Mathematical Theory of Elasticity

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PREFACE

The theory of elasticity, in its broad aspects, deals with a study of the behavior of those substances that possess the property of recovering their size and shape when the forces producing deformations are removed. In common with other branches of applied mathematics, the growth of this theory proceeded from a synthesis of special ideas and techniques devised to solve concrete problems. This resulted in a patchwork of theories treating isolated classes of problems, determined largely by the geometry of bodies under consideration. The embedding of such diverse theories in a unified structure, and the construction of the analytical tools for calculating stresses and deformations in a strained elastic body, are among the dominant concerns of the mathematical theory of elasticity.

This book represents an attempt to present several aspects of the theory of elasticity from a unified point of view and to indicate, along with the familiar methods of solution of the field equations of elasticity, some newer general methods of solution of the two-dimensional problems.

The first edition of this book, published in 1946, had its origin in a course of lectures I gave in 1941 and 1942 in the Program of Advanced Instruction and Research in Mechanics conducted by the Graduate School of Brown University. In those lectures I stressed the contributions to the theory by the Russian school of elasticians and, in particular, the relatively little-known work of great elegance and importance by N. I. Muskhelishvili. I planned to supplement that book by a companion volume dealing with effective methods of attack on the two-dimensional and anisotropic problems of elasticity. The developments in the intervening years, however, were so rapid that I was urged to publish instead a single volume containing an up-to-date treatment of material presented in the first edition and supplement it with new topics, in order to give a rounded idea of the current state of the subject.

The present edition differs from its predecessor by extensive additions and changes. Most of the material appearing in the last three chapters had no counterpart in the first edition. Throughout I have tried to give a clear indication of the frontiers of the developments, and I have constantly kept in mind those readers whose principal concern is with practical application of the theory. While no volume of this size can lay claim to an exhaustive list of references to research literature, I have

CHAPTER 3

EQUATIONS OF ELASTICITY

20. Hooke's Law. It has already been noted that the treatment contained in Chaps. 1 and 2 is applicable to all material media that can be represented with sufficient accuracy as continuous bodies; this chapter will be concerned with the characterization of elastic solids.

The first attempt at a scientific description of the strength of solids was made by Galileo. He treated bodies as inextensible, however, since at that time there existed neither experimental data nor physical hypotheses that would yield a relation between the deformation of a solid body and the forces responsible for the deformation. It was Robert Hooke who, some forty years after the appearance of Galileo's Discourses (1638), gave the first rough law of proportionality between the forces and displacements.[1] Hooke published his law first in the form of an anagram "ceiiinosssttuu" in 1676, and two years later gave the solution of the anagram: "*ut tensio sic vis*," which can be translated freely as "the extension is proportional to the force." To study this statement further, we discuss the deformation of a thin rod subjected to a tensile stress.

Consider a thin rod (of a low-carbon steel, for example), of initial cross-sectional area a_0, which is subjected to a variable tensile force F. If the stress is assumed to be distributed uniformly over the area of the cross section, then the *nominal stress* $T = F/a_0$ can be calculated for any applied load F. The *actual stress* is obtained, under the assumption of a uniform stress distribution, by dividing the load at any stage of the test by the actual area of the cross section of the rod at that stage. The difference between the nominal and the actual stress is negligible, however, throughout the elastic range of the material.

If the nominal stress T is plotted as a function of the extension e (change in length per unit length of the specimen), then for some ductile metals a graph like that in Fig. 14 is secured. The graph is very nearly a straight line with the equation

$$T = Ee \tag{20.1}$$

until the stress reaches the *proportional limit* (point P in Fig. 14). The position of this point, however, depends to a considerable extent upon the sensitivity of the testing apparatus. The constant of proportionality E is known as *Young's modulus*.

In most metals, especially in soft and ductile materials, careful observation will reveal very small permanent elongations which are the results of very small tensile forces. In many metals, however (steel and wrought iron, for example), if these very small permanent elongations are neglected (less than 1/100,000 of the length of a bar under tension), then the graph of stress against extension is a straight line, as noted above, and practically all the deformation disappears after the force has been removed. The greatest stress that can be applied without producing a permanent deformation is called the *elastic limit* of the material. When the applied force is increased beyond this fairly sharply defined limit, the material exhibits both elastic and plastic properties. The determination of this limit requires successive loading and unloading by ever larger forces until a permanent set is recorded. For many materials the proportional limit is very nearly equal to the elastic limit, and the distinction between the two is sometimes dropped, particularly since the former is more easily obtained.

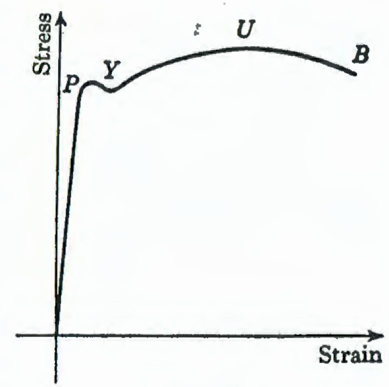


FIG. 14

When the stress increases beyond the elastic limit, a point is reached (Y on the graph) at which the rod suddenly stretches with little or no increase in the load. The stress at point Y is called the *yield-point stress.*

The nominal stress T may be increased beyond the yield point until the *ultimate stress* (point U) is reached. The corresponding force $F = Ta_0$ is the greatest load that the rod will bear. When the ultimate stress is reached, a brittle material (such as a high-carbon steel) breaks suddenly, while a rod of some ductile metal begins to "neck"; that is, its cross-sectional area is greatly reduced over a small portion of the length of the rod. Further elongation is accompanied by an increase in actual stress but by a decrease in total load, in cross-sectional area, and in nominal stress until the rod breaks (point B).

The elastic limit of low-carbon steels is about 35,000 lb per sq in.; the ultimate stress is about 60,000 lb per sq in. Hard steels may be prepared with an ultimate strength greater than 200,000 lb per sq in.

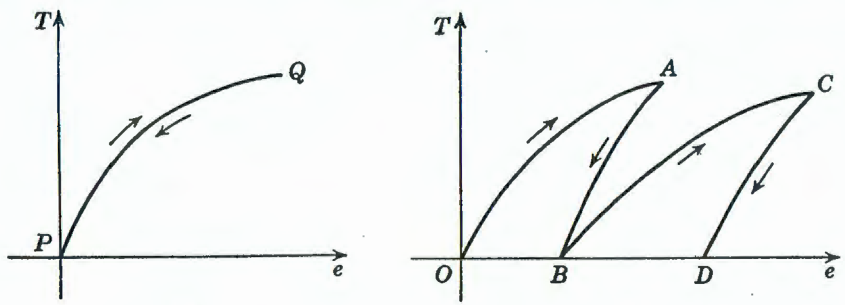
We shall consider only the behavior of elastic materials subjected to stresses below the proportional limit; that is, we shall be concerned only with those materials and situations in which Hooke's law, expressed by Eq. (20.1), or a generalization of it, is valid.[1]

[1] In order to give the reader some feeling regarding the magnitude of deformations with which the theory of elasticity deals, note that a 1-in.-long rod of iron with proportional limit of 25,000 lb per sq in., a yield point of 30,000 lb per sq in., and Young's mod-

Some materials subjected to tensile tests have an extremely small range of values of extensions e for which the law (20.1) is valid. In this case, the stress-strain curve above the proportional limit may have the appearance indicated in Fig. 15*a*. In the process of loading and unloading specimens made of such materials, the same curve PQ may be traced out, and if there is no residual deformation, the material is elastic with the stress-strain law of the form

$$T = f(e),$$

where f is a single-valued nonlinear function. More frequently, however, the loading-unloading diagram has the appearance shown in Fig. 15*b*. In this diagram the curve OA is associated with the loading of the specimen and AB with the unloading. In this instance there is a residual



(*a*) (*b*)

FIG. 15

deformation, represented by OB, which characterizes the plastic behavior. For plastic materials the relationship between T and e is no longer one-to-one, and after repeated loadings and unloadings a saw-tooth pattern indicated in Fig. 15*b* may be obtained.

