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ON-SHELL EFFECTIVE FIELD THEORY*,**

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I show how to organize the coefficients in an effective field theory so that the accuracy of the effective field theory calculation is less seriously degraded when there are ratios of mass scales in the problem that are not large. I show how this new effective field theory can be obtained automatically, by doing the matching calculation on the perturbative mass shell. I also speculate about how to do even better by consistently including non-local terms in the effective lagrangian.

1. Introduction

In an effective field theory calculation, one includes in the effective field theory at the scale μ, only those particles that have masses less than μ. Of course, once particles have been removed from the "full" renormalizable theory to produce an effective theory, there will be a tower of non-renormalizable interactions. These terms are usually organized in terms of a derivative expansion, in increasing powers of derivatives, or numbers of fields. The terms can all be calculated, in principle, if one knows the full theory from which the effective theory comes. However, in practice, most of them can be ignored. Thus an effective field theory has two parts. In addition to the usual stuff required to describe a renormalizable theory, specifying the fields and the symmetries, and enumerating the renormalizable parameters, an effective field theory involves two additional *rules*:

(i) A rule for estimating the size of any possible non-renormalizable term.
(ii) A rule for eliminating redundant terms in the effective lagrangian and writing the remaining terms in a canonical form.

Once these rules are specified, it is easy to check whether a given class of non-renormalizable interactions can be safely disregarded.

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Rule (i) has been discussed extensively elsewhere [1]. In sect. 2, I will briefly review how it works in the context of a general review of effective field theory. I will pay more attention to a precise description of the process of matching from one effective theory to the next. Rule (ii) is one of the major subjects of this paper. In sects. 3 and 4, I will describe such a rule in detail. It leads to an "on-shell" effective field theory that has many advantages over the conventional formulation. In sect. 5, I argue that this rule makes the effective field theory formalism more useful for precise calculations, and I speculate about how one might do even better, by consistently including non-locality in the effective lagrangian. This speculation is logically independent of rule (ii), but arises naturally from the discussion in sect. 2 of matching.

2. Effective field theory

In general, the size of a given non-renormalizable term depends on the scale at which it first appears, which in turn depends on the physics that produced it. An effective field theory calculation proceeds in the following way. One starts at a very large scale, that is with μ very large. In a strongly interacting theory or a theory with unknown physics at high energy, this starting scale should be sufficiently large that non-renormalizable interactions produced at higher scales are too small to be relevant. In a renormalizable, weakly interacting theory, one starts at a scale above the masses of all the particles, where the effective theory is given simply by the renormalizable theory, with no non-renormalizable terms. One then evolves the theory down to lower scales. As long as no particle masses are encountered, this evolution is described by the renormalization group. However, when μ goes below the mass Λ of one of the particles in the theory, we must change the effective theory to a new theory without that particle. In the process, the parameters of the the theory change, and new, non-renormalizable interactions may be introduced. Both the changes in the existing parameters, and the coefficients of the new interactions are computed by "matching" the physics just below the boundary in the two theories*. It is this process that determines the sizes of the non-renormalizable terms associated with the heavy particles. Because matching is done for $\mu \approx \Lambda$, the rule for the size of the coefficients of the new operators is simple for $\mu \approx \Lambda$. At this scale, all the new contributions scale with Λ to the appropriate power (set by dimensional analysis) up to factors of coupling constants, group theory or counting factors and loop factors (of $16\pi^2$, etc.). Then, when the new effective theory is evolved down to smaller μ, the renormalization group introduces additional factors into the coefficients. Thus a heavy particle mass appears in the parameters of an effective field theory in two ways. There is power

* In a sense, the renormalization group is simply the "matching" of the theory at the scale μ to the theory at the scale $\mu - \mathrm{d}\mu$ without changing the particle content.

dependence on the mass that arises from matching conditions. There is also logarithmic dependence that arises from the renormalization group.

This process of matching the two theories at a heavy particle threshold is sometimes described as "integrating out" the heavy particle. But in fact, the process is somewhat more involved. When the heavy particle is integrated out of the theory, what results is a non-local action. An extra step is required to get to the effective lagrangian in the low-energy theory. One must disentangle the short distance physics that is incorporated into the coefficients of the effective lagrangian from the long-distance physics that remains explicit in the low-energy theory. It is here that "matching" really comes in.

