7.4

T22

$$\frac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1$$

$$\frac{dy}{dx} = (e^x+1)(e^{-y}+1)$$

$$\frac{dy}{e^{-y}+1} = e^x+1\,dx$$

两边同时积分 $\int \frac{dy}{e^{-y}+1} = \int e^x+1\,dx$

$$\int \frac{dy}{e^{-y}+1} = \int \frac{e^y dy}{1+e^y} = \int \frac{de^y}{1+e^y} = \int \frac{d(e^y+1)}{e^y+1} = \ln(e^y+1) + C_1$$

$$\int e^x+1\,dx = e^x + x + C_2$$

故方程为 $\ln(e^y+1) + C_1 = e^x + x + C_2$

即 $\ln(e^y+1) = e^x + x + C$,

其中 C 是任意实数.

T41

a. 我们有 $H - 20 = (90-20)e^{-kt}$

由 $t=10$, $H=60$ 得.

$$60-20 = 70e^{-k10} \Rightarrow k = \frac{1}{10}\ln\frac{7}{4} \approx 0.056\ \text{min}^{-1}$$

故 $H = 20 + 70e^{-\frac{1}{10}\ln 7/4\, t}$

令 $H=35$ 得 $t = -10\frac{\ln 3/14}{\ln 7/4} \approx 27.53$ min

b. $35-(-15) = (90-(-15))\,e^{\frac{1}{10}\ln 7/4\, t}$

$$\Rightarrow t = -10\frac{\ln 10/21}{\ln 7/4} \approx 13.26\ \text{min}$$

7.5

T21

$$\lim_{x\to 0} \frac{x^2}{\ln(\sec x)}$$

$$= \lim_{x\to 0} \frac{2x}{\frac{1}{\sec x}\left(-\frac{-\sin x}{\cos^2 x}\right)}$$

$$= \lim_{x\to 0} \frac{2x\cos x}{\sin x}$$

$$= \lim_{x\to 0} \frac{2\cos x - 2x\sin x}{\cos x}$$

$$= 2$$

T31

$$\lim_{x\to\infty} \frac{\ln(x+1)}{\log_2 x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{\ln 2\cdot x}}$$

$$= \lim_{x\to\infty} \frac{\ln 2\cdot x}{x+1}$$

$$= \ln 2$$

T41

$$\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$$

$$= \lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x}$$

$$= \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{\ln x + \frac{x-1}{x}}$$

$$= \lim_{x\to 1^+} \frac{1-x}{x\ln x + x - 1}$$

$$= \lim_{x\to 1^+} \frac{-1}{\ln x + 1 + 1} = -\frac{1}{2}$$

T44

$$\lim_{h\to 0} \frac{e^h - (1+h)}{h^2}$$

$$= \lim_{h\to 0} \frac{e^h - 1}{2h}$$

$$= \lim_{h\to 0} \frac{e^h}{2}$$

$$= \frac{1}{2}$$

T55

$$\lim_{x\to 0^+} x^{-1/\ln x}$$

$$= \lim_{x\to 0^+} e^{-\frac{1}{\ln x}\ln x}$$

$$= e^{-1}$$

T65

$$\lim_{x\to 0^+} x\tan\left(\frac{\pi}{2} - x\right)$$

$$= \lim_{x\to 0^+} \frac{x}{\tan x}$$

$$= \lim_{x\to 0^+} \frac{1}{\frac{1}{\cos^2 x}}$$

$$= 1$$

T71

$$\lim_{x\to\infty} \frac{2^x - 3^x}{3^x + 4^x}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{2}{3}\right)^x - 1}{1 + \left(\frac{4}{3}\right)^x}$$

$$= 0$$

T80 Find a, b s.t.

$$\lim_{x\to 0} \frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x} = 0$$

原式 $= \lim_{x\to 0} \frac{\tan 2x + ax + x^2 \sin bx}{x^3}$

$$= \lim_{x\to 0} \frac{\frac{2}{\cos^2 2x} + a + 2x\sin bx + bx^2\cos bx}{3x^2}$$

为使极限存在，分子极限必须为0.

