Fiber bdls

Consider a topological grp G acting continously on a topo space F, $\rho: G\times F \to F$. Any action ρ induce a map $Ad_\rho: G \to Homeo(F)$ (It's a grp homo when condition is good). We assume ρ is an effective action, i.e., Ad_ρ is injective.

[Def] (Atlas for a fiber bdl with grp G and fiber F)
Given a conti map $\pi: E \to B$ an atlas for the structure of a fiber bdl with grp G and fiber F on π consists of the following data:

(a) an open cover $\{U_\alpha\}_\alpha$ of B.

(b) homeomorphisms (trivialization charts or local trivializations) $h_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times F$ s.t. diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "h_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\text{projection}"] \\ & U_\alpha & \end{tikzcd}$$

(c) conti maps (transition functions) $g_{\alpha\beta}: U_\alpha \cap U_\beta \to G$ so that the horizontal map in commutative diagram

$$\begin{tikzcd} & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "h_\alpha"'] \arrow[dr, "h_\beta"] & \\ (U_\alpha \cap U_\beta)\times F \arrow[rr, "h_\beta \circ h_\alpha^{-1}"] & & (U_\alpha \cap U_\beta)\times F \end{tikzcd}$$

is given by $(x, m) \mapsto (x, g_{\beta\alpha}(x)\cdot m)$

[Rmk] Since ρ is effective (the induced Ad_ρ is inj.), hence if $g_{\alpha\beta}$ exist, they are unique. Indeed, fix any x, $h_\beta \circ h_\alpha^{-1}$ given an element in Homeo F. The injectiveness of $Ad_\rho: G \to Homeo\, F$ means that this element corresponding to unique $g_x \in G$. Hence $g_{\alpha\beta}(x) = g_x$. When x varies, we obtain $g_{\alpha\beta}$.

[Def] Two atlas $\mathcal{A}$ and $\mathcal{B}$ on π are compatible if $\mathcal{A} \cup \mathcal{B}$ is an atlas.

[Def] (Fiber bdl with grp G and fiber F) A structure of fiber bdl with grp G and fiber F on $\pi: E \to B$ is a maximal atlas for $\pi: E \to B$.

[Rmk] What's relationship between "Fiber bdl with grp G and fiber F" and the "fiber bdl (locally trivial fiber bdl)"? A locally trivial fiber bdl is a fiber bdl with structure grp $\mathrm{Homeo}(F)$ and fiber F.

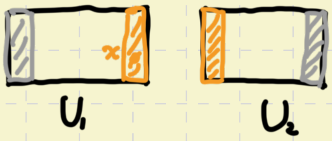
Fiber bdl:

Locally trivial

To construct a bdl, we need glue patches together. we focus on U_α & U_β and $x \in U_\alpha \cap U_\beta$. We glue them by Homeo F. $h: U_\alpha \cap U_\beta \longrightarrow \mathrm{Homeo}\,F$

$(x, m) \sim (x, h(x)m)$

e.g. Mobiüs band



With fiber $F = [-1, 1]$

are intersection of U_1 & U_2

$\forall x \in$, $h: \longrightarrow \mathrm{Homeo}\,F$
$x \longmapsto id$

$\forall x \in$, $h: \longrightarrow \mathrm{Homeo}\,F$
$x \longmapsto [m \mapsto -m]$

Fiber bdl with grp G and fiber F generalize Homeo F to $Ad_\rho : G \to \mathrm{Homeo}\,F$.

[Exp] $G = \{e\}$. $\pi: E \to B$ has structure G and fiber F. Then $E \simeq B \times F$ is a trivial bdl. Indeed, for any U_α, U_β with $U_\alpha \cap U_\beta \neq \phi$, $h_\alpha h_\beta^{-1}(x, m) = (x, id \cdot m) = (x, m)$. Hence $h_\alpha h_\beta^{-1} = id$, i.e., $h_\alpha = h_\beta$ on $U_\alpha \cap U_\beta$. Hence we can glue all local trivialization together to obtain a global trivial $B \times F \cong E$.

[Prop] transition functions have following properties

(a) $g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$, $\forall x \in U_\alpha \cap U_\beta \cap U_\gamma$

(b) $g_{\beta\alpha}(x) = g_{\alpha\beta}^{-1}(x)$, $\forall x \in U_\alpha \cap U_\beta$

(c) $g_{\alpha\alpha}(x) = e_G$.

pf:

(a)
$$\begin{tikzcd} & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "h_\alpha"'] \arrow[dr, "h_\beta"] & \\ (U_\alpha \cap U_\beta) \times F \arrow[rr, "h_\beta h_\alpha^{-1}"] \arrow[urr, phantom, "①"] & & (U_\alpha \cap U_\beta) \times F \end{tikzcd} \quad \overset{\text{restrict to}}{\underset{U_\alpha \cap U_\beta \cap U_\gamma}{\rightsquigarrow}} \quad \begin{tikzcd} & \pi^{-1}(U_\alpha \cap U_\beta \cap U_\gamma) \arrow[dl, "h_\alpha"'] \arrow[dr, "h_\beta"] & \\ (U_\alpha \cap U_\beta \cap U_\gamma) \times F \arrow[rr, "h_\beta h_\alpha^{-1}"] \arrow[urr, phantom, "②"] & & (U_\alpha \cap U_\beta \cap U_\gamma) \times F \end{tikzcd}$$

$$\begin{tikzcd} & \pi^{-1}(U_\gamma \cap U_\beta) \arrow[dl, "h_\beta"'] \arrow[dr, "h_\gamma"] & \\ (U_\alpha \cap U_\beta) \times F \arrow[rr, "h_\gamma h_\beta^{-1}"] \arrow[urr, phantom, "③"] & & (U_\alpha \cap U_\beta) \times F \end{tikzcd} \quad \overset{\text{restriction}}{\rightsquigarrow} \quad \begin{tikzcd} & \pi^{-1}(U_\alpha \cap U_\beta \cap U_\gamma) \arrow[dl, "h_\beta"'] \arrow[dr, "h_\gamma"] & \\ (U_\alpha \cap U_\beta \cap U_\gamma) \times F \arrow[rr, "h_\gamma h_\beta^{-1}"] \arrow[urr, phantom, "④"] & & (U_\alpha \cap U_\beta \cap U_\gamma) \times F \end{tikzcd}$$

where ① ③ are data of bdl with str G and fiber F.

Join ③ and ④ together we obtain

$$\begin{tikzcd}
& \pi^{-1}(U_\alpha \cap U_\beta \cap U_\gamma) \arrow[dl, "h_\alpha"'] \arrow[d, "h_\beta"] \arrow[dr, "h_\gamma"] \arrow[dl, phantom, shift left=6ex, "③"] \arrow[dr, phantom, shift right=6ex, "④"] & \\
(U_\alpha \cap U_\beta \cap U_\gamma)\times F \arrow[r, "h_\beta \circ h_\alpha^{-1}"'] & (U_\alpha \cap U_\beta \cap U_\gamma)\times F \arrow[r, "h_\gamma \circ h_\beta^{-1}"'] & (U_\alpha \cap U_\beta \cap U_\gamma)\times F
\end{tikzcd}$$

$(x,m) \longmapsto (x, g_{\beta\alpha}(x)\cdot m)$ $\quad$ $(x,m) \longmapsto (x, g_{\gamma\beta}(x)\cdot m)$.

conditions of fiber bdl with str grp G and fiber F.

Hence $(h_\gamma \circ h_\beta^{-1})\circ(h_\beta \circ h_\alpha^{-1}) = h_\gamma \circ h_\alpha^{-1}$ is given by $(x,m) \mapsto (x, g_{\gamma\beta}(x) g_{\beta\alpha}(x)\, m)$

On the other hand, $h_\gamma h_\alpha^{-1}$ is given by $(x,m) \mapsto (x, g_{\gamma\alpha}(x)\, m)$. Hence we have $(x, g_{\gamma\beta}(x) g_{\beta\alpha}(x) m) = (x, g_{\gamma\alpha}(x)\, m)$. Since $Ad_\rho : G \to \mathrm{Homeo}\, F$ is inj, $g_{\gamma\beta}(x) g_{\beta\alpha}(x) = g_{\gamma\alpha}(x)$ for $\forall x \in U_\alpha \cap U_\beta \cap U_\gamma$.

(b) We join two commt. diagram of transition maps together

$$\begin{tikzcd}
& \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "h_\alpha"'] \arrow[d, "h_\beta"] \arrow[dr, "h_\alpha"] & \\
(U_\alpha \cap U_\beta)\times F \arrow[r, "h_\beta h_\alpha^{-1}"] & (U_\alpha \cap U_\beta)\times F \arrow[r, "h_\alpha h_\beta^{-1}"] & (U_\alpha \cap U_\beta)\times F
\end{tikzcd}$$

$(x,m) \longmapsto (x, g_{\beta\alpha}(x) m)$ $\quad$ $(x,m) \longmapsto (x, g_{\alpha\beta}(x) m)$

Hence $(h_\alpha h_\beta^{-1})(h_\beta h_\alpha^{-1}) = id$ is given by $(x,m) \mapsto (x, g_{\alpha\beta}(x) g_{\beta\alpha}(x)\cdot m)$

On the other hand, id is given by $(x,m) \mapsto (x,m)$.

Therefore, $(x, g_{\alpha\beta}(x) g_{\beta\alpha}(x)\cdot m) = (x,m)$. By inj of Ad_ρ, $g_{\alpha\beta}(x) g_{\beta\alpha}(x) = e_G$.

So $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1} = g_{\beta\alpha}^{-1}(x)$.

(c) Take $\alpha=\gamma$ in (a), we obtain $g_{\alpha\beta}(x)\, g_{\beta\alpha}(x) = g_{\alpha\alpha}(x)$, $\forall x \in U_\alpha \cap U_\beta$.

By (b) we have $g_{\alpha\alpha}(x) = e_G$. $\square$

The following thm tells us transition functions determine a fiber bundle in a unique way. (Intuitively it's obvious, because transition maps determine a unique glueing.

[Thm] Given an open cover $\{U_\alpha\}$ of B and continuous functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to G$ satisfying props (a), (b), (c), there is a unique structure of a fiber bdl over B with grp G, given fiber F, and transition functions $\{g_{\alpha\beta}\}$.

pf: Let $\tilde{E} = \coprod_\alpha U_\alpha \times F \times \{\alpha\}$. Define a relation $(x, m, \alpha) \sim (x, g_{\alpha\beta}(x)\cdot m, \beta)$.

$\triangle$ Check $\sim$ is equivalence relation.

(i) $(x, m, \alpha) \sim (x, g_{\alpha\alpha}(x)\cdot m, \alpha) \overset{\text{prop (c)}}{=} (x, e_G \cdot m, \alpha) = (x, m, \alpha)$

(ii) for any $(x, m, \alpha) \sim (x, g_{\alpha\beta}(x)\cdot m, \beta)$, we have
$(x, g_{\alpha\beta}(x)\cdot m, \beta) \sim (x, g_{\beta\alpha}(x) g_{\alpha\beta}(x)\cdot m, \alpha) \overset{\text{prop (b)}}{=} (x, m, \alpha)$

(iii) for any $(x, m, \alpha) \sim (x, g_{\alpha\beta}(x)\cdot m, \beta) \sim (x, g_{\gamma\alpha}(x) g_{\alpha\beta}(x)\cdot m, \beta)$
so $(x, m, \alpha) \sim (x, g_{\gamma\beta}(x)\cdot m, \beta) = (x, g_{\gamma\alpha}(x) g_{\alpha\beta}(x)\cdot m, \beta)$.

$\triangle$ Define $E = \tilde{E}/\sim$. Define $\pi: E \to B$ locally by $[(x, m, \alpha)] \mapsto x$ for $x \in U_\alpha$.

i) Check π is well-defined. (fiberwise change)
$(x, m, \alpha) \sim (x, g_{\alpha\beta}(x)\cdot m, \beta)$ with $\pi(x, m, \alpha) = x = \pi(x, g_{\alpha\beta}(x)\cdot m, \beta)$

ii) Check π is continous

Topo of E is quotient topo. For any open $U \subseteq B$. $\pi^{-1}(U) = \{[(x, m, \alpha)] \mid x \in U\} \subseteq E$.
Let $p: \tilde{E} \to \tilde{E}/\sim = E$, $p^{-1}(\pi^{-1}(U)) = \coprod_\alpha (U \cap U_\alpha) \times F \times \{\alpha\}$ which is open in $\tilde{E} = \coprod_\alpha U_\alpha \times F \times \{\alpha\}$

iii) Check fiber of π is F. Obviously.

$\triangle$ Trivializations.
For $p: \tilde{E} \to \tilde{E}/\sim = E$, let $p_\alpha = p|_{U_\alpha \times F \times \{\alpha\}} : U_\alpha \times F \times \{\alpha\} \to \pi^{-1}(U_\alpha)$ which is homeo. Then let local trivializations $h_\alpha = p_\alpha^{-1}$.

$\triangle$ transition maps. transition map $l_{\beta\alpha}: U_\alpha \cap U_\beta \to G$ is the map satisfying $h_\beta h_\alpha^{-1}(x, m) = (x, l_{\beta\alpha}(x)\cdot m)$

$$
\begin{array}{ccccc}
\tilde{E} & & \tilde{E}/\sim & & \\
\cup\!\shortmid & & \cup\!\shortmid & & \\
(U_\beta \cap U_\alpha) \times F \times \{\alpha\} & \xrightarrow{p_\alpha} & \pi^{-1}(U_\alpha \cap U_\beta) & \xrightarrow{p_\beta^{-1}} & (U_\alpha \cap U_\beta) \times F \times \{\beta\} \\
(x, m, \alpha) & \longmapsto & [(x, m, \alpha)] & \longmapsto & (x, g_{\beta\alpha}(x)\cdot m, \alpha) \\
 & & \| & & \\
 & & [(x, g_{\beta\alpha}(x)\cdot m, \alpha)] & &
\end{array}
$$

Hence $\{g_{\beta\alpha}\}$ are transition maps

[Exp] Str grp $GL(n,\mathbb{R})$ with fiber $\mathbb{R}^n$, called rank n real vector bdls.