The way this disentangling works is instructive. Matching corrections are computed by comparing calculations of physical quantities in the high-energy theory ($\mu \geqslant \Lambda$) with calculations of the same quantities in the low-energy theory ($\mu < \Lambda$) and choosing the parameters in the effective theory so that the physics is the same at the boundary ($\mu = \Lambda$). Any interaction that is unchanged in the matching cancels out of the matching because it contributes in the same way in the two theories. As I will describe in detail below, the *change* in a parameter in the effective theory due to matching is related, order by order in perturbation theory, to a difference between the high-energy and low-energy calculations. In this difference, all effects of long-distance physics, infrared divergences, physical cuts, etc., disappear, because they are the same, by construction, in the high- and low-energy theories. Thus only the short-distance contributions are incorporated into the coefficients of the effective lagrangian.

Here is how it works in more detail. Suppose that at a matching scale $\mu = \Lambda$, the high-energy theory involves some heavy particle with mass Λ and interpolating field χ, and light particles with interpolating fields ϕ_j. The effective theory is described by the lagrangian

$$\mathscr{L}_H(\chi, \phi) + \mathscr{L}(\phi) \tag{1}$$

where the first term includes all the heavy particle fields.

The low-energy theory contains only the light fields and is described by the lagrangian

$$\mathscr{L}(\phi) + \delta\mathscr{L}(\phi). \tag{2}$$

I will describe the process of matching in perturbation theory, so I assume that the interactions in each of the theories are organized in powers of a small parameter, ϵ

$$\mathscr{L} = \sum_{n=0}^{\infty} \epsilon^n \mathscr{L}^n, \qquad \mathscr{L}_H = \sum_{n=0}^{\infty} \epsilon^n \mathscr{L}_H^n. \tag{3}$$

Now, what we want to do is to choose $\delta\mathscr{L}(\phi)$ so that the physics involving the light fields, ϕ, computed in the two theories are identical at $\mu = \Lambda$. The strong form of this equivalence, ignoring rule (ii) for the nonce, is obtained by requiring that the effective action as a function of the light fields (or equivalently, all one particle irreducible (1PI) functions with external light particles) be the same in the two theories. If we denote the effective action produced by a lagrangian $\mathscr{L}$ by

$$S_{\mathscr{L}}(\Phi) = \sum_{n=0}^{\infty} \epsilon^n S_{\mathscr{L}}^N(\Phi)\,, \tag{4}$$

then the matching condition is

$$S_{\mathscr{L}_H+\mathscr{L}}(\Phi) = S_{\mathscr{L}+\delta\mathscr{L}}(\Phi)\,. \tag{5}$$

The change, $\delta\mathscr{L}$ vanishes trivially when all interactions vanish, because

$$S^0_{\mathscr{L}_H+\mathscr{L}}(\Phi) = S^0_{\mathscr{L}+\delta\mathscr{L}}(\Phi) = \int \mathscr{L}^0(\Phi) \tag{6}$$

so that $\delta\mathscr{L}$ can be calculated in perturbation theory in ϵ starting in order ϵ,

$$\delta\mathscr{L} = \sum_{n=1}^{\infty} \epsilon^n \delta\mathscr{L}^n\,. \tag{7}$$

First consider the lowest-order term. It follows from general considerations that

$$S^1_{\mathscr{L}+\delta\mathscr{L}}(\Phi) = S^1_{\mathscr{L}}(\Phi) + \int \delta\mathscr{L}^1(\Phi)\,. \tag{8}$$

Thus the matching condition to first order in ϵ can be written

$$S^1_{\mathscr{L}_H+\mathscr{L}}(\Phi) - S^1_{\mathscr{L}}(\Phi) = \int \delta\mathscr{L}^1(\Phi)\,. \tag{9}$$

Because the effective action is the generating functional for the 1PI functions, we can rewrite this in terms of them. If we write the change in the lagrangian as

$$\delta\mathscr{L}^n(\phi) = \sum_j \phi^j \delta\gamma^{n,j}\,, \tag{10}$$

then

$$\delta\gamma^{1,j} = \Gamma^{1,j}_{\mathscr{L}_H+\mathscr{L}} - \Gamma^{1,j}_{\mathscr{L}} \tag{11}$$

where $\Gamma^{n,j}_{\mathscr{L}}$ is the 1PI j-point function in order n for the lagrangian $\mathscr{L}$.

