AE2-202 Numerical Analysis Coursework

Lecturer: Dr Ajit Panesar

Your name:

Sample Answer

Your CID:

Due date: Monday, 18 Feb 2019 (via Blackboard)

Each individual problem is marked out of a total of 100% and broken down as indicated in the right-hand margin.

You should submit the provided document for this coursework:

- Do not modify the margins, use a font Arial (12) and use a line spacing set at 1.15.
- Do not exceed the given page numbers.
- The pdf document should be self-contained and markable without reference to the entire code and without the need to run them. However, please provide your MATLAB code in the appendix (use font size 8).
- Clearly display the plot with large enough line width, font size, grid and axis labels.
- Do not present the information as a report. Be specific and provide evidence to your argument.
- Export file as a professionally-typed pdf document (no hand-writing, or smartphone photos of hand-written solutions).
- Note: the programs and question answers will be checked for similarities.

Question 1 (worth 40% of total)

a) An electric sports car has two 6-bit sensors installed. One reports the time (in minutes) for journey made and another the distance travelled (in kilometers), both are limited by the capacity of the on-board battery.

Time	0	1	0	1	0	0
Distance	1	1	0	1	1	0

It has been brought to your attention that the highlighted bits are faulty (i.e. for both Time & Distance measurements) and displays a random 0/1 value with equal probability.
What is the expectation for the computed speed (in km/min) and the associated error in its value. Justify your approach. [30%]

As those bits are faulty and have equal probabilities for 0/1 bit, we must choose the values mid-way between the range offered by the faulty bit.
Consequently, Time = 18±2 and Distance = 52±2.

Therefore, V = 52/18 = 2.88

Using relative errors; $\frac{\Delta V}{V} = \frac{\Delta D}{D} + \frac{\Delta T}{T} = 0.1495$

ΔV= ±0.43

b) Given a quadratic function y=f(x) (see table for values), report how many stencil points would be sufficient for obtaining an exact value for $\frac{dy}{dx}_{x=2}$ (employing finite difference scheme) and what would this derivative value be [10%]

x	1.8	2	2.1	2.2
y	2.64	3	3.21	3.44

Given this is a quadratic function, employing a central difference scheme would result in the error term associate with 1st derivative to depend upon 3rd derivative term which is 0. Therefore, using 2 stencil points, i.e. x=1.8 and 2.2 for central difference we can find 1st derivative accurately at x=2.

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{3.44 - 2.64}{2 * (2.2 - 2)} = 2$$

c) Find the approximation to $\frac{dy}{dx}$ for y = sin(x) at x = 0.625 using finite difference methods, specifically, forward difference (FD), backward difference (BD) and central difference (CD). You are allowed to evaluate the function value precisely but are limited to express the value of x with only 5-bits for the fractional part. Clearly show the decimal numbers you obtain for x, report the derivative values (7 decimal place accuracy) for the schemes employed and their errors from the analytical value. Comment on the trend you see in the errors, even for FD & BD. [20%]

0.65625	1	0	1	0	1
0.625	1	0	1	0	0
0.59375	1	0	0	1	1

FD	0.8016897	**0.0092733**
BD	0.8199725	**-0.0090094**
CD	0.8108311	**0.0001319**

It is better to use (1+.B) *2^-1

0.640625	0	1	0	0	1
0.625	0	1	0	0	0
0.609375	0	0	1	1	1

FD	0.8063591	**0.0046039**
BD	0.8155011	**-0.0045379**
CD	0.8109301	**0.0000329**

CD best as 2nd order accurate.

To estimate the rate of change of tangent, we look at 2nd derivative of function (~curvature) which is -sin(x).

For extent, we consider absolute values |-sin(x-h)| < |-sin(x+h)| hence rate of change of tangent is slower before x than later. FD less accurate than BD.

d) Consider the differential equation $\frac{dU}{dt} - \frac{d^4U}{dx^4} = 0$. Use the systematic method to construct the finite-difference approximation of the spatial derivative using the following stencil (-2,-1,0,1,2). Report the coefficients obtained and the order of the truncation error in this approximation. Comment under which circumstance will this approximation provide exact solution. [40%]

The systematic method of obtaining the spatial derivative is given by:

$$\frac{d^4U}{dx^4}(x_i) = a_{-2}U_{i-2} + a_{-1}U_{i-1} + a_0U_i + a_1U_{i+1} + a_2U_{i+2} + \epsilon_{trunc}$$

