Elastic constants

April 26, 2021

There are 9 independent components of stresses

There are 9 independent components of strains

$$\varepsilon_{ij}\left(i=1,2 \text{ or } 3, \quad \text{and } j=1,2 \text{ or } 3\right) \qquad \sigma_{kl}\left(k=1,2 \text{ or } 3, \quad \text{and } l=1,2 \text{ or } 3\right)$$

Strains are linearly proportional to the stresses

$$\varepsilon_{ij} = A_{ijkl}\sigma_{kl}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{33} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\ \vdots & & & & \ddots & & & & \vdots \\ \vdots & & & & & \ddots & & & \vdots \\ \vdots & & & & & & \ddots & & \vdots \\ \vdots & & & & & & & \ddots & \vdots \\ a_{91} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{99} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix}$$

81 independent components

Elastic constants

April 26, 2021

There are 9 independent components of stresses

There are 9 independent components of strains

$$\varepsilon_{ij}\left(i=1,2 \text{ or } 3, \quad \text{and } j=1,2 \text{ or } 3\right) \qquad \sigma_{kl}\left(k=1,2 \text{ or } 3, \quad \text{and } l=1,2 \text{ or } 3\right)$$

Strains are linearly proportional to the stresses

$$\sigma_{ij}=C_{ijkl}\varepsilon_{kl}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} & c_{18} & c_{19} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & c_{27} & c_{28} & c_{29} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} & c_{37} & c_{38} & c_{39} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} & c_{47} & c_{48} & c_{49} \\ \vdots & & & & \ddots & & & & \vdots \\ \vdots & & & & & \ddots & & & \vdots \\ \vdots & & & & & & \ddots & & \vdots \\ \vdots & & & & & & & \ddots & \vdots \\ c_{91} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & c_{99} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{33} \end{Bmatrix}$$

81 independent components

<u>Elastic constants</u>

Stresses and strains are symmetric tensors

$$\varepsilon_{ij} = \varepsilon_{ji}, \qquad \sigma_{kl} = \sigma_{lk}$$

There are 6 independent components of stresses

There are 6 independent components of strains

Engineering strains: $\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 2\varepsilon_{12} = 2\varepsilon_{21}, \gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 2\varepsilon_{23} = 2\varepsilon_{32},$

$$\gamma_{31} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 2\varepsilon_{31} = 2\varepsilon_{13}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}$$

36 independent components

Elastic constants

Strain energy density must be positive

$$U=\frac{1}{2}\sigma_{ij}\varepsilon_{ij}=\frac{1}{2}\left(\sigma_{11}\varepsilon_{11}+\sigma_{22}\varepsilon_{22}+\sigma_{33}\varepsilon_{33}+\sigma_{12}\gamma_{12}+\sigma_{23}\gamma_{23}+\sigma_{31}\gamma_{31}\right)$$

$$=\frac{1}{2}\begin{Bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{Bmatrix}^T\begin{Bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\gamma_{12}\\\gamma_{23}\\\gamma_{31}\end{Bmatrix}=\frac{1}{2}\begin{Bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{Bmatrix}^T\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}&a_{15}&a_{16}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}&a_{26}\\a_{31}&a_{32}&a_{33}&a_{34}&a_{35}&a_{36}\\a_{41}&a_{42}&a_{43}&a_{44}&a_{45}&a_{46}\\a_{51}&a_{52}&a_{53}&a_{54}&a_{55}&a_{56}\\a_{61}&a_{62}&a_{63}&a_{64}&a_{65}&a_{66}\end{bmatrix}\begin{Bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{Bmatrix}$$

$$>0$$

The quadratic form of $\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}&a_{15}&a_{16}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}&a_{26}\\a_{31}&a_{32}&a_{33}&a_{34}&a_{35}&a_{36}\\a_{41}&a_{42}&a_{43}&a_{44}&a_{45}&a_{46}\\a_{51}&a_{52}&a_{53}&a_{54}&a_{55}&a_{56}\\a_{61}&a_{62}&a_{63}&a_{64}&a_{65}&a_{66}\end{bmatrix}$ must be positive.

