

# **The behavior of structures composed of composite materials**

*By*

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Table 2.1  
Summary of the number of elastic coefficients involved for certain classes of materials

Class of Material	Number of nonzero coefficients	Number of independent coefficients
<i>Three-Dimensional Case</i>		
General anisotropic	36	21
One-plane of symmetry	20	13
Two-planes of symmetry	12	9
Transversely isotropic	12	5
Isotropic	12	2
<i>Two-Dimensional Case</i>		
General anisotropic	9	6
One-plane of symmetry	9	6
Two-planes of symmetry	5	4
Transversely isotropic	5	4
Isotropic	5	2

### 2.3. The Physical Meaning of the Components of the Orthotropic Elastic Tensor.

So far, the components of both the stiffness matrix,  $C_{ij}$ , and the compliance matrix,  $a_{ij}$ , are mathematical symbols to relate stresses and strains. By performing simple tensile and shear tests all of the components above can be related to physical or mechanical properties.

Consider a simple, standard tensile test in the  $X_1$  direction. The resulting stress and strain tensors are

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & -\nu_{12}\epsilon_{11} & 0 \\ 0 & 0 & -\nu_{13}\epsilon_{11} \end{bmatrix} \quad (2.10)$$

where the Poisson's ratio,  $\nu_{ij}$ , is very carefully defined as the negative of the ratio of the strain in the  $X_j$  direction to the strain in the  $X_i$  direction due to an applied stress in the  $X_i$  direction. In other words in the above it is seen that  $\epsilon_{22} = -\nu_{12}\epsilon_{11}$  or  $\nu_{12} = -\epsilon_{22}/\epsilon_{11}$ .

Also, the constant of proportionality between stress and strain is noted to be  $E_i$ , the modulus of elasticity in the  $X_i$  direction. Thus,

$$\epsilon_1 = a_{11}\sigma_1 = \frac{\sigma_1}{E_1}$$

$$\epsilon_2 = a_{21}\sigma_1 = -\nu_{12}\epsilon_1 = -\frac{\nu_{12}\sigma_1}{E_1}$$

$$\epsilon_3 = a_{31}\sigma_1 = -\nu_{13}\epsilon_1 = -\frac{\nu_{13}\sigma_1}{E_1}$$

Therefore,

$$a_{11} = 1/E_1, \quad a_{21} = -\nu_{12}/E_1, \quad a_{31} = -\nu_{13}/E_1. \quad (2.11)$$

For a simple tensile test in the  $X_2$  direction, it is found that

$$a_{12} = -\nu_{21}/E_2, \quad a_{22} = 1/E_2, \quad a_{32} = -\nu_{23}/E_2. \quad (2.12)$$

Likewise, a tensile test in the  $X_3$  direction yields

$$a_{13} = -\nu_{31}/E_3, \quad a_{23} = -\nu_{32}/E_3, \quad a_{33} = 1/E_3. \quad (2.13)$$

From the fact that  $a_{ij} = a_{ji}$ , it is seen that

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad (i, j = 1, 2, 3). \quad (2.14)$$

Next, consider a hypothetical simple shear test as shown in Figure 2.4. In this case the stress, strain, and displacement tensor components

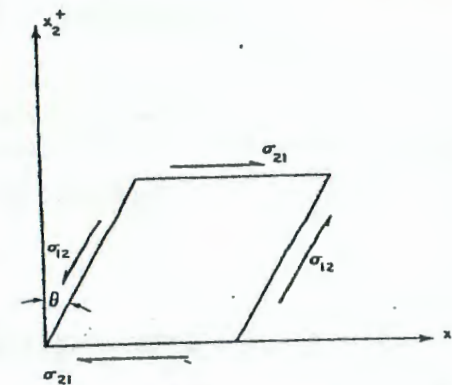


Figure 2.4. Shear Stresses and Strains.

are:

$$\sigma_{ij} = \begin{bmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \epsilon_{ij} = \begin{bmatrix} 0 & \epsilon_{12} & 0 \\ \epsilon_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad U_{i,j} = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_{21}/G_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In the above,  $U_i$  is the displacement and  $U_{i,j} = (\partial U_i)/(\partial X_j)$ . From elementary strength of materials the constant of proportionality between the shear stress  $\sigma_{21}$  and the angle  $\theta$  is  $G_{21}$ , the shear modulus in the  $X_1 - X_2$  plane.