The Taylor series expansion of U_{i+k} to degree m is:

$$U_{i+k} = U_i + \sum_{n=1}^{m} \frac{d^nU}{dx^n}(x_i)\frac{(k\Delta x)^n}{n!}$$

Construct a matrix $\mathbf{U}$ using the first 5 terms of Taylor expansion at 5 stencil points:

$$\mathbf{U}\begin{bmatrix} a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} U_i & U_i & U_i & U_i & U_i \\ -2\frac{dU}{dx}\Delta x & -\frac{dU}{dx}\Delta x & 0 & \frac{dU}{dx}\Delta x & 2\frac{dU}{dx}\Delta x \\ 4\frac{d^2U}{dx^2}\frac{\Delta x^2}{2} & \frac{d^2U}{dx^2}\frac{\Delta x^2}{2} & 0 & \frac{d^2U}{dx^2}\frac{\Delta x^2}{2} & 4\frac{d^2U}{dx^2}\frac{\Delta x^2}{2} \\ -8\frac{d^3U}{dx^3}\frac{\Delta x^3}{6} & -\frac{d^3U}{dx^3}\frac{\Delta x^3}{6} & 0 & \frac{d^3U}{dx^3}\frac{\Delta x^3}{6} & 8\frac{d^3U}{dx^3}\frac{\Delta x^3}{6} \\ 16\frac{d^4U}{dx^4}\frac{\Delta x^4}{24} & \frac{d^4U}{dx^4}\frac{\Delta x^4}{24} & 0 & \frac{d^4U}{dx^4}\frac{\Delta x^4}{24} & 16\frac{d^4U}{dx^4}\frac{\Delta x^4}{24} \end{bmatrix}\begin{bmatrix} a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{d^4U}{dx^4} \end{bmatrix}$$

Reduce matrix $\mathbf{U}$ to coefficient form gives:

$$\mathbf{U_c}\begin{bmatrix} a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{bmatrix} = \Delta x^4\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & 1/2 & 0 & 1/2 & 2 \\ -4/3 & -1/6 & 0 & 1/6 & 4/3 \\ 2/3 & 1/24 & 0 & 1/24 & 2/3 \end{bmatrix}\begin{bmatrix} a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The above system of equations can be solved using Gaussian elimination:

$$\mathbf{U_c}\begin{bmatrix} a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{bmatrix} = \Delta x^4\begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 0 & 0.5 & 1 & 1.5 & 2 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which gives $a_2 = 1/\Delta x^4, a_1 = -4/\Delta x^4, a_0 = 6/\Delta x^4, a_{-1} = -4/\Delta x^4$ and $a_{-2} = 1/\Delta x^4$

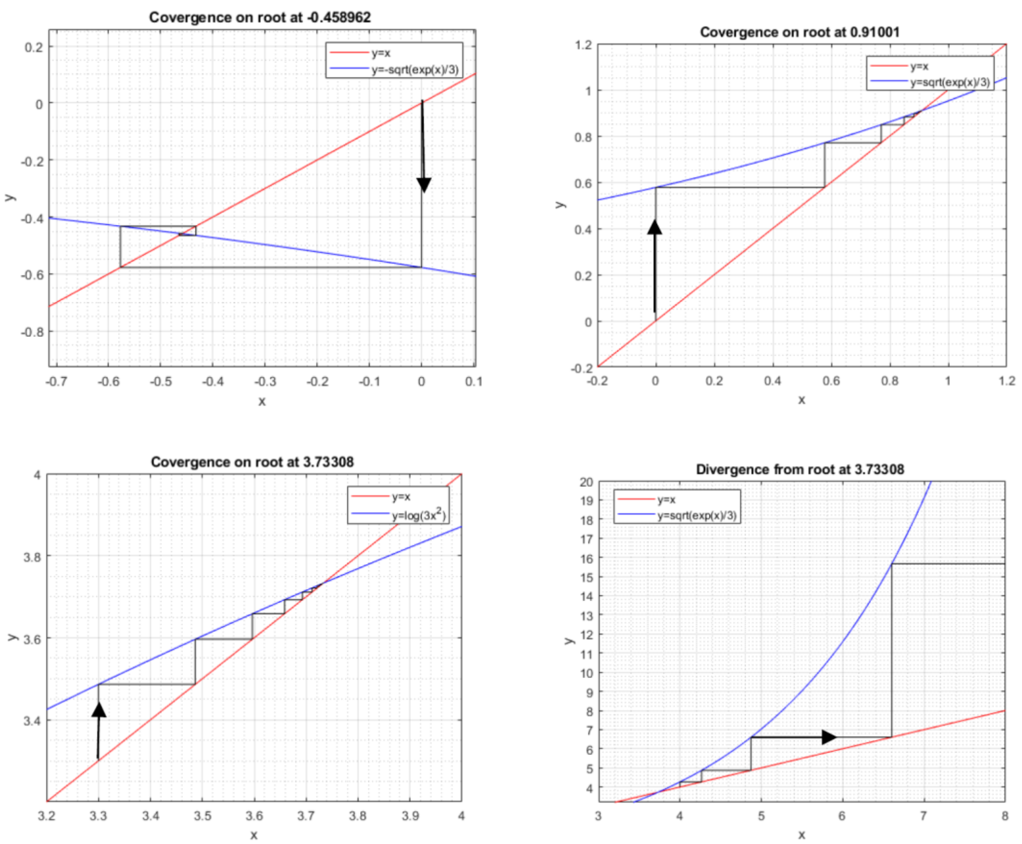
Finally, plug constant values back into the formula:

$$\frac{d^4U}{dx^4}(x_i) = \frac{U_{i-2} - 4U_{i-1} + 6U_i - 4U_{i+1} + U_{i+2}}{\Delta x^4} + \epsilon_{trunc}$$

where $\epsilon_{trunc} = O(\frac{d^6U}{dx^6}(x_i)\Delta x^2)$ since the term involving $\frac{d^5U}{dx^5}\frac{(k\Delta x)^5}{5!}$ is zero using the above coefficients a_i. The approximation is thus second order accurate. This approximation will become exact when $\frac{d^6U}{dx^6}(x) = 0$.

Question 2 (worth 30% of total)

a) Consider $f(x) = e^x - 3x^2 = 0$ which has 3 roots. An obvious arrangement to find roots using fixed-point method is $x = g(x) = \pm\sqrt{\frac{e^x}{3}}$
Show that convergence is to the root near -0.5 if we begin with $x_0 = 0$ and use the negative value of $g(x)$. Show also that convergence to a second root near 1.0 is obtained if $x_0 = 0$ and the positive value is used. Show, however, that this form does not converge to the third root near 4.0 even though a starting value very close to the root is used. Find a different arrangement that will converge to the root near 4.0. General hint: you may find that evaluating $g'(x)$ for x close to the required roots reveals important information. Use MATLAB plots to clearly show the root finding process [30%]



Function, $g(x)$	x_0	$g'(x_0)$	$\left\lvert\frac{E_{n+1}}{E_n}\right\rvert, \lvert g'(x_0)\rvert$	Root	Converges or Diverges?
$-\sqrt{e^x/3}$	0	$-1/2\sqrt{3}$	<1	-0.458962	converges
$\sqrt{e^x/3}$	0	$1/2\sqrt{3}$	<1	0.91001	converges
$\sqrt{e^x/3}$	4	2.13	>1	-	diverges
$\ln(3x^2)$	3.3	0.6061	<1	3.73308	converges

Here E_n is defined as the error in the n-th iterate.

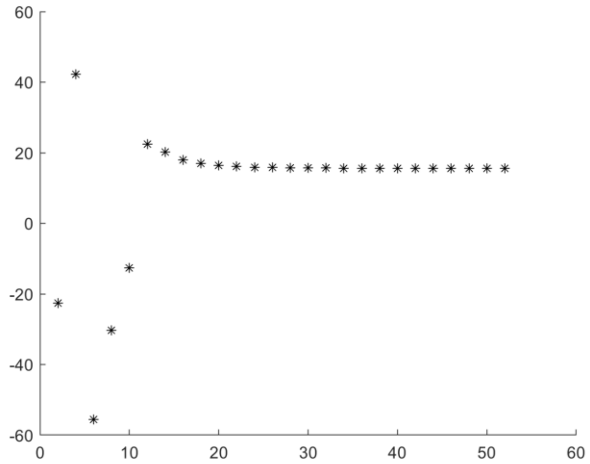
b) Find an approximation to the integral