<u>Elastic constants</u>

April 26, 2021

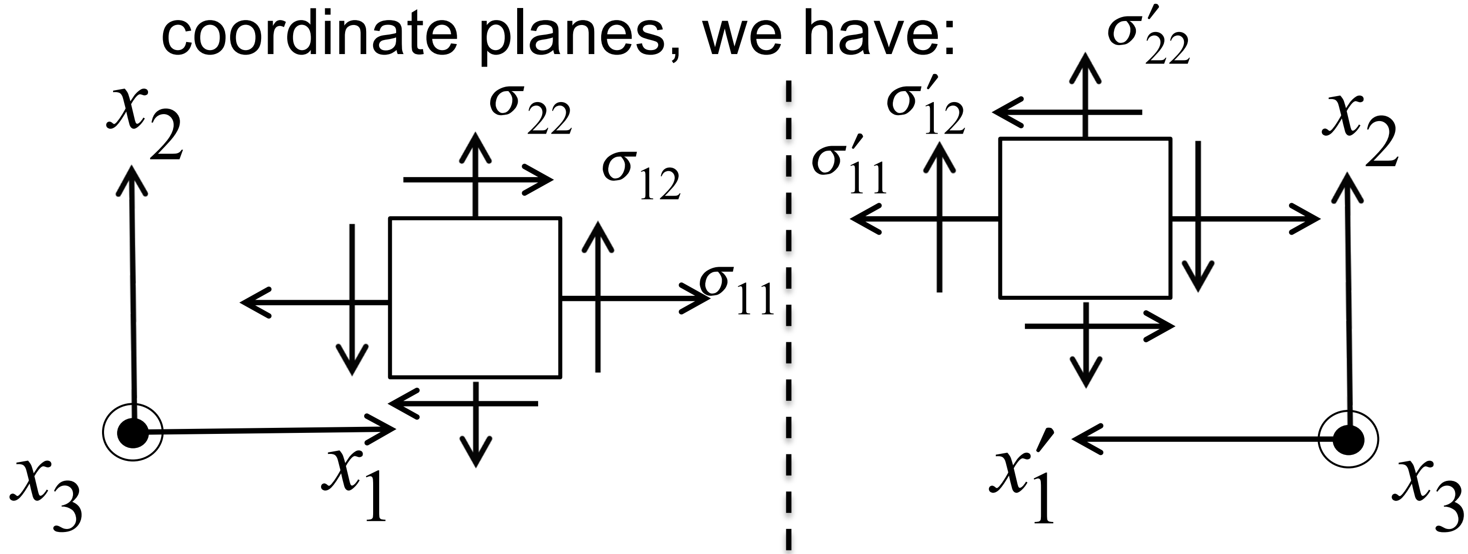
Strain energy density must be positive

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}$$

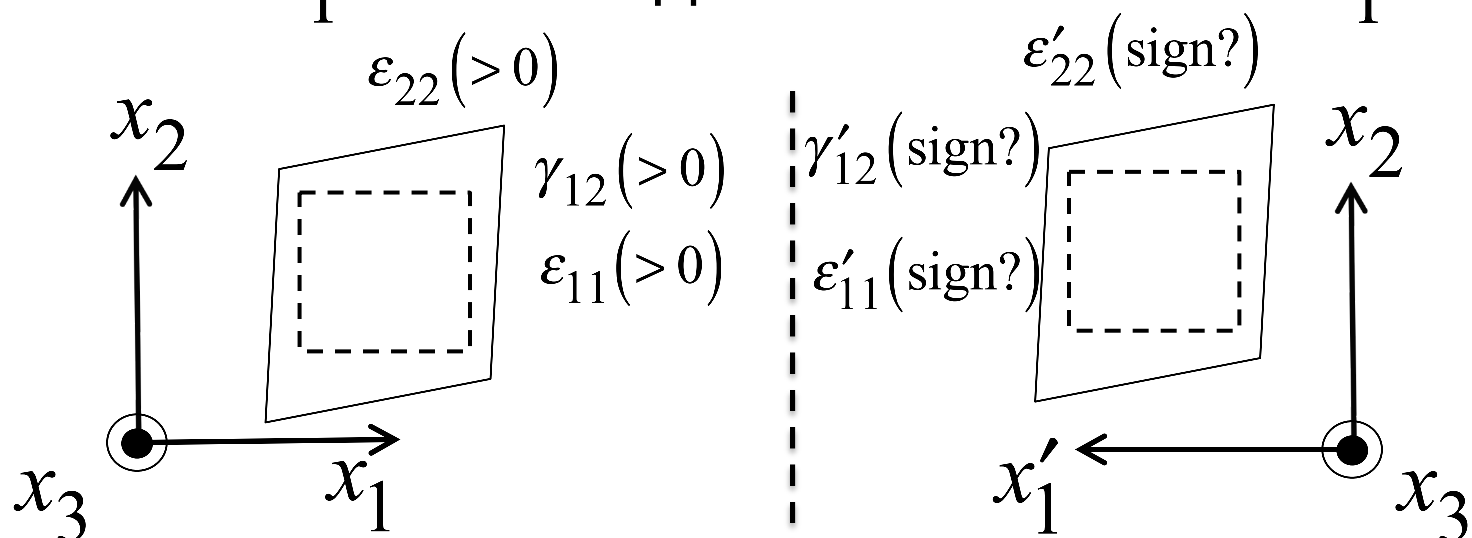
- The matrix must be symmetric and all the diagonal components must be positive!

