \dots\right] \psi\left(x - \int_{t_a}^{t} dt'\eta(t')\right).
\end{aligned} \tag{20.234}$$

In the limit $\epsilon \to 0$, only the linear terms in ϵ contribute, which are all due to the connected parts of the correlation functions of $\eta(t)$. The differential operators in the brackets can now be pulled out of the integral and we find the differential equation

$$\partial_t \psi(x,t) \;=\; \left[-c_1\partial_x + \frac{c_2}{2!}\partial_x^2 - \frac{c_3}{3!}\partial_x^3 + \frac{c_4}{4!}\partial_x^4 + \dots\right] \psi\,(x,t)\,. \tag{20.235}$$

We now replace $c_1 \to r_x$ and express using (20.141) the differential operators in brackets as Hamiltonian operator $-H_{r_x}(-i\partial_x)$. This leads to the Schrödinger-like equation [55]

$$\partial_t \psi(x,t) \;=\; -H_{r_x}(-i\partial_x)\,\psi\,(x,t)\,. \tag{20.236}$$

Due to the many derivatives in $H(i\partial_x)$, this equation is in general non-local. This can be made explicit with the help of the Lévy-Khintchine weight function $F(x)$ in the Fourier representation (20.173). In this case, the right-hand side

$$-H(-i\partial_x)\psi(x,t) = \int dx'\, e^{-x'\partial_x}\, F(x')\psi(x,t) = \int dx'\, F(x')\psi(x-x',t), \tag{20.237}$$

and the Fokker-Planck-like equation (20.236) takes the form of an integro-differential equation. Some people like to use the subtracted form (20.175) of the Lévy-Khintchine formula and arrive at the integro-differential equation

$$\partial_t\psi(x,t) = \left[-c_1\partial_x - \frac{c_2}{2}\partial_x^2\right]\psi(x,t) + \int dx'\, \bar{F}(x')\psi(x-x',t). \tag{20.238}$$

The integral term can then be treated as a perturbation to an ordinary Fokker-Planck equation.

The initial condition of the probability is, of course, always given by (20.187), as a consequence of the semigroup property (20.186).

20.1.24 Kramers-Moyal Equation

The Fokker-Planck-type equation (20.231) is a special case of the most general time evolution equation satisfied by a probability distribution $P(x_b, t_b, |x_a t_a)$ which fulfills the Chapman-Kolmogorov or Smoluchowski equation (20.186). For a short time interval $t_c - t_b = \epsilon$, that equation can be written as

$$P(x_c\, t_b{+}\epsilon|x_a t_a) = \int_{-\infty}^{\infty} dx_b\, P(x_c t_b + \epsilon|x_b t_b)P(x_b t_b|x_a t_a). \tag{20.239}$$

We now re-express $P(x_c t_b{+}\epsilon|x_b t_b)$ as a trivial integral

$$P(x_c t_b{+}\epsilon|x_b t_b) = \int dx\, \delta(x{-}x_b{+}x_b{-}x_c)P(x\, t_b{+}\epsilon|x_b t_b), \tag{20.240}$$

and expand the δ-function in powers of $x - x_b$ to obtain

$$P(x_c t_b{+}\epsilon|x_b t_b) = \int dx \sum_{n=0}^{\infty} \frac{(x-x_b)^n}{n!} P(x\, t_b{+}\epsilon|x_b t_b)\, \partial_{x_b}^n \delta(x_b - x_c). \tag{20.241}$$

Introducing the moments

$$C_n(x_b t_b) \equiv \int dx\, (x-x_b)^n \partial_{t_b} P(x\, t_b|x_b t_b), \tag{20.242}$$

and inserting (20.241) into Eq. (20.239) the right-hand side reads

$$P(x_c t_b|x_b t_b) + \epsilon \sum_{n=1}^{\infty} \int dx_b\, \frac{C_n(x_b t_b)}{n!} [\partial_{x_b}^n \delta(x_b - x_c)]\partial_{t_b} P(x_b t_b|x_a t_a) + \mathcal{O}(\epsilon^2). \tag{20.243}$$

Taking the $n = 0$ -term to the left-hand side of (20.239) and going to the limit $\epsilon \to 0$, we obtain the *Kramers-Moyal equation*:

$$\partial_{t_b} P(x_b t_b | x_a t_a) = -H(-i\partial_{x_b}, x_b, t_b) P(x_b t_b | x_a t_a). \tag{20.244}$$

with the Hamiltonian operator

$$H(-i\partial_{x_b}, x_b, t_b) \equiv -\sum_{n=1}^{\infty} (-\partial_{x_b})^n \frac{C_n(x_b t_b)}{n!}, \tag{20.245}$$

which is the generalization of the Hamiltonian operator in Eqs. (20.235) and (20.231).

There is a simple formal solution of the time evolution equation (20.244) with the initial condition (20.187). It can immediately be written down by recalling the similar time evolution equation (1.233):

$$P(x_b t_b | x_a t_a) = \hat{T} e^{-\int_{t_a}^{t_b} dt\, H(-i\partial_{x_b}, x_b, t)} \delta(x_b - x_a), \tag{20.246}$$

where $\hat{T}$ is the time-ordering operator (1.241). Using Dirac bra-ket states $\langle x_b|$ and $|x_a\rangle$ with the scalar product $\langle x_b | x_a \rangle = \delta(x_b - x_a)$ as in Eq. (18.462), and the rules (18.463), we can rewrite (20.246) as

$$P(x_b t_b | x_a t_a) = \langle x_b | \hat{T} e^{-\int_{t_a}^{t_b} dt\, H(-i\partial_{x_b}, x_b, t)} | x_a \rangle. \tag{20.247}$$

Due to the initial condition (20.187), the moments (20.242) can also be calculated from the short-time limit

$$C_n(x\, t) \underset{\epsilon \to 0}{=} \frac{1}{\epsilon} \int dx\, (x' - x)^n P(x'\, t{+}\epsilon | x\, t) \underset{\epsilon \to 0}{=} \frac{1}{\epsilon} \langle [x(t{+}\epsilon) - x(t)]^n \rangle, \ \ n \geq 1. \tag{20.248}$$

Another way of writing the expectation value on the right-hand side is

$$C_n(x\, t) \underset{\epsilon \to 0}{=} \frac{1}{\epsilon} \int_t^{t+\epsilon} dt_1 \cdots \int_t^{t+\epsilon} dt_n \ \langle \dot{x}(t_1) \cdots \dot{x}(t_n) \rangle, \quad n \geq 1. \tag{20.249}$$

If $x(t)$ follows the Langevin equation (20.169) with the correlation functions (20.155)–(20.262), we obtain

$$C_n(x\, t) = c_n, \tag{20.250}$$

and the Kramers-Moyal equation (20.248) reduces to out previous Fokker-Planck-type equation (20.231).

For a Langevin equation

$$\dot{x}(t) = r(x,t) + \sigma(x,t)\eta(t), \tag{20.251}$$

with a white noise variable $\eta(t)$, we find instead

$$C_1(x\, t) {=} a(x,t) {+} b'(x,t) b(x,t) {=} a(x,t) {+} \frac{\partial_x C_2(x\, t)}{2}, \ C_2(x\, t) = b^2(x,t). \tag{20.252}$$

According to a theorem due to Pawula [56], the positivity of the probability in the Kramers-Moyal equation (20.244) requires that the expansion (20.245) can only be truncated after the first or second terms, or must go on to infinity.

20.2 Itō-like Formula for Non-Gaussian Distributions

By a procedure similar to that in Subsection 18.13.3 it is possible to derive a generalization of the Itō-like expansions (18.412) and (18.432) for the fluctuations of functions containing non-Gaussian noise.

20.2.1 Continuous Time

As in (18.408) we expand $f(x(t+\epsilon))$:

$$
\begin{aligned}
f(x(t+\epsilon)) &= f(x(t)) + f'(x(t))\Delta x(t) \\
&\quad + \frac{1}{2} f''(x(t))[\Delta x(t)]^2 + \frac{1}{3!} f^{(3)}[\Delta x(t)]^3 + \dots \,. \qquad (20.253)
\end{aligned}
$$

where $\dot{x}(t) = \eta(t)$ is the stochastic differential equation with a nonzero expectation value $\langle \eta(t) \rangle = c_1$. In contrast to the expansion (18.426), which for $\epsilon \to 0$ had to be carried only up to the second order in $\Delta x(t) \equiv \int_t^{t+\epsilon} dt' \, \dot{x}(t')$, we must now keep *all* orders. Evaluating the noise averages of the multiple integrals on the right-hand side with the help of the correlation functions (20.155)–(20.262), we find the time dependence of the expectation value of an arbitrary function of the fluctuating variable $x(t)$

$$
\begin{aligned}
\langle f(x(t+\epsilon)) \rangle &= \langle f(x(t)) \rangle + \langle f'(x(t)) \rangle \epsilon c_1 + \frac{1}{2} \langle f''(x(t)) \rangle (\epsilon c_2 + \epsilon^2 c_1^2) \\
&\quad + \frac{1}{3!} \langle f^{(3)}(x(t)) \rangle (\epsilon c_3 + \epsilon^2 c_2 c_1 + \epsilon^3 c_1^3) + \dots \qquad (20.254) \\
&= \langle f(x(t)) \rangle + \epsilon \left[c_1 \partial_x + c_2 \frac{1}{2} \partial_x^2 + c_3 \frac{1}{3!} \partial_x^3 + \dots \right] \langle f(x(t)) \rangle + \mathcal{O}(\epsilon^2).
\end{aligned}
$$

After the replacement $c_1 \to r_x$, the function $f(x(t))$ obeys therefore the following equation:

$$
\langle \dot{f}(x(t)) \rangle = -H_{r_x}(i\partial_x) \langle f(x(t)) \rangle. \qquad (20.255)
$$

Separating the lowest-derivative term, this takes a form generalizing Eq. (18.412):

$$
\langle \dot{f}(x(t)) \rangle = \langle f'(x(t)) \rangle \langle \dot{x}(t) \rangle - \bar{H}_{r_x}(i\partial_x) \langle f(x(t)) \rangle. \qquad (20.256)
$$

This might be viewed as the expectation value of the stochastic differential equation

$$
\dot{f}(x(t)) = f'(x(t)) \dot{x}(t) - \bar{H}_{r_x}(i\partial_x) f(x(t)), \qquad (20.257)
$$

which, if valid, would be a simple direct generalization of Itō's Lemma (18.413). However, this conclusion is not allowed. The reason lies in the increased size of the fluctuations of the higher expansion terms $[\Delta x(t)]^n$ for $n \geq 2$. In the harmonic case of Subsection 18.13.3, these were negligible in comparison with the leading term $z_1(t)$ in Eq. (18.413). Let us see what happens in the present case, where all higher cumulants c_n in the Hamiltonian (20.141) may be nonzero. For the argument we proceed as in Eq. (18.405) by working with the stochastic differential equation

$$\dot{x}(t) = \langle \dot{x}(t) \rangle + \eta(t) = c_1 + \eta(t). \tag{20.258}$$

rather that (20.169). Then the correlation functions of $\eta(t)$ consist only of the connected parts of (20.155)–(20.262):

$$\begin{aligned}
\langle \eta(t_1) \rangle &= c_1, && (20.259)\\
\langle \eta(t_1)\eta(t_2) \rangle &= c_2\delta(t_1 - t_2) && (20.260)\\
\langle \eta(t_1)\eta(t_2)\eta(t_3) \rangle &= c_3\delta(t_1-t_2)\delta(t_1-t_3), && (20.261)\\
\langle \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) \rangle &= c_4\delta(t_1-t_2)\delta(t_1-t_3)\delta(t_1-t_4) && (20.262)\\
&+ c_2^2[\delta(t_1-t_2)\delta(t_3-t_4)+\delta(t_1-t_3)\delta(t_2-t_4)+\delta(t_1-t_4)\delta(t_2-t_3)].\\
&\ \vdots\ \ .
\end{aligned}$$

We now estimate the size of the fluctuations z_n of $[\Delta x(t)]^n$. The first contribution $z_{2,1}$ to z_2 defined in Eq. (18.415) is still negligible since it is smaller than the leading one $z_1(t) \equiv \int_t^{t+\epsilon} dt'\, \eta(t')$ by a factor ϵ. The second contribution $z_{2,3}$ to z_2 defined in Eq. (18.415), however, has now a larger variance, due to the c_4-term in (20.262), which contains one more δ-function than the c_2^2-term, so that

$$\left\langle [z_{2,2}(t)]^2 \right\rangle = \int_t^{t+\epsilon} dt_1 \int_t^{t+\epsilon} dt_2 \int_t^{t+\epsilon} dt_3 \int_t^{t+\epsilon} dt_4 \, \langle \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) \rangle = \epsilon c_4 + \epsilon^2 c_2^2. \tag{20.263}$$

This is larger than the harmonic estimate (18.420) by a factor $1/\epsilon$, which makes $z_{2,2}(t)$, in general, as large as the leading fluctuation $z_1(t)$ of $x_1(t)$. It can therefore *not* be ignored. The subtraction of the second term $\langle z_{2,2}(t) \rangle^2 = \epsilon^2 c_2^2$ in the variance does not help since that is ignorable.

A similar estimate holds for the higher powers. Take for instance $[\Delta x(t)]^3$:

$$[\Delta x(t)]^3 \;=\; \epsilon^3 c_1^3 + 3\epsilon^2 c_1^2 z_1(t) + \epsilon c_1 [z_1(t)]^2 + [z_1(t)]^3, \tag{20.264}$$

and calculate the variance of the strongest fluctuating last term in this: $\langle \{[z_1(t)]^3\}^2 \rangle - \langle [z_1(t)]^3 \rangle^2 \rangle$. The first contribution is equal to

$$\int_t^{t+\epsilon} dt_1 \int_t^{t+\epsilon} dt_2 \int_t^{t+\epsilon} dt_3 \int_t^{t+\epsilon} dt_4 \int_t^{t+\epsilon} dt_5 \int_t^{t+\epsilon} dt_6 \, \langle \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4)\eta(t_5)\eta(t_6) \rangle, \tag{20.265}$$

and becomes, after an obvious extension of the expectation values (20.155)–(20.262) to the six-point function:

$$\langle \{[z_1(t)]^3\}^2 \rangle = \epsilon c_6 + \mathcal{O}(\epsilon^2). \tag{20.266}$$

The second contribution $\langle [z_1(t)]^3 \rangle^2$ in (20.264) is equal to $\epsilon^2 c_3^2$ and can be ignored for $\epsilon \to 0$. Thus, due to (20.266), the size of the fluctuations $[z_1(t)]^3$ is of the same order as of the leading one $x_1(t)$ [compare again (18.421)].

This is an important result which is the reason why we cannot derive a direct generalization (20.257) of the Itō formula (18.413) to non-Gaussian fluctuations, but only the weaker formula (20.256) for the expectation value.

For an exponential function $f(x) = e^{Px}$ this implies the relation

$$\frac{d}{dt}\langle e^{Px(t)}\rangle = \left[P\langle \dot{x}(t)\rangle - \bar{H}_{r_x}(iP)\right]\left\langle e^{Px(t)}\right\rangle, \tag{20.267}$$

and it is not allowed to drop the expectation values.

A consequence of the weaker Eq. (20.267) for $P = 1$ is that the rate r_S with which the average of the stock price $S(t) = e^{x(t)}$ grows is given by formula (20.1), where the rate r_x is related to r_S by

$$r_S = r_x - \bar{H}(i) = r_x - [H(i) - iH'(0)] = -H_{r_x}(i), \tag{20.268}$$

which replaces the simple Itō relation $r_S = r_x + \sigma^2/2$ in Eq. (20.5). Recall the definition of $\bar{H}(p) \equiv H(p) - H'(0)p$ in Eq. (20.141). The corresponding version of the left-hand part of Eq. (20.4) reads

$$\left\langle \frac{\dot{S}}{S}\right\rangle = \langle \dot{x}(t)\rangle - \bar{H}(i) = \langle \dot{x}(t)\rangle - [H(i) - iH'(0)] = \langle \dot{x}(t)\rangle - r_x - H_{r_x}(i). \tag{20.269}$$

The forward price of a stock must therefore be calculated with the generalization of formula (20.8), in which we assume again $\eta(t)$ to fluctuate around zero rather than r_x:

$$\langle S(t)\rangle = S(0)e^{r_S t} = S(0)\langle e^{r_x t + \int_0^t dt'\,\eta(t')}\rangle = S(0)e^{-H_{r_x}(i)t} = S(0)e^{\{r_x t - [H(i) - iH'(0)]t\}}. \tag{20.270}$$

If $\eta(t)$ fluctuates around r_x this takes the simpler form

$$\langle S(t)\rangle = S(0)e^{r_S t} = S(0)\langle e^{\int_0^t dt'\,\eta(t')}\rangle = S(0)e^{-H_{r_x}(i)t}. \tag{20.271}$$

This result may be viewed as a consequence of the following generalization of the Gaussian Debye-Waller factor (18.425):

$$\left\langle e^{P\int_0^t dt'\,\eta(t')}\right\rangle = e^{-H_{r_x}(iP)t}. \tag{20.272}$$

Note that we may derive the differential equation (20.255) of an arbitrary function $f(x(t))$ from a simple mnemonic rule, expanding sloppily [similar to Eq. (18.426) but restricted to the expectation values]

$$\begin{aligned}\langle f(x(t+dt))\rangle &= \langle f(x(t))\rangle + \langle f'(x(t))\rangle\,\langle \dot{x}(t)\rangle\,dt + \frac{1}{2}\,\langle f''(x(t))\rangle\left\langle \dot{x}^2(t)\right\rangle dt^2\\ &\quad + \frac{1}{3!}\langle f^{(3)}(x(t))\rangle\langle \dot{x}^3(t)\rangle dt^3 + \dots\,,\end{aligned} \tag{20.273}$$

and replacing

$$\langle \dot{x}(t)\rangle dt \to c_1 dt, \quad \langle \dot{x}^2(t)\rangle dt^2 \to c_2 dt, \quad \langle \dot{x}^3(t)\rangle dt^3 \to c_3 dt, \dots\,. \tag{20.274}$$

In contrast to Eq. (18.429), this replacement holds now only on the average.

For the same reason, portfolios containing assets with non-Gaussian fluctuations cannot be made risk-free in the continuum limit $\epsilon \to 0$, as will be seen in Section 20.7.

20.2.2 Discrete Times

The prices of financial assets are recorded in the form of a discrete time series $x(t_n)$ in intervals $\Delta t = t_{n+1} - t_n$, rather than the continuous function $x(t)$. For stocks with a large turnover, or for market indices, the smallest time interval is typically $\Delta t = 1\,\text{minute}$. We have seen at the end of Section 18.13.3 that this makes Itō's Lemma (18.413) an approximate statement. Without the limit $\Delta t \to 0$, the fluctuations of the higher expansion terms in (20.253) no longer disappear but are merely suppressed by a factor $\sigma\sqrt{\Delta t_n}$. In quiet economic periods this is usually quite small for $\Delta t = 1\,\text{minute}$, so that the higher-order fluctuations can usually be neglected after all.

While this approximate validity is a disadvantage of the discrete time series over the continuous one, it is an advantage with respect to processes with non-Gaussian noise, at least as long as the financial markets are not in turmoil. Then the non-Gaussian higher-order fluctuations of the expansion terms in (20.253) are suppressed by the same order $\sigma\sqrt{\Delta t}$ as the corrections to Itō's rule in the Gaussian case [recall (18.432)]. As a typical example, take the Boltzmann distribution (20.77). Its cumulants c_n carry a factor T^n [see Eq. (20.80)] where the market temperature T is a small number of the order of a few percent in the natural time units of one minute used in this context [see Fig. (20.11)]. The smallness of T is, of course, a consequence that the minute data do not usually possess large volatility, except near a crash. In the natural units of minutes, the time interval ϵ in the calculations after Eq. (20.257) is unity, so that powers of ϵ can no longer be used for size estimates. This role is now taken over by T. In fact, since c_n is of order T^n, the fluctuations of $z_n(t)$ are of the order T^n. Thus, in non-Gaussian fluctuations of a discrete time series, the smallness of T leads to a suppression of the higher fluctuations after all. Moreover, the suppression is just as good as for time series with Gaussian fluctuations, where the corrections are of the order $(\sigma\sqrt{\Delta t})^{n-1}$ for $z_n(t)$ with $n \geq 2$. These have the size of T^{n-1} for the Boltzmann distribution [recall (20.198)]. A similar estimate holds for all non-Gaussian noise distributions with semi-heavy tails as defined on at the end of Subsection 20.1.1.

For all semi-heavy tails for which the cumulants c_n decrease with power T^n of a small parameter such as the temperature T, we may drop the expectations values of the Itō-like expansion (20.256), and remain with an approximate Itō formula for the fluctuating difference $\Delta f(x(t_n)) \equiv f(x(t_{n+1}))$:

$$\Delta f(x(t_n)) = f'(x(t_n))\Delta x(t_n) - \bar{H}_{r_x}(i\partial_x) f(x(t_n)) + \mathcal{O}(\sigma\sqrt{\Delta t}), \qquad (20.275)$$

which is a direct generalization of the discrete version (18.432) of Itō's Lemma (18.413).

A characteristic property of the Boltzmann distribution (20.77) and others with semi-heavy tails is that their Hamiltonian possesses a power series in p^2 of the type (20.141) with finite cumulants c_n of the order of σ^n, where σ is the volatility of the minute data. This property is violated only by power tails of price distributions which do exist in the data (see Fig. 20.10). These do not go away upon multiple convolution which turns the Boltzmann distribution more and more into a Gaussian as required by the central limit theorem (20.167). These heavy tails are caused by drastic price changes in short times observed in nervous markets near a crash. If these are taken into account, the discrete version (18.432) of Itō's Lemma (18.413) is no longer valid. This will be an obstacle to setting up a risk-free portfolio in Section 20.7.

20.3 Martingales

In financial mathematics, an often-encountered concept is that of a *martingale*. The name stems from a casino strategy in which a gambler doubles his stake each time a bet is lost. A stochastic variable $m(t)$ is called a martingale, if its expectation value is time-independent.

A trivial martingale is provided by any noise variable $m(t) = \eta(t)$.