A natural generalization of Hooke's law immediately suggests itself, namely, one can invoke the principle of superposition of effects and assume that at each point of the medium the strain components e_{ij} are linear functions of the stress components τ_{ij}. Such a generalization was made by Cauchy, and the resulting law is known as the *generalized Hooke's law*. We discuss it in the following section.

21. Generalized Hooke's Law. We saw in the preceding chapters that the state of stress in continuous media is completely determined by the stress tensor τ_{ij}, and the state of deformation by the strain tensor e_{ij}. We shall now assume that when an elastic medium is maintained at a

ulus of 30,000,000 lb per sq in. will elongate under a load of 13,000 lb per sq in. about 0.0004 in. Even if the rod is loaded to the yield point, the determination of the extension will require very refined measurements.

fixed temperature there is a one-to-one analytic relation

$$\tau_{ij} = F_{ij}(e_{11}, e_{22}, \ldots, e_{12}), \qquad (i, j = 1, 2, 3)$$

between the τ_{ij} and e_{ij} and that the τ_{ij} vanish when the strains e_{ij} are all zero. This last assumption implies that in the initial unstrained state the body is unstressed. Now, if the functions F_{ij} are expanded in the power series in e_{ij} and only the linear terms retained in the expansions, we get

$$\tau_{ij} = c_{ijkl}e_{kl} \qquad (i, j, k, l = 1, 2, 3). \tag{21.1}$$

The coefficients c_{ijkl}, in the linear forms (21.1), in general will vary from point to point of the medium. If, however, the c_{ijkl} are independent of the position of the point, the medium is called *elastically homogeneous*. Henceforth we confine our attention to those media in which the c_{ijkl} do not vary throughout the region under consideration. The law (21.1) is a natural generalization of Hooke's law, and it is used in all developments of the linear theory of elasticity.[1]

Inasmuch as the components τ_{ij} are symmetric, an interchange of the indices i and j in (21.1) does not alter these formulas, so that

$$c_{ijkl} = c_{jikl}.$$

Moreover, we can assume, without loss of generality, that the c_{ijkl} are also symmetric with respect to the last two indices. For if the constants c'_{ijkl} and c''_{ijkl} are defined by the formulas

$$c'_{ijkl} = \tfrac{1}{2}(c_{ijkl} + c_{ijlk}),$$
$$c''_{ijkl} = \tfrac{1}{2}(c_{ijkl} - c_{ijlk}),$$

then, clearly, $c'_{ijkl} = c'_{ijlk}$ and $c''_{ijkl} = -c''_{ijlk}$. Thus c_{ijkl} can be written as the sum

$$c_{ijkl} = c'_{ijkl} + c''_{ijkl},$$

in which the c'_{ijkl} are symmetric and the c''_{ijkl} are skew-symmetric with respect to k and l. Accordingly, the law (21.1) can always be written in the form

$$\tau_{ij} = c'_{ijkl}e_{kl} + c''_{ijkl}e_{kl}.$$

However, the double sum in the second term of this expression vanishes inasmuch as $e_{kl} = e_{lk}$ and $c''_{ijkl} = -c''_{ijlk}$. Thus,

$$\tau_{ij} = c'_{ijkl}e_{kl},$$

where the c'_{ijkl} are symmetric with respect to the first two and the last two indices.

[1] It is important to note that the generalized Hooke's law (21.1) is also used in some investigations where the strains are finite, in the sense of Sec. 11. For many materials a linear relationship (21.1) holds for an appreciable range of values of the e_{ij}. The linear theory of elasticity, however, is based on the use of the infinitesimal strains, defined in Sec. 7, and on the linear law (21.1).

We shall consider henceforth that the c_{ijkl} in (21.1) have been symmetrized, so that there are at most 36 independent constants in the general stress-strain law (21.1).

To avoid dealing with double sums, we can introduce the notation

$$\begin{array}{llllll} \tau_{11} = \tau_1, & \tau_{22} = \tau_2, & \tau_{33} = \tau_3, & \tau_{23} = \tau_4, & \tau_{31} = \tau_5, & \tau_{12} = \tau_6, \\ e_{11} = e_1, & e_{22} = e_2, & e_{33} = e_3, & 2e_{23} = e_4, & 2e_{31} = e_5, & 2e_{12} = e_6, \end{array}$$

and write the six equations (21.1) in the form

$$\begin{aligned} \tau_1 &= c_{11}e_1 + c_{12}e_2 + c_{13}e_3 + c_{14}e_4 + c_{15}e_5 + c_{16}e_6, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \tau_6 &= c_{61}e_1 + c_{62}e_2 + c_{63}e_3 + c_{64}e_4 + c_{65}e_5 + c_{66}e_6, \end{aligned}$$

or, more compactly,

$$\tau_i = c_{ij}e_j \qquad (i, j = 1, 2, \ldots, 6). \tag{21.2}$$

The relations between the τ_i and e_i must be reversible, hence $|c_{ij}| \neq 0$, and we can write

$$e_i = C_{ij}\tau_j.$$

The constants c_{ij} in (21.2) are called the *elastic constants*, or *moduli*, of the material. Inasmuch as the strains e_i are dimensionless, the c_{ij} have the same dimensions as the stress components.[1]

We have just remarked that the maximum number of the independent elastic constants is 36, but this number reduces to 21 whenever there exists a function

$$W = \tfrac{1}{2}c_{ij}e_ie_j, \tag{21.3}$$

with the property,

$$\frac{\partial W}{\partial e_i} = \tau_i. \tag{21.4}$$

For one can always suppose that the quadratic form (21.3) is symmetric, and it then follows from (21.4) that

$$\tau_i = c_{ij}e_j,$$

where $c_{ij} = c_{ji}$.

The potential function W was introduced by George Green,[2] and it is called the *strain-energy density function.* Its existence for the isothermal and adiabatic processes[3] has been argued on the basis of the first and

[1] See Sec. 14.

[2] G. Green, *Transactions of the Cambridge Philosophical Society*, vol. 7 (1839), p. 121.

[3] See references in this book to Lord Kelvin's papers in the *Historical Sketch.* The isothermal process corresponds to the case of slow loading and unloading involving no temperature changes of the medium. It is of interest in elastostatic problems. The adiabatic process is approximated in those dynamical problems where bodies execute small and rapid vibrations. The elastic constants c_{ij} in the two cases cannot be expected to be identical.

second laws of thermodynamics, and it is now generally accepted that, for the most general case of an anisotropic elastic body, the number of independent elastic constants in the generalized Hooke's law is 21. The matter of the number of elastic constants required to describe the stress-strain law of the form (21.1) was the subject of a lengthy controversy. Cauchy and Poisson argued,[1] on the basis of special mathematical models of molecular interaction, that the number of independent constants cannot exceed 15. Their arguments proved wanting and are in contradiction to experimental evidence.

If an elastic medium exhibits a geometrical symmetry of internal structure (crystallographic form, regular arrangement of fibers or molecules, etc.) then its elastic properties become identical in certain directions.[2] The geometric symmetry, however, is not equivalent to elastic symmetry because there may be certain other directions for which the elastic properties are the same but the geometric ones are not.

If the medium is elastically symmetric in certain directions, then the number of independent constants c_{ij} in (21.2) is further reduced. Because of their practical importance, we discuss in this section two particular types of elastic symmetry. These are (1) symmetry with respect to a plane (in which 13 independent elastic constants are involved), and (2) symmetry with respect to three mutually perpendicular planes (involving 9 independent constants c_{ij}). In the next section we prove that when the elastic properties of a body are identical in all directions, that is, if the body is elastically *isotropic*, the number of essential elastic constants reduces to 2.

It is obvious from (21.2) that the coefficients c_{ij}, in general, depend on the chosen reference frame inasmuch as the stress components τ_i and the strain components e_i vary with the choice of coordinate systems. For certain media the coefficients c_{ij} may remain invariant under a given transformation of coordinates, and it is this invariance which determines the elastic symmetry of the medium under consideration.[3]

[1] A. L. Cauchy, *Exercises de mathématique*, vol. 3 (1828*a*), p. 213; vol. 3 (1828*b*), p. 328.

S. D. Poisson, *Mémoires de l'académie*, Paris, vol. 8 (1829); vol. 18 (1842).

See also in this connection:

M. Born, Dynamik der Kristalgitter (1915) and Atomtheorie des festen Zustandes, 2d ed. (1923), and comments on Born's work by I. Stakgold, *Quarterly of Applied Mathematics*, vol. 8 (1950), pp. 169–186.

P. S. Epstein, *Physical Review*, vol. 70 (1946), pp. 915–922.

The arguments of Green and Lord Kelvin, in support of the 21-constant theory, are presented in Chap. III of Love's Treatise on the Mathematical Theory of Elasticity.

[2] This is the principle expressed by F. Neumann in Vorlesungen über die Theorie der Elastizität (1885). See also Love's Treatise (1927), p. 155.

[3] The reader familiar with the rudiments of tensor analysis will recognize that when

Consider a substance elastically symmetric with respect to the x_1x_2-plane. This symmetry is expressed by the statement that the c_{ij} are invariant under the transformation

$$x_1 = x_1', \qquad x_2 = x_2', \qquad x_3 = -x_3'.$$

The table of direction cosines of this transformation is

	x_1	x_2	x_3
x_1'	1	0	0
x_2'	0	1	0
x_3'	0	0	-1

and from formulas (5.9) and (16.4) it is seen that

$$\tau_i' = \tau_i, \qquad e_i' = e_i, \qquad (i = 1, 2, 3, 6),$$
$$\tau_4' = -\tau_4, \qquad e_4' = -e_4, \qquad \tau_5' = -\tau_5. \qquad e_5' = -e_5.$$

The first equation of (21.2) becomes

$$\tau_1' = c_{11}e_1' + c_{12}e_2' + c_{13}e_3' + c_{14}e_4' + c_{15}e_5' + c_{16}e_6',$$

or

$$\tau_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 - c_{14}e_4 - c_{15}e_5 + c_{16}e_6.$$

Comparison of this equation with the expression for τ_1 given by (21.2) shows that

$$c_{14} = c_{15} = 0.$$

Similarly, by considering $\tau_2', \ldots, \tau_6'$, we find that

$$c_{24} = c_{25} = c_{34} = c_{35} = c_{64} = c_{65} = 0,$$
$$c_{41} = c_{42} = c_{43} = c_{46} = c_{51} = c_{52} = c_{53} = c_{56} = 0.$$

For a material with one plane of elastic symmetry (which is taken to be the x_1x_2-plane), the matrix of the coefficients of the linear forms in (21.2) can be written as follows,

the law (21.1) is written in the form

$$\tau_{ij} = c_{ij}^{kl}e_{kl}, \qquad (i, j, k, l = 1, 2, 3),$$

valid in all coordinate systems, then it follows from the tensor character of the τ_{ij} and e_{kl} that the c_{ij}^{kl} are components of a tensor of rank 4. Consequently, under a transformation of coordinates from the system X to X', the c_{ij}^{kl} transform according to the law

$$(a) \qquad c'^{kl}_{ij} = \frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\beta}{\partial x_j'}\frac{\partial x_k'}{\partial x_\gamma}\frac{\partial x_l'}{\partial x_\delta}c_{\alpha\beta}^{\gamma\delta}.$$

If the c_{ij}^{kl} are invariant (so that $c'^{kl}_{ij} = c_{ij}^{kl}$) under a given coordinate transformation, then the transformation characterizes the nature of elastic symmetry. The $\dfrac{\partial x_i}{\partial x_j'}$ figuring in the law (a) are the direction cosines appearing in the tables of this section, because the systems X and X' are cartesian.

$$(21.5) \qquad \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{21} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{31} & c_{32} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{54} & c_{55} & 0 \\ c_{61} & c_{62} & c_{63} & 0 & 0 & c_{66} \end{pmatrix}$$

Such materials as wood, for example, have three mutually orthogonal planes of elastic symmetry and are said to be *orthotropic*. In considering such materials, we shall choose the axes of coordinates so that the coordinate planes coincide with the planes of elastic symmetry. In this case, some of the coefficients c_{ij} exhibited in the array (21.5) vanish. Besides the symmetry with respect to the x_1x_2-plane, expressed by (21.5), the elastic constants c_{ij} must also be invariant under the transformation of coordinates defined by the following table of direction cosines.

	x_1	x_2	x_3
x_1'	-1	0	0
x_2'	0	1	0
x_3'	0	0	1

This change of coordinates is a reflection in the x_2x_3-plane and leaves the τ_i and e_i unchanged with the following exceptions:

$$\tau_5' = -\tau_5, \qquad e_5' = -e_5, \qquad \tau_6' = -\tau_6, \qquad e_6' = -e_6.$$

From (21.5) we have

$$\tau_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 + c_{16}e_6.$$

This becomes

$$\tau_1' = c_{11}e_1' + c_{12}e_2' + c_{13}e_3' + c_{16}e_6',$$

or

$$\tau_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3 - c_{16}e_6,$$

from which it follows that $c_{16} = 0$. By considering in a similar way the transformed expressions for $\tau_2, \ldots, \tau_6$, we find that[1]

$$c_{26} = c_{36} = c_{45} = c_{54} = c_{61} = c_{62} = c_{63} = 0.$$

Thus, for orthotropic media the matrix of the c_{ij} takes the following form.

$$(21.6) \qquad \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}$$

[1] Note that elastic symmetry in the x_1x_2-plane and in the x_2x_3-plane implies elastic symmetry in the x_1x_3-plane.

If the coefficients c_{ij} are symmetric, that is,

$$c_{ij} = c_{ji}, \qquad (i, j = 1, 2, \ldots, 6), \tag{21.7}$$

we see that there are 13 essential constants in the array (21.5) and 9 in (21.6). This symmetry has not been assumed, however, in establishing the forms of the arrays of coefficients (21.5) and (21.6), nor will it be used in the next section, where the law (21.2) is specialized to that for an isotropic medium.

It is worth noting that the statement of the law (21.2) is not devoid of inconsistency. In the process of formulating the notion of the components of strain e_{ij}, it was assumed that the components of displacement u_i are functions of the coordinates (x_1, x_2, x_3) of the body in its undeformed state; that is, Lagrangian coordinates were used. On the other hand, Eulerian coordinates were employed in defining the components of the stress tensor τ_{ij}; that is, it was assumed that the τ_{ij} are functions of the coordinates (x_1', x_2', x_3') of the stressed (and hence deformed) medium. Of course, if the displacements u_i and their derivatives are small, then the values of $\tau_{ij}(x)$ and $\tau_{ij}(x')$ cannot differ by a great deal. As an indication of the order of approximation involved here, note that, if $x_k' = x_k + u_k$, then

$$\frac{\partial \tau_{ij}}{\partial x_k} = \frac{\partial \tau_{ij}}{\partial x_l'}\frac{\partial x_l'}{\partial x_k} = \frac{\partial \tau_{ij}}{\partial x_l'}\left(\delta_{kl} + \frac{\partial u_l}{\partial x_k}\right) = \frac{\partial \tau_{ij}}{\partial x_k'} + \frac{\partial \tau_{ij}}{\partial x_l'}\frac{\partial u_l}{\partial x_k}.$$

Hence, in writing $\dfrac{\partial \tau_{ij}}{\partial x_k} = \dfrac{\partial \tau_{ij}}{\partial x_k'}$, we assume that the displacement derivatives are small compared with unity. In what follows, it will be assumed that both the components of strain e_{ij} and the components of stress τ_{ij} are functions of the initial coordinates (x_1, x_2, x_3).

REFERENCES FOR COLLATERAL READING

A. E. H. Love: A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, London, Secs. 60–65, pp. 92–100.

Chap. VI of Love's treatise is given to a discussion of the equilibrium of non-isotropic elastic solids and contains further references on the subject. Voigt's Lehrbuch der Kristallphysik is a standard treatise on the subject.

L. Lecornu: Théorie mathématique de l'élasticité, Mémorial des sciences mathématiques, Gauthiers-Villars & Cie, Paris, pp. 12–18.

Contains a discussion of the theory of Poincaré regarding the number of elastic constants in the generalized Hooke's law.