- There are 21 independent components

- Symmetry about the coordinate planes
  - ✓ When the elastic body is symmetric about the coordinate planes, we have:

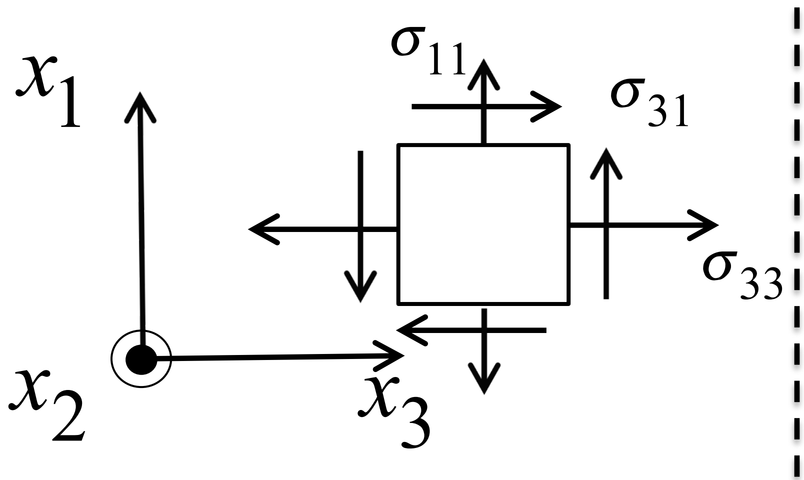


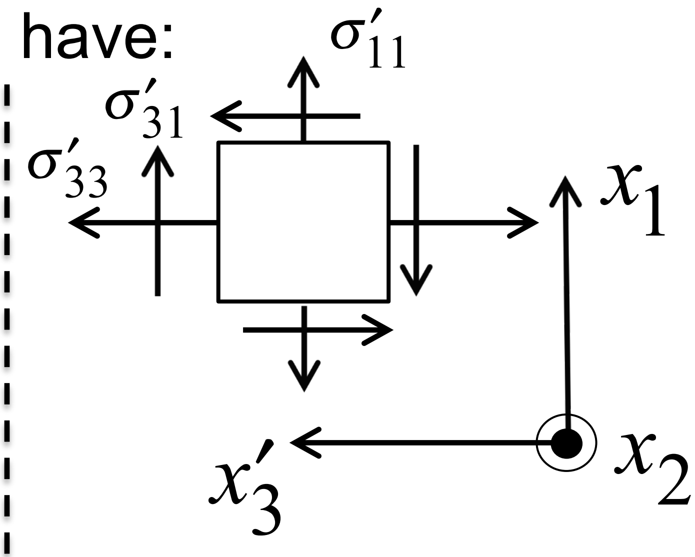
x'_1 is in the opposite direction from x_1



