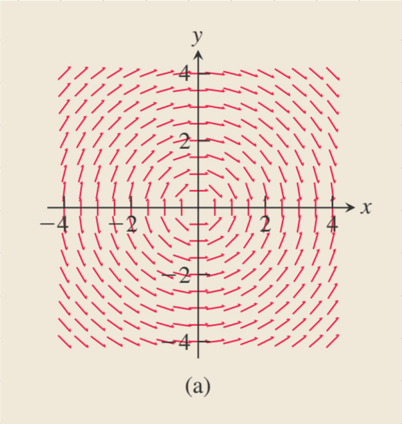
9.1

T3

$y' = -\frac{x}{y}$ （第一象限斜率为负）



(a)

T13

$y' = 2xy + 2y,\ y(0) = 3,\ dx = 0.2$

$y' = 2xy + 2y,\ y(0) = 3,\ dx = 0.2$

$y_1 = 3 + (2 \times 0 \times 3 + 2 \times 3) \times 0.2 = 4.2$

$y_2 = 4.2 + (2 \times 0.2 \times 4.2 + 2 \times 4.2) \times 0.2 = 6.216$

$y_3 = 6.216 + (2 \times 0.4 \times 6.216 + 2 \times 6.216) \times 0.2 = 9.6970$

9.2

T3

$xy' + 3y = \frac{\sin x}{x^2},\ x > 0$

$y' + \frac{3}{x}y - \frac{\sin x}{x^3} = 0$

$u\frac{dy}{dx} + \frac{3}{x}uy - u\frac{\sin x}{x^3} = 0$

$d(uy) - y\frac{du}{dx} + \frac{3}{x}uy - \frac{u\sin x}{x^3} = 0$

$d(uy) + \left(\frac{3u}{x} - \frac{du}{dx}\right)y - \frac{u\sin x}{x^3} = 0$

取 $u = x^3$

$d(x^3y) = \sin x$

$x^3y = -\cos x + C$

T19

$(x+1)\frac{dy}{dx} - 2(x^2+x)y = \frac{e^{x^2}}{x+1}$

$\frac{dy}{dx} - 2xy = \frac{e^{x^2}}{(x+1)^2}$

$u\frac{dy}{dx} - 2uxy = \frac{ue^{x^2}}{(x+1)^2}$

$\frac{d(uy)}{dx} - y\frac{du}{dx} - 2uxy = \frac{ue^{x^2}}{(x+1)^2}$

$$\left(\begin{array}{l} 2ux + \frac{du}{dx} = 0 \\ \Rightarrow \frac{du}{u} = -2xdx \\ \ln u = -x^2 + C \\ \text{取 } u = e^{-x^2} \end{array}\right)$$

$\frac{d(e^{-x^2}y)}{dx} = e^{-x^2}\frac{e^{x^2}}{(x+1)^2} = \frac{1}{(x+1)^2}$

$\Rightarrow d(e^{-x^2}y) = \frac{dx}{(x+1)^2}$

$e^{-x^2}y = \int \frac{dx}{(x+1)^2} = -(x+1)^{-1} + C$

$\Rightarrow y = -\frac{e^{x^2}}{(x+1)} + Ce^{x^2}$

T31.

HISTORICAL BIOGRAPHY
James Bernoulli
(1654–1705)

A **Bernoulli differential equation** is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

For example, in the equation

$$\frac{dy}{dx} - y = e^{-x}y^2$$

we have $n = 2$, so that $u = y^{1-2} = y^{-1}$ and $du/dx = -y^{-2}\,dy/dx$. Then $dy/dx = -y^2\,du/dx = -u^{-2}\,du/dx$. Substitution into the original equation gives

$$-u^{-2}\frac{du}{dx} - u^{-1} = e^{-x}u^{-2}$$

or, equivalently,

$$\frac{du}{dx} + u = -e^{-x}.$$

This last equation is linear in the (unknown) dependent variable u.

Solve the Bernoulli equations in Exercises 29–32.