From the theory of elasticity

$$\epsilon_{12} = \frac{1}{2}(U_{1,2} + U_{2,1}) = \frac{\sigma_{21}}{2G_{21}} = \frac{\tan \theta}{2} \quad (2.15)$$

From (2.7),  $\epsilon_6 = a_{66}\sigma_6$ , or

$$\epsilon_{12} = \frac{a_{66}\sigma_{21}}{2} = \frac{\sigma_{21}}{2G_{21}}$$

Hence,

$$a_{66} = \frac{1}{G_{21}} = \frac{1}{G_{12}} \quad (2.16)$$

Similarly,

$$a_{44} = \frac{1}{G_{23}} \quad \text{and} \quad a_{55} = \frac{1}{G_{13}} \quad (2.17)$$

Thus, all  $a_{ij}$  components have now been related to mechanical properties, and it is seen that to characterize a three dimensional orthotropic body, nine physical quantities — hence nine separate tests — are needed (that is  $E_1, E_2, E_3, G_{12}, G_{23}, G_{31}, \nu_{12}, \nu_{13}, \nu_{21}, \nu_{23}, \nu_{31}$  and  $\nu_{32}$ , and equation (2.14)).

There are several sets of equations for obtaining the composite elastic properties from those of the fiber and matrix materials. These include those of Halpin and Tsai [7], Hashin [8], and Christensen [9]. Recently, Hahn [10] codified certain results for fibers of circular cross section which are randomly distributed in a plane normal to the unidirectionally oriented fibers. For that case the composite is macroscopically transversely isotropic, that is  $(\ )_{12} = (\ )_{13}, (\ )_{22} = (\ )_{33}$  and  $(\ )_{44} = (\ )_{55}$ , where the parentheses could be  $E, G$  or  $\nu$ ; hence, the elastic properties involve only five independent constants, namely  $(\ )_{11}, (\ )_{22}, (\ )_{12}, (\ )_{44}$  and  $(\ )_{66}$ .

For several of the elastic constants, Hahn states that they all have the same functional form:

$$P = \frac{(P_f V_f + \eta P_m V_m)}{(V_f + \eta V_m)} \quad (2.18)$$

where for the elastic constant  $P$ , the  $P_f, P_m$  and  $\eta$  are given in the table below, and where  $V_f$  and  $V_m$  are the volume fractions of the fibers and matrix respectively (and whose sum equals unity):

Elastic Constant	$P$	$P_f$	$P_m$	$\eta$
$E_{11}$	$E_{11}$	$E_{11f}$	$E_m$	1
$\nu_{12}$	$\nu_{12}$	$\nu_{12f}$	$\nu_m$	1
$G_{12}$	$1/G_{12}$	$1/G_{12f}$	$1/G_m$	$\eta_6$
$G_{23}$	$1/G_{23}$	$1/G_{23f}$	$1/G_m$	$\eta_4$
$K_T$	$1/K_T$	$1/K_f$	$1/K_m$	$\eta_K$

The expressions for  $E_{11}$  and  $\nu_{12}$  are called the Rule of Mixtures. In the above  $K_T$  is the plane strain bulk modulus,  $K_f = [E_f/2(1 - \nu_f)]$  and  $K_m = [E_m/2(1 - \nu_m)]$ . Also, the  $\eta$ 's are given as follows:

$$\eta_6 = \frac{1 + G_m/G_{12f}}{2}$$

$$\eta_4 = \frac{3 - 4\nu_m + G_m/G_{23f}}{4(1 - \nu_m)}$$

$$\eta_K = \frac{1 + G_m/K_f}{2(1 - \nu_m)}$$

The shear modulus of the matrix material,  $G_m$ , if isotropic is given by  $G_m = E_m/2(1 + \nu_m)$ .

The transverse moduli of the composite,  $E_{22} = E_{33}$ , are found from the following equation:

$$E_{22} = E_{33} = \frac{4K_T G_{23}}{K_T + m G_{23}} \quad (2.19)$$

where

$$m = 1 + \frac{4K_T \nu_{12}^2}{E_{11}}$$

The equations above can be used for composites reinforced with anisotropic fibers such as graphite and aramid (Kevlar) fibers. If the fibers are isotropic, simplification of the above equations will result. In



that case also  $\eta_K$  becomes

$$\eta_K = \frac{1 + (1 - 2\nu_f)G_m/G_f}{2(1 - \nu_m)}$$

Hahn notes that for most structural composites,  $G_m/G_f < 0.05$ . Thus the  $\eta$  parameters are approximately:

$$\eta_6 \approx 0.5; \quad \eta_4 = \frac{3 - 4\nu_m}{4(1 - \nu_m)}; \quad \eta_K = \frac{1}{2(1 - \nu_m)}$$

Finally, noting that  $\nu_m = 0.35$  for most epoxies, then  $\eta_4 = 0.662$  and  $\eta_K = 0.77$ .