M is a differentiable real n-manifold. Then $\pi: TM \to M$ is rank n real vector bdls. If $\varphi_\alpha: U_\alpha \xrightarrow{\sim} \mathbb{R}^n$ are trivialization charts on M, the transi unctions for TM are given by $g_{\alpha\beta}(x) = d(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(x)}$.

$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dl, "\varphi_\beta"', "\wr"] \arrow[dr, "\varphi_\alpha", "\wr"'] & \\ \mathbb{R}^n \arrow[rr, "\varphi_\alpha \circ \varphi_\beta^{-1}"'] & & \mathbb{R}^n \end{tikzcd}$$

$$d(\varphi_\alpha \varphi_\beta^{-1})_{\varphi_\beta(x)} : \underset{\mathbb{R}^n}{T_{\varphi_\beta(x)}\mathbb{R}^n} \longrightarrow \underset{\mathbb{R}^n}{T_{\varphi_\alpha(x)}\mathbb{R}^n}$$

which is an element in $GL(n,\mathbb{R})$

Slogan: tangent bundle is glued by differential of trivialization.

[Rmk]

① $F=\mathbb{R}^n$, $G=O(n)$, we get vect bdls with a Riemannian str.

② $F=\mathbb{C}^n$, $G=GL(n,\mathbb{C})$ we get rank n complex vector bdls.

③ $F=\mathbb{C}^n$, $G=U(n)$, we get real vector bundles with a hermitian structure.

[Rmk] What's relationship between fiber bdl and fibration?

Fact: A fiber bdl has homotopy lifting property (HLP) w.r.t. all CW complexes (i.e., it's serre fibration). Fibration is always a serre fibration but the converse is not true. Moreover, fiber bdls over paracompact spaces are fibrations (HLP w.r.t. all spaces).

[Def] (Bdl homo) Fix a topo grp G acting effectively on a space F. $\pi: E \to B$ and $\pi': E' \to B'$ are bdls with grp str and fiber F. A homo between two bdls are a pair $(f, \hat{f})$ where $f: B' \to B$ and $\hat{f}: E' \to E$ s.t. ① the diagram commutes

$$\begin{tikzcd} E' \arrow[r, "\hat{f}"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

$x \qquad f(x)$

(This diagram commutes means $\hat{f}\big|_{\pi'^{-1}(x)} : \pi'^{-1}(x) \longrightarrow \pi^{-1}(f(x))$, i.e., f and $\hat{f}$ "preserve fiber")

② If $\{(U_\alpha, h_\alpha)\}$ is a trivialization atlas of π and $\{(V_\beta, H_\beta)\}$ is a trivializing atlas of π', then in the commutative diagram:

$$\begin{tikzcd}
(V_\beta \cap f^{-1}(U_\alpha)) \times F \arrow[dr, "p_1"'] & \pi'^{-1}(V_\beta \cap f^{-1}(U_\alpha)) \arrow[l, "H_\beta"'] \arrow[r, "\hat{f}"] \arrow[d, "\pi'"] \arrow[dl, phantom, "ⓐ"] & \pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[d, "\pi"] \arrow[dl, phantom, "ⓑ"] & U_\alpha \times F \arrow[dl, "p_1"] \arrow[dll, phantom, "ⓒ"] \\
& V_\beta \cap f^{-1}(U_\alpha) \arrow[r, "f"] & U_\alpha &
\end{tikzcd}$$

condition

there exist functions $\alpha_{\alpha\beta} : V_\beta \cap f^{-1}(U_\alpha) \longrightarrow G$ s.t.

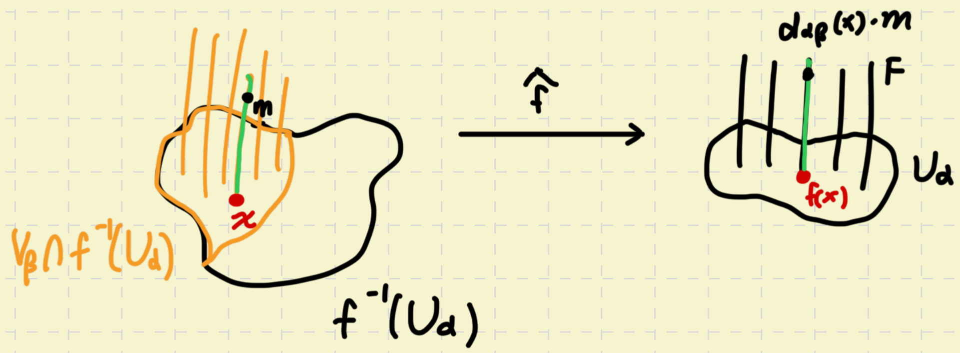
$$h_\alpha \circ \hat{f} \circ H_\beta^{-1}(x, m) = (f(x), \alpha_{\alpha\beta}(x) \cdot m) \quad \text{for } \forall x \in V_\beta \cap f^{-1}(U_\alpha), \forall m \in F.$$

[Rmk] ⓐ, ⓑ, ⓒ commutes because they are restriction of commutative diagrams.

[Rmk] Bundle homo preserves what?

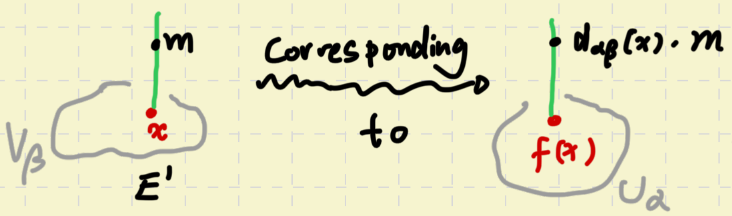
Condition ① means $\hat{f}(E'_p) \subseteq E_{f(p)}$ where $E_x := \pi^{-1}(x)$ is the fiber at x.

Condition ② means



$\pi'^{-1}(f^{-1}(U_\alpha))$ is not trivial bdl but on $\pi'^{-1}(V_\beta \cap f^{-1}(U_\alpha))$ it's trivial since $\pi'^{-1}(V_\beta)$ is trivial.

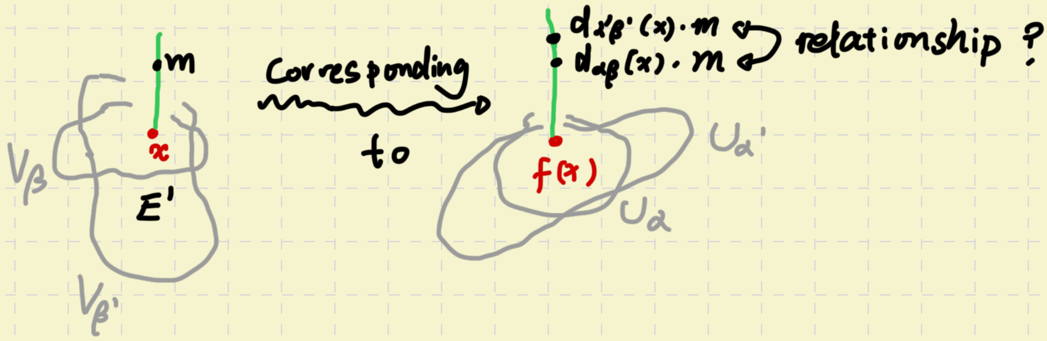
Homo of bdl means fibers of two bdls are "same", the map $\alpha_{\alpha\beta} : V_\beta \cap f^{-1}(U_\alpha) \longrightarrow G$ tells you this correspondence.



If $x \in V_\beta \cap \underbrace{f^{-1}(U_\alpha)}_{f(x) \in U_\alpha} \neq \phi$.

Bundle homo between local trivially bdl doesn't contain so much information because it doesn't have trivialization atlas. □

A natural question arises: what if we change trivialization?



The answer is the following prop.

[prop] Given $d_{\alpha\beta}: V_\beta \cap f^{-1}(U_\alpha) \to G$ and $d_{\alpha'\beta'}: V_{\beta'} \cap f^{-1}(U_{\alpha'}) \to G$ for different trivialization charts. Then for $\forall x \in V_\beta \cap V_{\beta'} \cap f^{-1}(U_\alpha \cap U_{\alpha'}) \neq \emptyset$, we have $d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(f(x))\, d_{\alpha\beta}\, g_{\beta\beta'}(x)$

pf: By definition $h_\alpha \hat{f} H_\beta^{-1}(x,m) = (f(x), d_{\alpha\beta}(x)\cdot m)$.

Hence $h_{\alpha'} \hat{f} H_{\beta'}^{-1}(x,m) = (f(x), d_{\alpha'\beta'}(x)\cdot m)$.

On the other hand, $h_{\alpha'} \hat{f} H_{\beta'}^{-1}(x,m) = \underbrace{h_{\alpha'} h_\alpha^{-1}}_{g_{\alpha'\alpha}} \underbrace{h_\alpha \hat{f} H_\beta^{-1}}_{d_{\alpha\beta}} \underbrace{H_\beta H_{\beta'}^{-1}}_{g_{\beta\beta'}}(x,m)$

$= (f(x), g_{\alpha'\alpha}(f(x))\, d_{\alpha\beta}(x)\, g_{\beta\beta'}(x))$

Then inj of Ad_ρ, $d_{\alpha'\beta'} = g_{\alpha'\alpha}(f(x))\, d_{\alpha\beta}(x)\, g_{\beta\beta'}(x)$.

[Prop] Given a map $f: B' \to B$ and bdls $E \xrightarrow{\pi} B$, $E' \xrightarrow{\pi'} B'$, a map of bdls $(f, \hat{f}): \pi' \to \pi$ exists iff there exist conti $\{d_{\alpha\beta}\}$ as above, satisfying $d_{\alpha'\beta'} = g_{\alpha'\alpha}(f(x))\, d_{\alpha\beta}(x)\, g_{\beta\beta'}(x)$ $\forall \alpha, \alpha', \beta, \beta'$.

pf: $\Rightarrow$ we've proved above.

$\Leftarrow$ Assume we have conti $\{d_{\alpha\beta}\}$ with $d_{\alpha'\beta'} = g_{\alpha'\alpha}(x)\, d_{\alpha\beta}(x)\, g_{\beta\beta'}(x)$.

For any $a \in E'$, if $\pi(a) \in V_\beta$, write $H_\beta(a) = (x, m)$.

Assume $x \in U_\alpha$, let $\hat{f}(a) = \hat{f} H_\beta^{-1}(x,m) = h_\alpha^{-1}(f(x), d_{\alpha\beta}(x)\cdot m)$.

It suffices to show $\hat{f}$ is well-defined.

Assume there is another choice of atlas, say, $\pi'(a) \in V_{\beta'}$, write $H_{\beta'}(a) = (x', m')$. ① $x' = x = \pi'(a)$. ② compute m'. $H_{\beta'}(a) = H_{\beta'} H_\beta^{-1} H_\beta(a) = (x, g_{\beta'\beta}(x)\cdot m)$

Hence, $m' = g_{\beta'\beta}(x)\cdot m$.

Assume $x \in U_{\alpha'}$. $\hat{f}(a) = \hat{f} H_{\beta'}^{-1}(x, g_{\beta'\beta}(x)\cdot m) = h_{\alpha'}^{-1}(f(x), d_{\alpha'\beta'}\, g_{\beta'\beta}(x)\cdot m)$

$= h_\alpha^{-1} \underset{\sim\sim\sim}{h_\alpha h_{\alpha'}^{-1}} (f(x), d_{\alpha'\beta'} g_{\beta'\beta}(x) \cdot m) = h_\alpha^{-1} (f(x), g_{\alpha\alpha'} d_{\alpha'\beta'} g_{\beta'\beta}(x) \cdot m) = h_\alpha^{-1}(f(x), d_{\alpha\beta}(x) \cdot m)$

Therefore $\hat{f}$ is well-defined.

[Thm] Every bdl map $\hat{f}$ over $f = id_B$ is an iso.

pf:

Δ Construct $\hat{g}: E \to E'$ over id_B

Let $\{d_{\alpha\beta}\}$ satisfying $d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(f(x))\, d_{\alpha\beta}(x)\, g_{\beta\beta'}(x)$ $(*)$ are maps given by $\hat{f}$. Since $f = id_B$, $d_{\alpha\beta}: V_\beta \cap U_\alpha \to G$, $d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(x)\, d_{\alpha\beta}(x)\, g_{\beta\beta'}(x)$.

Set $\overline{d_{\beta\alpha}}(x) = d_{\alpha\beta}(x)^{-1}$. Inverse $(*)$ in G we obtain

$$\overline{d_{\beta'\alpha'}}(x) = g_{\beta'\beta}(x)\, \overline{d_{\beta\alpha}}(x)\, g_{\alpha\alpha'}(x).$$

(Note that if $f \neq id_B$, $\overline{d_{\beta'\alpha'}}(x) = g_{\beta'\beta}(\underline{x})\, \overline{d_{\beta\alpha}}(x)\, g_{\alpha\alpha'}(\underline{f(x)})$ which is not right)

Hence $\{\overline{d_{\beta\alpha}}\}$ satisfies condition we want and by above prop there exists $\hat{g}: E \to E'$ over id_B.

Δ Check $\hat{g}$ is the inverse of $\hat{f}$.

$\hat{g} \circ \hat{f}\, H_\beta^{-1}(x, m) \underset{\uparrow}{=} \hat{g}\, h_\alpha^{-1}(x, d_{\alpha\beta}(x) \cdot m) = H_\beta^{-1}(x, \overline{d_{\beta\alpha}}(x)\, d_{\alpha\beta}(x) \cdot m)$

Independent of atlas, see proof of above prop

$= H_\beta^{-1}(x, m)$.

