PHYS 139 Midterm

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Problem 1. χ^2 Distribution

Prove that p(χ^2) is distributed with Chisquare (ν).

a. Prove for $x_i \sim Normal(0, \sigma)$

We know that since there is only one independent variable with zero mean value,

$$\chi^2 = (\frac{x_i}{\sigma})^2 = \frac{1}{\sigma^2} x_i^2$$
$$x_i \sim N(0, \sigma)$$
$$\chi^2 = Y = \frac{X^2}{\sigma^2}$$
$$F_Y(y) = P(Y \leq y) = P(-\sigma\sqrt{y} \leq |X| \leq \sigma\sqrt{y}) = CDF_X(\sigma\sqrt{y}) - CDF_X(-\sigma\sqrt{y})$$
$$p(\chi^2) = \frac{dF}{dy} = \frac{\sigma}{2\sqrt{y}} \cdot (PDF_X(\sigma\sqrt{y}) + PDF_X(-\sigma\sqrt{y})$$

The PDF of X is symmetric about x=0,

$$p(\chi^2) = \frac{\sigma}{\sqrt{y}} PDF_X(\sigma\sqrt{y}) = \frac{\sigma}{\sqrt{\chi^2}} \frac{1}{\sqrt{2\pi}\sigma} exp(-\frac{1}{2}\chi^2)$$
$$= \frac{1}{\sqrt{2\pi}} \chi^{-\frac{1}{2}} exp(-\frac{1}{2}\chi^2)$$

which is Chisquare(1).

b. Prove for an arbitrary number of $x_i \sim Normal(0, \sigma)$

The characteristic function of χ^2 is,

$$E[e^{it\chi^2}] = E[e^{it(Y_1+Y_2+...Y_N)}]$$
$$Y_i = \frac{x_i^2}{\sigma_i^2} = Y_i(x_i)$$

Expand this in integral form,

$$E[e^{it(Y_1+Y_2+...Y_N)}] = \int e^{it(Y_1+Y_2+...Y_N)} f(x_1, x_2, ...x_N)\ d(x_1, x_2, ...x_N)$$

Since these random variables are independent,

$$f(x_1, x_2, ...x_N) = \Pi_{i=0}^{N} f(x_i)$$
$$E[e^{it(Y_1+Y_2+...Y_N)}] = \Pi_{i=0}^{N} \int_{-\infty}^{\infty} e^{itY_i} f(x_i)\ dx_i$$

Compute characteristic function for single Y_i,

$$f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{1}{2}(\frac{x_i}{\sigma})^2) = \frac{1}{\sigma_i\sqrt{2\pi}} exp(-\frac{1}{2}Y_i) \tag{1}$$
$$x_i = \sigma_i\sqrt{Y_i} \tag{2}$$
$$dx_i = \sigma_i \frac{1}{2\sqrt{Y_i}} dY_i \tag{3}$$
$$\int_{-\infty}^{\infty} e^{itY_i} f(x_i)\ dx_i = \int_{-\infty}^{\infty} e^{itY_i} f(Y_i)\sigma_i \frac{1}{2\sqrt{Y_i}}\ dY_i \tag{4}$$
$$= \frac{\sigma_i}{2} \int_{-\infty}^{\infty} \frac{1}{\sigma_i\sqrt{2\pi}} exp(-\frac{1}{2}Y_i^2) \cdot exp[itY_i] \frac{1}{\sqrt{Y_i}}\ dY_i \tag{5}$$
$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} exp(-\frac{1}{2}Y_i) \cdot exp[itY_i] \frac{1}{\sqrt{Y_i}}\ dY_i \tag{6}$$
$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} exp[(-\frac{1}{2} + it)Y_i] \cdot \frac{1}{\sqrt{Y_i}}\ dY_i \tag{7}$$

At this point, it is more convenient to adopt the notation of x_i^2, and I realized it was completely useless to replace dx_i to dY_i.

$$\int_{-\infty}^{\infty} e^{itY_i} f(x_i)\ dx_i = \frac{1}{\sqrt{2\pi}\sigma_i} \int_{-\infty}^{\infty} exp[(-\frac{1}{2} + it)\frac{x_i^2}{\sigma^2}]\ dx_i$$
$$= \frac{1}{\sqrt{2\pi}\sigma_i} \sqrt{\frac{2\pi}{1-2it}}\sigma_i = (1-2it)^{-\frac{1}{2}}$$

The characteristic function of χ^2 is,

$$[(1-2it)^{-\frac{1}{2}}]^N$$

where $N = \nu$. This is just the characteristic function of chisquared distribution.

b. Prove that shifting mean values doesn't matter

Adopting the same notation of Y_i, now we get,

$$Y_i = (\frac{x_i - \mu_i}{\sigma_i})^2$$
$$x_i = \sigma_i\sqrt{Y_i} + \mu_i$$

We notice that this doesn't change equation 3, and therefore equation 7 is not affected and is still equal to $(1-2it)^{1/2}$. It follows that values of μ_i doesn't affect chisquared distribution. Chisquared distribution only depends on N, the number of independent random variables.

Problem 2

d. Calculate the confidence of b1 while marginalizing other b parameters

As taken from David's notes,

$$\rho(\vec{x}) = \frac{1}{\sqrt{(2\pi)^N|\Sigma|}} \cdot exp(-\frac{1}{2}\vec{x}^T(\Sigma)^{-1}\vec{x})$$

This is adapted to a probability density function of $\vec{b}$ with non-zero mean vector,

$$\rho(\vec{b}) = \frac{1}{\sqrt{(2\pi)^N|\Sigma|}} \cdot exp[-\frac{1}{2}(\vec{b}-\vec{b}_{max})^T(\Sigma)^{-1}(\vec{b}-\vec{b}_{max})]$$

With marginalization on every component except b1, this becomes,

$$\rho(b_1) = \frac{1}{\sqrt{2\pi|\Sigma_{11}|}} \cdot exp[-\frac{1}{2}(b_1 - b_1(max))^2(\Sigma_{11})^{-1}]$$

Therefore, the 68 % confidence interval becomes evaluations of the following two integrals,

$$\int_{-\infty}^{b_{lower}} \rho(b)\ db = 0.16$$
$$\int_{b_{higher}}^{\infty} \rho(b)\ db = 0.84$$

We notice that the probability density distribution function is just a normal distribution with $\sigma = \sqrt{\Sigma_{11}}$ and $\mu = b_1(max)$. Therefore, we know that the 68 % confidence interval is roughly,

$$[b(max) - \sqrt{\Sigma_{11}}, b(max) + \sqrt{\Sigma_{11}}]$$

The final result is saved in "output.txt".

e. Contour plot

We discovered that the distribution is just multivariate normal, with covariance matrix of (b3,b5). I used the $scipy.stats.multivariate_normal$ package to compute the contour plot. See "output.txt".