The above equations along with (2.14) provide the engineer with the wherewithal to estimate the elastic constants for a composite material, if the constituent properties and volume fractions are known.

#### 2.4. Thermal and Hygrothermal Considerations

In the previous two sections, the elastic relations developed pertain only to an anisotropic elastic body at one temperature, that temperature being the "stress free" temperature, that is the temperature at which the body is considered to be free of stress if it is under no mechanical static or dynamic loadings.

However, in both metallic and composite structures changes in temperature are commonplace both during fabrication and during structural usage. Changes in temperature result in two effects that are very important. First, most materials expand when heated, and contract when cooled, and in most cases this expansion is proportional to the temperature change. If, for instance, one had a long thin bar of a given material then with change in temperature, the ratio of the change in length of the bar,  $\Delta L$ , to the original length  $L$ , is related to the temperature of the bar  $T$ , as shown in Figure 2.5.

Mathematically, this can be written as

$$\epsilon_{\text{THERMAL}} = \frac{\Delta L}{L} = \alpha \Delta T \quad (2.20)$$

where  $\alpha$  is the coefficient of thermal expansion, i.e., the proportionality constant between the "thermal" strain ( $\Delta L/L$ ) and the change in temperature,  $\Delta T$ , from some reference temperature at which there are no thermal stresses or thermal strains.

The second major effect of temperature change relates to stiffness and strength. Most materials become softer, more ductile, and weaker as they

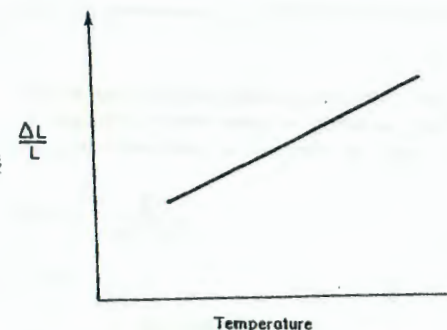


Figure 2.5. Change in Length of a Bar or Rod as a Function of Temperature.

are heated. Typical plots of ultimate strength, yield stress and modulus of elasticity with temperature are shown in Figure 2.6. In performing a stress analysis, determining the natural frequencies, or the buckling load of a heated or cooled structure one must use the strengths and the moduli of elasticity of the material at the temperature at which the structure is expected to perform.

In an orthotropic material, such as a composite, there will or could be three different coefficients of thermal expansion, and three different thermal strains, one in each of the orthogonal directions comprising the orthotropic material (equation 2.20 would then have subscripts of 1, 2 and 3 on both the strains and the coefficients of thermal expansion).

During the mid-seventies another physical phenomenon associated with polymer matrix composites was recognized as important. It was found that the combination of high temperature and high humidity caused a doubly deleterious effect on the structural performance of these composites. Engineers and material scientists became very concerned about these effects, and considerable research effort was expended in

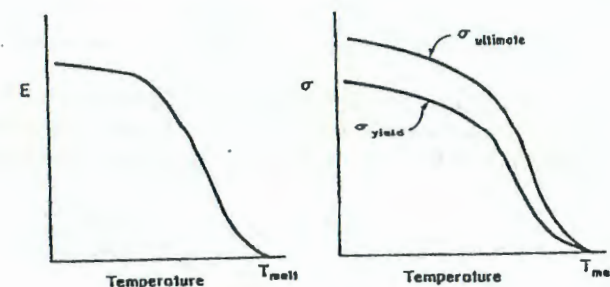


Figure 2.6. Modulus of Elasticity and Strengths as Functions of Temperature.