Hence $\hat{g} \circ \hat{f} = id$. Similarly $\hat{f} \circ \hat{g} = id$. $\square$

[Def] Gauge transformation of bundle $\pi: E \to B$ if a bundle homo $(id: B \to B,\ g: E \to E)$.

[Rmk] It's easy to show a gauge transformation is $g: E \to E$ with

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d] \\ B \arrow[r, "id"'] & B \end{tikzcd}$$

(Second condition automatically holds)

[prop] The set of all gauge transformations forms a group.

i) $id: E \to E$ is a gauge transformation, by diagram ①.

$$\begin{tikzcd} E \arrow[r, "id"] \arrow[d, "\pi"'] \arrow[dr, phantom, "①"] & E \arrow[d, "\pi"] \\ B \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\hat{f}_1"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r, "\hat{f}_2"] \arrow[d, "\pi"] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "\pi"] \\ B \arrow[r, "id"'] & B \arrow[r, "id"'] & B \end{tikzcd}$$

②

ii) associativity is satisfied by joining commutative diagrams, by ③.

iii) Any $\hat{f}$ over $f = id$ is an iso. Hence any gauge transformation has inverse.

Therefore the set of all gauge transformations form a grp. $\square$

One way fiber bdl homo arise is from the pullback construction.

[Def] Given a bdl $\pi : E \to B$ with grp G and fiber F and a conti $f : X \to B$.

Define $f^*(E) = \{(x,e) \in X \times E \mid f(x) = \pi(e)\}$

$f^*\pi : f^*E \longrightarrow X$, $(x,e) \mapsto x$

$\hat{f} : f^*E \longrightarrow E$, $(x,e) \mapsto e$,

$$\begin{tikzcd} (x,e) \arrow[rr, mapsto] \arrow[dd, mapsto] & & e \\ & f^*E \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "\pi"] \\ x & X \arrow[r, "f"'] & B \end{tikzcd}$$

s.t. the diagram commutes.

$f^*\pi$ is called the induced bdl under f or the pullback of π by f. $\square$

[Thm] (a) $f^*\pi : f^*E \longrightarrow X$ is a fiber bdl with grp G and fiber F.

(b) $(f, \hat{f}) : f^*\pi \longrightarrow \pi$ is a bundle map

Pf: (a) Let $\{(U_\alpha, h_\alpha)\}_\alpha$ be trivializing atlas of π. We want to construct k_α s.t. $\{(f^{-1}(U_\alpha), k_\alpha)\}_\alpha$ be trivializing atlas.

Restrict
$$\begin{tikzcd} f^*E \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$
to
$$\begin{tikzcd} f^*\pi^{-1}(f^{-1}(U_\alpha)) \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \pi^{-1}(U_\alpha) \arrow[d, "\pi"] \\ f^{-1}(U_\alpha) \arrow[r, "f"'] & U_\alpha \end{tikzcd}$$
, hence

$f^*\pi^{-1}(f^{-1}(U_\alpha)) = \{(x,e) \in f^{-1}(U_\alpha) \times \pi^{-1}(U_\alpha) \mid f(x) = \pi(e)\}$.

$$\begin{tikzcd} e \arrow[r, mapsto] & h_\alpha(e) & \\ \pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times F \arrow[dl, "p_1"] \arrow[dr, "p_2"] & \\ U_\alpha & & F \end{tikzcd}$$

Define $k_\alpha : (f^*\pi)^{-1}(f^{-1}(U_\alpha)) \longrightarrow f^{-1}(U_\alpha) \times F$

$(x, e) \longmapsto (x, p_2(h_\alpha(e)))$

△ Check $\{(f^{-1}(U_\alpha), k_\alpha)\}_\alpha$ is a trivializing atlas of $f^*\pi$.

It suffices to check k_α is homeo. We want to construct k_α^{-1}.

Idea: $k_\alpha(x,e) = (x, p_2 h_\alpha(e))$. Construct

$k_\alpha^{-1}(x, m) = (x, ?)$ s.t. when $m = p_2 h_\alpha(e)$,

we have $? = e$.

Consider

$$\begin{tikzcd}
 & e & h_\alpha e \\
 & \pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times F \arrow[dl, "p_1"'] \arrow[d, "p_2"] \\
f^{-1}(U_\alpha) \arrow[r, "f"] & U_\alpha & F \\
x & f(x) &
\end{tikzcd}$$

$p_2 h_\alpha e = m$

Hence we let $k_\alpha^{-1}(x, m) = (x, h_\alpha^{-1}(f(x), m))$.

Δ Find transition maps.

Let $f^* g_{\alpha\beta} := g_{\alpha\beta} \circ f : f^{-1}(U_\alpha \cap U_\beta) \to G$.

$$\begin{tikzcd}
 & f^*\pi^{-1} f^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "k_\alpha"'] \arrow[dr, "k_\beta"] & \\
f^{-1}(U_\alpha \cap U_\beta) \times F \arrow[rr, "k_\beta k_\alpha^{-1}"] & & f^{-1}(U_\alpha \cap U_\beta) \times F
\end{tikzcd}$$

$k_\beta k_\alpha^{-1}(x, m) = k_\beta(x, h_\alpha^{-1}(f(x), m)) = (x, p_2 h_\beta h_\alpha^{-1}(f(x), m))$

$= (x, p_2(f(x), g_{\beta\alpha}(f(x)) \cdot m)) = (x, g_{\beta\alpha}(f(x)) \cdot m) = (x, f^* g_{\beta\alpha}(x) \cdot m)$

(b) It suffices to find $\{d_{\alpha\beta}\}$ with $d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(f(x))\, d_{\alpha\beta}(x)\, f^* g_{\beta\beta'}(x)$.

Let $d_{\alpha\beta} : f^{-1}(U_\alpha) \cap \underbrace{f^{-1}(U_\beta)}_{V_\beta} \to G$ be $f^* g_{\alpha\beta}$.

Check: $g_{\alpha'\alpha}(f(x))\, f^* g_{\alpha\beta}(x)\, f^* g_{\beta\beta'}(x) = g_{\alpha'\alpha}(f(x))\, g_{\alpha\beta}(f(x))\, g_{\beta\beta'}(f(x)) = g_{\alpha'\beta'}(f(x))$

$= f^* g_{\alpha'\beta'}(x) = d_{\alpha'\beta'}(x)$. Hence $d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(f(x))\, d_{\alpha\beta}(x)\, f^* g_{\beta\beta'}(x)$. $\square$

[prop] If
$$\begin{tikzcd}
E' \arrow[r, "\bar{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\
B' \arrow[r, "f"] & B
\end{tikzcd}$$
is a bdl map, then $\pi' \cong f^*\pi$ as bdls over B'. (Uniqueness of pull back).

pf:
$$\begin{tikzcd}
E' \arrow[rr, "\bar{f}"] \arrow[dr, "h"] \arrow[dd, "\pi'"'] & & E \arrow[dd, "\pi"] \\
 & f^*E \arrow[ur] \arrow[dl] & \\
B' \arrow[rr, "f"'] & & B
\end{tikzcd}$$

Define $h : E' \to f^*E$ by setting $h(e') = (\pi'(e'), \bar{f}(e'))$.
It's well defined since $f\pi'(e') = \pi\bar{f}(e')$.
Obviously h is iso.

[prop] (1) $(f \circ g)^*\pi = g^* f^* \pi$ (2) $id_B^* \pi = \pi$

(3) pullback of a trivial bdl is a trivial bdl.

pf:

(1)

$$\begin{tikzcd} g^*f^*E \arrow[r, "\hat{g}"] \arrow[d, "g^*f^*\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & f^*E \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "\pi"] \\ X \arrow[r, "g"'] & Y \arrow[r, "f"'] & B \end{tikzcd}$$

together with uniqueness of pull back.

(2)

$$\begin{tikzcd} id^*E \arrow[r] \arrow[d, "id^*\pi"'] & E \arrow[d, "\pi"] \\ B \arrow[r, "id_B"'] & B \end{tikzcd}$$

e, b, b

$id^*E = \{(b,e) \mid \pi(e)=b\} \cong E$, then $id^*\pi = \pi$.

(3)

$$\begin{tikzcd} f^*X \arrow[r] \arrow[d, "f^*\pi"'] & E=B\times F \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

$(x,(b,m))$, (b,m), x, $f(x)=b$

$f^*X = \{(x,(b,m)) \mid f(x) = \pi(b,m) = b\}$

$\cong \{(x,m) \mid x\in X, m\in F\} = X\times F$

Then $f^*\pi : X\times F \to X$ is a trivial bdl.

[Fact] Given fiber bdl $\pi: E\to B$ with grp G and fiber F. For two homotopic maps $f\simeq g: X\to B$ there is an iso $f^*\pi \cong g^*\pi$ of bdles over X.

[Rmk] Induced bdls under homotopic maps are iso.

[Coro] Fiber bdl over a contractible space B is trivial.

pf: Since B is contractible, so $id_B \simeq i\circ C_b = [B \xrightarrow{C_b} b \xrightarrow{i} B]$ where $C_b: B \to b\in B$ is a const map. Since induced bdl over homotopy maps are iso, we have bdl iso $(iC_b)^*\pi \cong id_B^*\pi$.

Then $\pi \cong id_B^*\pi \cong (iC_b)^*\pi \cong C_b^* i^*\pi$. Note that the pullback of a trivial bdl is always trivial, it suffices to show $i^*\pi$ is trivial, and then $\pi \cong C_b^*(i^*\pi)$ is trivial. Since $i^*\pi$ is a bdl over a point b, hence $i^*\pi$ is trivial.

$$\left(\begin{tikzcd} i^*E \arrow[r] \arrow[d, "i^*\pi"'] & E \arrow[d, "\pi"] \\ b \arrow[r, "i"] & B \end{tikzcd}\right)$$

□

Next, we'll show the set of iso classes of bdls over S^n with grp G and fiber F is iso to $\pi_{n-1}(G)$,

i.e., $\left\{ \begin{tikzcd} F \arrow[r, hook, "\curvearrowright G"] & E \arrow[d] \\ & S^n \end{tikzcd} \right\} \simeq \pi_{n-1}(G)$.

Let $U_\pm$ obtained by S^n removing north, resp, south pole.

△ Show $U_\pm$ can provide trivializing atlas for $\pi: E \to S^n$.

$$\begin{tikzcd} i_\pm^* E \arrow[r] \arrow[d, "i_\pm^* \pi"'] & E \arrow[d, "\pi"] \\ U_\pm \arrow[r, "i_\pm"] & S^n \end{tikzcd}$$

Since $U_\pm$ is contractible, the restriction bundle over $U_\pm$, $i_\pm^* E$, is trivial.

△ Hence, the bdl π is completely determined the transition function $g_\pm : U_+ \cap U_- \cong S^{n-1} \longrightarrow G$, i.e., by an element in $\pi_{n-1}(G)$

Principal bdles

Let's first show the motivation. Generally, we aim to classify fiber bdls on a given topo space. Denote $B(X, G, F, \rho)$ the iso classes (over id_X) of fiber bdls on X with grp G and fiber F, and G-action ρ on F. The fact is, the fiber F doesn't play any essential role in the classification of fiber bdls. Hence, it's enough to understand the set

$$\mathcal{P}(X, G) := \mathcal{B}(X, G, G, m_G), \quad (m_G : G \times G \to G \text{ is grp multiplication})$$

which is the set of iso classes of fiber bdls with grp G and fiber G, and G-action m_G on G.

[Def] Elements in $\mathcal{P}(X, G)$ are called principal G-bdls.

[Exp] Any regular cover $p: E \to X$ is a principal G-bundle, with $G = \pi_1(X) / p_* \pi_1(E)$ and G is given discrete topo. In particular $\tilde{X} \to X$ is a principal $\pi_1(X)$-bdl. (Prop 1.40 in Hatcher)

[Thm] Let $\pi: E \to X$ be principal G-bdl. Then G acts freely and transitively on the right of E so that $E/G \cong X$. In particular, π is the quotient map. ($\pi: E \to X \cong E/G$)

[Rmk] Freely and transitively means G acts on a fiber is freely and transitively.

pf: Let $\{(U_\alpha, h_\alpha)\}$ be trivializing atlas of $\pi: E \to X$.

(Goal: Define right action of E locally and then glue them globally, i.e., it's independent of the choice of trivializing chart.)

△ Define right action of G locally on $\pi^{-1}(U_\alpha)$.

$$\pi^{-1}(U_\alpha) \times G \longrightarrow \pi^{-1}(U_\alpha)$$
$$(e, g) \longmapsto ?$$

① $? = e$. It's trivial action.

② $? = h_\alpha^{-1}(\pi(e), g)$ $\quad (\pi^{-1}(U_\alpha) \xrightarrow{h_\alpha} U_\alpha \times G)$

Suppose $e \in \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$. $h_\alpha^{-1}(\pi(e), g) \neq h_\beta^{-1}(\pi(e), g)$, since $(\pi(e), g_{\beta\alpha}(\pi(e)) \cdot g) \neq (\pi(e), g)$. Hence we can't glue.

It seems that we need act something depends on charts to eliminate $g_{\beta\alpha}(\pi(e)) \cdot g \neq g$. There comes the following definition.

Define
$$\pi^{-1}(U_\alpha) \times G \longrightarrow \pi^{-1}(U_\alpha)$$
$$(e, g) \longmapsto h_\alpha^{-1}(\pi(e), P_2 h_\alpha(e) \cdot g)$$

(G acts on the right on $e \in \pi^{-1}(U_\alpha)$ by acting on the right on $h = P_2 h_\alpha(e)$.)

△ Check we can glue this action globally.

