

$$\bullet V(S_1) \cup V(S_2) = V(S_1 \cap S_2)$$

$$\cap V(S_i) = V(\cup S_i)$$

闭集有有限个还是闭集. (-T: 2只是无穷维积; 没完)

- Any affine var can be written as zero locus of finite many polys.

That's because : A is Noetherian \Leftrightarrow every ideal I of A is finitely generated
 由 $k[x_1, \dots, x_n]$ 是 Noetherian ring,
 其商环 $A(x) = k[x_1, \dots, x_n]/I(x)$ 也是 Noetherian.

$$V(S) = V(\langle S \rangle) = V(\{p_1 f_1 + p_2 f_2 + \dots + p_k f_k \mid f_i \in S, p_i \in k[x_1, \dots, x_n]\})$$

$\langle S \rangle \trianglelefteq k[x_1, \dots, x_n]$ Noetherian, so $\langle S \rangle$ can be finitely generated.
 $= V(\langle f_1, \dots, f_k \rangle)$

$$\bullet V(\sqrt{J}) = V(J)$$

$$\text{Exp: } J = \langle (x-a_1)^{k_1}, (x-a_2)^{k_2}, \dots, (x-a_r)^{k_r} \rangle$$

$$V(J) = V(\sqrt{J}) = V((x-a_1), \dots, (x-a_r))$$

Ideals of all polys vanishing at a_1, \dots, a_r (with any order)

↓
affine var. 不仅住重数信息, 这是其
scheme 的重大区别.

$$V(J_1) \cup V(J_2) = V(J_1 \cap J_2)^* \subseteq V(J_1 \cap J_2)$$

$$V(J_1) \cap V(J_2) = V(J_1 \cup J_2) \underset{\text{as a set}}{=} V(\langle J_1 \cup J_2 \rangle) = V(J_1 + J_2)$$

pf for $*$: 理想的运算 $\langle J_1 \cup J_2 \rangle = J_1 + J_2$.

" \supseteq " $J_1 \cap J_2 \subseteq J_1 \cup J_2$ so $V(J_1 \cap J_2) \supseteq V(J_1 \cup J_2)$

" \subseteq " it suffices to show $V(J_1 \cap J_2) \supseteq V(\sqrt{J_1 \cap J_2}) = V(J_1 \cap J_2)$.

$\forall f \in J_1 \cap J_2$, we have $f^2 \in J_1 \cap J_2$. so $f \in \sqrt{J_1 \cap J_2}$. Hence $J_1 \cap J_2 \subseteq \sqrt{J_1 \cap J_2}$.

Then $V(J_1 \cap J_2) \supseteq V(\sqrt{J_1 \cap J_2})$.

$V(\cdot) \cup V(\cdot) = V(\cdot \cap \cdot)$ $V(\cdot) \cap V(\cdot) = V(\cdot \cup \cdot)$	More dual than set-version $\langle \cdot, \cdot \rangle$
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$\langle \cdot, \cdot \rangle$

- $V(I(X)) = X, IV(J) = \sqrt{J}$

$$\{ \text{affine var in } A^n \} \xleftrightarrow{1:1} \{ \text{radical ideals in } k[x_1, \dots, x_n] \}$$

$$X \longleftarrow I(x)$$

$$V(J) \longleftarrow J$$

Rmk: 要求 alg closed. 原因是 $\sqrt{J} \supseteq IV(J)$ 只有 k alg closed 才满足。
 k not closed 定理失真的例子:

$$J = x^2 + 1 \subseteq R[x] \quad V(J) = \emptyset. \quad IV(J) = R[x]. \quad IV(J) \supsetneq \sqrt{J}$$

Rmk: proof $VIX \subset X$ is interesting.

$X = V(J)$ for some $J \in k[x_1, \dots, x_n]$.

w.t.s. $VIV(J) \subset V(J) = V(\sqrt{J})$ and w.t.s. $\sqrt{J} \subset V(J)$
easy.

• $k[x]$ is PID. so $J \subseteq k[x]$ can only be $J = \langle f \rangle = \langle (x-a_1)^{k_1}, \dots, (x-a_r)^{k_r} \rangle$
 So $V(J) = \bigcup_{i=1}^r (x-a_i)$ and closed set in A^1 is a union
 of finite points.

• $\{ \text{points in } A^n \} \xleftrightarrow{1:1} \{ \text{maximal ideals in } k[x_1, \dots, x_n] \}$.

It's a restriction of the bijection in Hilbert Null thm.

△ a pt is minimal non-empty varieties in A^n . so $I(a)$ is maximal $\subseteq k[x_1, \dots, x_n]$

△ maximal ideal is radical $\underset{\text{maximal}}{\sqrt{J}} \subseteq \sqrt{J} \Rightarrow J = \sqrt{J}$

△ $\{ \text{maximal ideals} \} \subseteq \{ \text{radical ideals} \}$ fix \sqrt{J} 1-1 & P bijection

限制到 $\{ \text{pts in } A^n \}$ 上.

- $I(x_1 \cup x_2) = I(x_1) \cap I(x_2)$ trivial.

$$I(x_1 \cap x_2) = \sqrt{I(x_1) + I(x_2)}$$

$$I(x_1 \cap x_2) = I(VI(x_1) \cap VI(x_2))$$

$$= I(V(I(x_1) + I(x_2)))$$

$$= \sqrt{I(x_1) + I(x_2)}$$

Rmk1: Obtained by "1 to 1 correspondence"

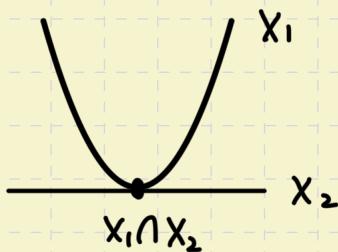
prop of $V(\cdot)$ and prop of $I(\cdot)$

$V_P(\cdot)$ $I_P(\cdot)$

$P(\cdot)$ $C(\cdot)$

Rmk2: Geometric picture

绘制实部如叶:



$x_1, x_2 \subseteq \mathbb{A}_{\mathbb{C}}^2$ with ideals

$$I(x_1) = \langle x_2 - x_1^2 \rangle \text{ and } I(x_2) = \langle x_2 \rangle$$

$$\Delta I(x_1 \cap x_2) = I(0) = \langle x_1, x_2 \rangle$$

$$(I(a) = \langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle)$$

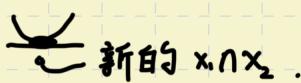
$$\Delta I(x_1) + I(x_2) = \langle I(x_1) \cup I(x_2) \rangle$$

$$= \langle x_2 - x_1^2, x_2 \rangle$$

$$= \langle x_1^2, x_2 \rangle \text{ it's not radical.}$$

△ 'Streck' — why $I(x_1) + I(x_2)$ is non-radical?

x_1 与 x_2 在线性近似下是 x_2 轴. 所以我们可以想象把交点沿 x_1 轴 stretch 无穷小量.



对于新 $x_1 \cap x_2$, x_1 不 vanish, 只有 x_1^2 才 vanish. 所以

$I(x_1) + I(x_2)$ 记录了交点附近的信息 (tangential 信息)

但 $\sqrt{I(x_1) + I(x_2)}$ 忘记这些 tangential 信息.

• $\sqrt{\langle 0 \rangle} = \langle 0 \rangle$ 意味着没有零元.

One way to compute $\sqrt{\langle 0 \rangle}$: $I(V(\langle 0 \rangle)) = I(A^n)$

• 求坐标环的相容性 $X \subset Y \subset A^n$.

用 $X \subset A^n$ 写 ' $A(X)$ ': $A(X) = k[x_1, \dots, x_n]/I(X)$

用 $X \subset Y$ 写 ' $A(Y)$ ': $A(Y) = A(X)/I_Y(X)$

二者相同, i.e., $A(X) \cong A_Y(X)$.

$A(X)$ 指 X 上所有 poly funcs, 这是一个内蕴的概念, 无论把 X 套在哪个 aff. var. 中都给出一样的坐标环.

• 尽管 $A(X)$ 为 coordinate ring, 我们总将其视作 k -alg.

$$k[x_1, \dots, x_n]/I(X).$$

Rmk: K closed. A poly. fun. in x_1, \dots, x_n is determined uniquely by its values at pts of var.

• Zariski拓扑不是积拓扑.

$\mathbb{A}^1 \times \mathbb{A}^1$ 若只考虑乘积拓扑 形如左图, 有限维横线竖线相关.

但 $\mathbb{A}^1 \times \mathbb{A}^1$ 中的闭集还包括 对角线 $V(x_2 - x_1)$

- $\{\text{zariski closed subset}\} \subseteq \{\text{classical closed set}\}$

e.g. $A_{\mathbb{C}}^1 = \{\text{finite pts}\}$ is closed sets.

$$A = \{x : A_{\mathbb{C}}^1 : |x| \leq 1\} \text{ not closed in } A_{\mathbb{C}}^1 \text{ and } \bar{A} = /A_{\mathbb{C}}^1$$

Closed subsets in Zariski topo is very small (finite points)

open subsets in Zariski topo is very big (以至于 $A^1 \times A^1$ 不 Hausdorff)

- 为什么 aff var 间的 mor 不单只考虑连绵映射，而考虑 mor of ringed space?

$f: A^1 \rightarrow A^1$ 的 map 可以非常 weird，但只要 f 是单射，Zariski topo 会判定它连续。

由于 Zariski topo 和 Classical topo 非常不同，导致很多经典拓扑定义的极点概念在 Zariski topo 中变得 useless。Zariski topo 下有新的、与 classical topo 对应的、更实用的概念。

$$\begin{array}{ccc} \Delta \text{ disconnected} & V(x_1) \cup V(x_1 - 1) & | \\ \downarrow & & | \\ \text{reducible} & V(xy) & + \end{array}$$

$$\begin{array}{l} \text{nontrivial} \\ X = X_1 \cup X_2, X_1 \cap X_2 = \emptyset \\ \text{nontrivial} \\ X = X_1 \cup X_2 \end{array}$$

Δ irr \Rightarrow connected.

ΔX is disconnected aff. var. with $X = X_1 \cup X_2$, $X_1, X_2 \subseteq X$. Then $A(X) \cong A(X_1) \times A(X_2)$

$$(I_i + R_j = R, \forall i \neq j, R/I_i = \mathbb{F}_p R/I_i)$$

disjoint
closed

$$\text{pf1: } f \mapsto (f|_{X_1}, f|_{X_2})$$

$$\text{pf2: } I(X_1) + I(X_2) = 1$$

$$A(X)/I(X_1 \cup X_2) \cong \frac{A(X)}{I(X_1) + I(X_2)} \cong \underbrace{A(X)/I(X_1)}_{A_X(X_1) = A(X_1)} \times \underbrace{A(X)/I(X_2)}_{A_X(X_2) = A(X_2)}$$

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Δ $\nexists X$ is irr iff $\begin{cases} A(X) \text{ is an integral domain} \\ I(X) \text{ is prime ideal} \end{cases}$

* $f_1, f_2 \neq 0$ and $X = V(0) = V(f_1, f_2) = V(f_1) \cup V(f_2)$ red. 因式分解对应 var 的分解。

Δ Since prime ideal is radical, so we have restriction to Hilbert null thm

$$\{ \text{non-empty irr aff var of } Y \} \xleftrightarrow{1:1} \{ \text{prime ideals in } A(Y) \}$$

• Aff. var. X given by linear eqns is irr. In particular $A^n = V(0)$ is irr.

$$X = V(a_1 x_1 + \dots + a_n x_n). \quad \text{so } A(X) = k[x_1, \dots, x_n] / \langle a_1 x_1 + \dots + a_n x_n \rangle$$

say $a_1 \neq 0$

(多条 linear eqns 一样，总可解出)
若干变量 (无解就是 $k[x_1, \dots, x_n]$)

$\cong k[x_2, \dots, x_n]$ is integral domain.

- $X \not\subseteq Y$ 则 $I_Y(X) \neq 0$. (由bijection, 若是 0 則 $X = V_Y I_Y X = V_Y(0) = Y$.
空间.)

- Noetherian space: 没有谓子集无穷降链.

infinite decreasing chain
of closed sets \Leftrightarrow infinite increasing chain
 $x_0 \supseteq x_1 \supseteq \dots$ $I(x_0) \subseteq I(x_1) \subseteq \dots$ \leftarrow impossible.
in $A(X)$. Noetherian. (USE DUALITY to prove.)

凡有 aff. var. 是 Noetherian space.

- Subspace of Noetherian space is Noetherian.

$A \cap x_0 \supseteq A \cap x_1 \supseteq A \cap x_2 \supseteq \dots$ \leftarrow chain of closed sets in A .



$x_0 \supseteq x_1 \supseteq x_2 \supseteq \dots$

$x \supseteq x_0 \cap x_1 \supseteq x_0 \cap x_1 \cap x_2 \supseteq \dots$

e.g. A (a set with multiple nested closed subsets)
 x_0 (a point in A)
 x_i, x_0 间 可以没有包含关系

$$\begin{aligned} & \left(\begin{array}{l} A \cap x_1 \supseteq A \cap x_2 \\ A \cap x_1 \cap x_2 \supseteq A \cap x_2 \cap x_1 \\ A \cap x_2 \supseteq A \cap (x_1 \cap x_2) \\ x_1 \supseteq x_2 \cap x_1 \end{array} \right) \quad (\text{矛盾}) \end{aligned}$$

- Every Noetherian topo space can be written as a finite union

$X = X_1 \cup \dots \cup X_r$ of non-empty irr closed subsets.

Assume $X_i \not\subseteq X_j$ for all $i \neq j$, X_1, \dots, X_r are unique (up to permutation)

Existence: 如果可以一直拆分, 则会得无穷降链与 Noetherian 矛盾矛盾.

Uniqueness: $X = X_1 \cup \dots \cup X_r = X'_1 \cup \dots \cup X'_s$.

$X_i \subset \bigcup_j X'_j$ so $X_i = \bigcup_j (X'_j \cap X_i)$. 由 X_i irr, $X_i = X'_j \cap X_i$, i.e., $X_i \subset X'_j$

for some j . In the other way $X'_j \subset X_k$ for some k .

$X_i \subset X'_j \subset X_k \xrightarrow{i \neq k \text{ and } X_i \not\subseteq X_k} X_i = X_k = X'_j$.

{non-empty irr aff var of Y } $\xleftrightarrow{1:1}$ {prime ideals in $A(Y)$ }

{Restriction}

{Maximal non-empty irr aff var of Y } $\xleftrightarrow{1:1}$ {Minimal prime ideals in $A(Y)$ }

irr components of Y

$U_1, U_2, U \stackrel{\text{open}}{\subseteq} X$ Then $U_1 \cap U_2 \neq \emptyset$, $\overline{U} = X$ (A irr \Leftrightarrow \overline{A} irr, so U irr)

$(U_1 \cap U_2)^c \neq X$

Hausdorff nonsense
in Zariski topo

$\forall Y \supseteq U$, $\overline{U} = Y \cup (X \setminus U)$ is trivial!

$X \setminus U \neq X$ so $Y = X$

Zariski: open sets are big!

- Δ The dimension $\dim X \in \mathbb{N} \cup \{\infty\}$ is the supremum over all $n \in \mathbb{N}$
s.t. there is a chain $\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n \subset X$
of length n of **irr closed** subsets Y_1, \dots, Y_n of X

* Idea of this def: if X irr. Any **closed** subset of X not equal to X
should have **smaller dimension**.

- $\Delta Y \subseteq X$ $\text{codim}_X Y$ is the supremum over all n s.t. there is a chain
 $\stackrel{\text{irr}}{\text{closed}}$ $Y \subset Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n \subset X$ of irr closed subsets

Δ dim for aff var, codim only defined for irr closed sets
of an aff var (irr sub aff var)

* ' $\subset X$ ', because X may not irr.

- $\Delta \dim X = \dim A(X)$, $\text{codim}_X X = \text{codim}_{A(X)} I_c(Y)$

In particular, since $A(X)$ is finite generated K -alg,
dim of aff var and codim of irr aff var are always finite

- $\Delta \underline{A \subseteq X}$ Then $\dim A \leq \dim X$
arbitrary set

- $\emptyset \neq X, Y$ irr aff. var.
 - ① $\dim X \times Y = \dim X + \dim Y$ ($\dim A^n = n \dim A^1 = n$)
 - ② If $Y \subseteq X$, we have $\dim X = \dim Y + \text{codim}_X Y$
($\text{codim}_X \{a\} = \dim X, \forall a \in X$)
 - ③ $\emptyset \neq f \in A(X)$, every irr component of $V(f)$ has codim 1 in X .



What if X is not irr? $X = X_1 \cup X_2 \cup \dots \cup X_r$.

- ① $\dim X = \max \{\dim X_1, \dots, \dim X_r\}$

idea: show $\dim X \geq n \Rightarrow \max \{\dim X_i\} \geq n$
 $\max \{\dim X_i\} \geq n \Rightarrow \dim X \geq n$. ← REASON: $\dim X \geq n \Rightarrow \max \{\dim X_i\} \geq n$
 $\dim X$ is sup $\sup_{i=1}^r \dim X_i \leq \max \{\dim X_i\}$

用此 idea ① 对 irr 易证.

$\sup_{i=1}^r \dim X_i \geq \dim X$ 上界

$\dim X \leq \max \{\dim X_i\}$

- ② $\dim X = \sup \{\text{codim}_X \{a\}; a \in X\}$ $\dim X$ is sup local dim!

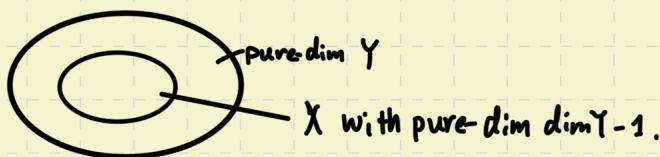
idea: show $\dim X \geq n \Rightarrow \sup \{\text{codim}_X \{a\}; a \in X\} \geq n \rightarrow \sup(\text{codim}_X \{a\}) \geq \sup n = \dim X$
 $\sup \{\text{codim}_X \{a\}; a \in X\} \geq n \Rightarrow \dim X \geq n \rightarrow \dim X \geq \sup n = \sup \{\text{codim}_X \{a\}; a \in X\}$

- ③ $\text{codim}_X \{a\} = \max \{\dim X_i; a \in X_i\}$ (经过 a 的不可约分支最大维数)

互为对方的 MAX

- A Noetherian topo space X is said to be pure dim n if every irr component of X has dim n .

- Hypersurface X is an aff var in a pure-dim aff. Y with pure-dim $\dim Y - 1$



Δ Thm: R is a Noetherian integral domain (e.g. $A(x)$ of an irr aff. var. X).

TFAE (a) Every prime ideal of codim 1 in R is principal

(b) R is a unique factorization domain (with no finite product of prime elements)

Δ Ideal of hypersurface.

* $f \neq 0, V(f) \subseteq X_{\text{irr}}$ then every irr component of $V(f)$ has codim 1, i.e., $V(f)$ is a hypersurface with pure dim $\dim X - 1$.

* hypersurface X in \mathbb{A}^n

① X irr.

$I(X) \triangleleft k[x_1, \dots, x_n]$ prime
 $\text{codim } I(X) = \text{codim } X = 1$
 $k[x_1, \dots, x_n]$ is U.F.D.
proceeding
thm $\Rightarrow I(X)$ is principal, i.e.,
 $I(X) = \langle f \rangle$. So $X = V(f)$.

② X red.

$$X = X_1 \cup X_2 \cup \dots \cup X_r$$

$\begin{cases} I(X_i) \triangleleft k[x_1, \dots, x_n] \text{ prime} \\ \text{codim } I(X_i) = \text{codim } X_i = 1 \\ k[x_1, \dots, x_n] \text{ is U.F.D.} \end{cases}$

$$\Rightarrow I(X_i) = \langle f_i \rangle \Rightarrow X_i = V(f_i)$$

$$X = UV(f_i) = V(f_1 \cdots f_r) =: V(f)$$

Hypersurface in \mathbb{A}^n (whether irr or red.) can be written as a zero locus of a single poly f . And f is unique up to units (U.F.D.)

Degree of f is called the degree of X .

• Def of regular functions: $U \subseteq \underset{\text{open}}{X}$ $\varphi: U \rightarrow K$, $\forall a \in U \exists f, g \in A(x)$

$a \in U_a \subseteq U$ s.t. $\varphi(x) = \frac{g(x)}{f(x)}$, $f(x) \neq 0, \forall x \in U_a$. The set of all such regular functions on U will be denoted $\mathcal{O}_x(U)$.

Δ we write " $\varphi = \frac{g}{f}$ on U_a " for simplicity. Note that $\frac{g}{f}$ is pointwise quotient of functions, NOT an element in a localized ring.

Δ By def we know 不同 U_a 相交处表达式相同 (因为都是 φ , 只是不同写法)

Δ Why we need so complex def?

What if we def regular functions as quotient of functions,

NOT Locally quotient of functions?

It's NOT useful. $\nrightarrow \varphi: U \rightarrow K$ 可能没有全局的 quotient of funs
表达式, 但有 locally quotient of functions 的表达式.

\downarrow

e.g. $X = V(x_1 x_4 - x_2 x_3) \subset \mathbb{A}^4$, $U = x_1 \setminus V(x_2, x_3)$

$$\varphi: U \rightarrow K, (x_1, x_2, x_3, x_4) \mapsto \begin{cases} x_1/x_2 & \text{if } x_2 \neq 0 \\ x_3/x_4 & \text{if } x_4 \neq 0 \end{cases}$$

φ 没有 global quotient.

* Good point of view: Identify $\mathbb{A}^4 \cong M_{2 \times 2}(K)$. Then $X = V(\det M)$,

i.e., $X = \{ M \in M_{2 \times 2}(k) \mid \text{rank } M \leq 1 \}$. (since $\det M = 0$)

Then $U = X \setminus V(x_2, x_4) = \left\{ M \in M_{2 \times 2}(k) \mid \begin{array}{l} \text{second column vector } \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \text{first column vector } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \end{array} \right\}$

Then $\varphi: U \rightarrow k$ $\varphi \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \lambda$.

$U \subseteq X$, $\varphi \in \mathcal{O}_X(U)$, $V(\varphi)$ is closed in U .

($\varphi \notin A(X)$, it's not obvious $V(\varphi)$ is closed.)

Idea: Decompose to small parts.

Pf: $\forall a \in U \exists a \in U_a \subseteq U$ s.t. $\varphi = \frac{g_a}{f_a}$ on U_a .

$U_a \setminus V(\varphi) = \{x \in U_a \mid \varphi(x) \neq 0\} = \{x \in U_a \mid g_a(x) \neq 0\} = U_a \setminus V_a$ open.

So $U \setminus V(\varphi) = \bigcup_a (U_a \setminus V(\varphi))$ is open.

$\varphi \in \mathcal{O}_X(U)$ is conti.

Pf: $\varphi: U \rightarrow k = \mathbb{A}_k^1$. w.t.s. $\varphi^{-1}(\{\text{finite pts}\}) = \bigcup_{pt} \varphi^{-1}(pt)$ is closed. It suffices to show $\varphi^{-1}(c)$ closed for any $c \in k$. It will reduce to show $\varphi^{-1}(0)$ closed, since let $\tilde{\varphi} = \varphi - c$, then $\tilde{\varphi}^{-1}(0) = \varphi^{-1}(c)$.

Idea: Decompose to small parts.

$\forall a \in U, \exists a \in U_a \subseteq U$ s.t. $\varphi = \frac{g_a}{f_a}$ on U_a . $U_a \setminus \varphi^{-1}(0) = \{x \in U_a \mid \frac{g_a(x)}{f_a(x)} \neq 0\} = \{x \in U_a \mid g_a(x) \neq 0\} = D(g_a) \cap U_a$ open in U

so $U \setminus \varphi^{-1}(0) = \bigcup_a U_a \setminus \varphi^{-1}(0)$ is open and thus $\varphi^{-1}(0)$ closed.

$\underbrace{U}_{\text{open}} \subseteq V \subseteq X$ irr. $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$. If $\varphi_1|_U = \varphi_2|_U$, then $\varphi_1 = \varphi_2$.

$(\varphi_1 - \varphi_2)|_U = 0 \Rightarrow U \subseteq (\varphi_1 - \varphi_2)^{-1}(0) \Rightarrow \overline{U} \subseteq (\varphi_1 - \varphi_2)^{-1}(0) \Rightarrow V$ open in irr X , so V irr
 closed by conti. closure of U in V

$\Rightarrow U$ open in irr V , so $\overline{U} = V \Rightarrow V = \overline{V} \subseteq (\varphi_1 - \varphi_2)^{-1}(0) \Rightarrow \varphi_1 - \varphi_2|_V = 0 \Rightarrow \varphi_1 = \varphi_2 \in \mathcal{O}_X(V)$

Gaga principle: 解析几何与代数几何的结论可以共用.
 在解析几何中有类似的 Identity thm

$\Delta D(f) \cap D(g) = D(fg)$ 开集有限交还是开集 无穷乘积没定义.

Δ 任何 aff. var. 可写作有限多项式零点. $V(f_1, \dots, f_n)$.

故任何开集 $U = X \setminus V(f_1, \dots, f_n) = X \setminus (\bigcap V(f_i)) = X \cap (\bigcap V(f_i))^c = \bigcup_i V(f_i)^c = \bigcup_i D(f_i)$
 Distinguish open set: 打石扑克基.

$\Delta \mathcal{O}_X(D(f)) \cong A(X)_f = \left\{ \frac{g}{f^n} \mid g \in A(X), n \in \mathbb{N} \right\}$ f diff. to reg fun 可以 全局地 写成多项式之商

set $f = 1$, $\mathcal{O}_X(X) \cong A(X)$. X 上的 regular fun 是 poly, 不是 quotient of poly.

* NOT TRUE for k not alg closed. e.g., $\frac{1}{x+1} \notin A(\mathbb{A}_{IR}^1) = k[x]$ BUT $\frac{1}{x+1} \in \mathcal{O}_{\mathbb{A}_R^1}(\mathbb{A}_R^1)$
 contradicting to $\mathcal{O}_X(X) \cong A(X)$.

* 用于判断一个开集是否可以写成单多项式的非零点.