20.3.1 Gaussian Martingales

For a harmonic noise variable, the exponential $m(t) = e^{\int_0^t dt' \eta(t') - \sigma^2 t/2}$ is a nontrivial martingale, due to Eq. (20.8). For the same reason, a stock price $S(t) = e^{x(t)}$ with $x(t)$ obeying the stochastic differential equation (20.140) can be made a martingale by a time-dependent multiplicative factor associated with the average growth rate r_S, i.e.,

$$e^{-r_S t} S(t) = e^{-r_S t} e^{x(t)} = e^{-r_S t} e^{r_x t + \int^t dt'\, \eta(t')} \tag{20.276}$$

is a martingale. The prefactor $e^{-r_S t}$ is referred to as a *discount factor* with the rate r_S. If we calculate the probability distribution (20.170) associated with the stochastic differential equation (20.140), which for harmonic fluctuations of standard deviation σ corresponds to a Hamiltonian

$$H(p) = i r_x p + \frac{\sigma^2}{2} p^2, \tag{20.277}$$

The integral representation (20.170) for the probability distribution,

$$P^{r_x}(x_b t_b | x_a t_a) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left[ip(x_b - x_a) - \Delta t \left(i r_x p + \sigma^2 p^2/2 \right) \right] \tag{20.278}$$

with $\Delta t = t_b - t_a$ has obviously a time-independent expectation value of $e^{-r_S t} S(t) = e^{-r_S t} e^{x(t)}$. Indeed, the expectation value at the time t_b is given by the integral over x_b

$$e^{-r_S t_b} \int dx_b\, e^{x_b} P^{r_x}(x_b t_b | x_a t_a) = e^{-r_S t_b} \int dx_b\, e^{x_b} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left[ip(x_b - x_a) - \Delta t\, \sigma^2 p^2/2 \right],$$

which yields

$$e^{-r_S t_b} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \delta(p - i) e^{-ipx_a} e^{\Delta t\, \sigma^2 p^2/2} = e^{-r_S t} e^{x_a} e^{(r_x - \sigma^2/2)\Delta t} = e^{-r_S t_a} e^{x_a}, \tag{20.279}$$

where we have used the Itō relation $r_S = r_x + \sigma^2/2$ of Eq. (20.5). The result implies that

$$\langle e^{-r_S t_b} S(t_b)\rangle^{r_x} = \langle e^{-r_S t_b} e^{x(t_b)}\rangle^{r_x} = e^{-r_S t_a} S(t_a). \tag{20.280}$$

Since this holds for all t_b we may drop the subscripts b, thus proving the time independence and thus the martingale nature of $e^{-r_S t} S(t)$.

In mathematical finance, where the expectation value in Eq. (20.280) is written according to the definition (20.189) as $\mathbb{E}[e^{-r_S(t_b-t_a)} e^{x_b}|x_a t_a]$, the martingale property of $e^{-r_S(t_b-t_a)} S(t)$ is expressed as

$$\mathbb{E}[e^{-r_S(t_b-t_a)} S(t_b)|x_a t_a] = S(t_a) \tag{20.281}$$

or as $\mathbb{E}[e^{-r_S(t_b-t_a)} e^{x_b}|x_a t_a] = e^{x_a}$.

For the martingale $f(x_b, t_b) = e^{-r_S(t_b-t_a)} e^{x_b}$, the expectation values on the right-hand side of the towering formula (20.191) reduces with the help of (20.281) and (20.192) to

$$\mathbb{E}[\mathbb{E}[f(x_b, t_b)|x\, t]|x_a t_a] = \mathbb{E}[\mathbb{E}[f(x, t)|x\, t]|x_a t_a] = \mathbb{E}[f(x, t)|x_a t_a]. \tag{20.282}$$

Using once more the relations (20.281) and (20.192) to leads to

$$\mathbb{E}[\mathbb{E}[f(x_b, t_b)|x\, t]|x_a t_a] = f(x_a, t_a). \tag{20.283}$$

Performing the momentum integral in (20.278) gives the explicit distribution

$$P^{r_x}(x_b t_b|x_a t_a) \equiv \frac{1}{\sqrt{2\pi\sigma^2(t_b-t_a)}} \exp\left\{-\frac{[x_b - x_a - r_x(t_b-t_a)]^2}{2\sigma^2(t_b-t_a)}\right\}. \tag{20.284}$$

We may incorporate the discount factor $e^{-r_S(t_b-t_a)}$ into the probability distribution (20.284) and define a martingale distribution for the stock price

$$P^{(M,r_x)}(x_b t_b|x_a t_a) = e^{-r_S(t_b-t_a)} P^{r_x}(x_b t_b|x_a t_a), \tag{20.285}$$

whose normalization falls off like $e^{-r_S(t_b-t_a)}$. If we define expectation values with respect to $P^{M,r_x}(x_b t_b|x_a t_a)$ by the integral (without normalization)

$$\langle f(x_b)\rangle^{(M,r_x)} \equiv \int dx_b\, f(x_b) P^{(M,r_x)}(x_b t_b|x_a t_a), \tag{20.286}$$

then the stock price itself is a martingale:

$$\langle e^{x_b}\rangle^{(M,r_x)} = e^{x_a}. \tag{20.287}$$

Note that there exists an entire family of equivalent martingale distributions

$$P^{(M,r)}(x_b t_b|x_a t_a) = e^{-r\Delta t} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left[ip(x_b - x_a) - \Delta t\, H_r(p)\right], \tag{20.288}$$

with an *arbitrary* rate r and $r_x \equiv r + \bar{H}(i)$. Indeed, multiplying this with e^{x_b} and integrating over x_b gives rise to a δ-function $\delta(p-i)$ and produces the same result e^{x_a} for any time difference $\Delta t = t_b - t_a$. Performing the integral yields

$$P^{(M,r)}(x_b t_b|x_a t_a) \equiv \frac{e^{-r(t_b-t_a)}}{\sqrt{2\pi\sigma^2(t_b-t_a)}} \exp\left\{-\frac{[x_b - x_a - r(t_b-t_a)]^2}{2\sigma^2(t_b-t_a)}\right\}. \tag{20.289}$$

These martingale distributions are called *risk-neutral.*

20.3.2 Non-Gaussian Martingale Distributions

For $S(t) = e^{x(t)}$ with an arbitrary non-Gaussian noise $\eta(t)$, there are many ways of constructing distributions which make the stock price a martingale.

Natural Martingale Distribution

The relation (20.268) allows us to write down immediately the simplest martingale distribution. It is given by an obvious generalization of the Gaussian expression (20.276):

$$e^{-r_S t} S(t) = e^{-r_S t} e^{r_x t + \int_0^t dt' \eta(t')} \tag{20.290}$$

where r_S and r_x are related by $r_S = r_x - \bar{H}(i) = -H_{r_x}(i)$. It is easy to prove this. The distribution function associated with the stochastic differential equation (20.140) with non-Gaussian fluctuations is given by

$$P^{r_x}(x_b t_b | x_a t_a) = \int \mathcal{D}\eta \int \mathcal{D}x \exp\left\{-\int_{t_a}^{t_b} dt\, \tilde{H}_{r_x}(\eta(t))\right\} \delta[\dot{x} - r_x(t_b - t_a)\eta], \tag{20.291}$$

where $\Delta t \equiv (t_b - t_a)$ and $r_x = r_S + \bar{H}(i)$. As explained in Subsection 20.1.23, the path integral is solved by the Fourier integral [compare (20.170)]

$$P^{r_x}(x_b t_b | x_a t_a) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left[ip(x_b - x_a) - \Delta t\, H_{r_x}(p)\right]. \tag{20.292}$$

Using this distribution we can again calculate the time independence of the expectation value (20.280), so that $P^{M,r_x}(x_b t_b | x_a t_a) = e^{-r_S(t_b - t_a)} P^{r_x}(x_b t_b | x_a t_a)$ is a martingale distribution which makes the stock price time-independent, as in Eq. (20.287).

Note that there exists an entire family of equivalent martingale distributions

$$P^{(M,r)}(x_b t_b | x_a t_a) = e^{-r\Delta t} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left[ip(x_b - x_a) - \Delta t\, H_{r_x}(p)\right], \tag{20.293}$$

with an *arbitrary* rate r and $r_x \equiv r + \bar{H}(i)$. Indeed, multiplying this with e^{x_b} and integrating over x_b gives rise to a δ-function $\delta(p - i)$ and produces the same result e^{x_a} for any time difference $\Delta t = t_b - t_a$.

For non-Gaussian fluctuations there exists an infinite set martingale distributions for the stock price of which we are now going to dicuss the one proposed by Esscher.

Esscher Martingales

In the literature on mathematical finance, much attention is paid to another family of equivalent martingale distributions. It has been used a long time ago to estimate risks of actuaries [67] and introduced more recently into the theory of option prices [68, 69] where it is now of wide use [70]–[76]. This family is constructed as follows. Let $\tilde{D}(z)$ be an arbitrary distribution function with a Fourier transform

$$\tilde{D}(z) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-H(p)} e^{ipz}, \tag{20.294}$$

and $H(0) = 0$, to guarantee a unit normalization $\int dz\tilde{D}(z) = 1$. We now introduce an *Esscher-transformed* distribution function. It is obtained by slightly tilting the initial distribution $\tilde{D}(z)$, multiplying it with an asymmetric exponential factor $e^{\theta z}$:

$$D^\theta(z) \equiv e^{H(i\theta)}\, e^{\theta z}\tilde{D}(z). \tag{20.295}$$

The constant prefactor $e^{H(i\theta)}$ is necessary to conserve the total probability. This distribution can be written as a Fourier transform

$$D^\theta(z) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{-H^\theta(p)}\, e^{ipz}, \tag{20.296}$$

with the Esscher-transformed Hamiltonian

$$H^\theta(p) \equiv H(p + i\theta) - H(i\theta). \tag{20.297}$$

Since $H^\theta(0) = 0$, the transformed distribution is properly normalized: $\int dzD^\theta(z) = 1$. We now define the Esscher-transformed expectation value

$$\langle F(z)\rangle^\theta \equiv \int dzD^\theta(z)F(z). \tag{20.298}$$

It is related to the original expectation value by

$$\langle F(z)\rangle^\theta \equiv e^{H(i\theta)}\,\langle e^{\theta z}F(z)\rangle. \tag{20.299}$$

For the specific function $F(z) = e^z$, Eq. (20.299) becomes

$$\langle e^z\rangle^\theta = e^{-H^\theta(i)} \equiv e^{H^\theta(i\theta)}\,\langle e^{(\theta+1)z}\rangle = e^{H^\theta(i\theta)-H^\theta(i\theta+i)}. \tag{20.300}$$

Applying the transformation (20.296) to each time slice in the general path integral (20.144), we obtain the *Esscher-transformed* path integral

$$P^\theta(x_bt_b|x_at_a) = e^{-r_S\Delta t}e^{H_{r_x}(i\theta)\Delta t}\int\mathcal{D}\eta\int\mathcal{D}x\exp\left\{\int_{t_a}^{t_b}dt\,\left[\theta\eta(t) - \tilde{H}_{r_x}(\eta(t))\right]\right\}\delta[\dot{x}-\eta], \tag{20.301}$$

where $\Delta t \equiv t_b - t_a$ is the time interval. This leads to the Fourier integral [compare (20.293)]

$$P^\theta(x_bt_b|x_at_a) = e^{-r_S\Delta t}e^{H_{r_x}(i\theta)\Delta t}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\exp\left[ip(x_b - x_a) - \Delta t\, H_{r_x}(p + i\theta)\right]. \tag{20.302}$$

Let us denote the expectation values calculated with this probability by $\langle\ \dots\ \rangle^\theta$. Then we find for $S(t) = e^{x(t)}$ the time dependence

$$\langle S(t)\rangle^\theta = e^{-H^\theta_{r_x}(i)t}. \tag{20.303}$$

This equation shows that the exponential of a stochastic variable $x(t)$ can be made a martingale with respect to any Esscher-transformed distribution if we remove the exponential growth factor $\exp(r^\theta \Delta t)$ with

$$r^\theta \equiv -H^\theta_{r_x}(i) = -H_{r_x}(i + i\theta) + H_{r_x}(i\theta). \tag{20.304}$$

Thus, a family of equivalent martingale distributions for the stock price $S(t) = e^{x(t)}$ is

$$P^{M\,\theta}(x_b t_b | x_a t_a) \equiv e^{-r^\theta t} P^\theta(x_b t_b | x_a t_a), \tag{20.305}$$

for any choice of the parameter θ.

For a harmonic distribution function (20.284), the Esscher martingales and the previous ones are equivalent. Indeed, starting from (20.284) in which $r_S = r_x + \sigma^2/2$, the Esscher transform leads, after a quadratic completion, to the family of natural martingales (20.284) with the rate parameter $r = r_x + \theta\sigma^2$.

Other Non-Gaussian Martingales

Many other non-Gaussian martingales have been discussed in the literature. Mathematicians have invented various sophisticated criteria under which one would be preferable over the others for calculating financial risks. Davis, for instance, has introduced a so-called *utility function* [77] which is supposed to select optimal martingales for different purposes. There exists also a so-called minimal martingale [83], but the mathematical setup in these discussions is hard to understand.

For the upcoming development of a theory of option pricing, only the initial *natural martingale* will be relevant.

20.4 Origin of Semi-Heavy Tails

We have indicated at the end of Subsection 20.1.1 that semi-heavy tails of the type used in the previous discussions may be viewed as a phenomenological description of a nontrivial Gaussian process with fluctuating volatility. This has been confirmed in Subsection 20.1.22 where we have expressed a number of non-Gaussian distributions as a superposition of Gaussian distributions. We have further seen that the weight functions $w_t(v)$ for the superpositions can be Laplace-transformed to find a Hamiltonian $H_v(p_v)$ from which we can derive which governs the noise distribution a stochastic differential equation for the volatility whose noise distribution is which governed by $H_v(ip_v)$.

There exist a simple solvable model due to Heston [78] which shows this explicitly.[8]

[8]The discussion in this section follows a paper by Drăgulescu and Yakovenko [79].

20.4.1 Pair of Stochastic Differential Equations

Starting point is a coupled pair of ordinary stochastic differential equations with Gaussian noise, one for the fluctuating logarithm $x(t)$ of the stock price, and one for the fluctuating time-dependent variance $\sigma^2(t)$. Since this will appear frequently in the upcoming discussion, we shall name it $v(t)$. For simplicity, we remove the growth rate of the share price. Replacing σ^2 in the stochastic differential equation (20.4) by the time-dependent variance $v(t)$ we obtain

$$\dot{x}(t) = -\frac{v(t)}{2} + \sqrt{v(t)}\eta(t), \tag{20.306}$$

where the noise variable $\eta(t)$ has zero average and *unit* volatility

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \delta(t-t'). \tag{20.307}$$

The variance is assumed to satisfy an equation with another noise variable $\eta_v(t)$ [7]

$$\dot{v}(t) = -\gamma[v(t) - \bar{v}] + \varepsilon\sqrt{v(t)}\eta_v(t). \tag{20.308}$$

The time in $\sqrt{v(t)}$ is supposed to lie slightly *before* the time in the noise $\eta_v(t)$. The parameter ε determines the volatility of the variance. The parameter $\bar{v}$ will become the average of the variance at large times. It is proportional to the variance σ^2 of the Gaussian distribution to which the distribution converges in the long-time limit by the central limit theorem. The precise relation will be given in Eq. (20.363).

In general, the noise variables $\eta(t)$ and $\eta_v(t)$ are expected to exhibit correlations which are accounted for by introducing an independent noise variable $\eta'(t)$ and setting

$$\eta_v(t) \equiv \rho\,\eta(t) + \sqrt{1-\rho^2}\eta'(t). \tag{20.309}$$

The pair correlation functions are then

$$\langle \eta(t)\eta(t') \rangle = \delta(t-t'), \quad \langle \eta(t)\eta_v(t') \rangle = \rho\delta(t-t'), \quad \langle \eta_v(t)\eta_v(t') \rangle = \delta(t-t'). \tag{20.310}$$

20.4.2 Fokker-Planck Equation

If $\boldsymbol{\eta}(t)$ denotes the pair of noise variables $(\eta(t), \eta_v(t))$, the probability distribution analogous to (18.324) for paths $x(t)$, $v(t)$ starting out at x_a, v_a and arriving at x, v is

$$P_{\boldsymbol{\eta}}(x\,v\,t\,|x_a v_a t_a) = \delta(x_\eta(t) - x)\delta(v_{\eta_v}(t) - v). \tag{20.311}$$

The time evolution of this is calculated using the differentiation rule (18.381) as

$$\partial_t P_{\boldsymbol{\eta}}(x\,v\,t\,|x_a v_a t_a) = -\left[\partial_x \dot{x}_\eta(t) + \partial_v \dot{v}_{\eta_v}(t)\right] P_{\boldsymbol{\eta}}(x\,v\,t\,|x_a v_a t_a), \tag{20.312}$$

or more explicitly

$$\partial_t P_{\boldsymbol{\eta}}(x\,v\,t\,|x_a v_a t_a) = \Bigg\{\partial_x \left[\frac{v(t)}{2} - \sqrt{v(t)}\eta(t)\right] + \partial_v \left[\gamma\,(v(t)-\bar{v}) - \varepsilon\sqrt{v(t)}\eta_v(t)\right]\Bigg\} P_{\boldsymbol{\eta}}(x\,v\,t\,|x_a v_a t_a). \quad (20.313)$$

We now take the noise average (18.325) with the distribution

$$P[\boldsymbol{\eta}] = \exp\left\{-\frac{1}{2}\int dt\left[\eta^2(t)+\eta'^2(t)\right]\right\} = \exp\left\{-\frac{1}{1-\rho^2}\int dt\left[\eta^2(t)+\eta_v^2(t)-2\rho\,\eta\eta_v\right]\right\}. \quad (20.314)$$

Applying the rules (18.385) and (18.386), we may replace $\eta(t)$ and $\eta_v(t)$ on the right-hand side of (20.313):

$$\eta(t) \;\to\; -\delta/\delta\eta(t) - \rho\,\delta/\delta\eta_v(t) \quad (20.315)$$

$$\eta_v(t) \;\to\; -\rho\,\delta/\delta\eta(t) - \delta/\delta\eta_v(t). \quad (20.316)$$

The functional derivatives are evaluated by analogy with (18.388) as

$$\frac{\delta}{\delta\eta(t')}\delta(x_\eta(t)-x)\delta(v_{\eta_v}(t)-v) = -\left[\frac{\delta x_\eta(t)}{\delta\eta(t')}\partial_x + \frac{\delta v_{\eta_v}(t)}{\delta\eta(t')}\partial_v\right]\delta(x_\eta(t)-x)\delta(v_{\eta_v}(t)-v), \quad (20.317)$$

with a similar equation for $\delta/\delta\eta_v(t)$. Under the above-made assumption that $\sqrt{v(t)}$ lies before $\eta_v(t)$, we find the evolution equation

$$\partial_t P(x\,v\,t\,|x_a v_a t_a) = -\hat{H}\,P(x\,v\,t\,|x_a v_a t_a), \quad (20.318)$$

where $\hat{H}$ is the Hamiltonian operator

$$\hat{H} = -\frac{1}{2}\partial_x^2\,v - \frac{1}{2}\partial_x\,v - \frac{\varepsilon^2}{2}\partial_v^2\,v - \gamma\partial_v(v-\bar{v}) - \rho\varepsilon\partial_x\partial_v\,v. \quad (20.319)$$

The probability distribution can thus be written as a path integral

$$P(x_b v_b t_b|x_a v_a t_a) = \int\mathcal{D}x\int\frac{\mathcal{D}p}{2\pi}\int\mathcal{D}v\int\frac{\mathcal{D}p_v}{2\pi} \times \exp\left\{\int_{t_a}^{t_b} dt\,\left[i(p\dot{x}+p_v\dot{v}) - H(p,p_v,v)\right]\right\}, \quad (20.320)$$

with the Hamiltonian

$$H(p,p_v,v) = \frac{p^2}{2}v - i\frac{1}{2}pv + \varepsilon^2\frac{p_v^2}{2}v - i\gamma p_v(v-\bar{v}) + \rho\varepsilon\,pp_v\,v - \frac{3\varepsilon^2}{4}ip_v - \frac{\gamma}{2} - \frac{i}{2}\rho\epsilon p. \quad (20.321)$$

The last two terms arise from the fact that the order of the operators in $\hat{H}$ obtained from the path integral is always symmetric in v and p_v, such that the order in Eq. (20.319) is reached only after the replacement of $p_v v \to (1/2)\{\hat{p}_v, v\} = \hat{p}_v v + i/2$

in the fourth and fifth terms of (20.321). Recall the discussions in Sections 10.5, 11.3, and Subsection 18.9.2.

The third-last term in (20.321) is present to ensure the appearance of the operator $\partial_v^2\, v$ in (20.319) with no extra ∂_v due to operator ordering. The term $p_v^2 v$ in (20.321) can be understood as a kinetic term $g_{\mu\nu}p^\mu p^\nu$ in a one-dimensional Riemannian space with a metric $g^{\mu\nu} = \delta^{\mu\nu}/v$. According to Subsection 11.1.1, this turns into the Laplace-Beltrami operator $\Delta = \sqrt{v}\partial_v^2\sqrt{v}\partial_v = v\partial_v^2 + \frac{1}{2}\partial_v = \partial_v^2 v - \frac{3}{2}\partial_v$. The extra ∂_v is canceled by the third-last term in (20.319).

The stochastic differential equation (20.308) can, in principle, lead to negative variances. Since this would be unphysical, we must guarantee that this does not happen. The condition for this is that the fluctuation width of the variance is sufficiently small, satisfying

$$\frac{\varepsilon^2}{2\gamma\bar{v}} \leq 1. \tag{20.322}$$

Then an initially positive $v(t)$ will always stay positive. Indeed, consider the partial differential equation (20.318) for $v = 0$:

$$(\partial_t + \alpha\partial_v)\, P(x\, v\, t\,|x_a v_a t_a) = (\gamma + \rho\varepsilon\partial_x)\, P(x\, v\, t\,|x_a v_a t_a), \tag{20.323}$$

where $\alpha \equiv \gamma\bar{v} - \varepsilon^2/2$. It is solved by some function

$$P(x\, v\, t\,|x_a v_a t_a) = f(v - \alpha t, x), \tag{20.324}$$

which shows that a nonvanishing f for positive v can never propagate to negative-v.[9]

The time dependence of the variance is given by the integral

$$P(v\, t\,|v_a t_a) \equiv \int dx\, P(x\, v\, t\,|x_a v_a t_a). \tag{20.325}$$

It satisfies the equation

$$\frac{\partial}{\partial t}P(v\, t\,|v_a t_a) = \left[\frac{\varepsilon^2}{2}\partial_v^2 v + \gamma\partial_v(v - \bar{v})\right] P(v\, t\,|v_a t_a), \tag{20.326}$$

which follows from (20.318) by integration over x. After a long time, the solution becomes stationary and reads

$$P^*(v) = \frac{\mu^\nu}{\Gamma(\nu)}v^{\nu-1}e^{-\mu v}, \quad \text{where} \quad \mu \equiv \frac{2\gamma}{\varepsilon^2}, \quad \nu \equiv \mu\bar{v}. \tag{20.327}$$

This is the Gamma distribution found in Fig. 20.3. Hence the pair of stochastic differential equations (20.306) and (20.308) produces precisely the type of volatility fluctuations observed in the previous financial data.