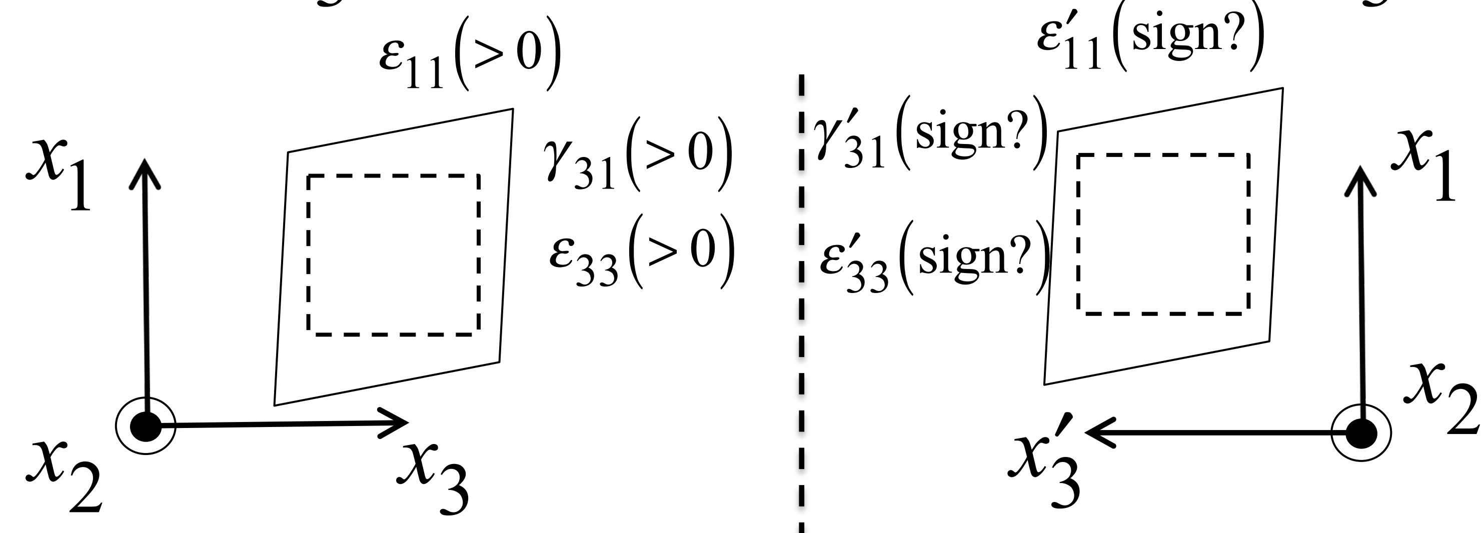
Symmetry about the coordinate planes

- When the elastic body is symmetric about the coordinate planes, we have:

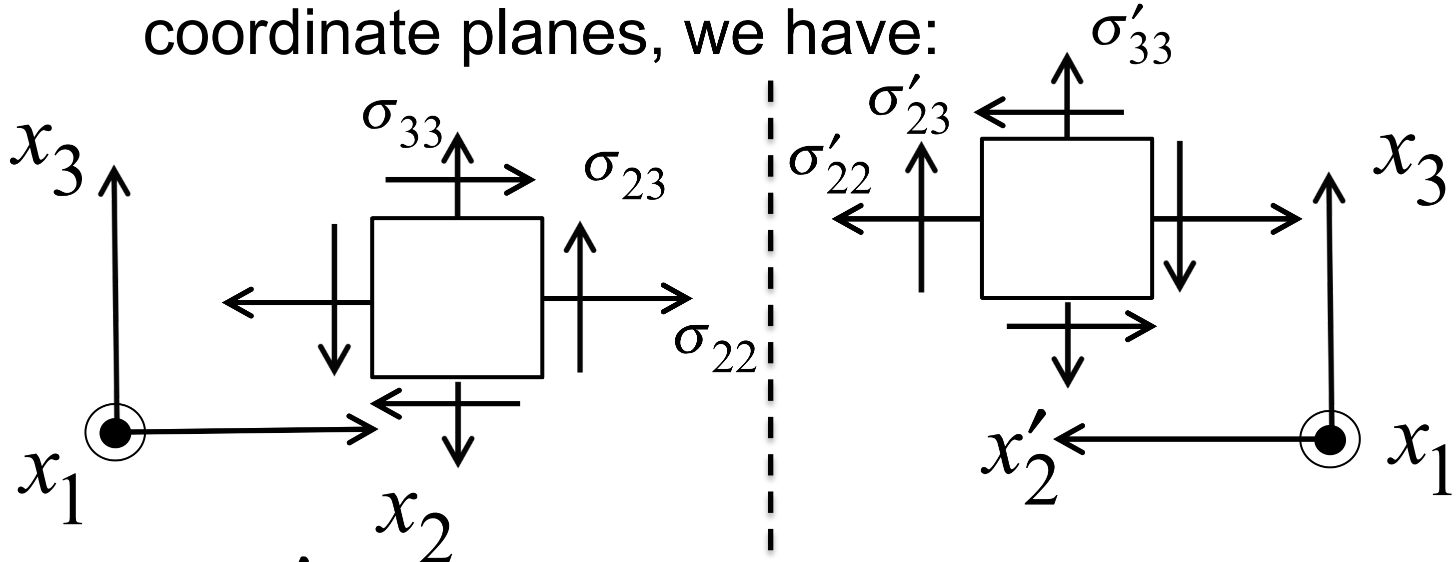




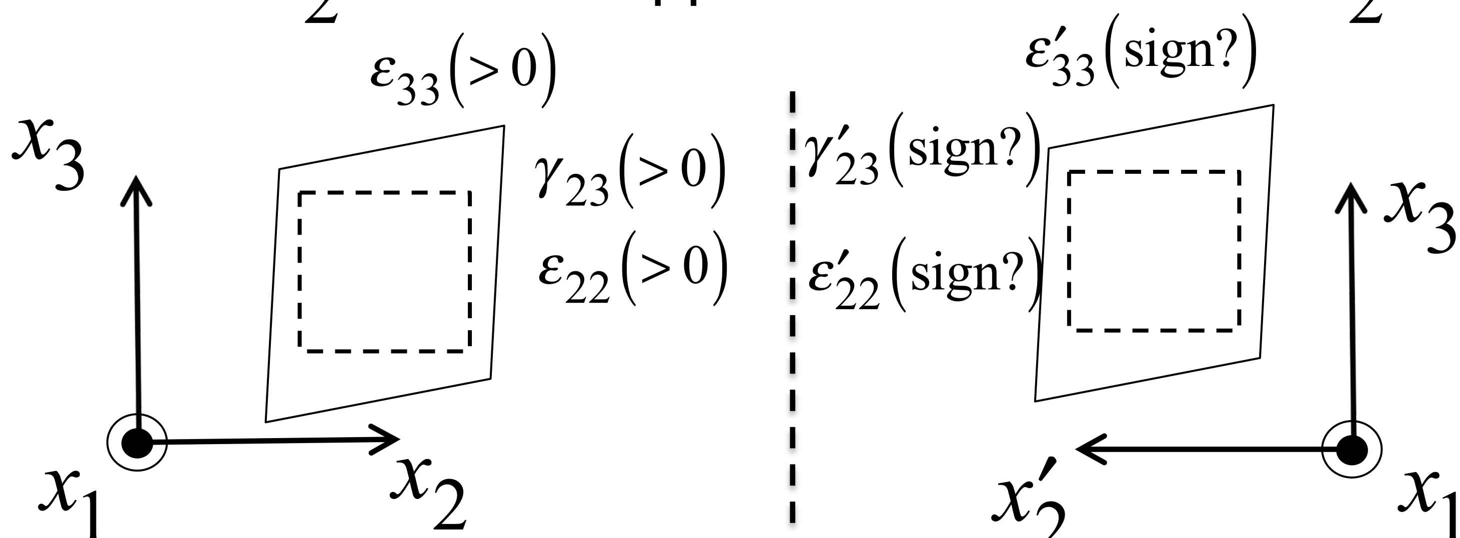
x'_3 is in the opposite direction from x_3



