4.7

T52

$$\int 2+\tan^2\theta \, d\theta$$

$$= \int 2 d\theta + \int \tan^2\theta \, d\theta$$

$$= 2\theta + \int \frac{\sin^2\theta}{\cos^2\theta} d\theta$$

$$= 2\theta + \int \frac{1-\cos^2\theta}{\cos^2\theta} d\theta$$

$$= 2\theta + \int \frac{1}{\cos^2\theta} d\theta - \int 1 d\theta$$

$$= 2\theta + \tan\theta - \theta + C$$

$$= \theta + \tan\theta + C$$

T89

$$\begin{cases} y^{(4)} = -\sin t + \cos t \\ y'''(0) = 7 \quad y''(0) = y'(0) = -1 \quad y(0) = 0 \end{cases}$$

$y'''(t) = \cos t + \sin t + C_1$, C_1 arbitrary

$y'''(0) = 7 \Rightarrow C_1 = 1 + C_1 = 7 \Rightarrow C_1 = 6.$

$y'''(t) = \cos t + \sin t + 6$

$y''(t) = \sin t - \cos t + 6t + C_2$

$y''(0) = -1 + C_2 = -1 \Rightarrow C_2 = 0$

$y''(t) = \sin t - \cos t + 6t$

$y'(t) = -\cos t - \sin t + 3t^2 + C_3$

$y'(0) = -1 + C_3 = -1 \Rightarrow C_3 = 0$

$y'(t) = -\cos t - \sin t + 3t^2$

$y(t) = -\sin t + \cos t + t^3 + C_4$

$y(0) = 1 + C_4 = 0 \Rightarrow C_4 = -1$

$\Rightarrow y(t) = -\sin t + \cos t + t^3 - 1$

5.2

T9

b.

b的第一项($k=0$)是 1

而a,b的第项是 -1

T28

$$\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$$

$$= \left(\frac{1+7}{2} \cdot 7\right)^2 - \frac{1}{4} \sum_{k=1}^{7} k^3$$

$$= 28^2 - \frac{1}{4}\left(\frac{(1+7) \cdot 7}{2}\right)^2$$

$$= 28^2 - \frac{1}{4} 28^2$$

$$= 588$$

T43

$f(x) = x + x^2$ over interval $[0,1]$

取 $\Delta x = \frac{1}{n}$. $c_i = i\Delta x = \frac{i}{n}$

$$\sum_{i=1}^{n} (c_i + c_i^2) \frac{1}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{i}{n} + \frac{i^2}{n^2}\right) \frac{1}{n}$$

$$= \sum_{i=1}^{n} \frac{i}{n^2} + \frac{i^2}{n^3}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} i + \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

$$= \frac{1}{n^2} \frac{1+n}{2} \cdot n + \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1+n}{2n} + \frac{2n^2+3n+1}{6n^2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}(c_i+c_i^2)\frac{1}{n}$$

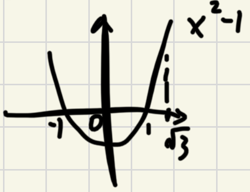
$$=\lim_{n\to\infty}\frac{1+n}{2n}+\frac{2n^2+3n+1}{6n^2}$$

$$=\lim_{n\to\infty}\frac{1/n+1}{2}+\frac{2+3/n+1/n^2}{6}$$

$$=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$$

5.3

T55

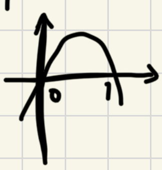


$$\frac{1}{\sqrt{3}}\int_0^{\sqrt{3}}x^2-1\,dx$$

$$=\frac{1}{\sqrt{3}}\left(\frac{x^3}{3}-x\right)\Big|_0^{\sqrt{3}}$$

$$=\frac{1}{\sqrt{3}}\left(\frac{3\sqrt{3}}{3}-\sqrt{3}\right)=0$$

T71



$a=0$, $b=1$ 时取到最大值.

T73

$\frac{1}{1+x^2}$ 在 $[0,1]$ 上单调递减.

故 $\frac{1}{2}\leq\frac{1}{1+x^2}\leq 1$.

$$\underbrace{\frac{1}{2}}_{\text{lower bound}}=\int_0^1\frac{1}{2}dx\leq\int_0^1\frac{1}{1+x^2}\,dx\leq\int_0^1 1\,dx=\underbrace{1}_{\text{upper bound}}$$

T85

$$\Delta x=\frac{\frac{\pi}{2}-0}{n}=\frac{\pi}{2n}$$

$$c_i=\frac{\pi i}{2n}.$$

$$\sum_{i=1}^{n}\sin c_i\,\Delta x$$

$$=\sum_{i=1}^{n}\left(\sin\frac{\pi i}{2n}\right)\cdot\frac{\pi}{2n}$$

$$=\frac{\pi}{2n}\sum_{i=1}^{n}\sin\frac{\pi i}{2n}$$

$$=\frac{\pi}{2n}\,\frac{\cos\left(\frac{\pi}{4n}\right)-\cos\left(\left(n+\frac{1}{2}\right)\frac{\pi}{2n}\right)}{2\sin\left(\frac{\pi}{4n}\right)}\quad(*)$$

$$\lim_{n\to\infty}(*)=\lim_{n\to\infty}\frac{\frac{\pi}{4n}}{\sin\left(\frac{\pi}{4n}\right)}\left[\cos\left(\frac{\pi}{4n}\right)-\cos\left(\frac{\left(1+\frac{1}{2n}\right)\pi}{2}\right)\right]$$

$$=1\cdot\left[\cos(0)-\cos\frac{\pi}{2}\right]$$

$$=1$$

5.4

T16

$$\int_0^{\pi/6}(\sec x+\tan x)^2\,dx$$

$$=\int_0^{\pi/6}\sec^2x+\tan^2x+2\sec x\tan x\,dx$$

$$=\underbrace{\int_0^{\pi/6}\sec^2x\,dx}_{\mathrm{I}}+\underbrace{\int_0^{\pi/6}\tan^2x\,dx}_{\mathrm{II}}+\underbrace{2\int\sec x\tan x\,dx}_{\mathrm{III}}.$$

$$\mathrm{I}=\int_0^{\pi/6}\sec^2x\,dx$$

$$=\tan x\Big|_0^{\pi/6}$$

$$Ⅱ = \int_0^{\pi/6} \tan^2 dx$$

$$= \int_0^{\pi/6} \frac{1-\cos^2 x}{\cos^2 x} dx$$

$$= \int_0^{\pi/6} \frac{1}{\cos^2 x} - 1 dx$$

$$= \tan x \Big|_0^{\pi/6} - x \Big|_0^{\pi/6}$$

$$Ⅲ = 2 \int_0^{\pi/6} \sec x \tan x \, dx$$

$$= 2 \int_0^{\pi/6} \frac{\sin x}{\cos^2 x} dx$$

$$= -2 \int_0^{\pi/6} \frac{1}{\cos^2 x} d\cos x$$

$$= 2 \frac{1}{\cos x} \Big|_0^{\pi/6}$$

$$原式 = 2\tan x \Big|_0^{\pi/6} - x \Big|_0^{\pi/6} + 2 \frac{1}{\cos x} \Big|_0^{\pi/6}$$

$$= 2\left(\tan \pi/6\right) - \pi/6 + 2\left(\frac{1}{\cos \pi/6} - 1\right)$$

$$= 2\sqrt{3} - \pi/6 - 2$$

T24

$$\int_1^8 \frac{(x^{1/3}+1)(2-x^{2/3})}{x^{1/3}} dx$$

$$\overset{令 x^{1/3}=t}{=} \int_1^2 \frac{(t+1)(2-t^2)}{t} 3t^2 dt$$

$$= \int_1^2 3(t+1)(2-t^2)\, t \, dt$$

$$= \int_1^2 -3t^4 - 3t^3 + 6t^2 + 6t \, dt$$

$$= -\frac{3}{5}t^5 - \frac{3}{4}t^4 + \frac{6}{3}t^3 + \frac{6}{2}t^2 \Big|_1^2$$

$$= -\frac{137}{20}$$

T31

$$\int_2^5 \frac{x\,dx}{\sqrt{1+x^2}}$$

(直接展示怎么积分)

$$= \int_2^5 \frac{\frac{1}{2} \cdot 2x\,dx}{\sqrt{1+x^2}}$$

$$= \frac{1}{2}\int_2^5 \frac{dx^2}{\sqrt{1+x^2}}$$

$$= \frac{1}{2}\int_2^5 \frac{d(1+x^2)}{\sqrt{1+x^2}}$$

$$= \frac{1}{2} \cdot \frac{(1+x^2)^{1/2}}{1/2} \Big|_2^5$$

$$= \sqrt{1+x^2} \Big|_2^5 = \sqrt{26} - \sqrt{5}$$

T36

a) $$\int_0^{\tan\theta} \sec^2 y \, dy$$

$$= \tan y \Big|_0^{\tan\theta}$$

$$= \tan \tan\theta$$

b) $$\frac{d}{d\theta} \tan \tan\theta$$

$$= \frac{1}{\cos^2 \tan\theta} \frac{1}{\cos^2\theta}$$

T46.

$$y = \int_{\tan x}^{0} \frac{dt}{1+t^2}$$

$$= \arctan t \Big|_{\tan x}^{0}$$

$$= 0 - x = -x$$

$$\frac{d}{dx} y = \frac{d}{dx}(-x) = -1$$

T71

$\frac{d}{dx} f(x) > 0.$ $f(1) = 0$

$g(x) = \int_0^x f(t)\,dt.$

$= F(x) - F(0).$

a. g is a differentiable function of x

$\frac{d}{dx} g(x) = f(x)$ True.

b. g 可导，因此 g 连续 True

c. $\frac{d}{dx} g(x)\Big|_{x=1} = f(1) = 0.$

g(x) 在 x=1 处斜率为0，故有水平切线. True.

d. $g''(x) = f'(x) > 0$

$g'(x)\Big|_{x=1} = f(1) = 0$

x=1 是 g(x) 的 local min. False

e. True.

f. $g''(1) > 0$ 故 $(1, g(1))$ 不是拐点 False.

g. $g'(1) = f(1) = 0.$ True

补充题

T2. f(t) 在 [a,b] 上恒正且连续.

$F(x) = \int_a^x f(t)\,dt + \int_b^x f(t)\,dt$

在 [a,b] 上连续、可导.

$F(a) = \int_a^a f(t)\,dt + \int_b^a f(t)\,dt$

$= -\int_a^b f(t)\,dt < 0$

$F(b) = \int_a^b f(t)\,dt + \int_b^b f(t)\,dt$

$= \int_a^b f(t)\,dt > 0.$

故 $F(x) = 0$ 在 (a,b) 内一定有解.

$F'(x) = 2f(x) > 0.$

故 F(x) 在 (a,b) 上单调递增.

因此 F(x) 在 (a,b) 上只有一解.

选 B.

T4

$$f(x) = \begin{cases} 2x & x \le 0 \\ \sin x & x > 0 \end{cases}$$

$F'(x) = f(x).$ $F(0) = 1.$

$x<0$ 部分的原函数: $f_1(x) = x^2 + C_1$

$x>0$ 部分的原函数: $f_2(x) = -\cos x + C_2$

$f_1(0) = f_2(0) = 1$

$$\Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 2 \end{cases}$$

$$F(x) = \begin{cases} x^2 + 1 & x \le 0 \\ -\cos x + 2 & x > 0 \end{cases}$$

T10

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\cos\frac{\pi}{n}}+\sqrt{1+\cos\frac{2\pi}{n}}+\cdots+\sqrt{1+\cos\frac{n\pi}{n}}\right)$$

$$=\int_0^1\sqrt{1+\cos\pi x}\,dx$$

$$=\int_0^1\frac{\sqrt{1+\cos\pi x}\sqrt{1-\cos\pi x}}{\sqrt{1-\cos\pi x}}dx$$

$$=\int_0^1\frac{\sqrt{1-\cos^2\pi x}}{\sqrt{1-\cos\pi x}}dx$$

$x\in[0,1]$时 $\sin\pi x>0$

$$=\int_0^1\frac{\sin\pi x}{\sqrt{1-\cos\pi x}}dx$$

$$=\frac{1}{\pi}\int_0^1\frac{\sin\pi x}{\sqrt{1-\cos\pi x}}\,d\pi x$$

$$=-\frac{1}{\pi}\int_0^1\frac{1}{\sqrt{1-\cos\pi x}}\,d\cos\pi x$$

$$=\frac{1}{\pi}\int_0^1\frac{1}{\sqrt{1-\cos\pi x}}\,d(1-\cos\pi x)$$

$$=\left.\frac{(1-\cos\pi x)^{1/2}}{\frac{1}{2}\pi}\right|_0^1=\left.\frac{2}{\pi}\sqrt{1-\cos\pi x}\right|_0^1$$

$$=\frac{2\sqrt{2}}{\pi}$$

T14 (1)

$$h(s)=\int_s^{s^2}\sqrt{1+x^2}\,dx$$

令 $F(x)=\int\sqrt{1+x^2}$

$$\frac{d}{ds}h(s)=\frac{d}{ds}(F(s^2)-F(s))$$

$$=\sqrt{1+s^4}\,2s-\sqrt{1+s^2}$$

T20

$$\int_0^{x^2-1}f(t)\,dt=x-1.\ 求 f(7)$$

令 $F(t)=\int f(t)dt$

$$\int_0^{x^2-1}f(t)dt=F(x^2-1)-F(0)=x-1.$$

$$\Rightarrow F(x^2-1)=x-1+F(0).$$

$$\frac{d}{dx}F(x^2-1)$$

$$=f(x^2-1)\,2x$$

$$=\frac{d}{dx}(x-1+F(0))$$

$$=1$$

故 $f(x^2-1)\,2x=1.$

令 $x=\sqrt{8}$ 得 $f(7)\,2\sqrt{8}=1$

$$\Rightarrow f(7)=\frac{1}{2\sqrt{8}}=\frac{1}{4\sqrt{2}}$$