The maximum of $P^*(v)$ lies slightly below $\bar{v}$ at $v_{\max} = (\nu - 1)/\mu = \bar{v} - \varepsilon^2/2\gamma$ [recall (20.72)]. If we define the w of $P^*(v)$ by the curvature at the maximum, we

[9]See also p. 67 in the textbook [8].

find $w = \sqrt{v_{\max}\varepsilon^2/2\gamma}$. The shape of $P^*(v)$ is characterized by the dimensionless ratio

$$\frac{v_{\max}}{w} = \sqrt{\frac{2\gamma\bar{v}}{\varepsilon^2} - 1}. \tag{20.328}$$

The size of the fluctuations of the variance are restricted by the condition (20.322) which guarantees positive $v(t)$ for all times. The distribution (20.327) is plotted in Fig. 20.21. As a cross check, we go to the limit $\varepsilon^2/2\gamma\bar{v} \to 0$ where the fluctuations of the variance are frozen out. The ratio $v_{\max}/w$ diverges and the distribution $P^*(v)$ tends to $\delta(v - \bar{v})$, as expected.

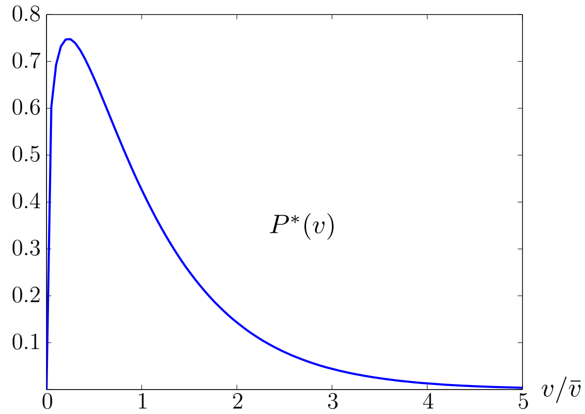

Figure 20.21 Stationary distribution (20.327) of variances for parameters of fits to the Dow-Jones data in Fig. 20.22 listed in Table 20.1. Compare with Fig. 20.3.

20.4.3 Solution of Fokker-Planck Equation

Since the Hamiltonian operator (20.319) does not depend explicitly on x, we can take advantage of translational invariance and write $P(x\,v\,t\,|x_a v_a t_a)$ as a Fourier integral

$$P(x\,v\,t\,|x_a v_a t_a) = \int \frac{dp}{2\pi} e^{ip(x-x_a)}\,\bar{P}_p(v\,t\,|v_a t_a). \tag{20.329}$$

The fixed-momentum probability satisfies the Fokker-Planck equation

$$\partial_t \bar{P}_p(v\,t\,|v_a t_a) = \left[\gamma\frac{\partial}{\partial v}(v-\bar{v}) - \frac{p^2 - ip}{2}v - i\rho\varepsilon p\frac{\partial}{\partial v}v - \frac{\varepsilon^2}{2}\frac{\partial^2}{\partial v^2}v\right]\bar{P}_p(v\,t\,|v_a t_a). \tag{20.330}$$

This partial differential equation is of second-order. The variable v occurs only linearly. It is therefore convenient to go to Fourier space also in v and write

$$\tilde{P}_p(v\,t\,|v_a t_a) = \int \frac{dp_v}{2\pi} e^{ip_v v}\,\tilde{P}_p(p_v\,t\,|v_a t_a), \tag{20.331}$$

which solves the first-order partial differential equation

$$\left[\frac{\partial}{\partial t}+\left(\Gamma p_v+\frac{i\varepsilon^2}{2}p_v^2+\frac{ip^2+p}{2}\right)\frac{\partial}{\partial p_v}\right]\tilde{P}_p(p_v\,t\,|v_at_a)=-i\gamma\bar{v}\,p_v\,\tilde{P}_p(p_v\,t\,|v_at_a), \tag{20.332}$$

where we have used the abbreviation

$$\Gamma(p)\equiv\gamma+i\rho\varepsilon p. \tag{20.333}$$

Since the initial condition for $\bar{P}_p(v\,t\,|v_at_a)$ is $\delta(v-v_a)$, the Fourier transform has the initial condition

$$\tilde{P}_p(p_v\,t_a|v_at_a)=e^{-ip_vv_a}. \tag{20.334}$$

The solution of the first-order partial differential equation (20.332) is found by the method of characteristics [9]:

$$\tilde{P}_p(p_v\,t|v_at_a)=\exp\left[-i\tilde{p}_v(t_a)v_a-i\gamma\bar{v}\int_{t_a}^{t}dt'\,\tilde{p}_v(t')\right], \tag{20.335}$$

where the function $\tilde{p}_v(t)$ is the solution of the characteristic differential equation

$$\frac{d\tilde{p}_v(t)}{dt}=\Gamma(p)\tilde{p}_v(t)+\frac{i\varepsilon^2}{2}\tilde{p}_v^2(t)+\frac{i}{2}(p^2-ip), \tag{20.336}$$

with the boundary condition $\tilde{p}_v(t_b)=p_v$. The differential equation is of the Riccati type with constant coefficients [10], and its solution is

$$\tilde{p}_v(t)=-i\frac{2\Omega(p)}{\varepsilon^2}\frac{1}{\zeta(p,p_v)e^{\Omega(p)(t_b-t)}-1}+i\frac{\Gamma(p)-\Omega(p)}{\varepsilon^2}, \tag{20.337}$$

where we introduced the complex frequency

$$\Omega(p)=\sqrt{\Gamma^2(p)+\varepsilon^2(p^2-ip)}, \tag{20.338}$$

and the coefficient are

$$\zeta(p,p_v)=1-i\frac{2\Omega(p)}{\varepsilon^2p_v-i[\Gamma(p)-\Omega(p)]}. \tag{20.339}$$

Taking the Fourier transform we obtain the solution of the original Fokker-Planck equation (20.318):

$$P(x\,v\,t\,|x_av_at_a)=\iint_{-\infty}^{+\infty}\frac{dp}{2\pi}\,\frac{dp_v}{2\pi}\,e^{ipx+ip_vv}$$
$$\times\exp\left\{-i\tilde{p}_v(t_a)v_a+\frac{\gamma\bar{v}[\Gamma(p)-\Omega(p)]}{\varepsilon^2}\Delta t-\frac{2\gamma\bar{v}}{\varepsilon^2}\ln\frac{\zeta(p,p_v)-e^{-\Omega(p)\Delta t}}{\zeta(p,p_v)-1}\right\}, \tag{20.340}$$

where $\Delta t\equiv t-t_a$ is the time interval.

20.4.4 Pure x-Distribution

We now show that upon averaging over the variance we obtain a non-Gaussian distribution of x with semi-heavy tails of the type observed in actual financial data. Thus we go over to the probability distribution

$$P(x\,t\,|x_a v_a t_a) \equiv \int_{-\infty}^{+\infty} dv\, P(x\,v\,t\,|x_a v_a t_a), \tag{20.341}$$

where the final variable v is integrated out. The integration of (20.340) over v generates the delta-function $\delta(p_v)$ which enforces $p_v = 0$. Thus we obtain

$$P(x\,t\,|x_a v_a t_a) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, e^{ipx - \frac{p^2 - ip}{\Gamma + \Omega \coth(\Omega\Delta t/2)} v_a + \frac{\gamma\Gamma\bar{v}}{\varepsilon^2}\Delta t - \frac{2\gamma\bar{v}}{\varepsilon^2}\ln\left(\cosh\frac{\Omega\Delta t}{2} + \frac{\Gamma}{\Omega}\sinh\frac{\Omega\Delta t}{2}\right)}, \tag{20.342}$$

where we have omitted the arguments of $\Gamma(p)$ and $\Omega(p)$, for brevity. As a cross check of this result we consider the special case $\varepsilon = 0$ where the time evolution of the variance is deterministic:

$$v(t) = \bar{v} + (v_a - \bar{v})e^{-\gamma\Delta t}. \tag{20.343}$$

Performing the integral over p in Eq. (20.342) we obtain

$$P(x\,t\,|x_a t_a v_a) = \frac{1}{\sqrt{2\pi(t-t_a)\overline{v}(t)}} \exp\left\{-\frac{[x + \overline{v}(t)(t-t_a)/2]^2}{2\overline{v}(t)}\right\}, \tag{20.344}$$

where $\overline{v}(t)$ denotes the time-averaged variance

$$\overline{v}(t) \equiv \frac{1}{\Delta t}\int_{t_a}^{t} dt'\, v(t'). \tag{20.345}$$

The distribution becomes Gaussian in this limit. The same result could, of course, have been obtained directly from the stochastic differential equation (20.306).

The result (20.342) cannot be compared directly with financial time series data, because it depends on the unknown initial variance v_a. Let us assume that v_a has initially a stationary probability distribution $P^*(v_a)$ of Eq. (20.327).[10] Then we evaluate the probability distribution $P(x\,t\,|x_a t_a)$ by averaging (20.342) over v_a with the weight $P^*(v_a)$:

$$P(x\,t\,|x_a t_a) = \int_0^{\infty} dv_a\, P^*(v_a)\, P(x\,t\,|x_a t_a v_a). \tag{20.346}$$

The final result has the Fourier representation

$$P(x\,t\,|x_a t_a) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, e^{ip\Delta x - \Delta t H(p,\Delta t)}, \tag{20.347}$$

[10] For a determination of the empirical probability distribution of volatilities for the S&P 500 index see Ref. [2].

where $\Delta x \equiv x - x_a$ and $H(p, \Delta t)$ is the Hamiltonian

$$H(p,\Delta t) = -\frac{\gamma\Gamma(p)\bar{v}}{\varepsilon^2} + \frac{2\gamma\bar{v}}{\varepsilon^2\Delta t}\ln\left[\cosh\frac{\Omega(p)\Delta t}{2} + \frac{\Omega^2(p)-\Gamma^2(p)+2\gamma\Gamma(p)}{2\gamma\Omega(p)}\sinh\frac{\Omega(p)\Delta t}{2}\right]. \tag{20.348}$$

In the absence of correlations between the fluctuations of stock price and variance, i.e. for $\rho = 0$, the second term in the right-hand side of Eq. (20.348) vanishes. The simplified result is discussed in Appendix 20B.

The general Hamiltonian (20.348) vanishes for $p = 0$. This guarantees the unit normalization of the distribution (20.347) at all times: $\int dx\, P(x\,t\,|x_a t_a) = 1$. The first expansion coefficient of $H(p, \Delta t)$ in powers of p is [recall the definition in Eq. (20.54)]

$$c_1(\Delta t) \;=\; -\frac{\bar{v}}{2} - \frac{\rho}{\epsilon\Delta t}\left(1 - e^{-\gamma\Delta t}\right). \tag{20.349}$$

We have added the argument Δt to emphasize the time dependence. By adding $ic_1(\Delta t)p$ to $H(p, \Delta t)$, we obtain the Hamiltonian $\bar{H}(p, \Delta t)$ which starts out quadratically in p in accordance with our general definition in Eq. (20.65):

$$\bar{H}(p,\Delta t) \equiv H(p,\Delta t) - ic_1(\Delta t)p = H(p,\Delta t) + i\left[\frac{\bar{v}}{2} + \frac{\rho}{\epsilon\Delta t}\left(1 - e^{-\gamma\Delta t}\right)\right]p. \tag{20.350}$$

The distribution $P(x\,t\,|x_a t_a)$ is real since $\mathrm{Re}\, H$ is an even function and $\mathrm{Im}\, H$ is an odd function of p.

In general, the integral in (20.347) must be calculated numerically. The interesting limit regimes $\Delta t \gg 1$ and $x \gg 1$, however, can be understood analytically. These will be treated in Subsections 20.4.5 and 20.4.6. In Fig. 20.22, the calculated distributions $P(x\,t\,|x_a t_a)$ with increasing time intervals are displayed as solid curves and compared with the corresponding Dow-Jones data indicated by dots. The technical details of the data analysis are explained in Appendix 20C. The figure demonstrates that with a fixed set of the five parameters γ, $\bar{v}$, ε, μ, and ρ, the distribution (20.347) with (20.348) reproduce extremely well the data for *all* time scales Δt. In the logarithmic plot the far tails of the distributions fall off linearly.

20.4.5 Long-Time Behavior

According to Eq. (20.308), the variance approaches the equilibrium value $\bar{v}$ within the characteristic relaxation time $1/\gamma$. Let us study the limit in which the time interval Δt is much longer than the relaxation time: $\gamma\Delta t \gg 1$. According to Eqs. (20.333) and (20.338), this condition also implies that $\Omega(p)\Delta t \gg 1$. Then Eq. (20.348) reduces to

$$H(p,\Delta t) \approx \frac{\gamma\bar{v}}{\varepsilon^2}[\Omega(p) - \Gamma(p)]. \tag{20.351}$$

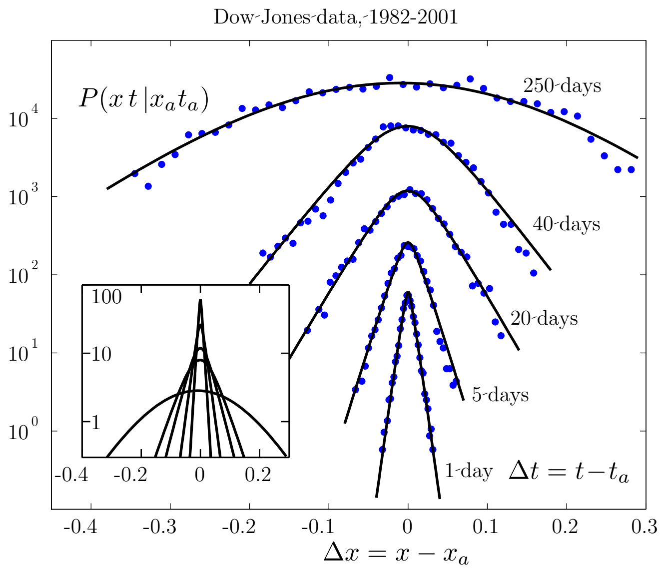

Figure 20.22 Probability distribution of logarithm of stock price for different time scales (from Ref. [79]). For a better comparison of the shapes, the data points for increasing Δt are shifted upwards by a factor 10 each. The unshifted positions are shown in the insert.

The Fourier integral (20.347) can easily be performed after changing the variable of integration to

$$p = C\,\tilde{p} + ip_0, \tag{20.352}$$

where

$$C \equiv \frac{\omega_0}{\varepsilon\sqrt{1-\rho^2}}, \quad \omega_0 \equiv \sqrt{\gamma^2 + \varepsilon^2(1-\rho^2)p_0^2}, \quad p_0 \equiv \frac{\varepsilon - 2\rho\gamma}{2\varepsilon(1-\rho^2)}. \tag{20.353}$$

Then the integral (20.347) takes the form

$$P(x\,t\,|x_a t_a) = \frac{C}{\pi} e^{-p_0\Delta x + \Lambda\Delta t} \int_0^\infty d\tilde{p}\, \cos(A\tilde{p}) e^{-B\sqrt{1+\tilde{p}^2}}, \tag{20.354}$$

where

$$A = C\left(\Delta x + \rho\frac{\gamma\bar{v}}{\varepsilon}\Delta t\right), \quad B = \frac{\gamma\bar{v}\omega_0}{\varepsilon^2}\Delta t, \tag{20.355}$$

and

$$\Lambda = \frac{\gamma\bar{v}}{2\varepsilon^2}\frac{2\gamma - \rho\varepsilon}{1-\rho^2}. \tag{20.356}$$

The integral in (20.354) is equal to[11] $BK_1(\sqrt{A^2+B^2})/\sqrt{A^2+B^2}$, where $K_1(y)$ is a modified Bessel function, such that the probability distribution (20.347) for $\gamma\Delta t \gg 1$ can be represented in the scaling form

$$P(x\,t\,|x_a t_a) = N(\Delta t)\, e^{-p_0\Delta x} F^*(y), \quad F^*(y) = K_1(y)/y, \tag{20.357}$$

[11]See I.S. Gradshteyn and I.M. Ryzhik, *op. cit.*, Formula 3.914.

with the argument

$$y \equiv \sqrt{A^2+B^2} = \frac{\omega_0}{\varepsilon}\sqrt{\frac{(\Delta x+\rho\gamma\bar{v}\Delta t/\varepsilon)^2}{1-\rho^2}+\left(\frac{\gamma\bar{v}\Delta t}{\varepsilon}\right)^2}, \tag{20.358}$$

and the time-dependent normalization factor

$$N(\Delta t) = \frac{\omega_0^2\gamma\bar{v}}{\pi\varepsilon^3\sqrt{1-\rho^2}}\Delta t\, e^{\Lambda\Delta t}. \tag{20.359}$$

Thus, up to the factors $N(\Delta t)$ and $e^{-p_0\Delta x}$, the dependence of $P(x\,t\,|x_at_a)$ on the arguments Δx and Δt is given by the function $F^*(y)$ of the single scaling argument y. When plotted as a function of y, the data for different Δx and Δt should collapse on the single universal curve $F^*(y)$. This does indeed happen as illustrated in Fig. 20.23, where the Dow-Jones data for different time differences Δt follows the curve $F^*(y)$ for seven decades.

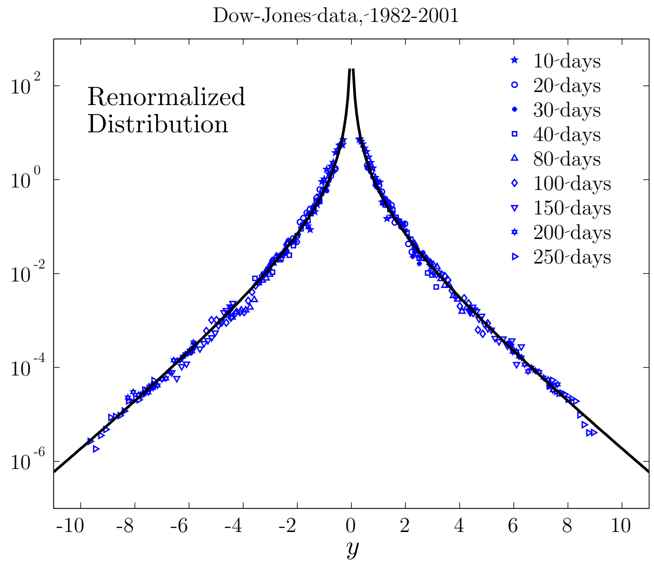

Figure 20.23 Renormalized distribution function $P(x\,t\,|x_at_a)e^{p_0\Delta x}/N(\Delta t)$ plotted as a function of the scaling argument y defined in Eq. (20.358). The solid curve shows the universal function $F^*(y) = K_1(y)/y$ (from Ref. [79]).

In the limit $y \gg 1$, we can use the asymptotic expression $K_1(y) \approx e^{-y}\sqrt{\pi/2y}$ and find the asymptotic behavior

$$\ln\frac{P(x\,t\,|x_at_a)}{N(\Delta t)} \approx -p_0\Delta x - y. \tag{20.360}$$

Let us examine this expression for large and small $|\Delta x|$. In the first case $|\Delta x| \gg \gamma\bar{v}\Delta t/\varepsilon$, and Eq. (20.358) shows that $y \approx \omega_0|\Delta x|/\varepsilon\sqrt{1-\rho^2}$, so that (20.360) becomes

$$\ln\frac{P(x\,t\,|x_at_a)}{N(\Delta t)} \approx -p_0\Delta x - c|\Delta x|. \tag{20.361}$$

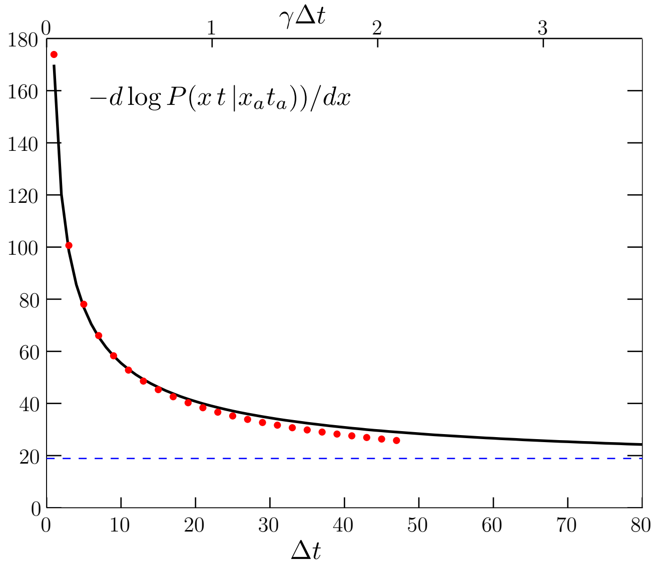

Figure 20.24 Solid curve showing slope $-d\log P(x\,t\,|x_a t_a)/dx$ of exponential tail of distribution as a function of time. The dots indicate the analytic short-time approximation (20.369) to the curve (from [79]).

This shows that the probability distribution $P(x\,t\,|x_a t_a)$ has exponential tails (20.361) for large $|\Delta x|$. Note that in the present limit $\gamma\Delta t \gg 1$ the slopes of the exponential tails are independent of Δt. The presence of p_0 causes the slopes for positive and negative Δx to be different, so that the distribution $P(x\,t\,|x_a t_a)$ is not up-down symmetric with respect to price changes. From the definition of p_0 in Eq. (20.353) we see that this asymmetry increases for a negative correlation $\rho < 0$ between stock price and variance.