studying this new phenomenon. Conferences [11] were held which discussed the problem, and both short range and long range research plans were proposed. The twofold problem involves the fact that the combination of high temperature and high humidity results in the entrapment of moisture in the polymer matrix, with attendant weight increase ( $\leq 2\%$ ) and more importantly, a swelling of the matrix. It was realized [12] that the ingestion of moisture varied linearly with the swelling so that in fact

$$\epsilon_{\text{HYGROTHERMAL}} = \frac{\Delta L}{L} = \beta \Delta m \quad (2.21)$$

where  $\Delta m$  is the increase from zero moisture measured in percentage weight increase, and  $\beta$  is the coefficient of hygrothermal expansion, analogous to the coefficient of thermal expansion, depicted in equation (2.20). This analogy is a very important one because one can see that the hygrothermal effects are entirely analogous mathematically to the thermal effect. The reader, however, should be aware that there exist some ambiguities in the literature regarding the coefficient of hygrothermal expansion.

The second effect also is similar to the thermal effect. These are shown qualitatively in Figure 2.7. Dry polymers have good properties until a temperature is reached, traditionally called by polymer chemists the "glass transition temperature," above which both strength and stiffness deteriorate rapidly. If the same polymer is saturated with moisture, not only are the mechanical properties degraded at any one temperature but the glass transition temperature for that polymer is significantly lower.

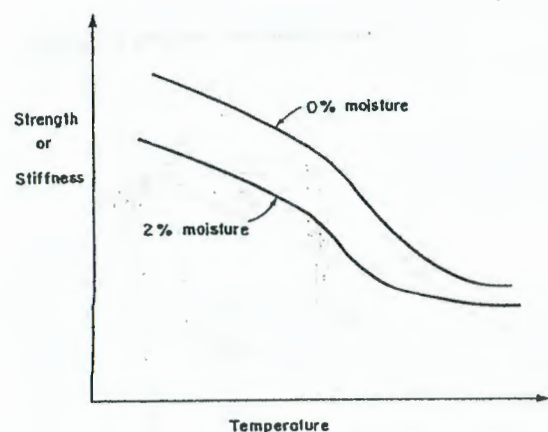


Figure 2.7. Mechanical Properties as a Function of Temperature and Moisture Absorption.

Thus, for modern polymer matrix composites one only the thermal effects but also the hygrothermal effects structure could be considerably underdesigned.

Thus, to deal with the real world of polymer composites, equation (2.9) must be modified to read

$$\epsilon_i = a_{ij}\sigma_j + \alpha_i\Delta T + \beta_i\Delta m \quad (i = 1, 2, 3) \quad (2.22)$$

$$\epsilon_i = a_{ij}\sigma_j \quad (i = 4, 5, 6) \quad (2.23)$$

$$\text{where in each equation } j = 1 - 6. \quad (2.24)$$

Two types of equations are shown above because both thermal and hygrothermal effects are dilatational only, that is, they cause an expansion or contraction, but do not affect the shear stresses or strains. This is important to remember.

## 2.5. Laminae of Composite Materials

Almost all practical composite material structures are thin in the thickness direction because their superior properties permit the use of thin-walled structures. Most polymeric matrix composites are made in the form of a uniaxial set of fibers surrounded by a polymeric matrix in the form of a tape several inches wide termed as "prepreg." The basic element in most long fiber composite structures is a lamina of fiber plus matrix, all fibers oriented in one direction, made by laying the prepreg tape of a certain length side by side. In the next section, stacking of various laminae to form a superior structure termed a laminate will be discussed.

To effect this, consider a small element of a lamina of constant thickness  $h$ , wherein the principal material axes are labelled 1 and 2, that is 1 direction is parallel to the fibers, the 2 direction is normal to them, and consider that the beam, plate or shell geometric axes are  $x$  and  $y$  as depicted in Figure 2.8.

This element has the positive directions of all stresses shown in a consistent manner [2,5,6]. If one does a force equilibrium study to relate  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_{xy}$  to  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_{12}$ , it is exactly analogous to the Mohr's circle analysis in basic strength of materials with the result that, in matrix form,

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = [T]_{CL} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} \quad (2.25)$$

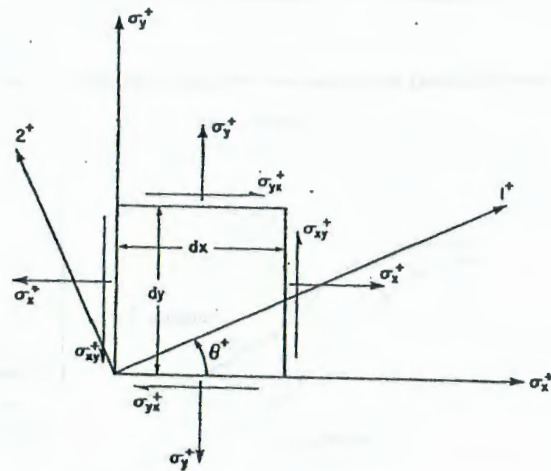


Figure 2.8. Laminar Coordinate Systems.

