PHYS 139 Final Project - Group Part

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The number of detection events (x) of gravitational waves during an interval is modeled with a Poisson distribution,

$$P(x, \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

We assume that the Poisson parameters λ_1, λ_2 are gamma-distributed,

$$p(\lambda) = \frac{1}{\Gamma(a)} b^a \lambda^{a-1} e^{-b\lambda}$$

where the mean is $\frac{a}{b}$, and the mode is $\frac{a-1}{b}$. The variance is $\frac{a}{b^2}$.
The change of λ happens at time n_0. A uniform prior distribution for n_0 is chosen, namely, $P(n_0) = \frac{1}{N}$.
Bayesian rule $p(X, Y) = P(Y|X) \cdot P(X)$ gives us,

$$p(n_0, \lambda_1, \lambda_2, x_{1:N}) = p(x_{1:N}|\lambda_1, \lambda_2, n_0)p(\lambda_1)p(\lambda_2)P(n_0)$$

Rewrite the above integral as,

$$p(x_{1:N}|\lambda_1, \lambda_2, n_0) = \prod_{n=1}^{n_0} P(x_n|\lambda_1) \prod_{n=n_0+1}^{N} P(x_n|\lambda_2)$$

$$p(n_0, \lambda_1, \lambda_2, x_{1:N}) = \prod_{n=1}^{n_0} P(x_n|\lambda_1) \prod_{n=n_0+1}^{N} P(x_n|\lambda_2)p(\lambda_1)p(\lambda_2)P(n_0)$$

Part A: Prove the conditional probability needed for Gibbs Sampling

Compute the term in $p(n_0, \lambda_1, \lambda_2, x_{1:N})$,

$$\prod_{n=1}^{n_0} P(x_n|\lambda_1) = \frac{\lambda_1^{x_1} e^{-\lambda_1}}{x_1!} \cdot \frac{\lambda_1^{x_2} e^{-\lambda_1}}{x_2!} \cdot ...$$

$$ln[\prod_{n=1}^{n_0} P(x_n|\lambda_1)] = ln(\lambda_1^{x_1+x_2+x_3+\cdots}) - n_0\lambda_1 - \sum_{n=1}^{n_0} ln(x_n!)$$

$$= ln\lambda_1 \sum_{n=1}^{n_0} x_n - n_0\lambda_1 - \sum_{n=1}^{n_0} ln(x_n!)$$

We can throw away the constant term $\sum_{n=1}^{n_0} ln(x_n!)$ for normalization. Similarly,

$$ln[\prod_{n=n_0+1}^{N} P(x_n|\lambda_2)] = ln\lambda_2 \sum_{n=n_0+1}^{N} x_n - n_0\lambda_2 - \sum_{n=n_0+1}^{N} ln(x_n!)$$

From Bayesian Rule, we know that,

$$p(\lambda_1|n_0, \lambda_2, x_{1:N}) \propto p(n_0, \lambda_2, x_{1:N}|\lambda_1) \cdot p(\lambda_1) \propto p(x_{1:n_0}|\lambda_1) \cdot p(\lambda_1)$$

$$p(\lambda_1) = p(\lambda_1|a, b) = \frac{1}{\Gamma(a)} b^a \lambda_1^{a-1} e^{-b\lambda_1}$$

$$p(x_{1:n_0}|\gamma_1) = \prod_{n=1}^{n_0} P(x_n|\lambda_1)$$

$$p(\lambda_1|n_0, \lambda_2, x_{1:N}) \propto \frac{1}{\Gamma(a)} b^a \lambda_1^{(a-1+\sum_{n=1}^{n_0} x_n)} e^{(-b-n_0)\lambda_1} \cdot \frac{1}{\prod_{n=1}^{n_0} x_n!}$$

Since $\Gamma_(n) = (n-1)!$,

$$\frac{1}{\prod_{n=1}^{n_0} x_n!} \cdot \frac{1}{\Gamma(a)} = \frac{1}{\Gamma(a + \sum_{n=1}^{n_0} x_n)}$$

From this we can see that,

$$p(\lambda_1|n_0, \lambda_2, x_{1:N}) \propto \frac{1}{\Gamma_(a_1)} \cdot \lambda_1^{a_1-1} e^{-\lambda_1 b_1}$$

where,

$$a_1 = a + \sum_{n=1}^{n_0} x_n$$

$$b_1 = b + n_0$$

After normalization, this just becomes the Gamma function$(\lambda_1|a_1, b_1)$
Using similar procedures, we can find that,

$$p(\lambda_2|n_0, \lambda_1, x_{1:N}) \propto p(x_{n_0:N}|\lambda_2) \cdot p(\lambda_2) \propto \frac{1}{\Gamma_(a_2)} \cdot \lambda_2^{a_2-1} e^{-\lambda_2 b_2}$$

where,

$$a_2 = a + \sum_{n_0+1}^{N} x_n$$
$$b_2 = b + (N - n_0)$$

The goal is to prove that the probability is proportional to Gamma function, and then since it's probability density function, it should be normalized and thus equal to the Gamma function, which is normalized.

Part B: Gibbs Sampling of the Markov Chain M.C.

Step 1: Choose an n_0 value.
Step 2: Select one γ_1 value from the distribution based on n_0.
Step 3: Select one γ_2 value from the distribution based on n_0.
Step 4: Compute a new value based on the new distribution of γ_1, γ_2.
Step 5: Repeat step 2-4.