In the second case $|\Delta x| \ll \gamma\bar{v}\Delta t/\varepsilon$, by Taylor-expanding y in (20.358) near its minimum and substituting the result into (20.360), we obtain

$$\ln \frac{P(x\,t\,|x_a t_a)}{N'(\Delta t)} \approx -p_0\Delta x - \frac{\omega_0(\Delta x + \rho\gamma\bar{v}\Delta t/\varepsilon)^2}{2(1-\rho^2)\gamma\bar{v}\Delta t}, \tag{20.362}$$

where $N'(\Delta t) = N(\Delta t)\exp(-\omega_0\gamma\bar{v}\Delta t/\varepsilon^2)$. Thus, for small $|\Delta x|$, the probability distribution $P(x\,t\,|x_a t_a)$ is a Gaussian, whose width

$$\sigma^2 = \frac{(1-\rho^2)\gamma\bar{v}}{\omega_0}\Delta t \tag{20.363}$$

grows linearly with Δt. The maximum of $P(x\,t\,|x_a t_a)$ lies at

$$\Delta x_m(t) = \Delta r_S \Delta t, \quad \text{with} \quad \Delta r_S \equiv -\frac{\gamma\bar{v}}{2\omega_0}\left[1 + 2\,\frac{\rho(\omega_0-\gamma)}{\varepsilon}\right]. \tag{20.364}$$

The position moves with a constant rate Δr_S which adds to the average growth rate r_S of $S(t)$ removed at the beginning of the discussion. The true final growth rate of $S(t)$ is $\bar{r}_S = r_S + \Delta r_S$.

The above discussion explains the property of the data in Fig. 20.22 that the logarithmic plots of $P(x\,t\,|x_at_a)$ are linear in the tails and quadratic near the peaks with the parameters specified in Eqs. (20.361) and (20.362).

As time progresses, the distribution broadens in accordance with the scaling form (20.357) and (20.358). In the limit $\Delta t \to \infty$, the asymptotic expression (20.362) is valid for all Δx and the distribution becomes a pure Gaussian, as required by the central limit theorem [22].

It is interesting to quantify the fraction $f(\Delta t)$ of the total probability contained in the Gaussian portion of the curve. This fraction is plotted in Fig. 20.25. The precise way of defining and calculating the fraction $f(\Delta t)$ is explained in Appendix 20B. The inset in Fig. 20.25 illustrates that the time dependence of the probability density at the maximum x_m approaches $\Delta t^{-1/2}$ for large time, a characteristic property of evolution of Gaussian distributions.

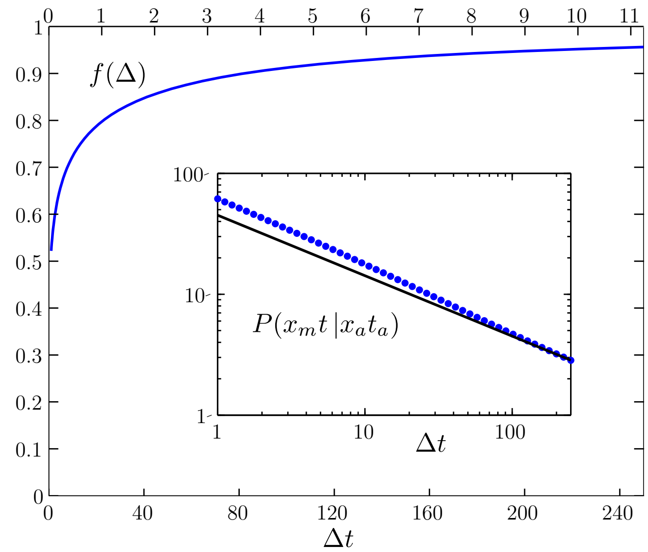

Figure 20.25 Fraction $f(\Delta t)$ of total probability contained in Gaussian part of $P(x\,t\,|x_at_a)$ as function of time interval Δt. The inset shows the time dependence of the probability density at maximum $P(x_mt\,|x_at_a)$ (points), compared with the falloff $\propto \Delta t^{-1/2}$ of a Gaussian distribution (solid curve).

20.4.6 Tail Behavior for all Times

For large $|\Delta x|$, the integrand in (20.347) oscillates rapidly as a function of p, so that the integral can be evaluated in the saddle point approximation of Section 4.2. As in the evaluation of the integral (17.9) we shift the contour of integration in the complex p-plane until it passes through the leading saddle point of the exponent $ip\Delta x - \Delta t H(p, \Delta t)$. To determine its position we note that the function $H(p, \Delta t)$ in Eq. (20.348) has singularities in the complex p-plane, where the argument of the

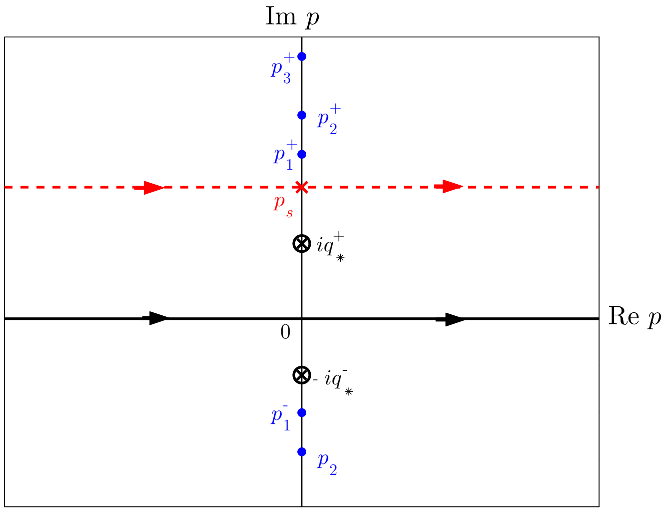

Figure 20.26 Singularities of $H(p, \Delta t)$ in complex p-plane (dots). Circled crosses indicate the limiting positions $\pm iq_*^{\pm}$ of the singularities for $\gamma\Delta t \gg 1$. The cross shows the saddle point p_s located in the upper half-plane for $\Delta x > 0$. The dashed line is the shifted contour of integration to pass through the saddle point p_s.

logarithm vanish. These points are located on the imaginary p-axis and are shown by dots in Fig. 20.26. The relevant singularities are those lying closest to the real axis. They are located at the points p_1^+ and p_1^-, where the argument of the second logarithm in (20.348) vanishes. Near these zeros, we can approximate $H(p, \Delta t)$ by the singular term:

$$\bar{H}(p, \Delta t) \approx \frac{2\gamma\bar{v}}{\varepsilon^2 \Delta t} \log(p - p_1^{\pm}). \tag{20.365}$$

With this approximation, the position of the saddle point $p_s = p_s(\Delta x)$ is determined by the equation

$$i\Delta x = \Delta t \frac{dH(p, \Delta t)}{dp}\bigg|_{p=p_s} \approx \frac{2\gamma\bar{v}}{\varepsilon^2} \times \begin{cases} \dfrac{1}{p_s - p_1^+}, & \Delta x > 0, \\ \dfrac{1}{p_s - p_1^-}, & \Delta x < 0. \end{cases} \tag{20.366}$$

The solutions are indicated in Fig. 20.26 by the cross. The approximation (20.366) is obviously applicable since for a large $|\Delta x|$ satisfying the condition $|\Delta x p_1^{\pm}| \gg \gamma\bar{v}/\varepsilon^2$, the saddle point p_s is very close to one of the singularities. Inserting the approximation (20.366) into the Fourier integral (20.347), we obtain the asymptotic expression for the probability distribution

$$P(x\,t\,|x_a t_a) \sim \begin{cases} e^{-\Delta x\, q_t^+}, & \Delta x > 0, \\ e^{-|\Delta x|\, q_t^-}, & \Delta x < 0, \end{cases} \tag{20.367}$$

where $q_t^\pm \equiv \mp i p_1^\pm(t)$ are real and positive. Thus the tails of the probability distribution $P(x\,t\,|x_a t_a)$ for large $|\Delta x|$ are exponential for all times t. The slopes of the logarithmic plots in the tails $q^\pm \equiv \mp\, d\log(x\,t\,|x_a t_a)/dx$ are determined by the positions $p_1^\pm$ of the singularities closest to the real axis.

These positions $p_1^\pm$ depend on the time interval Δt. For times much shorter than the relaxation time ($\gamma t \ll 1$), the singularities lie far away from the real axis. As time increases, the singularities approach the real axis. For times much longer than the relaxation time ($\gamma\Delta t \gg 1$), the singularities approach the limiting points $p_1^\pm \to \pm i q_*^\pm$, marked in Fig. 20.26 by circled crosses. The limiting values are

$$q_*{}^\pm = \pm p_0 + \frac{\omega_0}{\varepsilon\sqrt{1-\rho^2}} \quad \text{for} \quad \gamma\Delta t \gg 1. \tag{20.368}$$

The slopes $q^\pm(\Delta t)$ approach these limiting slopes monotonously from above. The behavior is shown in Fig. 20.24. The slopes (20.368) are of course in agreement with Eq. (20.361) in the limit $\gamma\Delta t \gg 1$. In the opposite limit of short time ($\gamma\Delta t \ll 1$), we find the analytic time behavior

$$q^\pm(\Delta t) \approx \pm p_0 + \sqrt{p_0^2 + \frac{4\gamma}{\varepsilon^2(1-\rho^2)\Delta t}} \quad \text{for} \quad \gamma\Delta t \ll 1. \tag{20.369}$$

This approximation is shown in Fig. 20.24 as dots.

20.4.7 Path Integral Calculation

Instead of solving the Fokker-Planck equation (20.330) we may also study directly the path integral for the probability distribution using the Hamiltonian (20.321). Note the extra two terms at the end in comparison with the operator expression (20.319). They account for the symmetric operator order implied by the path integral. The distribution $\bar{P}_p(v_b, t_b|v_a t_a)$ at fixed momentum introduced in Eq. (20.329) has the path integral representation

$$\bar{P}_p(v_b, t_b|v_a t_a) = \int \mathcal{D}v\, \frac{\mathcal{D}p_v}{2\pi}\, e^{\mathcal{A}_p[p_v, v]}, \tag{20.370}$$

with the action

$$\mathcal{A}_p[p_v, v] = \int_{t_a}^{t_b} dt\, [i p_v \dot{v} - H\,(p, p_v, v)]\,. \tag{20.371}$$

The path integral (20.370) sums over all paths $p_v(t)$ and $v(t)$ with the boundary conditions $v(t_a) = v_a$ and $v(t_b) = v_b$.

It is convenient to integrate the first term in the (20.371) by parts, and to separate $H\,(p, p_v, v)$ into a v-independent part $i\gamma\bar{v}p_v - i\gamma/2 - i\rho\epsilon p/2$ and a linear term $[\partial H\,(p, p_v, v)\,/\partial v]v$. Thus we write

$$\begin{aligned}\mathcal{A}_p[p_v, v] \;&=\; i[p_v(t_b)v_b - p_v(t_a)v_a] - i\gamma\bar{v}\int_{t_a}^{t_b} dt\, p_v(t)(t_b - t_a)\\ &\quad - \int_{t_a}^{t_b} dt\left[i\dot{p}_v(t) + \frac{\partial H}{\partial v(t)}\right] v(t).\end{aligned} \tag{20.372}$$

Since the path $v(t)$ appears linearly in this expression, we can integrate it out to obtain a delta-functional $\delta\left[p_v(t)-\tilde{p}_v(t)\right]$, where $\tilde{p}_v(t)$ is the solution of the Hamilton equation $\dot{p}_v(t)=i\partial H/\partial v(t)$. This, however, coincides exactly with the characteristic differential equation (20.336) which was solved by Eq. (20.337) with a boundary condition $\tilde{p}_v(t_b)=p_v$. Taking the path integral over $\mathcal{D}p_v$ removes the delta-functional, and we find

$$\bar{P}_p(v_b,t_b|v_a t_a)=\int_{-\infty}^{+\infty}\frac{dp_v}{2\pi}\,J\,e^{i[p_v v-\tilde{p}_v(t_a)v_a]-i\gamma\theta\int_0^t dt\,\tilde{p}_v(t)+(\gamma-i\rho\epsilon p)(t_b-t_a)/2},\qquad(20.373)$$

where J denotes the Jacobian

$$J=\mathrm{Det}^{-1}\left(i\partial_t+\frac{\partial^2 H(p,p_v,v)}{\partial p_v\partial v}\right).\qquad(20.374)$$

From (20.321) we see that

$$\frac{\partial^2 H(p,p_v,v)}{\partial p_v\partial v}=-i\gamma+\rho\epsilon p.\qquad(20.375)$$

According to Formula (18.254), this is equal to

$$J=e^{-(\gamma-i\rho\epsilon p)(t_b-t_a)/2},\qquad(20.376)$$

thus canceling the last term in the exponent of the integrand in (20.373). The result for $\bar{P}_p(v_b,t_b|v_a t_a)$ is therefore the same as the one obtained from the Fokker-Planck equation in Eq. (20.331) with the Fourier transform (20.335).

20.4.8 Natural Martingale Distribution

Let us calculate the natural martingales associated with the Hamiltonian (20.348). Reinserting the initially removed drift r_S, the total Hamiltonian reads

$$H^{\mathrm{tot}}(p,\Delta t)=H(p,\Delta t)+ir_S p.\qquad(20.377)$$

To construct the martingale according to the rule of Subsection 20.1.11, we need to know the value of $\bar{H}(p,\Delta t)$ at the momentum $p=i$. The expression is somewhat complicated, but the analysis of the Dow-Jones data in Appendix 20C shows that the parameter ρ determining the strength of correlations between the noise functions $\eta(t)$ and $\eta_v(t)$ [see (20.310)] is small. It is therefore sufficient to find only the first two terms of the Taylor series of $H(i,\Delta t)$ in powers of ρ:

$$H(i,\Delta t)\approx\frac{\bar{v}}{\epsilon\Delta t}\left(1-e^{-\gamma\Delta t}\right)\rho+\frac{\bar{v}}{4\gamma\Delta t}\left[3-2e^{-\gamma\Delta t}\left(1+2\gamma\Delta t\right)-e^{-2\gamma\Delta t}\right]\rho^2.\qquad(20.378)$$

From this we obtain

$$\bar{H}(i,\Delta t)=H(i,\Delta t)+c_1(\Delta t).\qquad(20.379)$$

The analog of the martingale (20.291) is given by a Fourier integral (20.292) with $H_{r_x}(p)$ replaced by $H^{\rm tot}_{r_x(\Delta t)}(p,\Delta t)$ of Eq. (20.350):

$$P^{(M,r_S)}(x_bt_b|x_at_a) = e^{-r(\Delta t)\Delta t}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\exp\left[ip(x_b-x_a)-\Delta t\, H^{\rm tot}_{r_x(\Delta t)}(p,\Delta t)\right]. \quad (20.380)$$

The linear term $ir_x(\Delta t)p$ in $H^{\rm tot}_{r_x(\Delta t)}(p,\Delta t)$ is now time-dependent:

$$r_x(\Delta t) = r_S + c_1(\Delta t) + \bar{H}(i,\Delta t). \quad (20.381)$$

This makes the rate in the exponential prefactor, by which the distribution becomes a martingale distribution, a time-dependent quantity:

$$r(\Delta t) = r_S + c_1(\Delta t). \quad (20.382)$$

By analogy with (20.293), there exists again an entire family of natural martingales, which is obtained by replacing r_S by an *arbitrary* growth rate r in which case $r(\Delta t)$ becomes equal to $r + c_1(\Delta t)$.

20.5 Time Series

True market prices are not continuous functions of time but recorded in discrete time intervals. For stocks which have a low turnover, only the daily prices are recorded. For others which are traded in large amounts, the prices change by the minute. They are listed as time series $S(t_n)$. In the year 2003, Robert Engle received the Nobel prize for his description of time series with the help of the so called *ARCH models* (autoregressive conditional heteroskedasticity model) and its generalization proposed by Tim Bollerslev [85], the *GARCH models*. The first contains an integer parameter q and is defined by a discrete modification of the pair of stochastic differential equations (20.306) and (20.308) for log-return and variance

$$x(t_n) = \sqrt{v(t_{n-1})}\eta(t_n), \quad v(t_n) = v_0 + \sum_{k=1}^{q}\alpha_k(t_{n-k})x^2(t_{n-k}), \quad (20.383)$$

with a white noise variable $\eta(t_n)$. The drift is omitted so that $\langle x(t_n)\rangle = 0$. This is called the ARCH(q) model. The second is a generalization of this in which the equation for the variance contains extra terms involving the past p values of σ^2:

$$v(t_n) = v_0 + \sum_{k=1}^{q}\alpha_k(t_{n-k})x^2(t_{n-k}) + \sum_{k=1}^{p}\beta_k(t_{n-k})v(t_{n-k}). \quad (20.384)$$

The ARCH(1) process has the time-independent expectations of variance and kurtosis

$$\sigma^2 = \langle v(t_n)\rangle = \frac{v_0}{1-\alpha_1}, \quad \kappa = \frac{\langle x^4(t_n)\rangle_c}{\langle x^2(t_n)\rangle_c^2} = \frac{6\alpha_1^2}{1-3\alpha_1^2}. \quad (20.385)$$

For the GARCH(1,1) process, these quantities are

$$\sigma^2 = \langle v(t_n)\rangle = \frac{v_0}{1-\alpha_1-\beta_1}, \quad \kappa = \frac{\langle x^4(t_n)\rangle_c}{\langle x^2(t_n)\rangle_c^2} = \frac{6\alpha_1^2}{1-3\alpha_1^2-2\alpha_1\beta_1-\beta_1^2}. \quad (20.386)$$

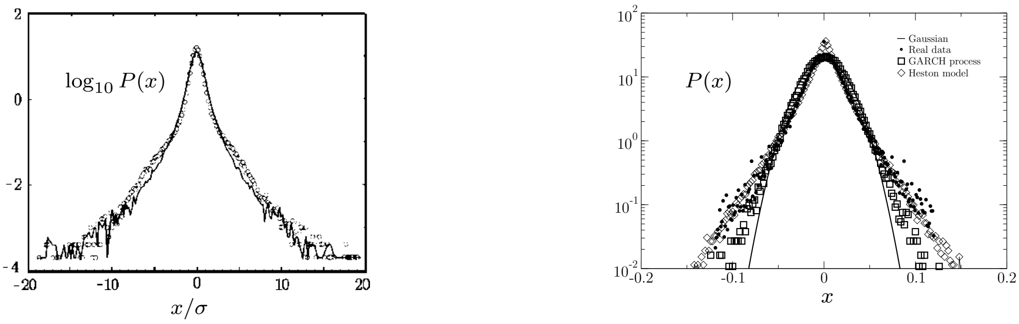

Figure 20.27 Left: Comparison of GARCH(1,1) process with $v_0 = 2.3 \times 10^{-5}$, $\alpha_1 = 0.091$, $\beta = 0.9$ with S&P 500 index (minute data). Right: Comparison of GARCH(1,1), Heston model, and Gaussian of the same σ with daily market data. Parameters of GARCH process are $v_0 = 7.7 \times 10^{-6}$, $\alpha_1 = 0.07906$, $\beta_1 = 0.90501$ and of Heston model $\gamma = 4.5 \times 10^{-2}$, $\bar{v} = 8.62 \times 10^{-8}$, $r_S = 5.67 \times 10^{-4}$, $\varepsilon = 10.3 \times 10^{-3}$ [79, 81, 82].

20.6 Spectral Decomposition of Power Behaviors

Plots of the log return curves taken over short time intervals reveal a richer structure than the simple model functions discussed so far. In fact, it seems reasonable to distinguish different types of stocks according to the characters of the investors. For instance, the bubble of the computer stocks was partly caused by quite inexperienced people who were not investing mainly to develop an industry but who wanted to earn fast money. There was, on the other hand also a substantial portion of institutional investors who poured in money more steadily. It appears that one should distinguish the sources of noise coming from the different groups of investors.

We therefore improve the stochastic differential equation (20.4) by extending it to

$$\dot{x}(t) = r_x + \sum_{\lambda} \eta_\lambda(t), \tag{20.387}$$

where the sum runs over different groups of investors, each producing a different a noise $\eta_\lambda(t)$ with Lévy distributions falling off with different powers $|x|^{-1-\lambda_i}$. Their probability distributions are

$$P_\lambda[\eta_\lambda] = \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}_\lambda(\eta_\lambda(t))\right] = \int \frac{\mathcal{D}p}{2\pi} \exp\left\{\int_{t_a}^{t_b} dt\, [ip(t)\eta_\lambda(t) - H_\lambda(p(t))]\right\}, \tag{20.388}$$

with a Hamiltonian [compare (20.10)]

$$H_\lambda(p) \equiv \frac{\sigma_\lambda^2\, |p|^\lambda}{2}. \tag{20.389}$$

The probability (20.144) of returns becomes now

$$P(x_b t_b|x_a t_a) = \prod_\lambda \left\{\int \mathcal{D}\eta_\lambda \int \mathcal{D}x \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}_\lambda(\eta_\lambda(t))\right]\right\} \delta[\dot{x} - \sum_\lambda \eta_\lambda]. \tag{20.390}$$

If we insert here a Fourier representation of the δ-functional, we obtain

$$P(x_bt_b|x_at_a) = \int_{-\infty}^{\infty} \frac{\mathcal{D}p}{2\pi} \prod_\lambda \left[\int \mathcal{D}\eta_\lambda \right] \int \mathcal{D}x \, e^{\int_{t_a}^{t_b} dt \left[ip(t)\dot{x}(t) - \tilde{H}_\lambda(\eta_\lambda(t))\right]} e^{-i\sum_\lambda \int_{-\infty}^{\infty} dt \, p(t)\eta_\lambda(t)}. \tag{20.391}$$

Performing the path integrals over $\eta_\lambda(t)$, this becomes

$$P(x_bt_b|x_at_a) = \int_{-\infty}^{\infty} \frac{\mathcal{D}p}{2\pi} \int \mathcal{D}x \, e^{\int_{t_a}^{t_b} dt \, [ip(t)\dot{x}(t) - H_\lambda(p)]}, \tag{20.392}$$

where the Hamiltonian is the sum of the group Hamiltonians

$$H(p) \equiv \sum_\lambda H_\lambda(p). \tag{20.393}$$

The continuous generalization of this is

$$H(p) = H(|p|) \equiv \int_0^\infty d\lambda \, \sigma_\lambda^2 |p|^\lambda. \tag{20.394}$$

The spectral function σ_λ^2 must be extracted from the data by forming the integral

$$\sigma_\lambda^2 = \int_{-i\infty}^{i\infty} \frac{d\log|p|}{2\pi i} \, p^{-\lambda} H(p). \tag{20.395}$$

20.7 Option Pricing

Historically, the most important use of path integrals in financial markets was made in the context of determining a fair price of financial derivatives, in particular options.[12]Options are an ancient financial tool. They are used for speculative purposes or for hedging major market transactions against unexpected changes in the market environment. These can sometimes produce dramatic price explosions or erosions, and options are supposed to prevent the destruction of large amounts of capital. Ancient Romans, Grecians, and Phoenicians traded options against outgoing cargos from their local seaports. In financial markets, options are contracted between two parties in which one party has the right but not the obligation to do something, usually to buy or sell some underlying asset. Having rights without obligations has a value, so option holders must pay a price for acquiring them. The price depends on the value of the associated asset, which is why they are also called *derivative assets* or briefly *derivatives*. *Call options* are contracts giving the option holder the right to buy something, while *put options* entitle the holder to sell something. The price of an option is called *premium*. Usually, options are associated with stock, bonds, or commodities like oil, metals or other raw materials. In the sequel we shall consider call options on stocks, to be specific.