$$\int_{-\frac{3}{2}\pi}^{0} \frac{\cos 7x}{e^x}\, dx$$

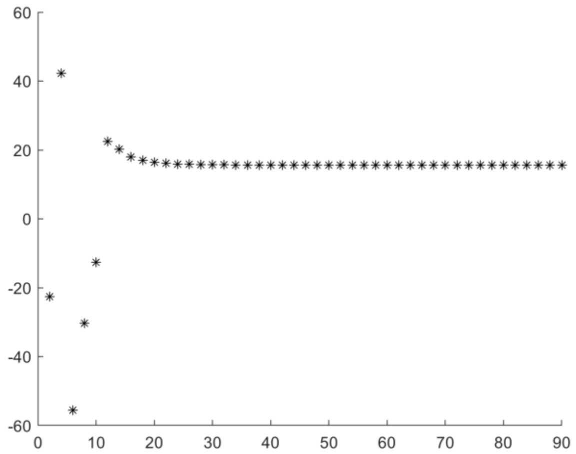
how many sub-intervals are required to have a relative error (take the approximate value) less than 0.1% and 0.01%, respectively? Starting with the minimum possible number of intervals. Show a MATLAB plot (displaying function value) for how the Simpson1/3 scheme converges. Also, specify the sub-interval number identified above where the criterion is met and the final value for integral.

(Use the expression $\frac{1}{50}(7e^{\frac{3\pi}{2}} - 1)$ for computing the exact integral) [30%]

for 0.1% error, 52 intervals & integral value = 15.5785

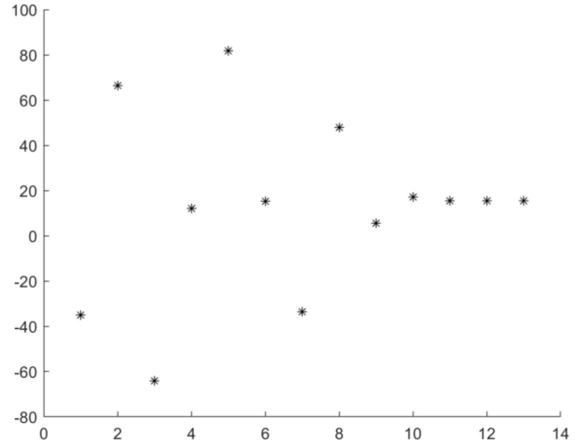


for 0.01% error, 90 intervals & integral value = 15.5660

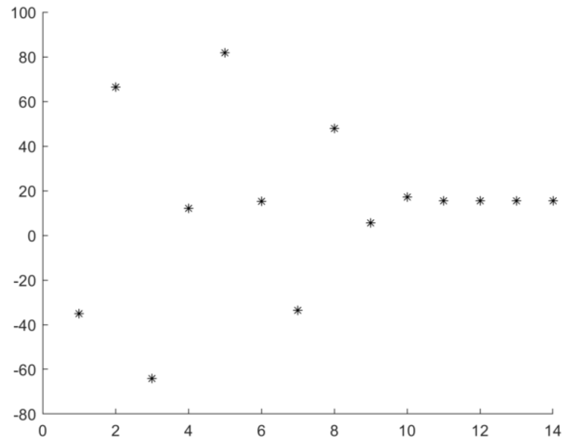


c) Vary the quadrature points in the Gauss-Legendre scheme to produce the same as requested in b). Contrast with b) and discuss why much fewer additional points are needed when considering relative error from 0.1% to 0.01% to obtain the approximation for this integral. [40%]

for 0.1% error, 13 gauss points & integral value = 15.572



for 0.01% error, 14 points & integral value = 15.5633

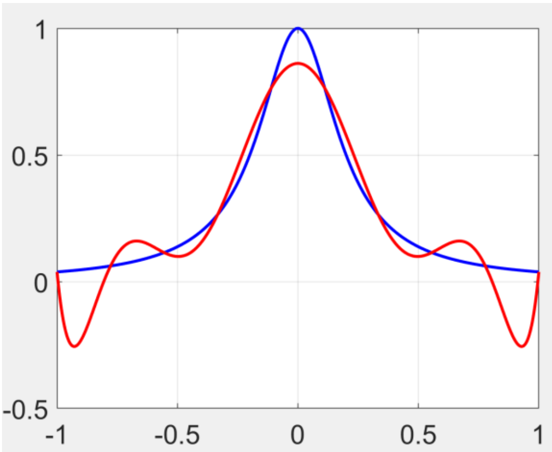


As Simpson relies of fitting a piece-wise quadratic polynomial, improvement in accuracy for an integral other than degree 2 requires sufficient refinement. Contrary, in Gauss-Legendre method, for n-points we can accurately integrate a 2n-1 degree polynomial. Therefore, by using 1 additional point instead of 13 gauss points, we obtain accuracy improvement by upto a 2 higher degree polynomial than the case with 13 gauss points.

Question 3 (worth 30% of total)

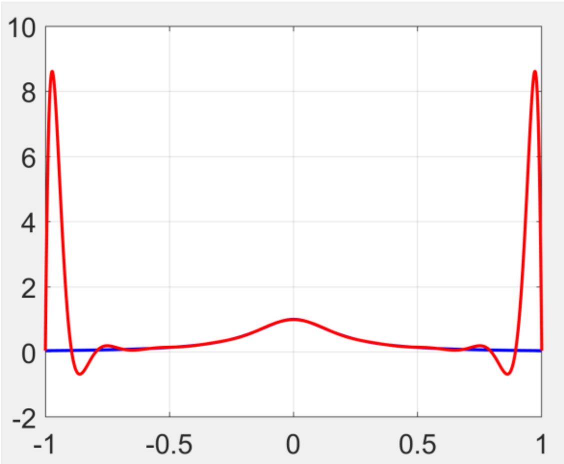
The following question is about constructing interpolatory polynomial approximations of arbitrary functions using Lagrange polynomials.

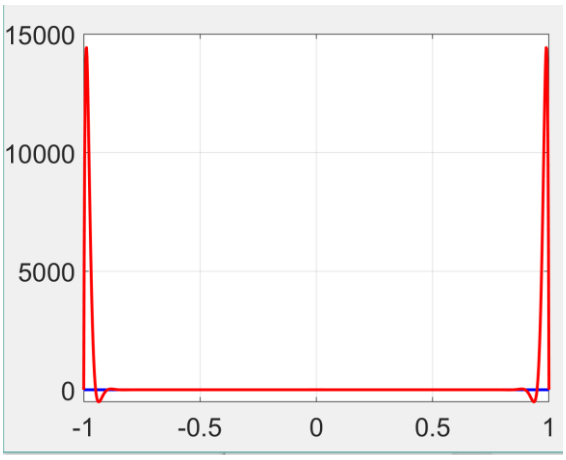
a) Produce the interpolatory polynomial approximation of $g(x) = \frac{1}{(1+25x^2)}$ in the standard interval -1 ≤ x ≤ 1 by sampling it at 10 equispaced points (include the end points). Plot the interpolatory polynomial approximation on top of the true function $g(x)$ using 1001 equispaced points in the given interval. Additionally, calculate the L_2 and L_∞ norm (check definition for vectors! - http://mathworld.wolfram.com/L2-Norm.html) of the error norm between the interpolatory polynomial approximation and $g(x)$ within the standard interval. Note: if you plotted $g(x)$ and the interpolatory function using 1001 points, your error vector should also be 1001 points long. [20%]



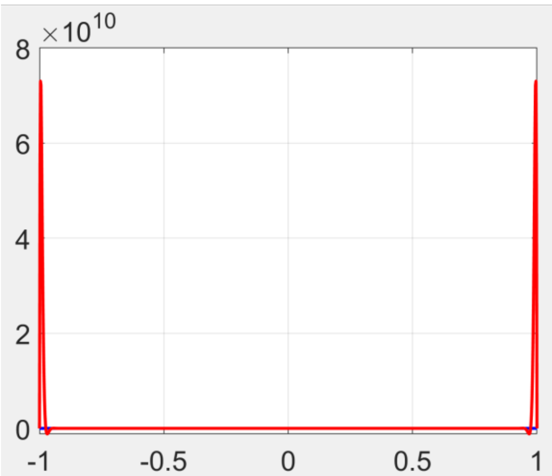
Using 10 equispaced points; L_2_norm = 3.4658 & L_inf_norm = 0.3

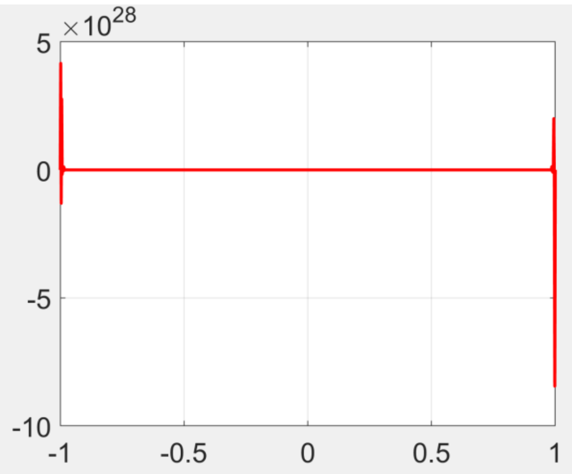
b) Now construct interpolatory polynomial approximations of g(x) in the standard interval by sampling it at 20, 40, 80, and 160 equispaced points respectively (include the end points). Plot the resulting interpolatory polynomial approximations such that they can be compared clearly with the true function g(x). Additionally, calculate the L_2 and L_∞ norm of the error between each interpolatory polynomial approximation and g(x) when evaluated at the 1001 points. Make plots to show how each of these norms vary as the number of points used to sample g(x) is increased. Comment on the results. [40%]



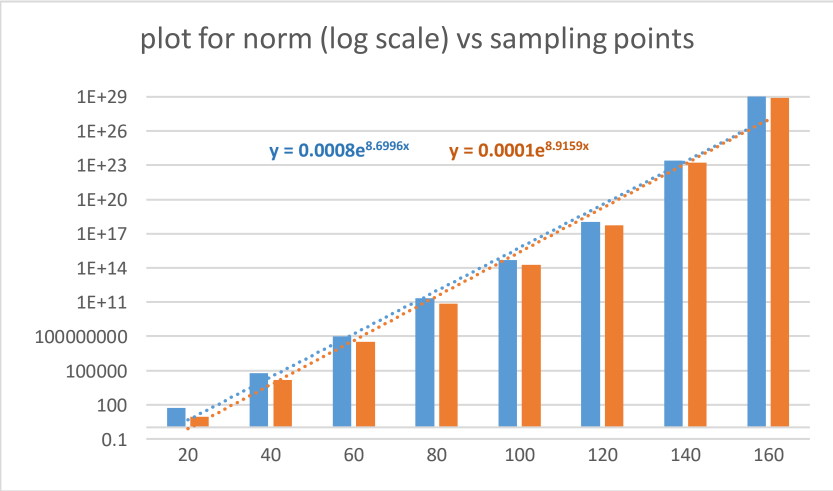


Using 20 equispaced points; L_2_norm = 56.2386 & L_inf_norm = 8.5756	Using 40 equispaced points; L_2_norm = 6.2097e+04 & L_inf_norm = 1.4428e+04





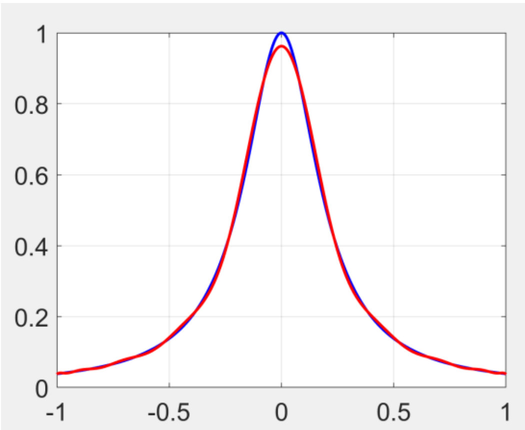
Using 80 equispaced points; L_2_norm = 2.1183e+11 & L_inf_norm = 7.3032e+10	Using 160 equispaced points; L_2_norm = 1.0112E+29 & L_inf_norm = 8.4497E+28

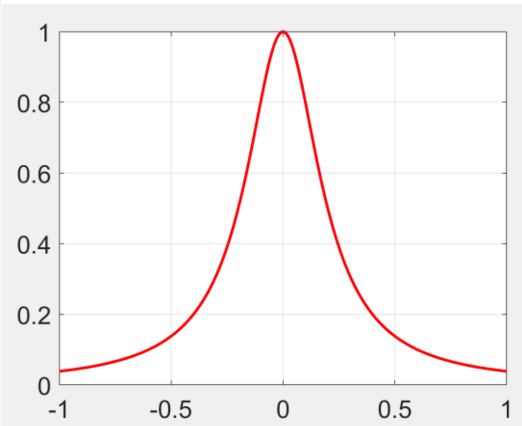


A clear exponential increasing trend for norms is observed when number of sampling points are increase from 20 to 100. This arises from the errors in interpolating values near the interval limits of -1 and 1.