在之前例子中, $U = V(x_1 x_4 - x_2 x_3) \setminus V(x_2, x_4)$. 若 $U = D(f)$, 则其上 regular function $\varphi \in \mathcal{O}_x(U) \cong A(x)_f$ 有全局的多项式表达式, 但事实上 φ 没有. 所以 U 不是 $D(f)$.

* Application: Every regular fun of $A^2 \setminus \{(0,0)\}$ can be extended to a regular fun on A^2 .

Idea: Decompose to small parts. $A^2 \setminus \{(0,0)\} = D(x_1) \cup D(x_2) \leftarrow \mathcal{O}_x(D(f)) \cong A(x)_f$!

For any $\varphi \in \mathcal{O}_{A^2}(A^2 \setminus \{(0,0)\})$, $\varphi|_{D(x_1)} \in \mathcal{O}_{A^2}(D(x_1)) \cong k[x_1, x_2]_{x_1}$, say

$$\varphi|_{D(x_1)} = \frac{f_1}{x_1^n}, f_1 \in k[x_1, x_2], x_1 \nmid f_1; \text{ Similarly, say } \varphi|_{D(x_2)} = \frac{f_2}{x_2^m}, x_2 \nmid f_2.$$

On $D(x_1) \cap D(x_2)$, $\frac{f_1}{x_1^n} = \frac{f_2}{x_2^m}$ so $x_2^m f_1 - f_2 x_1^n = 0$ on $D(x_1) \cap D(x_2)$.

By Identity thm for regular functions, $x_2^m f_1 - f_2 x_1^n = 0$ on A^2 .

So $x_2^m f_1 = f_2 x_1^n$ on A^2 . If $m \neq 0$, we have $x_2 \mid f_2$; If $n \neq 0$, we have $x_1 \mid f_1$, (与假设矛盾)

so $m=n=0$. Then $f_1=f_2$ on A^2 , $\varphi=f_1=f_2$ on A^2

* Step 1 show $\mathcal{O}_x(D(f)) = \left\{ \frac{g}{f^n} : g \in A(x), n \in \mathbb{N} \right\}$ (在集合层面等于 $A(x)_f$)

" \supseteq " obviously.

" \subseteq " $\forall \varphi \in \mathcal{O}_x(D(f))$, $\forall a \in D(f)$, $\exists a \in U_a \subseteq D(f)$, s.t. $\varphi = \frac{g_a}{f_a}$ on U_a for some $g_a, f_a \in A(X)$.

Rmk: We have open cover $D(f) = \bigcup_a U_a$. 但这样的open cover不好用, 我们希望得到元组如 $D(h_a)$ 的 open cover.

U_a 可以用一堆 $D(h_a)$ cover, 不妨设 $U_a = D(h_a)$. 则 $\varphi|_{U_a} = \frac{h_a g_a}{h_a f_a} = \frac{\overline{g}_a := h_a g_a}{\overline{f}_a := h_a f_a}$ on $D(h_a) = U_a$. It's well defined because $f_a \neq 0$ on U_a , $h_a \neq 0$ on $D(h_a)$, so $\overline{f}_a \neq 0$ on $D(h_a)$. Besides $D(h_a) = D(\overline{f}_a)$, because \overline{f}_a vanishes on $V(h_a)$ and not vanish on $D(h_a)$. As subspace of Noetherian space, $D(f)$ is Noetherian.

All Noetherian space is compact. So $D(f) = \bigcup D(h_a) = \bigcup D(\overline{f}_a)$ has finite cover

$$D(f) = \bigcup_{i=1}^s D(\overline{f}_{a_i}). \text{ 取补得 } V(f) = \bigcap_{i=1}^s V(\overline{f}_{a_i}) = V(\overline{f}_{a_1}, \dots, \overline{f}_{a_s})$$

$$\text{取 } \sqrt{f} = \sqrt{\langle \overline{f}_{a_1}, \overline{f}_{a_2}, \dots, \overline{f}_{a_s} \rangle}. \text{ so } f^n \in \langle \overline{f}_{a_1}, \dots, \overline{f}_{a_s} \rangle.$$

$$\Rightarrow f^n = p_{a_1} \overline{f}_{a_1} + \dots + p_{a_s} \overline{f}_{a_s}$$

于是我们可以把 φ globally 写成多项式之和: $\forall x \in D(\overline{f}_{a_1}) \cap D(\overline{f}_{a_2}) \dots \cap D(\overline{f}_{a_s})$

$$\varphi = \frac{\overline{g}_{a_1}}{\overline{f}_{a_1}} = \frac{\overline{g}_{a_2}}{\overline{f}_{a_2}} = \dots = \frac{\overline{g}_{a_s}}{\overline{f}_{a_s}} = \frac{p_{a_1} \overline{g}_{a_1} + p_{a_2} \overline{g}_{a_2} + \dots + p_{a_s} \overline{g}_{a_s}}{p_{a_1} \overline{f}_{a_1} + p_{a_2} \overline{f}_{a_2} + \dots + p_{a_s} \overline{f}_{a_s}} =: \frac{g}{f^n}$$

$\frac{g}{f^n}$ 这个表达式可以拓展到其余部分, 因为 $V(\overline{f}_{a_i})$ 上 $\begin{cases} \overline{g}_{a_i} = 0 \\ \overline{f}_{a_i} = 0 \end{cases}$

e.g. $x \in D(\overline{f}_{a_1}) \cap D(\overline{f}_{a_2})$ 但在其于 $V(\overline{f}_{a_j})$ 里.

$$\frac{g}{f^n} = \frac{p_{a_1} \overline{f}_{a_1} + p_{a_2} \overline{f}_{a_2}}{p_{a_1} \overline{f}_{a_1} + p_{a_2} \overline{f}_{a_2}} = \frac{\overline{g}_{a_1}}{\overline{f}_{a_1}} = \frac{\overline{g}_{a_2}}{\overline{f}_{a_2}}$$

留下的都是等比例相加;
没留下的分子分母是0.

* Step 2 证明 $\mathcal{O}_x(D(f)) \cong A(x)_f$ as $k\text{-alg}$.

$$A(x)_f \longrightarrow \mathcal{O}_x(D(f))$$

$$\frac{g}{f^n} \longmapsto \frac{g}{f^n}$$

Formal fraction

Quotient of poly funs.

well-defined:

$$\frac{g_1}{f^n} \sim \frac{g_2}{f^m} \text{ in } A(x)_f, \text{ i.e.,}$$

$$\exists f^k \text{ s.t. } f^k(g_1 f^m - g_2 f^n) = 0 \text{ in } A(x)$$

\Rightarrow on $D(f)$, $f \neq 0$, so $g_1 f^m - g_2 f^n = 0$

$$\Rightarrow \frac{g_1}{f^n} = \frac{g_2}{f^m}.$$

Surjective: Step 1.

$$\text{Injective: } \frac{g}{f^n} = 0 \text{ on } D(f) \Rightarrow g = 0 \text{ on } D(f) \Rightarrow fg = 0 \text{ on } X \Rightarrow f(g \cdot 1 - 0 \cdot f^n) = 0$$

$(X = \bigcup_{\substack{\text{open} \\ f=0}} U_i \cup D(f)) \Rightarrow \frac{g}{f^n} = \frac{0}{1}$

- presheaf \mathcal{J}^* of rings consists of following data

(1) $\forall U \subseteq X$, $\mathcal{J}^*(U)$ is a ring

(2) \forall inclusion $U \subseteq V$, a ring homo $P_{V,U} : \mathcal{J}^*(V) \rightarrow \mathcal{J}^*(U)$ called the restriction map satisfying

$$(1) \mathcal{J}^*(\emptyset) = 0$$

$$(2) P_{U,U} = \text{id}, \forall U \subseteq X$$

$$(3) \text{ Any inclusion } U \subseteq V \subseteq W, P_{V,U} \circ P_{W,V} = P_{W,U}$$

* Elements in $\mathcal{J}^*(U)$ are called sections of \mathcal{J}^* over U

* (Sheaf) A presheaf \mathcal{J}^* is called a sheaf if it satisfies glueing prop:

$U \subseteq X$, $U = \bigcup_{i \in I} U_i$ an open cover of U , $\varphi_i \in \mathcal{J}^*(U_i)$ fulfilling $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$

for all $i, j \in I$, then there is a unique $\varphi \in \mathcal{J}^*(U)$ s.t. $\varphi|_{U_i} = \varphi_i$ for $\forall i$.

* Local性质构成 sheaf. $\mathcal{J}^*(U) = \{\varphi : U \rightarrow \mathbb{R} \text{ continuous}\}$ 都是 sheaf.

Locally const

differentiable

Global概念不构成 sheaf, 13/140 const funs 无法结合起来, 只是 presheaf.

$\mathcal{J}^*(U) = \{\varphi : U \rightarrow \mathbb{R} \text{ const}\}$ is not a sheaf. Consider $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$

$$\begin{array}{ll} U_1 = \{x \mid x > 1\} & U_2 = \{x \mid x < 0\} \\ \varphi_1 : U_1 \rightarrow \mathbb{R} & \varphi_2 : U_2 \rightarrow \mathbb{R} \\ x \mapsto 1 & x \mapsto 0 \end{array}$$

不存在 $U = U_1 \cup U_2$ 上的 const fun φ s.t. $\varphi|_{U_1} = \varphi_1$, $\varphi|_{U_2} = \varphi_2$
不满足 glueing prop.

* $U \subseteq X$, \mathcal{J}^* is a presheaf on X . 则可构造 U 上的 presheaf $\mathcal{J}|_U : \mathcal{J}|_U(V) = \mathcal{J}^*(V)$

* When $U \subseteq X \subseteq Y$ open open aff. var. $\mathcal{O}_X(U) = \mathcal{O}_Y(U)$
By restriction

• Sheaf = Collection of stalks

* Def: (stalks of (pre-)sheaves) \mathcal{F} be a presheaf on X , $a \in X$
the **stalk** of \mathcal{F} at a is

$$\mathcal{F}_a := \{(U, \varphi) \mid U \subset X \text{ open with } a \in U, \varphi \in \mathcal{F}(U)\} / \sim$$

$$(U, \varphi) \sim (U', \varphi') \iff \exists \text{ open } q \in V \subset U \cap U' \text{ s.t. } \varphi|_V = \varphi'|_V$$

* \mathcal{F}_a inherits structure of k -alg of $\mathcal{F}(U)$.

* Elements of \mathcal{F}_a are called germs of \mathcal{F} at a . $(\mathcal{O}_X)_a = \mathcal{O}_{X,a}$

* \mathcal{F}_a : "all functions on a ".

* germs in stalks are something "local functions", 包含 local 信息.

e.g. sheaf of differentiable function over \mathbb{R} , a germ

at a pt $a \in \mathbb{R}$ allows to compute all the derivatives
at a .

* 数学描述 "local functions"

$$\mathcal{O}_{X,a} \cong A(X)_{I(a)} = \left\{ \frac{g}{f} : f, g \in A(X), \underbrace{f(a)}_{\neq 0} \right\}, \quad f \in I(a)$$

As a local ring, $\mathcal{O}_{X,a}$ has unique maximal ideal.

Rmk: ($\mathcal{O}_{X,a}$, a 上 fun $\frac{f(a)}{0}$ 为分子; 同样 $\mathcal{O}_x(D(f))$ 为分母)

Pf: (Routine check)

$$\begin{aligned} \rho: A(X)_{I(a)} &\longrightarrow \mathcal{O}_{X,a} \\ \frac{g}{f} &\longmapsto \frac{g}{(D(f), \frac{g}{f})} \end{aligned} \quad \text{← 模掉等价关系}$$

分子
分母
在 $\mathcal{O}_x(D(f))$

① Well defined

$$\frac{g_1}{f_1} \sim \frac{g_2}{f_2} \Rightarrow \exists h \in I(a) \text{ s.t. } h(g_1 f_2 - f_1 g_2) = 0 \text{ in } A(X)$$

↑
 $a \in D(h)$

Trick: 放到 $D(h)$ 上让式子为 0: $g_1 f_2 - f_1 g_2 = 0$

In $D(h) \cap D(f_1) \cap D(f_2) \subseteq D(f_1) \cap D(f_2)$, $h \neq 0$ so $g_1 f_2 - f_1 g_2 = 0$, i.e., $\frac{g_1}{f_1} = \frac{g_2}{f_2}$ on

$D(h) \cap D(f_1) \cap D(f_2)$.

② Surjective.

For any $(U, \varphi) \in \mathcal{O}_{X,a}$, 任何开集可以拆成有限个 $D(f_i)$ 之并, $U = \bigcup_{i=1}^n D(f_i)$

Pick an $D(f_i)$, $\varphi|_{D(f_i)} = \frac{g_i}{f_i}$. so $(U, \varphi) \sim (D(f_i), \frac{g_i}{f_i})$.

通过补集证明, 由 $V(f_i) = V(f_i')$, 得 $D(f_i) = D(f_i')$. 比较结构层的 map, 只需把这里替 $D(f_i)$

于是 $(U, \varphi) \sim (D(f_i'), \frac{g_i}{f_i'})$

③ Injective

$$[D(f), \frac{g}{f}] = 0 \Rightarrow \exists a \in V \subset D(f), \text{ s.t. } \frac{g}{f}|_V = 0.$$

$$\Rightarrow \exists a \in V \subset D(f), \text{ s.t. } g|_V = 0$$

\Rightarrow 在 g 前乘上 h , 构造 $hg=0$ on total X

open V 才成有限 $D(f)$ 之并, 取 $D(h) \subset V$.

$$\text{则 } hg=0 \text{ on } X.$$

$$\forall a \in V \subset D(f)$$

\Rightarrow 把构造的写成 $h(g \cdot 1 - a)$ 的形式

$$h(g \cdot 1 - a) = 0 \text{ on } X. \text{ 故 } \frac{g}{f} = \frac{a}{h} \in A(k)_{\{a\}}.$$

• Δ -Ringed space (X, \mathcal{O}_X)

$\xrightarrow{\text{topo space}}$ $\xrightarrow{\text{sheaf of rings, always called structure sheaf}}$

* An aff. var will always be considered as a ringed space (X, \mathcal{O}_X) where \mathcal{O}_X is sheaf of regular functions.

* $U \subseteq X$ will always be considered as a ringed space $(U, \mathcal{O}_X|_U)$.

Δ Mor of ringed space: $f: X \rightarrow Y$ conti $\xrightarrow{\text{make sure } f^{-1}(U) \text{ open so } \mathcal{O}_X(f^{-1}(U)) \text{ well-defined.}}$

$$\left\{ \begin{array}{l} \varphi \in \mathcal{O}_Y(U), \text{ pull-back } f^*\varphi = \varphi \circ f: f^{-1}(U) \rightarrow k \\ \text{such that } f^*\varphi \in \mathcal{O}_X(f^{-1}(U)) \end{array} \right.$$

This pulling back by f yields a k -alg homo

$$f^*: \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U)), \varphi \mapsto f^*\varphi = \varphi \circ f$$

• Glueing prop for morphisms: 在 open cover 的每一片上是 mor, 则整体是 mor.

$f: X \rightarrow Y$ be a map of ringed spaces. $\{U_i\}_{i \in I}$ is an open cover

of X s.t. $f|_{U_i}: U_i \rightarrow Y$ are mor for all i . Then f is a mor.

Pf: Idea: 证明 f is mor \Rightarrow (1) f is conti.

(2) $\forall \varphi \in \mathcal{O}_Y(V), f^*\varphi \in \mathcal{O}_X(f^{-1}(V))$ pullback sections

(1) f conti

$$f^{-1}(V) = \bigcup_i U_i \cap f^{-1}(V) = \bigcup_i \underbrace{f|_{U_i}^{-1}(V)}_{\substack{\text{conti} \\ \text{open}}} \underbrace{V}_{\text{open}}$$



(2) f pull-back sections $f|_{U_i}: U_i \rightarrow Y$ is a mor

$\forall \varphi \in \mathcal{O}_Y(V)$, we have

$$(f^*\varphi)|_{\bigcup_i f^{-1}(V)} = (f|_{f^{-1}(V) \cap U_i})^*\varphi \in \mathcal{O}_X(f|_{U_i})^{-1}(V) = \mathcal{O}_X(f^{-1}(V) \cap U_i)$$

每一片上都有 section

Glueing prop for sheaves: $\exists! \underbrace{f^*\varphi|_{f^{-1}(V)}}_{\text{构造出整个 open set 上的 section}} \in \mathcal{O}_X(f^{-1}(V))$

构造出整个 open set 上的 section

- Mor between aff. var. is an n -tuple of regular functions
 X, Y aff. var, $U \subseteq^{\text{open}} X$. Mor $f: U \rightarrow Y$ are exactly the maps of the form
 $f = (\varphi_1, \dots, \varphi_n) : U \rightarrow Y \subseteq \mathbb{A}^n$, $x \mapsto (\varphi_1(x), \dots, \varphi_n(x))$, with $\varphi_i \in \mathcal{O}_X(U)$

In particular, when $Y = \mathbb{A}^1$, $f \in \mathcal{O}_X(U)$.

RF:

- Cofunctor $A(\cdot)$ is fully faithful

X, Y are aff. vars. There is a bijection

$$\{\text{mor } X \rightarrow Y\} \xleftrightarrow{1:1} \{k\text{-alg homo } A(Y) \rightarrow A(X)\}$$

$$f \longmapsto f^*$$

In particular, iso \longleftrightarrow iso

- Why we need to consider aff. var as a **ringed space** (with structure sheaf of regular functions)? Why can't we just view them as a **topo space**?

Ex: $X = \mathbb{A}^1$

$\downarrow f$

$$Y = V(x_1^2 - x_2^3) \subseteq \mathbb{A}^2$$

$$\left\{ \begin{array}{l} f: \mathbb{A}^1 \rightarrow Y \quad t \mapsto (t^3, t^2) \\ f^{-1}: Y \rightarrow \mathbb{A}^1 \quad (x_1, x_2) \mapsto \begin{cases} \frac{x_1}{x_2} & x_2 \neq 0 \\ 0 & x_2 = 0 \end{cases} \end{array} \right.$$

conti (所有到 \mathbb{A}^1 的 map 都是 conti)

As Zariski topo space,

they are homeomorphic.

这是荒谬的, X 没有 singular 点而 Y 有, 因此仅考虑 topo structure 无法区分不同的空间.

f 作为 ringed space mor 不是 iso (这是符合预期的)

若 f 是 iso 则

$$\varphi: A(V(x_1^2 - x_2^3)) = k[x_1, x_2]/(x_1^2 - x_2^3) \rightarrow A(\mathbb{A}^1) = k[t]$$

是 iso

$$[p_1: (x_1, x_2) \mapsto x_1] \mapsto [t \mapsto t^3] \stackrel{\text{poly } t^3 \in k[t]}{\longleftarrow}$$

$$[p_2: (x_1, x_2) \mapsto x_2] \mapsto [t \mapsto t^2]$$

$$\boxed{\begin{array}{c} \mathbb{A}^1 \rightarrow V(x_1^2 - x_2^3) \xrightarrow{p_1} k \\ t \mapsto (t^3, t^2) \mapsto t^3 \end{array}}$$

$$\text{Im } \varphi = \overbrace{k[t^2, t^3]}^{\text{set}} \subseteq k[t], \text{ i.e., } t \text{ 不在 Im } \varphi \text{ 中, 即}$$

view function as a poly

Rmk: 涉及多项式与函数 map 及观点灵活转换.

- A product $X \times Y$ 满足 univ. prop

$$\begin{array}{ccc} Z & \xrightarrow{f_X} & X \\ \pi_Y \downarrow & \nearrow f_Y & \pi_X \\ Y & & \end{array}$$

Routine check.

Note: Need to show
 f we constructed is a mor

Rmk: 不选乘积 topo 的原因. 正因为 $X \times Y$ 赋予 zariski topo, 才会有这样的 univ. prop.

$$\Delta \quad A(X \times Y) = A(X) \otimes_k A(Y)$$

Pf: 由 $A(\cdot)$ 是 fully-faithful co-functor, 上面 univ. prop. 亦有 $A(\cdot)$ 版本, 一望便知.

Rmk: $A(\cdot)$ version 只用在 reduced ring 上 check univ. prop.

• Why def of aff. var. is a ringed space iso to zero locus of poly in A^n ?
 new def. old def.

① 这是内蕴的定义 (不是靠嵌入 A^n 来定义的)

② Consider exp.: 嵌入 A^n 方式不同但同构的例子. (因此定义不应该涉及对 A^n 的嵌入方式)

Idea: 构造同构 aff. var., 只需构造 $A(X) \cong A(Y)$. 我们选定 finitely generated reduced

$k\text{-alg } R$, 设 R 生成元为 a_1, a_2, \dots, a_n . 考虑.

No nilpotents

$$g: k[x_1, \dots, x_n] \longrightarrow R$$

$$f \longmapsto f(a_1, a_2, \dots, a_n)$$

$$g \text{ surj.}, \text{ so } k[x_1, \dots, x_n]/\ker g \cong R.$$

$\ker g$ is radical, because R is reduced.

$$(f^k \in \ker g \Rightarrow f^k(a_1, \dots, a_n) = 0 \stackrel{R \text{ reduced}}{\Rightarrow} f(a_1, \dots, a_n) = 0 \Rightarrow f \in \ker g)$$

$$\text{Then } X = V(\ker g) \text{ is an aff. var. with } A(X) = k[x_1, \dots, x_n]/\underbrace{\ker g}_{I(X)} = R.$$

Different choice of generators can get different aff. var. X .

But all these X has same coordinate ring so they're all iso.

$$R = k[t]. \quad \text{Choice 1: } R = \langle t \rangle$$

$$\rho: k[x_i] \longrightarrow R$$

$$f(x) \longmapsto f(t)$$

$$\ker \rho = 0 \text{ so } X = V(0) = A^1$$

$$\text{Choice 2: } R = \langle t, t^2, t^3 \rangle$$

$$\rho: k[x_1, x_2, x_3] \longrightarrow R$$

$$f \longmapsto f(t, t^2, t^3)$$

$$\ker \rho = \langle x_1^2 - x_2, x_1^3 - x_3 \rangle$$

$$X = V(\ker \rho) = \{(x_1, x_2, x_3) \mid x_i \in k\}$$

iso to A^1 . But it's in A^3 !

* This example shows

$$\{\text{aff. var.}\}/\text{iso} \xleftrightarrow{1:1} \{\text{finitely generated reduced } k\text{-alg}\}/\text{iso}$$

$$X \longmapsto A(X)$$

$$V(\ker \rho) \longleftrightarrow R$$

$$\text{where } \rho: k[x_1, \dots, x_n] \rightarrow R$$

$$f \longmapsto f(a_1, \dots, a_n)$$

a_1, \dots, a_n are generators of R .

△ $A(X) = \bigcup_{x \in X} (X)_x$, when we do not embedding X in A^n in our new def.

△ X aff. var., $f \in A(X)$, $D(f)$ is an aff. var. with coordinate ring

$$A(D(f)) = \bigcup_{D(f)} (D(f)) = \bigcup_x (D(f)) = A(X)_f$$

$$\text{In particular, } A(\underbrace{A^1 \setminus 0}_{A^2}) = A(D(x_1)) = A(A^1)_{x_1} = k[x_1]_{x_1}$$

* 任何 open U 可以被有限 aff. patch $D(f_i)$ cover

Pf: Idea: 通过 $x \in \mathbb{A}^2$: 增加一维, 把 $D(f) \subseteq X$ 嵌入到 $X \times \mathbb{A}^1$ 中写成 zero loci.

Let $Y = V(t f(x) - 1) \subseteq X \times \mathbb{A}^1$ $t f(x) - 1 = 0 \Rightarrow t = \frac{1}{f(x)} \Rightarrow Y$ 是 $D(f)$ 上的函数图像(graph),
 $\Rightarrow Y \cong D(f)$.
 $\underline{Y \cong D(f)}$ 故 $D(f)$ 是 aff. var.
 $\underline{\text{aff. var.}}$

$$g: Y \rightarrow D(f) \quad g^{-1}: D(f) \rightarrow Y \\ (x, t) \mapsto x \quad x \mapsto (x, \frac{1}{f(x)})$$

$\Delta \setminus \{0\}$ is not aff. Let $U = \mathbb{A}^2 \setminus \{0\}$, $X = \mathbb{A}^2$. Assume U aff. var.

$$A(U) = \mathcal{O}_U(U) = \mathcal{O}_X(U) = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) \underset{q}{=} \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) = k[x_1, x_2] = A(X) \Rightarrow X \cong U \rightarrow \text{从 } \mathbb{A}^2 \setminus \{0\} \text{ 上 regular fun 可以延拓到 } \mathbb{A}^2$$

- Manifold \Rightarrow Locally Euclidean

Prevariety \Rightarrow Locally affine

\Downarrow

Ringed space with finite open cover by aff. vars.

Note: prevariety 的定义只要求存在 finite open aff. cover, 而 open cover 本身并不是 prevariety 的 data.

* Open subset of aff. var. may not be aff. any more, but it can be covered by finite aff. open sets $D(f_i)$, so it's a prevar.

- 把一堆 prevar. 粘起来得到 prevar.

$\{X_i\}_{i \in I}$ be a set of prevar. We are given $U_{i,j} \subseteq X_i$ for all $i \neq j$ and

iso $f_{i,j}: U_{i,j} \rightarrow U_{j,i}$ s.t.

$$(a) f_{j,i} = f_{i,j}^{-1} \quad (b) U_{i,j} \cap f_{i,j}^{-1}(U_{j,k}) \subset U_{i,k} \text{ and } f_{i,k} \circ f_{i,j} = f_{j,k} \text{ on } U_{i,j} \cap f_{i,j}^{-1}(U_{j,k})$$

$$\begin{aligned} a \sim f_{i,j}(a) &\sim f_{j,i} f_{i,j}(a) \\ &= f_{j,i} f_{i,j}(a) = a \end{aligned}$$

(1) (2): 反身性

(2): 对称性

$$a \sim f_{i,j}(a), \quad f_{i,j}(a) \sim f_{j,k} f_{i,j}(a),$$

$$\text{then } a \sim f_{j,k}(a) = f_{j,k} f_{i,j}(a), \quad \text{传递性 满足}$$

(a) (b) 只为 } 说明 $a \sim f_{i,j}(a)$ 是等价关系)

We construct prevariety obtained by glueing X_i along the iso $f_{i,j}$ as following:

(1) As a set, $X = \coprod_{i \in I} X_i / \sim$, $a \sim b \Leftrightarrow \exists f_{i,j}, b = f_{i,j}(a)$

(b) As a topo space, it's defined by quotient topo, i.e., $\pi_i: X_i \rightarrow X$

$U \subseteq X$ open $\Leftrightarrow \pi_i^{-1}(U)$ open in X_i for all $i \in I$

(c) As a ringed space, $\mathcal{O}_X(U) = \{\varphi: U \rightarrow k \mid \varphi^* \varphi \in \mathcal{O}_{X_i}(\pi_i^{-1}(U)) \text{ for all } i\}$

Idea: 都是拉回到每一块上去定义

- Prevar. is Noetherian, 所有 Noetherian space 的结论它都满足.

