Elementary engineering fracture mechanics

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3 *The elastic crack-tip stress field*

3.1 The Airy stress function

Consider a coordinate system X, Y, Z in a stressed solid. In each point (x, y, z) one can define the stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz}. In a condition of plane stress $\sigma_z = \tau_{xz} = \tau_{yz} = 0$. In a condition of plane strain $\epsilon_z = 0$ from which it follows that $\sigma_z = \upsilon(\sigma_x + \sigma_y)$.

For plane problems the equilibrium equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \tag{3.1}$$

If the displacements in x and y direction are u en υ respectively, the expressions for the strains are:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial \upsilon}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x} \tag{3.2}$$

and the stress-strain relations:

$$E\varepsilon_x = \sigma_x - \upsilon\sigma_y, \quad E\varepsilon_y = \sigma_y - \upsilon\sigma_x, \quad \mu\gamma_{xy} = \tau_{xy} \tag{3.3}$$

where the shear modulus μ is related to Young's modulus, E, by $\mu = E/2$ $(1 + \nu)$ in which ν is Poisson's ratio.

The equilibrium equations (3.1) are automatically satisfied if

$$\sigma_x = \frac{\partial^2 \psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y}. \tag{3.4}$$

The function ψ is called the Airy stress function. Substitution of eqs (3.2) and (3.4) into (3.3), and differentiating twice leads to the compatibility equation:

$$\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0 \tag{3.5}$$

or:

$$\nabla^2(\nabla^2\psi)=0. \tag{3.6}$$

In general, a plane extensional problem in linear elasticity can be solved by finding a stress function ψ that satisfies eq (3.6). Also, the stresses calculated from eqs (3.4) must satisfy the boundary conditions of the problem. The stress function for a particular problem must be guessed on the basis of some experience. The approach is fully discussed in any text book on the theory of elasticity [*e.g.* 1].

3.2 Complex stress functions

One can define a complex function by

$$Z(z)=\mathrm{Re}\,Z+i\,\mathrm{Im}\,Z \quad \text{with } z=x+\mathrm{i}y. \tag{3.7}$$

For Z to be an analytic function, the derivative $\mathrm{d}Z/\mathrm{d}z$ must be defined unambiguously. This leads to the Cauchy-Riemann conditions:

$$\frac{\partial\,\mathrm{Re}\,Z}{\partial x}=\frac{\partial\,\mathrm{Im}\,Z}{\partial y}=\mathrm{Re}\frac{\mathrm{d}Z}{\mathrm{d}z}$$

$$\frac{\partial\,\mathrm{Im}\,Z}{\partial x}=-\frac{\partial\,\mathrm{Re}\,Z}{\partial y}=\mathrm{Im}\frac{\mathrm{d}Z}{\mathrm{d}z} \tag{3.8}$$

For the solution of crack problems several complex forms of the Airy stress function can be used [2–9]. In the case of mode I cracks it is convenient to use a function proposed by Westergaard [3]. It was shown by Sih [6] and by Eftis and Liebowitz [7] that the Westergaard function is not fully correct, but this does not affect the result as far as the singular terms of the stresses are concerned.

The Westergaard function is:

$$\psi=\mathrm{Re}\,\bar{\bar{Z}}+y\,\mathrm{Im}\,\bar{Z} \tag{3.9}$$

where $\bar{\bar{Z}}$, $\bar{Z}$ and Z' are given by:

$$\frac{\mathrm{d}\bar{\bar{Z}}}{\mathrm{d}z}=\bar{Z},\quad \frac{\mathrm{d}\bar{Z}}{\mathrm{d}z}=Z,\quad \frac{\mathrm{d}Z}{\mathrm{d}z}=Z'. \tag{3.10}$$

with the Cauchy-Riemann equations (3.8) it follows that

$$\nabla^2\,\mathrm{Re}\,Z=\nabla^2\,\mathrm{Im}\,Z=0 \tag{3.11}$$

which means that eq (3.9) automatically satisfies the compatibility equation (3.6).

By using eqs (3.4) the stresses can be determined as:

$$\sigma_x=\mathrm{Re}\,Z-y\,\mathrm{Im}\,Z',\quad \sigma_y=\mathrm{Re}\,Z+y\,\mathrm{Im}\,Z',$$

$$\tau_{xy}=-y\,\mathrm{Re}\,Z'. \tag{3.12}$$

Any analytic function $Z(z)$ will result in stresses defined by eqs (3.12). It remains to find a function $Z(z)$ that also satisfies the boundary conditions for the problem under consideration. As pointed out by Sih [6] and by Eftis and Liebowitz [7] eqs (3.12) have to be extended by constant terms if the corrected Westergaard function is used. The terms only vanish for rather special loading conditions, but they do not affect the stress singularity.