Modern option pricing techniques have their roots in early work by Charles Castelli who published in 1877 a book entitled *The Theory of Options in Stocks*

[12] An introduction is found on the site `http://bradley.bradley.edu/~arr/bsm/model.html`.

and Shares. This book presented an introduction to the hedging and speculation aspects of options. Twenty three years later, Louis Bachelier offered the earliest known analytical valuation for options in his dissertation at the Sorbonne [86]. Remarkably, Bachelier discovered the treatment of stochastic phenomena five years before Einstein's related but much more famous work on Brownian motion [87], and twenty-three years before Wiener's mathematical development [88]. The stochastic differential equations considered by him still had an important defect of allowing for negative security prices, and for option prices exceeding the price of the underlying asset. Bachelier's work was continued by Paul Samuelson, who wrote in 1955 an unpublished paper entitled *Brownian Motion in the Stock Market*. During that same year, Richard Kruizenga, one of Samuelson's students, cited Bachelier's work in his dissertation *Put and Call Options: A Theoretical and Market Analysis*. In 1962, a dissertation by A. James Boness entitled *A Theory and Measurement of Stock Option Value* developed a more satisfactory pricing model which was further improved by Fischer Black and Myron Scholes. In 1973 they published their famous *Black and Scholes Model* [89] which, together with the improvements introduced by Robert Merton, earned them the Nobel prize in 1997.[13]

As discussed before, the Gaussian distribution severely underestimates the probability of large jumps in asset prices and this was the main reason for the catastrophic failure in the early fall of 1998 of the hedge fund *Long Term Capital Management*, which had Scholes and Merton on the advisory board (and as shareholders). The fund contained derivatives with a notional value of 1,250 Billion US$. The fund collected 2% for administrative expenses and 25% of the profits, and was initially extremely profitable. It offered its shareholders returns of 42.8% in 1995, 40.8% in 1996, and 17.1% even in the disastrous year of the Asian crisis 1997. But in September 1998, after mistakenly gambling on a convergence in interest rates, it almost went bankrupt. A number of renowned international banks and Wall Street institutions had to bail it out with 3.5 Billion US$ to avoid a chain reaction of credit failures.

In spite of this failure, the simple model is still being used today for a rough but fast orientation on the fairness of an option price.

20.7.1 Black-Scholes Option Pricing Model

In the early seventies, Fischer Black was working on a valuation model for stock warrants and observed that his formulas resembled very much the well-known equations for heat transfer. Soon after this, Myron Scholes joined Black and together they discovered an approximate option pricing model which is still of wide use.

The Black and Scholes Model is based on the following assumptions:

1. The returns are normally distributed.

   The shortcomings of this assumption have been discussed above in Section 20.1. The appropriate improvement of the model will be developed below.

[13]For F. Black the prize came too late — he had died two years earlier.

2. Markets are efficient.

   This assumption implies that the market operates continuously with share prices following a continuous stochastic process without memory. It also implies that different markets have the same asset prices.

   This is not quite true. Different markets do in general have slightly different prices. Their differences are kept small by the existence of arbitrage dealers. There also exist correlations over a short time scale which make it possible, in principle, to profit without risk from the so-called *statistical arbitrage*. This possibility is, however, strongly limited by transaction fees.

3. No commissions are charged.

   This assumption is not satisfied. Usually, market participants have to pay a commission to buy or sell assets. Even floor traders pay some kind of fee, although this is usually very small. The fees payed by individual investors is more substantial and can distort the output of the model.

4. Interest rates remain constant and known.

   The Black and Scholes model assumes the existence of a *riskfree interest rate* to represent this constant and known rate. In reality there is no such thing as the riskfree rate. As an approximation, one uses the discount rate on U.S. Government Treasury Bills with 30 days left until maturity. During periods of rapidly changing interest rates, these 30 day rates are often subject to change, thereby violating one of the assumptions of the model.

5. The stock pays no dividends during the option's life.

   Most companies pay dividends to their share holders, so this is a limitation to the model since higher dividends lead to lower call premiums. There is, however, a simple possibility of adjusting the model to the real situation by subtracting the discounted value of a future dividend from the stock price.

6. European exercise terms are used.

   European exercise terms imply the exercise of an option only on the expiration date. This is in contrast to the American exercise terms which allow for this at any time during the life of the option. This greater flexibility makes an American option more valuable than the European one.

   The difference is, however, not dramatic in praxis because very few calls are ever exercised before the last few days of their life, since an early exercise means giving away the remaining time value on the call. Different exercise times towards the end of the life of a call are irrelevant since the remaining time value is very small and the intrinsic value has a small time dependence, barring a dramatic event right before expiration date.

Since 1973, the original Black and Scholes Option Pricing Model has been improved and extended considerably. In the same year, Robert Merton [90] included the effect of dividends. Three years later, Jonathan Ingersoll relaxed the assumption of no taxes or transaction costs, and Merton removed the restriction of constant interest rates. In recent years, the model has been generalized to determine the prices of options with many different properties.

The relevance of path integrals to this field was recognized in 1988 by the theoretical physicist J.W. Dash, who wrote two unpublished papers on the subject entitled *Path Integrals and Options I* and *II* [91]. Since then many theoretical physicists have entered the field, and papers on this subject have begun appearing on the Los Alamos server [13, 92, 93].

20.7.2 Evolution Equations of Portfolios with Options

The option price $O(t)$ has larger fluctuations than the associated stock price. It usually varies with a slope $\partial O(S(t),t)/\partial S(t)$ which is commonly denoted by $\Delta(S(t),t)$ and called the *Delta* of the option. If $\Delta(S(t),t)$ depends only weakly on $S(t)$ and t it is possible, in the ideal case of Gaussian price fluctuations, to guarantee a steady growth of a portfolio. One merely has to mix a suitable number $N_S(t)$ stocks with $N_O(t)$ options and short-term bonds whose number is denoted by $N_B(t)$. As mentioned before, these are typically U.S. Government Treasury Bills with 30 days left to maturity which have only small price fluctuations. The composition $[N_S(t), N_O(t), N_B(t)]$ is referred to as the *strategy* of the portfolio manager. The total wealth has the value

$$W(t) = N_S(t)S(t) + N_O(t)O(S,t) + N_B(t)B(t). \tag{20.396}$$

The goal is to make $W(t)$ grow with a smooth exponential curve *without fluctuations*

$$\dot{W}(t) \approx r_W W(t). \tag{20.397}$$

As an idealization, the short-term bonds are assumed to grow deterministically without any fluctuations:

$$\dot{B}(t) \approx r_B B(t). \tag{20.398}$$

The rate r_B is the earlier-introduced riskfree interest rate encountered in true markets only if there are no events changing excessively the value of short-term bonds.

The existence of arbitrage dealers will ensure that the growth rate r_W is equal to that of the short-term bonds

$$r_W \approx r_B. \tag{20.399}$$

Otherwise the dealers would change from one investment to the other.

In the decomposition (20.396), the desired growth (20.397) reads

$$\begin{aligned} N_S(t)\dot{S}(t) + N_O(t)\dot{O}(S,t) + N_B(t)\dot{B}(t) + \dot{N}_S(t)S(t) + \dot{N}_O(t)O(S,t) + \dot{N}_B(t)B(t) \\ = r_W \left[N_S(t)S(t) + N_O(t)O(S,t) + N_B(t)B(t)\right]. \end{aligned} \tag{20.400}$$

Due to (20.398) and (20.399), the terms containing $N_B(t)$ without a dot drop out. Moreover, if no extra money is inserted into or taken from the system, i.e., if stocks, options, and bonds are only traded against each other, this does not change the total wealth, assuming the absence of commissions. This so-called *self-financing strategy* is expressed in the equation

$$\dot{N}_S(t)S(t) + \dot{N}_O(t)O(S,t) + \dot{N}_B(t)B(t) = 0. \tag{20.401}$$

Thus the growth equation (20.397) translates into

$$\dot{W}(t) = N_S\dot{S} + N_O\dot{O} + N_B\dot{B} = r_W\left(N_S S + N_O O + N_B B\right). \tag{20.402}$$

Due to the equality of the rates $r_W = r_B$ and Eq. (20.398), the entire contribution of $B(t)$ cancels, and we obtain

$$N_S\dot{S} + N_O\dot{O} = r_W\left(N_S S + N_O O\right). \tag{20.403}$$

The important observation is now that there exists an optimal ratio between the number of stocks N_S and the number of options N_O, which is equal to the negative slope $\Delta(S(t),t)$:

$$\frac{N_S(t)}{N_O(t)} = -\Delta(S(t),t) = -\frac{\partial O(S(t),t)}{\partial S(t)}. \tag{20.404}$$

Then Eq. (20.403) becomes

$$N_S\dot{S} + N_O\dot{O} = N_O r_W\left(-\frac{\partial O}{\partial x} + O\right). \tag{20.405}$$

The two terms on the left-hand side are treated as follows: First we use the relation (20.404) to rewrite

$$N_S\dot{S} = -N_O\frac{\partial O(S,t)}{\partial S}\dot{S} = -N_O\frac{\partial O(S,t)}{\partial x}\frac{\dot{S}}{S}. \tag{20.406}$$

In the second term on the left-hand side of (20.405), we expand the total time dependence of the option price in a Taylor series

$$\begin{aligned}\frac{dO}{dt} &= \frac{1}{dt}\Big[O(x(t)+\dot{x}(t)\,dt, t+dt) - O(x(t),t)\Big]\\ &= \frac{\partial O}{\partial t} + \frac{\partial O}{\partial x}\,\dot{x} + \frac{1}{2}\frac{\partial^2 O}{\partial x^2}\,\dot{x}^2\,dt + \frac{1}{3!}\frac{\partial^3 O}{\partial x^3}\,\dot{x}^3\,dt^2 + \ldots\,. \end{aligned} \tag{20.407}$$

We have gone over to the logarithmic stock price variable $x(t)$ rather than $S(t)$ itself. In financial mathematics, the lowest derivatives on the right-hand side are all denoted by special Greek symbols. We have already introduced the name Delta for the slope $\partial O/\partial S$. The curvature $\Gamma \equiv \partial^2 O/\partial S^2 = (\partial^2 O/\partial x^2 - \partial O/\partial x)/S^2$ is called the *Gamma* of an option. Another derivative with a standard name is the

Vega $V \equiv \partial O/\partial\sigma$. The partial time derivative $\partial O/\partial t$ is denoted by Θ. The set of these quantities is collectively called the *Greeks* [94].

In general, the expansion (20.407) is carried to arbitrary powers of $\dot{x}$ as in (20.273). It is, of course, only an abbreviated notation for the proper expansion in powers of a stochastic variable to be performed as in Eq. (20.273). After inserting (20.407) and (20.406) on the left-hand side of Eq. (20.405), this becomes

$$\begin{aligned} N_S\dot{S} + N_O\dot{O} = & - N_O\frac{\partial O}{\partial x}\frac{\dot{S}}{S} \\ & + N_O\left(\frac{\partial O}{\partial t} + \frac{\partial O}{\partial x}\dot{x} + \frac{1}{2}\frac{\partial^2 O}{\partial x^2}\dot{x}^2 dt + \frac{1}{3!}\frac{\partial O}{\partial x}\dot{x}^3 dt + \ldots\right) \\ = & N_O\left[\frac{\partial O}{\partial t} + \left(\dot{x} - \frac{\dot{S}}{S}\right)\frac{\partial O}{\partial x} + \frac{1}{2}\frac{\partial^2 O}{\partial x^2}\dot{x}^2 dt + \frac{1}{3!}\frac{\partial^3 O}{\partial x^3}\dot{x}^3 dt + \ldots\right]. \end{aligned} \tag{20.408}$$

Replacing further the left-hand side by the right-hand side of (20.405) we obtain

$$\frac{\partial O}{\partial t} = -r_W O - \left(\dot{x} - \frac{\dot{S}}{S} + r_W\right)\frac{\partial O}{\partial x} - \frac{1}{2}\frac{\partial^2 O}{\partial x^2}\dot{x}^2 dt - \frac{1}{3!}\frac{\partial^3 O}{\partial x^3}\dot{x}^3 dt + \ldots = 0. \tag{20.409}$$

The crucial observation which earned Black and Scholes the Nobel prize is that for Gaussian fluctuations with a Hamiltonian $H(p) = \sigma^2 p^2/2$, the equation (20.409) becomes very simple. First, due to the Itō relation (20.4), the prefactor of $-\partial O/\partial x$ becomes a constant

$$r_W - \frac{\sigma^2}{2} \equiv r_{xW}, \tag{20.410}$$

where the notation r_{xW} is chosen by analogy with r_x in the Itō relation (20.5). Thus there are no more fluctuations in the prefactor of $\partial O/\partial x$.

Moreover, also the fluctuations of all remaining terms

$$-\frac{1}{2}\frac{\partial^2 O}{\partial x^2}\dot{x}^2\, dt - \frac{1}{6}\frac{\partial^3 O}{\partial x^3}\dot{x}^3\, dt^2 + \ldots \tag{20.411}$$

can be neglected due to the result (18.429) and the estimates (18.431), which make truncate the expansion (20.411) and make it equal to $-(\sigma^2/2)\,\partial^2 O/\partial x^2$. Thus Eq. (20.409) loses its stochastic character, and the fair option price $O(x,t)$ is found to obey the Fokker-Planck-like differential equation

$$\frac{\partial O}{\partial t} = r_W O - r_{xW}\frac{\partial O}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2 O}{\partial x^2}, \tag{20.412}$$

At the same time, the total wealth (20.402) loses its stochastic character and grows with the riskfree rate r_W following the deterministic Eq. (20.398). The cancellation of the fluctuations is a consequence of choosing the ratio between options and stocks according to Eq. (20.404). The portfolio is now hedged against fluctuation.

The Fokker-Planck equation (20.412) can be expressed completely in terms of the Greeks of the option, using the relation (20.410):

$$\Theta = r_W O - r_{xW} S\Delta - \frac{\sigma^2}{2}\left(S\frac{\partial O}{\partial S} + S^2\frac{\partial^2 O}{\partial S^2}\right) = r_W O - r_W S\Delta - \frac{\sigma^2}{2} S^2\Gamma. \qquad (20.413)$$

The strategy to make a portfolio riskfree by balancing the fluctuations of N_S stocks by N_O options to satisfy Eq. (20.404) is called Delta-hedging. Hedging is of course imperfect. Since $\Delta(S(t), t)$ depends on the stock price, a Delta hedge requires frequent re-balancing of the portfolio, even several times per day. After a time span δt, the Delta has changed by

$$\delta\Delta = \frac{d}{dt}\frac{\partial^2 O(S(t), t)}{\partial S(t)}\delta t = \left[\frac{\partial}{\partial t}\frac{\partial O(S(t), t)}{\partial S(t)} + \frac{\partial^2 O(S(t), t)}{\partial S(t)^2}\right]\delta t = \left[\frac{\partial \Theta}{\partial S} + \Gamma\right]\delta t. \qquad (20.414)$$

So one has to readjust the ratio of stocks and options in Eq. (20.404) by

$$\delta\frac{N_S}{N_O} = -\delta\Delta = -\left(\frac{\partial\Theta}{\partial S} + \Gamma\right)\delta t. \qquad (20.415)$$

This procedure is called *dynamical* Δ*-hedging*. Since buying and selling costs money, δt cannot be made too small, otherwise dynamical hedging becomes too expensive.

In principle it is possible to buy a third asset to hedge also the curvature Gamma. This procedure is not so popular, again because of the transaction costs.

For non-Gaussian noise, the differential equation (20.409) is still stochastic due to remaining fluctuations in the expansion terms (20.411). This is an obstacle to building a riskfree portfolio with deterministically growing total wealth $W(t)$. As in Eq. (20.255) we can only derive the equation of motion for the average value

$$\left\langle \frac{1}{2}\frac{\partial^2 O}{\partial x^2}\dot{x}^2\, dt + \frac{1}{6}\frac{\partial^3 O}{\partial x^3}\dot{x}^3\, dt^2 + \ldots \right\rangle = -\bar{H}(i\partial_x)O. \qquad (20.416)$$

Hence a fair option price can only be calculated on the average from the Fokker-Planck-like differential equation

$$\frac{\partial}{\partial t}\langle O\rangle = \left[r_W - r_{xW}\frac{\partial}{\partial x} + \bar{H}(i\partial_x)\right]\langle O\rangle, \qquad (20.417)$$

where we have defined, by analogy with (20.268) and (20.410), an auxiliary rate parameter

$$r_{xW} \equiv r_W + \bar{H}(i). \qquad (20.418)$$

However, as pointed out in Subsection 20.2.2, this limitation is not really stringent for discrete short-time data if the tails of the noise distribution are non-Gaussian with semi-heavy tails. Then the higher expansion terms in Eq. (20.411) are suppressed by a small factor $\sigma\sqrt{\Delta t}$, and the expectation values in Eqs. (20.416) and (20.417) can again be dropped approximately.

For Gaussian fluctuations, the Fokker-Planck equation (20.412) can easily be solved. If we rename $t \to t_a$, and $x \to x_a$, for symmetry reasons, the solution which starts out, at some time t_b, like $\delta(x_a - x_b)$, has the Fourier representation

$$P^{(M,r_x)}(x_b t_b|x_a t_a) = e^{-r_W(t_b-t_a)} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x_b-x_a)} e^{-\left(\sigma^2 p^2/2 + i r_{x_W} p\right)(t_b-t_a)}. \tag{20.419}$$

A convergent integral exists for $t_b > t_a$.

For non-Gaussian fluctuations with semi-heavy tails, there exists an approximate solution whose Fourier representation is

$$P^{(M,r_x)}(x_b t_b|x_a t_a) = e^{-r_W(t_b-t_a)} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x_b-x_a)} e^{-\left[\bar{H}(p) + i r_{x_W} p\right](t_b-t_a)}, \tag{20.420}$$

with $\bar{H}(p)$ of Eq. (20.141).

Recalling the discussion in Section 20.3, this distribution function is recognized as a member of the equivalent family of martingale distributions (20.293) for the stock price $S(t) = e^{x(t)}$. It is the particular distribution in which the discount factor r coincides with the riskfree interest rate r_W.

20.7.3 Option Pricing for Gaussian Fluctuations

For Gaussian fluctuations where $H(p) = \sigma^2 p^2/2$, the integral in (20.419) can easily be performed and yields for $t_b > t_a$ the martingale distribution

$$P^{(M,r_x)}(x_b t_b|x_a t_a) = \frac{e^{-r_W(t_b-t_a)}}{\sqrt{2\pi\sigma^2(t_b-t_a)}} \exp\left\{-\frac{[x_b - x_a - r_{x_W}(t_b-t_a)]^2}{2\sigma^2(t_b-t_a)}\right\}. \tag{20.421}$$

This is the risk-neutral martingale distribution of Eq. (20.289) for $r = r_x$.

This distribution is obviously the solution of the path integral

$$P^{(M,r_x)}(x_b t_b|x_a t_a) = e^{-r_W(t_b-t_a)} \int \mathcal{D}x \, \exp\left\{-\frac{1}{2\sigma^2} \int_{t_a}^{t_b} [\dot{x} - r_{x_W}]^2\right\}. \tag{20.422}$$

An option is written for a certain *strike price* E of the stock. The value of the option at its expiration date t_b is given by the difference between the stock price on expiration date and the strike price:

$$O(x_b, t_b) = \Theta(x_b - x_E)(e^{x_b} - e^{x_E}), \tag{20.423}$$

where

$$x_E \equiv \log E. \tag{20.424}$$

The Heaviside function in (20.423) accounts for the fact that only for $S_b > E$ it is worthwhile to execute the option.