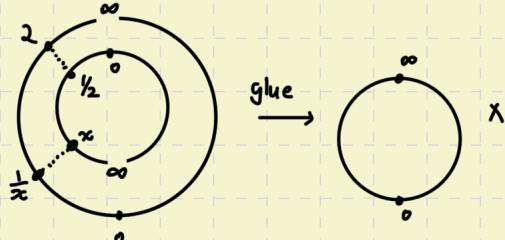
• EXAMPLE: $X_1 = X_2 = \mathbb{A}^1$ $U_{1,2} = U_{2,1} = \mathbb{A}^1 \setminus \{0\}$

不同 $f: U_{12} \rightarrow U_{21}$ 给出不同粘连规则, 从而得到不同 X .

Choice 1: $f: U_{12} \rightarrow U_{21}, x \mapsto \frac{1}{x}$

$$\begin{array}{c} x_1 \sim x_2 \Leftrightarrow x_2 = \frac{1}{x_1} \\ \uparrow \quad \uparrow \\ U_1 \quad U_2 \end{array}$$

$$\begin{aligned} \text{从图中看 } X &= \mathbb{A}^1 \cup \{\infty\} \\ &= \mathbb{A}^1 \cup \mathbb{P}^1 \\ &= \mathbb{P}^1 \end{aligned}$$



More details: ① When $\mathbb{A}^1 = \mathbb{A}_C^1$, $X = \mathbb{C} \cup \{\infty\}$, real pts is a circle.

② $x_1 \sim \frac{1}{x_1}$ 实际上在说 $[x_1, 1] \sim [1, \frac{1}{x_1}]$, 正是 \mathbb{P}^1 中的等价关系

Application of glueing prop for mors: 构造 $X \rightarrow X \xleftarrow{\text{glue prop}} \text{构造 } \{x_1 \rightarrow x \mid x_2 \rightarrow x\}$ (可以不满)

$\begin{cases} x_1 \rightarrow x_2, x \mapsto x \\ x_2 \rightarrow x_1, x \mapsto x \end{cases}$ glue to a mor $X \rightarrow X$

这个 mor 是 $X \rightarrow X, x \mapsto \frac{1}{x}$ ($\frac{1}{0} = \infty, \frac{1}{\infty} = 0, X = \mathbb{A}^1 \cup \{\infty\}$)

$$\begin{array}{ll} X_1 \rightarrow X_2 \text{ 实际上是 } & X_1 \rightarrow X_2 \\ x \mapsto x & (x)_{x_1} \mapsto (x)_{x_2} = (\frac{1}{x})_{x_1} \\ (0)_{x_1} \mapsto (0)_{x_2} = (\infty)_{x_1} & (x)_{x_2} \mapsto (x)_{x_1} = (\frac{1}{x})_{x_2} \\ \text{两边换到同一坐标} & (0)_{x_2} \mapsto (0)_{x_1} \\ \text{且通过 } x \mapsto \frac{1}{x} & (\infty)_{x_1} \end{array}$$

只看 x_i 坐标且记 $(\infty)_{x_1} = \infty$, 并认为 $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$, 则有 $x \mapsto \frac{1}{x}$

Choice 2: $f: U_{12} \rightarrow U_{21}, x \mapsto x$ be id.

$$\begin{array}{ccc} \frac{x}{x_1} \cdots \frac{0}{x_2} & \xrightarrow{\text{glue}} & \frac{\cdot}{\cdot} \end{array} \quad \text{It's an aff. line with double zeros.}$$

$$\begin{array}{l} \begin{cases} X_1 \rightarrow X_2 \subset X, x \mapsto x \\ X_2 \rightarrow X_1 \subset X, x \mapsto x \end{cases} \quad \begin{array}{c} X_1 \rightarrow X_2 \\ x \neq 0 \end{array} \quad \begin{array}{c} X_2 \rightarrow X_1 \\ x \neq 0 \end{array} \\ \text{构造 } X \rightarrow X \xleftarrow{\text{glue prop}} \text{构造 } \{x_1 \rightarrow x \mid x_2 \rightarrow x\} \end{array}$$

(可以不满足)

没有 X_1 中等价类, 没有 X_2 中等价类. 此 mor 是交换零点, 其余点不变

• What kind of subset in prevariety is prevariety?

① open subsets of prevar is prevar.

$$X = \bigcup_{i=1}^n X_i \text{ is aff. open cover. } U \stackrel{\text{open}}{\subseteq} X. \quad U = \bigcup_{i=1}^n X_i \cap U$$

open subsets of aff. var $X_i \cap U$ is prevar, so U is prevar

(as a finite union of prevar)

$\bigcup_U (V) := \bigcup_X (V)$ is nicely worked, because $V \stackrel{\text{open}}{\subseteq} U \stackrel{\text{open}}{\subseteq} X$ means $V \stackrel{\text{open}}{\subseteq} X$

② closed subsets of prevar is prevar.

$Y \stackrel{\text{closed}}{\subseteq} X$. $\mathcal{O}_Y(U) \neq \mathcal{O}_X(U)$, because $U \stackrel{\text{open}}{\subseteq} Y \stackrel{\text{closed}}{\subseteq} X$ doesn't

mean $U \stackrel{\text{open}}{\subseteq} X$. So we define

$$\mathcal{O}_Y(U) = \{ \varphi: U \rightarrow \mathbb{K} \mid \forall a \in U \exists a \in V \stackrel{\text{open}}{\subseteq} X, \varphi \in \mathcal{O}_X(V), \varphi = \varphi \text{ on } U \cap V \}$$

A regular fun on an open subset of closed subprevar Y of a prevar X is locally the restriction of a regular fun on X

③ Mix open and closed may not be a prevariety

$$X = \mathbb{A}^2 \quad U_1 = \mathbb{A} \times (\mathbb{A} \setminus 0) \quad \text{open}$$

$$U_2 = \{0\} \quad \text{closed}$$

$$U_1 \cup U_2$$



假如它是 prevar, 则 0 处必存在 aff. var. cover 它.
但 0 附近不存在 aff. var. cover, 因为 0 附近显然
没有闭子集 (极限点不包含在里面)

• Products of prevars:

* 若使用 glueing 来定义 product, 过程会非常复杂, 而且还要证明所定义的 product

与 glueing 无关. 这时就显出用 univ. prop. 定义 product 的简单了.

△ product of prevar X and Y is a prevar P 满足 univ prop

$$\begin{array}{ccc} Z & \xrightarrow{f_X} & X \\ f_Y \swarrow & \lrcorner \pi_Y \downarrow & \\ Y & & \end{array}$$

△ Existence and uniqueness of products

• Separate : Hausdorff nonsense 的解决办法.

△ Intuition picture: (1) In Classical topo, Hausdorff $\Leftrightarrow \Delta_X \stackrel{\text{closed}}{\subseteq} X \times X$

In classical topo we use 乘积拓扑

$$X = \frac{\bullet b}{\bullet a} \quad \text{aff. line with double zeros is not sep.}$$

$X \times X$ 有 4 个 zeroes $(a, b), (b, a), (a, a), (b, b)$

Δ_X 只含 $(a, a), (b, b)$, 从而 Δ_X 不 closed in $X \times X$

(2) Separate means every sequence has at most one limit

$$\lim_{n \rightarrow \infty} (x_n, y_n) = (a, b) \quad \& \quad \Delta_X \text{ 说明 } \Delta_X \not\subseteq X \times X$$

Δ_X 中一个序列的极限

△ Def: A prevar X is called a variety (or separated) if the diagonal $\Delta_X \stackrel{\text{closed}}{\subseteq} X \times X$ Zariski topo

• 一些 var 的例子

$$(1) \text{aff. vars are vars. } \Delta_x = V(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n) \stackrel{\text{closed}}{\subseteq} X \times X \\ (\underbrace{(x_1, \dots, x_n)}_{\substack{\text{open} \\ \text{or} \\ \text{closed}}}, \underbrace{(y_1, \dots, y_n)}_{\text{closed}})$$

(2) Open and closed subvars are vars, called open and closed subvars.

$$Y \subseteq X \Rightarrow Y \text{ is prevar} \Rightarrow Y \times Y \text{ prevar} \Rightarrow i: Y \times Y \rightarrow X \times X \\ \text{Unlabeled} \\ \Delta_Y$$

$$\Rightarrow \Delta_Y = i^{-1}(\Delta_X) \text{ is closed} \Rightarrow Y \text{ is var.}$$

(3) A var of pure dim 1 is called curve,
of pure dim 2 is called surface.

A pure dim closed subvar $Y \subseteq X$ with $\dim Y = \dim X - 1$,
then Y is called hypersurface

(4) $f, g: \underbrace{X}_{\text{prevar}} \rightarrow \underbrace{Y}_{\text{var}}$ mor of prevars.

(4.1) The graph $\Gamma_f := \{(x, f(x)) | x \in X\}$ is closed in $X \times Y$

By univ prop of product, there is $(f, id): X \times Y \rightarrow Y \times Y$
 $\text{mor } A \rightarrow X \times Y \Leftrightarrow \begin{cases} A \rightarrow X \\ A \rightarrow Y \end{cases} \quad (x, y) \mapsto (f(x), y)$

$\Gamma_f = (f, id)^{-1}(\Delta_Y)$ closed by conti.

(4.2) The set $\{x \in X : f(x) = g(x)\}$ is closed in X

By univ prop of product, $(f, g): X \rightarrow Y \times Y$
 $x \mapsto (f(x), g(x))$

The set equals to $(f, g)^{-1}(\Delta_Y)$ is closed in X .

(5) $X, Y \subseteq \mathbb{A}^n$ are pure-dim aff. var., then every irr component
of $X \cap Y$ has dim at least $\dim X + \dim Y - n$.

• Intuitive picture of Projective variety

值不定! 不是 P^n 上的函数! 只有零点固定
 $f(x_1, \dots, x_n) = 0 \Leftrightarrow f(x_1, \dots, x_n) = 0$

不齐次则这个齐次坐标系的多项式的零点不定

(1) 类似 aff. var \subseteq aff. space \mathbb{A}^n , proj var = 齐次多项式的零点 \subseteq proj space P^n

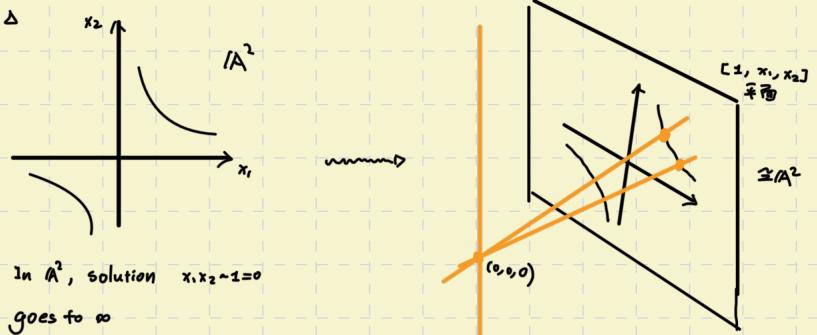
(2) P^n 是 vect space K^{n+1} 中 1-dim vect spaces 构成的 set. $P^n = (K^{n+1} \setminus \{0\}) / \sim$

(3) $[0, 0, 0] \notin P^n$: show a more well-defined 对必须已 check.
 不能作直线连来

(4) 把 ∞ 添进去, Local 描述: 一堆 aff. var. 通过 gluing 得 var.

globally 描述: proj. var.

projective var. 如何 include as pt? Global 描述 P^n 更简洁

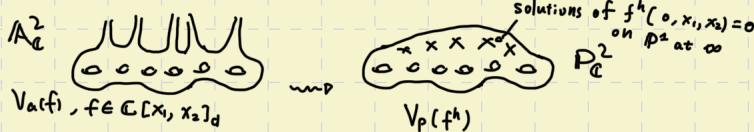


$$\Delta \mathbb{P}^n = \left\{ (1 : x_1 : \dots : x_n) \right\} \cup \left\{ (0 : x_1 : \dots : x_n) \right\} = \mathbb{A}^n \cup \mathbb{P}^{n-1}$$

\mathbb{A}^n open \mathbb{P}^{n-1} closed
 aff. part pts at infinity

互为补集.

$\Delta \mathbb{P}^n$ includes ∞ 的示意图



(5) 由于 proj var 添加了 ∞ pt, 在 proj var 中非常容易相交.

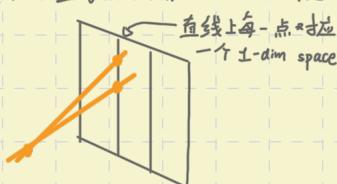
prop: $X, Y \subseteq \mathbb{P}^n$ are proj var. then $\dim X \cap Y \geq \dim X + \dim Y - n$

例 | 如 $X, Y \subseteq \mathbb{P}^2$, $\dim X = \dim Y = 1$. $1+1=2$, $\dim X \cap Y \geq 0$ X, Y 相交

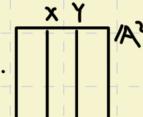
假设 X, Y 是在同一 aff. patch 上的直线

	$X = \{(1:0:x_2)\}$ $\lim_{x_2 \rightarrow \infty} (1:0:x_2) = \lim_{x_2 \rightarrow \infty} (\frac{1}{x_2}:0:1) = (0:0:1)$ $Y = \{(1:1:x_2)\}$ $\lim_{x_2 \rightarrow \infty} (1:1:x_2) = \lim_{x_2 \rightarrow \infty} (\frac{1}{x_2}, \frac{1}{x_2}:1) = (0:0:1)$
--	---

Rmk: 图中看起来 X, Y 是两条直线, 实际上在 \mathbb{P}^2 中是



It's not true in aff. var. \mathbb{A}^2 中两个 1-dim aff. var. 可以不交.



- Graded ring $\{R = \bigoplus_{d=0}^{\infty} R_d \Rightarrow f = \sum_d f_d \text{ (有限和)}$
- $f \in R_d, g \in R_e \Rightarrow fg \in R_{d+e}$

elements of $R_d \setminus \{0\}$ are said to be homogeneous of deg d

$$\text{Exp: } k[x_0, \dots, x_n] = \bigoplus_d k[x_0, \dots, x_n]_d$$

• Homogeneous ideal: special ideals in homogeneous ring

Δ Intuitive picture: It must contain non-homog elements
since $\text{齐次} \cdot \text{不齐次} = \text{不齐次} \in J$

$$\text{ideal } J \subset R$$

$$\text{e.g. } J = \langle x^2 \rangle \subseteq k[x] \quad x^2(2+x) = \underline{2x^2 + x^3} \in J$$

non-homog.

Δ Def: An ideal in a graded ring is called homogeneous if

it can be generated by homogeneous elements, i.e., $J = \langle f_1, \dots, f_s \rangle$, f_i homog.

• props for **homo** ideals

(1) The ideal J is **homo** iff $\forall f \in J$ with **homo** decomposition

$f = \sum_a f_a$ we also have $f_a \in J$ for all a .

Pf: \Rightarrow Let $J = \langle h^i : i \in I \rangle$, h^i homo $\xrightarrow{f \in J} f = \sum_{i \in I} g_i h^i$ for some $g_i \in R$

$g^i = \sum_{e \in N} g_e^i$ be **homo** decomposition of g^i , then degree d part of f is

$$f_d = \sum_{\substack{e \in I \\ e + \deg h^i = d}} g_e^i h^i \oplus \langle h^i \rangle \text{生成} \nexists f_d \in J$$

\Leftarrow Claim $J = \langle h_d : h \in J, d \in N \rangle$

' \subseteq ' $\forall f \in J$, f can generated by f_d .

' \supseteq ' $h_d \in J$, $\forall h \in J, d \in N$ by assumption.

(2) J_1, J_2 homo then $J_1 + J_2, J_1 J_2, J_1 \cap J_2, \sqrt{J_1}$ are also homo

Pf: $J_1 + J_2 = \langle J_1 \cup J_2 \rangle$ is also generated by **homo** elements $\Rightarrow J_1 + J_2$ is homo

(\Rightarrow \Leftarrow) $J_1 J_2 = \langle h_i g_j : h_i, g_j \text{ 分别是 } J_1, J_2 \text{ 的 } \text{homo} \text{ 生成元} \rangle \Rightarrow J_1 J_2$ is homo

For $f = \sum_a f_a$, $f_a \in J_1$, $f_a \in J_2$ by (1), so $f_a \in J_1 \cap J_2 \Rightarrow J_1 \cap J_2$ is homo

$f = \sum_a f_a \in \sqrt{J_1} \Rightarrow (f_0 + \dots + f_d)^n = \underbrace{f_0^n}_{\deg f_0 = d} + (\text{terms of lower degree}) \in J_1$

By (1), $f_0^n \in J_1 \Rightarrow f_0 \in \sqrt{J_1}$. $f - f_0 = f_1 + \dots + f_{d-1}$ lies in $\sqrt{J_1}$ as well, by induction we have $f_0, f_1, \dots, f_d \in \sqrt{J_1} \Rightarrow \sqrt{J_1}$ is homo

(3) Graded ring 的商环仍然 graded. $R/J = \bigoplus_{d \in N} R_d / (R_d \cap J)$ 由 **homo** ideal

Pf: w.t.s. $\forall \bar{f} \in R/J \exists!$ decomposition $\bar{f} = \sum_a \bar{f}_a$, $\bar{f}_a \in R_a \cap J$

存在性由 $f = \sum f_d \in R$ induce, 唯一性: 设 $\bar{f} = \sum_a \bar{f}_a = \sum_b \bar{g}_b$.

$$\overline{\sum (f_a - g_a)} = \sum \bar{f}_a - \sum \bar{g}_a = 0 \Rightarrow \sum (f_a - g_a) \in J \xrightarrow{J \text{ homo}} f_a - g_a \in J \Rightarrow \bar{f}_a = \bar{g}_a$$

$\Delta V_p(S)$: proj var. $S \subset k[x_0, \dots, x_n]$ a set of **homo** polys

$$V_p(J) = \{x \in \mathbb{P}^n \mid f(x) = 0, \forall f \in J\}$$

$\langle S \rangle = J$ is **homo** ideal. $V(J) \subseteq V(S)$

$\Delta V_p(f_1, \dots, f_r) \subseteq \mathbb{P}^n$, $f_1, \dots, f_r \in k[x_0, \dots, x_n]$ **homo** linear poly is a linear subspace of \mathbb{P}^n .

$\Delta I_p(X) = \langle f \in k[x_0, \dots, x_n] \text{ homo: } f(x) = 0 \forall x \in X \rangle$ is a **homo** ideal in $k[x_0, \dots, x_n]$

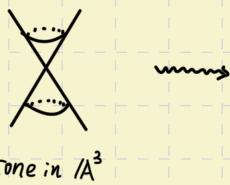
Δ Exp of sub proj. var. (省略下标)

$$Y = V(x_0 + x_1 + x_2) \subseteq \mathbb{P}^2$$

$$\begin{aligned} V_f(-x_0^2 + x_1^2 + x_2^2) &= V(x_0 + x_1 + x_2, -x_0^2 + x_1^2 + x_2^2) \\ &= V(x_0 + x_1 + x_2, \underbrace{-(x_1 - x_2)^2 + x_1^2 + x_2^2}_{\text{II}}) \\ &= \{(-1:0:1), (-1:1:0)\} \end{aligned}$$

• Cone Intuitive picture:

$$(1) V_a(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$$



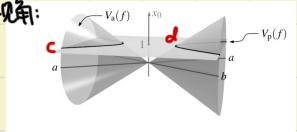
$$V_p(x^2 + y^2 - z^2) \subseteq \mathbb{P}_R^2$$



$Cone \subseteq \mathbb{A}^{n+1}$ 是 \mathbb{P}^n
中的 set 每一点长直线得到的

$$(2) f = x_1^2 - x_2^2 - x_0^2 \in \mathbb{C}[x_0, x_1, x_2]$$

\mathbb{A}^3 视角:



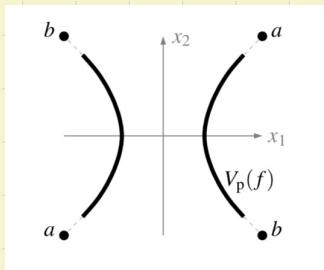
灰色部分是 \mathbb{A}^3 中的 cone

\mathbb{P}^2 视角: 这个 cone 上所有直线是 $V(f)$

c, d 是 cone 上每条直线取代表元

c, d, a, b 合起来是 $V(f)$. 容易看出

a, b 是无穷远点



• Cones and projection

Def: $\mathbb{A}^n \supseteq \text{Cone } X \setminus \{\text{origin } \mathbf{0} \in X\}$
a union of lines through $\mathbf{0}$. $\forall x \in X, \lambda \in \mathbb{K}, \lambda x \in X$

Def:

$$\pi: \mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{P}^n, (x_0, \dots, x_n) \mapsto (x_0: \dots : x_n)$$

(1) For a cone $X \subset \mathbb{A}^{n+1}$, we call

$$\mathbb{P}(X) := \pi(X \setminus \mathbf{0}) = \{(x_0: x_1: \dots: x_n) \in \mathbb{P}^n \mid (x_0, \dots, x_n) \in X\} \subseteq \mathbb{P}^n$$

the projection of X

(2) For a projective var $Y \subseteq \mathbb{P}^n$, we call

$$C(Y) := \{\mathbf{0}\} \cup \pi^{-1}(Y) = \{\mathbf{0}\} \cup \{(x_0, \dots, x_n) \mid (x_0: \dots: x_n) \in Y\} \subseteq \mathbb{A}^{n+1}$$

the cone over Y

• Every cone has the form $V_a(S)$ for some $S \subseteq \mathbb{K}[x_0, \dots, x_n]$ a set of homo poly

(Every aff. var. has form $V_a(H)$, $H \subseteq \mathbb{K}[x_0, \dots, x_n]$)
(Every proj. var. has form $V_p(S)$, $S \subseteq \mathbb{K}[x_0, \dots, x_n]$)

Pf: $\Leftarrow S \subseteq \mathbb{K}[x_0, \dots, x_n]$ a set of homo poly, then $V_a(S)$ is a cone

for any $x \in V_a(S) \Rightarrow \forall f \in V_a(S)$ satisfy $f(x) = 0 \Rightarrow \forall \lambda \in \mathbb{K}, f(\lambda x) = \lambda^{\deg f} f(x) = 0, \forall f \in V_a(S) \Rightarrow \lambda x \in V_a(S)$

\Rightarrow Let X be a cone, it's an aff. var so we can use tools in aff. var.

Show $X = V_a(S)$ is equiv to show $I(X) = \langle S \rangle$ is a homo ideal

For any $f \in I(X)$, $f = \sum_{d \in \mathbb{N}} f_d$. w.t.s. $f_d \in I(X)$ (By prop of homo ideal)

X is a cone, so $\forall \lambda \in \mathbb{K}, \forall x \in X, \lambda x \in X$. By $f \in I(X)$, $f(\lambda x) = 0$.

so $f(\lambda x) = \sum f_d(\lambda x) = \sum \lambda^d f_d(x) = 0$ holds for $\forall \lambda, \forall x \Rightarrow f_d(x) = 0 \quad \forall x \in X \Rightarrow f_d \in I(X)$

• Prop: $\dim C(X) = \dim X + 1$, $X \subseteq \mathbb{P}^n$ is a proj. var.

• Transform a proj var to an aff. var

(非常多问题要用 $C(\text{projvar})$ 化到 aff. var. 来证明)

Bijection: $\{ \text{Cones in } A^{n+1} \} \xleftrightarrow{1:1} \{ \text{proj. vars in } P^n \}$

$$X \longmapsto P(X)$$

$$C(Y) \longleftrightarrow Y$$

Pf: Idea: We've known forms of cones & proj. vars!

\forall cone $X \subseteq A^{n+1}$, $X = V_a(S)$ for some $S \subseteq k[x_0, \dots, x_n]$ homo

$$\{ \text{Cones} \} \rightarrow \{ \text{proj. vars} \} \rightarrow \{ \text{Cones} \}$$

$$\begin{aligned} V_a(S) &\mapsto V_p(S) \mapsto V_a(S) \\ " &\qquad\qquad\qquad " \\ P(V_a(S)) &\qquad\qquad C(V_p(S)) \end{aligned}$$

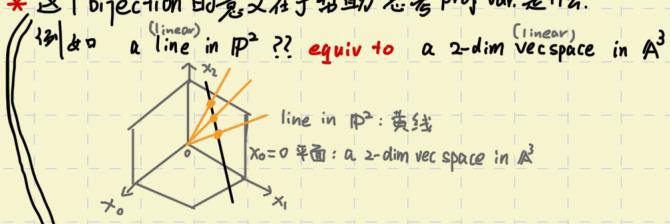
都是 identity
they are bijection

$$\{ \text{proj. vars} \} \rightarrow \{ \text{cones} \} \rightarrow \{ \text{proj. vars} \}$$

$$V_p(S) \mapsto V_a(S) \mapsto V_p(S)$$

$$C(V_p(S)) \qquad P(V_a(S))$$

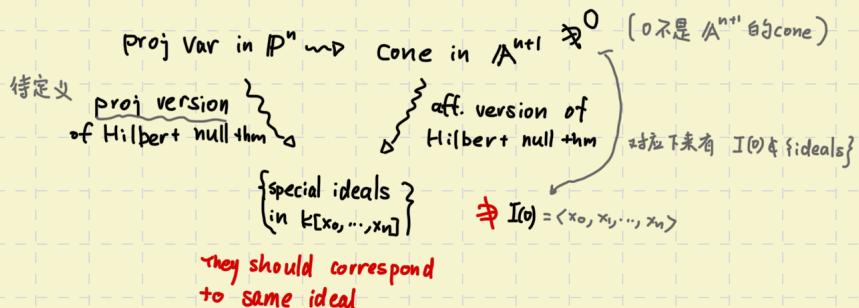
* 这个 bijection 的意义在于帮助思考 proj. var. 是什么.