3.3 Solution to crack problems

Consider the mode I crack problem of figure 3.1, representing an infinite plate under biaxial stress. The stress function for this case is

$$Z=\frac{\sigma z}{\sqrt{z^2-a^2}},\quad \text{where } z=x+\mathrm{i}y. \tag{3.13}$$

The function is analytic except for $(-a\leqslant x\leqslant a,\ y=0)$. The boundary stresses follow from eqs (3.12). At infinity, where $|z|\to\infty$, the result is $\sigma_x=\sigma_y=\sigma$ and $\tau_{xy}=0$, and on the crack surface $\sigma_y=\tau_{xy}=0$, which means that the boundary conditions are satisfied.

It is more convenient to convert to a coordinate system with the origin at the crack tip, hence z should be replaced by $(z+a)$. Turning then to

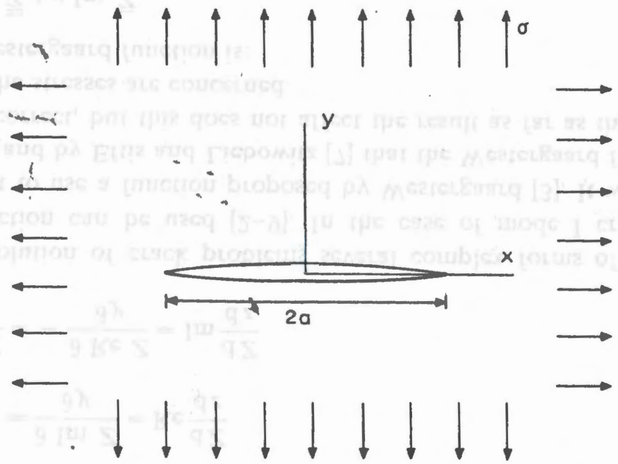

Figure 3.1. Mode I crack under bi-axial stress

the general problem (figure 3.2) where the boundary conditions are not yet specified, Z must take the form:

$$Z = \frac{f(z)}{\sqrt{z}} \tag{3.14}$$

where $f(z)$ is well behaved and must be real and a constant at the origin. Then according to eqs (3.12) both σ_y and τ_{xy} are zero at the crack surface, i.e. the crack edges are stress free. The required real and constant value of $f(z)$ at the crack tip is given the notation K_I, hence

$$Z_{|z| \to 0} = \frac{K_I}{\sqrt{2\pi z}}. \tag{3.15}$$

Taking polar coordinates from the origin (figure 3.2) with $z = re^{i\theta}$ the

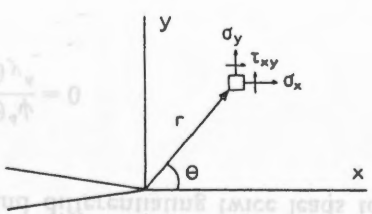

Figure 3.2. General mode I problem

stresses at the crack tip can be calculated from eqs (3.12) and (3.15) to be:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{3} \sin\frac{3\theta}{2}\right) - \sigma$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2}\right)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\frac{3\theta}{2} \tag{3.16}$$

or

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta).$$

The term $-\sigma$ results for the case of uniaxial tension if the Westergaard stress function is applied correctly, as shown by Sih [6] and Eftis and Liebowitz [7]. It is of no effect for the singular terms.

For plane stress $\sigma_z = 0$, for plane strain $\sigma_z = \upsilon(\sigma_x + \sigma_y)$. The parameter K in these equations is known as the stress intensity factor. For $r \to 0$ (at the very crack tip) the stresses become infinite. The stress intensity factor is then a measure for the stress singularity at the crack tip. Since the stresses are elastic they must be proportional to the external load. For the case of uniaxial tension with σ at infinity, it means that K_I must be proportional to σ. In order to give the proper dimension to the stresses in eq (3.16), K_I must also be proportional to the square root of a length. For an infinite plate the only characteristic length is the crack size, hence K_I must take the form:

$$K_I = c\sigma\sqrt{a}. \tag{3.17}$$

Returning now to the specific case of biaxial tension of figure 3.1, the stress function is given in eq (3.13). Displacement of the origin of the coordinate system to the crack tip modifies eq (3.13) to:

$$Z = \frac{\sigma(z+a)}{\sqrt{z(z+2a)}} \tag{3.18}$$

Comparison of eqs (3.15) and (3.18) shows that

$$K_{\mathrm{I}} = \sigma\sqrt{\pi a}. \tag{3.19}$$

Since it may be expected that the stress system parallel to the crack is not disturbed by the crack, the solution for the uniaxial case must be the same as for the biaxial case. Hence, the factor C in eq (3.17) is equal to $\sqrt{\pi}$ for a plate under uniaxial tension.

