An Introduction to Higgs Bundles

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Abstract

This note will introduce the basic knowledge necessary for Higgs bundles and the definition of Higgs bundles and non abelian Hodge equivalence, which is an application of the moduli space of Higgs bundles.

Contents

1 Introduction

Higgs bundles appear at the crossing of various topics in mathematics and physics. This note aims to introduce the Higgs bundle from basic knowledge and come to a picture comprehension of Higgs bundles. Sections 2,3,4 prepare

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preliminary knowledge, and the rest introduce Higgs bundles. For picture description, [8] is recommended, and readers can read [2] for further discussion. One can see recent applications in [10].

2 $\wedge^{p,q}$ for vector spaces

Let $V_{\mathbb{R}}$ be an $\mathbb{R}$-vector space. Consider a linear transformation $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$ fulfilling $J^2 = -id$; the eigenvalues are $\pm i \notin \mathbb{R}$. That motivates us to comlexifies $V_{\mathbb{R}}$ to $V_c = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 2.1. Let V be a $\mathbb{R}$-vector space. The complexification of $V_{\mathbb{R}}$ is a $\mathbb{R}$-linear map $f : V_{\mathbb{R}} \to V_c$, where V_c is a $\mathbb{C}$-vector space, such that for any $\mathbb{R}$-linear map $g : V_{\mathbb{R}} \to W_{\mathbb{C}}$, there exists a unique $\bar{f} : V_c \to W_{\mathbb{C}}$ rendering the following diagram commutes:

$$\begin{tikzcd}
V_{\mathbb{R}} \arrow[rr, "f"] \arrow[dr, "g"'] & & V_c \arrow[dl, dashed, "\bar{f}"] \\
& W_{\mathbb{C}} &
\end{tikzcd}$$

$\square$

Remark 2.2. There are two equivalent ways to construct the complexification of a real vector space.

- $f : V_{\mathbb{R}} \to V_{\mathbb{R}} \oplus V_{\mathbb{R}}$, $v \mapsto (v, 0)$ is acomplexification of V
- $g : V_{\mathbb{R}} \to V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, $v \mapsto v \otimes 1$ is a complexification of V

$\square$

Remark 2.3. One can read [1] for more discussion of complexification.

We can extend $\mathbb{R}$-linear $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$ to $\mathbb{C}$-linear $\tilde{J} : V_c \to V_c$ by setting $\tilde{J}(v \otimes a) = J(v) \otimes a$. Since $\tilde{J}^2(v \otimes a) = -v \otimes a$, we have $\tilde{J}^2 = -id$. So $\tilde{J}$ has eigenvalues $\pm i$ on V_c.

Construction 2.4. (a) Denote $V^{1,0} = E(\tilde{J}, i)$, eigenspace of i for linear transformation $\tilde{J}$. (b) Denote $V^{0,1} = E(\tilde{J}, -i)$

$\square$

Then we have $V_c = V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$. $V^{1,0} \simeq_{\mathbb{R}} V^{0,1}$ by conjugation $\overline{v \otimes a} = v \otimes \bar{a}$. Then we denote $V_J := V^{1,0}$, the $\mathbb{C}$-vector space obtained by J.

Assume $dim_{\mathbb{C}} V^{1,0} = n$. Next we consider exterior algebras of those $\mathbb{C}$-vector spaces: $\wedge V_J := \bigoplus_{p=1}^{n} \wedge^p V_J$, $\wedge V^{1,0} := \bigoplus_{p=1}^{n} \wedge^p V^{1,0}$, $\wedge V^{0,1} := \bigoplus_{p=1}^{n} \wedge^p V^{0,1}$.(They are graded algebra with product $\wedge$)

Remark 2.5. Here is a quick review for $\wedge$. Let V be a $\mathbb{K}$-vector space with basis $(e_1, e_2, \cdots, e_n)$ and V^* be dual space with basis $(de_1, de_2, \cdots, de_n)$ fulfilling $de_i(e_j) = \delta_i j$. An element $T \in \otimes^k V^*$ is a multi $\mathbb{K}$-linear map $V \times V \times \cdots V \to \mathbb{K}$.

Define a space

$$\wedge^k V = \{T \in \otimes^k V^* | T(v_1, \cdots, v_k) = (-1)^\sigma T(v_{\sigma(1)}, v_{\sigma(2), \cdots, v_{\sigma(k)}}), \forall \sigma S_k\}$$

For $T \in \otimes^k V^*$, $S \in \otimes^l V^*$, define wedge $T \wedge S = \frac{1}{k!l!} \sum_{\pi \in S_{k+l}} (-1)^\pi (T \otimes S)^\pi$, where $(T \otimes S)^\pi (v_1, \cdots, v_{k+l}) = T \otimes S(v_{\pi(1), \cdots, v_{\pi(k+l)}})$. Specially, $de_1 \wedge de_2 \wedge \cdots \wedge de_k = \sum_{\pi \in S_k} (-1)^\pi (de_1 \otimes de_2 \otimes \cdots \otimes de_k)^\pi$