* 所有 proj. var. in P^n 并不神秘, 它就是熟悉的 A^{n+1} 中的特殊 aff. var. — cone!

• proj version Hilbert Null thm

△ Only difference: (Irrelevant ideal)



因此, 借用 aff. var. 的观点, 我们猜到 $I(0) = \langle x_0, x_1, \dots, x_n \rangle$ 应该不会有 proj. var. 与之对应

Def: The radical homo ideal $I_0 = \langle x_0, x_1, \dots, x_n \rangle$ is called the irrelevant ideal.

△ Proj. version Hilbert Null thm

a) For any proj. var. $X \subseteq P^n$, we have $V_p I_p(X) = X$

b) For any homo ideal $J \subseteq k[x_0, \dots, x_n]$ with $\sqrt{J} \neq I_0$, we have $I_p V_p(J) = \sqrt{J}$

There is a bijection

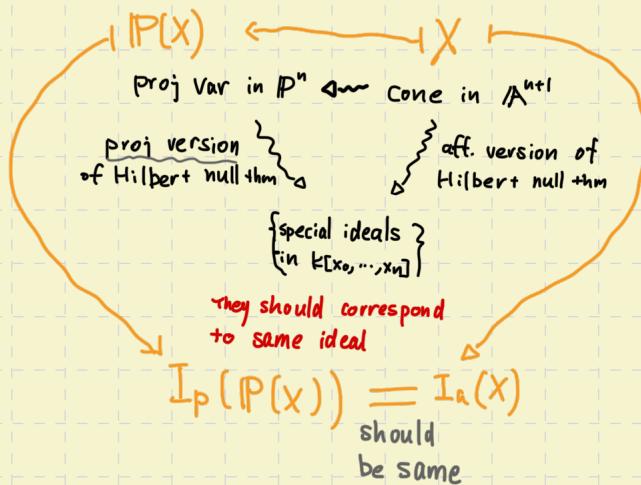
$$\{ \text{proj. var. in } P^n \} \xleftrightarrow{1:1} \{ \text{homo radical ideals in } k[x_0, \dots, x_n] \}$$

not equal to irrelevant ideal

$$X \longmapsto I_p(X)$$

$$V_p(J) \longmapsto J$$

pf: VERY useful bijection for helping consider proj. var. :



Claim: (1) $X \neq 0, I_p(P(X)) = I_a(X)$

(2) $P(V_a(J)) = V_p(J)$ we've proved it

$P(\phi) = \phi$ obviously

If the claim is true, then

$$\text{pf for (1): } \forall X \neq 0, V_p I_p(X) = V_p \underbrace{I_p P C(X)}_{\text{元中生有}} = V_p I_a C(X) = P V_a I_a C(X) = P C(X) = X$$

$$X = \phi \quad V_p I_p(X) = V_p I_p(\phi) = V_p(k[x_0, \dots, x_n]) = \phi$$

$$\text{pf for (2): } \forall \text{ homo radical ideal } J \text{ with } \sqrt{J} \neq I_0, I_p V_p(J) = \underbrace{I_p P V_a J}_{\text{把 } I_p, V_p \text{ 与 } I_a, V_a \text{ 联系起来}} = I_a V_a(J) = \sqrt{J}$$

Finally, pf for claim (1):

\subseteq $\forall f \in I_p(P(X)), f(x) = 0, \forall x \in P(X)$. f is homo, hence $f(ax) = 0, \forall a \in k, \forall x \in P(X)$.

Besides, cone $X \neq 0$, then $P(X) \neq \phi$. so $f \neq \text{常数}$. With f homo, $f(0) = 0$.

Since $\{\exists x \mid x \in P(X), x \in k\} \cup \{0\} = X$, we have $f(x) = 0, \forall x \in X$. So $f \in I_a(X)$.

$$\begin{aligned} \supseteq & \forall f \in I_a(X) \Rightarrow \begin{cases} f = \sum_{d \in N} f_d, f_d \in I_a(X) & (I_a(X) \text{ is homo ideal}) \\ f(x) = \sum_{d \in N} f_d(x) = 0, \forall x \in X & (f \in I_a(X)) \end{cases} \\ & \xrightarrow{\text{由 } f_d \text{ 是 } X \text{ 上的}} f_d(x) = 0, \forall d \in N, \forall x \in X \Rightarrow f_d(\bar{x}) = 0, \forall d \in N, \bar{x} \in P(X) \quad (\bar{x} \in X) \\ & \text{with } f_d \text{ homo} \Rightarrow f_d \in I_p(P(X)) \Rightarrow f = \sum_{d \in N} f_d \in I_p(P(X)) \end{aligned}$$

• some differences from aff. case

(1) Prop of V_p and I_p are same as V_a and I_a BUT

$$I_p(x_1 \wedge x_2) = \sqrt{I_p(x_1) + I_p(x_2)} = I_0 \text{ 则说明 } x_1 \wedge x_2 = \phi$$

$$\text{e.g. } X_1 = V_p(x_0), X_2 = V_p(x_1), \text{ then } X_1 \wedge X_2 = \phi \text{ and } \sqrt{I_p(x_1) + I_p(x_2)} = \sqrt{\langle x_0, x_1 \rangle} = \sqrt{\langle x_0, x_1 \rangle} = \langle x_0, x_1 \rangle$$

(2) $Y \subseteq \mathbb{P}^n$, coordinate ring $S(Y) = k[x_0, \dots, x_n]/I_p(Y)$

Can NOT be viewed as functions on Y .

(homo coord 变化则函数值变化)

- proj. var. 用 $V_p(J)$ 赋予 zariski topo, 立刻可以证明

\mathbb{P}^n is irr of dim- n

prop : $\{U_i\}_{i \in I}$ is an open cover of topo space X ,
then $\dim X = \sup \{\dim U_i | i \in I\}$

we have open cover $\mathbb{P}^n = \bigcup_{i=0}^n D(x_i)$, $D(x_i) \cong \mathbb{A}^n$, so $\dim D(x_i) = n$.

Hence by prop $\dim \mathbb{P}^n = n$.

- 下面许多关于 proj. var. 的命题证明都需要新工具：齐次化 & 去齐次化

dehomogenization (去齐次化) : $\forall \text{ homo } f \in k[x_0, \dots, x_n]$, dehomogenization $f^i := f(x_0=1)$

$\forall \text{ homo ideal } J \subset k[x_0, \dots, x_n]$, dehomogenization ideal $J^i := \{f^i : f \in J\}$

dehomogenization is a ring homo $(f, g)^i = f^i, g^i$ and $(f+g)^i = f^i + g^i$

homogenization (齐次化) : $\forall f \in k[x_0, \dots, x_n]$, homogenization $f^h := x_0^d f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$

$\forall J \subset k[x_0, \dots, x_n]$, homogenization ideal $J^h = \langle f^h : 0 \neq f \in J \rangle$

homogenization is not a ring homo $(f, g)^h = f^h, g^h$ and $(f+g)^h \neq f^h + g^h$

- $\mathbb{A}^n \cong D(x_0) \hookrightarrow \mathbb{P}^n$ 有子空间拓扑 } 二者一样吗?
 \mathbb{A}^n 用 $V(J)$ 定义 Zariski topo.

Claim : (1) $V_p(J) \cap \mathbb{A}^n = V_a(J^i)$ $J \cong^{\text{homo}} k[x_0, \dots, x_n]$

(2) $V_a(J) = V_p(J^h) \cap \mathbb{A}^n$ $J \subset k[x_0, \dots, x_n]$

(Note that $\mathbb{A}^n \cong D(x_0) \hookrightarrow \mathbb{P}^n$)

If the claim is true, then Zariski close $V_a(J)$ is also
subtopo closed $V_p(J^h) \cap \mathbb{A}^n$. Subtopo closed $V_p(J) \cap \mathbb{A}^n$
is also Zariski closed $V_a(J^i)$. Hence two topo are same.

$$\begin{aligned} \text{pf for (1)} \quad V_p(J) \cap \mathbb{A}^n &= \mathbb{P}V_a(J) \cap \mathbb{A}^n = \mathbb{P}V_a(J) \cap D(x_0) \\ &= \left\{ (x_0 : \dots : x_n) \mid (x_0, \dots, x_n) \in V_a(J) \right\} \cap D(x_0) \\ &\stackrel{\text{把 } I_p, V_p \text{ 与 } I_a, V_a \text{ 关联起来}}{=} \left\{ (1 : x_1 : \dots : x_n) \mid (1, x_1, \dots, x_n) \in V_a(J) \right\} \\ &\stackrel{\mathbb{P} V_a = V_p}{=} \left\{ (x_1, \dots, x_n) \mid (x_0, x_1, \dots, x_n) \in V_a(J) \right\} = V_a(J^i) \end{aligned}$$

$$\begin{aligned} V_p(J^h) \cap \mathbb{A}^n &= \left\{ (x_0 : \dots : x_n) \in \mathbb{P}^n \mid (x_0, \dots, x_n) \in V_p(J^h) \right\} \cap D(x_0) \\ &= \left\{ (1 : x_1 : \dots : x_n) \in \mathbb{P}^n \mid (1, x_1, \dots, x_n) \in V_p(J^h) \right\} \\ &= \left\{ (1 : x_1, \dots, x_n) \in \mathbb{P}^n \mid \forall f(x_1, \dots, x_n) \in J, f^h(x_0, x_1, \dots, x_n)_{|_{x_0=1}} = 0 \right\} \\ &= \left\{ (x_1, \dots, x_n) \in \mathbb{A}^n \mid \forall f(x_1, \dots, x_n) \in J, f^{hi} = f(x_1, \dots, x_n) = 0 \right\} \\ &= V_a(J) \end{aligned}$$

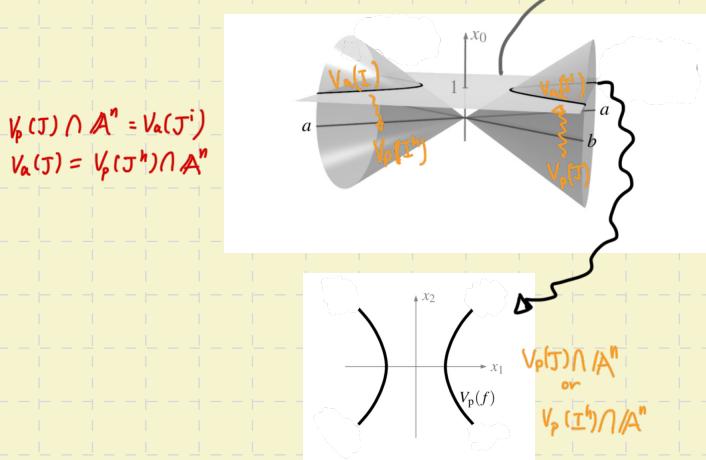
* Geometric picture of homogenization & dehomogenization?

Why we need these tools?

$V_p(J)$ 与 $C(J)$ 给出 $\{\text{proj var in } \mathbb{P}^n\}$ 与 $\{\text{cone in } A^{n+1}\}$ 之间的 bijection

J^h 与 J^i 给出 $\{\text{proj var in } \mathbb{P}^n\}$ 与 $\{\text{aff. var in } A^n\}$ 之间的映射 (不一定 bijective)

考慮 proj var. 的 aff. 截面
 $\text{aff. part } D(x_0) \cong \mathbb{A}^2$



$$\begin{array}{c} \{\text{proj var in } \mathbb{P}^n\} \\ \downarrow \\ \{\text{proj var. of } J^i\} \\ \text{aff. part?} \end{array} \quad \begin{array}{c} V_p(J) \quad V_p(I^h) \\ \downarrow \quad \uparrow \\ V_a(J^i) \quad V_a(I) \end{array}$$

* $J = \langle f_1, f_2, \dots, f_n \rangle$. $J^h \neq \langle f_1^h, f_2^h, \dots, f_n^h \rangle$
 这只对主理想正确. $J = \langle f \rangle$, then $J^h = \langle f^h \rangle$

Pf: \subseteq $J^h = \langle \varphi^h \mid \varphi \in J = \langle f \rangle \rangle$

$$\begin{aligned} &= \langle (fg)^h \mid g \in k[x_1, \dots, x_n] \rangle \\ &= \langle f^h g^h \mid g \in k[x_1, \dots, x_n] \rangle \\ &\subseteq \langle f^h \rangle \end{aligned}$$

* $\forall f^h g \in \langle f^h \rangle$, $f \in J^h$ is an ideal
 $\therefore f^h g \in J^h$. then $\langle f^h \rangle \subseteq J^h$

• 双曲线 & 抛物线都是圆

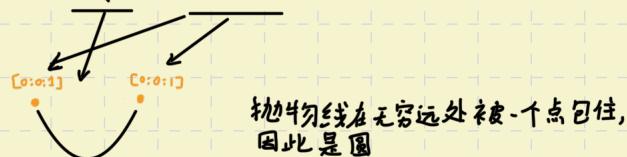
抛物线: $J = \langle x_2 - x_1^2 \rangle \trianglelefteq k[x_1, x_2]$ $J^h = \langle x_2 x_0 - x_1^2 \rangle$

$V_p(J^h)$: 一个 $D(x_0)$ 截面是 $V_a(J)$ 双曲线活在 \mathbb{P}^2 中的 proj. var.

$$V_p(J^h) = V_a(J) \cup \underbrace{[0:0:1]}_{x_0=1} \quad \underbrace{[0:0:1]}_{x_0=0}$$

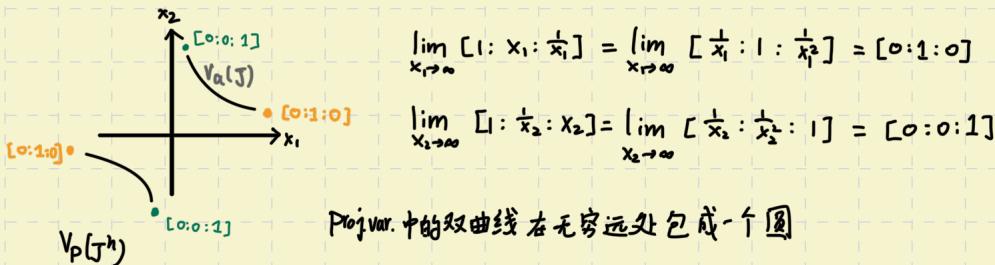
Observe that $\lim_{x_1 \rightarrow \infty} [1 : x_1 : x_1^2] = \lim_{x_1 \rightarrow \infty} [\frac{1}{x_1^2} : \frac{1}{x_1} : 1] = [0 : 0 : 1]$
双曲线上自己的点

$$V_p(J^h) = V_a(J) \cup [0:0:1]$$



双曲线: $J = \langle x_1 x_2 - 1 \rangle \trianglelefteq k[x_1, x_2]$ $J^h = \langle x_1 x_2 - x_0^2 \rangle$

$$V_p(J^h) = V_a(J) \cup \underbrace{[0:0:1]}_{x_0=1} \cup \underbrace{[0:1:0]}_{x_0=0}$$



- Another application of homogenization and dehomogenization:

Computation of the projective closure

Intuitive picture: (1) 把 proj closure 近似于 把射影光源找

到并添加回去的过程. 射影光源一般是无穷远点, 找 closure 也可视作 添加无穷远点

(2) 上面 $V_a(J) \subseteq A^n \subseteq P^n$ 并不 closed, 但 $V_p(J^h)$ closed, 就是添加了一些无穷远点.

prop: (Computation) $J \subseteq k[x_1, \dots, x_n]$ Consider $X = V_a(J) \subseteq A^n \subseteq P^n$, compute its closure \bar{X} in P^n .

$$(a) \bar{X} = V_p(J^h)$$

$$(b) \text{ If } I = \langle f \rangle, f \neq 0, \text{ then } \bar{X} = V_p(f^h)$$

Pf:

$$(a) \leq'' X = V_a(J) = V_p(J^h) \cap A^n \subseteq V_p(J^h). V_p(J^h) \text{ closed}, \text{ so } \bar{X} \subseteq V_p(J^h)$$

$$\geq'' \bar{X} \text{ closed, so } \bar{X} = V_p(I) \text{ for some } I \stackrel{\text{homo}}{\subseteq} k[x_0, x_1, \dots, x_n] \text{ neutral}$$

$$\text{w.t.s. } V_p(J^h) \subseteq V_p(I), \text{ i.e., w.t.s. } \sqrt{J^h} \supseteq I.$$

$\forall g \in I \subseteq k[x_0, x_1, \dots, x_n]$ is an U.F.D., so $g = x_0^{s_0} h$ s.t. h 不含 x_0 .
于是 h 满足 $(h^i)^h$.

By $\bar{X} = V_p(I)$, we have $g(x_0, x_1, \dots, x_n) = 0, \forall (x_0, x_1, \dots, x_n) \in \bar{X}$

In particular $g(x_1, \dots, x_n) = 0, \forall (x_1, \dots, x_n) \in X \subseteq A^n$,

i.e., $g^i(x_1, \dots, x_n) = 0, \forall (x_1, \dots, x_n) \in X$. By $g = x_0^{s_0} h$,
 $h^i = g^i = 0$ on X , so $h^i \in I(X) = \sqrt{V_a(J)} = \sqrt{J}$.

Then, $\exists n \in \mathbb{N}$ s.t. $(h^i)^n \in J$, thus $((h^i)^n)^h \in J^h$

$$((h^i)^n)^h = ((h^n)^i)^h = h^n \in J^h. \text{ So } g^n = x_0^{ns_0} h^n \in J^h$$

i is ring homo Hence $g \in \sqrt{J^h}$.

(b) 由 $J = \langle f \rangle$ 则 $J^h = \langle f^h \rangle$ 易得.

• X is hypersurface in P^n . $I(X)$ is principal ideal

1st way: $Y := X \cap A^n$ is hypersurface in A^n . $I(Y) = \langle g \rangle$.

$$X = \bar{X} = V_p(I(Y)^h) = V_p(\langle g^h \rangle) \Rightarrow I(X) = \langle g^h \rangle$$

2nd way: $I_p(X) = \bigcup_{\alpha} C(X)$ when $X \neq \emptyset$. $I_p(X) = I_p(C(X)) = \langle f \rangle$

• $\text{Aut}(P^n) \cong \text{PGL}(n+1, k) = GL(n+1, k)/\sim$ ($A \sim \lambda A$ for $\lambda \in k^\times$) (类比 $[A] \cdot [B] = [A \cdot B]$)

类比于 $\text{Aut}(A^n) = GL(n, k)$

于是, $\forall a \in P^n, \exists A \in \text{Aut}(P^n)$, s.t. $Aa = [1 : 0 : \dots]$ (Simplify question) ($\text{Aut}(P^n)$ 只有线性元)

• Proj var.s are prevars. (proj. var. 是有 global description 的 pre var.)

△ Structure sheaf: $U \stackrel{\text{open}}{\subseteq} \bigcup_{\text{Proj var}} X$. A regular fun on U is a map

$\phi: U \rightarrow k$ with the following prop: For every $a \in U$, there are

$d \in \mathbb{N}$ and $f, g_a \in S(X)_d$ with $f(x) \neq 0$ on open $U_a \ni a$, s.t.

$$\phi(x) = \frac{g_a(x)}{f_a(x)}$$

We construct a sheaf of regular fun (locally defined, so it's a sheaf)

* 下标 d 表示 d deg elements in $S(X)$. 上下 deg 相同, 使得 φ 真的是一个函数:

$$\frac{f(x)}{g(x)} = \frac{x^d f(x)}{x^d g(x)} = \frac{f(x)}{g(x)}$$

* $\mathcal{O}_X(U)$ is a subring of k -alg { function $\varphi: U \rightarrow k$ }

△ 要证 $X \subseteq \mathbb{P}^n$ is prevar, 只要证 X 有 finite open aff. cover.

显而易见 $X = \bigcup_{i=0}^n (X \cap D(x_i))$ is an open cover of X , it suffices to show $X \cap D(x_i) = \{(x_0 : x_1 : \dots : x_n) \mid x_i \neq 0\}$ aff.

Let $X = V_p(J)$. 下证 $X \cap D(x_0) =: V_0 \cong V_a(J^0) =: Y$, 其余 V_1, \dots, V_n 同理

$$F: Y \longrightarrow V_0 \subseteq \mathbb{P}^n$$

$$(x_1, \dots, x_n) \mapsto [1, x_1, \dots, x_n]$$

把 $V_a(J) \subseteq \mathbb{A}^n$ 嵌到 \mathbb{P}^n 的
aff. part $D(x_0)$ 中

$$F^{-1}: V_0 \longrightarrow Y \subseteq \mathbb{A}^n$$

$$[x_0 : \dots : x_n] \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right)$$

把 X 的 aff. part V_0 放到 aff. space \mathbb{A}^n

对于 $V_0 = X \cap D(x_0)$

$$F^{-1}: V_0 \rightarrow Y$$

$$[1 : x_1 : \dots : x_n] \mapsto (x_1, \dots, x_n)$$

a) Check F, F^{-1} are conti.

For closed set $V_p(J') \cap V_0$ in V_0 , $F^{-1}(V_p(J') \cap V_0) = V_a(J'^0)$ closed

For closed set $V_a(I) \cap Y$ in Y , $(F^{-1})^{-1}(V_a(I) \cap Y) = V_p(I^0) \cap V_0$ closed

b) Check pullback.

$\forall \varphi \in \mathcal{O}_{V_0}(U)$, w.t.s. $F^*\varphi \in \mathcal{O}_Y(F^{-1}(U))$

$\forall a \in F^{-1}(U)$, \exists open $U_{F(a)}$ containing $F(a)$, s.t. $\varphi = \frac{g_a(1 : x_1 : \dots : x_n)}{f_a(1 : x_1 : \dots : x_n)}$ on $U_{F(a)}$.

Then on $U_a = F^{-1}(U_{F(a)}) \stackrel{\text{open}}{\subseteq} F^{-1}(U)$, $F^*\varphi = \frac{g_a^h(x_1, \dots, x_n)}{f_a^h(x_1, \dots, x_n)}$ on U_a

$\forall \psi \in \mathcal{O}_Y(U)$, w.t.s. $F^{-1}\psi \in \mathcal{O}_{V_0}(F(U))$.

$\forall a \in F(U)$, \exists open $U_{F^{-1}(a)}$ containing $F^{-1}(a)$, s.t.

$$\psi = \frac{g_a(x_1, \dots, x_n)}{f_a(x_1, \dots, x_n)} \text{ on } U_{F^{-1}(a)}.$$

Then on $U_a := F(U_{F^{-1}(a)})$, $F^{-1}\psi = \frac{g_a(1 : x_1 : \dots : x_n)}{f_a(1 : x_1 : \dots : x_n)} = \frac{g_a^h(x_1, \dots, x_n)}{f_a^h(x_1, \dots, x_n)}$ (通过整
体乘 x_0^m s.t.
 $\deg g_a^h = \deg f_a^h$)

• A special type of mors of proj. var.

* In aff. var. case, 所有 mor 形如 $(\varphi_1(x), \dots, \varphi_n(x))$, $\varphi_i \in \mathcal{O}_X(U)$.

但在 proj. var. case 没有这样简单的结论, 只有如下更弱的结论.

Let $x \in \mathbb{P}^n$ be proj. var. and $f_0, \dots, f_m \in S(X)$. Then on the open subset $U := X \setminus V_p(f_0, \dots, f_m)$, these elements define a mor:

$$f: U \rightarrow \mathbb{P}^m, x \mapsto (f_0(x) : \dots : f_m(x))$$

* f_0, \dots, f_m have same degree is necessary.

$$[x_0 : x_1] \mapsto [x_0^2 : x_1] \quad \begin{matrix} \downarrow \\ [x_0 : x_1] \mapsto [x^2 x_0^2 : x_1] \end{matrix} \quad \text{二者值不同.}$$

Pf: Idea: 用 Glueing prop for mor.

$$\text{open set } U = X \cap (\bigcap_i V_p(f_i))^c = X \cap \left(\bigcup_i V_p(f_i) \right)^c = \bigcup_{i=1}^m X \cap D_p(f_i) =: \bigcup_{i=1}^m U_i$$

U_i open in X . It suffices to show $f|_{U_i}$ is a mor, by glueing prop.

$$f|_{U_i}: X \cap D_p(f_i) \longrightarrow \mathbb{P}^m$$

$$x \mapsto \left[\frac{f_0(x)}{f_i(x)}, \frac{f_1(x)}{f_i(x)}, \dots, 1, \dots, \frac{f_m(x)}{f_i(x)} \right]$$

$$\text{观察得 } \begin{cases} \text{Im } f_{U_i} \subseteq A^n \hookrightarrow P^m \\ X \cap D(f_i) \subseteq A^n \end{cases} \text{ 故 } f_{U_i}: X \cap D(f_i) \xrightarrow{A^n} A^m. \quad (\text{化归到 aff. case})$$

由于 f_{U_i} 满足 aff. var. 间 mor 的条件, 有 f_{U_i} 是 mor.

* 但 Every mor $P^n \rightarrow P^m$ has the form

$$f: P^n \rightarrow P^m, x \mapsto (f_0(x), \dots, f_m(x))$$

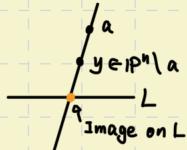
$$f_0, f_1, \dots, f_m \in k[x_0, \dots, x_n], V_p(f_0, \dots, f_m) = \emptyset$$

- Projection from $a \in P^n$ to the linear subspace $L \cong P^{n-1}$

$$\text{Exp: } a = (1:0:0:\dots:0) \in P^n, L = V_p(x_0) \cong P^{n-1}$$

则可构造 projection $P^n \setminus a \rightarrow L$ as following

$$\forall y = (y_0: y_1: \dots: y_n) \in P^n. L \text{ is the line passing } a \& y.$$



line L in $P^n \Leftrightarrow$ 2-dim vect space in A^{n+1}

$$L = \{ \text{2-dim vect space } + \text{的点} \}$$

$$= \{ sa + ty | (s, t) \in P^1 \}$$

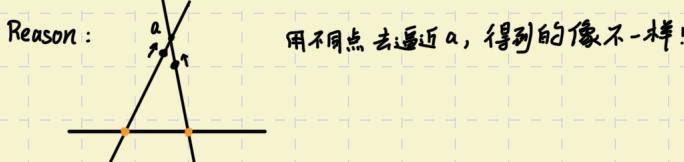
$$= \{ (sy_0: ty_1: \dots: ty_n) | (s: t) \in P^1 \}$$

$$\text{Then } L \cap L = L \cap V_p(x_0) = \{ (0: ty_1: \dots: ty_n) | sty_0 = 0, (s: t) \in P^1 \} \\ = \{ (0: y_1: \dots: y_n) \}$$

$$\text{So projection is } f: P^n \setminus a \rightarrow L \cong P^{n-1}$$

$$(y_0: \dots: y_n) \mapsto (0: y_1: \dots: y_n) \mapsto (y_1: \dots: y_n)$$

Question: 这个 $P^n \setminus a \rightarrow P^{n-1}$ 的 projection 无法延拓至 a .