where

$$[T]_{CL} = \begin{bmatrix} m^2 & n^2 & +2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & (m^2 - n^2) \end{bmatrix} \quad (2.26)$$

where  $m = \cos \theta$ ,  $n = \sin \theta$ , and  $\theta$  is defined positive as shown in Figure 2.8, and where the subscripts CL refer to the classical two-dimensional case only, that is, in the 1-2 plane or the  $x$ - $y$  plane only.

Analogously, a strain relationship also follows for the classical isothermal case

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_{12} \end{bmatrix} = [T]_{CL} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{bmatrix} \quad (2.27)$$

However, these classical two-dimensional relationships must be modified to treat a composite material to include thermal effects, hygrothermal effects, and the effects of transverse shear deformation treated in detail elsewhere [6,13-15]. The effects of transverse shear deformation, shown through the inclusion of the  $\sigma_4 - \epsilon_4$  and  $\sigma_5 - \epsilon_5$  relations shown below in equations (2.28) and (2.30), must be included in composite materials, because in the fiber direction the composite has many of the mechanical properties of the fiber itself (strong and stiff) while in the

thickness direction the fibers are basically ineffective and the shear properties are dominated by the weaker matrix material. Similarly, because quite often the matrix material has much higher coefficients of thermal and hygrothermal expansion ( $\alpha$  and  $\beta$ ), thickening of the thin lamina cannot be ignored in some cases. Hence, without undue derivation, the equations (2.25) through (2.27) are modified to be:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = [T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = [T] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{bmatrix} \quad (2.28)$$

wherein

$$[T] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & (m^2 - n^2) \end{bmatrix} \quad (2.29)$$

For completeness

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} \quad (2.30)$$

where \*

$$[T]^{-1} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & -2mn \\ n^2 & m^2 & 0 & 0 & 0 & 2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & n & 0 \\ 0 & 0 & 0 & -n & m & 0 \\ mn & -mn & 0 & 0 & 0 & (m^2 - n^2) \end{bmatrix} \quad (2.31)$$

\*  $[T]^{-1}$  can be found by replacing  $\theta$  by  $(-\theta)$  in  $[T]$



If one systematically uses these expressions, and utilizes Hooke's Law relating stress and strain, and includes the thermal and hygrothermal effects, one can produce the following overall general equations for a lamina of a fiber reinforced composite material in terms of the principal material directions (1, 2, 3); see equations (2.5) through (2.8).

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 - \alpha_1 \Delta T - \beta_1 \Delta m \\ \epsilon_2 - \alpha_2 \Delta T - \beta_2 \Delta m \\ \epsilon_3 - \alpha_3 \Delta T - \beta_3 \Delta m \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{12} \end{Bmatrix} \quad (2.32)$$

In the above, the  $Q_{ij}$  quantities are used for the stiffness matrix quantities because modern composite materials technology uses them in all literature, but they are identical to the  $C_{ij}$  quantities of classical elasticity discussed in earlier sections, and can be obtained directly from (2.5) through (2.17). One should also remember that  $\epsilon_{23} = (1/2G_{23})\sigma_4$ ,  $\epsilon_{31} = (1/2G_{31})\sigma_5$  and  $\epsilon_{12} = (1/2G_{12})\sigma_6$ , hence the coefficients of "two" appearing in the  $Q$  matrix. Using the notation of Sloan [15]

$$\begin{aligned} Q_{11} &= E_{11}(1 - \nu_{23}\nu_{32})/\Delta, & Q_{22} &= E_{22}(1 - \nu_{31}\nu_{13})/\Delta \\ Q_{33} &= E_{33}(1 - \nu_{12}\nu_{21})/\Delta, & Q_{44} &= G_{23}, & Q_{55} &= G_{13}, & Q_{66} &= G_{12} \\ Q_{12} &= (\nu_{21} + \nu_{31}\nu_{23})E_{11}/\Delta = (\nu_{12} + \nu_{32}\nu_{13})E_{22}/\Delta \\ Q_{13} &= (\nu_{31} + \nu_{21}\nu_{32})E_{11}/\Delta = (\nu_{13} + \nu_{12}\nu_{23})E_{22}/\Delta \\ Q_{23} &= (\nu_{32} + \nu_{12}\nu_{31})E_{22}/\Delta = (\nu_{23} + \nu_{21}\nu_{13})E_{33}/\Delta \\ \Delta &= 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13} \end{aligned} \quad (2.33)$$