PROBLEMS

1. Are the principal axes of strain coincident with those of stress for an anisotropic medium with Hooke's law expressed by Eq. (21.2)? For a medium with one plane of elastic symmetry? For an orthotropic medium? *Hint:* Take the coordinate axes along the principal axes of strain so that $e_4 = e_5 = e_6 = 0$.

2. Show directly from the generalized Hooke's law [Eq. (21.2)] that in an isotropic body the principal axes of strain coincide with those of stress. *Hint:* Take the coordinate axes along the principal axes of strain ($e_4 = e_5 = e_6 = 0$), and consider the effect on τ_{23} and τ_{31} of a rotation of axes by 180° about the x_3-axis.

22. Homogeneous Isotropic Media.

Most structural materials are formed of crystalline substances, and hence very small portions of such materials cannot be regarded as being isotropic. Nevertheless, the assumption of isotropy and homogeneity, when applied to an entire body, often does not lead to serious discrepancies between the experimental and theoretical results.[1] The reason for this agreement lies in the fact that the dimensions of most crystals are so small in comparison with the dimensions of body and they are so chaotically distributed that, in the large, the substance behaves as though it were isotropic.

From the definition of the isotropic medium, it follows that its elastic properties are independent of the orientation of coordinate axes. In particular, the coefficients c_{ij} must remain invariant when we introduce new coordinate axes x_1', x_2', x_3', obtained by rotating the x_1, x_2, x_3-system through a right angle about the x_1-axis. By considering the transformed stress components τ_{ij}', in exactly the same way as was done in the preceding section, it is found that

$$c_{12} = c_{13}, \qquad c_{31} = c_{21}, \qquad c_{32} = c_{23}, \qquad c_{33} = c_{22}, \qquad c_{66} = c_{55}.$$

Similarly, a rotation of axes through a right angle about the x_3-axis leads to the relations

$$c_{21} = c_{12}, \qquad c_{22} = c_{11}, \qquad c_{23} = c_{13}, \qquad c_{31} = c_{32}, \qquad c_{55} = c_{44}.$$

We introduce, finally, the coordinate system x_1', x_2', x_3', got from the x_1, x_2, x_3-system by rotating the latter through an angle of 45° about the x_3-axis. In this case, we have

$$\tau_{12}' = -\tfrac{1}{2}\tau_{11} + \tfrac{1}{2}\tau_{22}, \qquad e_{12}' = -\tfrac{1}{2}e_{11} + \tfrac{1}{2}e_{22},$$

or, noting the definitions on page 60,

$$\tau_6' = -\tfrac{1}{2}\tau_1 + \tfrac{1}{2}\tau_2, \qquad e_6' = -e_1 + e_2.$$

From (21.6) and the relation $c_{66} = c_{44}$, we have

$$\tau_6 = c_{44}e_6.$$

When referred to the x_1', x_2', x_3'-axes, this becomes $\tau_6' = c_{44}e_6'$ or

$$-\tfrac{1}{2}\tau_1 + \tfrac{1}{2}\tau_2 = c_{44}(-e_1 + e_2). \tag{22.1}$$

Now from (21.6)

$$\tau_1 = c_{11}e_1 + c_{12}e_2 + c_{13}e_3,$$
$$\tau_2 = c_{21}e_1 + c_{22}e_2 + c_{23}e_3,$$

[1] Many cast metals are notable exceptions. The processes of rolling and drawing frequently produce a definite orientation of crystals, so that many rolled and drawn metals are anisotropic.

and from the relations given above, namely,

$$c_{22} = c_{11}, \qquad c_{23} = c_{13} = c_{21} = c_{12},$$

we get

$$-\tfrac{1}{2}\tau_1 + \tfrac{1}{2}\tau_2 = \tfrac{1}{2}(c_{11} - c_{12})(-e_1 + e_2).$$

Comparison of this equation with (22.1) yields the result

$$c_{44} = \tfrac{1}{2}(c_{11} - c_{12}) \equiv \mu, \tag{22.2}$$

so that

$$\tau_6 = \mu e_6.$$

We shall find it convenient to write the generalized Hooke's law for an isotropic body in terms of the two constants λ and μ, where μ is defined by (22.2) and where we put

$$c_{12} = \lambda.$$

From (21.6) we can now write

$$\begin{aligned}\tau_{11} &= c_{11}e_{11} + c_{12}e_{22} + c_{12}e_{33}\\ &= c_{12}(e_{11} + e_{22} + e_{33}) + (c_{11} - c_{12})e_{11}\\ &= \lambda\vartheta + 2\mu e_{11}.\end{aligned}$$

Thus, the generalized Hooke's law for a homogeneous isotropic body can be written in the following form:

$$\tau_{ij} = \lambda\delta_{ij}\vartheta + 2\mu e_{ij}, \qquad (i, j = 1, 2, 3). \tag{22.3}$$

Equation (22.3) yields a simple relation connecting the invariants $\vartheta = e_{ii}$ and $\Theta = \tau_{ii}$.

Putting $j = i$ in (22.3) and noting that $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$, one finds that

$$\Theta \equiv \tau_{ii} = 3\lambda\vartheta + 2\mu e_{ii},$$

or

$$\Theta = (3\lambda + 2\mu)\vartheta. \tag{22.4}$$

Equations (22.3) can now be solved easily for the strains e_{ij} in terms of the stresses τ_{ij}. We have

$$e_{ij} = \frac{-\lambda}{2\mu}\,\delta_{ij}\vartheta + \frac{1}{2\mu}\,\tau_{ij},$$

or

$$e_{ij} = \frac{-\lambda\delta_{ij}}{2\mu(3\lambda + 2\mu)}\,\Theta + \frac{1}{2\mu}\,\tau_{ij}. \tag{22.5}$$

It is clear from (22.5) that we must require that $\mu \neq 0$ and $3\lambda + 2\mu \neq 0$.

The constants λ and μ were introduced by G. Lamé and are called the *Lamé constants.*

We have shown that the stress-strain law for isotropic media involves no more than two elastic constants. The fact that no further reduction is possible is physically obvious from the simple tensile tests, but an

analytic proof of this, utilizing the properties of isotropic tensors, can be constructed.[1]

If the axes x_i are directed along the principal axes of strain, then $e_{23} = e_{31} = e_{12} = 0$. But from (22.3) we see that in this case τ_{23}, τ_{31}, and τ_{12} also vanish. Hence the axes x_i must lie along the principal axes of stress, and we have the result that *the principal axes of stress are coincident with the principal axes of strain if the medium is isotropic.* This property was used by Cauchy to define the isotropic elastic medium.

Henceforth no distinction will be made between the principal axes of strain and those of stress, and such axes will be referred to simply as the *principal axes.*

23. Elastic Moduli for Isotropic Media. Simple Tension. Pure Shear. Hydrostatic Pressure. In order to gain some insight into the physical significance of elastic constants entering in formulas (22.3), we consider the behavior of elastic bodies subjected to simple tension, pure shear, and hydrostatic pressure.

Assume that a right cylinder with the axis parallel to the x_1-axis is subjected to the action of longitudinal forces applied to the ends of the cylinder. If the applied forces give rise to a uniform tension T in every cross section of the cylinder, then

$$\tau_{11} = T = \text{const}, \qquad \tau_{22} = \tau_{33} = \tau_{12} = \tau_{23} = \tau_{31} = 0. \tag{23.1}$$

Since the body forces are not present, the state of stress determined by (23.1) satisfies the equilibrium equations (15.3) in the interior of the cylinder, and equations (13.3) show that the lateral surface of the cylinder is free of tractions.

The substitution from (23.1) in (22.5) yields the appropriate values of strains, namely,[2]

$$\begin{cases} e_{11} = \dfrac{(\lambda + \mu)T}{\mu(3\lambda + 2\mu)}, \qquad e_{22} = e_{33} = \dfrac{-\lambda T}{2\mu(3\lambda + 2\mu)},\\ e_{12} = e_{23} = e_{31} = 0,\end{cases} \tag{23.2}$$

which clearly satisfy the compatibility equations (10.9). Accordingly, the state of stress (23.1) actually corresponds to the one that can exist in a deformed elastic body.