Solution: 把 a 放到一个 proj. var. 上, 让 projection 限制在这个 proj. var. 上
这样逼近 a 的路线唯一就可以延拓出 a 处的投影映射了.

Exp: (Extension projection to a) Restrict the projection to a suitable proj. var. $X = V_p(x_0x_2 - x_1^2)$

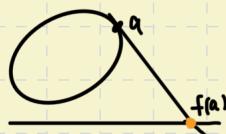
$$f: X \rightarrow P^1 \quad (x_0: x_1: x_2) \mapsto \begin{cases} (x_1: x_2) & \text{if } (x_0: x_1: x_2) \neq (1:0:0) \text{ (Restriction)} \\ (x_0: x_1) & \text{if } (x_0: x_1: x_2) = (0:0:1) \end{cases}$$

打到 P^1 而不是 L ,
避免了光谱是否
在 L 上的讨论

猜的, 需让相交部分像相同

$$x_0x_2 = x_1^2 \Rightarrow \frac{x_1}{x_2} = \frac{x_0}{x_1}$$

f 是两片 mor 粘出来的, 因此是 mor, 而且在 a 处有定义 (几何上是切线与 P^1 交点)



f is bijective. w.t.s. $\forall y \in L$, the line passing through a and y intersects X in two points. If it's true, the one is a , the other one is $f^{-1}(y)$. Indeed,

$$L = \{ s(0: y_1: y_2) + t(1: 0: 0) \mid (s: t) \in P^1 \}$$

$$= \{ (t: sy_1: sy_2) \mid (s: t) \in P^1 \}$$

pts in $\mathbb{P}^1 \times \mathbb{P}^1$ fulfilling $t_1 s y_2 - t_2 s^2 y_1^2 = 0$
 there are only two solutions (1) $s=0$ and a
 $(2) \frac{s}{t} = \frac{y_1}{y_2} \Rightarrow f^{-1}(y)$

Moreover, f is an iso:

$$f^{-1}: \mathbb{P}^1 \longrightarrow X$$

$$(y_1: y_2) \longmapsto (y_1^2: y_1 y_2: y_2^2) \quad (\text{用 } \frac{s}{t} = \frac{y_1}{y_2})$$

f^{-1} 每个分量是 2:2 poly, 故是 mor. 于是 f 是 iso.

* 不是所有 proj 是 iso. $X = V_p(x_0^2 - x_1 x_2) \subseteq \mathbb{P}^2$

$(1:0:0) \notin X$ 于是可定义忘掉 x_0 的 mor. $\varphi: X \rightarrow \mathbb{P}^1$ $(x_0: x_1: x_2) \mapsto (x_1: x_2)$ well-def.

We find $\varphi(1:1:1) = \varphi(-1:1:1)$.

- Segre embedding $\mathbb{P}^n \times \mathbb{P}^m$ 同构到 \mathbb{P}^N 中一个 proj. var., 描述更容易.

$$f: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N, N = (n+1)(m+1) - 1$$

$$([x_0: x_1: \dots: x_n], [y_0: \dots: y_m]) \longmapsto (x_i y_j) = (z_{ij}) = (x_0 y_0, x_0 y_1, \dots, x_0 y_m, \\ x_1 y_m, \dots, x_n y_0, \dots, x_n y_m)$$

* Segre embedding is well-defined

$$\bar{x} \neq 0, \exists x_i = 1. \quad \bar{y} \neq 0, \exists y_j = 1. \quad \text{So } z_{ij} = 1, (z_{ij}) \neq 0.$$

idea: 证像集是想要的形式 (2) 证 mor 是 iso

△ The image $X = f(\mathbb{P}^n \times \mathbb{P}^m)$ is a proj. var.

$$S := V_p(z_{ij} z_{kl} - z_{il} z_{kj}: 0 \leq i, k \leq n, 0 \leq j, l \leq m \\ i, k \text{ 不动}, j, l \text{ 变换. 因此 } j=l \text{ 或 } i=k \text{ 则等于 0})$$

$$\text{pf: } X \subseteq S: z_{ij} = x_i y_j, \text{ 显然有 } \frac{x_i y_j}{z_{ij}} \frac{x_k y_l}{z_{kl}} - \frac{x_i y_l}{z_{il}} \frac{x_k y_j}{z_{kj}} = 0$$

$$S \subseteq X: \forall z = [z_{00}, \dots, z_{mn}] \in S$$

只要证 z 是像. 只要把 \bar{x} 和 \bar{y} 找到. Assume $z_{00} \neq 0$ (假设非 0 部分是零除掉它)

$$\text{由 } z \in S = V(z_{ij} z_{kl} - z_{il} z_{kj}) \text{ 得 } z_{00} z_{kl} - z_{0l} z_{k0} = 0.$$

$$\text{两边同除 } z_{00}^2 \text{ 得 } \frac{z_{kl}}{z_{00}} = \frac{z_{0l}}{z_{00}} \frac{z_{k0}}{z_{00}}, \text{ 令 } x_l = \frac{z_{0l}}{z_{00}}, y_k = \frac{z_{k0}}{z_{00}}$$

$$x = (x_1: x_2: \dots: x_n), y = (y_1: y_2: \dots: y_m) \\ x_0 = \frac{z_{00}}{z_{00}} = 1$$

$$f(x, y) = (x_i y_j) = \frac{z_{ij}}{z_{00}} \text{ 和预期的只差一个整体的系数}$$

△ $f: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow X$ is an iso

pf: f 是 bijection. $\forall z \in X$, 不妨设 $z_{00}=1$, 则 $x_0 \neq 0, y_0 \neq 0$ 且 $x_0 = y_0 = 1$.

由 $z_{00} = x_0 y_0 = x_0, z_{0l} = x_0 y_l = y_l$ 可知原像 \bar{x}, \bar{y} 被我们

暴力解完了, 因此原像唯一. 原像存在(proceding process) 且唯一, 即 f 是 bijection.

还需证 $f \& f^{-1}$ 是 mor. 不是用 glueing prop, 比 section 2 回是 section

简单. 用 glueing prop 最好拆成 aff. parts, 因为 aff. parts 上 mor 的

形式比 proj. var. 要更清楚.

$U_i = D_p(x_i) \subseteq \mathbb{P}^n, V_j = D_p(y_j) \subseteq \mathbb{P}^m$, 则 $U_i \times V_j$ are aff. open cover of $\mathbb{P}^n \times \mathbb{P}^m$ (非常常见的折法)

(别证 $f|_{U_i \times V_j}$, 写起来太繁琐, 证 $f|_{U_0 \times V_0}$ 上是 mor 即可!)

$$f|_{U_0 \times V_0} ([1:x_1:\dots:x_n], [1:y_1:\dots:y_n]) = (1:y_1:\dots:y_n : x_1 : x_2 y_1 : \dots : x_n y_n)$$

$U_0 \times V_0 \subseteq \mathbb{A}^n \times \mathbb{A}^m \rightarrow D(z_{00}) \cong \mathbb{A}^n$ 每个分量是 $(x_1, \dots, x_n, y_1, \dots, y_n)$ 的 regular fun.

同理 $D(z_{ij}) \cap X$ 是 X 上 aff. open cover.

$$f^{-1}|_{D(z_{00})} (z_{00}) = ([1:z_{10}:\dots:z_{n0}], [1:z_{01}:\dots:z_{0m}])$$

aff. map 每个分量是 z_{ij} regular fun.

- Segre embedding application I: Every proj var. is a var.

pf: diagonal 出现 $\mathbb{P}^n \times \mathbb{P}^n$, 所以用 $[z_{ij}]$ 坐标会很方便.

$$\Delta_{\mathbb{P}^n} = \{([x_0:\dots:x_n], [y_0:\dots:y_n]) \mid (x_0:\dots:x_n) = (y_0:\dots:y_n)\}$$

★ $(x_0:\dots:x_n) = (y_0:\dots:y_n)$ 的化归 \Leftrightarrow MATRIX $\begin{cases} \text{RANK} \\ \text{Det} \end{cases}$

$$\begin{aligned} \text{rank} \begin{pmatrix} x_0 & x_1 & \dots & x_n \\ y_0 & y_1 & \dots & y_n \end{pmatrix} = 1 &\Leftrightarrow \text{所有 } 2 \times 2 \text{ minors vanish} \\ &\Leftrightarrow x_i y_j - x_j y_i = 0 \quad \forall i, j \\ &\Leftrightarrow z_{ij} - z_{ji} = 0 \end{aligned}$$

$V(z_{ij} - z_{ji} : \forall i, j)$ closed in $\mathbb{P}^{(n+1)^2-1}$, 自然 closed
in $\mathbb{P}^n \times \mathbb{P}^n = V(z_{ij} z_{kl} - z_{il} z_{kj} : \forall i, j, k, l)$.

Rank: (VERY USEFUL) $M \in M_{n \times m}(\mathbb{K})$.

$\text{rank}(M) \leq k \Leftrightarrow \exists \text{任意} (k+1) \times (k+1) \text{ minors vanish.}$

$\text{rank } M \geq k \Leftrightarrow \exists k \times k \text{-minor} \neq 0$

$\text{rank } M = k \Leftrightarrow \forall k \times k \text{-minor} \neq 0$
 $\forall (k+1) \times (k+1) \text{-minor} = 0$

- $X \subset \mathbb{P}^m$, $Y \subset \mathbb{P}^n$ are proj. var. then $X \times Y$ is proj. var.

pf: X, Y closed in $\mathbb{P}^m, \mathbb{P}^n$, resp. then $X \times Y$ closed in $\mathbb{P}^m \times \mathbb{P}^n$

$\mathbb{P}^m \times \mathbb{P}^n$ is proj. var. closed in $\mathbb{P}^{(m+1)(n+1)-1}$, so $X \times Y$ is also closed in $\mathbb{P}^{(m+1)(n+1)-1}$ and thus $X \times Y$ is a proj. var.

- Closed maps

Motivation: Noetherian spaces are all compact, hence $\mathbb{P}_{\mathbb{C}}^n$ and $\mathbb{A}_{\mathbb{C}}^n$ are compact under zariski-topo. BUT $\mathbb{P}_{\mathbb{C}}^n$ is compact and $\mathbb{A}_{\mathbb{C}}^n$ is uncompact in classical topo. 所以需要引入新概念区分紧性.

Def: A map $f: X \rightarrow Y$ between topo space is called closed if $f(A) \subseteq Y$ closed for any closed $A \subseteq X$.

- prop: projection map $\pi: \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$ is closed.

* 需要用到结论: For $V_p(f_1, \dots, f_n)$ f_i homo, 总可将 f_i 造成

齐相多项式. key point: Check $V_p(f) = V_p(x_0^s f, x_1^s f, \dots, x_n^s f)$

$$\text{R.H.S.} = \bigcap_i V_p(x_i^s f) = \bigcap_i (V_p(x_i^s) \cup V_p(f)) = (\bigcap_i V_p(x_i^s)) \cup V_p(f) = V_p(f)$$

e.g. $V_p(f_1, f_2)$, $\deg f_1 < \deg f_2$. Let $s = \deg f_2 - \deg f_1$.

$$V_p(f_1, f_2) = V_p(x_0^s f_1, x_1^s f_1, \dots, x_n^s f_1, f_2)$$

Pf: Idea: 把某个条件用代数几何的语言描述出来

Any closed set X in $\mathbb{P}^n \times \mathbb{P}^m$ can be written as $X = V(f_1, \dots, f_r)$ where f_i are homo poly in Segre coordinates (z_{ij}) with same degree (上面已证能力到)

(一个关于 Segre coord 的 poly 观察 (x_i) or (y_i) 的 poly degree 相同, 因为出 现一次 x_i 必出现一次 y_j .)

Let a be a pt.

$$a = (a_0 : \dots : a_m) \in \pi(X) \Leftrightarrow \exists (x, y) \in V(f_1, \dots, f_r), s.t. \pi(x, y) = y = a$$

$$\Leftrightarrow \exists x, f_i(x, a) = 0, \forall i.$$

令 $g_i = f_i(1, a)$. 则 g_i 是子 x 的 poly, $\deg g_i = \deg f_i$ over $x = d$

$$a \in \pi(X) \Leftrightarrow \exists x, g_i(x) = 0, \forall i \Leftrightarrow V(g_1, \dots, g_r) \neq \emptyset$$

trick: $V(\dots) \neq \emptyset$ 不好用, $V(\dots) = \emptyset$ 才可以用 Hilbert null thm. 因此换成 如下等价形式:

$$a \notin \pi(X) \Leftrightarrow V(g_1, \dots, g_r) = \emptyset \Leftrightarrow \sqrt{\langle g_1, \dots, g_r \rangle} = \langle 1 \rangle \quad (\text{用 } \langle \cdot \rangle \text{ 表示全空间})$$

$$\text{or } \sqrt{\langle g_1, \dots, g_r \rangle} = \langle x_0, \dots, x_n \rangle \text{ (relevant ideal)}$$

$$\Leftrightarrow \exists k_i \in \mathbb{N} \text{ s.t. } x_i^{k_i} \in \langle g_1, \dots, g_r \rangle \text{ for all } i$$

$$\Leftrightarrow \exists k \text{ [取 } k = k_1 + k_2 + \dots + k_r], \text{ s.t. } k[x_0, \dots, x_n]_k \subseteq \langle g_1, \dots, g_r \rangle$$

$$\Leftrightarrow \exists k, k[x_0, \dots, x_n]_k = \underbrace{\langle g_1, \dots, g_r \rangle}_n \text{ (因为 } \langle g_1, \dots, g_r \rangle \subseteq k[x_0, \dots, x_n]_k \text{.)}$$

$$\Leftrightarrow \varphi: \widehat{k[x_0, \dots, x_n]_{k-d}} \times \dots \times \widehat{k[x_0, \dots, x_n]_{k-d}} \rightarrow k[x_0, \dots, x_n]_k$$

(视作 k -vect space) $(h_1, \dots, h_r) \mapsto \sum g_i h_i$
 $\deg d \leftarrow \deg k-d$

is surjective as a k -linear map between vect space

$$\begin{aligned} & \left(\text{空间的基是每个分量放单项式} \quad \dim k[x_0, \dots, x_n]_k = \binom{n+k}{k} \right) \\ & \dim \left(k[x_0, \dots, x_n]_{k-d} \times \dots \times k[x_0, \dots, x_n]_{k-d} \right) = r \binom{n+k-d}{k-d} \end{aligned}$$

只要写出 linear map 矩阵就知道 φ 是否 surj.

$a \notin \pi(X) \Leftrightarrow$ matrix of φ has rank $\binom{n+k}{k}$ (满秩)

$\pi(X)$ 的补集 $\Leftrightarrow \exists$ minor of $\binom{n+k}{k} \times \binom{n+k}{k}$ with determinate $\neq 0$
 矩阵的每一个 entry 是 a_0, \dots, a_m 的 poly with deg d.

e.g. 取左边一个特殊的基 $(x_0^{k-d}, 0, 0, \dots, 0)$

$$\varphi(x_0^{k-d}, 0, \dots, 0) = g_0 x_0^{k-d} = f_0(0, a) x_0^{k-d}$$

是与 a 有关的多项式.

$$\Leftrightarrow \det(a_0, \dots, a_m) \neq 0 \quad \text{是开条件}$$

- Prop: The projection map $\pi: \mathbb{P}^n \times Y \rightarrow Y$ is closed for any var. Y

Pf: 分步证明 (aff 以及 var 构成 aff.)

①假设 $Y \subseteq \mathbb{A}^n$ 是 aff. 设 $Z \subseteq \overset{\text{closed}}{\mathbb{P}^n \times Y}$

Idea: aff. var. 嵌入到 \mathbb{P}^m 中从而用 $\pi: \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$ 是 closed 来证.

$$Z \subseteq \overset{\text{closed}}{\mathbb{P}^n \times Y} \subseteq \mathbb{P}^n \times \mathbb{A}^m \subseteq \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\pi'} \mathbb{P}^m$$

Z 不是这里的闭集.

自己制造闭集. \bar{Z} is closure of Z in $\mathbb{P}^n \times \mathbb{P}^m$. 则 $\pi'(\bar{Z})$ closed.

由 $Z = \bar{Z} \cap \mathbb{P}^n \times Y$ ($A \subseteq B \subseteq X$
 $\bar{A}_B = \bar{A} \times_B \bar{B}$ 包含 A 的最小 B 中闭集), VERY USEFUL

得 $\pi(Z) = \pi(\bar{Z} \cap \mathbb{P}^n \times Y) = \pi(\bar{Z}) \cap Y$ 是 Y 中的闭集

$\pi: \mathbb{P}^n \times Y \rightarrow Y$

$\pi': \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$

② 假设 Y is not aff. $Y = \bigcup_i Y_i$, Y_i aff.

$\forall Z$ closed in $\mathbb{P}^n \times Y$, then $Z \cap \mathbb{P}^n \times Y_i$ is closed in $\mathbb{P}^n \times Y_i$

so $\pi(Z \cap \mathbb{P}^n \times Y_i)$ is closed in Y_i . Then

$\pi(Z) \cap Y_i = \pi(\bar{Z}) \cap \pi(\mathbb{P}^n \times Y_i) = \pi(\bar{Z} \cap \mathbb{P}^n \times Y_i)$ is closed in Y_i

Hence $\pi(Z) = \bigcup_i \pi(\bar{Z}) \cap Y_i$ closed in Y

Hence $Y \setminus \pi(Z) = \bigcup_i \underbrace{Y_i \setminus (\pi(\bar{Z}) \cap Y_i)}_{\text{open in } Y_i}$ \Rightarrow open in $Y \Rightarrow \pi(Z)$ closed.

Y_i open in Y , so
it's open in Y .

- Complete var: A var is called **complete** if the projection $\pi: X \times Y \rightarrow Y$ is closed for any var Y .

Ex 1) \mathbb{P}^n is complete.

2) In general topo, 紧空间闭子集是紧的.

In Zariski topo, complete space 的闭子集是 complete.

$Z \subseteq \underset{\text{closed}}{X}_{\text{complete}}$ w.r.t. $\pi: Z \times Y \rightarrow Y$ closed.

$\pi_1: Z \times Y \xrightarrow{\cong} X \times Y \xrightarrow{\pi_X} Y$, $\pi = \pi_1 \circ \pi_2$. π_1 is closed, π_2 is closed,
so π is closed. (Why? π_1 is closed? $Z \subseteq \underset{\text{closed}}{X}$, $Y \subseteq \underset{\text{closed}}{Y}$ so $Z \times Y \subseteq \underset{\text{closed}}{X \times Y}$
then any closed subset of $Z \times Y$ is still closed in $X \times Y$)

(3) Every proj. var. is complete (complete 紧闭子集 complete),

BUT NOT all complete var is proj. var.

(4) A **complex variety** is compact in classical topo iff it's complete

(5) \mathbb{A}^2 is not complete (图像上显然, \mathbb{A}^2 没有无穷远点.)

$\pi: \mathbb{A}^2 \times \mathbb{A}^2 \rightarrow \mathbb{A}^2$

" \mathbb{A}^2 " $\pi(V(xy=1)) = \mathbb{A}^2 \setminus \{(0,0)\}$ is not closed

$V(xy=1)$
closed in
 $\mathbb{A}^2 \times \mathbb{A}^2$

把 \mathbb{A}^2 视为 \mathbb{P}^2 aff. patch 才显然了.

- Prop: $f: X \rightarrow Y$ is mor of var, X is complete then $f(X)$ is complete closed in Y . (In general topo, 紧集的像也是紧的)

* 在 Zariski topo 下, $f(X)$ 不仅 complete, 而且 closed!

(1) Show $f(X)$ closed.

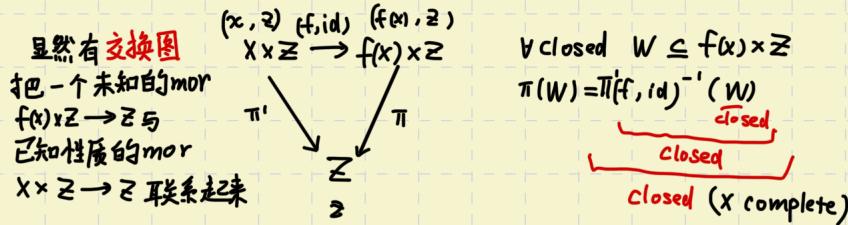
$X \xrightarrow{(\text{id}, f)} X \times Y \xrightarrow{\pi} Y$

$x \mapsto (x, f(x)) \mapsto f(x)$

$\mathcal{F}_f = \{(x, f(x)) \in X \times Y\}$ is closed in $X \times Y$

so $f(X)$ is closed in Y by π closed

(2) Show $f(X)$ complete, i.e., check $f(X) \times Z \rightarrow Z$ closed for $\forall Z$.



- Prop: X : connected complete var. Then $\mathcal{O}_X(X) = k$

Pf: $\forall f \in \mathcal{O}_X(X)$ has the form $f: X \rightarrow \mathbb{A}^1$.

X complete so $f(X)$ closed in \mathbb{A}^1 . Closed sets in \mathbb{A}^1 has the form {finite pts}. So $f(X)$ is finite pts. X connected so $f(X)$ connected, hence $f(X)$ is a singlet, i.e., f is a const.

- Veronese embedding 把小proj space嵌到大proj space.

pick $n, d \in \mathbb{N}_{>0}$.

Veronese embedding $F: \mathbb{P}^n \longrightarrow \mathbb{P}^N$

$$x \longmapsto (f_0(x); \dots; f_N(x))$$

where $f_0, \dots, f_N \in k[x_0, \dots, x_n]$ 是所有 deg d monomials, 共有 $\binom{n+d}{d}$ 个。
由于是从0开始计数, $N = \binom{n+d}{d} - 1$

* Well-defined: f_0, \dots, f_N 包含 x_0^d, \dots, x_n^d , 因此无公共零点。

* It's a mor: 每个分量是齐次多项式

* $F(\mathbb{P}^n)$ is a proj. var. in \mathbb{P}^N : \mathbb{P}^n complete, so image of mor $F(\mathbb{P}^n) =: X$ closed in \mathbb{P}^N thus it's proj. var in \mathbb{P}^N .

* $F: \mathbb{P}^n \longrightarrow X$ is an iso

Key point: 找到 F^{-1} 再证 F 与 F^{-1} 在每个 aff. patch 上是 mor (routine)

Consider open subset $D(x_i)$ of X , i.e., on this patch X has the form

$$\text{So } \frac{x_j}{x_i} = \frac{z_{(d-1)e_i + e_j}}{z_{de_i}} = \frac{x_j x_i^{d-1}}{x_i^d} \quad (\frac{f_0}{x_i^d}, \frac{f_1}{x_i^d}, \dots, \underset{i\text{-th}}{1}, \dots, \frac{f_m}{x_i^d})$$

* e.g. $\mathbb{P}^1 \longrightarrow \mathbb{P}^{\binom{1+d}{d}-1} = \mathbb{P}^d$

$$(x_0: x_1) \longmapsto (x_0^d: x_0^{d-1} x_1: \dots: x_1^d)$$

* Importance of Veronese embedding:

Coord in \mathbb{P}^n

degree-d poly

Coord in \mathbb{P}^N

linear poly

// example

- Prop: Let $X \subset \mathbb{P}^n$ be a proj var. and $f \in S(X)$ be homogeneous and non constant. Then $X \setminus V(f)$ is an aff. var.

* \mathbb{P}^n 挖掉 hypersurface 是 aff.

Pf: Let $F: \mathbb{P}^n \longrightarrow \mathbb{P}^N$ be degree d Veronese map e.g. $V(a_0 x_0^d + \dots + a_n x_n^d)$, 则它在 F 下的像是 $V(a_0 z_{de_0} + \dots + a_n z_{de_n})$ 这是 \mathbb{P}^N 中 hyperplane, Veronese embedding 的好处在于把方程变线性.

Veronese embedding 把 deg d hyperplane map 到 deg 1 hyperplane.
 在复合 Aut(P^n) 中元素后, 总可假设 map 后的 hyperplane 满足 $z_0 = 0$.
 于是 $F(P^n) \setminus \text{hyperplane} \subseteq D(z_0)$ is an aff.

- Grassmannians $G(k, n)$: k -dim linear spaces in K^n

Goal : Show $G(k, n)$ is a proj. var.

Idea : find embedding $G(k, n) \rightarrow P$?

To construct this embedding we introduce **alternating tensor products**

- Alternating tensor product

Def : V is a k -vector space. $k \in \mathbb{N}$. A k -fold **multilinear map**

$f: V^k \rightarrow W$ is called **alternating** if any $v_i = v_j$ for some $i \neq j$ we

have $f(v_1, \dots, v_k) = 0$

* 任何两个分量相同会被打到0.

* 等价定义, - 次对换多一个负号 $f(\dots, v_j, \dots, v_i, \dots) = -f(\dots, v_i, \dots, v_j, \dots)$

i.e., $f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{sgn}(\sigma) f(v_1, \dots, v_k)$

Def: (Alternating tensor product)

a k -fold tensor product of V is a vect space T with an k -fold alternating multilinear map $\tau: V^k \rightarrow T$ satisfying univ. prop:

$$\begin{array}{ccc} V^k & \xrightarrow{\quad f \text{ alt. multilinear} \quad} & W \\ \tau \downarrow & \swarrow \text{big} & \\ T & & \end{array}$$

* The image $\tau(v_1, \dots, v_k) =: v_1 \wedge v_2 \wedge \dots \wedge v_k$

* alt. tens. prod. of V exists, 可以证明 $T = \Lambda^k V$, $\Lambda^k V$ 是 V

{ $1 \leq j_1 < j_2 < \dots < j_k \leq n$ } $v_{j_1} \wedge v_{j_2} \wedge \dots \wedge v_{j_k}$ 为基的 linear space,

其中 $\{v_i\}_{i \in [1, n]}$ 是 V 的 basis.