- Symmetry about the coordinate planes
  - ✓ When the elastic body is symmetric about the coordinate planes, we have:



x'_2 is in the opposite direction from x_2



Elastic constants

The results of symmetries about the coordinate planes

An example: x_1' is in the opposite direction from x_1

The same relationship must hold

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & Symm. & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon'_{11} \\ \varepsilon'_{22} \\ \varepsilon'_{33} \\ \gamma'_{12} \\ \gamma'_{23} \\ \gamma'_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & Symm. & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{23} \\ \sigma'_{31} \end{Bmatrix}$$

But:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ -\gamma_{12} \\ \gamma_{23} \\ -\gamma_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & Symm. & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -\sigma_{12} \\ \sigma_{23} \\ -\sigma_{31} \end{Bmatrix}$$

Elastic constants

The results of symmetries about the coordinate planes

An example: x_2' is in the opposite direction from x_2

The same relationship must hold

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & Symm. & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{11}' \\ \varepsilon_{22}' \\ \varepsilon_{33}' \\ \gamma_{12}' \\ \gamma_{23}' \\ \gamma_{31}' \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & Symm. & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11}' \\ \sigma_{22}' \\ \sigma_{33}' \\ \sigma_{12}' \\ \sigma_{23}' \\ \sigma_{31}' \end{Bmatrix}$$

But:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ -\gamma_{12} \\ -\gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & Symm. & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -\sigma_{12} \\ -\sigma_{23} \\ \sigma_{31} \end{Bmatrix}$$

Elastic constants

April 26, 2021

The results of symmetries about the coordinate planes

An example: x_3' is in the opposite direction from x_3

The same relationship must hold

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & Symm. & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{11}' \\ \varepsilon_{22}' \\ \varepsilon_{33}' \\ \gamma_{12}' \\ \gamma_{23}' \\ \gamma_{31}' \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & Symm. & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11}' \\ \sigma_{22}' \\ \sigma_{33}' \\ \sigma_{12}' \\ \sigma_{23}' \\ \sigma_{31}' \end{Bmatrix}$$