29. $y' - y = -y^2$ **30.** $y' - y = xy^2$

31. $xy' + y = y^{-2}$ **32.** $x^2y' + 2xy = y^3$

参考这个.

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^{-2}}{x}$$

$n=-2$. 故令 $u = y^{1-(-2)} = y^3$

于是原方程化为

$$\frac{du}{dx} + \frac{3}{x}u = \frac{3}{x}$$

$$v\frac{du}{dx} + \frac{3v}{x}u = \frac{3v}{x}$$

$$\frac{d(vu)}{dx} - u\frac{dv}{dx} + \frac{3v}{x}u = \frac{3v}{x}$$

$$\Rightarrow \frac{3v}{x} - \frac{dv}{dx} = 0 \Rightarrow 取 v = x^3$$

$$\frac{d(x^3u)}{dx} = 3x^2$$

$$x^3u = x^3 + C$$

$$\Rightarrow u = 1 + \frac{C}{x^3}$$

故方程解是 $y^3 = 1 + \frac{C}{x^3}$

9.3

T1

(a) 总质量 $66+7=73$kg, 由 Eq(1) 得

$$v = 9e^{-3.9/73\,t}$$

$$s(t) = \int 9e^{-3.9t/73}dt = -\frac{2190}{13}e^{-3.9t/73} + C$$

由 $s(0)=0$, $C = \frac{2190}{13}$.

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty}\frac{2190}{13}\left(1-e^{-3.9t/73}\right) = \frac{2190}{13} \approx 168.5$$

(b) $1 = 9e^{-3.9t/73} \Rightarrow \frac{3.9t}{73} = \ln 9$

$$\Rightarrow t = \frac{73\ln 9}{3.9} \approx 41.13 \text{ sec.}$$

T13.

13. Salt mixture A tank initially contains 400 L of brine in which 20 kg/L of salt are dissolved. A brine containing 0.2 kg/L of salt runs into the tank at the rate of 20 L/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 16 L/min.

a. At what rate (kilograms per minute) does salt enter the tank at time t?

b. What is the volume of brine in the tank at time t?

c. At what rate (kilograms per minute) does salt leave the tank at time t?

d. Write down and solve the initial value problem describing the mixing process.

e. Find the concentration of salt in the tank 25 min after the process starts.

a. $0.2 \times 20 = 4$ kg/min

b. 每分钟净流入 4L.

故为 $400+4t$ L

c. 设 tank 在 t 时刻盐质量是 $Q(t)$ kg.

盐离开速率是 $\frac{16Q(t)}{400+4t}$ kg/min $= \frac{4Q(t)}{100+t}$

d. $$\frac{dQ(t)}{dt} = 4 - \frac{4Q(t)}{100+t}$$

$$\dot{Q} + \frac{4}{100+t}Q - 4 = 0$$

$$e^{\int\frac{4}{100+t}dt} = e^{4\int\frac{d(t+100)}{100+t}} = e^{4\ln|100+t|} = (100+t)^4$$

$$\Rightarrow \frac{d((100+t)^4 \cdot Q)}{dt} = 4(100+t)^4$$

故 $Q(t) = \frac{4}{5}(100+t) + \frac{C}{(100+t)^4}$

由 $Q(0)=20$, $C = -6\times10^9$.

故 $Q(t) = \frac{4}{5}(100+t) + \frac{-6\times10^9}{(100+t)^4}$

e. $$Q(25) = \frac{4}{5}\times125 + \frac{-6\times10^9}{125^4} \approx 0.1508 \text{ kg/L}$$

9.4

T14

14. Controlling a population The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition.

由常识, $m<M$

a. Discuss the reasonableness of the following model for the growth rate of the deer population as a function of time:

$$\frac{dP}{dt} = rP(M-P)(P-m),$$

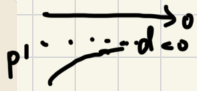
where P is the population of the deer and r is a positive constant of proportionality. Include a phase line.

b. Explain how this model differs from the logistic model $dP/dt = rP(M-P)$. Is it better or worse than the logistic model?

c. Show that if $P > M$ for all t, then $\lim_{t\to\infty} P(t) = M$.

d. What happens if $P < m$ for all t?

e. Discuss the solutions to the differential equation. What are the equilibrium points of the model? Explain the dependence of the steady-state value of P on the initial values of P. About how many permits should be issued?



c. $P>M$ 则 $\frac{dP}{dt}<0$

单调递减且有下界的函数必存在极限.