即 $2 + a = 0 \Rightarrow a = -2$

原式 $: \lim_{x\to 0} \frac{-2(2)\cos^{-3}2x(-2\sin 2x) + 2\sin bx + 2xb\cos bx + b2x\cos bx - bx^2 b\sin bx}{6x}$

$$= \lim_{x\to 0} \frac{8\sin 2x\cos^{-3}2x + (2 - b^2x^2)\sin bx + 4bx\cos bx}{6x}$$

$$= \lim_{x\to 0} \frac{8}{3}\frac{\sin 2x}{2x}\cos^{-3}2x + \frac{2-b^2x^2}{6/b}\frac{\sin bx}{bx} + \frac{4b}{6}\cos bx$$

$$= \frac{8}{3} + b/3 + \frac{4}{6}b = 0 \Rightarrow b = -8/3$$

T88

求 $f'(x)$. $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f'(x) = 3e^{-1/x^2}x^{-3}, \quad x \neq 0$

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{e^{-1/h^2} - 0}{h}$$

$$= \lim_{h\to 0} e^{-1/h^2}/h.$$

$$= \lim_{h\to 0} \frac{1}{h}/e^{1/h^2}$$

$$= \lim_{h\to 0} \frac{1}{h}/e^{1/h^2}$$

$$= \lim_{h\to 0} \frac{-\frac{1}{h^2}}{e^{\frac{1}{h^2}}(-2\frac{1}{h^3})}$$

$$= \frac{1}{2}\lim_{h\to 0} \frac{h}{e^{\frac{1}{h^2}}}$$

$$= \frac{1}{2}\lim_{h\to 0} h e^{-\frac{1}{h^2}}$$

$$= 0$$

7.6

T11

$\tan(\sin^{-1}(-\frac{1}{2}))$

$= \tan(\pi + \pi/6)$

$= -\sqrt{3}/3$

T14 $\lim_{x\to -1^+} \cos^{-1}x = \pi$

T41

$y = x\sin^{-1}x + \sqrt{1-x^2}$

$\frac{dy}{dx} = \sin^{-1}x + x\frac{1}{\sqrt{1-x^2}} + \frac{-2x}{2\sqrt{1-x^2}}$

$= \sin^{-1}x$

T54

$$\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$$

$$= \int_{-2/3}^{-\sqrt{2}/3} \frac{d3y}{3y\sqrt{(3y)^2-1}}$$

令 $3y = t$

$$= \int_{-2}^{-\sqrt{2}} \frac{dt}{t\sqrt{t^2-1}}$$

$$= \sec^{-1}|t|\Big|_{-2}^{-\sqrt{2}}$$

$$= \sec^{-1}(\sqrt{2}) - \sec^{-1}(2)$$

$$= \pi/4 - \frac{\pi}{3} = -\frac{\pi}{12}$$

T80

$$\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$$

$$= \int \frac{d(x-2)}{(x-2)\sqrt{(x-2)^2-1}}$$

$$= \sec^{-1}|x-2| + C$$

T87

$$\int_{\sqrt{2}} \frac{\sec^2(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}}$$

$$= \int_{\sqrt{2}}^{2} \frac{x^2}{x\sqrt{x^2-1}}\,dx$$

$$= \frac{1}{2}\int_{\sqrt{2}}^{2} \frac{2x}{\sqrt{x^2-1}}\,dx$$

$$= \frac{1}{2}\int_{\sqrt{2}}^{2} \frac{d(x^2-1)}{\sqrt{x^2-1}}$$

$$= \frac{1}{2}\,\frac{(x^2-1)^{1/2}}{1/2}\Big|_{\sqrt{2}}^{2}$$

$$= \sqrt{3} - 1$$

T97

$$\lim_{x\to 0^+} \frac{(\tan^{-1}\sqrt{x})^2}{x\sqrt{x+1}}$$

$$= \lim_{x\to 0^+} \frac{2\tan^{-1}\sqrt{x}\,\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2}x^{-1/2}}{\sqrt{x+1} + x\,\frac{1}{2}(x+1)^{-1/2}}$$

$$= \lim_{x\to 0^+} \frac{\tan^{-1}\sqrt{x}\,\frac{1}{\sqrt{x}}\,\frac{1}{1+x}}{\sqrt{x+1} + \frac{x}{2\sqrt{x+1}}}$$

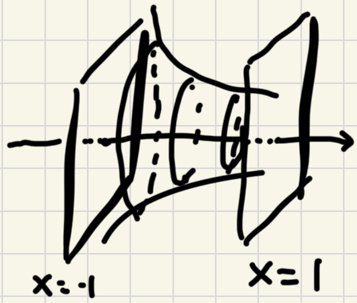
$$= \lim_{x\to 0} \frac{\tan^{-1}\sqrt{x}}{\sqrt{x(x+1)}\,(1+\frac{3}{2}x)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x}\,\frac{1}{2\sqrt{x}}}{\frac{2x+1}{2\sqrt{x(x+1)}}(1+\frac{3}{2}x)+\sqrt{x(x+1)}\,\frac{3}{2}}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+1}\,(2x+1)(1+\frac{3}{2}x) - 3x(x+1)\sqrt{x+1}}$$

$$= 1$$

T123



a. $$\int_{-1}^{1} \pi\left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx$$

$$= \pi\int_{-1}^{1} \frac{1}{1+x^2}\,dx$$

$$= \pi \arctan x\Big|_{-1}^{1}$$

$$= \pi\left(\frac{\pi}{4} - (-\pi/4)\right) = \frac{\pi^2}{2}$$

b.

$$\int_{-1}^{1}\left(\frac{2}{\sqrt{1+x^2}}\right)^2 dx$$

$$= 4\int_{-1}^{1}\frac{1}{1+x^2}dx$$

$$=4\arctan x\Big|_{-1}^{1}$$

$$=4\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right)=2\pi$$

7.8

T10

a. $$\lim_{x\to\infty}\frac{\frac{1}{x+3}}{\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{x}{x+3}$$

$$=\lim_{x\to\infty}\frac{1}{1+3/x}=1$$

True.

b. $$\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^2}}{\frac{1}{x}}$$

$$=\lim_{x\to\infty} 1+\frac{1}{x}=1$$

True.

c. $$\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x^2}}{\frac{1}{x}}$$

$$=\lim_{x\to\infty} 1-\frac{1}{x}=1\neq 0$$

False

d.

$$\frac{2+\cos x}{2}\leq\frac{2+1}{2}=\frac{3}{2}\quad \text{True}$$

e. $$\lim_{x\to\infty}\frac{e^x+x}{e^x}=\lim_{x\to\infty}\frac{e^x+1}{e^x}=\lim_{x\to\infty}\frac{e^x}{e^x}=1$$

True.

f.

$$\lim_{x\to\infty}\frac{x\ln x}{x^2}$$

$$=\lim_{x\to\infty}\frac{\ln x}{x}$$

$$=\lim_{x\to\infty}\frac{1}{x}\Big/1=\lim_{x\to\infty}\frac{1}{x}=0\quad \text{True}$$

g. $$\lim_{x\to\infty}\frac{\ln(\ln x)}{\ln x}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{1}{\ln x}=0\quad \text{True}$$

h.

$$\lim_{x\to\infty}\frac{x}{\ln(x^2+1)}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x^2+1}(2x)}$$

$$=\lim_{x\to\infty}\frac{x^2+1}{2x^2}$$

$$=\lim_{x\to\infty}\frac{1+1/x^2}{2}=1/2\quad \text{False}$$

T22

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1}{x}\Big/1=0$$

而 $x=O(f(x))$, $f(x)$为任意非常数多项式

故 $\ln x$ 比 任何多项式增长慢

补充题

CH1 T1

$$\lim_{x\to0^+} f(x)$$
$$=\lim_{x\to0^+}\frac{1+e^{1/x}}{-1+e^{1/x}}$$
$$=\lim_{x\to0^+}\frac{-\frac{1}{x^2}e^{1/x}}{-\frac{1}{x^2}e^{1/x}}=1$$

$$\lim_{x\to0^-} f(x)$$
$$=\lim_{x\to0^-}\frac{1+e^{1/x}}{-1+e^{1/x}}=-1$$

$\lim\limits_{x\to0^+}f(x)\ne\lim\limits_{x\to0^-}f(x)$，选 A.

T4

(1)
$$\lim_{x\to\infty}\left(\frac{x^2}{(x-1)(x+3)}\right)^x$$
$$=\lim_{x\to\infty}\left(\frac{x^2}{x^2+2x-3}\right)^x$$
$$=\lim_{x\to\infty}\left[\frac{x^2+2x-3-(2x-3)}{x^2+2x-3}\right]^x$$
$$=\lim_{x\to\infty}\left(1-\frac{2x-3}{x^2+2x-3}\right)^x$$
$$=\lim_{x\to\infty}\left(1-\frac{2x-3}{x^2+2x-3}\right)^{x\cdot\left(-\frac{x^2+2x-3}{2x-3}\right)\left(-\frac{2x-3}{x^2+2x-3}\right)}$$
$$=e^{\lim\limits_{x\to\infty}-\frac{(2x-3)x}{x^2+2x-3}}$$
$$=e^{\lim\limits_{x\to\infty}\frac{3x-2x^2}{x^2+2x-3}}$$
$$=e^{\lim\limits_{x\to\infty}\frac{3/x-2}{1+2/x-3/x^2}}$$
$$=e^{-2}$$

(4)
$$\lim_{x\to\infty}\left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}$$
$$=\lim_{x\to\infty}e^{\frac{1}{\ln x}\ln\left(\frac{\pi}{2}-\arctan x\right)}$$

$$\lim_{x\to\infty}\frac{\ln\left(\frac{\pi}{2}-\arctan x\right)}{\ln x}$$
$$=\lim_{x\to\infty}\frac{\frac{1}{\frac{\pi}{2}-\arctan x}\left(-\frac{1}{1+x^2}\right)}{1/x}$$
$$=\lim_{x\to\infty}-\frac{x}{(1+x^2)\left(\frac{\pi}{2}-\arctan x\right)}$$
$$=\lim_{x\to\infty}-\frac{\frac{x}{1+x^2}}{\frac{\pi}{2}-\arctan x}$$
$$=\lim_{x\to\infty}-\frac{\frac{1+x^2-x\cdot2x}{(1+x^2)^2}}{-\frac{1}{1+x^2}}$$
$$=\lim_{x\to\infty}\frac{1-x^2}{1+x^2}=\lim_{x\to\infty}\frac{-2x}{2x}=-1$$

故原式 $=e^{-1}$

(8)
$$\lim_{x\to1}\frac{x-x^x}{1-x+\ln x}$$
$$=\lim_{x\to1}\frac{1-\left(e^{x\ln x}\right)'}{-1+1/x}$$
$$=\lim_{x\to1}\frac{1-x^x(1+\ln x)}{-1+1/x}$$
$$=\lim_{x\to1}\frac{-x^x(1+\ln x)^2-x^x\frac{1}{x}}{-1/x^2}$$
$$=\lim_{x\to1}x^{x+2}(1+\ln x)^2+x^{x+1}$$

$= 1+1=2$

(14)

$$\lim_{x\to 0^+}(e^x - x - 1)^{\frac{1}{\ln x}}$$

$$= \lim_{x\to 0^+} e^{\frac{1}{\ln x}\ln(e^x - x - 1)}$$

$$\lim_{x\to 0^+} \frac{\ln(e^x - x - 1)}{\ln x}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{e^x - x - 1}(e^x - 1)}{\frac{1}{x}}$$

$$= \lim_{x\to 0^+} \frac{xe^x - x}{e^x - x - 1}$$

$$= \lim_{x\to 0^+} \frac{e^x + xe^x - 1}{e^x - 1}$$

$$= \lim_{x\to 0^+} \frac{e^x + e^x + xe^x}{e^x}$$

$$= \lim_{x\to 0^+} 2 + x = 2$$

原式 $= e^2$