From (20.423) we calculate the option price at an arbitrary earlier time using the time evolution probability (20.421)

$$O(x_a, t_a) = \int_{-\infty}^{\infty} dx_b \, O(x_b, t_b) \, P^{(M, r_W)}(x_b t_b | x_a t_a). \tag{20.425}$$

Inserting (20.423) we obtain the sum of two terms

$$O(x_a, t_a) = O_S(x_a, t_a) - O_E(x_a, t_a), \tag{20.426}$$

where

$$O_S(x_a, t_a) = \frac{e^{-r_W(t_b - t_a)}}{\sqrt{2\pi\sigma^2(t_b - t_a)}} \int_{x_E}^{\infty} dx_b \, e^{x_b} \exp\left\{-\frac{[x_b - x_a - r_{xW}(t_b - t_a)]^2}{2\sigma^2(t_b - t_a)}\right\}, \tag{20.427}$$

and

$$O_E(x_a, t_a) = E e^{-r_W(t_b - t_a)} \frac{1}{\sqrt{2\pi\sigma^2(t_b - t_a)}} \int_{x_E}^{\infty} dx_b \, \exp\left\{-\frac{[x_b - x_a - r_{xW}(t_b - t_a)]^2}{2\sigma^2(t_b - t_a)}\right\}. \tag{20.428}$$

In the second integral we set

$$x_- \equiv x_a + r_{xW}(t_b - t_a) = x_a + \left(r_W - \frac{1}{2}\sigma^2\right)(t_b - t_a), \tag{20.429}$$

and obtain

$$O_E(x_a, t_a) = E \frac{e^{-r_W(t_b - t_a)}}{\sqrt{2\pi\sigma^2(t_b - t_a)}} \int_{x_E - x_-}^{\infty} dx_b \, \exp\left\{-\frac{x_b^2}{2\sigma^2(t_b - t_a)}\right\}. \tag{20.430}$$

After rescaling the integration variable $x_b \to -\xi\sigma\sqrt{t_b - t_a}$, this can be rewritten as

$$O_E(x_a, t_a) = e^{-r_W(t_b - t_a)} E \, N(y_-), \tag{20.431}$$

where $N(y)$ is the cumulative Gaussian distribution function

$$N(y) \equiv \int_{-\infty}^{y} \frac{d\xi}{\sqrt{2\pi}} e^{-\xi^2/2} = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{y}{\sqrt{2}}\right)\right], \tag{20.432}$$

evaluated at

$$\begin{aligned} y_- \;&\equiv\; \frac{x_- - x_E}{\sqrt{\sigma^2(t_b - t_a)}} = \frac{\log[S(t_a)/E] + r_{xW}(t_b - t_a)}{\sqrt{\sigma^2(t_b - t_a)}} \\ &=\; \frac{\log[S(t_a)/E] + \left(r_W - \frac{1}{2}\sigma^2\right)(t_b - t_a)}{\sqrt{\sigma^2(t_a - t_b)}}. \end{aligned} \tag{20.433}$$

The integral in the first contribution (20.427) to the option price is found after completing the exponent in the integrand quadratically as follows:

$$\begin{aligned} x_b - & \frac{[x_b - x_a - r_{xW}(t_b - t_a)]^2}{2\sigma^2(t_b - t_a)} \\ &= -\frac{[x_b - x_a - (r_{xW} + \sigma^2)(t_b - t_a)]^2 - 2r_W\sigma^2(t_b - t_a) - 2x_a\sigma^2(t_b - t_a)}{2\sigma^2(t_b - t_a)}. \end{aligned} \tag{20.434}$$

Introducing now

$$x_+ \equiv x_a + \left(r_{xW} + \sigma^2\right)(t_b - t_a) = x_a + \left(r_W + \frac{1}{2}\sigma^2\right)(t_b - t_a), \tag{20.435}$$

and rescaling x_b as before, we obtain

$$O_S(x_a, t_a) = S(t_a)N(y_+), \tag{20.436}$$

with

$$\begin{aligned} y_+ &\equiv \frac{x_+ - x_E}{\sqrt{\sigma^2(t_a - t_b)}} = \frac{\log[S(t_a)/E] + (r_{xW} + \sigma^2)(t_b - t_a)}{\sqrt{\sigma^2(t_a - t_b)}} \\ &= \frac{\log[S(t_a)/E] + \left(r_W + \frac{1}{2}\sigma^2\right)(t_b - t_a)}{\sqrt{\sigma^2(t_a - t_b)}}. \end{aligned} \tag{20.437}$$

The combined result

$$O(x_a, t_a) = S(t_a)N(y_+) - e^{-r_W(t_b - t_a)}E\,N(y_-) \tag{20.438}$$

is the celebrated *Black-Scholes formula* of option pricing.

In Fig. 20.28 we illustrate how the dependence of the call price on the stock price varies with different times to expiration $t_b - t_a$ and with different volatilities σ.

Floor dealers of stock markets use the Black-Scholes formula to judge how expensive options are, so they can decide whether to buy or to sell them. For a given riskfree interest rate r_W and time to expiration $t_b - t_a$, and a set of option, stock, and strike prices O, S, E, they calculate the volatility from (20.438). The result is called the *implied volatility*, and denoted by $\Sigma(x - x_E)$. As typical plot as a function of $x - x_E$ is shown in Fig. 20.29.

If the Black-Scholes formula were exactly valid, the data should lie on a horizontal line $\Sigma(x - x_E) = \sigma$. Instead, they scatter around a parabola which is called the *smile* of the option. The smile indicates the presence of a nonzero kurtosis in the distribution of the returns, as we shall see in Subsection 20.7.8.

It goes without saying that if the integral (20.427) is carried out over the entire x_b axis, it becomes independent of time, due to the martingale character of the risk-neutral distribution $P^{(M,r_W)}(x_b t_b | x_a t_a)$.

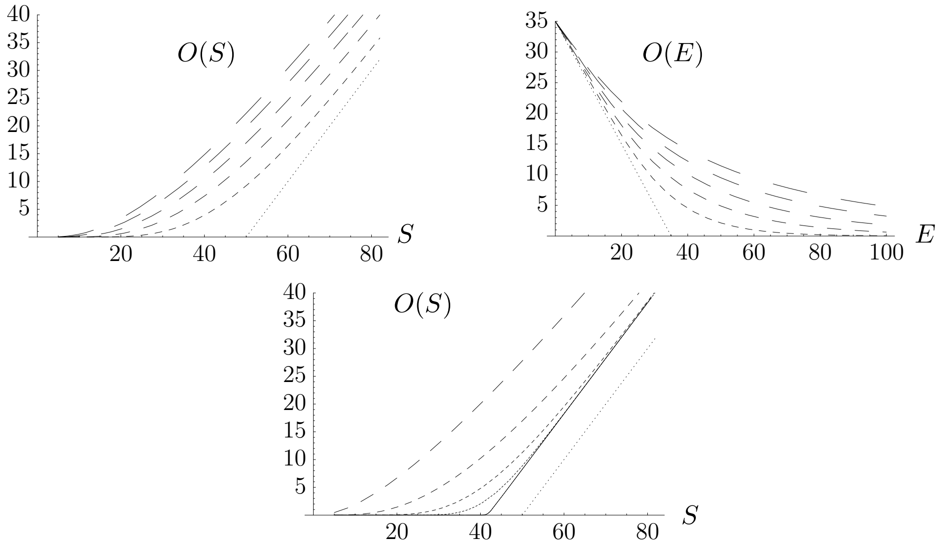

Figure 20.28 Left: Dependence of call price O on stock price S for different times before expiration date (increasing dash length: $1, 2, 3, 4, 5$ months). The parameters are $E = 50$ US\$, $\sigma = 40\%$, $r_W = 6\%$ per month. Right: Dependence on the strike price E for fixed stock price 35 US\$ and the same times to expiration (increasing with dash length). Bottom: Dependence on the volatilities (from left to right: $80\%, 60\%, 20\%, 10\%, 1\%$) at a fixed time $t_b - t_a = 3$ months before expiration.

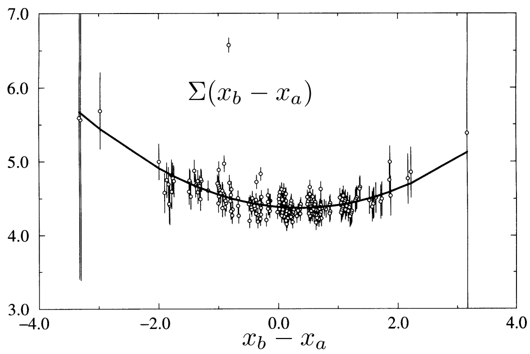

Figure 20.29 Smile deduced from options (see [13]).

20.7.4 Option Pricing for Asian Option

In an Asian option one does not bet on an option price at the expiration date but on an average option price over the option life. The calculation is very simple if we recall the time amplitude amplitude (3.200) of a particle at a fixed average x_0. The stochastic version of the quantum-mechanical expression (3.200) is

$$P(x_b t_b | x_a t_a)^{x_0} = \frac{\sqrt{3}}{\pi\sigma^2(t_b - t_a)} \exp\left\{ -\frac{1}{2\sigma^2(t_b - t_a)} \left[(x_b - x_a)^2 + 12 \left(x_0 - \frac{x_b + x_a}{2} \right)^2 \right] \right\}. \tag{20.439}$$

For the growth rate r_W, the associated martingale becomes

$$P^{(M,r_x)}(x_bt_b|x_at_a)^{x_0}=e^{-r_W(t_b-t_a)}\frac{\sqrt{3}}{\pi\sigma^2(t_b-t_a)}$$
$$\times\exp\left\{-\frac{1}{2\sigma^2(t_b-t_a)}\left[(x_b-x_a-r_{x_W}(t_b-t_a))^2+12\left(x_0-\frac{x_b+x_a}{2}\right)^2\right]\right\},\quad(20.440)$$

whose integral over x_0 yields back the previous martingale distribution (20.421)

$$\int_{-\infty}^{\infty}dx_0\,P^{(M,r_x)}(x_bt_b|x_at_a)^{x_0}=P^{(M,r_x)}(x_bt_b|x_at_a).\qquad(20.441)$$

Instead of (20.423), the option price at expiration is now

$$O(x_b,t_b)=\Theta(x_b-x_0)(e^{x_b}-e^{x_0}),\qquad(20.442)$$

so that stead of (20.426) is replaced by the difference between

$$O_S(x_a,t_a)=\int_{-\infty}^{\infty}dx_0\int_{x_0}^{\infty}dx_b\,e^{x_b}P^{(M,r_x)}(x_bt_b|x_at_a)^{x_0},\qquad(20.443)$$

which is the same as (20.427), due to (20.441), and

$$O_{\bar{x}}(x_a,t_a)=\int_{-\infty}^{\infty}dx_0\,e^{x_0}\int_{x_0}^{\infty}dx_b\,P^{(M,r_x)}(x_bt_b|x_at_a)^{x_0},\qquad(20.444)$$

which replaces (20.430). Since both expressions involve integrals over errorfunctions it is useful to represent the δ-function in (20.442) as a Fourier integral and rewrite the option price at t_a as

$$\begin{aligned}O(x_a,t_a)&=\int_{-\infty}^{\infty}\frac{dt}{2\pi i}\frac{e^{i(x_b-x_0)t}}{t-i\eta}\\&\times\int_{-\infty}^{\infty}dx_0\int_{-\infty}^{\infty}dx_b\,(e^{x_b}-e^{x_0})P^{(M,r_x)}(x_bt_b|x_at_a)^{x_0},\end{aligned}\qquad(20.445)$$

The result of the double integral in the second line is simply

$$e^{-A(t)}-e^{-A_0(t)},\qquad(20.446)$$

with

$$\begin{aligned}A(t)&=-r_W(t_b-t_a)-\frac{i}{2}\left(r_W+\frac{\sigma^2}{2}\right)t(t_b-t_a)+\frac{\sigma^2t^2(t_b-t_a)}{6},\\A_0(t)&=\left(-r_W+\frac{\sigma^2}{6}\right)\frac{t_b-t_a}{2}-\frac{i}{2}\left(r_W-\frac{\sigma^2}{6}\right)t(t_b-t_a)+\frac{\sigma^2t^2(t_b-t_a)}{6}.\end{aligned}\qquad(20.447)$$

Now we use the integral formula

$$\int_{-\infty}^{\infty}\frac{dt}{2\pi i}\frac{e^{-a^2t^2/2+ibt}}{t-i\eta}=N[b/a].\qquad(20.448)$$

and find [95]

$$O(x_a,t_a) = S(t_a)[N(z_+) - e^{-(r_W+\sigma^2/6)(t_b-t_a)/2}N(z_-)], \tag{20.449}$$

where

$$z_+ = \sqrt{\frac{3(t_b-t_a)}{4\sigma^2}}\left(r_W + \frac{\sigma^2}{2}\right), \qquad z_- = \sqrt{\frac{3(t_b-t_a)}{4\sigma^2}}\left(r_W - \frac{\sigma^2}{6}\right). \tag{20.450}$$

20.7.5 Option Pricing for Boltzmann Distribution

The above result can easily be extended to price the options for assets whose returns obey the Boltzmann distribution (20.204). Since this is a superposition of Gaussian distributions, we merely have to perform the same superposition over the Black-Scholes formula (20.438). Thus we insert (20.438) into the integral (20.204) and obtain the option price for the Boltzmann-distributed assets of variance σ^2:

$$O^B(x_a,t_a) = \left(\frac{t}{\sigma^2}\right)^t \frac{1}{\Gamma(t)}\int_0^\infty \frac{dv}{v}\, v^t e^{-tv/\sigma^2} O^v(x_a,t_a), \tag{20.451}$$

where $O^v(x_a,t_a)$ is the Black-Scholes option price (20.438) of variance v. The superscript indicates that the variance σ^2 in the variables of y_+ and y_- in (20.433) and (20.437) is now exchanged by the integration variable v.

Since the Boltzmann distribution for the minute data turns rapidly into a Gaussian (recall Fig. 20.15), the price changes with respect to the Black-Scholes formula are relevant only for short-term options running less than a week. The changes can most easily be estimated by using the expansion (20.209) to write

$$O^B(x_a,t_a) = O^v(x_a,t_a) + \frac{[2T^2(1-1/t)]^2}{2t}\partial_v^2 O^v(x_a,t_a) + \dots\ . \tag{20.452}$$

20.7.6 Option Pricing for General Non-Gaussian Fluctuations

For general non-Gaussian fluctuations with semi-heavy tails, the option price must be calculated numerically from Eqs. (20.425) and (20.423). Inserting the Fourier representation (20.419) and using the Hamiltonian

$$H_{r_{xW}}(p) \equiv \bar{H}(p) + i r_{xW} p \tag{20.453}$$

defined as in (20.141), this becomes

$$\begin{aligned} O(x_a,t_a) &= \int_{x_E}^\infty dx_b\,(e^{x_b} - e^{x_E})P(x_b t_b|x_a t_a) \\ &= e^{-r_W(t_b-t_a)}\int_{x_E}^\infty dx_b\,(e^{x_b} - e^{x_E})\int_{-\infty}^\infty \frac{dp}{2\pi} e^{ip(x_b-x_a) - H_{r_{xW}}(p)(t_b-t_a)}. \end{aligned} \tag{20.454}$$

The integrand can be rearranged as follows:

$$O(x_a,t_a)=e^{-r_W(t_b-t_a)}\int_{x_E}^{\infty}dx_b\int_{-\infty}^{\infty}\frac{dp}{2\pi}\left[e^{x_a}e^{i(p-i)(x_b-x_a)}-e^{x_E}e^{ip(x_b-x_a)}\right]e^{-H_{r_{xW}}(p)(t_b-t_a)}. \tag{20.455}$$

Two integrations are required. This would make a numerical calculation quite time consuming. Fortunately, one integration can be done analytically. For this purpose we change the momentum variable in the first part of the integral from p to $p+i$ and rewrite the integral in the form

$$O(x_a,t_a)=e^{-r_W(t_b-t_a)}\int_{x_E}^{\infty}dx_b\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ip(x_b-x_a)}f(p), \tag{20.456}$$

with

$$f(p)\equiv e^{x_a}e^{-H_{r_{xW}}(p+i)(t_b-t_a)}-e^{x_E}e^{-H_{r_{xW}}(p)(t_b-t_a)}. \tag{20.457}$$

We have suppressed the arguments x_a, x_E, t_b-t_a in $f(p)$, for brevity. The integral over x_b in (20.456) runs over the Fourier transform

$$\tilde{f}(x_b-x_a)=\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ip(x_b-x_a)}f(p) \tag{20.458}$$

of the function $f(p)$. It is then convenient to express the integral $\int_{x_E}^{\infty}dx_b$ in terms of the Heaviside function $\Theta(x_b-x_E)$ as $\int_{-\infty}^{\infty}dx_b\,\Theta(x_b-x_E)$ and use the Fourier representation (1.312) of the Heaviside function to write

$$\int_{x_E}^{\infty}dx_b\,\tilde{f}(x_b-x_a)=\int_{-\infty}^{\infty}dx_b\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{i}{q+i\eta}e^{-iq(x_b-x_E)}\tilde{f}(x_b-x_a). \tag{20.459}$$

Inserting here the Fourier representation (20.458), we can perform the integral over x_b and obtain the momentum space representation of the option price

$$O(x_a,t_a)=e^{-r_W(t_b-t_a)}\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ip(x_E-x_a)}\frac{i}{p+i\eta}f(p). \tag{20.460}$$

For numerical integrations, the singularity at $p=0$ is inconvenient. We therefore employ the decomposition (18.54)

$$\frac{1}{p+i\eta}=\frac{\mathcal{P}}{p}-i\pi\delta(p), \tag{20.461}$$

to write

$$O(x_a,t_a)=e^{-r_W(t_b-t_a)}\left[\frac{1}{2}f(0)+i\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{e^{ip(x_E-x_a)}f(p)-f(0)}{p}\right]. \tag{20.462}$$

We have used the fact that the principal value of the integral over $1/p$ vanishes to subtract the constant $f(0)$ from $e^{ip(x_E-x_a)}f(p)$. After this, the integrand is regular and does not need any more the principal-value specification, and allows for a numerical integration.

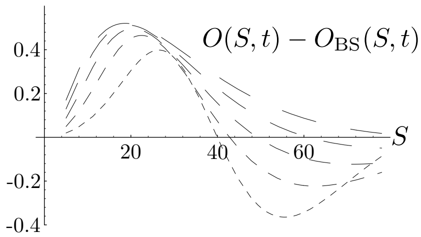

Figure 20.30 Difference between call price $O(S,t)$ obtained from truncated Lévy distribution with kurtosis $\kappa = 4$ and Black-Scholes price $O_{\rm BS}(S,t)$ with $\sigma^2 = \bar{v}$ as function of stock price S for different times before expiration date (increasing dash length: $1,2,3,4,5$ months). The parameters are $E = 50\,$US\$, $\sigma = 40\%$, $r_W = 6\%$ per month.

For x_a very much different from x_E, we may approximate

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{e^{ip(x_E - x_a)} f(p) - f(0)}{p} \approx \frac{1}{2} \epsilon(x_a - x_E) f(0), \tag{20.463}$$

where $\epsilon(x) \equiv 2\Theta(x) - 1$ is the step function (1.315), and obtain

$$O(x_a, t_a) \approx \frac{e^{-r_W(t_b - t_a)}}{2} \left[1 + \Theta(x_a - x_E)\right] f(0). \tag{20.464}$$

Using (20.418) we have $e^{-H_{r_{xW}}(i)} = e^{r_W}$, and since $e^{-H_{r_{xW}}(0)} = 1$ we see that $O(x_a, t_a)$ goes to zero for $x_a \to -\infty$ and has the large-x_a behavior

$$O(x_a, t_a) \approx e^{x_a} - e^{x_E} e^{-r_W(t_b - t_a)} = S(t_a) - e^{-r_W(t_b - t_a)} E. \tag{20.465}$$

This is the same behavior as in the Black-Scholes formula (20.438).

In Fig. 20.30 we display the difference between the option prices emerging from our formula (20.462) with a truncated Lévy distribution of kurtosis $\kappa = 4$, and the Black-Scholes formula (20.438) for the same data as in the upper left of Fig. 20.28.

For truncated Lévy distributions, the Fourier integral in Eq. (20.454) can be expressed directly in terms of the original distribution function which is the Fourier transform of (20.52):

$$\tilde{L}_{\sigma^2}^{(\lambda,\alpha,\beta)}(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx - H(p)}. \tag{20.466}$$

By inspecting Eq. (20.53) we see that the factor $t_b - t_a$ multiplying $H_{r_{xW}}(p)$ in (20.454) can be absorbed into the parameters $\sigma, \lambda, \alpha, \beta$ of the truncated Lévy distributions by replacing

$$\sigma^2 \to \sigma^2(t_b - t_a), \qquad r_{xW} \to r_{xW}(t_b - t_a). \tag{20.467}$$

Let us denote the truncated Lévy distribution with zero average by $\bar{L}_{\sigma^2}^{(\lambda,\alpha,\beta))}(x)$. It is the Fourier transform of $e^{-\bar{H}(p)}$:

$$\bar{L}_{\sigma^2}^{(\lambda,\alpha,\beta)}(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx - \bar{H}(p)}. \tag{20.468}$$

The Fourier transform of $e^{-\bar{H}(p)(t_b-t_a)}$ is then simply given by $\bar{L}^{(\lambda,\alpha,\beta))}_{\sigma^2(t_b-t_a)}(x)$. The additional term r_{xW} in the exponent of the integral (20.454) via (20.453) leads to a drift r_{xW} in the distribution, and we obtain

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx-[\bar{H}(p)+ir_{xW}p](t_b-t_a)} = \tilde{L}^{(\lambda,\alpha,\beta))}_{\sigma^2(t_b-t_a)}(x - r_{xW}(t_b-t_a)). \qquad (20.469)$$

Inserting this into (20.419), we find the riskfree martingale distribution to be inserted into (20.454):

$$P(x_b t_b|x_a t_a) = e^{-r_W(t_b-t_a)} \bar{L}^{(\lambda,\alpha,\beta)}_{\sigma^2(t_b-t_a)}(x_b - x_a - r_{xW}(t_b-t_a)). \qquad (20.470)$$

The result is therefore a truncated Lévy distribution of increasing width and uniformly moving average position. Since all expansion coefficients c_n of $H(p)$ in Eq. (20.40) receive the same factor $t_b - t_a$, the kurtosis $\kappa = c_4/c_2^2$ decreases inversely proportional to $t_b - t_a$. As time proceeds, the distribution becomes increasingly Gaussian, this being a manifestation of the central limit theorem of statistical mechanics. This is in contrast to the pure Lévy distribution which has no finite width and therefore maintains its power falloff at large distances.

Explicitly, the formula (20.454) for the option price becomes

$$O(x_a, t_a) = e^{-r_W(t_b-t_a)} \int_{x_E}^{\infty} dx_b\, (e^{x_b} - e^{x_E}) \bar{L}^{(\lambda,\alpha,\beta)}_{\sigma^2(t_b-t_a)}(x_b - x_a - r_{xW}(t_b-t_a)). \qquad (20.471)$$

This and similar equations derived from any of the other non-Gaussian models lead to fairer formulas for option prices.

20.7.7 Option Pricing for Fluctuating Variance

If the fluctuations of the variance are taken into account, the dependence of the price of an option on $v(t)$ needs to be considered in the derivation of a time evolution equation for the option price. Instead of Eq. (20.407) we write the time evolution as

$$\begin{aligned}\frac{dO}{dt} &= \frac{1}{dt}\Big[O(x(t)+\dot{x}(t)\,dt, v(t)+\dot{v}(t)\,dt, t+dt) - O(x(t),v(t),t)\Big] \\ &= \frac{\partial O}{\partial t} + \frac{\partial O}{\partial x}\dot{x} + \frac{\partial O}{\partial v}\dot{v} + \frac{1}{2}\frac{\partial^2 O}{\partial x^2}\dot{x}^2\,dt + \frac{\partial^2 O}{\partial x \partial v}\dot{x}\dot{v}\,dt + \frac{1}{2}\frac{\partial^2 O}{\partial v^2}\dot{v}^2 dt + \dots . \qquad (20.472)\end{aligned}$$

The expansion can be truncated after the second derivative due to the Gaussian nature of the fluctuations. We use Itō's rule to replace

$$\dot{x}^2 \longrightarrow v(t), \qquad \dot{v}^2 \longrightarrow \epsilon^2 v(t), \qquad \dot{x}\dot{v} \longrightarrow \rho\epsilon v(t). \qquad (20.473)$$

These replacements follow directly from Eqs. (20.306) and (20.308) and the correlation functions (20.310). Thus we obtain

$$\begin{aligned}\frac{dO}{dt} &= \frac{1}{dt}\Big[O(x(t)+\dot{x}(t)\,dt, v(t)+\dot{v}(t)\,dt, t+dt) - O(x(t),v(t),t)\Big] \\ &= \frac{\partial O}{\partial t} + \frac{\partial O}{\partial x}\dot{x} + \frac{\partial O}{\partial v}\dot{v} + \frac{1}{2}\frac{\partial^2 O}{\partial x^2} v + \frac{\partial^2 O}{\partial x \partial v}\rho\epsilon v + \frac{1}{2}\frac{\partial^2 O}{\partial v^2}\epsilon^2 v. \qquad (20.474)\end{aligned}$$

This is inserted into Eq. (20.403). If we adjust the portfolio according to the rule (20.404), and use the Itō relation $\dot{S}/S = \dot{x} + v/2$, we obtain the equation [compare (20.409)]

$$\begin{aligned} N_O r_W \left(-\frac{\partial O}{\partial x} + O \right) &= - N_O \frac{\partial O}{\partial x} \left(\dot{x} + \frac{v^2}{2} \right) + N_O \dot{O} \\ &= N_O \left[-\frac{v^2}{2} \frac{\partial O}{\partial x} + \frac{\partial O}{\partial v} \dot{v} + \frac{\partial O}{\partial t} + \frac{1}{2} \frac{\partial^2 O}{\partial x^2} v + \frac{\partial^2 O}{\partial x \partial v} \rho \epsilon v + \frac{1}{2} \frac{\partial^2 O}{\partial v^2} \epsilon^2 v + \dots \right]. \end{aligned} \quad (20.475)$$

As before in Eq. (20.409), the noise in $\dot{x}$ has disappeared. In contrast to the single-variable treatment, however, the noise variable η_v remains in the equation. It can only be removed if we trade a financial asset V whose price is equal to the variance directly on the markets. Then we can build a *riskfree portfolio* containing four assets

$$W(t) = N_S(t) S(t) + N_O(t) O(S, t) + N_V(t) V(t) + N_B(t) B(t), \quad (20.476)$$

instead of (20.396). Indeed, by adjusting

$$\frac{N_V(t)}{N_O(t)} = -\frac{\partial O(S(t), v(t), t)}{\partial v(t)}, \quad (20.477)$$

we could cancel the term $\dot{v}$ in Eq. (20.475). There is definitely need to establish trading in such an asset. Without this, we can only reach an approximate freedom of risk by ignoring the noise $\eta_v(t)$ in the term $\dot{v}$ and replacing $\dot{v}$ by the deterministic first term in the stochastic differential equation (20.308):

$$\dot{v}(t) \longrightarrow -\gamma [v(t) - \bar{v}]. \quad (20.478)$$

In addition, we can account for the fact that the option price rises with the variance as in the Black-Scholes formula by adding on the right hand side of (20.478) a phenomenological correction term $-\lambda v$ called *price of volatility risk* [78, 11]. Such a term has simply the effect of renormalizing the parameters γ and $\bar{v}$ to

$$\gamma^* = \gamma + \lambda, \qquad \text{and} \qquad \bar{v}^* = \gamma \bar{v} / \gamma^*. \quad (20.479)$$

Thus we find the Fokker-Planck-like differential equation [compare (20.417)]

$$\frac{\partial O}{\partial t} = r_W O - \left(r_W - \frac{v}{2} \right) \frac{\partial O}{\partial x} + \gamma^* [v(t) - \bar{v}^*] \frac{\partial O}{\partial v} - \frac{v}{2} \frac{\partial^2 O}{\partial x^2} - \rho \epsilon v \frac{\partial^2 O}{\partial x \partial v} - \frac{\epsilon^2 v}{2} \frac{\partial^2 O}{\partial v^2}. \quad (20.480)$$

On the right-hand side we recognize the Hamiltonian operator (20.319), with γ and $\bar{v}$ replaced by γ^* and $\bar{v}^*$, in terms of which we can write

$$\frac{\partial O}{\partial t} = r_W \left(O - \partial_x O \right) + \left(\hat{H}^* + \gamma^* + \rho \epsilon \partial_x + \epsilon^2 \partial_v \right) O. \quad (20.481)$$

The solution of this equation can easily be expressed as a slight modification of the solution $P(x_b, v_b, t_b|x_a v_a t_a)$ in Eq. (20.340) of the differential equation (20.318). Since it contains only additional first-order derivatives with respect to (20.318), we simply find, with $x \equiv x_a$ and $t \equiv t_a$, the solution $P^v(x_b, v_b, t_b|x_a v_a t_a)$ satisfying the initial condition

$$P^v(x_b, v_b, t_b|x_a v_a t_a) = \delta(x_b - x_a)\delta(v_b - v_a) \tag{20.482}$$

as follows:

$$P^v(x_b, v_b, t_b|x_a v_a t_a) = e^{-(r_W+\gamma^*)\Delta t} P_{\rm sh}(x_b, v_b, t_b|x_a v_a t_a), \tag{20.483}$$

where the subscript sh indicates that the arguments $x_b - x_a$ and $v_b - v_a$ are shifted:

$$x_b - x_a \to x_b - x_a - (r_W - \rho\epsilon), \qquad v_b - v_a \to v_b - v_a - \epsilon^2. \tag{20.484}$$

The distribution (20.482) may be inserted into an equation of the type (20.425) to find the option price at the time t_a from the price (20.423) at the expiration date t_b. If we assume the variance v_a to be equal to $\bar{v}$, and the remaining parameters to be

$$\gamma^* = 2, \quad \bar{v} = 0.01, \quad \epsilon = 0.1, r_W = 0, \tag{20.485}$$

the price of an option with strike price $E = 100$ one half year before expiration with the stock price $S = E$ (this is called an option *at-the-money*) is 2.83 US$ for $\rho = -0.5$ and 2.81 US$ for $\rho = 0.5$. The difference with respect to the Black-Scholes price is shown in Fig. 20.31.

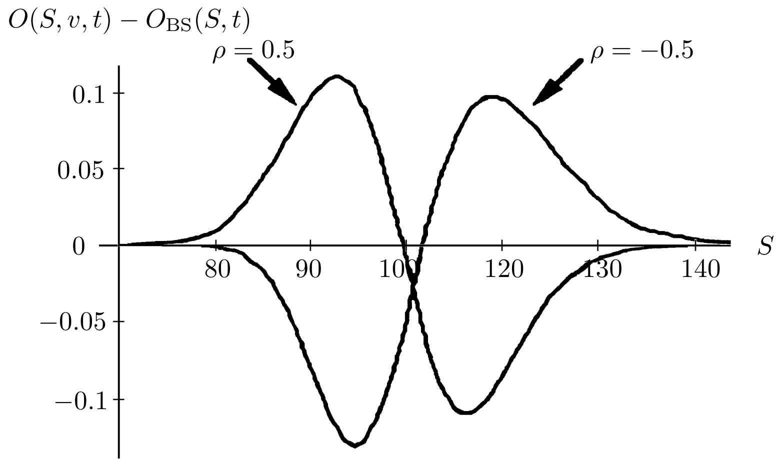

Figure 20.31 Difference between option price $O(S, v, t)$ with fluctuating volatility and Black-Scholes price $O_{\rm BS}(S, t)$ with $\sigma^2 = \bar{v}$ for option of strike price 100 US$. The parameters are given in Eq. (20.485). The noise correlation parameter is once $\rho = -0.5$ and once $\rho = 0.5$. For an at-the-money option the absolute value is 2.83 US$ for $\rho = -0.5$ and 2.81 US$ for $\rho = 0.5$ (after Ref. [78]).

20.7.8 Perturbation Expansion and Smile

A perturbative treatment of any non-Gaussian distributions $\tilde{D}(x)$, which we assume to be symmetric, for simplicity, starts from the expansion

$$D(p) = \left[1 + \frac{a_4}{4!}p^4 - \frac{a_6}{6!}p^6 + \frac{c_8}{8!}p^8 - \ldots - \ldots\right] e^{-\sigma^2 p^2/2}, \tag{20.486}$$

where

$$a_4 = c_4, \quad a_6 = c_6, \quad a_8 = c_8 + 35c_4^2, \quad a_{10} = c_{10} + 210c_4c_6, \ldots\ , \tag{20.487}$$

which can also be expressed as a series

$$D(p) = \left[1 + \frac{a_4}{4!}2^2\left(\frac{\partial}{\partial\sigma^2}\right)^2 + \frac{a_6}{6!}2^3\left(\frac{\partial}{\partial\sigma^2}\right)^3 + \frac{a_8}{8!}2^4\left(\frac{\partial}{\partial\sigma^2}\right)^4 + \ldots\right] e^{-\sigma^2 p^2/2}. \tag{20.488}$$

By taking the Fourier transform we obtain the expansion of the distributions in x-space

$$\begin{aligned}
\tilde{D}(x) &= \left[1 + \frac{a_4}{4!}2^2\left(\frac{\partial}{\partial\sigma^2}\right)^2 + \frac{a_6}{6!}2^3\left(\frac{\partial}{\partial\sigma^2}\right)^3 + \frac{a_8}{8!}2^4\left(\frac{\partial}{\partial\sigma^2}\right)^4 + \ldots\right] \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \\
&= \left[1 + \frac{\bar{c}_4}{8} - \frac{\bar{c}_6}{48} + \frac{35\bar{c}_4^2}{384} + \frac{\bar{c}_8}{384} + \frac{x^2}{\sigma^2}\left(-\frac{\bar{c}_4}{4} + \frac{\bar{c}_6}{16} - \frac{35\bar{c}_4^2}{96} - \frac{\bar{c}_8}{96}\right)\right. \\
&\quad + \frac{x^4}{\sigma^4}\left(\frac{\bar{c}_4}{24} - \frac{\bar{c}_6}{48} + \frac{35\bar{c}_4^2}{192} + \frac{\bar{c}_8}{192}\right) + \frac{x^6}{\sigma^6}\left(\frac{\bar{c}_6}{720} - \frac{35\bar{c}_4^2}{1440} - \frac{\bar{c}_8}{1440}\right) \\
&\quad \left. + \frac{x^8}{\sigma^8}\left(\frac{35\bar{c}_4^2}{40320} + \frac{\bar{c}_8}{40320}\right) + \ldots\right] \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}.
\end{aligned} \tag{20.489}$$

The quantities $\bar{c}_n$ contain the kurtosis κ, in the case of a truncated Lévy distribution with the powers $\varepsilon^{n/2-1}$. If the distribution is close to a Gaussian, we may re-expand all expressions in powers of the higher cumulants. In the case of a truncated Lévy distribution, we may keep systematically all terms up to a certain maximal power of ε and find

$$\begin{aligned}
\tilde{D}(x) &= \left\{1 - \frac{x^2}{2\sigma^2}\left(\frac{\bar{c}_4}{2} - \frac{\bar{c}_6}{8} + \frac{2\bar{c}_4^2}{3} + \frac{\bar{c}_8}{48} + \frac{5\bar{c}_4\bar{c}_6}{192} - \frac{33\bar{c}_4^3}{256}\right)\right. \\
&\quad + \frac{x^4}{\sigma^4}\left(\frac{\bar{c}_4}{24} - \frac{\bar{c}_6}{48} + \frac{17\bar{c}_4^2}{96} + \frac{\bar{c}_8}{192} + \frac{\bar{c}_4\bar{c}_6}{288} - \frac{239\bar{c}_4^3}{9216}\right) \\
&\quad + \frac{x^6}{\sigma^6}\left(\frac{\bar{c}_6}{720} - \frac{7\bar{c}_4^2}{288} - \frac{\bar{c}_8}{1440} - \frac{\bar{c}_4\bar{c}_6}{5760} + \frac{7\bar{c}_4^3}{2304}\right) \\
&\quad \left. + \frac{x^8}{\sigma^8}\left(\frac{\bar{c}_4^2}{1152} + \frac{\bar{c}_8}{40320} - \frac{\bar{c}_4^3}{9216}\right) + \ldots\right\} \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma_1^2}},
\end{aligned} \tag{20.490}$$

where we have introduced the modified width

$$\tilde{\sigma}_1^2 \equiv \sigma^2\left(1 - \frac{\bar{c}_4}{4} + \frac{\bar{c}_6}{24} - \frac{13\bar{c}_4^2}{96} - \frac{\bar{c}_8}{192} - \frac{\bar{c}_4\bar{c}_6}{64} - \frac{31\bar{c}_4^3}{512} + \ldots\right). \tag{20.491}$$

The prefactor can be taken into the exponent yielding

$$\begin{aligned}
\tilde{D}(x) &= \exp\left\{-\frac{x^2}{2\sigma^2}\left(1 + \frac{\bar{c}_4}{2} + \frac{2\,\bar{c}_4^2}{3} - \frac{33\,\bar{c}_4^3}{256} - \frac{\bar{c}_6}{8} + \frac{5\,\bar{c}_4\,\bar{c}_6}{192} + \frac{\bar{c}_8}{48}\right)\right. \\
&\quad + \frac{x^4}{\sigma^4}\left(\frac{\bar{c}_4}{24} + \frac{7\,\bar{c}_4^2}{48} - \frac{1007\,\bar{c}_4^3}{9216} - \frac{\bar{c}_6}{48} + \frac{11\,\bar{c}_4\,\bar{c}_6}{576} + \frac{\bar{c}_8}{192}\right) \\
&\quad + \frac{x^6}{\sigma^6}\left(-\frac{\bar{c}_4^2}{72} + \frac{43\,\bar{c}_4^3}{768} + \frac{\bar{c}_6}{720} - \frac{23\,\bar{c}_4\,\bar{c}_6}{2880} - \frac{\bar{c}_8}{1440}\right) \\
&\quad \left. + \frac{x^8}{\sigma^8}\left(-\frac{101\,\bar{c}_4^3}{9216} + \frac{7\,\bar{c}_4\,\bar{c}_6}{5760} + \frac{\bar{c}_8}{40320}\right) + \ldots\right\} \frac{1}{\sqrt{2\pi\sigma_1^2}}.
\end{aligned} \tag{20.492}$$

Introducing a second modified width

$$\tilde{\sigma}_2^2 \equiv \sigma^2\left(1-\frac{\bar{c}_4}{2}+\frac{\bar{c}_6}{8}-\frac{5\bar{c}_4^2}{12}-\frac{\bar{c}_8}{48}-\frac{29\bar{c}_4\bar{c}_6}{192}+\frac{515\bar{c}_4^3}{768}+\dots\right), \tag{20.493}$$

the exponential can be brought to the form

$$\begin{aligned}\tilde{D}(x) &= \exp\left\{-\frac{x^2}{\tilde{\sigma}_2^2}+\frac{x^4}{\tilde{\sigma}_2^4}\left(\frac{\bar{c}_4}{24}-\frac{\bar{c}_6}{48}+\frac{5\bar{c}_4^2}{48}+\frac{\bar{c}_8}{192}+\frac{29\bar{c}_4\bar{c}_6}{576}-\frac{2576\bar{c}_4^3}{9216}\right)\right.\\ &\quad+\frac{x^6}{\tilde{\sigma}_2^6}\left(\frac{\bar{c}_6}{720}-\frac{\bar{c}_4^2}{72}-\frac{8\bar{c}_8}{1440}-\frac{29\bar{c}_4\bar{c}_6}{2880}+\frac{59\bar{c}_4^3}{768}\right)\\ &\quad\left.+\frac{x^8}{\tilde{\sigma}_2^8}\left(\frac{\bar{c}_8}{40320}+\frac{7\bar{c}_4\bar{c}_6}{5760}-\frac{101\bar{c}_4^3}{9216}\right)+\dots\right\}\frac{1}{\sqrt{2\pi\tilde{\sigma}_1^2}}.\end{aligned} \tag{20.494}$$

The exponential can be re-expressed compactly in the quasi-Gaussian form

$$\tilde{D}(x)=\frac{1}{\sqrt{2\pi\tilde{\sigma}_1^2}}\ \exp\left[-\frac{x^2}{2\Sigma^2(x)}\right], \tag{20.495}$$

where we have defined an x-dependent width

$$\begin{aligned}\tilde{\Sigma}(x) &\equiv \sigma_2^2\left[1+\frac{x^2}{\sigma_2^2}\left(\frac{\bar{c}_4}{12}+\frac{5\,\bar{c}_4^2}{24}-\frac{2575\,\bar{c}_4^3}{4608}-\frac{\bar{c}_6}{24}+\frac{29\,\bar{c}_4\,\bar{c}_6}{288}+\frac{\bar{c}_8}{96}\right)\right.\\ &+\frac{x^4}{\sigma_2^4}\left(\frac{-\bar{c}_4^2}{48}+\frac{217\,\bar{c}_4^3}{1152}-\frac{1375\,\bar{c}_4^4}{27648}+\frac{\bar{c}_6}{360}-\frac{13\,\bar{c}_4\,\bar{c}_6}{480}-\frac{\bar{c}_4^2\,\bar{c}_6}{1728}+\frac{\bar{c}_6^2}{576}-\frac{\bar{c}_8}{720}+\frac{\bar{c}_4\,\bar{c}_8}{576}\right)\\ &\left.+\frac{x^6}{\sigma_2^6}\left(-\frac{359\,\bar{c}_4^3}{13824}+\frac{127\,\bar{c}_4^4}{6912}+\frac{5\,\bar{c}_4\,\bar{c}_6}{1728}-\frac{13\,\bar{c}_4^2\,\bar{c}_6}{17280}-\frac{\bar{c}_6^2}{4320}+\frac{\bar{c}_8}{20160}-\frac{\bar{c}_4\,\bar{c}_8}{4320}\right)+\dots\right].\end{aligned} \tag{20.496}$$

For small x the deviation of $\Sigma^2(x)$ from a constant σ^2 is dominated by the quadratic term. If one plots the fluctuation width $\Sigma(x)$ for the logarithms of the observed option prices one finds the smile parabola shown before in Fig. 20.29. On the basis of the expansion (20.488) it is possible to derive an approximate option price formula for assets fluctuating according to the truncated Lévy distribution. We simply apply the differential operator in front of the Gaussian distribution

$$\mathcal{O}\equiv\left[1+\frac{a_4}{4!}2^2\left(\frac{\partial}{\partial\sigma^2}\right)^2+\frac{a_6}{6!}2^3\left(\frac{\partial}{\partial\sigma^2}\right)^3+\frac{a_8}{8!}2^4\left(\frac{\partial}{\partial\sigma^2}\right)^4+\dots\right] \tag{20.497}$$

to the Gaussian distribution (20.421). For $\lambda=3/2$, the coefficients are

$$\begin{aligned}a_4&=\frac{\varepsilon}{M^2},\ a_6=\frac{5\cdot7\,\varepsilon^2}{3M^3},\ a_8=\frac{5\cdot7\cdot11\,\varepsilon^2}{M^4}+\frac{5\cdot7\,\varepsilon^2}{M^4},\\ a_{10}&=\frac{5^2\cdot7\cdot11\cdot13\,\varepsilon^4}{3M^5}+\frac{2^2\cdot5\cdot7\cdot17\,\varepsilon^4}{3M^5},\dots\ .\end{aligned} \tag{20.498}$$

The operator $\mathcal{O}$ winds up in front of the Black-Scholes expression (20.438), such that we obtain the formal result

$$O^L(x_a,t_a)=\mathcal{O}O(x_a,t_a)=\mathcal{O}\left[S(t_a)N(y_+)-e^{-(r_{xW}+\sigma^2/2)(t_b-t_a)}E\,N(y_-)\right], \tag{20.499}$$

where we have used (20.418) to exhibit the full σ-dependence on the right-hand side. This expression may now be expanded in powers of the kurtosis κ. The term proportional to a_4 yields a first correction to the Black-Scholes formula, linear in ε,

$$\begin{aligned}O_1(x_a,t_a)=-\frac{\varepsilon}{12M^2}\Bigg\{&\left(Se^{-y_+^2/2}-e^{-r_W(t_b-t_a)}Ee^{-y_-^2/2}\right)\frac{y_m}{\sqrt{2\pi\sigma^2}}\\ &-e^{-r_W(t_b-t_a)}(t_b-t_a)EN(y_-)\Bigg\}.\end{aligned} \tag{20.500}$$

The term proportional to a_6 adds a correction proportional to ε^2:

$$O_2(x_a,t_a) = \frac{35\varepsilon^2}{3M^3}\frac{1}{2^3\cdot 3^2\cdot 5}\Big\{\Big[Se^{-y_+^2/2}\Big(y_+y_-^2-3y_++4\sqrt{\sigma^2(t_b-t_a)}\Big)$$
$$-\ e^{-r_W(t_b-t_a)}Ee^{-y_-^2/2}\left(y_-^3-3y_--2\sigma^2(t_b-t_a)y_-\right)\Big]\frac{1}{\sqrt{2\pi\sigma^4}}$$
$$+\ e^{-r_W(t_b-t_a)}(t_b-t_a)^2EN(y_-)\Big\}. \tag{20.501}$$

The next term proportional to a_8 adds corrections proportional to ε^2 and ε^3:

$$O_3(x_a,t_a) = \frac{385\varepsilon^3+35\varepsilon^2}{M^4}\frac{1}{2^6\cdot 3^2\cdot 5\cdot 7}$$
$$\times\Big\{\Big[Se^{-y_+^2/2}\Big(-y_-^3y_+^2+9y_-y_+^2+y_-^3-15y_++18+\sqrt{\sigma^2(t_b-t_a)}(12y_-y_++18)\Big)$$
$$-\ e^{-r_W(t_b-t_a)}Ee^{-y_-^2/2}\Big(y_-^5-10y_-^3+15y_-+\sigma^2(t_b-t_a)(3y_-^3-9y_-)$$
$$+\sigma^4(t_b-t_a)^2\,3y_-\Big)\Big]\frac{1}{\sqrt{2\pi\sigma^6}}+e^{-r_W(t_b-t_a)}(t_b-t_a)^3EN(y_-)\Big\}. \tag{20.502}$$

Since the expansion is asymptotic, an efficient resummation scheme will be needed for practical applications.