* $\dim \Lambda^k V = \binom{n}{k}$

* $\Lambda^k V = \otimes^k V / L$, L is vector subspace of $V^{\otimes k}$ generated by $\{v_i \otimes \dots \otimes v_i \mid v_i = v_j \text{ for some } i \neq j\}$

* $\Lambda^0 V \cong K$ $\Lambda^1 V \cong V$

• Observation: $\dim \Lambda^n K^n = \binom{n}{n} = 1$ so $\Lambda^n K^n \cong K$

$\dim \Lambda^2 K^3 = \binom{3}{2} = 3$ so $\Lambda^2 K^3 \cong K^3$.

其中 $\Lambda^n K^n \cong K$ given by determinant

任何 $\Lambda^n K^n$ 中元 $v_1 \wedge v_2 \wedge \dots \wedge v_n = \det[v_1 \dots v_n] e_1 \wedge \dots \wedge e_n$

$\Lambda^3 K^3 \cong K^3$ given by cross product

$$\begin{aligned} \text{任何 } \Lambda^k \mathbb{K}^3 \text{ 元元 } V_1 \wedge V_2 &= (a_1 e_1 + a_2 e_2 + a_3 e_3) \wedge (b_1 e_1 + b_2 e_2 + b_3 e_3) \\ &= (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 + (a_2 b_3 - a_3 b_2) e_1 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_2 \wedge e_3 \end{aligned}$$

系数来自于 V_1 与 V_2 组成矩阵的 2×2 -minor

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

从这个 observation 出发。

Prop: Let $V_1, V_2, \dots, V_k \in \mathbb{K}^n$. Assume $V_i = \sum_{j=1}^n a_{ij} e_j$

Then $V_1 \wedge V_2 \wedge \dots \wedge V_k = \sum \text{ 对应余子式 } e_{j_1} \wedge \dots \wedge e_{j_k}$

Pf:

$$V_1 \wedge V_2 \wedge \dots \wedge V_k = (a_{11} e_1 + \dots + a_{nn} e_n) \wedge (a_{12} e_1 + \dots + a_{2n} e_n) \wedge \dots \wedge (a_{k1} e_1 + \dots + a_{kn} e_n)$$

$$= \sum a_{1j_1} a_{2j_2} \dots a_{kj_k} e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}$$

$$\begin{aligned} \text{只要} j_1, j_2, \dots, j_k \text{ 不相} \\ \text{同项求和} \end{aligned} = \sum_{\substack{1 \leq j_1 < j_2 < \dots < j_k \leq n}} a_{1j_1} a_{2j_2} \dots a_{kj_k} e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}$$

$$= \sum_{1 \leq j_1 < j_2 < \dots < j_k} \sum_{\sigma \in S_k} \text{sgn}(\sigma) a_{1j_{\sigma(1)}} \dots a_{kj_{\sigma(k)}} e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}$$

行列式的表达式。

$$\sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} a_{1j_1} a_{2j_2} \dots a_{kj_k} e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k} \text{ 是 } j_1, \dots, j_k \text{ 的全部可能排列}$$

也就取定从小到大排列的一组数

$$\sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} a_{1j_{\sigma(1)}} a_{2j_{\sigma(2)}} \dots a_{kj_{\sigma(k)}} e_{j_{\sigma(1)}} \wedge e_{j_{\sigma(2)}} \wedge \dots \wedge e_{j_{\sigma(k)}} \quad \underbrace{\text{用 } \text{sgn}(\sigma) \text{ 整理成按序排}}$$

$$= \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} a_{1j_{\sigma(1)}} a_{2j_{\sigma(2)}} \dots a_{kj_{\sigma(k)}} \sum_{\sigma \in S_k} \text{sgn}(\sigma) e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}$$

Slogan: Wedge 基展开系数是原向量矩阵 minors.

• Plücker embedding $G(k, n) \rightarrow \mathbb{P}^{n \choose k} - 1$

Lemma 1, 2 对于构造 Plücker embedding 非常重要。

△ Lemma 1: $V_1, \dots, V_k \in \mathbb{K}^n$ for some $k \leq n$. Then

$V_1 \wedge V_2 \wedge \dots \wedge V_k = 0$ iff V_1, \dots, V_k are Lin. dep.

Pf: Idea: Use minors to compute wedge.

Let $V_i = \sum_{j=1}^n a_{ij} e_j, i \in [1, k]$

$$V_1 \wedge \dots \wedge V_k = 0 \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{bmatrix} = A \text{ 的所有 } k \text{ 阶 minors } \Rightarrow 0 \Leftrightarrow \text{rank } A < k \Leftrightarrow V_1, \dots, V_k \text{ Lin. dep.}$$

△ Lemma 2: $V_1, \dots, V_k \in \mathbb{K}^n$ and $w_1, \dots, w_k \in \mathbb{K}^n$ both be Lin. ind.

Then

$V_1 \wedge \dots \wedge V_k$ and $w_1 \wedge \dots \wedge w_k$

Lin. ind. in $\Lambda^k \mathbb{K}^n$

(基等价)

$\Leftrightarrow \text{Lin}(V_1, \dots, V_k) = \text{Lin}(w_1, \dots, w_k)$

Pf:

\Rightarrow Assume $w_1 \wedge \dots \wedge w_k \in \text{Lin}(v_1, \dots, v_k)$

正推不容易，我们去找等价条件。

$v_i \in \text{Lin}(w_1, \dots, w_k) \iff v_i, w_1, \dots, w_k \text{ lin. dep.} \iff v_i \wedge w_1 \wedge \dots \wedge w_k = 0$

$\iff v_i \wedge v_1 \wedge \dots \wedge v_k \cdot c = 0$ 出现两个 v_i , 恒成立。 $\Rightarrow \text{Lin}(v_1, \dots, v_k) \subseteq \text{Lin}(w_1, \dots, w_k)$

\Leftarrow Assume $\text{Lin}(v_1, \dots, v_k) = \text{Lin}(w_1, \dots, w_k)$

Then $v_1 = c_{11}w_1 + \dots + c_{1k}w_k$

\vdots

$v_k = c_{k1}w_1 + \dots + c_{kk}w_k$

$$\text{and } v_1 \wedge \dots \wedge v_k = \begin{vmatrix} c_{11} & \dots & c_{1k} \\ \vdots & & \vdots \\ c_{k1} & \dots & c_{kk} \end{vmatrix} w_1 \wedge \dots \wedge w_k$$

矩阵可逆, $\det \neq 0$ 故二者 Lin. dep.

* Slogan: Wedge 判定 ① 是否 Lin. ind. (wedge $\neq 0$)
② 是否张成相同空间 (wedge 等于非零数)

△ Plücker embedding: $f: G(k, n) \rightarrow \mathbb{P}^{\binom{n}{k}-1}$

$$\text{Lin}(v_1, \dots, v_k) \mapsto v_1 \wedge \dots \wedge v_k \in \wedge^k K^n \cong K^{\binom{n}{k}}$$

$\text{Lin}(v_1, \dots, v_k) \in G(k, n)$ 在 $\mathbb{P}^{\binom{n}{k}-1}$ 中的 plücker coordinate 是

$k \times k$ -minors of matrix whose rows are v_1, v_2, \dots, v_k 用基展开的系数。
(maximal minors)

* Well-defined: If $v_1 \wedge \dots \wedge v_k = 0 \Rightarrow v_1, \dots, v_k$ lin. dep. $\Rightarrow \dim \text{Lin}(v_1, \dots, v_k) < k$

$\Rightarrow \text{Lin}(v_1, \dots, v_k) \notin G(k, n)$

If $\text{Lin}(w_1, \dots, w_k) = \text{Lin}(v_1, \dots, v_k) \Rightarrow w_1 \wedge \dots \wedge w_k = \lambda v_1 \wedge \dots \wedge v_k, \lambda \in K^*$

$$\Rightarrow w_1 \wedge \dots \wedge w_k = v_1 \wedge \dots \wedge v_k \text{ in } \mathbb{P}^{\binom{n}{k}-1} = K^{\binom{n}{k}} / K^*$$

* f inj: $v_1 \wedge \dots \wedge v_k = w_1 \wedge \dots \wedge w_k \Rightarrow \text{Lin}(v_1 \wedge \dots \wedge v_k) = \text{Lin}(w_1 \wedge \dots \wedge w_k)$

只要再证 $\text{Im } f$ closed in $\mathbb{P}^{\binom{n}{k}-1}$ 即可证明 $G(k, n) \cong$ proj. var.

• $G(k, n)$ is a proj. var.

* Examples convinced this fact:

$$G(1, n) \rightarrow \mathbb{P}^{\binom{n}{1}-1} = \mathbb{P}^{n-1}$$

$$\text{Lin}(a_1e_1 + \dots + a_ne_n) \mapsto (a_1 : \dots : a_n) \quad \text{and } G(1, n) = \mathbb{P}^{n-1}$$

* Idea of showing $\text{Im } f$ closed: $\text{Im } f = \text{"pure wedges"} \leftarrow \text{find equation to describe pure wedge}$

△ Lemma: For a fixed non-zero $w \in \wedge^k K^n$ with $k < n$, consider

k -linear map $g: K^n \rightarrow \wedge^{k+1} K^n, v \mapsto v \wedge w$

Then $\text{rk } g \geq n-k$. The equation holds iff $w = v_1 \wedge \dots \wedge v_k$ for some $v_1, \dots, v_k \in K^n$

w is "pure" or in image of plücker embedding

Pf:

$$\boxed{\text{rk } g + \dim \ker g = n}$$

Assume v_1, \dots, v_r are basis of $\ker g$. Extend them to basis of K^n

$v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n$. $w \in \wedge^k K^n$, so we assume

$$w = \sum_{1 \leq j_1 < \dots < j_k \leq n} a_{j_1} \dots a_{j_k} v_{j_1} \wedge \dots \wedge v_{j_k}$$

由于 $w \neq 0$, 存在 $a_{j_1} \dots a_{j_k} \neq 0$.

由 $v_i \in \ker g$, 我们有 $v_i \wedge w = 0 = \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ i \in \{j_1, \dots, j_k\}}} a_{j_1} \dots a_{j_k} v_i \wedge v_{j_1} \wedge \dots \wedge v_{j_k}$

When (j_1, \dots, j_k) ranges, there are two cases (1) $i \in \{j_1, \dots, j_k\}$
(2) $i \notin \{j_1, \dots, j_k\}$

(1) If $i \in \{j_1, \dots, j_k\}$, then $v_i \wedge v_{j_1} \wedge \dots \wedge v_{j_k} = 0$. a_{j_1, \dots, j_k} can be nonzero

(2) $v_i \wedge w = 0 \underset{i \notin \{j_1, \dots, j_k\}}{\sum} a_{j_1, \dots, j_k} \frac{v_i \wedge v_{j_1} \wedge \dots \wedge v_{j_k}}{\text{basis in } \Lambda^{k+1} k^n}$ so they are all lin. ind.!

Hence $a_{j_1, \dots, j_k} = 0$ for $\forall (j_1, \dots, j_k) \ni i$

以上分析对 $r \in [1, t]$ 都成立. 故

Conclusion: 只有满足 $\{1, \dots, t\} \subseteq \{j_1, \dots, j_k\}$ 的 (j_1, \dots, j_k) 才有

$a_{j_1, \dots, j_r} \neq 0$. 已知 $\exists a_{j_1, \dots, j_r} \neq 0$, 故 $\{j_1, \dots, j_k\} \supseteq \{1, \dots, t\}$

于是 $r = \dim \ker g \leq k$. 即 $\operatorname{rk} g = n - \dim \ker g = n - r \geq n - k$.

若等号成立, $r = \dim \ker g = k \Rightarrow \exists \{j_1, \dots, j_k\} = \{1, \dots, t\}$.

则 $w = a_{1, \dots, k} v_1 \wedge \dots \wedge v_k$ pure.

key pt: w 只留下 $\{j_1, \dots, j_k\} \supseteq \{1, \dots, t\}$ terms.

Exp: $k=2, n=4$

(1) $w = e_1 \wedge e_2$.

$$\begin{aligned} g(a_1e_1 + \dots + a_4e_4) &= (a_1e_1 + \dots + a_4e_4) \wedge e_1 \wedge e_2 \\ &= a_3 \underline{e_1 \wedge e_2 \wedge e_3} + a_4 \underline{e_1 \wedge e_2 \wedge e_4} \\ rk g &= 2 = n - k \quad \text{只留下含 } e_1, e_2 \text{ terms. (此时 } \ker g = \text{Lie}(e_1, e_2) \text{)} \end{aligned}$$

(2) $w = e_1 \wedge e_2 + e_3 \wedge e_4$

$$\begin{aligned} g(a_1e_1 + \dots + a_4e_4) &= (a_1e_1 + \dots + a_4e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4) \\ &= a_1e_1 \wedge e_3 \wedge e_4 + a_2e_2 \wedge e_3 \wedge e_4 + a_3e_3 \wedge e_2 \wedge e_3 + a_4e_4 \wedge e_2 \wedge e_4 \\ rk f &= 4 > n - k \end{aligned}$$

△ The image of plucker embedding is closed in $\mathbb{P}^{\binom{n}{k}-1}$

Pf: $\forall w \in \Lambda^k k^n \cong k^{\binom{n}{k}}$. $\underline{[w]} \in \mathbb{P}^{\binom{n}{k}-1}$.

$\underline{[w]} \in \operatorname{Im} f \Leftrightarrow w \text{ is pure} \Leftrightarrow g: k^n \rightarrow \Lambda^{k+1} k^n \quad v \mapsto v \wedge w \text{ has } \operatorname{rk} g = n - k$

\Leftrightarrow all $(n-k+1)$ -minors of matrix g are zeros, since $\operatorname{rk} g \geq n - k$

They are all homo polys

So those w forms a closed set, i.e., plucker embedding is closed.

Exp: $G(2,4) = V_p(?)$ in $\mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$

$g: k^4 \rightarrow \Lambda^3 k^4$

$v \mapsto v \wedge w$

$$w = \underbrace{a_{12}e_1 \wedge e_2}_1 + \underbrace{a_{13}e_1 \wedge e_3}_2 + \underbrace{a_{14}e_1 \wedge e_4}_3 + \underbrace{a_{23}e_2 \wedge e_3}_4 + \underbrace{a_{24}e_2 \wedge e_4}_5 + \underbrace{a_{34}e_3 \wedge e_4}_6$$

Coordinates of $G(2,4)$ in \mathbb{P}^5

★ Key: 把 \mathbb{P}^5 视作 $\Lambda^3 k^4 / k^*$

$$e_i \mapsto e_i \wedge w = a_{23}e_1 \wedge e_2 \wedge e_3 + a_{24}e_1 \wedge e_2 \wedge e_4 + a_{34}e_1 \wedge e_3 \wedge e_4$$

$$\begin{aligned}
V &\mapsto V \wedge W \\
e_1 &\mapsto e_1 \wedge e_2 \wedge e_3 + e_2 \wedge e_1 \wedge e_4 + e_3 \wedge e_1 \wedge e_4 \\
e_2 &\mapsto -e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_2 \wedge e_4 + e_3 \wedge e_1 \wedge e_4 \\
e_3 &\mapsto e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_2 \wedge e_4 - e_2 \wedge e_3 \wedge e_4 \\
e_4 &\mapsto e_1 \wedge e_2 \wedge e_4 + e_1 \wedge e_3 \wedge e_4 + e_2 \wedge e_3 \wedge e_4
\end{aligned}
\quad \text{Andreas Gathmann}$$

$$\begin{bmatrix}
e_1 \wedge e_2 \wedge e_3 & e_1 \wedge e_2 \wedge e_4 & e_1 \wedge e_3 \wedge e_4 & e_2 \wedge e_3 \wedge e_4 \\
e_2 \wedge e_3 \wedge e_4 & 0 & e_1 \wedge e_3 \wedge e_4 & e_2 \wedge e_3 \wedge e_4 \\
e_3 \wedge e_4 & e_1 \wedge e_3 \wedge e_4 & 0 & e_1 \wedge e_2 \wedge e_4 \\
e_4 \wedge e_1 \wedge e_2 & e_1 \wedge e_2 \wedge e_3 & e_1 \wedge e_2 \wedge e_4 & 0
\end{bmatrix}$$

这个矩阵所有 $(n-k+1)-minors$ 为 0
共有 16 个 equs.

$$\text{故 } G(2,4) = V_p(\text{all 3-minors})$$

事实上这 16 个 equs 是冗余的，只要一条方程就可以。这带来一个麻烦：用这种方法并不能清楚地看到 $\dim G(k,n) = ?$

- Compute $\dim G(k,n) = k(n-k)$

Idea: Note that we have this prop:

If $\{U_i : i \in I\}$ is an open cover of X then $\dim X = \sup \{\dim U_i : i \in I\}$

我们用 $G(k,n)$ 用 open aff. patch cover, 就可以计算 $\dim G(k,n)$.

$$G(k,n) \subseteq P^{k-1} = \bigcup_i U_i \leftarrow \text{has aff. cover}$$

$$\text{so } G(k,n) = \bigcup_i U_i \cap G(k,n) \text{ also an aff. cover of } G(k,n).$$

Let's look more carefully on these aff. patch.

$v_1 \wedge \dots \wedge v_k \in f(G(k,n)) \subseteq P^{k-1}$ 的 plücker coord 是

$(a_{1,2, \dots, k} : a_{1,2, \dots, k-1, k+1} : \dots)$ 第 1, 2, ..., k-1, k+1 列组成的子矩阵的行列式

$$\begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix} = \begin{bmatrix} a_{11} \dots a_{1n} \\ \vdots \\ a_{kn} \dots a_{nn} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

因此 $f(G(k,n))$ 的 aff. patch 是某个 k -minor 非 0.

规定 U_0 是第 1 至 k 列组成的子矩阵的行列式. 则 $f(G(k,n))$ 的 patch 是 $[A \ B]$.
with A invertible. 转到 $G(k,n)$ 中,

$G(k,n)$ 的 aff. patch U_0 是 $\text{Lin} \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix} = \text{row span of } [A \ B]$

$$= \text{row span of } k \begin{bmatrix} I & C \\ 0 & 0 \end{bmatrix} \text{ where } C = A^{-1}B \in M_{(n-k) \times k}$$

$$\text{Hence } U_0 \cong A^{k(n-k)}$$

$G(k,n)$ 的每一片 patch 都同构于 $A^{k(n-k)}$, 因此 $\dim G(k,n) = k(n-k)$.

* Moreover $G(k,n)$ is irr.

Fact: $\{U_i\}_{i \in I}$ be an open cover of topo space X and $U_i \cap U_j \neq \emptyset$ for all i, j . Then if U_i is irr for all $i \in I$, we have X is irr.

Since $A^{k(n-k)}$ is irr (open set in irr $P^{k(n-k)}$) so $G(k,n)$ is irr.

* $G(k,n) = \{ \text{full rank } k \times n \text{ matrix} \} / GL(k)$; 表示 row transformation.

$$\text{Lin} \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix} = \text{row span } [A \ B] \xrightarrow[\text{rank } k]{} \text{row span } \text{齐次型矩阵}$$

$$\text{Exp: } G(1,2) = \{ \text{full rank } 1 \times 2 \text{ matrix} \} / GL(1) = \{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} | * \in K \}$$

$V_0 = \{ (I \ C) | C \in M_{(n-k) \times k} \}$ is an aff. space patch of $G(k,n)$

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \right\} \text{ is also an aff. space of } G(4,6)$$

* 注意 $G(k,n)$ 不只会 $\{(I \ C)\}$, 这只是一个 patch!

- For all $0 \leq k < n$, we have $G(k, n) \cong G(n-k, n)$

Idea: Construct iso.

$$\text{Pf: } \varphi: G(k, n) \longrightarrow G(n-k, n)$$

$$L = (I_k C) \longmapsto L^\perp \quad (L \text{ 的正交补})$$

bijection is obvious: $(L^\perp)^\perp = L$. 证 φ 是 mor (同理可证 φ^{-1} 也是 mor, 只要把 k 换成 $n-k$). 需重写 L^\perp 的形式算出来.

$$\text{经过硬凑, 发现 } (I_{n-k} C) \begin{pmatrix} -C \\ I_{n-k} \end{pmatrix} = 0. \text{ 故 } L^\perp = (-C^T \ I_{n-k})$$

$$\varphi \text{ 可以看作 } \mathbb{A}^{k(n-k)} \rightarrow \mathbb{A}^{k(n-k)} \text{ 的 map}$$

$$C \longmapsto -C^T$$

$-C^T$ 的每一个分量是 C_{ij} 的 deg 1 poly, 因此是 mor.

每一片 patch 上 iso 粘出整体 iso.

- Birational maps: Almost the same (contain iso open sets though var are not iso)

* Exps for noniso vars containing iso open sets (examples for birational)

$$(1) \mathbb{P}^n \times \mathbb{P}^m \not\cong \mathbb{P}^{n+m}. \quad \mathbb{P}^n \times \mathbb{P}^m \supseteq \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m} \subseteq \mathbb{P}^{n+m}$$

$$(2) G(k, n) \supseteq \mathbb{A}^{k(n-k)} \subseteq \mathbb{A}^{k(n-k)}$$

(3)

$$\overbrace{\mathbb{V}(x_1^2 - x_2^2)} \neq \mathbb{A}^2$$

把 singular pt 拿掉 iso to $\mathbb{A}^2 \setminus 0$

- Rational maps 非常有用, 开集相交非空

△ Def: X, Y : irr var. A rational map f from X to Y , written $f: X \dashrightarrow Y$, is a mor $f: U \rightarrow Y$. We say $f_1, f_2: X \dashrightarrow Y$ defined on U_1 resp. U_2 are the same if $f_1 = f_2$ on a non-empty open subset of $U_1 \cap U_2$.

* $f: X \dashrightarrow Y$ defined on U 也记作 $f: U \rightarrow Y$.

$$\begin{aligned} * \text{Exp: } f_1: \mathbb{A}^1 \setminus \{0\} &\longrightarrow \mathbb{A}^1, x \mapsto \frac{1}{x} \\ f_2: \mathbb{A}^1 \setminus \{0, 1\} &\longrightarrow \mathbb{A}^1, x \mapsto \frac{1}{x} \end{aligned} \quad f_1 = f_2 \text{ as rational maps.}$$

* 这是等价关系. 反身性、对称性显然. Check 传递性:

$f_1: U_1 \rightarrow Y$ agree with $f_2: U_2 \rightarrow Y$ on an open set $U_{12} \stackrel{\text{open}}{\subseteq} U_1 \cap U_2$

$f_2: U_2 \rightarrow Y$ agree with $f_3: U_3 \rightarrow Y$ on an open set $U_{23} \stackrel{\text{open}}{\subseteq} U_2 \cap U_3$

$U_{12} \cap U_{23} \stackrel{\text{open}}{\subseteq} U_1 \cap U_2 \cap U_3 \subseteq U_1 \cap U_3$. 只要 $U_{12} \cap U_{23} \neq \emptyset$.

X irr, $U_1 \cap U_3 \stackrel{\text{open}}{\subseteq} X \Rightarrow U_1 \cap U_3$ irr. $U_{12} \stackrel{\text{open}}{\subseteq} U_1 \cap U_3$, $U_{23} \stackrel{\text{open}}{\subseteq} U_1 \cap U_3$

$\underline{U_1 \cap U_3 \text{ irr}} \Rightarrow U_{12} \cap U_{23} \neq \emptyset$.

* $f_1: U_1 \rightarrow Y, f_2: U_2 \rightarrow Y$ are the same $\Rightarrow f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$.

Pf: Let $W = \{x \in U_1 \cap U_2 \mid f_1(x) = f_2(x)\}$.

$= i^{-1}(\Delta_Y)$ is closed in $U_1 \cap U_2$, where $i: U_1 \cap U_2 \xrightarrow{(f_1, f_2)} Y \times Y$
 $x \longmapsto (f_1(x), f_2(x))$

f_1 and f_2 are the same $\Rightarrow \exists$ open set $U \subseteq U_1 \cup U_2$, $f_1|_U = f_2|_U$

Obviously $U \subseteq W \stackrel{\text{closed}}{\subseteq} U_1 \cup U_2$.

Since $W \stackrel{\text{closed}}{\subseteq} U_1 \cup U_2$, $\bar{U} \subseteq W$. However U is an open set in $U_1 \cup U_2$, so $\bar{U} = U_1 \cup U_2$. Then $U_1 \cup U_2 = \bar{U} \subseteq W$ meaning that $U_1 \cup U_2 \subseteq W$. So $W = U_1 \cup U_2$, i.e., $f_1|_{U_1 \cup U_2} = f_2|_{U_1 \cup U_2}$.

• Define birational

* Intuitively, X and Y are birational if

$\exists f: X \dashrightarrow Y$, $g: Y \dashrightarrow X$ with $g \circ f = \text{id}_Y$, $f \circ g = \text{id}_X$

But it may run to problems in some case:

$f: X \dashrightarrow Y$ with $f(x)$ closed in Y and

$g: Y \dashrightarrow X$ is defined on $Y \setminus f(x)$. and $g \circ f$ is meaningless!

Hence we need concept "dominant" just ensure composition is well-def.

△ Def: (Dominant) $f: X \dashrightarrow Y$ is called dominant

- if (TFAE)
 - $f(x)$ dense in Y for all representations
 - $f(x)$ contains a nonempty open subset of Y for all rep.

$$U \xrightarrow{f} Y \xrightarrow{g} Z$$

• f dominant, so $W \cap V \neq \emptyset$,
 $\exists V \subseteq f(U)$, $V \stackrel{\text{open}}{\subseteq} Y$ since Y irr

• assume g defined
on $W \stackrel{\text{open}}{\subseteq} Y$

△ A rational map $f: X \dashrightarrow Y$ is called birational if f is dominant

and \exists dominant $g: Y \dashrightarrow X$ with $g \circ f = \text{id}_Y$ and $f \circ g = \text{id}_X$

* We say X and Y are birational if \exists birational $f: X \dashrightarrow Y$

* If X is birational to a proj. space P^n we call X is rational.