Noting that

$$\frac{e_{22}}{e_{11}} = \frac{-\lambda}{2(\lambda + \mu)},$$

we introduce the abbreviations

$$\sigma \equiv \frac{\lambda}{2(\lambda + \mu)}, \qquad E \equiv \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \tag{23.3}$$

[1] H. Jeffreys, Cartesian Tensors (1931).

[2] The integration of Eqs. (23.2), yielding the displacements u_i, is carried out in Sec. 30.

Then Eqs. (23.2) can be written in the form

$$(23.4)\qquad \begin{cases} e_{11} = \dfrac{1}{E}T, & e_{22} = e_{33} = \dfrac{-\sigma}{E}T = -\sigma e_{11}, \\ e_{12} = e_{23} = e_{31} = 0. \end{cases}$$

If the stress T represents tension, so that $T > 0$, then a tensile stress will produce an extension in the direction of the axis of the cylinder and a contraction in its cross section. Accordingly, for $T > 0$, we have $e_{11} > 0$, $e_{22} < 0$, $e_{33} < 0$. It follows that E and σ are both positive.

Physical interpretations of the elastic moduli E and σ are easily obtained. It follows from the first of the formulas (23.4) that the quantity

$$E = \frac{T}{e_{11}}$$

represents the ratio of the tensile stress T to the extension e_{11} produced by the stress T. Again, from (23.4), it is seen that

$$\sigma = \left|\frac{e_{22}}{e_{11}}\right| = \left|\frac{e_{33}}{e_{11}}\right|;$$

thus σ denotes the ratio of the contraction of the linear elements perpendicular to the axis of the cylinder to the longitudinal extension of the rod. The quantity E is known as *Young's modulus*, and the number σ is called the *Poisson ratio.*

It is easy to verify that one can express the constants λ and μ in terms of Young's modulus and Poisson's ratio as

$$(23.5)\qquad \lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \qquad \mu = \frac{E}{2(1+\sigma)}.$$

Consider next the state of pure shear characterized by the stress components

$$\tau_{23} = T = \text{const}, \qquad \tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = \tau_{31} = 0.$$

Substituting these values in (22.5) yields

$$(23.6)\qquad e_{23} = \frac{1}{2\mu}T, \qquad e_{11} = e_{22} = e_{33} = e_{12} = e_{31} = 0.$$

These formulas show that a rectangular parallelepiped $OPQR$, whose faces are parallel to the coordinate planes, is sheared in the x_2x_3-plane (see Fig. 4) so that the right angle between the edges of the parallelepiped parallel to the x_2- and x_3-axes is diminished, for $T > 0$, by the angle $\alpha_{23} = 2e_{23}$. From (23.6) we have

$$\mu = \frac{T}{\alpha_{23}}.$$

Thus the number μ represents the ratio of the shearing stress T to the change in angle α_{23} produced by the shearing stress. For this reason the quantity μ is called the *modulus of rigidity*, or the *shear modulus.* Since E and σ are both positive, it follows from the second of Eqs. (23.5) that μ is also positive.

Finally consider a body τ of arbitrary shape subjected to a hydrostatic pressure of uniform intensity p distributed over its surface. The components $\overset{\nu}{T}_i$ of the stress vector acting on the surface are then

$$\overset{\nu}{T}_i = -p\nu_i,$$

where ν_i are the direction cosines of the normal $\mathbf{v}$ to the surface.

The system of stresses

$$(23.7)\qquad \begin{cases} \tau_{11} = \tau_{22} = \tau_{33} = -p, & \tau_{12} = \tau_{23} = \tau_{31} = 0, \\ \Theta = \tau_{11} + \tau_{22} + \tau_{33} = -3p, \end{cases}$$

satisfies the equilibrium equation in the interior of τ and on its surface. From (22.5) we deduce the expressions[1]

$$(23.8)\qquad e_{11} = e_{22} = e_{33} = -\frac{p}{3\lambda + 2\mu}, \qquad e_{12} = e_{23} = e_{31} = 0,$$

which, clearly, satisfy the compatibility equations (10.9). The cubical compression $\vartheta = e_{ii}$ can be obtained either from (23.8) or from the general relations (22.4) and (23.7). We get

$$\vartheta = e_{11} + e_{22} + e_{33} = -\frac{p}{\lambda + \frac{2}{3}\mu},$$

which can be written as

$$\vartheta = -\frac{p}{k}, \qquad \text{or} \qquad k = -\frac{p}{\vartheta},$$

by introducing the abbreviation

$$(23.9)\qquad k = \lambda + \tfrac{2}{3}\mu.$$

[1] If the substitution from (23.5) in (23.8) is made, we find that

$$e_{11} = e_{22} = e_{33} = \frac{-p(1-2\sigma)}{E}, \qquad e_{ij} = 0 \text{ for } i \neq j.$$

Since $u_{i,j} + u_{j,i} = 2e_{ij}$, we have for the determination of displacements the system of equations,

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = \frac{\partial u_3}{\partial x_3} = -p\frac{1-2\sigma}{E}, \qquad \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 0, \qquad i \neq j.$$

The integration of these equations yields [cf. Sec. 30]

$$u_i = -\frac{p}{3k}x_i + \alpha_{0i} + \alpha_{ij}x_j, \qquad k = \frac{E}{3(1-2\sigma)},$$

where $\alpha_{ij} = -\alpha_{ji}$ and the α_{0i} are the integration constants. These integration constants are associated with the rigid body motion. If we fix the point $x_i = 0$ (assumed to be in the body) and impose the condition that the rotation vector ω_i (Sec. 7) vanishes, we get

$$u_i = -\frac{p}{3k}x_i.$$

Thus, the quantity k represents the ratio of the compressive stress to the cubical compression, and for this reason it is called the *modulus of compression.* Since for all physical substances a hydrostatic pressure tends to diminish the bulk, it is clear that k is positive. Substituting in (23.9) the expressions for λ and μ from (23.5) gives

$$k = \frac{E}{3(1 - 2\sigma)}.$$

Since k is positive for all physical substances, *it follows that σ is less than one-half, and hence* [see (23.5)] λ *is positive.* For most structural materials, the value of σ does not deviate much from one-third. If the material is highly incompressible (rubber, for example), σ is nearly one-half and $\mu \doteq E/3$.

The stress-strain relations (22.5), when written by making the substitutions from (23.5), assume the simple form

$$e_{ij} = \frac{1 + \sigma}{E} \tau_{ij} - \frac{\sigma}{E} \delta_{ij}\Theta, \tag{23.10}$$

where $\Theta = \tau_{ii}$. If we recall the notation of Sec. 14, these relations can also be given in the following form:

$$\left\{\begin{aligned} e_{xx} &= \frac{1}{E}[\tau_{xx} - \sigma(\tau_{yy} + \tau_{zz})],\\ e_{yy} &= \frac{1}{E}[\tau_{yy} - \sigma(\tau_{zz} + \tau_{xx})],\\ e_{zz} &= \frac{1}{E}[\tau_{zz} - \sigma(\tau_{xx} + \tau_{yy})],\\ e_{yz} &= \frac{1+\sigma}{E}\tau_{yz}, \qquad e_{zx} = \frac{1+\sigma}{E}\tau_{zx}, \qquad e_{xy} = \frac{1+\sigma}{E}\tau_{xy}. \end{aligned}\right. \tag{23.11}$$

The following table gives average values of E, μ, and σ for several elastic materials; the moduli E and μ are given in millions of pounds per square inch.[1]

	E	μ	σ (experimental)	$\sigma = \frac{E}{2\mu} - 1$
Carbon steels	29.5	11.5	0.29	0.283
Wrought iron	28.0	11.0	0.28	0.273
Cast iron	16.5	6.5	0.25	0.269
Copper (hot-rolled)	15.0	5.6	0.33	0.339
Brass, 2:1 (cold-drawn)	13.0	4.9	0.33	0.327
Glass	8.0	3.2	0.25	0.250
Spruce (along the grain)	1.5	0.08		

[1] In the engineering literature, the modulus of shear is often denoted by G, and the reciprocal of Poisson's ratio σ is denoted by m; that is, $m = 1/\sigma$.

REFERENCES FOR COLLATERAL READING

A. E. H. Love: A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, London, Secs. 69–71.

E. Trefftz: Handbuch der Physik, Verlag von Julius Springer, Berlin, vol. 6, Secs. 11–12.

PROBLEMS

1. Show that Hooke's law in the form (23.11) can be obtained by the following argument: An elementary rectangular parallelepiped subjected to tensile stresses τ_{xx} on opposite faces will experience a longitudinal extension $e_{xx} = \tau_{xx}/E$ and lateral contractions $e_{yy} = e_{zz} = -\sigma e_{xx}$. Now consider the effect of stresses τ_{xx}, τ_{yy}, τ_{zz}, and superpose the resulting strains to get Eq. (23.11).

2. Use Hooke's law to show that the stress invariant $\Theta = \tau_{jj}$ and the strain invariant $\vartheta = e_{jj}$ are connected by the relation $\Theta = 3k\vartheta$, where k is the modulus of compression.

3. Show that a stress vector cannot cross a free surface (one on which there is no external load). *Hint:* Let $\mathbf{v}$ be the normal to the free surface. Then $\overset{\nu}{\mathbf{T}} = 0$ and, from (16.1), $\overset{\nu'}{\mathbf{T}} \cdot \mathbf{v} = \overset{\nu}{\mathbf{T}} \cdot \mathbf{v}' = 0$.

4. Derive the following relations between the Lamé coefficients λ and μ, Poisson's ratio σ, Young's modulus E, and the bulk modulus k:

$$\begin{aligned}
\lambda &= \frac{2\mu\sigma}{1 - 2\sigma} = \frac{\mu(E - 2\mu)}{3\mu - E} = k - \frac{2}{3}\mu = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}\\
&= \frac{3k\sigma}{1+\sigma} = \frac{3k(3k - E)}{9k - E},\\
\mu &= \frac{\lambda(1 - 2\sigma)}{2\sigma} = \frac{3}{2}(k - \lambda) = \frac{E}{2(1+\sigma)} = \frac{3k(1 - 2\sigma)}{2(1+\sigma)}\\
&= \frac{3kE}{9k - E},\\
\sigma &= \frac{\lambda}{2(\lambda + \mu)} = \frac{\lambda}{3k - \lambda} = \frac{E}{2\mu} - 1 = \frac{3k - 2\mu}{2(3k + \mu)}\\
&= \frac{3k - E}{6k},\\
E &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} = \frac{\lambda(1+\sigma)(1-2\sigma)}{\sigma} = \frac{9k(k - \lambda)}{3k - \lambda}\\
&= 2\mu(1+\sigma) = \frac{9k\mu}{3k + \mu} = 3k(1 - 2\sigma),\\
k &= \lambda + \frac{2}{3}\mu = \frac{\lambda(1+\sigma)}{3\sigma} = \frac{2\mu(1+\sigma)}{3(1-2\sigma)}\\
&= \frac{\mu E}{3(3\mu - E)} = \frac{E}{3(1 - 2\sigma)}.
\end{aligned}$$

24. Equilibrium Equations for an Isotropic Elastic Solid. The complete system of equations of equilibrium of a homogeneous isotropic elastic solid is made up of the following equations:

a. Equations of Equilibrium. From (15.3)

$$\tau_{ij,j} + F_i = 0, \qquad (i, j = 1, 2, 3); \tag{24.1}$$

b. Stress-Strain Relations. From (22.3)

$$\tau_{ij} = \lambda\, \delta_{ij}\vartheta + 2\mu e_{ij}, \tag{24.2}$$

where

$$\vartheta = e_{ii},$$

and [from (7.5)]

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{24.3}$$

The systems of Eqs. (24.1) and (24.2) must be satisfied at every interior point of the body τ, and on the surface Σ of the body τ the stresses must fulfill the equilibrium conditions (13.3)

$$\tau_{ij}\nu_j = \overset{\nu}{T}_i, \tag{24.4}$$

where the ν_i are the direction cosines of the exterior normal $\mathbf{v}$ to the surface Σ, and $\overset{\nu}{\mathbf{T}}$ is the stress vector acting on the surface element with normal $\mathbf{v}$. To these equations one must adjoin the equations of compatibility [from (10.9)]

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0. \tag{24.5}$$

It will be shown in Sec. 27 that the system of Eqs. (24.1) and (24.2), subject to the conditions of equilibrium on the surface (24.4), is complete in the sense that, if there exists a solution of the system, then that solution is unique. There are nine equations in the system on the set of nine unknown functions τ_{ij}, u_i $(i, j = 1, 2, 3)$. Once the displacements u_i are determined, the strain components e_{ij} entering into (24.2) are readily calculated with the aid of the formulas (24.3). We have assumed that the displacements u_i are continuous functions of class C^3 throughout the region τ, and a reference to (24.2) shows that the components of stress τ_{ij} are continuous of class C^2 in the same region. The equations of equilibrium (24.1) contain the components F_i of the body force $\mathbf{F}$, and they are assumed to be prescribed functions of the coordinates x_i of the undeformed body. Typical examples of the body forces $\mathbf{F}$, occurring in practical applications, are centrifugal forces and forces of gravitation. Furthermore, the components $\overset{\nu}{T}_i$ of the external surface force $\overset{\nu}{\mathbf{T}}$ are assumed to be prescribed functions of the coordinates x_i of the undeformed surface Σ of the body.

In order that the solution of the problem may exist, it is clear that one cannot prescribe the body force $\mathbf{F}$ and the surface force $\overset{\nu}{\mathbf{T}}$ in a perfectly arbitrary manner, inasmuch as Eqs. (24.1) were established on the hypothesis that the body is in equilibrium. Hence one must demand that the distribution of the forces $\mathbf{F}$ and $\overset{\nu}{\mathbf{T}}$, acting on the body τ, be such that the resultant force and the resultant moment vanish.[1]

[1] That is, $\mathbf{F}$ and $\overset{\nu}{\mathbf{T}}$ must be sufficiently regular and satisfy, for the body as a whole, the equations immediately preceding (15.1), and (15.4).

It is clear from physical considerations that, instead of prescribing the distribution of the surface force $\overset{\nu}{\mathbf{T}}$ acting on Σ, one could prescribe the displacements u_i on the surface Σ and that the state of stress established in the interior of the body by deforming its surface Σ must also be characterized in a unique way. Thus, we are led to consider the following fundamental boundary-value problems of elasticity:

Problem 1. *Determine the distribution of stress and the displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the distribution of the forces acting on the surface of the body is known.*

Problem 2. *Determine the distribution of stress and the displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the displacements of the points on the surface of the body are prescribed functions.*

In many applications, it is important to consider a problem resulting from the combination of the problems stated above. Thus, one may have the displacements of the points on part of the surface prescribed and the distribution of forces specified over the remaining portion. Such a problem will be referred to as a mixed boundary-value problem.

It should be noted that in Prob. 1 the external forces are assigned over the initial, or undeformed, surface of the body, while the equilibrium under these forces is reached when the body is in the final deformed state. Since the displacements are small, the error introduced in this approximation has the order of magnitude implicit in the formulation of the stress-strain relations, as stated in the concluding paragraph of Sec. 21.

The formulation of the fundamental boundary-value problems of elasticity given above suggests the desirability of expressing the differential equations for Prob. 1 entirely in terms of stresses and those for Prob. 2 entirely in terms of displacements. This is not difficult to do.

Let us first obtain the equations in terms of displacements u_i by substituting in (24.1) the expressions for stresses in terms of displacements. Making use of the formulas (24.3), we can write the system (24.2) in the form

$$\tau_{ij} = \lambda\, \delta_{ij} u_{k,k} + \mu(u_{i,j} + u_{j,i}). \tag{24.6}$$

Substituting the values of the stress components (24.6) in the equilibrium equations (24.1) gives

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + F_i = 0,$$

or

$$\mu \nabla^2 u_i + (\lambda + \mu)\frac{\partial \vartheta}{\partial x_i} + F_i = 0 \tag{24.7}$$

where

$$\vartheta = e_{ii} = u_{i,i} = \operatorname{div} \mathbf{u}.$$

Equations (24.7) are associated with the name of Navier.

Note that we need not adjoin the compatibility equations (10.9), for the only purpose of the latter is to impose restrictions on the strain components that shall ensure that the e_{ij} yield single-valued continuous displacements u_i, when the region τ is simply connected.

It is clear that Prob. 2 is completely solved if one obtains the solution of the system (24.7) subject to the boundary conditions

$$u_i = f_i(x_1, x_2, x_3), \qquad (i = 1, 2, 3),$$

where the f_i are prescribed continuous functions on the boundary of the undeformed solid. From the knowledge of the functions u_i, one can determine the strains, and hence the stresses by making use of the relations (24.2).

We now turn our attention to the first boundary-value problem. It was noted earlier that not every solution of the system of three equations of equilibrium (24.1) corresponds to a possible state of strain in an elastic body, because the components of strain, defined by the system of Eqs. (23.10), must satisfy the equations of compatibility (24.5). We proceed to derive the compatibility equations in terms of the stresses. If the expressions (23.10)

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\Theta$$

are inserted in the compatibility equations (24.5)

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0,$$

we obtain

$$(24.8) \quad \tau_{ij,kl} + \tau_{kl,ij} - \tau_{ik,jl} - \tau_{jl,ik} = \frac{\sigma}{1+\sigma}(\delta_{ij}\Theta_{,kl} + \delta_{kl}\Theta_{,ij} - \delta_{ik}\Theta_{,jl} - \delta_{jl}\Theta_{,ik}).$$

Since the indices i, j, k, l assume values 1, 2, 3, there are $3^4 = 81$ equations in the system (24.5), but not all these are independent, for an interchange of i and j or of k and l obviously does not yield new equations. Also for certain values of the indices (such as $i = j = k = l$), Eqs. (24.5) are identically satisfied, and, as already noted in Sec. 10, the set of Eqs. (24.5) contains only six independent equations obtained by setting

$$\begin{aligned}
&k = l = 1, \quad && i = j = 2; \\
&k = l = 2, && i = j = 3; \\
&k = l = 3, && i = j = 1; \\
&k = l = 1, && i = 2, \quad j = 3; \\
&k = l = 2, && i = 3, \quad j = 1; \\
&k = l = 3, && i = 1, \quad j = 2.
\end{aligned}$$

Inasmuch as Eqs. (23.10) establish one-to-one correspondence between the e_{ij} and the τ_{ij}, the set of 81 equations (24.8) likewise contains only 6

independent equations. If we combine Eqs. (24.8) linearly by setting $k = l$ and summing with respect to the common index, we get

$$\tau_{ij,kk} + \tau_{kk,ij} - \tau_{ik,jk} - \tau_{jk,ik} = \frac{\sigma}{1+\sigma}(\delta_{ij}\Theta_{,kk} + \delta_{kk}\Theta_{,ij} - \delta_{ik}\Theta_{,jk} - \delta_{jk}\Theta_{,ik}).$$

This is a set of 9 equations of which only 6 are independent because of the symmetry in i and j. Consequently, in combining Eqs. (24.8) linearly, the number of independent equations is not reduced, and hence the resultant set of equations is equivalent to the original one.

Noting that

$$\tau_{ij,kk} = \nabla^2\tau_{ij}$$

and

$$\tau_{kk} = \Theta,$$

the foregoing equations can be written as

$$(24.9) \quad \nabla^2\tau_{ij} + \Theta_{,ij} - \tau_{ik,jk} - \tau_{jk,ik} = \frac{\sigma}{1+\sigma}(\delta_{ij}\nabla^2\Theta + 3\Theta_{,ij} - 2\Theta_{,ij}),$$

if we make use of the continuity of the second derivatives of Θ.

Equations (24.9) can be written more neatly by utilizing the equations of equilibrium (24.1)

$$\tau_{ik,k} + F_i = 0.$$

Thus, differentiating (24.1) with respect to x_j, we get

$$(24.10) \qquad \tau_{ik,kj} = -F_{i,j}$$

and since $\tau_{ik,kj} = \tau_{ik,jk}$, we can rewrite (24.9) in the form

$$(24.11) \quad \nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} - \frac{\sigma}{1+\sigma}\delta_{ij}\nabla^2\Theta = -(F_{i,j} + F_{j,i}).$$

This set of 6 independent equations can be further simplified by expressing an invariant $\nabla^2\Theta$ in terms of the derivatives of the body force $\mathbf{F}$. This may be done as follows:

If we set $k = i$ and $l = j$ in (24.8) and sum with respect to the common indices, we get

$$2\tau_{ij,ij} - \tau_{ii,jj} - \tau_{jj,ii} = \frac{\sigma}{1+\sigma}(2\,\delta_{ij}\Theta_{,ij} - \delta_{ii}\Theta_{,jj} - \delta_{jj}\Theta_{,ii}).$$

But

$$\tau_{ii} = \tau_{jj} = \Theta, \qquad \delta_{ij}\Theta_{,ij} = \Theta_{,ii} = \nabla^2\Theta,$$

and

$$\delta_{ii}\Theta_{,jj} = \delta_{jj}\Theta_{,ii} = 3\nabla^2\Theta.$$

The foregoing equation can be written as

$$\tau_{ij,ij} - \nabla^2\Theta = \frac{\sigma}{1+\sigma}(\nabla^2\Theta - 3\nabla^2\Theta)$$

or

$$\tau_{ij,ij} = \frac{1-\sigma}{1+\sigma}\nabla^2\Theta. \tag{24.12}$$

The differentiation of the equilibrium equation

$$\tau_{ij,i} = -F_j$$

gives

$$\tau_{ij,ij} = -F_{j,j},$$

and inserting this in the left-hand member of (24.12) yields the formula

$$\nabla^2\Theta = -\frac{1+\sigma}{1-\sigma}F_{j,j} \equiv -\frac{1+\sigma}{1-\sigma}\operatorname{div}\mathbf{F}. \tag{24.13}$$

Substituting from (24.13) in (24.11) gives the final form of the compatibility equation in terms of stresses,

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = -\frac{\sigma}{1-\sigma}\delta_{ij}\operatorname{div}\mathbf{F} - (F_{i,j} + F_{j,i}). \tag{24.14}$$

Equations (24.14), when written out in unabridged notation, yield the following 6 equations of compatibility:

$$\left\{\begin{aligned}
\nabla^2\tau_{xx} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x^2} &= -\frac{\sigma}{1-\sigma}\operatorname{div}\mathbf{F} - 2\frac{\partial F_x}{\partial x},\\
\nabla^2\tau_{yy} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y^2} &= -\frac{\sigma}{1-\sigma}\operatorname{div}\mathbf{F} - 2\frac{\partial F_y}{\partial y},\\
\nabla^2\tau_{zz} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} &= -\frac{\sigma}{1-\sigma}\operatorname{div}\mathbf{F} - 2\frac{\partial F_z}{\partial z},\\
\nabla^2\tau_{yz} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y\,\partial z} &= -\left(\frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial y}\right),\\
\nabla^2\tau_{zx} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z\,\partial x} &= -\left(\frac{\partial F_z}{\partial x} + \frac{\partial F_x}{\partial z}\right),\\
\nabla^2\tau_{xy} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x\,\partial y} &= -\left(\frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x}\right).
\end{aligned}\right. \tag{24.15}$$

Equations (24.15) were obtained by Michell in 1900 and, for the case when the body forces are absent, by Beltrami in 1892. They are known as the Beltrami-Michell *compatibility equations*. Thus, in order to determine the state of stress in the interior of an elastic body, one must solve the system of equations consisting of (24.1) and (24.15) subject to the boundary conditions (24.4).

The system of Eqs. (24.1) and (24.15) is equivalent to the system consisting of Eqs. (24.1), (24.2), and (24.5).

If the field of body force $\mathbf{F}$ is conservative, so that

$$\mathbf{F} = \nabla\varphi$$

or

$$F_j = \varphi_{,j},$$

then

$$\operatorname{div}\mathbf{F} \equiv F_{j,j} = \varphi_{,jj} \equiv \nabla^2\varphi,$$

and

$$F_{i,j} = \varphi_{,ij}, \qquad F_{j,i} = \varphi_{,ji} = \varphi_{,ij},$$

so that (24.14) can be written as

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = -\frac{\sigma}{1-\sigma}\delta_{ij}\nabla^2\varphi - 2\varphi_{,ij}. \tag{24.16}$$

We shall consider two particular cases of body forces, namely, the case in which $\mathbf{F}$ is a constant vector and that in which the potential function φ is harmonic (that is, $\operatorname{div}\mathbf{F} = \nabla^2\varphi = 0$).

If $\mathbf{F}$ is constant, then φ is a linear function. In this case the right-hand member of (24.16) vanishes, and we obtain the equations of Beltrami,

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = 0. \tag{24.17}$$

From (24.13) it follows that in this case

$$\nabla^2\Theta = 0,$$

so that $\Theta = \tau_{ii}$ is a harmonic function. Equation (22.4) shows that the strain invariant $\vartheta = e_{ii}$ is also harmonic; that is,

$$\nabla^2\vartheta = 0$$

whenever Θ is harmonic. From (24.17) it is seen that, if the τ_{ij} are of class C^4, the components of stress satisfy the *biharmonic* equation

$$\nabla^2\nabla^2\tau_{ij} \equiv \nabla^4\tau_{ij} = 0,$$

and since the strain components e_{ij} are linear functions of the τ_{ij}, we have

$$\nabla^4 e_{ij} = 0.$$

A function V of class C^4, and satisfying the equation $\nabla^4 V = 0$, is called a *biharmonic function*.

If the body force $\mathbf{F}$ is derived from a harmonic potential function, so that

$$\operatorname{div}\mathbf{F} = \nabla^2\varphi = 0,$$

then from (24.13) and (22.4) we see that

$$\nabla^2\Theta = 0, \qquad \text{and} \qquad \nabla^2\vartheta = 0.$$

We can thus enunciate a theorem.

THEOREM: *When the components of the body force $\mathbf{F}$ are constant, the invariants Θ and ϑ are harmonic functions and the stress components τ_{ij} and strain components e_{ij} are biharmonic functions.*

When **F** *is derived from a harmonic potential function, the invariants* Θ *and* ϑ *are also harmonic.*

It will be shown, with the aid of some general theorems to be established in Sec. 26, that Probs. 1 and 2 have essentially unique solutions. Before proceeding to derive these theorems, however, we may note that, on account of the linear character of Eqs. (24.1), (24.2), and (24.3), the principle of superposition is applicable to the fundamental problems of elasticity.

Thus, suppose that one finds a set of nine functions

$$\tau_{ij}^{(1)},\ u_i^{(1)}, \qquad (i, j = 1, 2, 3),$$

which satisfy the systems (24.1) and (24.2) with prescribed body forces $F_i^{(1)}$. Also let a set of functions

$$\tau_{ij}^{(2)},\ u_i^{(2)}, \qquad (i, j = 1, 2, 3) \tag{24.18}$$

be the solutions of the systems corresponding to the choice of the body forces $F_i^{(2)}$. Then it is obvious that the solution

$$\tau_{ij} = \tau_{ij}^{(1)} + \tau_{ij}^{(2)}, \qquad u_i = u_i^{(1)} + u_i^{(2)}, \qquad (i = j = 1, 2, 3) \tag{24.19}$$

will correspond to the choice of the body force whose components are $F_i^{(1)} + F_i^{(2)}$. If the set of functions (24.18) represents a solution of the homogeneous system, that is, when $F_i^{(2)} = 0$, then the expressions (24.19) represent a solution of the problem corresponding to the choice of the body force with components $F_i^{(1)}$.

PROBLEMS

1. Show that the following stress components are not the solution of a problem in elasticity, even though they satisfy the equations of equilibrium with zero body forces:

$$\begin{aligned}
\tau_{xx} &= c[y^2 + \sigma(x^2 - y^2)], \qquad c \neq 0,\\
\tau_{yy} &= c[x^2 + \sigma(y^2 - x^2)],\\
\tau_{zz} &= c\sigma(x^2 + y^2),\\
\tau_{xy} &= -2c\sigma xy,\\
\tau_{yz} &= \tau_{zx} = 0.
\end{aligned}$$

2. The solutions of many problems in elasticity are either exactly or approximately independent of the value chosen for Poisson's ratio. This fact suggests that approximate solutions may be found by so choosing Poisson's ratio as to simplify the problem. Show that, if one takes $\sigma = 0$, then

$$\lambda = 0, \qquad \mu = \tfrac{1}{2}E, \qquad k = \tfrac{1}{3}E,$$

and Hooke's law is expressed by

$$\tau_{ij} = Ee_{ij} = \tfrac{1}{2}E(u_{i,j} + u_{j,i}).$$

Show by differentiation of these equations that

[illegible]

(no sum on repeated subscripts). That is, the six stress components are connected, in this case, by the three equilibrium equations

$$\tau_{ij,j} + F_i = 0$$

and by three compatibility equations, namely,

$$\frac{\partial^2\tau_{xy}}{\partial x\,\partial y} = \frac{1}{2}\left(\frac{\partial^2\tau_{xx}}{\partial y^2} + \frac{\partial^2\tau_{yy}}{\partial x^2}\right),$$

and two similar equations obtained by cyclic interchange of x, y, z. Derive these compatibility conditions from Eq. (24.8) by setting $\sigma = 0$, $k = i$, $l = j$.

A. and L. Föppl have discussed[1] the simplification of the equations of elasticity obtained by choosing for Poisson's ratio $\sigma = 0$ or $\sigma = \frac{1}{2}$. Westergaard[2] has treated the problem of obtaining the general solution from a solution for a particular choice of Poisson's ratio.

3. Define the stress function S by

$$\tau_{ij} = S_{,ij} \equiv \frac{\partial^2 S}{\partial x_i\,\partial x_j}$$

and consider the case of zero body force. Show that, if Poisson's ratio σ is assumed to vanish, then the equilibrium and compatibility equations given in the preceding problem reduce to

$$\nabla^2 S = \text{const.}$$

4. Show that, if Poisson's ratio σ has the value $\frac{1}{2}$, then

$$\mu = \tfrac{1}{3}E, \qquad \lambda = \infty, \qquad k = \infty, \qquad \vartheta \equiv e_{jj} = u_{i,i} = 0.$$

Interpret physically the situation described by these elastic coefficients. From Hooke's law (23.10) deduce the relations

$$\begin{aligned}
\tau_{ij} &= 2\mu e_{ij} + \tfrac{1}{3}\,\delta_{ij}\Theta\\
&= \mu(u_{i,j} + u_{j,i}) + \tfrac{1}{3}\,\delta_{ij}\Theta.
\end{aligned}$$

Show that in this case

$$u_{j,ij} = \frac{\partial\vartheta}{\partial x_i} = 0$$

and that the equilibrium equations (24.1) can be written in the form

$$\nabla^2 u_i + \frac{1}{\mu}\left(\frac{1}{3}\Theta_{,i} + F_i\right) = 0.$$

That is, putting $u_1 = u$, $u_2 = v$, etc., the four functions u, v, w, Θ are to be determined from the four equations

$$\begin{aligned}
\nabla^2 u + \frac{1}{\mu}\left(\frac{1}{3}\frac{\partial\Theta}{\partial x} + F_x\right) &= 0,\\
\nabla^2 v + \frac{1}{\mu}\left(\frac{1}{3}\frac{\partial\Theta}{\partial y} + F_y\right) &= 0,\\
\nabla^2 w + \frac{1}{\mu}\left(\frac{1}{3}\frac{\partial\Theta}{\partial z} + F_z\right) &= 0,\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}$$

This case ($\sigma = \frac{1}{2}$) has been discussed at length by A. and L. Föppl.[1]

[1] A. and L. Föppl, Drang und Zwang, vol. 1, Sec. 3.

[2] H. M. Westergaard, "Effects of a Change of Poisson's Ratio Analyzed by Twinned Gradients," *Journal of Applied Mechanics*, vol. 62 (1940), pp. A-113–A-116.