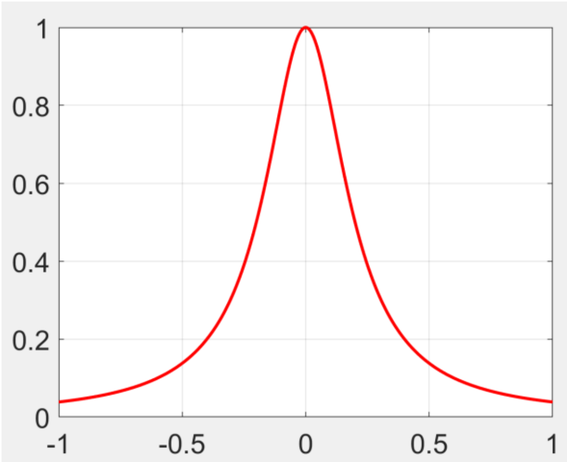
Using simple algebra it can be shown that L_2_norm > L_inf norm.

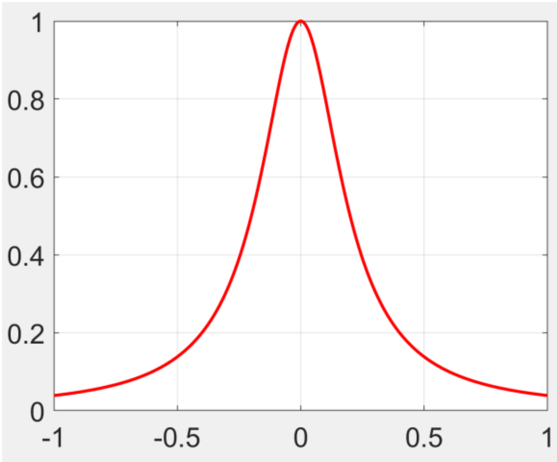
c) Repeat part (b). However, this time sample g(x) at Chebyshev points instead of equispaced points. What happens now? [40%]



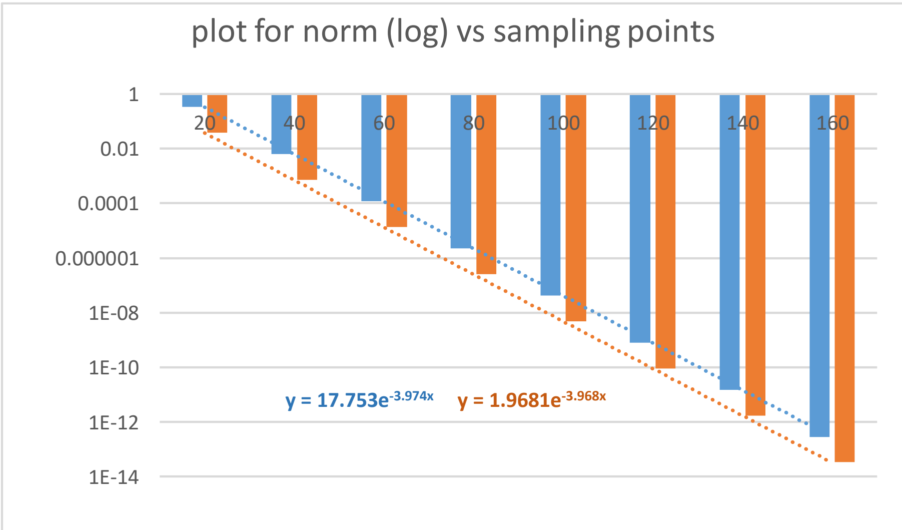


Using 20 Chebyshev points; L_2_norm = 0.3331 & L_inf_norm = 0.0376	Using 40 Chebyshev points; L_2_norm = 0.0063 & L_inf_norm = 7.0702e-04





Using 80 Chebyshev points; L_2_norm = 2.2114e-06 & L_inf_norm = 2.4994e-07	Using 160 Chebyshev points; L_2_norm = 2.7654e-13 & L_inf_norm = 3.3307e-14



A clear exponential decreasing trend for norms is observed when number of sampling points are increase from 20 to 100. This arises from the good fit of the interpolating polynomial with the original curve. As expected, L_2_norm > L_inf norm.

Q2) Please include your MATLAB codes here (you may use font size of 9)

Q3) Please include your MATLAB codes here (you may use font size of 9)