The fact is , $\wedge^k V$ is a vector space with basis $\{de_{i_1} \wedge de_{i_2} \cdots \wedge de_{i_k} | 1 \geqslant i_1 < i_2 < \cdots i_k \geqslant n\}$ and thus dim $\wedge^k V = \begin{pmatrix} n \\ k \end{pmatrix}$

□

Definition 2.6. Denote $\wedge^{p,q} V := Lin(u \wedge w : u \in \wedge^p V^{1,0}, w \in \wedge^q V^{0,1})$, where $Lin(\bullet)$ means spanned vector space.

□

Finally, we can state the decomposition of differential forms of (p, q)-type:

Property 2.7. $\wedge V_J = \sum_{r=0}^{2n} \sum_{p+q=r} \wedge^{p,q} V$

2.1 Differentrial forms of (p, q)-type of manifold

The almost complex structure of a manifold assigns a complex structure to each tangent space (fiber of a tangent bundle).

Definition 2.8. Let X be a differentialble manifold of dimension 2n and $J : TX \to TX$ a differentiable vector bundle iso such that $J_x : T_x X \to T_x X$ is a complex structur for $T_x X$, i.e., $J_x^2 = -id_{T_x M}$ for each $x \in X$. We call J an almost complex structure for differentiable manifolds, and (X, J) is called an almost complex manifold.

Fact 2.9. A complex manifold X induces an almost complex structure on underlying differentiable manifolds.

Construction 2.10. Let (X, J) be an almost complex manifold. Let $(TX)_c$ denotes the bundle with fiber $(T_x X)_c = T_x X \otimes_{\mathbb{R}} \mathbb{C}$. J can extends to a $\mathbb{C}$-linear bundle map $J : TX_c \to TX_c$ fiber wisely(we still denote this $\mathbb{C}$-linear map by J).

- Denote $TX^{1,0}$ be the bundle with fiber $T_x X^{1,0} = E(J_x, i)$
- Denote $TX^{0,1}$ be the bundle with fiber $T_x X^{0,1} = E(J_x, -i)$

Then the decomposiiton is $TX_c = TX^{1,0} \oplus TX^{0,1}$

□

Similarly, we have cotangent bundle $T^* X_c = T^* X^{1,0} \oplus T^* X^{0,1}$, and $\wedge^{p,q} T^* X$ be the bundle with fiber $\wedge^{p,q} T_x^* X = Lin(u \wedge w : u \in \wedge^p T_x^* X^{0,1}, w \in \wedge^q T_x^* X^{1,0})$

Definition 2.11. Let $\pi : E \to X$ is a vector bundle. $\Omega(X, E) = \{f : X \to E | \pi f = id_X\}$ The elements in Ω are called *sections*.

The following special cases of sections are important.

Definition 2.12. $\Omega^r(X) = \Omega(X, \wedge^r T^* X)$. These sections are called differential r-forms.

$\Omega^{p,q}(X) = \Omega(X, \wedge^{p,q} T^* X)$. These sections are called the differential forms of type (p, q) on X

Property 2.13. $\Omega^r(X) = \sum_{p+q=r} \Omega^{p,q}(X)$

2.2 Differential forms valued in vector spaces

Definition 2.14. $\Omega^k(X, E) := \Omega(X, \wedge^k T^* X \otimes_{\mathbb{C}} E)$ is called differential forms of degree k valued in E.

Fact 2.15. For any $\eta \in \Omega^k(X, E)$, η has the form $\eta = \Sigma a_{ij} \omega_i \otimes \eta_j$ with $a_{ij} \in \Omega^0(X)$, $\omega_i \in \Omega^k(X)$, $\eta_j \in \Omega(X, E)$

Remark 2.16. For $\alpha \in \Omega^k(X)$, $\beta \in \Omega^l(X)$, we have $\alpha land \beta \in \Omega^{k+l}(X)$. For E-valued k-forms, we cannot define wedge product, but replaced by wedge action:

$$\wedge : \Omega^k(X \times \Omega^l(X, E) \to \Omega^{k+l}(X))$$

$$(\omega_1, \omega_2 \otimes s) \mapsto \omega_1 \wedge (\omega_2 \otimes s) := (\omega_1 \wedge \omega_2) \otimes s$$

Therefore, $\Omega^*(X, E)$ is a graded module over graded algebra $\Omega^*(X)$.

Remark 2.17. One can read [12] for more details.

3 Operators

This section defines six operators: d, ∂, $\bar{\partial}$, connection, curvature, and Hodge-star. Reference of this section are [14],[4].