设 $\lim_{t\to\infty} P(t)=c$

当 $P>M$ 时 $P''>0$, 即 P' 单调增, P' 有上界 0,

故 P' 有极限. 设 $\lim_{t\to\infty}\frac{dP}{dt}=d$. 若 $d\neq 0$ (i.e., $d<0$)

$\frac{dP}{dt}<d$, 则 $\exists A$, s.t. $t>A$ 有 $P<M$.

故 $\lim_{t\to\infty}\frac{dP}{dt}=0$.

即 $\lim_{t\to\infty} rP(M-P)(P-m) = rc(M-c)(c-m)=0$

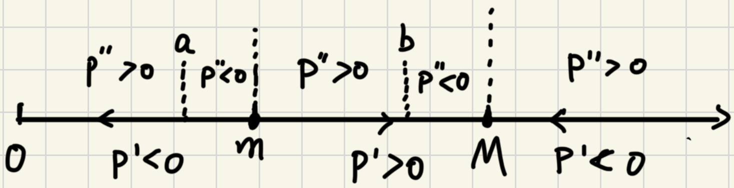
$\Rightarrow c=M$. 故 $\lim_{t\to\infty} P(t)=M$.

d. 与c同理,略

a.

$$\frac{d^2P}{dt^2} = -r\frac{dP}{dt}\left[3P^2-(2M+2m)P+Mm\right]$$

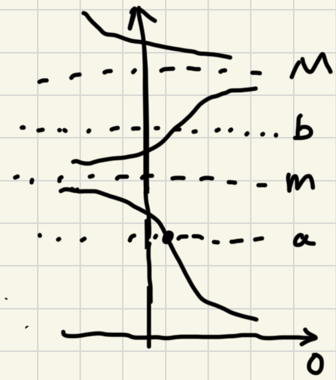
有两根为 $\frac{1}{3}\left[M+m\pm\sqrt{M^2-mM+m^2}\right]$ 分别记作 a,b.



b. logistic没考虑灭绝的情况,

此模型更符合现实情况.

e.

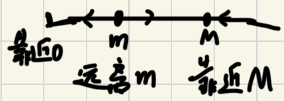


平衡点是使 $\frac{dP}{dt}=0$ 的点,

即 $P=0, M, m$.

稳定点是 $P=0, M$, 不稳定点是 $P=m$

(看图像.)



Applications and Examples

15. Skydiving If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of velocity, then the body's velocity t seconds into the fall satisfies the equation $\to v(t)>0$

$$m\frac{dv}{dt} = mg - kv^2, \qquad k > 0$$

where k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is too short to be affected by changes in the air's density.)

a. Draw a phase line for the equation.

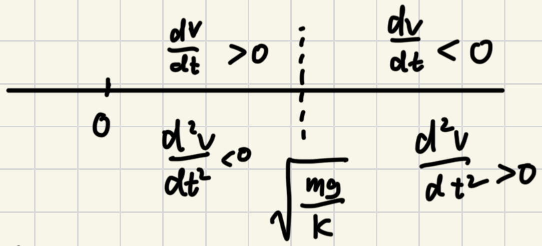
b. Sketch a typical velocity curve.

c. For a 45-kg skydiver ($mg = 441$) and with time in seconds and distance in meters, a typical value of k is 0.15. What is the diver's terminal velocity? Repeat for an 80-kg skydiver.