[Trick]: It's often used that : glue xx globally $\Leftrightarrow$ check result is same on the intersection of two charts

same result by two charts

Assume $e \in \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$. w.t.s.
$$h_\alpha^{-1}(\pi(e), P_2 h_\alpha(e) \cdot g) = h_\beta^{-1}(\pi(e), P_2 h_\beta(e) \cdot g)$$

i.e., w.t.s. $(\pi(e), P_2 h_\alpha(e) \cdot g) = (\pi(e), g_{\alpha\beta}(\pi(e)) P_2 h_\beta(e) \cdot g)$

$$\begin{tikzcd}
& \pi^{-1}(U_\alpha \cap U_\beta) \ni e \arrow[dl, "h_\beta"'] \arrow[dr, "h_\alpha"] & \\
(U_\alpha \cap U_\beta) \times G \arrow[rr, "h_\alpha h_\beta^{-1}"] & & (U_\alpha \cap U_\beta) \times G
\end{tikzcd}$$
$$(\pi(e), P_2 h_\beta(e)) \longmapsto (\pi(e), P_2 h_\alpha(e))$$

The condition of fiber bdl with grp G and fiber G requires that $P_2 h_\alpha(e) = g_{\alpha\beta}(\pi(e)) \cdot P_2 h_\beta(e)$. Hence we prove that there is a right action of G on E.

△ Show the action is free, transitive and $U_\alpha \times G / G \cong U_\alpha$ locally.

$$\pi^{-1}(U_\alpha) \times G \longrightarrow \pi^{-1}(U_\alpha)$$
$$(e, g) \longmapsto e \cdot g := h_\alpha^{-1}(\pi(e), P_2 h_\alpha(e) \cdot g)$$

Free: If $h_\alpha^{-1}(\pi(e), P_2 h_\alpha(e) \cdot g) = e$, i.e., $h_\alpha(e) = (\pi(e), P_2 h_\alpha(e) \cdot g)$

On the other hand, $h_\alpha(e) = (\pi(e), p_2 h_\alpha(e))$. Since action is effective, $g = e_G$.

Transitive: let $a, b \in \pi^{-1}(U_\alpha)$ with $\pi(a) = \pi(b)$, we want to find g s.t. $h_\alpha^{-1}(\pi(a), p_2 h_\alpha(a) g) = b$, i.e. $h_\alpha(b) = (\pi(a), p_2 h_\alpha(a) g)$. On the other hand, $h_\alpha(b) = (\pi(b), p_2 h_\alpha(b))$. So we need to find g s.t. $p_2 h_\alpha(b) = p_2 h_\alpha(a) \cdot g$

It suffices to set $g = (p_2 h_\alpha(a))^{-1}(p_2 h_\alpha(b))$.

$U_\alpha \times G/_G \cong U_\alpha$: Since G acts on $U_\alpha \times G \cong \pi^{-1}(U_\alpha)$ transitively, $\pi^{-1}(a)$ (a fiber) represents same elements for any $a \in U_\alpha$. Hence $U_\alpha \times G/G \cong U_\alpha$.

Δ It's easy to pass above props to global.

[Rmk] Converse of the thm holds in special cases

① E: compact Hausdorff, G: compact Lie grp, $G \overset{\text{freely}}{\curvearrowright} E$

Then orbit map $E \to E/G$ is a principal G-bdl

② G: Lie grp. H: compact subgrp.

Then $\pi: G \to G/H$ is a principal H-bdl

[Def] Given a principal G-bdl $\pi: E \to X$, then G acts freely on the right on E by previous thm. Hence we define a left G-action on $E \times F$ by:

$g \cdot (e, f) \mapsto (e \cdot g^{-1}, g \cdot f)$. Let $E \times_G F := E \times F/G$. The projection $\omega := \pi \times_G F : E \times_G F \to X \cong E/G$ fitting into a commutative diagram

$$\begin{tikzcd} & E \times F \arrow[dl, "p_1"'] \arrow[dr] & \\ E \arrow[dr, "\pi"'] & \circlearrowright & E \times F/G \arrow[dl, "\omega"] \\ & X & \end{tikzcd}$$

is called the associated bdl with fiber F.

The following thm justifies terminology in detail.

[Thm] $\omega: E \times_G F \to X$ is a fiber bdl with grp G, fiber F, and same transition functions as π.

Pf: Restrict commt. diagram
$$\begin{tikzcd} & E \times F \arrow[dl, "p_1"'] \arrow[dr] & \\ E \arrow[dr, "\pi"'] & & E \times F/G \arrow[dl, "\omega"] \\ & X & \end{tikzcd}$$
in def

to U_α, a trivialization of π on X.

$$\begin{tikzcd} & \pi^{-1}(U_\alpha)\times F \arrow[dl, "p_1"'] \arrow[dr] & \\ \pi^{-1}(U_\alpha) \arrow[dr] & & ? \arrow[dl, "\omega"] \\ & U_\alpha & \end{tikzcd}$$

path I : U_α $\quad ? = \omega^{-1}(U_\alpha)$

path II : U_α $\quad ? = \pi^{-1}(U_\alpha)\times F/(e,f)\sim(e\cdot g^{-1}, g\cdot f)$

Hence $\omega^{-1}(U_\alpha) = \pi^{-1}(U_\alpha)\times F/(e,f)\sim(e\cdot g^{-1}, g\cdot f)$

$= U_\alpha \times G\times F/(u,h,f)\sim(u, h\cdot g^{-1}, g\cdot f)$

use $\pi^{-1}(U_\alpha)\cong U_\alpha\times G$ and $e\cdot g \Leftrightarrow (u, h\cdot g)$ where $h_\alpha e = (u,h)$

Δ Define $k_\alpha : \omega^{-1}(U_\alpha) \longrightarrow U_\alpha\times F$

$[(u,h,f)] \longmapsto (u, h\cdot f)$

Since $k_\alpha(u, h\cdot g^{-1}, g\cdot f) = (u, h\cdot f) = k_\alpha(u,h,f)$, so k_α well-def.

Δ Show k_α has inverse and thus $\{(U_\alpha, k_\alpha)\}_\alpha$ are trivializing charts.

$l_\alpha : U_\alpha\times F \longrightarrow \omega^{-1}(U_\alpha)$

$(u,f) \longmapsto [(u, e_G, f)]$.

Then $l_\alpha k_\alpha[(u,h,f)] = l_\alpha(u, h\cdot f) = [(u, e_G, h\cdot f)] = [(u,h,f)]$

$k_\alpha l_\alpha(u,f) = k_\alpha[(u, e_G, f)] = (u,f)$. Hence $l_\alpha = k_\alpha^{-1}$.

Δ Show ω and π have same transition functions.

ω and π have same trivializing opens.

The following I use subindex denote coordinate.

Consider $(x,m)_\beta \in U_\beta\times F$, $(x,m)_\beta = (x, g_{\alpha\beta}(x)\cdot m)_\alpha$.

$k_\alpha k_\beta^{-1}(x,m) = k_\alpha[((x, e_G)_\beta, m)] = k_\alpha[((x, g_{\alpha\beta}(x))_\alpha, m)] = (x, g_{\alpha\beta}(x)\cdot m)$

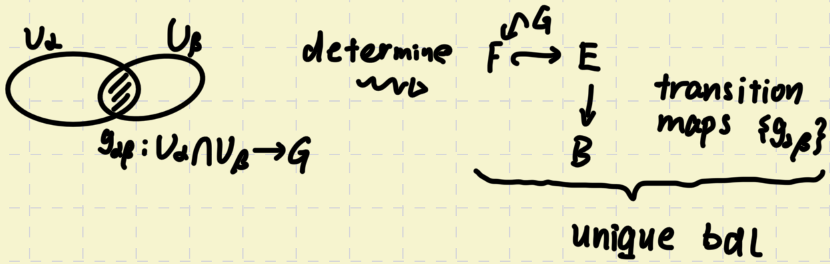
Hence the transition maps of ω are the same as π. $\square$

[prop] $\mathcal{P}(X,G) \longrightarrow \mathcal{B}(X,G,F,\rho)$ is one-to-one correspondence.

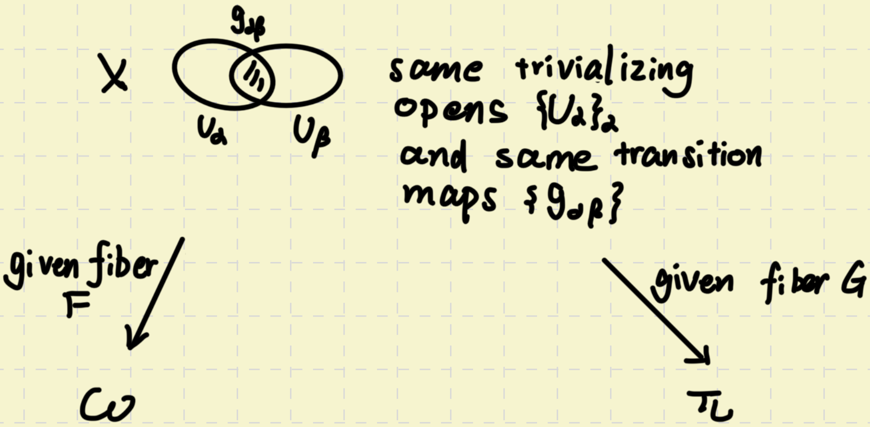
$\pi \longmapsto \omega := \pi\times_G F \to X$

[Rmk] This is the correspondence we state at begining. To study $\mathcal{B}(X,G,F,\rho)$, we only need to study principal G-bdls $\mathcal{P}(X,G)$.

pf: Recall the thm we've proved "Given an open cover $\{U_\alpha\}$ of B and conti functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to G$ satisfying props (a), (b), (c), there is a unique str of fiber bdl over B with grp G, given fiber F, and transition functions $\{g_{\alpha\beta}\}$."

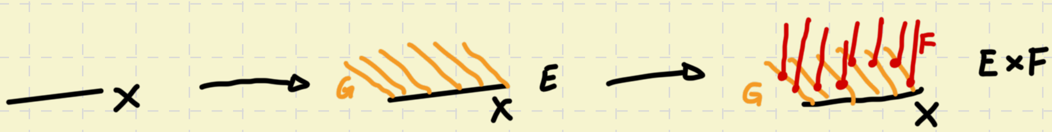


Hence $P(X,G) \xleftrightarrow{1:1} B(X,G,F,\rho)$ is given by



Given ω, we know $\{U_\alpha\}_\alpha$ and $\{g_{\alpha\beta}\}$. Then there is unique π with fiber G.
Given π, we know $\{U_\alpha\}_\alpha$ and $\{g_{\alpha\beta}\}$. Then there is unique ω with fiber F.
By previous prop we find bdl $\omega := \pi \times_G F$ satisfies conditions and by uniqueness it's the bdl in $B(X,G,F,\rho)$ corresponding to π. $\square$

[Rmk] Picture for associated bdl $E \times_G F \to E/G \cong X$



Why we need $E \times F/G$? Because we consider bld over $E/G \cong X$.
Hence we view an orbit an element, i.e., $(x, e_G) = (x, e_G) \cdot g = (x, g)$.
Hence the fiber over (x, e_G) and fiber over (x, g) should have a one to one correspondence —— the correspondence is $((x,g), g^{-1}f) \sim ((x, e_G), f)$
Hence, we should quotient equivalence relationship $(e,f) \sim (e \cdot g^{-1}, gf)$. $\square$

[Prop] (Associated bdl is functorial) If $(\hat{f}, f)$ is a map of principal G-bdls, then there is an induced map of associated bdls with fiber F.

$$\begin{tikzcd} E' \arrow[r, "\hat{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd} \quad \overset{\text{induces}}{\Rightarrow} \quad \begin{tikzcd} E' \times_G F \arrow[r, "\hat{f} \times_G id_F"] \arrow[d, "\omega'"] & E \times_G F \arrow[d, "\omega"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

□

[Exp] $\pi: S^1 \to S^1$, $z \mapsto z^2$ can be regarded as a principal $\mathbb{Z}/2$-bdl.

When $F = [-1,1]$, $\rho: \underset{\{-1,1\}}{\underset{\shortparallel}{\mathbb{Z}_2}} \times F \to F$ is multiplication,

$P(S^1, \mathbb{Z}_2) \longrightarrow B(S^1, \mathbb{Z}_2, F, \rho)$ $\quad S^1 \times_{\mathbb{Z}_2} [-1,1] = S^1 \times [-1,1] / (x,t) \sim (-x,-t)$

$\pi \longmapsto$ Mobiüs band

When $F = S^1$, the image of π is klein bottle. □

[Construction] Let $E_1 \xrightarrow{\pi_1} X$ and $E_2 \xrightarrow{\pi_2} Y$ be two principal G-bdls. G acts on E_1 and E_2 on the right. Define a left action on E_2 by $g \cdot e_2 := e_2 \cdot g^{-1}$. We obtain an associated bdl of π_1 with fiber E_2, namely

$\omega: E_1 \times_G E_2 \longrightarrow X$.

[Thm] Bdl maps from π_1 to π_2 are in one-to-one correspondence to sections of ω.

Pf: $\left(\begin{array}{l} \text{Idea: Given bdl maps, construct sections.} \\ \qquad\quad\ \text{Given sections, construct bdl maps.} \end{array}\right)$ $\quad \begin{tikzcd} E_1 \arrow[r, "\hat{f}"] \arrow[d] & E_2 \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$

Δ Given a bdl map $(f, \hat{f})$, we want to construct a section. Since section can be defined locally and then glue, so we reduce to the case of trivial bdl. Let $U \overset{\text{open}}{\subseteq} Y$ s.t. $E_2|_U$ is trivial. We can pick $V \overset{\text{open}}{\subseteq} f^{-1}(U)$ (e.g. $V = V_\alpha \cap f^{-1}(U)$ where V_α is some trivialization open in X) s.t. $E_1|_V$ is trivial. Hence, the bdl map diagram restricts to the following diagram:

$$\begin{tikzcd} V \times G \arrow[r, "\hat{f}"] \arrow[d, "\pi_1"'] & U \times G \arrow[d, "\pi_2"] \\ V \arrow[r, "f"'] & U \end{tikzcd} . \qquad \begin{tikzcd} (V \times G) \times_G (U \times G) \arrow[d, "\omega"] \\ V \end{tikzcd}$$

We shall define section σ of ω over V.