Eq. (2.9) (or equivalently, eq. (11)) is the mathematical statement of the words above about disentangling the long- and short-distance physics, at least to lowest order in ϵ. Let us pause to consider what it means. Both $S^1_{\mathscr{L}_H+\mathscr{L}}(\Phi)$ and $S^1_{\mathscr{L}}(\Phi)$ are non-local functions of Φ in which the non-locality is determined by the long-distance physics. If there are massless particles in the theory, the 1PI functions may not even be analytic in the momenta p as $p \to 0$. However, in the difference, all the long-distance physics that gives rise to this non-analyticity cancels. All the remaining non-locality is due to the propagation of the heavy particles and the result can be expanded in powers of p/Λ with no large coefficients. This gives the leading contribution to the change in the effective lagrangian.

The notation here is slightly imprecise, but intentionally so. The change in the "lagrangian", $\int \delta\mathscr{L}^1(\Phi)$, is actually still non-local, because it depends on p/Λ in some complicated way. However, it is *analytic* in p/Λ everywhere in the region relevant to the low-energy theory. Thus it can be expanded in powers with the higher-order terms steadily decreasing in importance and dealt with as a local lagrangian. However, we will speculate below that it may be possible to define the low-energy theory directly in terms of the non-local action $\int(\mathscr{L}+\delta\mathscr{L})$ – that locality is not required, just analyticity. One problem with this speculation is that there is no satisfactory name for this object. It would be natural to call it the effective action, but that name has already been taken for something else. I will call $(\mathscr{L}+\delta\mathscr{L})$ the "non-local effective lagrangian". This mildly oxymoronic name will, I hope, be a reminder that because of its analyticity in momentum this object can be thought of, either as a non-local function of momenta, or as a power series in p/Λ.

It is straightforward to go beyond leading order. The only subtlety is that now the lower-order changes in $\delta\mathscr{L}$ can also contribute to the long-distance physics, and must be subtracted. This can be done using the following:

$$S^n_{\mathscr{L}+\delta\mathscr{L}}(\Phi) = S^n_{\mathscr{L}+\delta\mathscr{L}_{n-1}}(\Phi) + \int \delta\mathscr{L}^n(\Phi)\,, \tag{12}$$

where

$$\delta\mathscr{L}_k \equiv \sum_{n=1}^{k} \delta\mathscr{L}^n\,. \tag{13}$$

Thus the matching condition to nth order in ϵ can be written

$$\int \delta^n_{\mathscr{L}}(\Phi) = S^n_{\mathscr{L}_H+\mathscr{L}}(\Phi) - S^n_{\mathscr{L}+\delta\mathscr{L}_{n-1}}(\Phi)\,, \tag{14}$$

or in terms of 1PI functions,

$$\delta\gamma^{n,j} = \Gamma^{n,j}_{\mathscr{L}_H+\mathscr{L}} - \Gamma^{n,j}_{\mathscr{L}+\delta\mathscr{L}_{n-1}}\,. \tag{15}$$

As expected in perturbation theory, we need the lower-order results to compute the higher-order results consistently. Eqs. (14) and (15) simply require that the lower-order changes be incorporated into the lagrangian before the next order is calculated. As in the lowest-order calculation, the rapid momentum dependence from the low-energy physics cancels between the two terms on the right hand sides of eqs. (14) and (15). The $\delta\gamma^{n,j}$ functions are analytic in the momenta for $p/\Lambda < 1$, where the effective theory is appropriate.

This completes the review of the general matching condition. Next let us see how we can use our freedom to redefine fields to simplify the resulting low-energy effective theory.