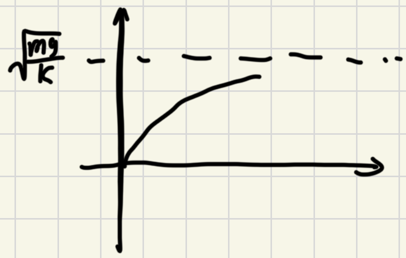
a.

$$\frac{dv}{dt} = g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{mg}{k}}.$$

$$\frac{d^2v}{dt^2} = -2\frac{k}{m}v\frac{dv}{dt} = -2\frac{k}{m}v\left(g - \frac{k}{m}v^2\right)$$



b.



falling from rest ⇒ 初值是0.

c. $\sqrt{\frac{441}{0.15}} = 54.22\ m/s$

$\sqrt{\frac{80\times 9.8}{0.15}} = 72.30\ m/s$

补充题

19年 T12

$$g(0)=\frac{2g(0)}{1-g(0)^2} \Rightarrow g(0)=0 \text{或} \ 1=\frac{2}{1-g(0)^2}$$

易知 $1=\frac{2}{1-g(0)^2}$ 无解，故 $g(0)=0$.

由导数定义

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{g(x)+g(h)}{1-g(x)g(h)}-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{g(x)+g(h)-g(x)+g(h)g^2(x)}{h\,(1-g(x)g(h))}$$

$$=\lim_{h\to 0}\frac{g(h)}{h}\,\frac{g^2(x)+1}{1-g(x)g(h)}$$

$$=g^2(x)+1$$

即 $\frac{dg}{dx}=g^2+1 \Rightarrow \frac{dg}{g^2+1}=dx$

$\Rightarrow \arctan g+C=x$

$\Rightarrow g(x)=\tan(x-C)$

由 $g(0)=0$ 得 $C=0$. 故 $g(x)=\tan x$

20年 T5.

(1) $$\lim_{n\to+\infty}\left(\frac{n}{2n^2+3n+1^2}+\frac{n}{2n^2+6n+2^2}-\cdots+\frac{n}{2n^2+3nk+k^2}+\cdots+\frac{n}{2n^2+3n^2+n^2}\right)$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{2n^2+3nk+k^2}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{2(n+\frac{3k}{4})^2-\frac{1}{8}k^2}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2}\frac{n}{(n+\frac{3k}{4}+\frac{k}{4})(n+\frac{3k}{4}-\frac{k}{4})}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2}\frac{n}{(n+k)(n+\frac{k}{2})}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{n}{n+\frac{k}{2}}-\frac{n}{n+k}\right)\frac{1}{k}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{n+\frac{k}{2}-\frac{k}{2}}{n+\frac{k}{2}}-\frac{n+k-k}{n+k}\right)\frac{1}{k}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}-\frac{1}{2n+k}+\frac{1}{n+k}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(-\frac{n}{2n+k}+\frac{n}{n+k}\right)$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(-\frac{1}{2+\frac{k}{n}}+\frac{1}{1+\frac{k}{n}}\right)$$

$$=\int_0^1-\frac{1}{2+x}+\frac{1}{1+x}\,dx$$

$$=-\ln(2+x)+\ln(1+x)\Big|_0^1$$

$$=-\ln 3+2\ln 2$$

(2) $\lim\limits_{x\to 0}\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{e^x-1}}$

$= \lim\limits_{x\to 0} e^{\frac{\ln\frac{\ln(1+x)}{x}}{e^x-1}}$ ($\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1$. 因此指数部分可以洛必达)

$= e^{\lim\limits_{x\to 0}\left(\frac{\ln\ln(1+x)-\ln x}{e^x-1}\right)}$

$\overset{洛}{=} e^{\lim\limits_{x\to 0}\frac{\frac{1}{\ln(1+x)}\cdot\frac{1}{1+x}-\frac{1}{x}}{e^x}}$

$\lim\limits_{x\to 0}\frac{\frac{1}{\ln(1+x)}\cdot\frac{1}{1+x}-\frac{1}{x}}{e^x}$