Recall that $(V\times G)\times_G(U\times G) = \{(e_1,e_2)\in (V\times G)\times(U\times G)\}/(e_1,e_2)\sim(e_1\cdot g,\ g^{-1}\cdot e_2)$

Define $\sigma: V \longrightarrow (V\times G)\times_G(U\times G)$

$x \longmapsto [(e_1, \hat{f}(e_1))]$ with $\pi_1(e_1)=x$

(from $x\in V$ to find an element in $(V\times G)\times_G(U\times G)$. It's natural to let $\sigma(x)=[(\underset{V\times G}{e_1}, \underset{U\times G}{\hat{f}(e_1)})]$ with $\pi_1(e_1)=x$.)

* Check it's independent of e_1. Trick: principal G-bdl G acts transversely on the fiber, hence when fix $e_1\in\pi^{-1}(x)$, $\pi^{-1}(x)$ can be written as $\pi^{-1}(x)=\{e_1\cdot g \mid g\in G\}$. When we have concrete form, anything is not difficult any more.

$$[(e_1\cdot g, \hat{f}(e_1\cdot g))] \underset{\uparrow}{=} [(e_1\cdot g, \hat{f}(e_1)\cdot g)] \underset{\text{By def}}{=\!=} [(e_1\cdot g,\ g^{-1}\cdot\hat{f}(e_1))] = [(e_1, \hat{f}(e_1))]$$

bdl maps between principal G-bdl is G-equivariant.

* Omit to show σ is continous (by point-set topo)

* Recall that $\omega: (V\times G)\times_G(U\times G) \longrightarrow (V\times G)/G \cong V$

$[(e_1, e_2)] \longmapsto \pi_1(e_1)$

$$\omega\sigma(x) = \omega[(e_1, \hat{f}(e_1))] = \pi_1(e_1) = x.$$

with $\pi_1(e_1)=x$

* Pass local to global. For any pt $x\in X$, we can find such V with $x\in V$. $\sigma(x)=[(e_1, \hat{f}(e_1))]$ with $\pi_1(e_1)=x$, so it's independent of the choice of trivialization (We use global information such as π_1, $\hat{f}$ to define σ, not local information, such as transition functions).

Δ Given a section σ of $\omega: E_1\times_G E_2 \longrightarrow X$, then we'll define a bdl map.

Recall $\omega: E_1\times_G E_2 \longrightarrow X$

$[(e_1, e_2)] \longmapsto \pi_1(e_1)$

Define $\hat{f}: E_1 \longrightarrow E_2$

$e_1 \longmapsto e_2$ with $\sigma(\pi_1(e_1)) = [(e_1, e_2)]$

$$\begin{array}{ccc} E_1\times_G E_2 & & [(e_1,e_2)] \\ \sigma\uparrow\downarrow\omega & & \downarrow \\ X & & \pi_1(e_1) \end{array}$$

Rmk: Can we let $\hat{f}(e_1) = P_2\sigma(\pi_1(e_1))$? No. Because $\sigma(\pi_1(e_1)) = [(e_1\cdot g, g^{-1}\cdot e_2)]$ which is not well-defined. But e_2 with $\sigma(\pi_1(e_1)) = [(e_1, e_2)]$ can completely determine $\hat{f}(e_1)$.

* Check it's G equivariant.

$$\hat{f}(e_1\cdot g) = g^{-1}\cdot e_2 \quad \text{with } \sigma(\pi_1(e_1\cdot g)) = [(e_1\cdot g,\, g^{-1}e_2)]$$
$$= e_2\cdot g = \hat{f}(e_1)\cdot g$$

We omit showing $\pi_2\hat{f} = f\pi_1$ and there exists $d_{\alpha\beta}$.

[Lemma] Let $\pi: E \to X\times I$ be a bdl and let $\pi_0 := i_0^*\pi : E_0 \to X$ be the pullback of π under $i_0 : X \to X\times I$, $x \mapsto (x,0)$. Then, $\pi \cong p_1^*\pi_0 \cong \pi_0\times id_I$, where $p_1 : X\times I \to X$ is the projection map.

pf:

Δ Show $p_1^*\pi_0 \cong \pi_0 \times id_I$. (Easy)

$$\begin{tikzcd} E_0 = i_0^*E \arrow[r] \arrow[d, "\pi_0 = i_0^*\pi"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "i_0"'] & X\times I \end{tikzcd}$$

$E_0 = i_0^*E = \{(x,e)\in X\times E \mid (x,0) = \pi(e)\} \cong E$

$(x,e) \longmapsto e$

Then $\pi_0 : E_0 \cong E \longrightarrow X$. $\quad (x,e) \longmapsto x$, $\pi(e) = (x,0)$

$$\begin{tikzcd} p_1^*E_0 \arrow[r] \arrow[d, "p_1^*\pi_0"'] & E_0 \arrow[d, "\pi_0"] \\ X\times I \arrow[r, "p_1"'] & X \end{tikzcd}$$

It's easy to check: $\pi_0(\underline{x},e) = p_1\pi(\underline{e})$

correspondence $E_0 \to E$

So $p_1^*E_0 = \{((x,t),(x_0,e_0)) \in (X\times I)\times E_0 \mid x = \pi_0(x_0,e_0) = p_1\pi(e_0)\}$

$\cong \{((p_1\pi e_0, t), (x_0,e_0)) \mid (x_0,e_0)\in E_0,\ t\in I\} \cong E_0\times I.$

$$\begin{tikzcd} p_1^*\pi_0 : p_1^*E_0 \arrow[r] \arrow[d, "2"] & X\times I \arrow[d, equal] \\ E_0\times I \arrow[r] & X\times I \end{tikzcd}$$

$((p_1\pi e_0, t),(x_0,e_0)) \mapsto (p_1\pi e_0, t)$

$((x_0,e_0),t)$ $\qquad (\pi_0(x_0,e_0),t)$

Hence the bottom line is
$\pi_0\times id_I : E_0\times I \longrightarrow X\times I$
$((x_0,e_0),t) \mapsto (\pi_0(x_0,e_0),t)$

Δ Show $\pi \cong p_1^*\pi_0$.

Since pull back is unique up to iso, it suffices to find bdl map $\hat{p}_1 : E \to E_0$ s.t. diagram commutes:

$$
\begin{tikzcd}
E \arrow[r, dotted, "\hat{\varphi}_1"] \arrow[d] & E_0 \arrow[d, "\pi_0"] \\
X \times I \arrow[r, "\varphi_1"'] & X
\end{tikzcd}
$$

(It's what we've proved, Prop 3.1.19 in Ref. bdl map contains enough information, s.t. when there is a bdl map, there is a pullback.)

The existence of $(\varphi_1, \hat{\varphi}_1)$ is equivalent to the enistence of section σ of $\omega: E \times_G E_0 \longrightarrow X \times I$. In summary, we need to find section $s: X \times I \to E \times_G E_0$ in the diagram:

$$
\begin{tikzcd}
 & E \times_G E_0 \arrow[d, "\omega"] \\
X \times I \arrow[r, "id"'] \arrow[ur, dotted, "s"] & X \times I
\end{tikzcd}
$$

. In many cases ω is a fibration(e.g., fiber bdl over a paracompact space is a fibration. Fiber bdls have Homotopy lifting property for CW complexes), so ω has H.L.P. Then the existence of s is equivalent to existence of $\sigma: X \times \{0\} \longrightarrow E \times_G E_0$ in the diagram

$$
\begin{tikzcd}
X \times \{0\} \arrow[r, dashed, "\sigma_0"] \arrow[d, "i"'] & E \times_G E_0 \arrow[d, "\omega"] \\
X \times I \arrow[r, "id"'] \arrow[ur, dotted, "s"] & X \times I
\end{tikzcd}
$$

.

The existence of section σ_0 is equivalent to the existence of the bdl map f in the diagram when "?" is appropriate.

$$
\begin{tikzcd}
E \arrow[r, dashed, "f"] \arrow[d, "?"'] & E_0 \arrow[d, "?"] \\
X \times \{0\} \arrow[r, "?"'] & ?
\end{tikzcd}
\quad \xrightarrow[X \times \{0\} \cong X]{\text{With } E \cong E_0} \quad
\begin{tikzcd}
E_0 \arrow[r, dashed, "f"] \arrow[d, "?"'] & E_0 \arrow[d, "?"] \\
X \arrow[r, "?"] & ?
\end{tikzcd}
$$

It's obviouse that the diagram should be:

$$
\begin{tikzcd}
E_0 \arrow[r, "id"] \arrow[d, "\pi_0"'] & E_0 \arrow[d, "\pi_0"] \\
X \arrow[r, "id_X"'] & X
\end{tikzcd}
$$

. We complete the proof. $\square$

Finally we'll introduce an application of principal G-bdl — prove the important fact that pullback bdl is iso under homotopy maps.

[Thm] Let $\pi: E \to Y$ be a fiber with grp G and fiber F, and let $f \simeq g: X \to Y$ be homotopic maps. Then $f^*\pi \cong g^*\pi$ over id_X.

[Rmk]
$$\begin{tikzcd} f^*\pi \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} g^*\pi \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

They are construction of $f^*\pi$ and $g^*\pi$

$$\begin{tikzcd} f^*\pi \arrow[r, "\sim"] \arrow[d] & g^*\pi \arrow[d] \\ X \arrow[r, "id"'] & X \end{tikzcd}$$

This is $f^*\pi \cong g^*\pi$ over id_X.

pf: By one-to-one correspondence between $\mathcal{P}(X,G)$ and $\mathcal{B}(X,G,F,\rho)$, we can only prove the thm in the case of principal G-bdls.

Let $H: X \times I \to Y$ be homotopy between f and g with $\begin{cases} H(-,0)=f \\ H(-,1)=g \end{cases}$

Recall above lemma

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d, "\pi_0"'] & E \arrow[r] \arrow[d, "\pi"] & E_0 \arrow[d, "\pi_0"] \\ X \arrow[r, "i_0"'] & X \times I \arrow[r, "p_1"'] & X \end{tikzcd}$$

then $p_1^*\pi_0 = \pi$.

we will have

$$\begin{tikzcd} f^*E = i_0^* H^*E \arrow[r] \arrow[d, "i_0^*H^*\pi"'] & H^*E \arrow[r] \arrow[d, "H^*\pi"] & E \arrow[d, "\pi"] \\ X \arrow[r, "i_0"'] & X \times I \arrow[r, "H"'] & Y \end{tikzcd}$$

Since $f = H(-,0) = H i_0$, $f^*\pi = i_0^* H^*\pi$.

$$\begin{tikzcd} g^*E = i_1^* H^*E \arrow[r] \arrow[d, "i_1^*H^*\pi"'] & H^*E \arrow[r] \arrow[d, "H^*\pi"] & E \arrow[d, "\pi"] \\ X \arrow[r, "i_1"'] & X \times I \arrow[r, "H"'] & Y \end{tikzcd}$$

Since $g = H(-,1) = H i_1$, $g^*\pi = i_1^* H^*\pi$.

By above lemma

$$H^*\pi = p_1^* i_0^* H^*\pi = p_1^* f^*\pi$$
$$H^*\pi = p_1^* i_1^* H^*\pi = p_1^* g^*\pi$$
$$\Longrightarrow f^*\pi = i_0^* H^*\pi = i_0^* p_1^* g^*\pi = g^*\pi.$$

□

The following is an important theorem.

[Thm] A principal G-bdl $\pi: E \to X$ is trivial iff π has a section.

pf: Consider the pull back of $\pi': G \to pt$ along $c_*: X \to pt$.

$$\begin{tikzcd} c_t^* G \arrow[r] \arrow[d] & G \arrow[d, "\pi'"] \\ X \arrow[r, "c_t"'] & pt \end{tikzcd} \quad . \quad c_t^* G = \{(x,g) \in X \times G \mid \underbrace{c_t x = \pi'(g)}_{\text{always hold}}\} = X \times G.$$

Hence, $\pi : E \to X$ is trivial $\Leftrightarrow$ E is the pullback of $\pi' : G \to pt$ along $c_t : X \to pt$ $\Leftrightarrow$ $\exists$ a bdl map $h : E \to G$ in the following diagram.

$$\begin{tikzcd} E \arrow[r, dashed, "h"] \arrow[d, "\pi"'] & G \arrow[d, "\pi'"] \\ X \arrow[r, "c_t"'] & pt \end{tikzcd} \quad \overset{\text{By above}}{\underset{\text{thm}}{\Longleftrightarrow}} \exists \text{ a section of } \omega : E \times_G G \longrightarrow X$$

$$\overset{\omega \cong \pi}{\Longleftrightarrow} \exists \text{ a section of } \pi : E \to X.$$

pf: To show two bdls are iso, we can prove they are iso locally.

Let U_α be a trivialization open. $E \times_G G$ locally be

$$\pi^{-1}(U_\alpha) \times_G G = \pi^{-1}(U_\alpha) \times G/\sim \; = U_\alpha \times G \times G \,/\, (u, g_1, g_2) \sim (u, g_1 g^{-1}, g g_2) \cong U_\alpha \times G.$$
$$[(u, g_1, g_2)] \longmapsto (u, g_1 g_2)$$

Classification of principal G-bdls

[Def] For a grp G, a principal G-bdl $\pi_G : EG \to BG$ with EG contractible is called universal principal G-bdl. BG is called classifying space.

[Rmk] Apply L.E.S. to principal G-bdl $G \to EG \to BG$, we obtain $\pi_n(BG) \cong \pi_{n-1}(G)$.