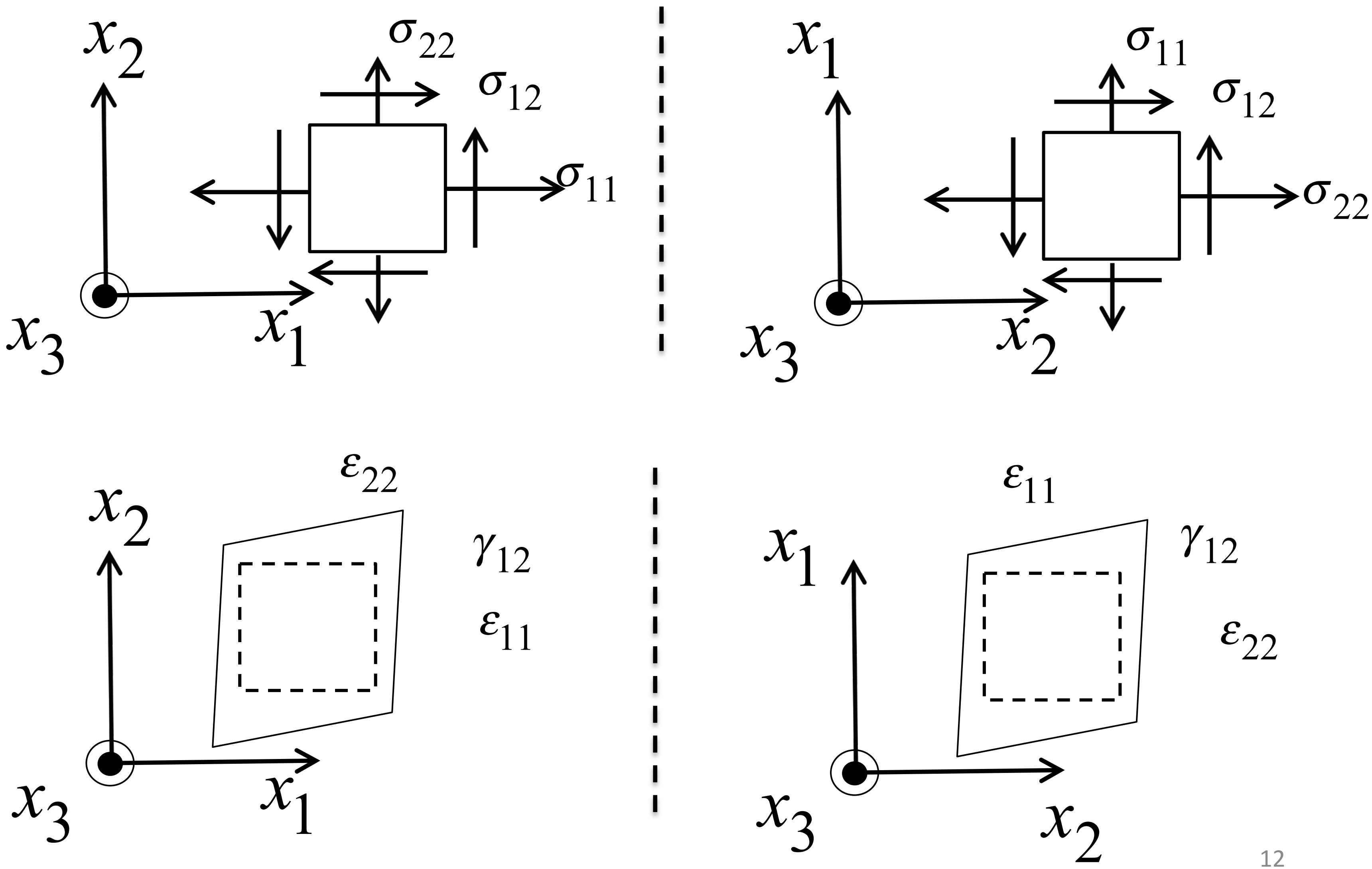
But:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ -\gamma_{23} \\ -\gamma_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & Symm. & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ -\sigma_{23} \\ -\sigma_{31} \end{Bmatrix}$$

<u>Elastic constants</u>

April 26, 2021

- The coordinate axes were exchanged
- The relationships stay the same



Elastic constants

- The coordinate axes were exchanged
- The relationships stay the same

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & a_{33} & 0 & 0 & 0 \\ & & & a_{44} & 0 & 0 \\ & Symm. & & & a_{55} & 0 \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}$$

Caution!

$$\begin{Bmatrix} \varepsilon_{22} \\ \varepsilon_{11} \\ \varepsilon_{33} \\ \gamma_{12} \\ \boxed{\begin{matrix} \gamma_{31} \\ \gamma_{23} \end{matrix}} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & a_{33} & 0 & 0 & 0 \\ & & & a_{44} & 0 & 0 \\ & Symm. & & & a_{55} & 0 \\ & & & & & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{22} \\ \sigma_{11} \\ \sigma_{33} \\ \sigma_{12} \\ \boxed{\begin{matrix} \sigma_{31} \\ \sigma_{23} \end{matrix}} \end{Bmatrix}$$

Elastic constants

- The coordinate axes were exchanged
- The relationships stay the same

This assumption gives:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{12} & 0 & 0 & 0 \\ & a_{11} & a_{12} & 0 & 0 & 0 \\ & & a_{11} & 0 & 0 & 0 \\ & & & a_{44} & 0 & 0 \\ & Symm. & & & a_{44} & 0 \\ & & & & & a_{44} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}$$

There are 3 independent constants

<u>Elastic constants</u>

April 26, 2021

The coordinate axes rotated

The relationships stay the same