Incidentally in the above expressions, if the lamina is transversely isotropic, i.e., same properties in both the 2 and 3 direction, then  $\nu_{12} = \nu_{13}$ ,  $G_{12} = G_{13}$ ,  $E_{22} = E_{33}$  with resulting simplification.

For preliminary calculation in design or where great accuracy is not needed, simpler forms [6] for some of the expressions in (2.33) can be used, as shown below:

$$\begin{aligned} Q_{11} &= E_{11}/(1 - \nu_{12}\nu_{21}), & Q_{22} &= E_{22}/(1 - \nu_{12}\nu_{21}) \\ Q_{12} &= Q_{21} = -\nu_{21}E_{11}/(1 - \nu_{12}\nu_{21}) = \nu_{12}E_{22}/(1 - \nu_{12}\nu_{21}) \\ Q_{66} &= G_{12} \end{aligned} \quad (2.34)$$

If these simpler forms are used then one would use the classical form of the constitutive relations instead of (2.32), neglecting transverse shear deformation and transverse normal stress, namely

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 - \alpha_1 \Delta T - \beta_1 \Delta m \\ \epsilon_2 - \alpha_2 \Delta T - \beta_2 \Delta m \\ \epsilon_{12} \end{Bmatrix} \quad (2.35)$$

where one should remember also that  $2\epsilon_{12} = \epsilon_6$ , hence the appearance of two before  $Q_{66}$ .

Now, to relate these relationships to the  $x$ - $y$ - $z$  coordinate system, one utilizes equations (2.30) through (2.32). The result is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} & 0 & 0 & 2\bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} & 0 & 0 & 2\bar{Q}_{26} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} & 0 & 0 & 2\bar{Q}_{36} \\ 0 & 0 & 0 & 2\bar{Q}_{44} & 2\bar{Q}_{45} & 0 \\ 0 & 0 & 0 & 2\bar{Q}_{45} & 2\bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{36} & 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x - \alpha_x \Delta T - \beta_x \Delta m \\ \epsilon_y - \alpha_y \Delta T - \beta_y \Delta m \\ \epsilon_z - \alpha_z \Delta T - \beta_z \Delta m \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} - \frac{1}{2}\alpha_{xy}\Delta T - \frac{1}{2}\beta_{xy}\Delta m \end{Bmatrix} \quad (2.36)$$

where  $[\bar{Q}] = [T]^{-1}[Q][T]$ , or more explicitly,

$$\begin{aligned} \bar{Q}_{11} &= Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4 \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(m^4 + n^4) \\ \bar{Q}_{13} &= Q_{13}m^2 + Q_{23}n^2 \\ \bar{Q}_{16} &= -mn^3Q_{22} + m^3nQ_{11} - mn(m^2 - n^2)(Q_{12} + 2Q_{66}) \\ \bar{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}m^4 \\ \bar{Q}_{23} &= n^2Q_{13} + m^2Q_{23} \\ \bar{Q}_{33} &= Q_{33} \\ \bar{Q}_{26} &= -m^3nQ_{22} + mn^3Q_{11} + mn(m^2 - n^2)(Q_{12} + 2Q_{66}) \\ \bar{Q}_{36} &= (Q_{13} - Q_{23})mn \\ \bar{Q}_{44} &= Q_{44}m^2 + Q_{55}n^2 \\ \bar{Q}_{45} &= (Q_{55} - Q_{44})mn \\ \bar{Q}_{55} &= Q_{55}m^2 + Q_{44}n^2 \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12})m^2n^2 + Q_{66}(m^2 - n^2)^2 \end{aligned}$$



$$\begin{aligned}
\alpha_x &= \alpha_1 m^2 + \alpha_2 n^2 & \beta_x &= \beta_1 m^2 + \beta_2 n^2 \\
\alpha_y &= \alpha_2 m^2 + \alpha_1 n^2 & \beta_y &= \beta_2 m^2 + \beta_1 n^2 \\
\alpha_z &= \alpha_3 & \beta_z &= \beta_3 \\
\alpha_{xy} &= (\alpha_1 - \alpha_2)mn & \beta_{xy} &= (\beta_1 - \beta_2)mn.
\end{aligned}$$

It should be remembered that although the coefficients of both thermal and hygrothermal expansion are purely dilatational in the material coordinate system 1-2, rotation into the structural coordinate system  $x$ - $y$ , results in an  $\alpha_{xy}$  and a  $\beta_{xy}$ .