3. On-shell effective field theory

One might worry that if a theory contains two scales that are not very different, the effective field theory formalism would be useless because the effect of higher-dimension operators from the higher scale would not be suppressed very much and we would have to include operators of very high dimension to get accurate results. If this were true, the effective field theory formalism would not be of much use for precise calculations. We can improve the situation somewhat by an intelligent application of rule (ii), for eliminating redundant terms in the effective lagrangian and putting the effective lagrangian into a canonical form. A part of rule (ii) is familiar from renormalizable theories, where we often perform an integration by parts to put a term in the lagrangian into a canonical form. This is entirely harmless in perturbation theory. However, in non-renormalizable theories, we have a much more important option. We are free to make non-linear redefinitions of the fields without changing the physics (that is, the S-matrix elements).

The first important point is that except for the quadratic or linear kinetic energy terms, any term in the effective lagrangian with ∂^2 acting on a field can be removed by a non-linear field-redefinition, in favor of a term without the ∂^2, plus terms with more fields and more powers of coupling constants. This can be done order by order in the couplings in the effective theory, because the change in the free field kinetic energy term under a change $\delta\phi$ in a field ϕ is

$$\delta\mathscr{L}^0 = -\delta\phi\mathscr{D}\phi\,, \tag{16}$$

where

$$\mathscr{D}\phi = (\partial^2 + m^2)\phi = 0\,, \tag{17}$$

is the free field equation of motion. Thus we can give an inductive proof of the assertion. Suppose that we have removed all terms involving $\partial^2\phi$ up to nth order in ϵ. Then if there is a term of the form

$$\epsilon^{n+1}G(\phi,\dots)\partial^2\phi, \tag{18}$$

for some function of the fields and derivatives $G(\phi,\dots)$, it can be changed by the field redefinition

$$\delta\phi = -\epsilon^{n+1}G(\phi,\dots), \tag{19}$$

into

$$-\epsilon^{n+1}G(\phi,\dots)m^2\phi. \tag{20}$$

In other words, by the field redefinition (19), we can replace $-\partial^2$ by m^2. Note that the transformation (19) does not effect the form of the effective lagrangian in order ϵ^n and lower. Furthermore, as long as $G(\phi,\dots)$ is quadratic or higher degree in ϕ, the redefinition (19) does not affect the S-matrix. Thus we can eliminate ∂^2 acting on any field in any term in the lagrangian of degree higher than 2.

Before I discuss the situation for the quadratic terms in the lagrangian, I will make some general remarks.

We almost never make use of this freedom to make non-linear field redefinitions in a renormalizable theory, because such a redefinition would make the theory look non-renormalizable. But in the effective theory, *all possible interaction terms are already there*! Thus the non-linear redefinition of the field simply changes the coefficient of a term that we had to consider anyway. It does no harm at all! The upshot is that we are completely free to adopt, as part of our definition of the effective theory, the procedure of using the free field equation of motion to eliminate redundant terms to all orders*.

The result of this re-organization (once we have included the quadratic terms with the discussion below) is a new lagrangian that has no ∂^2 acting on fields except in the kinetic energy term and that gives the same physics in the sense that 1PI graphs computed on-shell are the same in the two theories (because they can be extracted from physical S-matrix elements). Furthermore, the on-shell 1PI functions are sufficient to determine the lagrangian. One does not need to know the p^2 dependence of amplitudes away from $p^2=m^2$, because all reference to this is eliminated in the effective lagrangian in going to the canonical form. It follows that this rule for organizing the effective lagrangian can be implemented very simply in a perturbative calculation with Feynman diagrams. As I show explicitly below, it amounts to performing the matching for coefficients of the operators in

* A related argument with a slightly weaker conclusion is given in ref. [2]. See also ref. [3].

the effective theory with external momenta on the perturbative "mass" shell. This sets $p^2 = m^2$ on each external particle leg, which is the momentum space version of the argument above. That is why I call this new organization an "on-shell effective theory". Mass is in quotes here because of the renormalization group. The masses run, and the right prescription is to set $p^2 = m(\mu)^2$ at the scale of the matching.