△ two irr var are birational \Leftrightarrow they contain iso. non-empty open sets.

$\Rightarrow f: X \dashrightarrow Y$ birational defined on U with inverse $g: Y \dashrightarrow X$
defined on V , then $U \cap f^{-1}(V)$ iso to $V \cap g^{-1}(U)$ by f and g .

$\Leftarrow X \stackrel{\text{open}}{\subseteq} U \stackrel{\text{open}}{\subseteq} V \subseteq Y$ $f: X \dashrightarrow Y$ defined on U is birational.

△ Fact: Birational irr var have the same dimension.

△ Example of birational maps — Rational functions and function fields

* A rational function on X is a rational function $\varphi: X \dashrightarrow \mathbb{A}^1$

given by $\varphi \in \mathcal{O}_X(U)$ on some $U \stackrel{\text{open}}{\subseteq} X$. The set of all rational functions on X will be denoted by $K(X)$.

a field $\varphi_1 + \varphi_2, \varphi_1 \cdot \varphi_2$ defined on $U_1 \cap U_2$

* $\emptyset \neq U \stackrel{\text{open}}{\subseteq} X_{\text{irr}}$, then $K(U) \cong K(X)$ is obvious.

我们总可取 aff. open $U \subseteq X$, 于是问题变成 $K(\text{aff.})$ 是什么?

U aff. open in irr $X \Rightarrow U$ is irr $\Rightarrow A(U)$ is a domain

Then $K(U)$ 是 $A(U)$ 的分式域. (Reason: U 可以写成 $D(f_i)$ 之并.)

$K(U) \cong K(D(f))$. $D(f)$ 上 function 永远可以写成 $\frac{f}{g}$)

- Blow up : a general procedure 把 irr var 变成与原来 birational 的新 irr. var.

Def: Let $X \subseteq \mathbb{A}^n$ an aff. var. For some given $f_1, \dots, f_r \in A(X)$
set $U = X \setminus V(f_1, \dots, f_r)$, there is a well-defined mor

$$f: U \rightarrow \mathbb{P}^{r-1}, x \mapsto (f_1(x), f_2(x), \dots, f_r(x))$$

Consider $\tilde{U}_f = \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{P}^{r-1}$ which is closed in $U \times \mathbb{P}^{r-1}$

The closure of \tilde{U}_f in $X \times \mathbb{P}^{r-1}$ is called blow up of X at f_1, \dots, f_r
denoted by \tilde{X} . There is projection $\pi: \tilde{X} \rightarrow X$

△ (1) $\tilde{U}_f = \tilde{X} \cap U \times \mathbb{P}^{r-1}$

(2) $\pi: \tilde{X} = \tilde{U}_f \cup \tilde{U}_f \rightarrow X = U \cup X \setminus U$

isomorphic
not isomorphic

★ $\tilde{X} = \overline{U} = \overline{\tilde{U}_f}$

* $\tilde{U}_f \cong U$ be a dense open subset.

如我们所愿，我们造了一个 \tilde{X} birational to X (iso on an open subset)

Note that : we sometimes identify \tilde{U}_f as V (view $U \subseteq \tilde{X}$)

- △ Blow up of subvar. Let Y be a closed var of X .

$U \cap Y = Y \setminus V(f_1, \dots, f_r)$ \tilde{Y} is the closure of $\tilde{U}_{f|_{U \cap Y}}$ in $Y \times \mathbb{P}^{r-1}$,

where $f|_{U \cap Y}: U \cap Y \rightarrow \mathbb{P}^{r-1}$

(1) \tilde{Y} is closed in \tilde{X} . It is called strict transform of Y in \tilde{X} .

$\tilde{Y} \stackrel{\text{closed}}{\subseteq} Y \times \mathbb{P}^{r-1} \stackrel{\text{closed}}{\subseteq} X \times \mathbb{P}^{r-1} \Rightarrow \tilde{Y} \text{ closed in } X \times \mathbb{P}^{r-1}$

$\Rightarrow \tilde{Y} = \tilde{Y} \cap \tilde{X} \stackrel{\text{closed}}{\subseteq} \tilde{X}$

closed in $X \times \mathbb{P}^{r-1}$

Moreover

(2) \tilde{Y} is closure of $Y \cap U$ in \tilde{X} .

$$\overline{Y \cap U}_{\tilde{X}} = \overline{Y \cap U}_{\tilde{Y}} = \tilde{Y}$$

\tilde{Y} closed in \tilde{X}

* If $X = \bigcup_i X_i$ — irr decomposition, $\tilde{X} = \bigcup_i \tilde{X}_i$

\tilde{X}_i is closure of $X_i \cap U$ in \tilde{X} . 取 closure 与取有限并对易.

- △ Exp: Compute by taking closure

t=1. $U = X \setminus V(f_1)$. $\tilde{X} \subseteq X \times \mathbb{P}^0 \cong X$.

so \tilde{X} is closure of U in X (under iso)

$\Leftrightarrow f_1 \neq 0$. $\overline{U} = X = \tilde{X}$.

(2) $f_1 = 0$. $U = X \setminus V(0) = \emptyset$. $\tilde{X} = \overline{U} = \emptyset$.

• Computation — Description not refer to taking closures.

△ Lemma: The blow up \tilde{X} of an aff. var X at $f_1, \dots, f_r \in A(X)$

Satisfies $\tilde{X} \subseteq \{(x, y) \in X \times \mathbb{P}^{r-1} \mid y_i f_j(x) = y_j f_i(x) \text{ for all } i, j = 1 \dots r\}$
 $=: Z$

Pf: $Z_f = \{(x, [f_1(x), \dots, f_r(x)]) \mid \text{显然满足方程 } y_i f_j(x) = y_j f_i(x)\}$

Z closed in $X \times \mathbb{P}^{r-1}$ so $\tilde{X} = \overline{Z}_f \subseteq Z$.

△ Exp: Blow up of A^n at the coord. funs.

* Lemma 1: $X \overset{\text{closed}}{\subseteq} Y, X, Y \text{ irr}, \dim X = \dim Y \Rightarrow X = Y$

pf If $X \subsetneq Y$, then by def of dim, we have

$$\dots \subseteq X_n \subseteq Y \quad \dim Y \geq \dim X + 1 \iff$$

* Lemma 2: If X is irr, then \tilde{X} is irr.

$X = \overline{U}^*$, X irr so U is irr. Then $\overline{U}^* = \tilde{X}$ irr. (U irr $\Leftrightarrow \overline{U}$ irr)

* $X = A^n, f_1 = x_1, f_2 = x_2, \dots, f_n = x_n$.

$\tilde{A}^n \subseteq \{(x, [y_1, \dots, y_n]) \in X \times \mathbb{P}^{n-1} \mid y_i x_i = x_j y_j, \forall i, j\} =: Z$

Z has an aff. open cover: $Z \cap \{y_i \neq 0\}$.

We can pick $y_i = 1$, then $Z \cap \{y_i \neq 0\} = \{(x, [y_1, \dots, 1, \dots, y_n]) \in X \times \mathbb{P}^{n-1} \mid$

$$y_j x_i = x_j, \forall j\}$$

从 equ 中梳理真正的变量

$$U_i := Z \cap \{y_i \neq 0\} = \left\{ (y, x_i, y_2 x_i, \dots, y_n x_i), \left[y, \underset{i}{1}, \dots, y_n \right] \right\}$$

$$\cong \{(x_i, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)\} \cong A^n$$

$$\begin{cases} U_i \cap U_j \neq \emptyset \\ Z = \bigcup U_i \text{ is an open cover} \Rightarrow \dim Z = \sup \dim U_i = n \end{cases}$$

$$\begin{cases} U_i \cap U_j \neq \emptyset \\ Z = \bigcup U_i \text{ is an open cover} \Rightarrow Z \text{ is irr} \\ U_i \cong A^n \text{ irr} \end{cases}$$

A^n birational to \tilde{A}^n so $\dim \tilde{A}^n = \dim A^n = n$.

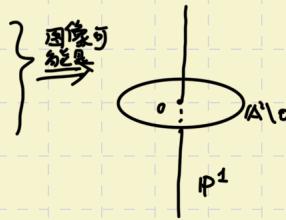
A^n is irr so \tilde{A}^n is irr. Then $\begin{cases} \tilde{A}^n \overset{\text{closed}}{\subseteq} Z \\ \tilde{A}^n, Z \text{ irr} \\ \dim \tilde{A}^n = n = \dim Z \end{cases} \Rightarrow \tilde{A}^n = Z$

Question: What's the picture of \tilde{A}^2 ?

* Consider $\pi: \tilde{A}^2 \rightarrow A^2$, where $\tilde{A}^2 = \mathbb{Z} = \{(x, y) \in A^2 \times \mathbb{P}^1 \mid y_2 x_1 = y_1 x_2\}$

$$\begin{aligned} i) \quad \pi^{-1}(0) &= \{(0, y) \in A^n \times \mathbb{P}^{n-1} \mid y_2 \cdot 0 = y_1 \cdot 0\} \\ &= \{(0, y) \in A^n \times \mathbb{P}^{n-1} \} \cong \mathbb{P}^{n-1}. \end{aligned}$$

$$ii) \quad \tilde{A}^2 \setminus \pi^{-1}(0) \cong A^2 \setminus 0$$



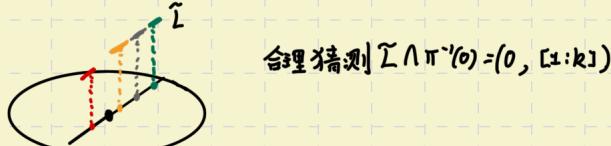
But it's wrong: A^2 irr $\Rightarrow \tilde{A}^2$ irr, BUT in the right picture it's reducible.

Correct picture: To obtain correct picture,

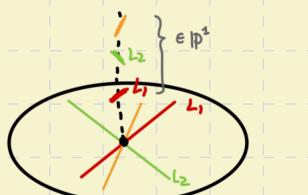
1) Let's consider line $L \subseteq A^2$

2) Compute strict transformation of L in \tilde{A}^2

设 L 的斜率是 k . 每点 on L can be represented by (x_1, kx_1)
So the image is $(x_1, kx_1), [x_1 : kx_1] \in A^2 \setminus 0 \times \mathbb{P}^1$

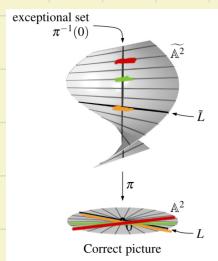


3) 不同 line L 与 $\pi^{-1}(0)$ 的交.



univ Exceptional set parametrize
the direction (L_1, L_2, L_3) in \tilde{A}^2

\tilde{A}^2 示意图:



☆ 原来相交的线分开了
Blow up后分开了

☆ 这是 Möbius 带

- prop: The blow up of an aff. var X at $f_1, \dots, f_r \in A(x)$ depends only on the ideal $\langle f_1, \dots, f_r \rangle \trianglelefteq A(x)$, i.e., if $\langle f_1, \dots, f_r \rangle = \langle f'_1, \dots, f'_s \rangle$, and $\pi: \tilde{X} \rightarrow X$ and $\pi': \tilde{X}' \rightarrow X$ are corresponding blow ups, there is an iso $F: \tilde{X} \rightarrow \tilde{X}'$ with $\begin{cases} F \circ \pi = \pi' \\ \pi \circ F = \pi' \end{cases}$

* Blow up 只与 ideal 有关, independent of 生成元选取.

Pf: (Routine check) Construct F . Trick: 两套生成元必可互相表示.

Assume $f_i = \sum g_{il} f'_l$, $f'_l = \sum h_{lk} f_k$.

$$F: \tilde{X} \rightarrow \tilde{X}'$$

$$(x, [y_1, \dots, y_r]) \mapsto (x, [y_1, \dots, \sum h_{lk} f_k, \dots])$$

Idea: 已知在 $X \setminus V(f_1, \dots, f_r) \subseteq \tilde{X}$ 上的点

坐标系 $(x, [f_1(x), f_2(x), \dots, f_r(x)])$. 我们希望有

$$F|_{X \setminus V(f_1, \dots, f_r)} : X \setminus V(f_1, \dots, f_r) \rightarrow \tilde{X}'$$

$$(x, [f_1(x), \dots, f_r(x)]) \mapsto (x, [\dots, \underset{\text{II}}{\boxed{\sum_k h_{jk} f_k}}, \dots])$$

$\underset{k\text{-th component}}{\text{k}}$

下面只需证 ① F 是 mor ② F 是 iso.

③ F 是 iso 非常容易, 只要构造

$$G : \tilde{X}' \rightarrow X$$

$$(x, [y_1, \dots, y_s]) \mapsto (x, [\dots, \sum g_{il} f_l, \dots])$$

如果证明 F 是 mor, 则 G 是 mor. 易知 $G = F'$, 故 F 是 iso.

It suffices to show ① F is a mor.

$$\begin{cases} \text{1) } F \text{ is well-defined} & \{(1.1) [\dots, \sum h_{jk} f_k, \dots] \neq [0, 0, \dots, 0] \\ & (1.2) F(\tilde{x}) \subseteq \tilde{X}' \end{cases}$$

(1.1)

Idea: $U = X \setminus V(f_1, \dots, f_r)$. $\tilde{X} = \bar{U}$. 从取闭包的 space X 手.

已知 $\tilde{X}|_U = \{(x, [f_1(x), \dots, f_r(x)])\}$. 显然 $\tilde{X}|_U \subseteq V((x, y) : y_i = \sum_k g_{ij} h_{jk} y_k)$ closed

$\tilde{X} = \bar{U}$, so $\tilde{X} \subseteq V((x, y) : y_i = \sum_k g_{ij} h_{jk} y_k)$. If $\exists (x, y) \in \tilde{X}$ with $\sum h_{jk} y_k = 0$ by j

then $y_i = 0 \forall i$, i.e., $y = 0$. It's impossible because $y \in \mathbb{P}^{r-1}$.

(1.2)

Idea: continue map 取闭才保闭集 w.t.s. $\tilde{X} \subseteq F'(\tilde{X}')$.

Idea: 出现闭集从取闭包的 space U 手. $\tilde{X} = \bar{U}$. w.t.s. $J_f \subseteq F'(\tilde{X}')$

$J_f = \{(x, [f_1(x), \dots, f_r(x)])\}$ 表达式已知, check $F(J_f) \subseteq \tilde{X}'$ 更容易.

$$F(J_f) = \{(x, [\dots, \sum h_{jk} f_k(x), \dots])\} = \{(x, [f'_j, \dots])\} = J_{f'} \subseteq \tilde{X}'$$

So we complete the proof.

• Blow up 只由 ideal 决定 \Rightarrow 把对多项式作 Blow up 改成 对 多项式 生成理想 的 blow up.

1. $J \trianglelefteq A(x)$ be an ideal. We define blow up of x at J to be the blow up of X at any set of generators of J .

2. $Y \subseteq X$ be closed subvariety. We define blow up of X at Y to be the blow up of X at $I(Y) \trianglelefteq A(x)$. and Unique up to isomorphism

3. X is an arbitrary variety. $Y \subseteq X$ be a closed subvariety.

Question: How to define blow up of X at Y ?

Pick an affine open cover $\{U_i : i \in I\}$ of X . $\tilde{U}_i :=$ blow up of U_i at closed subvariety $U_i \cap Y$. 把 \tilde{U}_i 拼起来得到 blow up of X at Y .

4. Special case: Blow up at a point a . If aff. open cover $\{U_i : i \in I\}$ of X .

$$\tilde{U}_i = \text{blow up of } U_i \text{ at } a = \begin{cases} \text{blow up } a \in U_i & \text{if } a \in U_i \\ U_i & a \notin U_i \end{cases}$$

Compute $\tilde{U}_0 \Rightarrow$ Glue $X \setminus a$ to \tilde{U}_0 along common open set $U_0 \setminus \{a\}$.

Slogan: Blow up 是 local 事情. Blow up at a point 时总可取 aff. open set cover 住 this point. 因此总可以认为 X is an aff. var.

5. Blow up of projective var.

Slogan: Blow up of proj. var. is still proj. var.

Let X be a proj. var., $f_1, \dots, f_r \in S(X)$ are homo with same degree.

Blow up of X at f_1, \dots, f_r is \widetilde{X} in $X \times \mathbb{P}^{r-1}$ where \widetilde{X} is:

$$\widetilde{X} = \{(x, [f_1(x), f_2(x), \dots, f_r(x)]) : x \in U := X \setminus V(f_1, \dots, f_r)\} \subseteq U \times \mathbb{P}^{r-1}$$

\widetilde{X} closed in proj var. $X \times \mathbb{P}^{r-1}$ so $\widetilde{X} = \widetilde{X}$ is proj. var.

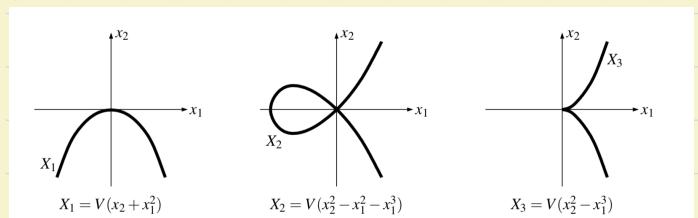
- Exceptional set of the blow up of a general variety X at point $a \in X$ parametrizes the tangent direction of X at a .

- Tangent cones. X : a variety. \widetilde{X} : blow up of X at a point $a \in X$.

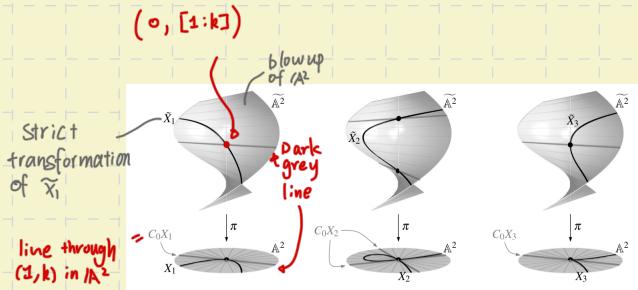
$$\begin{cases} \pi^{-1}(a) \subseteq \widetilde{X} \times \mathbb{P}^{r-1} \cong \mathbb{P}^{r-1} \\ a \text{ closed} \Rightarrow \pi^{-1}(a) \text{ closed} \end{cases} \Rightarrow \pi^{-1}(a) \text{ is a projective variety.}$$

Cone over $\pi^{-1}(a)$ is called
Tangent Cone

Exp 1 Intuitive picture.



Consider strict transformation of X_1, X_2, X_3 at origin:



- Exp 2,3 Convinced you:
- \circ a 光滑, tangent cone 是切空间
 - \circ a 不光滑, tangent cone 不是切空间, 会包含附近信息.
 - \circ tangent cone 与光滑性研究有关.

Exp 2: (a 光滑) $X = V(x_2 - x_1^2) \subseteq \mathbb{A}^2$ $a = (0,0)$. Compute $C_a(X)$.

Idea: $\widetilde{X} \subseteq Z = \{(x, [y_1, y_2]) \in X \times \mathbb{P}^1 \mid y_1 x_2 = y_2 x_1\} \subseteq X \times \mathbb{P}^1$.

has two aff. patch \leftarrow has two aff. patch \leftarrow has two aff. patch

$\widetilde{X} = \widetilde{X}_f$ consider \widetilde{X}_f has two aff. patch

Trick: 用 more 来说明 Z 的 aff. patch. $Z = U_1 \cup U_2$.

$\varphi: \mathbb{A}^1 \xrightarrow{\sim} U_1 \subseteq Z$, pick $y_1 = 1$

$(x_1, y_1) \mapsto ((x_1, y_2 x_1), [1 : y_2])$

$\psi: \mathbb{A}^1 \xrightarrow{\sim} U_2 \subseteq Z$, pick $y_2 = 1$

$(x_2, y_1) \mapsto ((y_1 x_2, x_2), [y_1 : 1])$

$$\begin{aligned}
\Gamma_f \cap U_1 &= \left\{ (x_1, y_2 x_1), [x_1 : y_2 x_1] \mid (x_1, y_2 x_1) \in X \setminus 0 \right\} \\
&= \left\{ (x_1, y_2 x_1), [1 : y_2] \mid \begin{array}{l} (y_2 x_1) - x_1^2 = 0 \\ x_1 \neq 0 \end{array} \right\} \\
&= \left\{ (x_1, x_1 y_2), [x_1, x_1 y_2] \mid \begin{array}{l} (x_1, y_2) \in X \setminus 0 \\ y_2 - x_1 \neq 0 \end{array} \right\} \\
&\stackrel{\text{结合 more}}{\cong} V(x_2 - x_1^2) \cap D(x_1) \subseteq \mathbb{A}^2 \cong U_1
\end{aligned}$$

Closure of Γ_f in $Z = U_1 \cup U_2$ is $\tilde{X} = \overline{\Gamma_f}$.

Closure of $\Gamma_f \cap U_1$ in $Z \cap U_1 = U_1$ is $\tilde{X} \cap U_1$.

$$\begin{aligned}
\Gamma_f \cap U_1 &= V(x_2 - x_1^2) \cap D(x_1) \stackrel{\text{open}}{\subseteq} V(x_2 - x_1^2) \Rightarrow \Gamma_f \cap U_1 \text{ irr. in } V(x_2 - x_1^2) \\
&\quad \text{as sub var.}
\end{aligned}$$

If $A \subseteq B \subseteq X$

+then $\bar{A}^X = \bar{A}^B$

pf:

$$\bar{A}^X \cap B = \bar{A}^B \Rightarrow \bar{A}^B \subseteq \bar{A}^X$$

It suffices to show $\bar{A}^X \subseteq \bar{A}^B$.

$$\begin{aligned}
\bar{A}^B &\stackrel{\text{closed}}{\subseteq} B \stackrel{\text{closed}}{\subseteq} X \Rightarrow A \subseteq \bar{A}^B \stackrel{\text{closed}}{\subseteq} X \\
&\Rightarrow \bar{A}^X \subseteq \bar{A}^B
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \overline{\Gamma_f \cap U_1} = V(x_2 - x_1^2) \\
&\text{as closure in } V(x_2 - x_1^2)
\end{aligned}$$

\Rightarrow closure of $\Gamma_f \cap U_1$ in U_1

$$= \overline{\Gamma_f \cap U_1} V(x_2 - x_1^2) = V(x_2 - x_1^2)$$

$$\Gamma_f \cap U_2 = \left\{ (y_1, x_2, x_2), [y_1 : 1] \in X \setminus 0 \times \mathbb{P}^1 \right\}$$

$$= \left\{ (y_1, x_2, x_2), [y_1 : 1] \mid \frac{x_2 - y_1 x_2}{x_2} = 0 \right\}$$

$$= \left\{ (y_1, x_2, x_2), [y_1 : 1] \mid \frac{1 - y_1^2 x_2}{x_2} = 0 \right\}$$

$$= V(1 - y_1^2 x_2) \cap D(x_2) \subseteq U_2 \cong \mathbb{A}^2$$

$$\overline{\Gamma_f \cap U_2}^{U_2} = \tilde{X} \cap U_2.$$

$$\begin{aligned}
\Gamma_f \cap U_2 &\stackrel{\text{open}}{\subseteq} V(1 - y_1^2 x_2) \Rightarrow \overline{\Gamma_f \cap U_2}^{V(1 - y_1^2 x_2)} = V(1 - y_1^2 x_2)
\end{aligned}$$

$$\begin{aligned}
V(1 - y_1^2 x_2) &\stackrel{\text{closed}}{\subseteq} U_2 \cong \mathbb{A}^2 \Rightarrow \overline{\Gamma_f \cap U_2}^{U_2} = \overline{\Gamma_f \cap U_2}^{V(1 - y_1^2 x_2)} = V(1 - y_1^2 x_2)
\end{aligned}$$

$$\begin{cases} \tilde{X} \cap U_1 = V(y_2 - x_1) \\ \tilde{X} \cap U_2 = V(1 - y_1^2 x_2) \end{cases}$$

$$\pi^{-1}(a) \cap U_1 = \left\{ (a, [1 : y_2]) \in \tilde{X} \cap U_1 \right\}$$

$$= \left\{ (a, [1 : y_2]) \in V(y_2 - \frac{a_1}{a_2}) \right\}$$

$$= \left\{ (0, [1 : 0]) \right\} \stackrel{\text{viewed as}}{=} [1 : 0] \in \mathbb{P}^1$$

$$So \quad \pi^{-1}(a) = \{[1:0]\} \quad (\pi^{-1}(a) = V(x_2) \subseteq \mathbb{A}^2). \quad \text{即 } \cup \text{ 为 } \mathbb{A}^2.$$

Exp 3: (a 不光滑) $x = V(x_2^2 - x_1^2 - x_1^3)$.

$$\Gamma_f \cap U_1 = \{(x_1, x_1 y_2), [1, y_2]\} \in V(x_2^2 - x_1^2 - x_1^3) \setminus 0 \in \mathbb{P}^1\}$$

$$= \{(x_1, x_1 y_2), [1, y_2] \mid \frac{(x_1 y_2)^2 - x_1^2 - x_1^3}{x_1} = 0\}$$

$$\cong V(y_2^2 - x_1 - 1) \cap D(x_1)$$

$$\text{同理 } \widetilde{X} \cap U_1 = V(y_2^2 - x_1 - 1)$$

$$\text{同理 } \widetilde{X} \cap U_2 = V(1 - y_1^2 - x_2 y_1^3)$$

$$\pi^{-1}(a) \cap U_1 = \{(a, [1:y_2]) \in \widetilde{X} \cap U_1\}$$

$$= \{(a, [1:y_2]) \in V(y_2^2 - x_1 - 1)\}$$

$$= \{(0, [1:1]), (0, [1:-1])\}$$

$$\pi^{-1}(a) \cap U_2 = \{(a, [y_1:1]) \in V(1 - y_1^2 - x_2 y_1^3)\}$$

$$= \{(0, [1:1]), (0, [1:-1])\}$$

$$So \quad \pi^{-1}(a) = \{[1:1], [1:-1]\} \in \mathbb{P}^1.$$

$$\pi^{-1}(a) = V(x_2^2 - x_1^2) \quad \begin{array}{c} \times \\ (1,1) \\ (1,-1) \end{array}$$

• Two props about dimension.