Appendix 20A Large-x Behavior of Truncated Lévy Distribution

Here we derive the divergent asymptotic expansion in the large-x regime. Using the variable $y\equiv p/\alpha$ and the constant $a\equiv -s\alpha^\lambda$, we may write the Fourier integral (20.24) as

$$\tilde L_{\sigma^2}^{(\lambda,\alpha)}(x) = \alpha e^{-2a}\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{i\alpha yx}e^{a[(1-iy)^\lambda+(1+iy)^\lambda]}. \tag{20A.1}$$

Expanding the last exponential in a Taylor series, we obtain

$$\begin{aligned}\tilde L_{\sigma^2}^{(\lambda,\alpha)}(x) &= \alpha e^{-2a}\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{iy\alpha x}\sum_{n=0}^{\infty}\frac{a^n}{n!}[(1-iy)^\lambda+(1+iy)^\lambda]^n = \\ &= \alpha e^{-2a}\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{i\alpha xy}\sum_{n=0}^{\infty}\frac{a^n}{n!}\sum_{m=0}^{n}\binom{n}{m}(1-iy)^{\lambda(n-m)}(1+iy)^{\lambda m}\end{aligned} \tag{20A.2}$$

containing binominal coefficients. Changing the order of summations yields

$$\tilde L_{\sigma^2}^{(\lambda,\alpha)}(x) = \alpha e^{-2a}\sum_{n=0}^{\infty}\frac{a^n}{n!}\sum_{m=0}^{n}\binom{n}{m}\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{i\alpha xy}(1-iy)^{\lambda(n-m)}(1+iy)^{\lambda m}. \tag{20A.3}$$

This can be written with the help of the Whittaker functions (20.30) as

$$\tilde L_{\sigma^2}^{(\lambda,\alpha)}(x) = \alpha e^{-2a}\sum_{n=0}^{\infty}\frac{a^n}{n!}\sum_{m=0}^{n}\binom{n}{m}\frac{(\alpha x)^{-1-\lambda n/2}2^{\lambda n/2}}{\Gamma(-\lambda m)}W_{\lambda n/2-\lambda m,(\lambda n+1)/2}(2\alpha x). \tag{20A.4}$$

After converting the Gamma functions of negative arguments into those of positive arguments, this becomes

$$\tilde{L}^{(\lambda,\alpha)}_{\sigma^2}(x) = -\frac{\alpha}{\pi}e^{-2a}\sum_{n=1}^{\infty}a^n\sum_{m=0}^{n}\frac{2^{\lambda n/2}\Gamma(1+\lambda m)\sin(\pi\lambda m)}{(\alpha x)^{1+\lambda n/2}m!(n-m)!}W_{\lambda n/2-\lambda m,(\lambda n+1)/2}(2\alpha x). \tag{20A.5}$$

The Whittaker functions $W_{\lambda,\gamma}(x)$ have the following asymptotic expansion

$$W_{\lambda,\gamma}(x) = e^{-x/2}x^{\lambda}\left\{1+\sum_{k=1}^{\infty}\frac{1}{k!x^k}\prod_{j=1}^{k}\left[\gamma^2-(\lambda-j+1/2)^2\right]\right\}. \tag{20A.6}$$

For $\gamma \equiv (\lambda n+1)/2$ and $\lambda \equiv \lambda(n-2m)/2$, the product takes the form

$$\begin{aligned}\prod_{j=1}^{k}[\gamma^2-(\lambda-j+1/2)^2] &= \prod_{j=1}^{k}\left\{\left(\frac{\lambda n+1}{2}\right)^2-\left[\frac{\lambda(n-2m)}{2}-j+\frac{1}{2}\right]^2\right\}\\ &= \prod_{j=1}^{k}(\lambda m+j)(\lambda n+1-\lambda m-j).\end{aligned} \tag{20A.7}$$

Inserting this into (20A.6) and the result into (20A.5), we obtain the asymptotic expansion for large x:

$$\begin{aligned}\tilde{L}^{(\lambda,\alpha)}_{\sigma^2}(x) &= -\frac{1}{\pi}e^{-2a}\frac{e^{-\alpha x}}{x}\sum_{n=1}^{\infty}a^n\,2^{\lambda n}\sum_{m=1}^{n}\frac{\Gamma(1+\lambda m)\sin(\pi\lambda m)}{m!(n-m)!}(2\alpha x)^{-\lambda m}\\ &\quad\times\left[1+\sum_{k=1}^{\infty}\frac{\prod_{j=1}^{k}(\lambda m+j)(\lambda n+1-\lambda m-j)}{k!(2\alpha x)^k}\right].\end{aligned} \tag{20A.8}$$

We have raised the initial value of the index of summation m by one unit since $\sin(\pi\lambda m)$ vanishes for $m=0$. If we define a product of the form $\prod_{j=1}^{0}...$ to be equal to unity, we can write the term in the last bracket as

$$\sum_{k=0}^{\infty}\frac{1}{k!(2\alpha x)^k}\prod_{j=1}^{k}(j+\lambda m)(1-\lambda m-j+\lambda n). \tag{20A.9}$$

Rearranging the double sum in (20A.8), we write $\tilde{L}^{(\lambda,\alpha)}_{\sigma^2}(x)$ as

$$\begin{aligned}\tilde{L}^{(\lambda,\alpha)}_{\sigma^2}(x) &= -\frac{1}{\pi}e^{-2a}\frac{e^{-\alpha x}}{x}\sum_{m=1}^{\infty}\frac{\Gamma(1+\lambda m)\sin(\pi\lambda m)}{m!(2\alpha x)^{\lambda m}}\sum_{n=m}^{\infty}\frac{a^n\,2^{\lambda n}}{(n-m)!}\\ &\quad\times\sum_{k=0}^{\infty}\frac{1}{k!(2\alpha x)^k}\prod_{j=1}^{k}(j+\lambda m)(1-\lambda m-j+\lambda n),\end{aligned} \tag{20A.10}$$

and further as

$$\begin{aligned}\tilde{L}^{(\lambda,\alpha)}_{\sigma^2}(x) &= -\frac{1}{\pi}e^{-2a}\frac{e^{-\alpha x}}{x}\sum_{m=1}^{\infty}\frac{\Gamma(1+\lambda m)\sin(\pi\lambda m)}{m!}(2\alpha x)^{-\lambda m}\\ &\quad\times\sum_{n=0}^{\infty}(2^{\lambda}a)^{n+m}\frac{1}{n!}\sum_{k=0}^{\infty}\frac{1}{k!(2\alpha x)^k}\prod_{j=1}^{k}(j+\lambda m)(1-j+\lambda n) =\end{aligned}$$

$$= -\frac{1}{\pi}e^{-2a}\frac{e^{-\alpha x}}{x}\sum_{k=0}^{\infty}\frac{1}{k!(2\alpha x)^k}\sum_{m=1}^{\infty}\frac{\Gamma(1+\lambda m)\sin(\pi\lambda m)(2^\lambda a)^m}{(2\alpha x)^{\lambda m}m!}$$
$$\times\prod_{j'=1}^{k}(j'+\lambda m)\sum_{n=0}^{\infty}\frac{(2^\lambda a)^n}{n!}\prod_{j=1}^{k}(\lambda n+1-j). \tag{20A.11}$$

The last sum over n in this expression can be re-expressed more efficiently with the help of a generating function

$$f^{(k)}(y^\lambda) \equiv \frac{d^k}{dy^k}e^{y^\lambda}, \tag{20A.12}$$

whose Taylor series is

$$f^{(k)}(y^\lambda) = \frac{d^k}{dy^k}\sum_{n=0}^{\infty}\frac{y^{\lambda n}}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\prod_{j=1}^{k}(\lambda n-j+1)y^{\lambda n-k}, \tag{20A.13}$$

leading to

$$\tilde{L}_{\sigma^2}^{(\lambda,\alpha)}(x) = \frac{1}{\pi}e^{-2a}\frac{e^{-\alpha x}}{x}\sum_{k=0}^{\infty}\frac{a^{k/\lambda}2^k f^{(k)}(2^\lambda a)}{k!(2\alpha x)^k}$$
$$\times\sum_{m=1}^{\infty}\frac{\Gamma(1+\lambda m)\sin(\pi\lambda m)(2^\lambda a)^m}{m!}(2\alpha x)^{-\lambda m}\prod_{j=1}^{k}(\lambda m+j). \tag{20A.14}$$

This can be rearranged to

$$\tilde{L}_{\sigma^2}^{(\lambda,\alpha)}(x) = -\frac{e^{-2a}}{\pi}e^{-\alpha x}\sum_{k=0}^{\infty}\frac{(-s)^{k/\lambda}f^{(k)}(-s(2\alpha)^\lambda)}{k!}$$
$$\times\sum_{m=1}^{\infty}\frac{\Gamma(1+\lambda m+k)\sin(\pi\lambda m)(-s)^m}{m!x^{1+\lambda m+k}}. \tag{20A.15}$$

Thus we obtain the asymptotic expansion

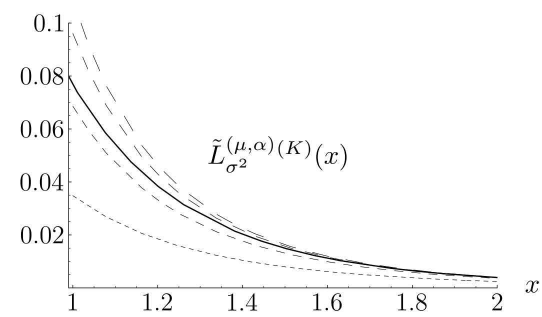

Figure 20.32 Comparison of large-x expansions containing different numbers of terms (with $K = 0, 1, 2, 3, 4, 5$, with increasing dash length) with the tails of the truncated Lévy distribution for $\lambda = \sqrt{2}$, $\sigma = 0.5$, $\alpha = 1$.

$$\tilde{L}_{\sigma^2}^{(\lambda,\alpha)}(x) = -\frac{e^{-2a}}{\pi}\frac{e^{-\alpha x}}{x}\sum_{k=0}^{\infty} A_k \sum_{m=1}^{\infty} B_{km}\frac{(-s)^m}{x^{\lambda m+k}}, \tag{20A.16}$$

where

$$A_k = \frac{(-s)^{k/\lambda} f^{(k)}(-s(2\alpha)^\lambda)}{k!}, \qquad B_{km} = \frac{\Gamma(1+\lambda m+k)\sin(\pi\lambda m)}{m!}. \tag{20A.17}$$

We shall denote by $\tilde{L}_{\sigma^2}^{(\lambda,\alpha)(K)}(x)$ the approximants in which the sums over K are truncated after the Kth term. To have a definite smallest power in $|x|$, the sum over m is truncated after the smallest integer larger than $(K-k+\lambda)/\lambda$. The leading term is

$$\begin{aligned}\tilde{L}_{\sigma^2}^{(\lambda,\alpha)(0)}(x) &= -\frac{e^{-2a}}{\pi}e^{-\alpha x}f^{(0)}(-s(2\alpha)^\lambda)\,\frac{\Gamma(1+\lambda)\sin(\pi\lambda)(-s)}{x^{1+\lambda}}\\ &= s\,\frac{e^{\alpha^\lambda s(2-2^\lambda)}}{\pi}\,\frac{\Gamma(1+\lambda)\sin(\pi\lambda)}{x^{1+\lambda}}\,e^{-\alpha x}.\end{aligned} \tag{20A.18}$$

The large-x approximations $\tilde{L}_{\sigma^2}^{(\lambda,\alpha)(0)}(x)$ are compared with the numerically calculated truncated Lévy distribution in Fig. 20.32.

Appendix 20B Gaussian Weight

For simplicity, let us study the Gaussian content in the final distribution (20.347) only for the simpler case $\rho = 0$. Then we shift the contour of integration in (20.347) to run along $p + i/2$, and we study the Fourier integral

$$P(x\,t\,|x_a t_a) = e^{-\Delta x/2}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\,e^{ip\Delta x-\bar{H}(p,\Delta t)}, \tag{20B.1}$$

where

$$\bar{H}(p,\Delta t) = -\frac{\gamma^2\bar{v}t}{\kappa^2} + \frac{2\gamma\bar{v}}{\kappa^2}\ln\left[\cosh\frac{\Omega t}{2} + \frac{\Omega^2+\gamma^2}{2\gamma\Omega}\sinh\frac{\Omega t}{2}\right], \tag{20B.2}$$

and

$$\Omega = \sqrt{\gamma^2+\kappa^2(p^2+1/4)}. \tag{20B.3}$$

The function $\bar{H}(p,\Delta t)$ is real and symmetric in p. The integral (20B.1) is therefore a symmetric function of Δx. The only source of asymmetry of $P(x\,t\,|x_a t_a)$ in Δx is the exponential prefactor in (20B.1).

Let us expand the integral in (20B.1) for small Δx:

$$P(x\,t\,|x_a t_a) \approx e^{-\Delta x/2}\left[\mu_0 - \frac{1}{2}\mu_2(\Delta x)^2\right] \approx \mu_0\, e^{-\Delta x/2}e^{-\mu_2(\Delta x)^2/2\mu_0}, \tag{20B.4}$$

where the coefficients are the first and the second moments of $\exp[\bar{H}(p,\Delta t)]$

$$\mu_0(\Delta t) = \int_{-\infty}^{+\infty}\frac{dp}{2\pi}\,e^{-\bar{H}(p,\Delta t)}, \quad \mu_2(\Delta t) = \int_{-\infty}^{+\infty}\frac{dp}{2\pi}\,p^2 e^{-\bar{H}(p,\Delta t)}. \tag{20B.5}$$

If we ignore the existence of semi-heavy tails and extrapolate the Gaussian expression on the right-hand side to $\Delta x \in (-\infty,\infty)$, the total probability contained in such a Gaussian extrapolation will be the fraction

$$f(\Delta t) = \int_{-\infty}^{+\infty} d\Delta x\,\mu_0\, e^{-\Delta x/2-\mu_2\Delta x^2/2\mu_0} = \sqrt{\frac{2\pi\mu_0^3}{\mu_2}}\,e^{\mu_0/8\mu_2}. \tag{20B.6}$$

This is always less than 1 since the integral (20B.6) ignores the probability contained in the semi-heavy tails. The difference $1 - f(\Delta t)$ measures the relative contribution of the semi-heavy tails. The parameters $\mu_0(\Delta t)$ and $\mu_2(\Delta t)$ are calculated numerically and the resulting fraction $f(\Delta t)$ is plotted in Fig. 20.25 as a function of Δt. For $\Delta t \to \infty$, the distribution becomes Gaussian, whereas for small Δt, it becomes a broad function of p.

Appendix 20C Comparison with Dow-Jones Data

For the comparison of the theory in Section 20.4 with actual financial data shown in Fig. 20.22, the authors of Ref. [79] downloaded the daily closing values of the Dow-Jones industrial index for the period of 20 years from 1 January 1982 to 31 December 2001 from the Web site of Yahoo [110]. The data set contained 5049 points $S(t_n)$, where the discrete time variable t_n parametrizes the days. Short days before holidays were ignored. For each t_n, they compiled the log-returns $\Delta x(t_n) = \ln S(t_{n+1})/S(t_n)$. Then they partitioned the x-axis into equally spaced intervals of width Δx and counted the number of log-returns $\Delta x(t_n)$ falling into each interval. They omitted all intervals with occupation numbers less than five, which they considered as too few to rely on. Only less than 1% of the entire data set was omitted in this way. Dividing the occupation number of each bin by Δr and by the total occupation number of all bins, they obtained the probability density for a given time interval $\Delta t = 1$ day. From this they found $P^{(DJ)}(x\,t\,|x_a t_a)$ by replacing $\Delta x \to \Delta x - r_S \Delta t$.

Assuming that the system is ergodic, so that ensemble averaging is equivalent to time averaging, they compared $P^{(DJ)}(x\,t\,|x_a t_a)$ with the calculated $P(x\,t\,|x_a t_a)$ in Eq. (20.347). The parameters of the model were determined by minimizing the mean-square deviation $\sum_{\Delta x,\Delta t} |\log P^{(DJ)}(x\,t\,|x_a t_a) - \log P(x\,t\,|x_a t_a)|^2$, with the sum taken over all available Δx and over $\Delta t = 1$, 5, 20, 40, and 250 days. These values of Δt were selected because they represent different regimes: $\gamma\Delta t \ll 1$ for $t = 1$ and 5 days, $\gamma\Delta t \approx 1$ for $t = 20$ days, and $\gamma\Delta t \gg 1$ for $t = 40$ and 250 days. As Figs. 20.22 and 20.23 illustrate, the probability density $P(x\,t\,|x_a t_a)$ calculate from the Fourier integral (20.347) with components (20.348) agrees with the data very well, not only for the selected five values of time t, but for the whole time interval from 1 to 250 trading days. The comparison cannot be extended to Δt longer than 250 days, which is approximately 1/20 of the entire range of the data set, because it is impossible to reliably extract $P^{(DJ)}(x\,t\,|x_a t_a)$ from the data when Δt is too long.

The best fits for the four parameters γ, $\bar{v}$, ε, μ are given in Table 20.1. Within the scattering of the data, there are no discernible differences between the fits with the correlation coefficient ρ being zero or slightly different from zero. Thus the correlation parameter ρ between the noise terms for stock price and variance in Eq. (20.310) is practically zero. This conclusion is in contrast with the value $\rho = -0.58$ found in [111] by fitting the leverage correlation function introduced in [112]. Further study is necessary to understand this discrepancy. All theoretical curves shown in the above figures are calculated for $\rho = 0$, and fit the data very well.

The parameters γ, $\bar{v}$, ε, μ have the dimensionality of 1/time. One row in Table 20.1 gives their values in units of 1/day, as originally determined in our fit. The other row shows the annualized values of the parameters in units of 1/year, where one year is here equal to the average number of 252.5 trading days per calendar year. The relaxation time of variance is equal to $1/\gamma = 22.2$ trading days = 4.4 weeks $\approx$ 1 month, where 1 week = 5 trading days. Thus one finds that the variance has a rather long relaxation time, of the order of one month, which is in agreement with an earlier conclusion in Ref. [111].

Using the numbers given in Table 20.1, the value of the parameter $\varepsilon^2/2\gamma\bar{v}$ is ≈ 0.772 thus satisfying the smallness condition Eq. (20.322) ensuring that v never reaches negative values.

The stock prices have an apparent growth rate determined by the position $x_m(t)$ where the probability density is maximal. Adding this to the initially subtracted growth rate r_S we find that the apparent growth rate is $\bar{r}_S = r_S - \gamma\bar{v}/2\omega_0 = 13\%$ per year. This number coincides with the apparent average growth rate of the Dow-Jones index obtained by a simple fit of the data points S_{t_n} with an exponential function of t_n. The apparent growth rate $\bar{r}_S$ is comparable to the

Table 20.1 Parameters of equations with fluctuating variance obtained from fits to Dow-Jones data. The fit yields $\rho \approx 0$ for the correlation coefficient and $1/\gamma = 22.2$ trading days for the relaxation time of variance.

Units	γ	$\bar{v}$	ε	μ
1/day	4.50×10^{-2}	8.62×10^{-5}	2.45×10^{-3}	5.67×10^{-4}
1/year	11.35	0.022	0.618	0.143

average stock volatility after one year $\sigma = \sqrt{\bar{v}} = 14.7\%$. Moreover, the parameter (20.328) which characterizes the width of the stationary distribution of variance is equal to $v_{\max}/w = 0.54$. This means that the distribution of variance is broad, and variance can easily fluctuate to a value twice greater than the average value $\bar{v}$. As a consequence, even though the average growth rate of the stock index is positive, there is a substantial probability of $\int_{-\infty}^{0} d\Delta x P(x\,t\,|x_a t_a) \approx 17.7\%$ to have negative growth for $\Delta t = 1$ year.

According to (20.368), the asymmetry between the slopes of exponential tails for positive and negative Δx is given by the parameter p_0, which is equal to 1/2 when $\rho = 0$ [see also the discussion of Eq. (20B.1) in Appendix 20B]. The origin of this asymmetry can be traced back to the transformation from $\dot{S}(t)/S(t)$ to $\dot{x}(t)$ using Itô's formula. This produces a term $v(t)/2$ in Eq. (20.306), which leads to the first term in the Hamiltonian operator (20.319). For $\rho = 0$ this is the only source of asymmetry in Δx of $P(x\,t\,|x_a t_a)$. In practice, the asymmetry of the slopes $p_0 = 1/2$ is quite small (about 2.7%) compared to the average slope $q_*^{\pm} \approx \omega_0/\varepsilon = 18.4$.

Notes and References

Option pricing beyond Black and Scholes via path integrals was discussed in Ref. [113], where strategies are devised to minimize risks in the presence of extreme fluctuations, as occur on real markets. See also Ref. [114].
Recently, a generalization of path integrals to functional integrals over surfaces has been proposed in Ref. [115] as an alternative to the Heath-Jarrow-Morton approach of modeling yield curves (see `http://risk.ifci.ch/00011661.htm`). Applications of the Duru-Kleinert transformation to financial markets are described in Ref. [21] of Chapter 14.

The individual citations refer to

[1] The development of the Dow Jones industrial index up to date can be found on internet sites such as `http://stockcharts.com/charts/historical/djia1900.html`. Alternatively they can be plotted directly using Stephen Wolfram's program *Mathematica* (v.6) using the small program

```
t=FinancialData["^DJI",All];l = Length[t]
tt = Table[t[[k]][[1]][[1]]+(t[[k]][[1]][[2]]-1)/12,t[[k]][[2]], {k,1,l}]
ListLogPlot[tt]
```

Note that the last stagnation period was *predicted* in the 3rd edition of this book in 2004.

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