To show how the construction of the on-shell effective lagrangian works in detail, I will repeat the analysis of matching in sect. 2, using only the on-shell 1PI functions. I assume that the lagrangian, $\mathscr{L}$, is already in canonical form. The required incantations are then precisely analogous to those of sect. 2. For the leading term, we get the change in the effective lagrangian simply by setting $p^2 \to m^2$ in $\delta\gamma^{n,j}(p)$. This is obviously right because the one-shell 1PI functions in the two theories agree by construction. But now we can get the next-order term by comparing the high-energy theory with the low-energy *on-shell* effective theory, including the effects of $\delta\tilde{\gamma}^{n,j}(p)$. Then the same argument determines the next-order terms, etc. Thus the change in the properly organized terms in the on-shell effective field theory lagrangian can be computed as follows:

$$\delta\tilde{\gamma}^{n,j}(p) = \lim_{p^2 \to m^2} \delta\hat{\gamma}^{n,j}(p) \tag{21}$$

for $j \geqslant 3$, where

$$\delta\hat{\gamma}^{n,j}(p) = \left[\Gamma^{n,j}_{\mathscr{L}_H + \mathscr{L}} - \Gamma^{n,j}_{\mathscr{L} + \delta\tilde{\mathscr{L}}_{n-1}}\right], \tag{22}$$

$$\delta\tilde{\mathscr{L}}_k \equiv \sum_{n=1}^{k} \delta\tilde{\mathscr{L}}^n, \tag{23}$$

with

$$\delta\tilde{\mathscr{L}}^n(\phi) = \sum_j \phi^j \delta\tilde{\gamma}^{n,j}. \tag{24}$$

The $\delta\hat{\gamma}^{n,j}$'s are constructed directly from the 1PI functions in the two effective theories, defined to the appropriate order in ϵ. They are perfectly well-defined for any p^2, but we only use information for $p^2 = m^2$ to construct the change, $\delta\tilde{\mathscr{L}}^n$, in the lagrangian on-shell effective theory. This change is then fed back into the analysis to compute the next-order term.

Note that this prescription of going onto the mass shell is actually possible for the coefficients in the effective lagrangian. We have removed all the long-distance physics that could give rise to infrared divergences or other problems with the limit in eq. (21). The new term in the effective lagrangian, $\delta\tilde{\gamma}^{n,j}(p)$ depends on the external momenta only through the independent Lorentz invariants, $(p_j p_l)$. Thus for example the coefficient of a 3-point function is just a constant because all the invariants are fixed by the mass shell condition, while 4-point functions and higher have a tower of coefficients involving powers of $(p_j p_l)/\Lambda^2$.

4. Quadratic terms

The change in the quadratic terms in the effective lagrangian is more complicated because the transformation, (19), does not necessarily leave the S-matrix invariant if G is linear in ϕ. To see what kind of redefinition is allowed, we must examine the 2-point functions in more detail. In particular, consider the first-order change in the 1PI 2-point function, $\delta\hat{\gamma}^{1,2}$ from eq. (22) for $j=2$. $\delta\hat{\gamma}^{1,2}$ depends on a single momentum p. I will assume that the Lorentz structure has been extracted explicitly, so that it depends only on p*. It also has two flavor indices, so I denote it as follows:

$$\delta\hat{\gamma}^{1,2}_{jk}(p^2)\,. \tag{25}$$

What we are allowed to do without changing the S-matrix is to make a transformation of the form

$$\delta\phi = \epsilon L(-\partial^2)(\partial^2+m^2)\phi\,, \tag{26}$$

where L is an arbitrary matrix function of $-\partial^2$ and m is the mass matrix for the light fields. This doesn't change the S-matrix because the explicit factor ∂^2+m^2 kills the pole contribution from (26), so that $\delta\phi$ is eliminated in the LSZ reduction. Note also that because of the explicit factor ∂^2+m^2, the effect of (26) on n-point functions for $n>2$ can be removed by the redefinition (19). When we make the change (26), the change in $\delta\hat{\gamma}^{1,2}_{jk}(p^2)$ is

$$\delta\hat{\gamma}^{1,2}_{jk}(p^2) \to \delta\hat{\gamma}^{1,2}_{jk}(p^2) - \left(m_j^2-p^2\right)L_{jk}(p^2)\left(m_k^2-p^2\right). \tag{27}$$