$= \lim\limits_{x\to 0}\frac{1}{e^x}\lim\limits_{x\to 0}\frac{x-\ln(1+x)\cdot(1+x)}{\ln(1+x)(1+x)x}$

$\overset{洛}{=} \lim\limits_{x\to 0}\frac{1-\frac{1+x}{1+x}-\ln(1+x)}{\frac{1}{1+x}(1+x)x+\ln(1+x)\cdot x+\ln(1+x)(1+x)}$

$= \lim\limits_{x\to 0}\frac{-\ln(1+x)}{x+\ln(1+x)(2x+1)}$

$\overset{洛}{=} \lim\limits_{x\to 0}\frac{-\frac{1}{1+x}}{1+\frac{2x+1}{1+x}+2\ln(1+x)}$

$= \frac{-1}{1+1} = -1/2$

原式 $= e^{-1/2}$

2.1 T7

$\int_0^\infty \frac{e^{-x}}{x^p}dx$ 什么时候收敛?

① 当 $p=0$ $\int_0^\infty \frac{e^{-x}}{x^0}dx = -e^{-x}\Big|_0^\infty$

$= 0-1=-1$ 收敛.

② 当 $p<0$. 记 $k=-p>0$.

原式 $=\int_0^\infty e^{-x}x^k\,dx$

易知当 x 充分大 $e^{-x}x^k<\frac{1}{x^2}$ 恒成立,

不妨设这个数为 L, 即 $x>L$ 时, $e^{-x}x^k<\frac{1}{x^2}$ 恒成立

$\int_0^\infty \frac{e^{-x}}{x^p}dx = \int_0^L e^{-x}x^k dx + \int_L^\infty e^{-x}x^k dx$

$\le \underbrace{\int_0^L e^{-x}x^k dx}_{某个常数} + \underbrace{\int_L^\infty x^{-2}dx}_{收敛}$

因此由比较判别法, $\int_0^\infty \frac{e^{-x}}{x^p}dx$ 收敛.

$p>0$

③.1 $p>1$.

$\int_0^\infty \frac{e^{-x}}{x^p}dx = \int_0^1 \frac{e^{-x}}{x^p}dx + \int_1^\infty \frac{e^{-x}}{x^p}dx$

$\lim\limits_{x\to 0}\frac{e^{-x}x^{-p}}{x^{-p}}=1$.

因此由 $\int_0^1 x^{-p}dx$ 发散可知

$\int_0^1 e^{-x}x^{-p}dx$ 发散.

又 $\int_1^\infty \frac{e^{-x}}{x^p}dx < \int_1^\infty \frac{1}{x^p}dx$ 收敛,

故 $\int_0^\infty \frac{e^{-x}}{x^p}dx$ 发散.

③.2 $p=1$.

$\int_0^\infty \frac{e^{-x}}{x}dx > \int_0^1 \frac{e^{-x}}{x}dx$ 发散

故 $\int_0^\infty \frac{e^{-x}}{x}dx$ 发散.

3.3 $0<p<1$

$$\int_0^\infty \frac{e^{-x}}{x^p}dx = \int_0^1 \frac{e^{-x}}{x^p}dx + \int_1^\infty \frac{e^{-x}}{x^p}dx$$

$\downarrow e^{-x}<1 \ (x>0)$ $\quad\downarrow \frac{1}{x^p}<1 \ (x>1)$

$$< \int_0^1 \frac{1}{x^p}dx + \int_1^\infty e^{-x}dx$$

收敛 收敛

21T9.