[Fact] Let G be a locally compact topo grp. Then a universal principal G-bdl $\pi_G : EG \to BG$ exists, and the construction is functorial in the sense that a conti grp homo $\mu : G \to H$ induces a bdl map $(B_\mu, E_\mu) : \pi_G \longrightarrow \pi_H$.

$$\begin{tikzcd} EG \arrow[r, "E_\mu"] \arrow[d, "\pi_G"'] \arrow[dr, phantom, "\circlearrowright"] & EH \arrow[d, "\pi_H"] \\ BG \arrow[r, "B_\mu"'] & BH \end{tikzcd}$$

$\square$

Universal principal G-bdls play an essential role in the classifying of principal G-bdls. The following we assume the universal principal G-bdl exists.

[Thm$\clubsuit$] If X is a CW-complex, there exists a bijection

$$\Phi : [X, BG] \longrightarrow P(X,G)$$
$$f \longmapsto f^*\pi_G$$

$$\begin{tikzcd} f^*\pi_G \arrow[r] \arrow[d, "f^*\pi_G"'] \arrow[dr, phantom, "\circlearrowleft"] & EG \arrow[d, "\pi_G"] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$

Before prove thm $\clubsuit$, we show some corollaries and examples.

[Exp] $B(S^n, G, F, \rho) \cong P(S^n, G) \cong [S^n, BG] = \pi_n(BG) \cong \pi_{n-1}(G)$ which we've proved by studying transition functions.

[Coro] The classifying space BG is unique up to homotopy.

pf: Let $\pi_G : EG \to BG$, $\pi_G' : EG' \to BG'$ be two universal principal bdls. Regard π_G as principal G-bdl, $P(X,G) \xrightarrow{\sim} [X, BG]$, $\pi_G' \longmapsto f$

s.t. $\pi_G' = f^*\pi_G$. Regard π_G' as principal G-bdl, $P(X,G) \xrightarrow{\sim} [X, BG']$, $\pi_G \longmapsto g$

s.t. $\pi_G \cong g^*\pi_G'$. Hence $\pi_G \cong g^*\pi_G' \cong g^*f^*\pi_G \cong (f\circ g)^*\pi_G$.

On the other hand, $\pi_G \cong id_{BG}^*\pi_G$, so $f\circ g \simeq id_{BG}$. Similarly $g\circ f \simeq id_{BG'}$ and thus $f : BG' \to BG$ is a homotopy equivalence. $\square$

[Exp] Fiber bdl $O(n) \hookrightarrow V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty)$ with $V_n(\mathbb{R}^\infty)$ contractible.

With uniqueness of classifying space, we have $BO(n) \cong G_n(\mathbb{R}^\infty)$.

A principal $O(n)$-bdl is a rank n $\mathbb{R}$-vector bdl with Riemann str. $G_n(\mathbb{R}^\infty)$ is the classifying space of rank n $\mathbb{R}$-vector bdls with Riemann str.

Fiber bdl $U(n) \longrightarrow V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty)$ with $V_n(\mathbb{C}^\infty)$ contracitible.

Hence $BU(n) \cong G_n(\mathbb{C}^\infty)$ is the classifying space for rank n $\mathbb{C}$-vec bdls with Hermitian str. $\square$

The following construction relates bdls and cohomology.

[Construction]

$$H^n(K(G,n);G) \cong \mathrm{Hom}(H_n(K(G,n)), G) \oplus \mathrm{Ext}(H_{n-1}(K(G,n)), G)$$

Let α be the inverse of the Hurewicz iso

$$G = \pi_n(K(G,n)) \longrightarrow H_n(K(G,n)),$$ we obtain

$$\alpha \in \mathrm{Hom}(H_n(K(G,n)), G) \subseteq H^n(K(G,n); G).$$

Given any $[f] \in [X, K(G,n)]$, it induces a map on cohomology

$$f^n: H^n(K(G,n);G) \longrightarrow H^*(X,G).$$
$$\alpha \longmapsto f^*(\alpha) \quad .$$
$\square$

With above construction, we have following useful fact.

[Fact] G is ab. grp. and X is a CW complex. There is a natural bijection
$$T: [X, K(G,n)] \longrightarrow H^n(X,G)$$
$$[f] \longmapsto f^*(\alpha)$$

[Exp] (Classification of real line bdls). Let X be any CW complex.

$\{\text{Real line bdls over } X\} = B(X, GL(1,\mathbb{R}), \underset{\dim\text{-}1}{V}, \rho) = P(X, GL(1,\mathbb{R})) = P(X, \mathbb{Z}_2)$

$= [X, B\mathbb{Z}_2] \underset{\uparrow}{=} [X, \mathbb{RP}^\infty] \underset{\uparrow}{=} [X, K(\mathbb{Z}_2, 1)] \underset{\uparrow}{\cong} H^1(X;\mathbb{Z}_2)$

$\mathbb{Z}_2$-bdl: $\mathbb{Z}_2 \to \underset{\text{contractible}}{\underset{\sim}{S^\infty}} \to \mathbb{RP}^\infty$ $\qquad \mathbb{RP}^\infty = K(\mathbb{Z}_2, 1)$ $\qquad$ By above fact

Given a bdl π in $P(X, \mathbb{Z}_2)$, say it corresponds to a classifying map $f_\pi: X \to \mathbb{RP}^\infty$.

$$\begin{tikzcd} E \arrow[d, "\pi"] & E\mathbb{Z}_2 \arrow[d, "\pi_{\mathbb{Z}_2}"] \\ X \arrow[r, "f_\pi"'] & B\mathbb{Z}_2 \end{tikzcd}$$

$$[X, K(\mathbb{Z}_2, 1)] \longrightarrow H^1(X;\mathbb{Z}_2)$$
$$f_\pi \longmapsto f_\pi^*(w) =: w_1(\pi)$$

the cohomology class $w_1(\pi)$ is called the Stiefel-Whitney class of π.

Because $P(X,\mathbb{Z}_2) \xrightarrow{\simeq} H^1(X,\mathbb{Z}_2)$ is given by $\pi \mapsto w_1(\pi)$, the real line bdl on X are classified by their first Stiefel-Whitney class.

w is "the inverse of Hurewicz mor", but it's not very clear.

Actually, w is a generator of $H^1(\mathbb{RP}^\infty, \mathbb{Z}_2)$ s.t. $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[w]$.

[Exp] (Classification of complex line bdls)

$B(X, GL(1,\mathbb{C}), V, \rho) = P(X, GL(1,\mathbb{C})) = P(X, S^1)$.

Consider $S^1 \to S^\infty \to \mathbb{CP}^\infty$, since S^∞ contractible, $BS^1 \simeq \mathbb{CP}^\infty$.

Since $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$, we get

$P(X, S^1) = [X, BS^1] = [X, K(\mathbb{Z}, 2)] = H^2(X;\mathbb{Z})$.

Similarly, $H^2(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[c]$ for some generator $c \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$.

We obtain a well-defined $c_1(\pi) := f_\pi^*(c) \in H^2(X, \mathbb{Z})$ called the first Chern class of π. The bijection $P(X, S^1) \xrightarrow{\cong} H^2(X, \mathbb{Z})$ is given by $\pi \mapsto c_1(\pi)$. So the complex line bdls are classified by the first Chern class.

If X is orientable closed oriented surface, $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$.

If X is non-orientable closed surface, $H^2(X, \mathbb{Z}) \cong \mathbb{Z}_2$, so there are only two iso classes of complex line bdls on such a surface.

[Rmk] Note that we use $GL(1, \mathbb{R}) = \mathbb{R}^* \simeq \mathbb{Z}_2$, $GL(1, \mathbb{C}) = \mathbb{C}^* \simeq S^1$ above.

Finally, we come back to the proof of thm☆.

[Lemma] Let X be a CW complex and $\pi: E \longrightarrow X \in B(X, G, F, \rho)$ with $\pi_i(F) = 0$ for all $i \geq 0$. Assume $A \subseteq X$ is a subcomplex, then every section of π over A extends to a section over X. In particular, π has a section. Moreover, any two sections of π are homotopic.

Pf: △ Let $\sigma_0 : A \to E$ be a section of $\pi|_A$. We extend σ_0 to a section over X by induction on dim of cells in $X - A$. Assume $X = A \cup_\phi e^n$ where e^n is an n-cell with attaching map $\phi: S^{n-1} \to A$. Since e^n is contractible, $\pi|_{e^n}$ is trivial. Let $h: \pi^{-1}(e^n) \to e^n \times F$ be trivialization. Consider following commutative diagram:

$$\begin{tikzcd}
 & \pi^{-1}(e^n) \arrow[r, "h", "\sim"'] \arrow[d, "\pi"] & e^n \times F \arrow[dl, "p_1"] \\
\partial e^n \arrow[ur, "\sigma_0"] \arrow[r, hook] & e^n &
\end{tikzcd}$$

We can write $\sigma_0(x) = (x, \tau_0(x))$ for some $\tau_0 : \partial e^n \cong S^{n-1} \longrightarrow F$.

Since $\pi_{n-1}(F) = 0$, τ_0 can extend to $\tau: e^n \to F$.

Hence σ_0 can extend to $\sigma : A \cup_\phi e_n \to E$ where $\sigma|_{e_n}(x) = (x, \tau(x))$.

△ Pick A be a trivialization open of X, A admits a section and this section can extend to global section. Therefore, π has a section.

△ Let σ and σ' be two sections of π.

To show σ homotopic to σ', it's equivalent to construct a section of $\pi \times id_I : E \times I \longrightarrow X \times I$. (Section can be written as $(\sigma_t(x), t)$, so σ_t is homotopy of σ and σ') σ is a section of $\pi \times id_I$ over $X \times \{0\}$. σ' is a section of $\pi \times id_I$ over $X \times \{1\}$. By the discussion in the first part, we can obtain a section of $\pi \times id_I$ by given σ and σ'.

[Rmk] $\pi_i(F) = 0$ for all i, then section exists. It's useful.

[Thm#] If X is a CW-complex, there exists a bijection

$$\Phi : [X, BG] \longrightarrow P(X, G)$$
$$f \longmapsto f^*\pi_G$$

$$\begin{tikzcd} f^*\pi_G \arrow[r] \arrow[d, "f^*\pi_G"'] \arrow[dr, phantom, "\circlearrowleft"] & EG \arrow[d, "\pi_G"] \\ X \arrow[r, "f"'] & BG \end{tikzcd}$$

Pf: Δ Show Φ is surj. Given any $\pi \in P(X, G)$, we want to find $f \in [X, BG]$ s.t. $\pi = f^*\pi_G$. Equivalently, we want to find a bdl map $(f, \hat{f})$. Equivalently, we want to show the existence

$$\left(\begin{tikzcd} E \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] & EG \arrow[d, "\pi_G"] \\ X \arrow[r, "f"'] & BG \end{tikzcd}\right)$$

of a section of $\varphi : E \times_G EG \to X$. φ is a bdl with contractible fiber EG, i.e., $\pi_i(EG) = 0$ for all $i \geq 0$. Hence $E \times_G EG \to X$ admits a section.

Δ Show Φ is inj. Assume $f^*\pi_G \cong g^*\pi_G \cong \pi$, we want to show $f \simeq g$. We have commutative diagram ①, ②. Put ①, ② together, we have ③

$$\begin{tikzcd} E \cong f^*EG \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "①"] & EG \arrow[d, "\pi_G"] \\ X \arrow[r, "f"'] & BG \end{tikzcd} \qquad \begin{tikzcd} E \cong g^*EG \arrow[r, "\hat{g}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "②"] & EG \arrow[d, "\pi_G"] \\ X \arrow[r, "g"'] & BG \end{tikzcd}$$

$$\begin{tikzcd} E \times I \arrow[d, "\pi \times id"'] & E \times \{0,1\} \arrow[l, hook'] \arrow[r, "\hat{\alpha} = (\hat{f}, 0) \cup (\hat{g}, 1)"] \arrow[d, "\pi \times \{0,1\}"] & EG \arrow[d, "\pi_G"] \\ X \times I & X \times \{0,1\} \arrow[l, hook'] \arrow[r, "\alpha = (f, 0) \cup (g, 1)"'] & BG \end{tikzcd}$$

③

To show $f \sim g$, it suffices to show $(\alpha, \hat{\alpha})$ can extend to $(H, \hat{H}) : \pi \times id \to \pi_G$ and then $H : X \times I \to BG$ is the homotopy between f and g. The existence of the bdl map is equivalent to the existence of a section of bdl $(E \times I) \times_G EG \to X \times I$. Since $(\alpha, \hat{\alpha})$ corresponds to a section $(E \times \{0,1\}) \times_G EG \to X \times \{0,1\}$, it can extend to global section, which corresponds to $(H, \hat{H})$. $\square$

Chern classes of complex vector bdls

[Prop] $H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \cdots, c_n]$ with $\deg c_i = 2i$.

[Def] The generators $c_1, \cdots, c_n$ of $H^*(BU(n); \mathbb{Z})$ are called the universal Chern classes of $U(n)$-bdls.

[Def] The i-th Chern class of the $U(n)$-bdl $\pi : E \to X$ with classifying map $f_\pi : X \to BU(n)$ is defined as $c_i(\pi) := f_\pi^*(c_i) \in H^{2i}(X; \mathbb{Z})$

[Rmk] If π is a $U(n)$-bdl, then $c_i(\pi)=0$ if $i>n$. (we only have generators $c_1, c_2, \cdots, c_n$).

The following are important props for Chern classes.

[Prop] Let ε denote the trivial $U(n)$-bdl on X, then $c_i(\varepsilon)=0$, $\forall i>0$.

pf: Trivial bdl ε is classified by const map $Ct: X \to BU(n)$.

$c_i(\varepsilon) = Ct^*(c_i) = 0$, where $Ct^*: H^*(BU(n)) \to H^*(X)$.