With (27), we can eliminate almost all the p^2 dependence of $\delta\hat{\gamma}^{1,2}_{jk}(p^2)$, but because of the two factors (m^2-p^2), we will, in general, be left both with a constant term and a term proportional to P^2**. Thus we choose

$$\delta\tilde{\gamma}^{1,2}_{jk}(p^2) = A^1_{jk} + p^2 B^1_{jk} = \delta\hat{\gamma}^{1,2}_{jk}(p^2) - \left(m_j^2-p^2\right)L_{jk}(p^2)\left(m_k^2-p^2\right), \tag{28}$$

where A^1_{jk} and B^1_{jk} are *constants*, independent of p^2.

Now clearly, after we have done this redefinition for the ϵ^1 terms, we can feed A^1 and B^1 into our effective lagrangian, through eqs. (22)–(24), and then we can

* Fermions are a little different, of course. For most purposes, to apply these considerations to spin-1/2 fermions, just replace p^2 by $\not{p}$ everywhere in the analysis.

**You can see this, for example, by expanding in powers of p^2, in which case you can solve iteratively for the required $L(p^2)$.

go on and do the same thing for ϵ^n for $n > 1$, to get

$$\delta\tilde{\gamma}_{jk}^{n,2}(p^2) = A_{jk}^n + p^2 B_{jk}^n = \delta\hat{\gamma}_{jk}^{n,2}(p^2) - (m_j^2 - p^2) L_{njk}(p^2)(m_k^2 - p^2), \quad (29)$$

where the $\delta\hat{\gamma}^{n,2}$ includes the effects of the lower-order $\delta\tilde{\gamma}^{n,j}$ terms for $j \geqslant 2$, that is with both the quadratic and higher-degree terms computed on-shell.

To actually calculate the functions A and B in terms of $\delta\hat{\gamma}_{jk}^{n,2}(p^2)$, the procedure is different for the diagonal ($j = k$) and off-diagonal terms. For $j \neq k$, we can obtain two relations that do not depend on the unknown function L by setting $p^2 = m_j^2$ and $p^2 = m_k^2$ to obtain

$$\begin{aligned} A_{jk}^N + m_j^2 B_{jk}^n &= \delta\hat{\gamma}_{jk}^{n,2}(m_j^2), \\ A_{jk}^n + m_k^2 B_{jk}^n &= \delta\hat{\gamma}_{jk}^{n,2}(m_k^2), \end{aligned} \quad (30)$$

or

$$A_{jk}^n = \frac{m_j^2 \delta\hat{\gamma}_{jk}^{n,2}(m_k^2) - m_k^2 \delta\hat{\gamma}_{jk}^{n,2}(m_j^2)}{m_j^2 - m_k^2}, \quad (31)$$

$$B_{jk}^n = \frac{\delta\hat{\gamma}_{jk}^{n,2}(m_j^2) - \delta\hat{\gamma}_{jk}^{n,2}(m_k^2)}{m_j^2 - m_k^2}. \quad (32)$$

For $j = k$, we can obtain one relation by setting $p^2 = m_j^2$,

$$A_{jj}^n + m_j^2 B_{jj}^n = \delta\hat{\gamma}_{jj}^{n,2}(m_j^2), \quad (33)$$

and another by differentiating with respect to p^2, and then setting $p^2 = m_j^2$,

$$B_{jj}^n = \delta\gamma_{jj}^{n,2\prime}(m_j^2). \quad (34)$$

Combining, we get

$$A_{jj}^n = \delta\hat{\gamma}_{jj}^{n,2}(m_j^2) - m_j^2 \delta\gamma_{jj}^{n,2\prime}(m_j^2). \quad (35)$$

The general construction of the on-shell effective theory thus proceeds as follows: To a given order, you first compute the $\delta\hat{\gamma}$ functions from the 1PI functions in the high- and low-energy theory, according to eq. (22). From these, the cubic and higher correction terms in the on-shell effective theory are computed using eq. (21). The quadratic terms are computed using eqs. (30)–(35). The results are then fed back into eqs. (23) and (24), and the process is repeated for the next-order terms.