(3)
$$\int_1^\infty \frac{1}{x^6(x^5+4)}dx$$

$$= \int_1^\infty \frac{1}{x^5+4}\, d\frac{x^{-5}}{-5}$$

令 $u=x^{-5}$
$$= -\frac{1}{5}\int_1^0 \frac{1}{\frac{1}{u}+4}du$$

$$= \frac{1}{5}\int_0^1 \frac{u+\frac{1}{4}-\frac{1}{4}}{1+4u}du$$

$$= \frac{1}{5}\int_0^1 \frac{1}{4} - \frac{1}{4}\frac{1}{1+4u}du$$

$$= \frac{1}{5}\frac{u}{4}\Big|_0^1 - \frac{1}{80}\int_0^1 \frac{1}{1+4u}d(4u+1)$$

$$= \frac{u}{20}\Big|_0^1 - \frac{1}{80}\ln(1+4u)\Big|_0^1$$

$$= \frac{1}{20} - \frac{1}{80}\ln 5$$

(4)
$$\int \frac{1}{(1+x+x^2)^2}dx$$

$$= \int \frac{1}{\left((x+\frac{1}{2})^2+\frac{3}{4}\right)^2}d(x+\tfrac{1}{2})$$

$u=x+\frac{1}{2}$
$$= \int \frac{du}{(u^2+3/4)^2}$$

公式 (见Week13 习题课)
$$\begin{cases} I_k=\displaystyle\int\frac{du}{(u^2+a^2)^k}, \quad I_1=\frac{1}{a}\arctan\frac{u}{a}+C' \\ I_{k+1}=\dfrac{1}{2ka^2}\dfrac{u}{(a^2+u^2)^k}+\dfrac{2k-1}{2ka^2}I_k \end{cases}$$

故 $I_{k+1}=\displaystyle\int\frac{du}{(u^2+3/4)^2} \overset{k=1}{\underset{a^2=3/4}{=}} \frac{1}{2\times 3/4}\frac{u}{3/4+u^2}+\frac{1}{2\times 3/4}\frac{1}{\sqrt{3}/2}\arctan\frac{u}{\sqrt{3}/2}+C$

$$=\frac{2}{3}\frac{u}{\frac{3}{4}+u^2}+\frac{4}{3}\frac{1}{\sqrt{3}}\arctan\frac{2\sqrt{3}u}{3}+C$$

$$=\frac{8u}{9+12u^2}+\frac{4}{3\sqrt{3}}\arctan\frac{2\sqrt{3}u}{3}+C$$

$$=\underbrace{\frac{8(x+\frac{1}{2})}{9+12(x+\frac{1}{2})^2}}_{=\frac{2x+1}{3x^2+3x+3}}+\frac{4}{3\sqrt{3}}\arctan\frac{2\sqrt{3}(x+\frac{1}{2})}{3}+C$$

22年

T8 设函数 $f(x)$ 在 $[0,1]$ 上连续，$(0,1)$ 内可导，
$f(0)=f(1)=0$，$f(\frac{1}{2})=1$.

证明 (1) 存在 $c\in(\frac{1}{2},1)$，s.t. $f(c)=c$

(2) $\forall k\in\mathbb{R}$，$\exists\xi\in(0,c)$ s.t. $f'(\xi)-k[f(\xi)-\xi]=1$

(1) pf

$g(x)=f(x)-x$.

$g(\frac{1}{2})=f(\frac{1}{2})-\frac{1}{2}=1-\frac{1}{2}=\frac{1}{2}>0$

$g(1)=f(1)-1=-1<0$

故存在 $c\in(\frac{1}{2},1)$，s.t. $g(c)=0$

即 $f(c)=c$

(2)

即证 $\forall k\in\mathbb{R}$，$\exists\xi$，s.t. $g'(\xi)=kg(\xi)$.

令 $h(x)=\dfrac{g(x)}{e^{kx}}$ 由 $h(0)=\dfrac{f(0)-0}{e^0}=0$

$h(c)=\dfrac{g(c)}{e^{kc}}=0=h(0)$.

故 $\exists\xi\in(0,c)$，s.t. $h'(\xi)=0$

即 $h'(\xi)=\dfrac{g'(\xi)-kg(\xi)}{e^{kx}}=0 \Rightarrow g'(\xi)=kg(\xi)$