Apart from the stresses, the displacements also can be determined. It follows from eqs (3.2) and (3.12) that for plane strain:

$$\begin{aligned} v &= \frac{1+\nu}{E}\left[2(1-\nu)\,\mathrm{Im}\,\bar{Z} - y\,\mathrm{Re}\,Z\right] \\ u &= \frac{1+\nu}{E}\left[(1-2\nu)\,\mathrm{Re}\,\bar{Z} - y\,\mathrm{Im}\,Z\right] \end{aligned} \tag{3.20}$$

which leads to:

$$\begin{aligned} u &= 2(1+\nu)\frac{K_{\mathrm{I}}}{E}\sqrt{\frac{r}{2\pi}}\,\cos\frac{\theta}{2}\left[1-2\nu+\sin^2\frac{\theta}{2}\right] \\ v &= 2(1+\nu)\frac{K_{\mathrm{I}}}{E}\sqrt{\frac{r}{2\pi}}\,\sin\frac{\theta}{2}\left[2-2\nu-\cos^2\frac{\theta}{2}\right]. \end{aligned} \tag{3.21}$$

The equations (3.16) for the stress field are the exact solution for the region $r \approx 0$. They can be used in the area where r is small compared to the crack size. In the general solution higher order terms of $f(z)$ have also to be included. The general solution is

$$\sigma_{ij} = C_1\left(\frac{r}{a}\right)^{-\frac{1}{2}} f_{1ij}(\theta) + C_2\left(\frac{r}{a}\right)^{0} f_{2ij}(\theta) + C_3\left(\frac{r}{a}\right)^{\frac{1}{2}} f_{3ij}(\theta) + \ldots \tag{3.22}$$

or

$$\sigma_{ij} = \frac{C_1}{\sqrt{r}} f_{1ij}(\theta) + \sum_{n=1}^{\infty} C_n r^{(n-1)/2} f_{nij}(\theta), \tag{3.23}$$

The term with r^0 ensures that σ_x and σ_y approach the external stress σ at a large distance from the crack. In the vicinity of the crack tip the higher order terms can be neglected and eqs (3.16) are obtained as:

$$\sigma_{ij} = \frac{C_1}{\sqrt{r}} f_{ij}(\theta) \quad \text{with } C_1 = \frac{K_{\mathrm{I}}}{\sqrt{2\pi}}. \tag{3.24}$$

The general analysis on the basis of figure 3.2 and eq (3.14) shows that the stress fields surrounding mode I crack tips are always of the same form. It only remains to find K_{I} for a particular configuration.

More or less similar procedures can be used to analyse mode II and mode III crack problems. The solutions can be found in the relevant literature [4,7].

The results are:

Mode II:

$$\begin{aligned} \sigma_x &= \frac{-K_{\mathrm{II}}}{\sqrt{2\pi r}}\,\sin\frac{\theta}{2}\left[2+\cos\frac{\theta}{2}\,\cos\frac{3\theta}{2}\right] \\ \sigma_y &= \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\,\sin\frac{\theta}{2}\,\cos\frac{\theta}{2}\,\cos\frac{3\theta}{2} \\ \tau_{xy} &= \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\,\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\,\sin\frac{3\theta}{2}\right] \\ \sigma_z &= \nu(\sigma_x+\sigma_y), \quad \tau_{xz} = \tau_{yz} = 0. \end{aligned} \tag{3.25}$$

For an infinite cracked plate with uniform in-plane shear τ at infinity:

$$K_{\mathrm{II}} = \tau\sqrt{\pi a} \tag{3.26}$$

and similarly for mode III

$$\begin{aligned} \tau_{xz} &= \frac{-K_{\mathrm{III}}}{\sqrt{2\pi r}}\,\sin\frac{\theta}{2}, \quad \tau_{yz} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}}\,\cos\frac{\theta}{2} \\ \sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0. \end{aligned} \tag{3.27}$$

Stress intensity factors have been calculated for many configurations. Procedures for such calculations are discussed in chapter 13.

3.4 The effect of finite size

Cracks in plates of finite size are of great practical interest, but for these cases no closed form solutions are available. The problems are difficult because of the boundary conditions. An approximate solution can be

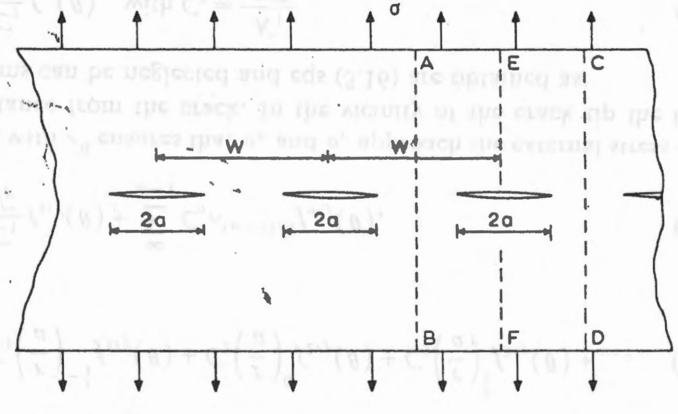

Figure 3.3. Infinite plate with collinear cracks

obtained for a strip of finite width loaded in tension and containing an edge crack or a central crack.

First consider an infinite sheet with an infinite row of evenly spaced collinear cracks as depicted in figure 3.3. Solutions for this case were given by Westergaard [3], Irwin [10] and Koiter [11]. The result is:

$$K_{\mathrm{I}} = \sigma\sqrt{\pi a}\left(\frac{W}{\pi a}\tan\frac{\pi a}{W}\right)^{\frac{1}{2}} \tag{3.28}$$

If the plate is cut along the lines AB and CD one obtains a strip of finite width W, containing a central crack $2a$. It is likely that the solution of eq (3.28) is approximately valid for the strip. In the case of the collinear cracks a strip of width W bears stresses (note that shear stresses are zero because of symmetry) along its edges AB and CD (figure 3.4), whereas the edges of a plate of finite size are stress free. Supposedly, the stresses parallel to the crack do not contribute much to K and consequently eq (3.28) can be used as an approximate solution for the strip of finite size. It appears that eq (3.28) reduces to $K_{\mathrm{I}} = \sigma\sqrt{\pi a}$ if a/W approaches zero. This means that the finite strip behaves as an infinite plate if the cracks are small.

Isida [12] has developed mapping functions to derive stress concentration factors. These can be used [4] to compute stress intensity factors for finite plates to any degree of accuracy. Usually the result is presented as:

$$K = Y\sigma\sqrt{a} \quad \text{or} \quad K = \beta\sigma\sqrt{\pi}\,\alpha \tag{3.29}$$

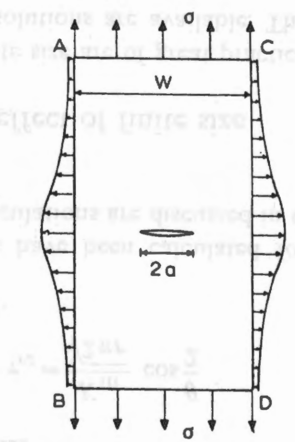

Figure 3.4. Stresses on the edges of strip cut from infinite plate with collinear cracks

where Y is a polynomial in a/W. The factor $\sqrt{\pi}$ is incorporated in β, with $\beta = Y/\sqrt{\pi}$. Feddersen [13] discovered that the solution of Isida is very closely approximated by $\sqrt{\sec \pi a/W}$. Therefore a convenient formula for the stress intensity factor for a strip in tension is

$$K_{\mathrm{I}} = \sigma\sqrt{\pi a}\,\sqrt{\sec \pi a/W}\,. \tag{3.30}$$

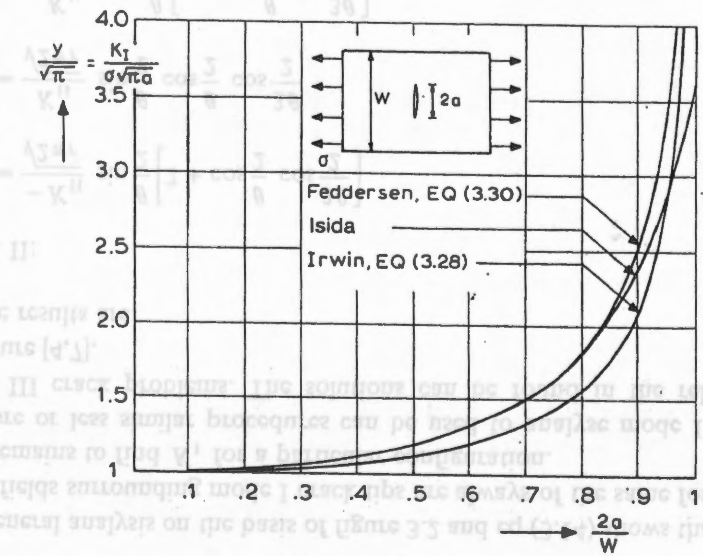

Figure 3.5. Finite width corrections for center cracked plate