The relations (31)–(35) are particularly simple when one of the particles is massless. A particularly important case is the two particle system of the photon A and the massive Z of the electroweak model. Here the relations are

$$A''_{\mathrm{AZ}} = \delta\hat{\gamma}^{n,2}_{\mathrm{AZ}}(0) = 0,$$

$$B''_{\mathrm{AZ}} \frac{1}{M_{\mathrm{Z}}^2}\left[\delta\hat{\gamma}^{n,2}_{\mathrm{AZ}}(M_{\mathrm{Z}}^2) - \delta\hat{\gamma}^{n,2}_{\mathrm{AZ}}(0)\right] = \frac{\delta\hat{\gamma}^{n,2}_{\mathrm{AZ}}(M_{\mathrm{Z}}^2)}{M_{\mathrm{Z}}^2}, \tag{36}$$

$$A''_{\mathrm{AA}} = \delta\hat{\gamma}^{n,2}_{\mathrm{AA}}(0) = 0, \qquad B''_{\mathrm{AA}} = \delta\hat{\gamma}^{n,2\prime}_{\mathrm{AA}}(0), \tag{37}$$

where the 0's follow from electromagnetic gauge invariance, and

$$A''_{\mathrm{ZZ}} = \delta\hat{\gamma}^{n,2}_{\mathrm{ZZ}}(M_{\mathrm{Z}}^2) - M_{\mathrm{Z}}^2\delta\hat{\gamma}^{n,2\prime}_{\mathrm{ZZ}}(M_{\mathrm{Z}}^2), \qquad B''_{\mathrm{ZZ}} = \delta\hat{\gamma}^{n,2\prime}_{\mathrm{ZZ}}(M_{\mathrm{Z}}^2). \tag{38}$$

5. Conclusions

In the on-shell effective field theory, the quadratic and cubic terms in the effective lagrangian are determined by a few constants, completely uncontaminated by higher-dimension operators, and thus reliably calculable in perturbation theory even if the ratios of mass scales in the theory are not large. Only in the quartic terms and higher do we find a tower of terms with higher and higher dimension. But because these terms involve more fields than the quadratic and cubic terms, they are generally higher-order in the couplings. This is one reason why the on-shell effective theory is less sensitive to the higher-dimension operators than the conventional analysis.

It is attractive to speculate that one could do even better than this, by explicitly keeping the full momentum dependence of the quartic and higher terms in the on-shell effective lagrangian, and using the $\delta\tilde{\gamma}^{n,j}(p)$ directly as the terms in a non-local effective lagrangian. This is not obviously crazy because these objects, unlike the terms in the effective action, are analytic in momentum for $p < \Lambda$. It may therefore be possible to manipulate them more or less like ordinary local terms in the lagrangian. This requires the development of a generalized renormalization group analysis. Also, the issue of gauge invariance is quite confusing. But if these difficulties can be surmounted, such a scheme could give rise to a version of effective field theory calculations with every bit as much precision as their conventional counterparts.

What is going on here is closely related to the analysis of Lynn et al. and Lynn and Kennedy [4], in which the oblique corrections in the standard electroweak theory are discussed in terms of the two-point function $\Pi_{ab}(k^2)$ of the gauge fields. However, although Kennedy and Lynn call their formalism an effective lagrangian, *it is not*. Their $*$-functions contain both short-distance physics and long-distance physics and furthermore they are not real, since for timelike momenta they incorporate physical imaginary parts, so they cannot be trivially used as lagrangian parameters. In the true effective lagrangian calculation that I have described, only the short-distance physics has been incorporated into parameters of the effective lagrangian, and they are real functions of the renormalization scale μ, not functions of momentum. On the other hand, one could perhaps use some version of the $*$-scheme in a real effective lagrangian calculation that I have outlined above.

I want to close by stressing that I am not suggesting any change in the renormalization prescription of the effective theory. It can (and should) be done in a convenient mass-independent scheme such as $\overline{\mathrm{MS}}$. I simply reorganize the renormalized theory to produce the on-shell effective theory. Thus we retain all the nice properties of a mass-independent renormalization scheme [5].

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