[prop] (Functoriality of Chern class) Let $f: Y \to X$ be a conti map, $\pi: E \to X$ be a $U(n)$-bdl, then $c_i(f^*\pi) = f^* c_i(\pi)$, $\forall i$.

pf: Consider the commutative diagram

$$\begin{tikzcd} f^*E \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r, "\hat{f}_\pi"] \arrow[d, "\pi"] \arrow[dr, phantom, "\circlearrowleft"] & EU(n) \arrow[d] \\ Y \arrow[r, "f"'] & X \arrow[r, "f_\pi"'] & BU(n) \end{tikzcd}$$

The composition of bdl maps $\hat{f}$ and $\hat{f}_\pi$, $\hat{f}_\pi \hat{f}$ still a bdl map, hence $f^*\pi$ is classified by $f_\pi f$.

$c_i(f^*\pi) = (f_\pi f)^* c_i = f^*(f_\pi^* c_i) = f^*(c_i(\pi))$

[Def] $c(\pi) = c_0(\pi) + c_1(\pi) + \cdots + c_n(\pi) = 1 + c_1(\pi) + \cdots + c_n(\pi) \in H^*(X;\mathbb{Z})$ is called the total Chern class of a $U(n)$-bdl.

[Def] (Whitney sum) Let $\pi_1 \in P(X, U(n))$, $\pi_2 \in P(X, U(m))$, we have product bdl $U(n)\times U(m) \to E_1 \times E_2 \xrightarrow{\pi_1\times\pi_2} X \times X$, i.e., $\pi_1 \times \pi_2 \in P(X\times X, U(n)\times U(m))$. By regarding $U(n)\times U(m)$ as $U(n+m)$ with inclusion $U(n)\times U(m) \to U(n+m)$, $(A,B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Define Whitney sum of π_1 and π_2 is defined as

$$\pi_1 \oplus \pi_2 := \underbrace{\Delta^*}_{\text{pullback}}(\pi_1 \times \pi_2),$$

where $\Delta: X \to X\times X$ is diagonal map.

$$\left(\begin{tikzcd} \Delta^*(\pi_1\times\pi_2) \arrow[r] \arrow[d] & E_1 \times E_2 \arrow[d, "\pi_1\times\pi_2"] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}\right)$$

[Rmk] $\pi_1 \oplus \pi_2$ is the $U(n+m)$-bdl with transition maps $\left(\begin{array}{c|c} g^{\pi_1}_{\alpha\beta} & \\ \hline & g^{\pi_2}_{\alpha\beta} \end{array}\right)$.

[prop] $B(G\times H) \cong BG \times BH$.

pf: We can take product of universal bdl and obtain the following bdl

$$G\times H \to EG\times EH \to BG \times BH$$

Since $\pi_i(EG\times EH) = \pi_i(EG)\times \pi_i(EH) = 0$, this is the universal bdl for $G\times H$, and thus $B(G\times H)\cong BG\times BH$.

[prop] Let $\pi_1 \in \mathcal{P}(X, U(n))$, $\pi_2 \in \mathcal{P}(X, U(m))$, then $c(\pi_1\oplus\pi_2) = c(\pi_1)\cup c(\pi_2)$.

Equivalently, $c_k(\pi_1\oplus\pi_2) = \sum_{i+j=k} c_i(\pi_1)\cup c_j(\pi_2)$.

pf: The inclusion $U(n)\times U(m) \to U(n+m)$ induces a map

$$\omega : B(U(n)\times U(m)) \cong BU(n)\times BU(m) \longrightarrow BU(n+m)$$

($X\to Y$ induces $BX \to BY$)

Fact: $\omega^* c_k = \sum_{i+j=k} c_i \times c_j$ (Use künneth formula, see Milnor's book P164)

Then $\underset{\text{def}}{c_k(\pi_1\oplus\pi_2) =} c_k(\underbrace{\Delta^*(\pi_1\times\pi_2)}_{\text{pullback}}) = \Delta^* c_k(\pi_1\times\pi_2) = \underbrace{\Delta^*}_{\text{pullback}} \underbrace{f^*_{\pi_1\times\pi_2}}_{\text{classifying map}}(c_k)$

pullback commutes with c_i

Consider the commut. diagram:

$$\begin{tikzcd}
E_1\times E_2 \arrow[r] \arrow[d, "\pi_1\times\pi_2"'] \arrow[dr, phantom, "①"] & EU(n)\times EU(m) \arrow[r] \arrow[d] \arrow[dr, phantom, "②"] & EU(n+m) \arrow[d] \\
X\times X \arrow[r, "f_{\pi_1}\times f_{\pi_2}"'] & BU(n)\times BU(m) \arrow[r, "\omega"'] & BU(n+m)
\end{tikzcd}$$

① : product

② functoriality

Hence $f^*_{\pi_1\times\pi_2} = \omega\circ(f_{\pi_1}\times f_{\pi_2})$.

Then $c_k(\pi_1\oplus\pi_2) = \Delta^* f^*_{\pi_1\times\pi_2}(c_k) = \Delta^*(f_{\pi_1}\times f_{\pi_2})^*\omega^*(c_k)$

$= \Delta^*(f^*_{\pi_1}\times f^*_{\pi_2})\sum_{i+j=k} c_i\times c_j = \sum_{i+j=k}\Delta^* f^*_{\pi_1}(c_i)\times f^*_{\pi_2}(c_j) = \sum_{i+j=k}\Delta^*(c_i(\pi_1)\times c_j(\pi_2))$

use fact $\omega^* c_k = \sum_{i+j=k} c_i\times c_j$

$= \bigcup_{i+j=k} c_i(\pi_1)\smile c_j(\pi_2)$

cup product

$\Delta^*: H^*(X\times X) \longrightarrow H^*(X)$

$c_i(\pi_1)\times c_j(\pi_2) \mapsto (c_i(\pi_1)\times c_j(\pi_2))\circ\Delta$

$= c_i(\pi_1)\smile c_j(\pi_2)$

cup product

□

[Coro] Let ε' be trivial $U(1)$-bdl. Then $c(\pi \oplus \varepsilon') = c(\pi)$.

Pf: $c(\pi \oplus \varepsilon') = c(\pi) \cup c(\varepsilon') = c(\pi)$ $\square$

Stiefel-Whitney classes of real vector bdls

[Prop] $H^*(BO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots, w_n]$ with $\deg w_i = i$.

[Def] The generators $w_1, \cdots, w_n$ are called the universal Stiefel-Whitney classes of $O(n)$-bdls.

[Def] The i-th Stiefel-Whitney class of the $O(n)$-bdl $\pi: E \to X$ with classifying map $f_\pi: X \to BO(n)$ is defined as $w_i(\pi) := f_\pi^*(w_i) \in H^i(X; \mathbb{Z}_2)$

Define the total Stiefel-Whitney class of π by $w(\pi) = 1 + w_1(\pi) + \cdots + w_n(\pi) \in H^*(X; \mathbb{Z}_2)$

[Rmk] (1) If π is a $O(n)$-bdl, then $w_i(\pi) = 0, \forall i > n$

(2) $w_i(\varepsilon) = 0, \forall i > 0$ for ε trivial bdl.

$\left.\begin{array}{l} (3)\ w_i(f^*\pi) = f^* w_i(\pi) \\ (4)\ w(\pi_1 \oplus \pi_2) = w(\pi_1) \cup w(\pi_2) \end{array}\right\}$ Similar as Chern class

Stiefel-Whitney classes of manifolds and applications

Note that when M smooth, its tangent bdl TM is an $O(n)$-bdl. That means we can use Stiefel-Whitney class.

[Def] The Stiefel-Whitney classes of a sm mf M are defined as

$$w_i(M) := w_i(TM).$$

[Thm] Stiefel-Whitney classes are homotopy invariants, i.e., $h: M_1 \to M_2$ is a homotopy equivalence, then $h^* w_i(M_2) = w_i(M_1)$, for $i \geq 0$

($h^*: H^i(M_2; \mathbb{Z}_2) \to H^i(M_1; \mathbb{Z}_2)$ is an iso)

One may think Stiefel-Whitney classes can only solve questions of classification, but it's useful for many problems. This section will state some of applications.

<Embedding problem> Given n-dim sm mf M, find minimal integer k s.t. M can be embedded/immersed in $\mathbb{R}^{n+k}$?

dimension

Let $f: M^{m} \to N^{m+k}$ be an embedding of sm mf. Then $f^*TN = TM \oplus \nu$

pullback

$$\begin{tikzcd} f^*TN \arrow[r] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Let ν be the normal bdl of M in N. ← "normal complete bdl"

Since $\operatorname{rank} \nu = k$, $w_i(\nu) = 0$ for $i > k$.

$w(f^*TN) = w(TM \oplus \nu) = w(TM) \smile w(\nu) = w(M) \smile w(\nu)$

$f^* w(TN) =: f^* w(N)$ (Recall $w_i(\pi) = f_\pi^*(w_i)$. w_i is generator so w_i^{-1} exists. $f_\pi^*(w_i^{-1}) = f_\pi^*(w_i)^{-1}$.)

Since $w(M) = 1 + w_1(M) + \dots + w_m(M)$ is invertible in $H^*(M;\mathbb{Z}_2)$, we have $w(\nu) = w(M)^{-1} \smile f^* w(N)$. When $N = \mathbb{R}^{n+k}$, TN is trivial since N contractible. Hence $w(N) = 0$ and thus $f^* w(N) = 0$. Therefore $w(\nu) = w(M)^{-1}$.

What if when $f: M^m \to N^{m+k}$ is only an immersion?

[Lemma] Let $i: E_1 \to E_2$ be linear monomorphism of vector bdls E_1, E_2 over X, i.e., locally i is given by $U \times \mathbb{R}^n \to U \times \mathbb{R}^m$ $(n \le m)$, $(u, v) \mapsto (u, l(u)v)$ where $l(u)$ is a linear map of rank n for $\forall u \in U$. Then there exists a vector bdl $\pi_1^\perp : E_1^\perp \to X$ s.t. $\pi_2 \cong \pi_1 \oplus \pi_1^\perp$. $\square$

By this lemma, normal space ν exists and we can apply same consideration and obtain $w(\nu) = w(M)^{-1} \smile f^* w(N)$.

The following we'll consider a special case — embedding (immersing) $\mathbb{RP}^m$ into $\mathbb{R}^{m+k}$. The main idea is, solve $w(\nu)$ by ① $w(\nu) = w(M)^{-1} \smile f^* w(N)$ ② $w_i(\nu) = 0$ for all $i > k$.

[Fact] $w(\mathbb{RP}^m) = (1+x)^{m+1}$, where $x \in H^1(\mathbb{RP}^m;\mathbb{Z}_2)$ is a generator.

[Exp] $\mathbb{RP}^9 \to \mathbb{R}^{9+k}$. $w(\mathbb{RP}^9) = (1+x)^{10} = (1+x)^8 (1+x)^2 = (1+x^8)(1+x^2)$

$(1+x)^p = 1 + x^p \pmod p$

By induction $(1+x)^{p^m} = 1 + x^{p^m} \pmod p$. In our case, $p=2$

$= (1+x^8)(1+x^2) = 1 + x^2 + x^8$ (Note that $x^{10} = 0$ in $H^*(\mathbb{RP}^9;\mathbb{Z}_2) = \mathbb{Z}_2[d]/d^{10}$, $|d|=1$)

$w(\nu) = w(\mathbb{RP}^9)^{-1} = 1 + x^2 + x^4 + x^6$ $((1+x^2+x^8)\cdot(1+x^2+x^4+x^6) = 1 + 2(x^2+x^4+x^6+x^8) = 1)$

Since $x^6 \in H^6(\mathbb{RP}^9;\mathbb{Z}_2)$, $w_6(\nu) \neq 0$. Recall we must have $w_i(\nu) = 0$ for $i > k$, so $k \geq 6$. Hence $\mathbb{RP}^9$ can be embedded to $\mathbb{R}^{9+k}$, $k \geq 6$, i.e., $\mathbb{RP}^9$ cannot be embedded to $\mathbb{R}^{14}$.

[Exp] Consider $\mathbb{RP}^{2^r} \to \mathbb{R}^{2^r+k}$.

$w(\mathbb{RP}^{2^r}) = (1+x)^{2^r+1} = (1+x)^{2^r}(1+x) = (1+x^{2^r})(1+x) \overset{x^{2^r+1}=0}{=} 1 + x + x^{2^r}$

For embedding (or immersion) $\mathbb{RP}^{2^r} \hookrightarrow \mathbb{R}^{2^r+k}$ with normal bdl ν, $w(\nu) = w(\mathbb{RP}^{2^r})^{-1} = 1 + x + x^2 + \dots + x^{2^r-1}$. So $k \geq 2^r - 1$. e.g., $\mathbb{RP}^8 \to \mathbb{R}^{8+k}$, $k \geq 7$. So $\mathbb{RP}^8$ can not embedded in $\mathbb{R}^{14}$. e.g., $\mathbb{RP}^2 \to \mathbb{RP}^{2+k}$, $k \geq 1$. So $\mathbb{RP}^2$ can not embedded in $\mathbb{R}^2$. Actually, one can immersed to $\mathbb{RP}^{2^{r}-1}$.

[Fact] (Whitney) An m-dim sm mf can be embedded in $\mathbb{R}^{2m}$ and immersed in $\mathbb{R}^{2m-1}$.

[Summary] One can compute $w(v)$ and compute $\mathbb{R}P^m \to \mathbb{R}P^{m+k}$ the k should at least be what. But one can not easily prove when $k=?$ it's a immersion (or embedding).

[Def] A sm mf M is called parallelizable if TM is trivial.

[Exp] All Lie grp is parallelizable.

[Thm] $w(\mathbb{R}P^m)=1$ iff $m+1=2^r$ for some r. In particular, if $\mathbb{R}P^m$ is parallelizable, $m+1=2^r$ for some r.