Again, for preliminary design purposes or for approximate calculations one can use the simpler classical form of

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 2\bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & 2\bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & 2\bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x - \alpha_x \Delta T - \beta_x \Delta m \\ \epsilon_y - \alpha_y \Delta T - \beta_y \Delta m \\ \epsilon_{xy} - \frac{1}{2} \alpha_{xy} \Delta T - \frac{1}{2} \beta_{xy} \Delta m \end{Bmatrix} \quad (2.37)$$

where the  $\bar{Q}_{ij}$  are defined in (2.36), but one must use the  $Q_{ij}$  of (2.34) instead of (2.33) for consistency.

One interesting variation of the above classical quantities of (2.37) resulted when Tsai and Pagano [16] rewrote many of the above quantities in a manner that is very useful in comparing various material systems in the design of a composite structure.

$$\bar{Q}_{11} = U_1 + U_2 \cos(2\theta) + U_3 \cos(4\theta)$$

$$\bar{Q}_{22} = U_1 - U_2 \cos(2\theta) + U_3 \cos(4\theta)$$

$$\bar{Q}_{12} = U_4 - U_3 \cos(4\theta)$$

$$\bar{Q}_{66} = U_5 - U_3 \cos(4\theta)$$

$$\bar{Q}_{16} = +\frac{1}{2} U_2 \sin(2\theta) + U_3 \sin(4\theta)$$

$$\bar{Q}_{26} = +\frac{1}{2} U_2 \sin(2\theta) - U_3 \sin(4\theta)$$

where

$$U_1 = \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})$$

$$U_2 = \frac{1}{2}(Q_{11} - Q_{22})$$

$$U_3 = \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})$$

$$U_4 = \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})$$

$$U_5 = \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66})$$

In the above, the  $U_i$  quantities are invariant with respect to axis rotation and therefore are truly composite lamina properties.

At this point, given a lamina of a unidirectional composite of known elastic properties, if used in a plate or panel, with the 1-2 material axis at an angle  $\theta$  from the plate or panel  $x$ - $y$  axes, all stiffness quantities  $Q_{ij}$  and  $\bar{Q}_{ij}$  can be determined relating stresses and strains in either coordinate system.

## 2.6. Laminate Analysis

In the previous section the generalized constitutive equations for one lamina of a composite material were formulated. In reality, any structure of composite materials is comprised of numerous laminae which are bonded and/or cured together. In fact, over and above the superior properties in strength and stiffness that composites possess, the ability to stack laminae one on the other in a varied but unique fashion to result in the optimum laminate material properties for a given structural size and set of loadings is one of the biggest advantages that composites have over metallic or plastic structures.

Consider a laminate composed of  $N$  laminae. For the  $k^{\text{th}}$  lamina of the laminate, equation (2.36) can be written

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix}_k = [\bar{Q}]_k \begin{Bmatrix} \epsilon_x - \alpha_x \Delta T - \beta_x \Delta m \\ \epsilon_y - \alpha_y \Delta T - \beta_y \Delta m \\ \epsilon_z - \alpha_z \Delta T - \beta_z \Delta m \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} - \frac{1}{2} \alpha_{xy} \Delta T - \frac{1}{2} \beta_{xy} \Delta m \end{Bmatrix}_k \quad (2.38)$$

where all matrices must have the subscript  $k$  due to the orientation of the particular lamina with respect to the plate or shell  $x$ - $y$  coordinates and its unique  $\bar{Q}$ ,  $\alpha$ , and  $\beta$ .

For any elastic body the strain-displacement equations, i.e., those kinematic relations describing the functional relations between the elastic strains in the body and its displacements, are given by

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (2.39)$$

where  $i, j = x, y, z$  in a Cartesian coordinate frame, and the comma denotes partial differentiation with respect to the coordinate denoted by the symbol after the comma. Explicitly, the relations are: