type of body force in this book. The second type of force is a surface force, which acts from the fluid outside the material control volume on the fluid inside and vice versa. In reality, there may exist short-range forces of molecular origin in the fluid. With the crude scale of resolution inherent in the use of the continuum approximation, these will appear as surface force contributions in the basic balance (2–20). In addition, the surface force terms will always include an *effective* surface force contribution to simulate the transport of momentum across the boundaries of the material control volume due to random differences between the continuum velocity **u** and the actual molecular velocities. As explained earlier, the necessity for these latter *effective* surface force contributions is entirely a consequence of the crude scale of resolution inherent in the continuum approximation, coupled with the discrete nature of any real fluid. Indeed, these latter contributions would be zero if the material were truly an indivisible continuum. We shall see that the main difficulties in obtaining pointwise differential equations of motion from (2–20) all derive from the necessity for including surface forces (whether real or effective) whose molecular origin is outside the realm of the continuum description.

With the necessity for body and surface forces thus identified, a mathematical statement of Newton's second law can be written with the material control volume chosen as the moving "body," that is,

$$\boxed{\frac{D}{Dt}\int_{V_m(t)}\rho\mathbf{u}dV=\int_{V_m(t)}\rho\mathbf{g}dV+\int_{S_m(t)}\mathbf{t}dS.} \tag{2–21}$$

The left-hand side is just the time rate of change of linear momentum of all the fluid within the specified material control volume. The first term on the right-hand side is the net body force due to gravity (other types of body force will not be considered in this book). The second term is the net surface force, with the local surface force per unit area being symbolically represented by the vector **t**. We call **t** the *stress vector.* It is the vector sum of all surface force contributions per unit area acting at a point on the surface of $V_m(t)$. We may recall that the flux of momentum across the surface of a material control volume due to the continuum level fluid motion **u** is identically zero, since $\mathbf{u}\cdot\mathbf{n}=0$ by definition at each point on the surface. Thus, any change in linear momentum is due to the action of surface or body forces, as indicated in (2–21). This may be contrasted with the situation that arises for a fixed control volume, where momentum is transported through the surface by convection due to **u**, and an additional source of changes in momentum is thus any imbalance between the mean influx and efflux of momentum due to the fluid's overall motion.

We may now attempt to simplify (2–21) to a differential form, as we did for the mass conservation equation (2–6). The basic idea is to express all terms in (2–21) as integrals over $V_m(t)$, leading to the requirement that the sum of the integrands is zero since $V_m(t)$ is initially arbitrary. However, it is immediately apparent that this scheme will fail unless we can say more about the surface stress vector **t**. Otherwise, there is no way to express the surface integral of **t** in terms of an equivalent volume integral over $V_m(t)$.

We note first of all that **t** is not only a function of position and time, as is the case

with **u,** but also of the orientation of the differential surface element through **x** on which it acts. The reader may well ask how this is known in the absence of a direct molecular derivation of a theoretical expression for **t** (the latter being outside the realm of continuum mechanics, even if it were possible in principle). The answer is that certain general properties of **t**, including its orientation dependence, can be either deduced or derived from (2–21) by considering the limit as the material control volume is decreased progressively toward zero while holding the geometry (shape) of V_m constant. Let us denote a characteristic linear dimension of V_m as l, with l^3 defined to be equal to V_m. An estimate for each of the integrals in (2–21) can be obtained in terms of l using the mean value theorem. A useful preliminary step is to apply the Reynolds transport theorem to the left-hand side. Although this might, at first sight, seem to present new difficulties since $\rho\mathbf{u}$ is a vector, whereas the Reynolds transport theorem was originally derived for a scalar, the result given by (2–9) carries over directly, as may be seen by applying it to each of the three scalar components of $\rho\mathbf{u}$, and then adding the results. Thus, (2–21) can be rewritten in the form

$$\int_{V_m(t)} \left[\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \rho\mathbf{g} \right] dV = \int_{S_m(t)} \mathbf{t}\, dS. \tag{2–22}$$

Now, denoting the mean value over V_m or S_m by the symbol $< >$, we can express (2–22) in the symbolic form

$$< > l^3 = < > l^2.$$

It is evident that as $l \to 0$, the volume integral of the momentum and body force terms vanishes more quickly than the surface integral of the stress vector. Hence, in the limit as $l \to 0$, (2–22) reduces to the form

$$\boxed{\lim_{l\to 0} \frac{1}{S_m} \int_{S_m(t)} \mathbf{t}\, dS \to 0.} \tag{2–23}$$

This result is sometimes called *the principle of stress equilibrium*, because it shows that the surface forces must be in local equilibrium for any arbitrarily small volume element centered at any point **x** in the fluid. This is true independent of the source or detailed form of the surface forces.

Now, it is clear that the stress vector at (or arbitrarily close to) a point **x** must depend not only on **x** but also on the orientation of the surface through **x** on which it acts, since otherwise the equilibrium condition (2–23) could not be satisfied. At first this dependence on orientation may seem to suggest that one would need a triply infinite set of numbers to specify **t** for all possible orientations of a surface through each point **x**. Not only is this clearly impossible, but (2–23) shows that it is not necessary. Let us consider a surface with completely arbitrary orientation, specified by a unit normal **n**, that passes near to point **x** but not precisely through it. Then using this surface as one side, we construct a tetrahedron, illustrated in Figure 2–5, centered around point **x**, whose remaining three sides are mutually perpendicular. In the limit as the volume of this tetrahedral volume element goes to zero, the surface stress equilibrium principle applies, and it is obvious that the surface stress vector on the arbitrarily-oriented sur-