3.1 Operators d, ∂ and $\bar{\partial}$

d, ∂, $\bar{\partial}$ are related to decomposition $\Omega^r(X) = \sum_{p+q=r} \Omega^{p,q}(X)$.

Definition 3.1. Define $\pi_{p,q} : \Omega^r(X) \to \Omega^{p,q}(X)$ where $p + q = r$.

□

Note that if we define a map $d : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$, then we can define operator $\bar{\partial}$ and ∂.

Definition 3.2. Define $\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$ by $\partial = \pi_{p+1,q} d$

Define $\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$ by $\bar{\partial} = \pi_{p,q+1} d$

□

Hence, the left question is defining $d : \Omega^{p,q}(X) \to \Omega^{p+q+1}(X)$. We can define d locally.

Definition 3.3. A local frame of a vector bundle $\pi : E \to X$ over open set $U \subset X$ is a set of sections $\{s_1, s_2, \cdots, s_n\}$ such that $\{s_1(x), s_2(x), \cdots, s_n(x)\}$ is a basis for $E_x := \pi^{-1}(x)$, for any $x \in U$.

□

Construction 3.4. Let $\{w_1, w_2, \cdots, w_n\}$ be local sections of $T^*X^{1,0}$ over U. Then we can write local frames of $T^*X^{1,0}$ and $\wedge^{p,q}T^*X$:

- $\{\overline{w_1}, \overline{w_2}, \cdots, \overline{w_n}\}$ is a local frame of $T^*X^{1,0}$ over U.
- $\{w^I \wedge w^J || I| = p, |J| = q\}$

Hence, any section $s \in \Omega^{p,q}(X)$ can be represented in U as $s = \sum'_{|I|=p,|J|=q} a_{IJ} w^I \wedge \bar{w}^J$ where $a_{IJ} \in \Omega^0(X)$

Remark 3.5. I is an ordered set $(k_1, k_2, \cdots, k_p)$ with $1 \geqslant k_i \geqslant n$. Define $|I|$ be ita cardinate p. Denote $w^I = w_{k_1} \wedge w_{k_2} \wedge \cdots \wedge w_{k_p}$. Denote Σ' as a summation for (I, J) ordered from small to big.

□

Definition 3.6. Define $d : \Omega^{p,q}(X) \to \Omega^{p+q+1}$, $s = \sum'_{|I|=p,|J|=q} a_{IJ} w^I \wedge \bar{w}^J \mapsto ds := \sum'_{|I|=p,|J|=q} da_{IJ} \wedge w^I \wedge \bar{w}^J + a_{IJ} d(w^I \wedge \bar{w}^J)$

Remark 3.7. The second d in $d(w^I \wedge \bar{w}^J)$ is the common dofferential for k-forms $d : \Omega^k(X) \to \Omega^{k+1}(X)$ locally be $d(\sum_{|I|=k} a_I w^I) = \sum_{|I|=k} \frac{\partial a_I}{\partial x^i} dx^i \wedge w^I$. The meaning of the symbol "d" can be clarified from the context.

Property 3.8. (a)$d^2 = 0$

(b)On a complex manifold, $d = \partial + \bar{\partial}$

□

3.2 Connection and curvature

Definition 3.9. ([4]) A connection D on a $\mathbb{C}$-bundle $E \to S$ is a differential operator $D : \Omega^k(S, E) \to \Omega^{k+1}(S, E)$ satisfying $D(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^p \alpha \wedge D\sigma$ where $\alpha \in \Omega^k(S, \mathbb{C})$, and $\sigma \in \Omega(S, E)$.

Definition 3.10. The curvature of a connection D is the operator $F_D = D^2 : \Omega^k(S, E) \to \Omega^{k+2}(S, E)$

Remark 3.11. After choosing a frame, connections and curvatures can be described by matrices. For details, one can read ChIII.1,2 in [14]

3.3 Hodge-star

Let V be a real vector space of dimension d equipped with an inner product. It induces the inner product on $\wedge^p V$ for any p. Namely, if $\{e_1, \cdots, e_d\}$ is an orthonormal basis for V, then $\{e_{i_1} \wedge \cdots e_{i_p} : 1 \geqslant i_1 < i_2 < \cdots < i_p \geqslant d\}$ is an orthonormal basis for $\wedge^p V$.

Definition 3.12. An orientation on V is a choice of the ordering basis up to an even permutation.

Remark 3.13. An orientation is specifying a d-form and let it be positive. Then, any odd permutation is the minus of this d-form.

Definition 3.14. (**Volume element**) Volume element is $e_1 \wedge \cdots \wedge e_d$, $\{e_1, \cdots, e_d\}$ is an orthonormal basis. We denote the volume element by vol.

Remark 3.15. vol is an orientation of V.

Definition 3.16. (**Hodge $\star$-operator**) Define a mapping $\star : \wedge^p V \to \wedge^{d-p} V$ by setiing

$$\star(e_{i_1} \wedge \cdots e_{i_p}) = \begin{cases} e_{j_1} \wedge \cdots \wedge e_{j_{d-p}}, \, \sigma \text{ is even} \\ -e_{j_1} \wedge \cdots \wedge e_{j_{d-p}}, \, \sigma \text{ is odd} \end{cases}$$

where $j_1, \cdots, j_{d-p}$ is the complement of $\{i_1, \cdots, i_p\}$ in $\{1, \cdots, d\}$ and $\sigma = \{i_1, \cdots, i_p, j_1, \cdots, j_{d-p}\}$.

Remark 3.17. Hodge $\star$ is defined so that $e_{i_1} \wedge \cdots e_{i_p} \wedge \star(e_{i_1} \wedge \cdots e_{i_p}) = e_1 \wedge \cdots \wedge e_d =: vol$.

Property 3.18. For $\alpha, \beta \in \wedge^p V$, $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \, vol$

Remark 3.19. The above property uses an obvious result $e_I \wedge \star e_J = \delta_I J vol$.

Property 3.20. Consider Hodge $\star$ on Riemannian surface (dimension 2). By definition, it is easy to show the following:

(a)$\star(dx) = dy$, $\star(dy) = -dx$

(b)$\star^2 = -id$

(c)$\star(dz) = id\bar{z}$, $\star(d\bar{z}) = -idz$

(d)For any $\alpha \in \wedge^{1,0}(\Sigma)$, $\star(\alpha) = -i\alpha$; For any $\beta \in \wedge^{0,1}(\Sigma)$, $\star\beta = i\beta$.

(e)$\star$ does not depend on holomorphic coordinates, since $\alpha = \alpha^{1,0} + \alpha^{0,1}$ and both $\alpha^{1,0}$ and $\alpha^{0,1}$ does not depend on holomorphic charts.

Then we can generalize Hodge $\star$ to $\Omega^k(\Sigma, Hom(E_1, E_2))$.

Definition 3.21. Let $\Psi = \sum_{i=1}^k \alpha_i \otimes \psi_i \in \Omega^k(\Sigma, Hom(E_1, E_2))$ where $\alpha_i \in \Omega^k(\Sigma, \mathbb{C})$ and $\psi_i \in \Gamma(\Sigma, Hom(E_1, E_2))$.

Define $\star\Psi = sum_{i=1}^k (\star\alpha) \otimes \psi^* \in \Omega^{2-k(\Sigma, Hom(E_2, E_1))}$, where psi_i^* means adjoint of ψ under Hermitian structure.

Remark 3.22. $\star\phi = \begin{cases} i\phi^*, \, \phi \in \Omega^{1,0}(\Sigma, Hom(E_1, E_2)) \\ -i\phi^*, \, \phi \in \Omega^{0,1}(\Sigma, Hom(E_1, E_2)) \end{cases}$

4 Holomorphic vector bundles

4.1 Holomorphic functions on manifold

This section aims to define holomorphic function over Euclidean spaces and generalize it to manifolds.

Definition 4.1. Let D be an open set of $\mathbb{C}^n$. A complex-valued function f is a holomorphic function on D if and only if it is complex analysis, i.e., near each point $x \in D$, f can be written as a convergent power series

$$f(z_1, z_2, \cdots, z_n) = \sum_{a_1, a_2, \cdots, a_n = 0}^{\infty} k_{a_1, a_2, \cdots, a_n} (z_1 - x_1)^{a_1} (z_2 - x_2)^{a_2} \cdots (z_n - x_n)^{a_n}$$

□

To define holomorphic functions on manifolds, we need an "instructive book" which specifies holomorphic functions on manifolds.

Definition 4.2. A *holomorphic structure on a* $\mathbb{C}$*-valued manifold* M is a family of $\mathbb{C}$-valued continous functions defined on the open sets of M, denoted by $\mathcal{O}(M)$, such that:

- (define on local trivialzation by **Definition 4.1**) For any $p \in M$, there exists an open neiborhood U_p and a homeomorphism $h : U_p \to U$, where U is open in $\mathbb{C}^n$, such that

  $$f : V \to \mathbb{C} \in \mathcal{O}(M) \text{ if and only if } fh^{-1} \in \mathcal{O}(h(V))$$

  where $\mathcal{O}(h(V))$ is the set of holomorphic functions on $h(V)$
- (define on general open sets by restricting to each trivialization opens) Let $f : U \to \mathbb{C}$ and assume $U = \cup_i U_i$ where U_i open in M. Then $f \in \mathcal{O}(M)$ if and only if $f|_{U_i} \in \mathcal{O}(M)$

□

Remark 4.3. One may think that a holomorphic structure is a sheaf. It must be corrected because the holomorphic structure is just an instruction book. When we have this instruction book, we can tell which function is holomorphic, and then we can define the sheaf of holomorphic functions on manifolds.

4.2 Two characteristics for holomorphic bundles

We have defined holomorphic structures on manifolds. In this section, we will define holomorphic structure on vector bundles.

Definition 4.4. A vector bundle is a continous map $\pi : E \to X$ between two topological spaces such that: for each $x \in X$, there exists a neighberhood U_x

and a homeomorphism $h : \pi^{-1}(U_x) \to U_x \times V$ where V is a given vector space of dimension n such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_x) \arrow[rr, "h"] \arrow[dr, "\pi"'] & & U_x \times V \arrow[dl, "p_1"] \\ & U_x & \end{tikzcd}$$

Terminologies: π is called a vector bundle of rank n. E is called total space. X is called base space. V is called fiber. □

There are two equivalent definitions of holomorphic structure on vector bundles. One is defined by [4]; it is simple to write but not easy to understand. The other is represented by [14]; it is simple to understand but complex to write.

Definition 4.5. (Definition 2.8 in [4]) A holomorphic structure on a complex vector bundle E over Riemannian surface Σ is a differential operator $\bar{\partial}_E : \Omega^{p,q}(\Sigma, E) \to \Omega^{p,q+1}(\Sigma, E)$ satisfying the leibniz rule:

if $\alpha \in \Omega^{p,q}(\Sigma, E), \sigma \in \Omega^{k,l}(\Sigma, E)$, we have $\bar{\partial}_E(\alpha \wedge \sigma) = (\bar{\partial}_E \alpha) \wedge \sigma + (-1)^{p+q} \alpha \wedge \bar{\partial}_E \sigma$.

We call a section σ of E *holomorphic* if $\bar{\partial}_E \sigma = 0$

□

Remark 4.6. Riemannian surface is a 2-dimensional differential manifold whose local transformations are holomorphic.

Remark 4.7. (a) $\bar{\partial}_E$ is called the Dolbeault operator in some references.

(b) Note that $\bar{\partial}_E$ is different from $\bar{\partial}$.

□

Remark 4.8. Why does the Leibneiz rule appear very often? One can see one of the reasons in the Definition 2.1.4 of [13]. Roughly speaking, those linear operators(concepts in algebra) satisfying the Leibneiz rule describe the "tangent vectors"(a concept in geometry).

□

Holomorphic bundles are vector bundles equipped with a holomorphic structure.

Definition 4.9. [14] A $\mathbb{C}$-vector bundle is called a holomorphic vector bundle if E and X are holomorphic manifolds (manifolds equipped with holomorphic structure), π is a holomorphic morphism, and the local trivializations are holomorphic isomorphisms.

Remark 4.10. A holomorphic bundle is a vector bundle with any morphism that appears holomorphic.

Two characteristics describe holomorphic vector bundles. The first characteristic is rank, the dimension of the fibers (vector space). The other characteristic is degree, the "twistedness" of the bundles (I recommend an interesting online page[7])

Definition 4.11. The degree of a complex vector bundle $E \to \Sigma$ is:

$$degE = \int_S c_1(E)$$

Remark 4.12. One can find the definition of Chern class in [14]. Note that $c_1(E) \in H^2(\Sigma, \mathbb{C})$. Any 2-cochain in $H^2(\Sigma, \mathbb{C})$ acts on an area (2-chain) is a number. That is the meaning of the integral.

Property 4.13. (a) $deg(E^*) = -deg(E)$
(b) $deg(E_1 \otimes E_2) = deg(E_1)rank(E_2) + rank(E_1)deg(E_2)$

The two characteristics can determine many things. Vector bundles over compact connected Riemann surfaces are classified by degree and rank since the following result holds.

Theorem 4.14. Let Σ_g be a compact connected Riemann surface. There is a one-to-one correspondence:

$$\{\text{isomorphisim classes of vector bundles over } \Sigma_g\} \to \mathbb{Z}_+ \times \mathbb{Z}$$

$$E \mapsto (rank(E), deg(E))$$

□

Remark 4.15. One can read [9] for more discussion of degree.

Remark 4.16. We have an algebraic geometry version of degree; see [6].

5 Higgs bundles and the picture

Let S be a closed orientable surface of genus $g \geqslant 2$ and Σ be a Riemann surface structure on S.

Definition 5.1. The canonical bundle of Σ is the cotangent bundle, denoted by K.

Definition 5.2. (Definition 2.10 in[4]) A rank n *Higgs bundle* over Σ is a pair (E, ϕ) where E is a holomorphic vector bundle of rank n and $\phi \in H^0(\Sigma, End(E) \otimes K)$, called the *Higgs field*.

What does $H^0(\Sigma, End(E) \otimes K)$ mean?

$End(E)$ is a bundle with fiber $End(E_x)$, K is a bundle with fiber $T^*_x\Sigma$, and then we have tensor of vector bundles $End(E) \otimes K$ with fiber $End(E_x) \otimes K_x = End(E_x) \otimes T^*_x\Sigma$.

The bundle $End(E) \otimes K$ induces a sheaf of sections, i.e., sheaf $End(E) \otimes K$ with $End(E) \otimes K(U)$ be sections of bundle $End(E) \otimes K$ over U.

$H^0(\Sigma, End(E) \otimes K)$ is a cohomology group with coefficients of sheaf. Theorem 3.11 in [14] shows $H^0(\Sigma, End(E) \otimes K) = End(E) \otimes K(\Sigma)$.

Remark 5.3. Note that $End(E) \otimes K(\Sigma) \neq End(E)(\Sigma) \otimes K(\Sigma)$. The reason is that: Presheaf $\mathcal{H}$:$U \mapsto \mathcal{A} \otimes \mathcal{B}(U)$($\mathcal{H}$ is not a sheaf, where $\mathcal{A}$, $\mathcal{B}$ be two sheaves, and U is an open set of X. $\mathcal{A} \otimes \mathcal{B}$ is the sheafification of the presheaf $\mathcal{H}$.

Although $End(E) \otimes K(\Sigma) \neq End(E)(\Sigma) \otimes K(\Sigma)$, we can consider on the stalk level because sheafification remains stalks unchanged, i.e., we have the following property:

Property 5.4. Let $\mathcal{F}$ be a presheaf over X and X be its sheafification. Then $\mathcal{F}_x = \tilde{\mathcal{F}}_x$ for any $x \in X$.

Hence for Higgs field $\phi \in End(E) \otimes K(\Sigma)$, we view the section ϕ as family of stalks $\{\phi_x \in End(E_x) \otimes K_x\}$. Equivalently, a family of morphisms $\{\phi_x : E_x \to E_x \otimes K_x\}$. Indeed, we have isomorphism $End(E_x) \otimes K_x \overset{(1)}{=\!=} End(E_x)^* \otimes K_x \overset{(2)}{=\!=} Hom(Hom(E_x, E_x), K_x)$. (1) is because $End(V) = Hom(V, V) = V^* \otimes V = V \otimes V^* = (V^* \otimes V)^* = EndV^*$. (2) is because $Hom(V_1, V_2) = V_1^* \otimes V_2$.

$\phi \in End(E) \otimes K(\Sigma)$ is a family of morphisms $\{\phi_x = E_x \to E_x \otimes K_x\}$. Tensor K_x means a twist of fiber E_x. View picture in [7].

We will end this section by showing a picture for Higgs bundles[3]: After choosing a frame of E, ϕ is a matrix of holomorphic one-forms with eigenvalues valued in K. Indeed, at each point $x \in X$, $\phi_x \in EndE_x \otimes K_x$ is a matrix of holomorphic one-form valued in $K_x \subset K$.

- Fig 1(a) Let E be a Higgs bundle over Σ with fiber E_x at $x \in \Sigma$
- Fig1(b) At each point $x \in \Sigma$ we can "draw" its eigenvalues, and finally, we can obtain a graph of eigenvalues, denoted by $\tilde{\Sigma}$. $\tilde{\Sigma}$ is a branched cover over Σ(may degenerate, so may not be a cover).
- Fig1(c) $L \to \tilde{\Sigma}$ is a line bundle with each point in $\tilde{\Sigma}$ (an eigenvalue of ϕ_x for some $x \in \Sigma$) assign its eigenvector.
- Fig1(d) Recall that a vector space can split into a direct sum of eigenspaces by a linear transformation. Hence, at a regular point b, $E_b = \oplus_i L_{p_i}$

We have introduced a single Higgs bundle. However, moduli spaces of Higgs bundles (a collection of Higgs bundles) are more important than a single Higgs bundle. One tool to study the moduli space of Higgs bundles is Hitchin fibration, see [4]. One can read [11] for more details of moduli spaces. The following section provides an example of the application of moduli spaces of Higgs bundles.

6 Non abelian Hodge equivalence

6.1 More structures

Representations

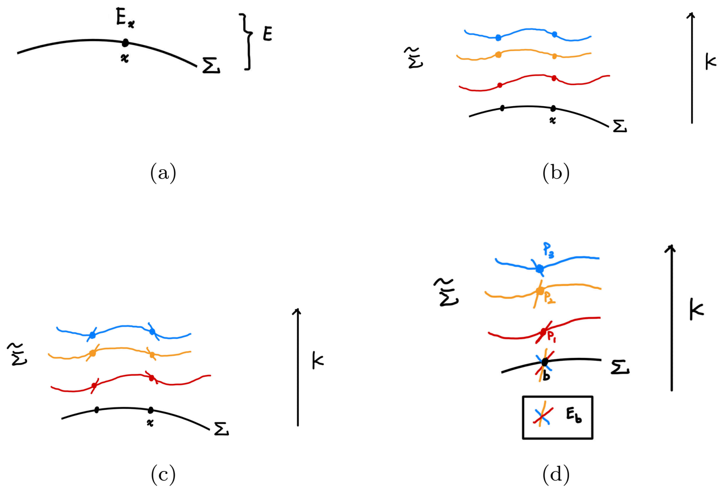

Figure 1: pictures for Higgs bundle

Definition 6.1. (irreducible and reductive) A representation $\rho : \pi_1(S) \to SL(n, \mathbb{C})$ is called irreducible (resp. reductive) if the induced representation on $\mathbb{C}^n$ is irreducible (resp. completely reducible)

Remark 6.2. It is a definition analog to the representation of groups.

Connections

Let D be a connection on a complex bundle E. Let O be the trivial line bundle over S. Let $\mathcal{O}$ be the trivial holomorphic line bundle over Σ.

Remark 6.3. Note that, as a Riemannian surface, Σ is surface S equipped with a holomorphic structure.

Definition 6.4. (irreducible and reductive)

(a) D is irreducible if there exists no proper D-invariant subbundle.

(b) D is reductive if $(E, D) = \bigoplus_{i=1}^{k} (E_i, D_i)$ where each D_i is an irreducible connection on E_i.

Remark 6.5. (Determinant bundle $detE$) Let $E \to X$ be vector bundle of rank n with fiber V. $detE$ is the vector bundle with fiber $\wedge^n V$. Since $dim \wedge^n V = \begin{pmatrix} n \\ n \end{pmatrix} = 1$, $detE$ is a line bundle.

Definition 6.6. (flat) A connection is falt if curvature vanishes, i.e., $F_D = D^2 = 0$

Definition 6.7. ($SL(n, \mathbb{C})$) Assume E satisfies $detE \simeq O$. D is called $SL(n, \mathbb{C})$ connection if its induced connection on the trivial line bundle $detE$ is d.

Remark 6.8. d is a connection on a trivial bundle.

For any connection D, the $D^{0,1}$ in the decomposition $D = D^{1,0} + D^{0,1}$ is a holomorphic structure on E. Conversely, there are many connection D satisfying $D^{0,1}$ equals to given $\bar{\partial}_E$. It motivates us to pick a special connection-Chern connection.

Definition 6.9. Let $(E, \bar{\partial}_E, H)$ be a holomorphic bundle with Hermitian metric H. There exists a unique connection $\nabla_{\bar{\partial}_E, H}$ such that
(a)$\nabla^{0,1}_{\bar{\partial}_E, H} = \bar{\partial}_E$
(b)$\nabla_{\bar{\partial}_E, H}$ is unitary
Such a connection is called a Chern connection.

Hermitian metric

Definition 6.10. (Hermitian metric) A hermitian metric H on bundle E is assigning hermitian metric to each fiber smoothly.

There are two equivalent definitions for harmonic metrics.

Property 6.11. A connection on a Hermitian bundle (E, H) decomposes uniquely $D = D_H + \Psi_H$ such that D_H is unitary and $\Psi_H \in \Omega^1(\Sigma, End(E))$ is self-adjoint.

Remark 6.12. A connection D is unitary if for any two sections $s, t \in \Gamma(S, E)$ we have

$$d(H(s,t)) = H(Ds,t) + H(s,Dt)$$

Unitary connection is a connection compatible with Hermitian metric.

Definition 6.13. (Energy functional) Fix flat $SL(n,\mathbb{C})$-vector bundle (E, D) and a conformal Riemannian metric g_0 on Σ. Define: $E(H) = \int_\Sigma \langle \Psi_H, \Psi_H \rangle \omega$ be energy functional.

Definition 6.14. (Harmonic metric) Hermitian metric H on (E, D) is called harmonic if it is a critical point of $E(H)$.

Remark 6.15. There is the least action principle in physics. We always need to minimize energy in the real physical world, which helps us to choose a special Hermitian metric.

Definition 6.16. (Harmonic metric, equivalently) Hermitian metric H on (E, D) is called harmonic if $D_H(\star\Psi_H) = 0$.

Bundles

Definition 6.17. ($SL(n,\mathbb{C})$-Higgs bundle) A $SL(n,\mathbb{C})$-Higgs bundle is a Higgs bundle satisfying $detE$ is a trivial line bundle over Σ and $tr\phi = 0$.

Definition 6.18. (**ϕ-invariant**) Let $\phi \in H^0(\Sigma, End(E) \otimes K)$ be a Higgs field of a Higgs bundle. A subbundle F of E is ϕ-invariant if $\phi(F) \subset F \otimes K$

Definition 6.19. Define ratio $\mu(E)$ of bundle E as $\mu(E) = degE/rankE$

Definition 6.20. (**semistable,stable,polystable**)

(a) A Higgs bundle (E, ϕ) is semistable if each proper ϕ-invariant subbundle F satisfies $\mu(F) \geqslant \mu(E)$.

(b)A Higgs bundle (E, ϕ) is stable if each proper ϕ-invariant subbundle F satisfies $\mu(F) < \mu(E)$.

(c) A Higgs bundle (E, ϕ) is polystable if it is a direct sum of stable Higgs bundles of the same ratio.

Harmonic maps

The map f is harmonic if it is a critical point of the energy functional $E(f)$. One can find the concrete formula for $E(f)$ in [4].

Property 6.21. Hermitian metric H being harmonic (minimizing $E(H)$) is equivalent to $f : (\tilde{S}, \tilde{g_0}) \to N$ being harmonic (minimizing the energy of f), where N is some subspace contained in $M_n(\mathbb{C})$. Details can be seen in [4].

6.2 Correspondence

Non abelian Hodge equivalence is about one-to-one correspondences between Higgs bundles, connections, harmonic maps, and representations:

$$\begin{tikzcd}
\text{Higgs bundles} \arrow[r, leftrightarrow, "(a)"] & \text{Connections} \arrow[r, leftrightarrow, "(b)"] \arrow[d, leftrightarrow, "(c)"] & \text{Harmonic maps} \\
& \text{Representations} &
\end{tikzcd}$$

(a) Theorem in Fig2 and Fig3 are form [4]

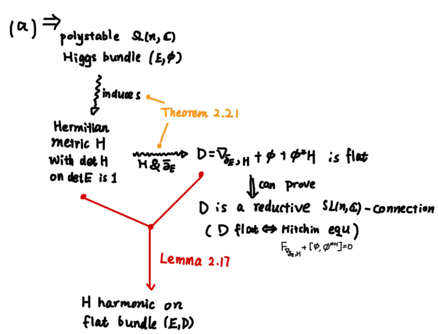

Figure 2: obtain a flat connection from a Higgs bundle

(b)

Fact 6.22. Let D be a reductive flat $SL(n, \mathbb{C})$-connection on E. There exists a harmonic metric H on E such that the induced metric $detH$ on $detE \simeq O$ is 1. If D is irreducible, the harmonic metric is unique.

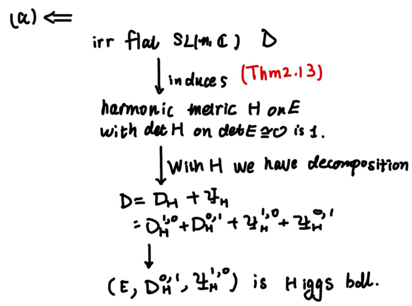

Figure 3: obtain a Higgs bundle from a flat connection

By **Property 6.21**, the following result holds:

Fact 6.23. Let D be a flat irreducible $SL(n, \mathbb{C})$-connection on a vector bundle E over Σ with holonomy representation $\rho : \pi_1(S) \to SL(n, \mathbb{C})$, there exists a unique ρ-equivariant harmonic maps $f : \tilde{\Sigma} \to SL(n, \mathbb{C})/SU(n)$

(c)
There is a one-to-one correspondence:

$$(E, D) \mapsto [\rho : \pi_1(S) \to SL(n, \mathbb{C})]$$

$[l] \mapsto$ **holonomy of D, i.e., parallel transport along loop l defines an element in $SL(n, \mathbb{C})$**

If D is flat, the holonomy of D only depends on the homotopy class of loops; thus, ρ is well-defined.

Definition 6.24. (Definition 2.23 in [4])
(1) The space of gauge equivalence classes of polystable $SL(n, \mathbb{C})$-Higgs bundles is called the moduli space of $SL(n, \mathbb{C})$-Higgs bundles, and we denote it by $\mathcal{M}_{Higgs}(SL(n, \mathbb{C}))$.
(2)The space of gauge equivalence classes of reductive flat $SL(n, \mathbb{C})$-connections is called the de Rham moduli space and we denote it by $\mathcal{M}_{deRham}(SL(n, \mathbb{C}))$.
(3)The space of conjugacy classes of reductive representations from $\pi_1(S)$ into $SL(n, \mathbb{C})$ is called the representation variety, and we denote it by $Rep(\pi_1(S), SL(n, \mathbb{C}))$
(4)The space of equivariant harmonic maps from $\tilde{\Sigma}$ to N modulo isometries in N is denoted by $\mathcal{H}$

□

We have the following one-to-one correspondence, called *non-abelian Hodge correspondence.*

$$\mathcal{M}_{deRham}(SL(n, \mathbb{C})) \simeq \mathcal{H} \simeq \mathcal{M}_{deRham}(SL(n, \mathbb{C})) \simeq Rep(\pi_1(S), SL(n, \mathbb{C}))$$

$$(E, \phi) \mapsto (f : \tilde{\Sigma} \to N) \mapsto D \mapsto \text{the holonomy of D}$$

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