(1) $\pi: \widetilde{X} \rightarrow X$ be the blow up of an irr aff. var X at $f_1, \dots, f_r \in A(X)$. Then every irr component of exceptional set $\pi^{-1}(V(f_1, \dots, f_r))$ has codim 1 in \widetilde{X} . (It's called exceptional hypersurface of the blow up)

Idea: 证 $\pi^{-1}(V(f_1, \dots, f_r))$ 的每个 aff. patch 有 codim = 1 ("pure codim = 1")

$$\begin{aligned} \widetilde{X} \subseteq \mathbb{Z} &= \{(x, [y_1, \dots, y_r]) \in X \times \mathbb{P}^{r-1} \mid y_i f_j(x) = y_j f_i(x), \forall i, j\} \\ &= \bigcup_i U_i \quad \text{where } U_i = \{y_i = 1\} \cap \mathbb{Z} = \{(x, [y_1, \dots, 1, \dots, y_r]) \in X \times \mathbb{P}^{r-1} \mid y_k f_j(x) = y_j f_k(x), \forall i, j\} \end{aligned}$$

$$U_i \cap \pi^{-1}(V(f_1, \dots, f_r)) = \{(x, [y_1, \dots, y_r]) \in \widetilde{X} \mid x \in V(f_1, \dots, f_r), y_i f_j(x) = y_j f_i(x), \forall i, j\}$$

$$Claim: U_i \cap \pi^{-1}(V(f_1, \dots, f_r)) = V_{u_i}(f_i) \quad \begin{array}{l} \text{只考虑 } f_i \\ \text{因为 } x \in V(f_1, \dots, f_r) \end{array}$$

If the claim is true, we have:

$$\begin{cases} X \text{ irr} \Rightarrow \widetilde{X} \text{ irr} \Rightarrow \widetilde{X} \cap U_i \subseteq \widetilde{X} : \text{is irr} \xrightarrow[\text{Thm}]{\text{Krull dim}} \text{ irr component of } \\ f_i \neq 0 \quad V_{\widetilde{X}}(f_i) \text{ has codim 1} \end{cases} \quad \text{in } \widetilde{X} \cap U_i.$$

i.e., A irr component of $U_i \cap \pi^{-1}(V(f_1, \dots, f_r))$ has dimension $\dim \widetilde{X} \cap U_i - 1 = \dim \widetilde{X} - 1$

So $U_i \cap \pi^{-1}(V(f_1, \dots, f_r))$ has pure dimension $\dim \widetilde{X} - 1$

$\{U_i \cap \pi^{-1}(V(f_1, \dots, f_r))\}$ is open cover of $\pi^{-1}(V(f_1, \dots, f_r)) \Rightarrow \dim \pi^{-1}(V(f_1, \dots, f_r)) = \sup \{\dim(U_i \cap \pi^{-1}(V(f_1, \dots, f_r)))\} = \dim \widetilde{X} - 1$.

pf for the claim: $U_i \cap \pi^{-1}(V(f_1, \dots, f_r)) \subseteq U_i$ 是 U_i 上的 aff. var.

$\forall (x, y) \in U_i \cap \pi^{-1}(V(f_1, \dots, f_r))$, we have $x \in V(f_1, \dots, f_r) \subseteq V(f_i)$.

Hence $U_i \cap \pi^{-1}(V(f_1, \dots, f_r)) \subseteq V_{U_i}(f_i)$.

$\forall (x, y) \in V_{U_i}(f_i)$, since $f_i(x) = y, f_j(x) = y_j, f_i(x) = 0$.

So $(x, y) \in V(f_1, f_2, \dots, f_r) \cap U_i$.

(2) $a \in \underset{\text{var}}{\tilde{X}}$. $\dim C_a X = \text{codim}_X a?$

Pf: $X = X_1 \cup X_2 \cup \dots \cup X_s$ are irr. component decomposition.

Trick: Assume $a \in X_i, \forall i$. If $a \notin X_i$, by blow up 的 local 性质
可以把 X_i 合并为 $X \setminus X_i$.

$$\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2 \cup \dots \cup \tilde{X}_s \Rightarrow C_a X = C_a X_1 \cup C_a X_2 \cup \dots \cup C_a X_s \\ \Rightarrow \dim C_a X = \max \dim C_a X_i$$

Every irr component of $\pi|_{X_i}^{-1}(a)$ has dimension $\dim X_i - 1$
 { take cone, \mathbb{R}^1 } \dim 同时加 1

Every irr component of $C_a X_i$ has dimension $\dim X_i - 1 = (\dim X_i + 1) - 1$
 $= \dim X_i$

So $\dim C_a X = \max \dim C_a X_i = \max \dim X_i = \text{codim}_X a$

Local dimension $\text{codim}_X a$ 是 cover a 时 irr comp. 最大值数.

~~Q:~~ $\pi^{-1}(a) = \bigcup_i \pi|_{X_i}^{-1}(a) \Rightarrow C\pi^{-1}(a) = \bigcup_i C\pi|_{X_i}^{-1}(a) \Rightarrow C_a X = \bigcup_i C_a X_i$
 $(C(A \cup B) = CA \cup CB)$
 \mathbb{P}^{r-1} complete so $p: X \times \mathbb{P}^{r-1} \rightarrow X$ is closed. Since $\tilde{X}_i \subseteq X \times \mathbb{P}^{r-1}$ closed,
 we have $p(\tilde{X}_i)$ closed. $p|_{\tilde{X}_i} = \pi$, so $\pi(\tilde{X}_i)$ closed in X_i .

$X_i \setminus a \stackrel{\text{open}}{\subseteq} \tilde{X}_i \Rightarrow X_i \setminus a$ irr and $\overline{X_i \setminus a} = X_i$.

$\Rightarrow X_i = \overline{X_i \setminus a} \subseteq \pi(\tilde{X}_i) \Rightarrow a \in \pi(\tilde{X}_i) \Rightarrow \pi|_{X_i}^{-1}(a) \neq \emptyset$
 $\leftarrow X_i \setminus a \subseteq \underbrace{\pi(\tilde{X}_i)}_{\text{closed}} \Rightarrow C_a X_i = C(\pi|_{X_i}^{-1}(a))$ is well-def.

• Blowing up 可以用来 extend mor.

* X : aff. var. Consider $f: U \longrightarrow \mathbb{P}^{r-1}$ where $U = X \setminus V(f_1, \dots, f_r)$.
 $x \mapsto (f_1(x) : \dots : f_r(x))$

We can extend it after blowing up. \tilde{X} is the blow up of X at $\langle f_1, \dots, f_r \rangle$.

$\tilde{f}: \tilde{X} \longrightarrow \mathbb{P}^{r-1}$ is projection ($\tilde{X} \subseteq X \times \mathbb{P}^{r-1}$) with $\tilde{f}|_U = f$. \tilde{f} is the extension.

△ $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in one point is iso to \mathbb{P}^2 blown up in two points.

• Tangent space and tangent cone

- i) We can approximate a variety by tangent cone, e.g. 
- ii) But in practice one often wants to approximate by a linear space rather than cone.

iii) Tangent space: a linear space, take linear terms of poly.

Tangent cone: 既不是线性的, 取 initial terms of poly.

Def: Let $a \in X$. By choosing aff. n.b.h. of a , assume $X \subseteq \mathbb{A}^n$ and $a=0$.

$T_a X := V(f_i : f \in I(X))$ is called tangent space of X at a .

f_i : linear term of f .

Note that: ① Why we want $a=0$? $0=a \in X$, $\forall f \in I(X)$, $f(a)=0$ & $f(0)=0$.
于是 $I(X)$ 中任何 poly 常数项为 0.

② $T_a X = V(f_i : f \in I(X)) = V(f_i : f \in S)$ where $I(X) = \langle S \rangle$

$$\forall f \in I(X) = \langle S \rangle, \text{ we have } f = \sum_{g_i \in S} a_i g_i \Rightarrow f_i = \sum_{g_i \in S} a_i(0) \cdot (g_i)_1 + (a_i)_1 \cdot \underbrace{\sum_{\tilde{g}_i} a_i(\tilde{g}_i) \cdot (\tilde{g}_i)_1}_{0} = \sum_{g_i \in S} a_i(0) \cdot (g_i)_1$$

$$\Rightarrow f_i \in \langle (g_i)_1 : g_i \in S \rangle$$

③ 意思是 $V(f_i : f \in I(X))$, 而不是 $V(f_i : f \in J)$ where $V(J)=X$.

例: 在 \mathbb{A}^2 , $V(x) = V(x^2) = 0$, 但 $V(f_i : f \in \langle x \rangle) \subseteq V(f_i : f \in \{x\}) = V(x) = 0$
 $V(f_i : f \in \langle x^2 \rangle) \supseteq V(f_i : f \in \{x^2\}) = V(0) = \mathbb{A}^2$.
 在 generator 上说已经.

④ 常数项加, 则 initial term 和 linear term 合出的方程更多, 因此 $T_a X \supseteq C_a X$.

* Tangent space is a linear space containing tangent cone.

于是 $\dim T_a X \geq \dim C_a X = \text{codim}_X \{a\}$.

Excellent examples:

$$X_1 = V(x_2 + x_1^2)$$

$$C_0 X_1 = V(x_2)$$

$$T_0 X_1 = V(0) = \mathbb{A}^2$$

$$X_2 = V(x_2^2 - x_1^2 - x_1^3)$$

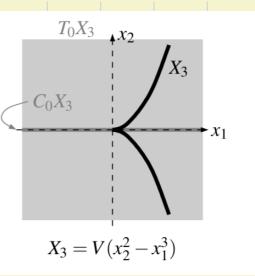
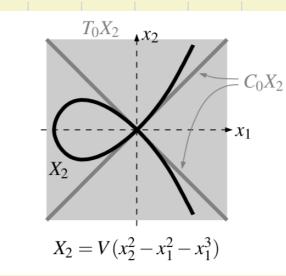
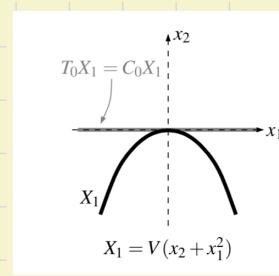
$$C_0 X_2 = V(x_2^2 - x_1^2)$$

$$T_0 X_2 = V(0) = \mathbb{A}^2$$

$$X_3 = V(x_2^2 - x_1^3)$$

$$C_0 X_3 = V(x_2^2)$$

$$T_0 X_3 = V(0) = \mathbb{A}^2$$



$T_a X$ 可以包含子等于 $C_a X$.

Tangent space 是包含 tangent cone $V(x_2^2 - x_1^3)$, 且是 $C_a X$ 张成的.
 是 linear space, 且是 \mathbb{A}^2 .

Tangent space 由内蕴描述.

Lemma: $X \subseteq \mathbb{A}^n$ aff. var. $a=0 \in X$. $I(a) = \langle \bar{x}_1, \dots, \bar{x}_n \rangle \leq \mathbb{A}(X) = k[x_1, \dots, x_n]/I(X)$

Then we have $I(a)/I(a)^2 \cong \text{Hom}_k(T_a X, k)$

* $\text{Hom}_k(T_a X, k)$ is the dual of tangent space.

Slogan: vector space $T_a X$ dual to $I(a)/I(a)^2$.

Pf: Idea: Check " $\ker\varphi = I(a)^2$ ". Routine check
 Let $\varphi: I(a) \rightarrow \text{Hom}_k(T_a X, k)$, w.t.s $\begin{cases} (1) \varphi \text{ well-defined} \\ (2) \varphi \text{ surj.} \\ (3) \ker\varphi = I(a)^2 \end{cases}$
 $\bar{f} \mapsto f_i|_{T_a X}$

Trick:
 ① 描述 $I(a)$ 中的元素只需要放到 $A(x) = k[x_1, \dots, x_n]/I(x)$ 中
 $\bar{f} \in I(a) \Leftrightarrow A(\bar{f}) = k[x_1, \dots, x_n]/I(x)$. $f \in k[x_1, \dots, x_n]$, \bar{f} 是等价类.
 ② $\varphi(\bar{f}) = ?$ $T_a X = V(f_i : f \in I(x)) \subseteq A^n$. $\text{Hom}_k(T_a X, k)$ 是 $T_a X$ 上的 linear function,
 B.P linear poly. 这里唯一出现 poly 是 f , 因此构造 linear poly f_i .

(1) Show φ well-defined.

For $\bar{f} = \bar{g} \in I(a)$, it means $f - g \in I(x)$. $\Rightarrow f_i - g_i|_{T_a X} = 0 \Rightarrow f_i|_{T_a X} = g_i|_{T_a X}$

(2) Show φ is surj. Let $h \in \text{Hom}_k(T_a X, k)$, i.e., f is a linear function on $T_a X$.

$T_a X \subseteq A^n$. We extend f to a linear function on A^n . We have $\bar{f} \in I(a) \cap A(x)$ be preimg.

(3) Show $\ker\varphi = I(a)^2$.

" \subseteq " $\forall h \in I(a)$ with $h|_{T_a X} = 0$, since $h_i|_{V(f_i : f \in I(x))} = 0$, the claim is reasonable.
 $\forall h \in \ker\varphi$

Claim: $\exists g \in I(x)$ s.t. $g_i = h_i$. If the claim is true, then $g-h$ has no const or linear term. Hence we have $\bar{h} = \bar{h-g} \in I(a)^2$

pf the claim: Dimension is really important when consider vector space.

Let $S = \{f_i : f \in I(x)\}$ and $W = \{\text{linear forms vanishing at } T_a X\}$. It suffices to show $S = W$. (则 $h_i \in W$ 可写成 $g_i, g \in I(x)$.) $T_a X = V(S)$, so $W \supseteq S$. We view S as sublinear space of $k[x_1, \dots, x_n]$ with dimension k , i.e., there're k linear eqns. So $T_a X = V(S)$ has dimension $n-k$. So $\dim W = n - (n-k) = k$.
 $\dim W = \dim S$ and $S \subseteq W$. So $S = W$.

\hookrightarrow is a linear transformation.

$T_a X = \ker A$. rank $A = n - \dim \ker A = n - (n-k) = k$.

" \supseteq " Let $\bar{f}, \bar{g} \in I(a)$, then $f(0) = g(0) = 0$ (f, g vanish at $a=0$)

For $\bar{f}, \bar{g} \in I(a)^2$, $\varphi(\bar{f}\bar{g}) = (\bar{f}\bar{g})_i = \underbrace{\bar{f}(0)}_0 \cdot \underbrace{g_i}_0 + \underbrace{f_i \cdot \bar{g}(0)}_0 = 0$. So $I(a)^2 \subseteq \ker\varphi$.

△ Intrinsic description of tangent space

To obtain intrinsic description, we need transfer $A(x)$ to $\mathcal{O}_{x,a}$

For general variety would require choice of an aff. coord that sends a to origin.
 (any point can have coord. (x_1, \dots, x_n) when choosing an aff. patch.)

Def3.19 in Gathmann: $\mathcal{O}_{x,a} \cong A(x)_{I(a)}$.

Besides $\mathcal{O}_{x,a}$ is a local ring with unique max ideal

$$I_a := \{(U, \varphi) \in \mathcal{O}_{x,a} \mid \varphi(a) = 0\}$$

$$\cong I(A)_{I(a)} = \left\{ \frac{g}{f} \mid g \in I(A), f \notin I(a) \right\}$$

$\mathcal{O}_{x,a}$ is independent of choice of aff. var.

prop: $I(a)/I(a)^2 \cong I_a/I_a^2$.

pf. If $f \in I(a)$, we have $f(a) \neq 0$. Trick: $f(a) \neq 0$ means $\frac{1}{f(a)}$ make sense.

$\Leftrightarrow A(x)/I(a) \rightarrow k$ is an iso. (Intuitively, $A(x)/I(a) = A_x(a)$ is functions on a , which is k .)
 $\bar{h} \mapsto h(a)$ rigorous check is also easy

$f(a) \neq 0 \Rightarrow \exists \frac{1}{f(a)} \in k$ By def An element denoted by $\frac{1}{f}$ in $A(x)/I(a)$ corresponds to $\frac{1}{f(a)} \in k$

Claim: $I(a)/I(a)^2 \cong S^{-1}(I(a)/I(a)^2)$, where $S = A(x) \setminus I(a)$.

If the claim: For any $\frac{g}{f} \in S^{-1}(I(a)/I(a)^2)$, i.e., $\frac{f}{g} \in A(x) \setminus I(a) \xrightarrow{\text{def}} \frac{1}{f} \in A(x)/I(a)$ corresponds to $\frac{g}{f} \in I(a)/I(a)^2$.

$\frac{g}{f} = \frac{1}{f(a)} g \in I(a)/I(a)^2$ so localization at S doesn't change $I(a)/I(a)^2$, since elements of S already invertible.

* $T_a X = (I(a)/I(a)^2)^*$ $\cong (I_a/I_a^2)^*$ where $I_a = I(a)_{\text{red}} = S^{-1}I(a)$ is maximal ideal of $\mathcal{O}_{X,a}$. Hence $T_a X$ is independent of choice of aff. patch.

• Smooth

△ Motivation — When $C_a X$ and $T_a X$ agree.

tangent cone $C_a X$: may Not linear, dimension = $\text{codim}_X \{a\}$.

tangent space $T_a X$: Linear space, dimension may bigger ($\text{codim}_X \{a\}$).

∴ Hence, we should pay attention to the case when two notations agree, i.e., X can be approximated around a by a linear space whose dimension is $\text{codim}_X \{a\}$.

△ Def: point $a \in X$ is called smooth, regular or nonsingular if $T_a X = C_a X$.

Otherwise is called a singular point of X .

△ $a \in X$ is a point. TFAE

(a) a is smooth

(b) $\dim T_a X = \text{codim}_X a$

∴ $\dim T_a X \leq \text{codim}_X a$

Pf: (a) \Rightarrow (b) a sm $\Rightarrow T_a X = C_a X \Rightarrow \dim T_a X = \dim C_a X = \text{codim}_X \{a\}$

(b) \Rightarrow (c) obviously.

(c) \Rightarrow (a) $\begin{cases} T_a X \supseteq C_a X \Rightarrow \dim T_a X \geq \dim C_a X = \text{codim}_X a \\ T_a X \subseteq \text{codim}_X a \end{cases} \Rightarrow \dim T_a X = \text{codim}_X a = \dim C_a X$

Since $T_a X$ is irr, $T_a X = C_a X$.

• Useful prop: a smooth, 至多只有 1 个 irr.分支 covera.

△ $T_a X \cong (I_a/I_a^2)^*$ $\Rightarrow \dim I_a/I_a^2 = \dim (I_a/I_a^2)^* = \dim T_a X \stackrel{a \text{ sm}}{=} \text{codim}_X a$

where I_a is the max ideal of local ring $\mathcal{O}_{X,a}$. Ring $\mathcal{O}_{X,a}$ with this prop is called regular local ring. Hence a sm $\Rightarrow \mathcal{O}_{X,a}$ regular $\Rightarrow \mathcal{O}_{X,a}$ integrand domain (Commutative alg.)

△ $\mathcal{O}_{X,a}$ is a domain means the variety is locally irr at every sm a . (Intuitively)

(两个 irr component 之和 都是 singular)

- Aff. Jacobi criterion $a \in X$ iff. Let $I(X) = \langle f_1, \dots, f_r \rangle$. Then X is sm at a iff $\text{rank } J \geq n - \text{codim}_X a$ (actually, when a sm $\text{rank } J = n - \text{codim}_X a$.)

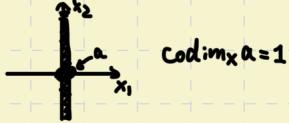
Pf: We can always assume $a=0$ when X aff. (Let $y = x-a$, a shift). For any generator f_j , $T_a X = V(f_j)_1 : j=1, \dots, r$. By Taylor expansion, $f_j = \sum_i \frac{\partial f_j}{\partial x_i} \cdot x_i$.

Hence $T_a X = \{x \mid \sum_i \frac{\partial f_j}{\partial x_i} x_i = 0, \forall j=1, 2, \dots, r\} = \ker J$.

a sm $\Rightarrow \dim \ker J = \dim T_a X \leq \text{codim}_X a$. Since $\dim \ker J = n - \text{rank } J$, $\text{rank } J = n - \text{codim}_X a$ (or \geq)

Note that it's not true for generators f'_1, \dots, f'_r for $X = V(f'_1, \dots, f'_r)$

Suppose $X = V(x^2)$ $\text{rank } J = \text{rank}(2x, 0)|_0 = 0 \neq 2-1=1$



- $V(f_1, \dots, f_r)$ version of Jacobi criterion.

$$a \in X = V(f_1, \dots, f_r) \quad J_{ij} = \frac{\partial f_i}{\partial x_j}$$

(a) If $\text{rank } J \geq n - \text{codim}_X a$, then a sm (Evidently X is sm)

(b) If $\text{rank } J = r$ then $\begin{cases} X \text{ is sm at } a \\ \text{codim}_X a = n-r. \end{cases}$

Pf:

(a) extend generator $J(f_1, \dots, f_r)$ to generators of $\sqrt{J} = I(V(J)) = I(X)$

as $f_1, \dots, f_r, f_{r+1}, \dots, f_s$.

$$\text{Then } \text{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{i \in [1, s] \\ j \in [1, n]}} \geq \text{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{i \in [1, r] \\ j \in [1, n]}} = \text{rank } J \geq n - \text{codim}_X a \Rightarrow a \text{ sm}$$

By assumption

(b) $\text{codim}_X a = \max_{\text{irr } X; \exists a} \dim X_i$. Trick: $X = V(f_1, \dots, f_r) \Rightarrow$ irr component has dimension $n-r$.

$$\text{So } \text{codim}_X a \geq n-r \xrightarrow{\text{rank } J=r} \text{codim}_X a \geq n - \text{rank } J \xrightarrow{\text{rank } J=r} \text{rank } J \geq n - \text{codim}_X a \xrightarrow{\text{By (a)}} a \text{ sm}$$

$$\text{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{s \times n} \geq \text{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{r \times n} \geq n - \text{codim}_X a \Rightarrow \text{rank} \left(\underbrace{\frac{\partial f_i}{\partial x_j}}_{j \in [1, n]} \right)_{r \times n} = n - \text{codim}_X a$$

or

\Downarrow

$$\text{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{s \times n} \geq \text{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{r \times n} \geq n - \text{codim}_X a \Rightarrow \text{rank} \left(\underbrace{\frac{\partial f_i}{\partial x_j}}_{j \in [1, n]} \right)_{r \times n} = n - \text{codim}_X a$$

$$\text{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{s \times n} \geq \text{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{r \times n} \geq n - \text{codim}_X a \Rightarrow \text{rank} \left(\underbrace{\frac{\partial f_i}{\partial x_j}}_{j \in [1, n]} \right)_{r \times n} = n - \text{codim}_X a$$

* Relation between 極值定理.

$a \in X = V(f_1, \dots, f_r)$ with $\text{rank} \left(\frac{\partial f_i}{\partial x_j} \right) = r$. So we can view X as a graph of continuously differentiable function (it doesn't have 'corners'). The following example show difference between Implicit Function thm.

Ex: $f(x_1, x_2) = x_2 - x_1^2$. $V(f)$ at $(1, 1)$ is sm ($\text{rank}(-2, 1) = 1 = 2 - \text{codim}_X a = 1$)

But $f(x_1, x_2) = 0$ cannot be solved for x_1 by a regular function — It can only be solved by continuous function $x_1 = \sqrt{x_2}$

- Blowing up makes singular points "nicer"

Consider $X = V(x_2^2 - x_1^3)$. We've prove ∂X singular. ($T_a X \neq C_a X$)

$$\text{③ } \text{rank}(-3x_1^2, 2x_2) \geq 2-1=1 \Leftrightarrow (x_1, x_2) \in \mathbb{A}^2 \setminus 0$$

So \mathbb{A}^2 is sm at $x \neq 0$.

Consider blow up \tilde{X} on aff. patch $\{(x_1, x_2, y_1, 1 : y_2)\}$, by Exe 9.22

$$\tilde{X} \text{ is given by } V(g(x_1, x_2)) \text{ where } g = y_1^2 - x_1. \begin{pmatrix} \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial y_1} \end{pmatrix} = \begin{pmatrix} -1 & 2y_1 \end{pmatrix}_0 = (-1, 0)$$

so \tilde{X} is sm at a .

Slogan: Good blow up remove all singularities.

- Projective version Jacobicriterion. (Same as aff. version)

$$I(X) = \langle f_1, \dots, f_r \rangle. \text{ Then } X \text{ is sm at } a \Leftrightarrow \text{rank}_{(n+1) \times (n+1)} \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{i,j} \text{ has rank } \geq n - \text{codim}_a X$$

Ex: Fermat hypersurface $X = V_p(x_0^d + \dots + x_n^d) \subseteq \mathbb{P}^n$.

X is sm for all n, d, k .

$$J = (dx_0^{d-1} \dots dx_n^{d-1}).$$

i) If $\text{char } k \nmid d$, $\text{rank } J = 1$. X is sm.

ii) If $\text{char } k \mid d$. Let $\text{char } k = p$ and say $d = p^r k$.

Since $x_0^d + \dots + x_n^d = (x_0^k + \dots + x_n^k)^{p^r}$, we have $I(X) = \langle x_0^k + \dots + x_n^k \rangle$.

$$J = (kx_0^{k-1}, \dots, kx_n^{k-1}). \text{ So } \text{rank } J = 1. X \text{ is sm.}$$

- The set of sm pts in a variety X is open

Step 1: $a \in X$ sm, we w.t.s. there is a n.b.h. of a which all pts in it are sm.

* Idea: 令 N 是 a 的邻域且由 sm pts 组成. 证明 $x \in N \Leftrightarrow x$ 满足开条件

a sm \Rightarrow 只有唯一 irr aff. component cover a , 记作 U

say $I(U) = \langle f_1, \dots, f_r \rangle$. So by Jacobi criterion, let $J = \frac{\partial f_i}{\partial x_j}$

$\text{rank } J \geq n - \text{codim}_U a \Leftrightarrow \exists (n - \text{codim}_U a) - m \text{ minors with determinant nonzero.}$

$\det \neq 0$ 是开条件 (write as $D(\dots)$, where $D(\dots)$ is an open set)

Step 2: 还要证明一定存在 sm pts. Sketch: 转化到 hypersurface 问题.

