6.6

T6.

$M=\delta\int_0^2 y-(y^2-y)dy$

$=\delta(y^2-\frac{1}{3}y^3)\Big|_0^1=\frac{4}{3}\delta$

$M_x=\int_0^2 y(y-(y^2-y))\,\delta dy$

$=\delta(\frac{2}{3}y^3-\frac{1}{4}y^4\Big|_0^2)=\frac{4}{3}\delta$

$M_y=\int_0^2 \frac{1}{2}(y^2-y+y)\quad \delta(y-(y^2-y))dy$

$=\frac{\delta}{2}\int_0^2 \quad y^2(2y-y^2)dy=\frac{4}{5}\delta$

$x_c=\frac{M_y}{M}=\frac{3}{5}$

$y_c=\frac{M_x}{M}=1$

质心坐标是$(\frac{3}{5},1)$.

T25

$M_y=\int_0^{\pi} a\cos\theta \quad a k\sin\theta\, d\theta=0$

$M_x=\int_0^{\pi} a\sin\theta\, a k\sin\theta d\theta$

$=a^2k\,\frac{\pi}{2}$

$M=\int_0^{\pi} a k\sin\theta\ d\theta$

$=ak\ \cos\theta\Big|_{\pi}^{0}=2ak$

$x_c=\frac{M_y}{M}=0$

$y_c=\frac{M_x}{M}=\frac{\pi a}{4}$

质心坐标是$(0,\frac{\pi a}{4})$

T29

$M_y=\int_0^2 x\ (x^2-x^2(x-1))\ \delta\ dx$

$=\ \frac{2}{4}x^4-\frac{x^5}{5}\Big|_0^2\ \delta=\frac{8}{5}\delta$

$M_x=\int\frac{x^2+x^2(x-1)}{2}\left[x^2-(x^2(x-1))\right]\delta dx$

$=\frac{32}{21}\delta$

$M=\delta\int_0^2\ x^2-x^2(x-1)\ dx=\frac{4}{3}\delta$

$x_c=\frac{M_y}{M}=\frac{6}{5}$

$y_c=\frac{M_x}{M}=\frac{8}{7}$

质心坐标是$(\frac{6}{5},\frac{8}{7})$

T39

易知 $M=\frac{\delta}{2}\pi ab$

$$\int_{-a}^{a}\pi\left(\frac{b}{a}\right)^2(a^2-x^2)\,dx=V=\frac{4}{3}\pi ab^2$$

故 $M_x=\frac{\delta}{2}\int_{-a}^{a}\left(\frac{b}{a}\right)^2(a^2-x^2)\,dx$

$$=\frac{\delta}{2}\ \frac{4}{3}ab^2$$

$$M_y=\delta\int x\frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{\delta}{2}\int_{-a}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,dx^2$$

$$=\frac{\delta b}{2a}\int_{-a}^{a}\sqrt{a^2-x^2}\,d(a^2-x^2)$$

$$=\frac{\delta b}{2a}\ \frac{(a^2-x^2)^{3/2}}{3/2}\Bigg|_{a}^{-a}=0$$

故 $x_c=\frac{M_y}{M}=0$

$$y_c=\frac{M_x}{M}=\frac{4b}{3\pi}$$

7.1

T33.

$$f^{-1}:\underset{\text{domain}}{\{y\in\mathbb{R}\mid y\geq -1\}}\longrightarrow\underset{\text{range}}{\{x\in\mathbb{R}\mid x\leq 1\}}$$

$$y\longmapsto 1-\sqrt{1+y}$$

T42

$f(x)=x^2-4x-5,\ x>2.$

$$\left.\frac{df^{-1}}{dx}\right|_{x=0}=\frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(0)}}=\frac{1}{\left.\frac{df}{dx}\right|_{x=5}}$$

$$=\frac{1}{f'(5)}=\left.\frac{1}{2x-4}\right|_{x=5}$$

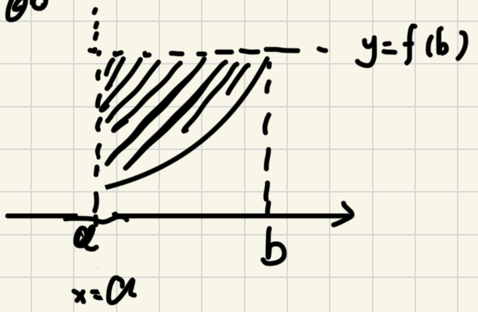
$$=\frac{1}{6}$$

T57 f的range落在g的domain内，f, g都是单射，则 $f\circ g$ 是单射.

证明：
任意 x_1,x_2 满足
$$fg(x_1)=fg(x_2)$$
由于f是单射，$g(x_1)=g(x_2)$.

由于g是单射 $x_1=x_2$.

T60



Shell: $V=\int_a^b 2\pi x\left(f(b)-f\right)dx$

Washer: $V=\int_{f(a)}^{f(b)} \pi\left(f^{-1}(y)^2-a^2\right)dy$

令 $W(t)=\int_{f(a)}^{f(t)} \pi\left(f^{-1}(y)^2-a^2\right)dy$

$S(t)=\int_a^t 2\pi x\left(f(t)-f(x)\right)dx$

$W(a)=S(a)=0.$

$W'(t)=\pi\left[f^{-1}(f(t))^2-a^2)\right]f'(t)=\pi(t^2-a^2)f'(t)$

$S'(t)=f(t)\int_a^t 2\pi x\,dx-\int_a^t 2\pi x f(x)\,dx$

$=f'(t)\int_a^t 2\pi x\,dx+f(t)(2\pi t)-2\pi t f(t)$

$=f'(t)\,2\pi\,\frac{x^2}{2}\Big|_a^t$

$=\pi f'(t)(t^2-a^2)$

故 $W(t)=S(t)$, $\forall t\in[a,b]$.

特别地, $W(b)=S(b)$

7.2

T22

$$y = \frac{x\ln x}{1+\ln x}$$

$$y' = \frac{(x\ln x)'(1+\ln x) - x\ln x(1+\ln x)'}{(1+\ln x)^2}$$

$$= \frac{(\ln x+1)^2 - x\ln x \cdot \frac{1}{x}}{(1+\ln x)^2}$$

$$= \frac{(\ln x)^2+1+\ln x}{(1+\ln x)^2}$$

T36

$$y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t \, dt .$$

$$\frac{dy}{dx} = \ln\sqrt[3]{x} \cdot (\sqrt[3]{x})' - \ln\sqrt{x}\,(\sqrt{x})'$$

$$= \frac{1}{3}\ln x \cdot \frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{2}\ln x \cdot \frac{1}{2}x^{-1/2}$$

$$= \ln x\left(\frac{1}{9}x^{-2/3} - \frac{1}{4}x^{-1/2}\right)$$

T53

$$\int \frac{dx}{2\sqrt{x}+2x}$$

令 $\sqrt{x} = t$. $\frac{1}{2\sqrt{x}}dx = dt \rightsquigarrow dx = 2t\,dt$

$$原式 = \int \frac{2t\,dt}{2t+2t^2}$$

$$= \int \frac{dt}{1+t} = \int \frac{d(1+t)}{1+t} = \ln|1+t| + C$$

$$= \ln|1+\sqrt{x}| + C$$

T67

$$y = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$$

$$= e^{\frac{1}{3}\ln\frac{x(x-2)}{x^2+1}}$$

$$= e^{\frac{1}{3}[\ln|x^2-2x| - \ln(x^2+1)]}$$

$$= e^{\frac{1}{3}[\ln x + \ln(x-2) - \ln(x^2+1)]}$$

$$y' = \sqrt[3]{\frac{x(x-2)}{x^2+1}} \cdot \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right]$$

T77

a.

$$\frac{dy}{dx} = \frac{x}{4} - \frac{1}{x}$$

$$L = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_4^8 \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\,dx$$

$$= \int_4^8 \sqrt{1+\frac{x^2}{16}+\frac{1}{x^2}-\frac{1}{2}}\,dx$$

$$= \int_4^8 \sqrt{\left(\frac{x}{4}\right)^2+\left(\frac{1}{x}\right)^2+\frac{x}{4}\cdot\frac{1}{x}\cdot 2}\,dx$$

$$= \int_4^8 \left|\frac{x}{4}+\frac{1}{x}\right|dx$$

$$= \int_4^8 \frac{x}{4}+\frac{1}{x}\,dx$$

$$= \frac{x^2}{8} + \ln|x|\Big|_4^8 = 6+\ln 2$$

b.

$$\frac{dx}{dy} = 2\left(\frac{y}{4}\right)\cdot\frac{1}{4} - 2/y$$

$$= \frac{y}{8} - \frac{2}{y}$$

$$L = \int_4^{12} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_4^{12} \sqrt{1+\left(\frac{y}{8}-\frac{2}{y}\right)^2}\, dy$$

$$= \int_4^{12} \sqrt{\frac{y^2}{64}+\frac{4}{y^2}+\frac{1}{2}}\, dy$$

$$= \int_4^{12} \sqrt{\left(\frac{y}{8}+\frac{2}{y}\right)^2}\, dy$$

$$= \int_4^{12} \frac{y}{8}+\frac{2}{y}\, dy$$

$$= \frac{y^2}{16}+2\ln|y| \Big|_4^{12}$$

$$= 8+2\ln 3$$

1.3

T19

$$y = \ln\left(\frac{e^\theta}{1+e^\theta}\right)$$

$$= \ln e^\theta - \ln(1+e^\theta)$$

$$= \theta - \ln(1+e^\theta)$$

$$y' = 1 - \frac{e^\theta}{1+e^\theta}$$

T23 $\quad y = \int_0^{\ln x} \sin e^t\, dt.$

$$y' = \sin e^{\ln x} \cdot \frac{1}{x}$$

$$= \frac{\sin x}{x}.$$

T27 求 $\frac{dy}{dx}$.

$$e^{2x} = \sin(x+3y)$$

$$2e^{2x} = \cos(x+3y)\left(1+3\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left[\frac{2e^{2x}}{\cos(x+3y)} - 1\right]\frac{1}{3}$$

T50.

$$\int \frac{dx}{1+e^x}$$

$$= \int \frac{e^{-x/2}\,dx}{e^{-x/2}+e^{x/2}}$$

$$= -2\int \frac{e^{-x/2}\,d(-\frac{x}{2})}{e^{-x/2}+e^{x/2}}$$

$$= -2\int \frac{de^{-x/2}}{e^{-x/2}+e^{x/2}}$$

令 $e^{-x/2}=t$

$$原式 = -2\int \frac{dt}{t+\frac{1}{t}}.$$

$$= -2\int \frac{t\,dt}{t^2+1}$$

$$= -\int \frac{dt^2}{t^2+1}$$

$$= -\ln|t^2+1|+C$$

$$= -\ln|e^{-x}+1|+C$$

T74.

$$y = \log_5 \sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}$$

$$= \frac{\ln 5}{2}\log_5\left(\frac{7x}{3x+2}\right)$$

$$= \frac{\ln 5}{2}\,\frac{\ln\frac{7x}{3x+2}}{\ln 5}$$

$$= \frac{1}{2}\ln\frac{7x}{3x+2}$$

$$= \frac{1}{2}\left(\ln 7x - \ln(3x+2)\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\,\frac{7}{7x} - \frac{1}{2}\,\frac{3}{3x+2}$$

$$= \frac{1}{2x} - \frac{3}{6x+4} = \frac{1}{x(3x+2)}$$

T96

$$\int_1^e x^{(\ln 2)-1}\,dx$$

$$= \left.\frac{x^{\ln 2}}{\ln 2}\right|_1^e$$

$$= \frac{1}{\ln 2}(2-1) = \frac{1}{\ln 2}$$

T115

$$y=(\sin x)^x$$

$$= e^{x\ln\sin x}$$

$$\frac{dy}{dx} = e^{x\ln\sin x}\cdot\left(\ln\sin x + x\frac{\cos x}{\sin x}\right)$$

$$= (\sin x)^x\left(\ln\sin x + x\frac{\cos x}{\sin x}\right)$$

T123

$f(x) = x^2 \ln \frac{1}{x}$ 的最大值.

$$f(x) = -x^2 \ln x$$

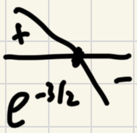
$$f'(x) = -2x\ln x - x^2 \cdot \frac{1}{x}$$

$$= -2x\ln x - x$$

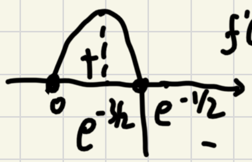
$$= -x(2\ln x + 1)$$

$$f''(x) = -2(1+\ln x) - 1$$

$$= -2\ln x - 3$$



$f''(x)$

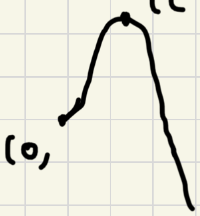


$f'(x)$

$(x>0)$

$(e^{-1/2}, f(e^{-1/2}) = \frac{1}{2e})$

↑ max



T136

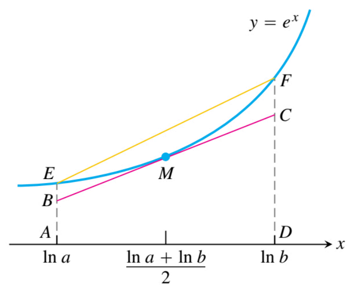
a. 令 $f(x) = e^x$.

$f''(x) = e^x > 0$.

故 e^x 是 concave up.

b. C.

$$e^{(\ln a + \ln b)/2} \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x\,dx < \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln b - \ln a).$$



计算易得 E、B、F、C 的坐标为

$E(\ln a, a) \quad F(\ln b, b)$

$B(\ln a, \sqrt{ab}\ln\sqrt{\frac{a}{b}} + \sqrt{ab})$

$C(\ln b, \sqrt{ab} + \sqrt{ab}\ln\sqrt{\frac{b}{a}})$

BC下的面积为

$$\frac{1}{2}\left(\sqrt{ab}\ln\sqrt{\frac{a}{b}} + \sqrt{ab} + \sqrt{ab} + \sqrt{ab}\ln\sqrt{\frac{b}{a}}\right)(\ln b - \ln a)$$

$$= \sqrt{ab}(\ln b - \ln a)$$

EF下的面积为

$$\frac{a+b}{2}(\ln b - \ln a)$$

故有 $\sqrt{ab}(\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x\,dx < \frac{a+b}{2}(\ln b - \ln a)$

即 $\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$

补充

CH5 T23

$$F(x) = \int_a^x f(t)(x-t)\,dt.$$

$$= x\int_a^x f(t)\,dt - \int_a^x f(t)t\,dt$$

$$F'(x) = \int_a^x f(t)\,dt + xf(x) - f(x)x$$

$$= \int_a^x f(t)\,dt$$

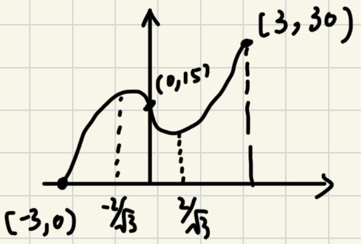
$$F''(x) = f(x).$$

CH6

T4.

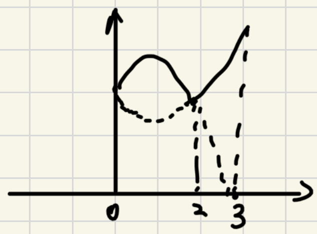
对 $f(x)=x^3-4x+15$ 求导分析

可得$f(x)$图像如下.



$$(-x)^3-4(-x)+15 = x^3-4x+15$$

解得 $x=0, \pm 2$.



要求此图形绕y轴转体积

只需求 $f_1(x)=-x^3+4x+15,\ x\in[0,2]$

与 $f_2(x)=x^3-4x+15,\ x\in[2,3]$

绕y轴旋转所得体积.

$$V_1=\int_0^2 2\pi x f_1(x)\,dx$$

$$=\int_0^2 2\pi x(-x^3+4x+15)\,dx$$

$$=2\pi\int_0^2 -x^4+4x^2+15x\,dx$$

$$=2\pi\left(-\frac{x^5}{5}+\frac{4}{3}x^3+\frac{15}{2}x^2\Big|_0^2\right)=\frac{1028}{15}\pi$$

$$V_2=\int_2^3 2\pi x f_2(x)\,dx$$

$$=2\pi\int_2^3 x(x^3-4x+15)\,dx$$

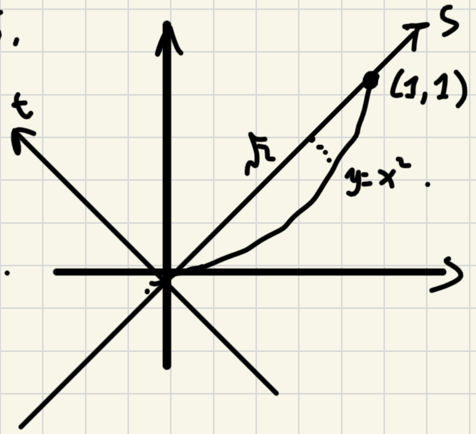
$$=2\pi\int_2^3 x^4-4x^2+15x\,dx$$

$$=2\pi\left(\frac{x^5}{5}-\frac{4}{3}x^3+\frac{15}{2}x^2\Big|_2^3\right)$$

$$=\frac{1631}{15}\pi$$

$$V=V_1+V_2=\frac{2659\pi}{15}$$

T8.



$$[s \quad t] = [x \quad y]\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

$$s = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$$

$$t = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$$

$$\Rightarrow \quad y = \frac{s+t}{\sqrt{2}}, \quad x = \frac{s-t}{\sqrt{2}}$$

$y=x^2$ 在s,t坐标下化为

$$\frac{s+t}{\sqrt{2}} = \frac{(s-t)^2}{2}$$

即 $s+t = \dfrac{(s-t)^2}{\sqrt{2}}$

解得 $t = \dfrac{\sqrt{2}+2s \pm \sqrt{2+8\sqrt{2}s}}{2}$

由 $t<0$

得 $t = \dfrac{\sqrt{2}+2s-\sqrt{2+8\sqrt{2}s}}{2}$

$$= s + \frac{1}{\sqrt{2}} - \sqrt{\frac{1}{2}+2\sqrt{2}s}$$

$$V = \int_0^{\sqrt{2}} \pi\, t^2(s)\, ds$$

$$= \pi\int_0^{\sqrt{2}} \left(s + \frac{1}{\sqrt{2}} - \sqrt{\frac{1}{2}+2\sqrt{2}s}\right)^2 ds$$

$$= \pi\int_0^{\sqrt{2}} s^2 + \frac{1}{2} + \frac{1}{2} + 2\sqrt{2}s + \sqrt{2}s - 2s\sqrt{\frac{1}{2}+2\sqrt{2}s} - \sqrt{2}\sqrt{\frac{1}{2}+2\sqrt{2}s}\, ds$$

$$= \pi\int_0^{\sqrt{2}} \underbrace{s^2+1+3\sqrt{2}s}_{\mathrm{I}} - \underbrace{s\sqrt{2+8\sqrt{2}s}}_{\mathrm{II}} - \underbrace{\sqrt{1+4\sqrt{2}s}}_{\mathrm{III}}\, ds$$

计算II: $\displaystyle\int_0^{\sqrt{2}} s\sqrt{2+8\sqrt{2}s}\, ds$

令 $\sqrt{2+8\sqrt{2}s} = u$

则 $s = \dfrac{u^2-2}{8\sqrt{2}}$, $ds = \dfrac{2u}{8\sqrt{2}}du$

原式 $= \displaystyle\int_{\sqrt{2}}^{3\sqrt{2}} \frac{u^2-2}{8\sqrt{2}}\, u\, \frac{2u}{8\sqrt{2}} du$

$$= \frac{1}{64}\int_{\sqrt{2}}^{3\sqrt{2}} (u^2-2)u^2\, du$$

原式 $= \frac{1}{64}\int_{\sqrt{2}}^{3\sqrt{2}} u^4 - 2u^2 \, du$

$= \frac{1}{64}\left(\frac{u^5}{5} - \frac{2}{3}u^3 \Big|_{\sqrt{2}}^{3\sqrt{2}}\right)$

$= \frac{1}{64}\left(\frac{243\times 4\sqrt{2} - 4\sqrt{2}}{5} - \frac{2\times(27\times 2\sqrt{2} - 2\sqrt{2})}{3}\right)$

$= \frac{\sqrt{2}}{16}\left(\frac{242}{5} - \frac{26}{3}\right)$

$= \frac{149\sqrt{2}}{60}$

计算 Ⅲ $= \int_0^{\sqrt{2}} \sqrt{1+4\sqrt{2}s} \, ds$

令 $\sqrt{1+4\sqrt{2}s} = u$

原式 $= \int_1^3 u \frac{2u}{4\sqrt{2}} du$

$= \frac{1}{2\sqrt{2}}\int_1^3 u^2 du$

$= \frac{1}{2\sqrt{2}} \frac{u^3}{3}\Big|_1^3$

$= \frac{26}{6\sqrt{2}} = \frac{13}{3\sqrt{2}} = \frac{13\sqrt{2}}{6}$

计算 Ⅰ $= \int_0^{\sqrt{2}} s^2 + 1 + 3\sqrt{2}s \, ds$

$= \frac{s^3}{3} + s + \frac{3\sqrt{2}}{2}s^2 \Big|_0^{\sqrt{2}}$

$= \frac{2\sqrt{2}}{3} + \sqrt{2} + 3\sqrt{2} = \frac{14}{3}\sqrt{2}$

$V = \pi\left(\frac{14}{3}\sqrt{2} - \frac{149\sqrt{2}}{60} - \frac{13}{6}\sqrt{2}\right)$

$= \pi \frac{280 - 149 - 130}{60}\sqrt{2}$

$= \frac{\sqrt{2}}{60}\pi$

T9.

(1) $V_1=\int_a^2 \pi(2x^2)^2\ dx$

$= 4\pi \left.\frac{x^5}{5}\right|_a^2 = \frac{4\pi}{5}(32-a^5)$

$V_2 = \int_0^a 2\pi x\cdot 2x^2\ dx$

$= 4\pi \left.\frac{x^4}{4}\right|_0^a = \pi a^4$

(2)
令$f(a)=\pi a^4 - \frac{4\pi}{5}a^5 + \frac{4\pi}{5}32 = V_1+V_2$.

$f'(a)= 4\pi a^3 - 4\pi a^4 = 0 \Rightarrow a=1$或$0$. ($0<a<2$, $a=0$舍去).

$f''(a) = 12\pi a^2 - 16\pi a^3\big|_{a=1} < 0$.

故$a=1$时$f(a)$取最大值.