Pf: $\Leftarrow$ Assume $m+1=2^r$. $w(\mathbb{R}P^m)=(1+x)^{m+1}=(1+x)^{2^r}=1+x^{2^r}$

$=1+x^{m+1}=1$ $(x^{m+1}=0)$

$\Rightarrow$ Assume $m+1\neq 2^r$, w.t.s. $w(\mathbb{R}P^m)\neq 1$. Trick: let $m+1=2^r k$, $k>1$ odd

$w(\mathbb{R}P^m)=((1+x)^{2^r})^k=(1+x^{2^r})^k=1+kx^{2^r}+\cdots$

k odd, so $k\neq 0$. $x^{2^r}<x^{m+1}=x^{2^r k}$, $x^{2^r}\neq 0$. Hence $w(\mathbb{R}P^m)\neq 1$.

In particular, $\mathbb{R}P^m$ is parallelizable. Then $T(\mathbb{R}P^m)$ is trivial and thus $w(\mathbb{R}P^m)=1$. Hence $m+1=2^r$ for some r. □

More powerful result is:

[Fact] (Adams) $\mathbb{R}P^m$ is parallelizable iff $m\in\{1,3,7\}$

[Rmk] X is parallelizable $\underset{\not\Leftarrow}{\Rightarrow}$ $w(X)=1$

<Boundary problem> Given an n-dim sm mf M, is there an $(n+1)$-dim sm mf W s.t. $\partial W=M$?

Let M be a closed mf M^n and $\mu_M\in H_n(M;\mathbb{Z}_2)$ be the fundamental class.

[Def] (Stiefel-Whitney number) Let $\alpha=(\alpha_1,\dots,\alpha_n)$ be a tuple of non-negative integers s.t. $\sum_{i=1}^n i\alpha_i=n$. Define:

$$w^{[\alpha]}(M):=\bigcup_{i=1}^n w_i(M)^{\alpha_i}=w_1(M)^{\alpha_1}\cup w_2(M)^{\alpha_2}\cup\cdots\cup w_n(M)^{\alpha_n}\in H^n(M;\mathbb{Z}_2)$$

Define the Stefel-Whitney number of M with index α as

$w_{[\alpha]}(M):=\langle w^{[\alpha]}(M),\mu_M\rangle\in\mathbb{Z}_2$ where

$\langle -,-\rangle: H^n(M;\mathbb{Z}_2)\times H_n(M;\mathbb{Z}_2)\longrightarrow\mathbb{Z}_2$ is the kronecker evaluation pairing.

(cochains can act on chains)

[Thm] (Pontrjagin-Thom) A closed n-dim mf M is the boundary of a sm compact $(n+1)$-dim mf W iff all Stiefel-Whitney numbers of M vanish, i.e., $w_{(\alpha)}(M)=0$ for any $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\sum_{i=1}^{n} i\alpha_i = n$.

Pf: We only show one direction. Assume $M=\partial W$, $i: M\hookrightarrow W$ be boundary embedding. Then

$$i^*TW \cong TM \oplus \nu' ,$$ where ν' is the normal bdl (of rank 1).

Assume TW has Euclidean metric, then ν' can trivialized to trivial bdl ε^1 by picking inward unit normal vector. So $i^*TW \cong TM \oplus \varepsilon^1$ and then $w(i^*TW) = w(TM)$, i.e., $i^*w(W) = w(M)$. In particular, we have $i^*w_k(W) = w_k(M)$.

Hence $i^* w^{[\alpha]}(W) = \mathop{\smile}\limits_{i=1}^{n} i^*w_i(W)^{\alpha_i} = \mathop{\smile}\limits_{i=1}^{n} w_i(M)^{\alpha_i} = w^{[\alpha]}(M)$.

By alg topo $\partial(\mu_W)=\mu_M$ where μ_W is the fundamental class of (W,M) and μ_M be fundamental class of M. Let $\delta: H^n(M;\mathbb{Z}_2)\to H^{n+1}(W,M;\mathbb{Z}_2)$ be map adjoint to ∂, i.e. $\langle x, \partial y\rangle = \langle \delta x, y\rangle$.

$$w_{(\alpha)}(M) = \langle w^{[\alpha]}M, \mu_M\rangle = \langle i^* w^{[\alpha]}W, \partial\mu_W\rangle$$

$$= \langle \delta i^* w^{[\alpha]}W, \mu_W\rangle = \langle 0, \mu_W\rangle = 0$$

$\delta i^* = 0$ by L.E.S. of (W,M) :

$$\cdots \to H_i(W) \xrightarrow{i^*} H_i(W,M) \xrightarrow{\delta} H_{i+1}(M) \to \cdots$$

[Exp] Consider $X\sqcup X$, where X is a closed n-dim mf. For any α,

$w_{(\alpha)}(X\sqcup X) = 2w_{(\alpha)}(X) = 0$ $\mathbb{Z}_2$-coefficients. Hence $X\sqcup X$ must be boundary of some mf. Actually $X\sqcup X = \partial(X\times[0,1])$.

$$w_{(\alpha)}(X\sqcup X) = \langle w^{[\alpha]}(X\sqcup X), \mu_{X\sqcup X}\rangle = \langle \mathop{\smile}\limits_i w_i(X\sqcup X)^{\alpha_i}, \mu_{X\sqcup X}\rangle$$

$$= 2\langle \mathop{\smile}\limits_i w_i(X)^{\alpha_i}, \mu_X\rangle$$

[Exp] Every $\mathbb{RP}^{2k-1}$ is a boundary. $w(\mathbb{RP}^{2k-1}) = (1+x)^{2k} = (1+x^2)^k$

$= \binom{k}{0} + \binom{k}{1}x^2 + \binom{k}{2}x^4 + \cdots + \binom{k}{k}x^{2k}$. Hence $w_i(\mathbb{RP}^{2k-1}) = 0$ when i is odd. Let α be any triple with $\sum i\alpha_i = 2k-1$. If all i is even, $\sum i\alpha_i$ is even, leading to a contradiction. Hence, there exists odd i in $\alpha = (\alpha_i)$. So $w^{[\alpha]}(\mathbb{RP}^{2k-1}) = \mathop{\smile}\limits_{i=1}^{2k-1} w_i(\mathbb{RP}^{2k-1})^{\alpha_i} = 0$, and then

$$w_{(\alpha)}(\mathbb{RP}^{2k-1}) = \langle w^{[\alpha]}(\mathbb{RP}^{2k-1}), \mu_{\mathbb{RP}^{2k-1}}\rangle = \langle 0, \mu_{\mathbb{RP}^{2k-1}}\rangle = 0$$

□

[Exp] Every $\mathbb{RP}^{2k}$ is not a boundary.

$$w(\mathbb{RP}^{2k}) = (1+x)^{2k+1} = \binom{2k+1}{0} + \binom{2k+1}{1}x + \binom{2k+1}{2}x^2 + \cdots + \binom{2k+1}{2k}x^{2k} + \binom{2k+1}{2k+1}\underset{\overset{\parallel}{0}}{x^{2k+1}}$$

In particular, $w_{2k}(\mathbb{RP}^{2k}) = x^{2k}$. For $\alpha = (0,0,\dots,1)$, we have:

$$w_{(\alpha)}(\mathbb{RP}^{2k}) = \underset{i}{\smile} w_i(\mathbb{RP}^{2k})^{\alpha_i} = w_{2k}(\mathbb{RP}^{2k}) \neq 0.$$

<Pontrjagin classes>

In this section, we denote π be real vector bdls (or $O(n)$-bdl) and ω be complex vect bdls (or $U(n)$-bdls) over a topo space X.

[Construction] (Complexification, realization, conjugation)

Δ π is a real vect bdl. Denote its transition maps $g_{\alpha\beta}: U_\alpha \cap U_\beta \to O(n)$.
Define its complexification, denoted by $\pi\otimes\mathbb{C}$, be a bdl with transition maps $g'_{\alpha\beta}: U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} O(n) \hookrightarrow U(n)$. Hence, $\pi\otimes\mathbb{C}$ is a complex vect bdl. The fiber of $\pi\otimes\mathbb{C}$ is $\mathbb{R}^n\otimes\mathbb{C} \simeq \mathbb{C}^n$.
($U(n)$ acts on $\mathbb{C}^n$)

Δ ω is a complex vect bdl. Denote its transition maps $g_{\alpha\beta}: U_\alpha \cap U_\beta \to U(n)$.
* Define its realization, denoted by $\omega_{\mathbb{R}}$, be a bdl with transition maps $g'_{\alpha\beta}: U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} U(n) \longrightarrow O(2n)$
$\omega_{\mathbb{R}}$ is a real vect bdl with fiber $\mathbb{R}^{2n}$. ($O(2n)$ acts on $\mathbb{R}^{2n}$)
* Define its conjugation $\bar{\omega}$ be a bdl with transition maps

$$\overline{g_{\alpha\beta}}: U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} U(n) \xrightarrow{\text{conjugate}} U(n)$$

$\bar{\omega}$ is a complex bdl with same fiber F as ω, but conjugate action, i.e.

$$\rho_{\bar{\omega}}: U(n) \times F \longrightarrow F$$
$$((a+ib), f) \mapsto \rho_\omega(a-ib, f).$$

[Rmk] What's relationship between ω and $\bar{\omega}$?
$F \to \omega \to X$, $F \to \bar{\omega} \to X$. They have opposite complex str on its fibers). They have same underlying real vect space. Indeed,

$U(n) \hookrightarrow O(2n)$ is given by $A+iB \mapsto \begin{bmatrix} A & \\ & B \end{bmatrix}$, since

$$\underset{\mathbb{C}\text{-basis}}{(A+iB)[\alpha]} = A\alpha + iB\alpha = \begin{bmatrix} A & \\ & B \end{bmatrix}\underset{\mathbb{R}\text{-basis}}{\begin{bmatrix} \alpha \\ i\alpha \end{bmatrix}}.$$

So we have:

$$\overline{\omega}: U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} U(n) \xrightarrow{\text{conjugate}} U(n) \hookrightarrow O(2n)$$
$$A+iB \longmapsto A-iB \longmapsto \begin{bmatrix} A & \\ & -B \end{bmatrix}$$
, and

$$\omega: U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} U(n) \longrightarrow O(2n)$$
$$A+iB \longmapsto \begin{bmatrix} A & \\ & B \end{bmatrix}$$

For $\overline{\omega}$, $\rho_{\overline{\omega}}: U(n) \times F \longrightarrow F$
$$(A+iB, f) \longmapsto \rho_{\omega}(A-iB, f)$$

View it as $\underset{\sim}{O(2n)}$ act on $\begin{bmatrix} d \\ id \end{bmatrix}$
$$U(n) \to O(2n)$$
$$A+iB \mapsto \begin{bmatrix} A & \\ & -B \end{bmatrix}$$

$$\rho_{\overline{\omega}}: O(2n) \times F \longrightarrow F$$
$$\left(\begin{bmatrix} A & \\ & -B \end{bmatrix}, f\right) \longmapsto \rho_{\omega}\left(\begin{bmatrix} A & \\ & B \end{bmatrix}, f\right)$$

For ω, $\rho_\omega: U(n) \times F \longrightarrow F$
$$(A+iB, f) \mapsto \rho_\omega(A+iB, f)$$

View it as $\underset{\sim}{O(2n)}$ act on $\begin{bmatrix} d \\ id \end{bmatrix}$
$$U(n) \to O(2n)$$
$$A+iB \mapsto \begin{bmatrix} A & \\ & B \end{bmatrix}$$

$$\rho_\omega: O(2n) \times F \longrightarrow F$$
$$\left(\begin{bmatrix} A & \\ & B \end{bmatrix}, f\right) \mapsto \rho_\omega\left(\begin{bmatrix} A & \\ & B \end{bmatrix}, f\right)$$

same element $A+iB$ act on same f and then obtain $\rho_\omega\left(\begin{bmatrix} A & \\ & B \end{bmatrix}, f\right)$, same result.

[Prop] Let ω be complex vect bdl, then $\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}$

[prop] $\overline{\pi \otimes \mathbb{C}} \cong \pi \otimes \mathbb{C}$

Pf: transition functions of $\pi \otimes \mathbb{C}$ are given by $g_{\alpha\beta}: U_\alpha \cap U_\beta \to O(n) \to U(n)$

so $g_{\alpha\beta}$ are real-valued. So conjugation has nothing impact. $\square$

[prop] Let ω be a rank n complex vect bdl. The Chern class of $\overline{\omega}$ are: $c_k(\overline{\omega}) = (-1)^k c_k(\omega)$, for $k = 1, 2, \cdots, n$.

[Coro] Let π be real vect bdl. $c_k(\pi \otimes \mathbb{C}) = (-1)^k c_k(\pi \otimes \mathbb{C})$.

Pf $c_k(\pi \otimes \mathbb{C}) = c_k(\overline{\pi \otimes \mathbb{C}}) = (-1)^k c_k(\pi \otimes \mathbb{C})$

[Rmk] For any odd k, $C_k(\pi\otimes\mathbb{C}) = -C_k(\pi\otimes\mathbb{C})$, meaning that $C_k(\pi\otimes\mathbb{C}) \in H^{2k}(X;\mathbb{Z})$ has order 2.

[Def] Let $\pi: E \to X$ be $\mathbb{R}$-vec bdl of rank n. The i-th Pontryagin class of π is defined as:

$$P_i(\pi) := (-1)^i C_{2i}(\pi\otimes\mathbb{C}) \in H^{4i}(X;\mathbb{Z})$$

Let ω be a $\mathbb{C}$-vec bdl of rank n. The i-th Pontryagin class of ω is defined as:

$$P_i(\omega) := P_i(\omega_{\mathbb{R}}) = (-1)^i C_{2i}(\omega\oplus\overline{\omega}) \qquad (\omega_{\mathbb{R}}\otimes\mathbb{C} = \omega\oplus\overline{\omega})$$

[Rmk] $P_i(\pi)=0$ for all $i > \frac{n}{2}$ $\quad$ ($C_i(\pi\otimes\mathbb{C})=0$ for all $i>n$)

[Def] Let π be $\mathbb{R}$-vec bdl on X. Define its total Pontryagin class as $p(\pi) = P_0 + P_1 + \cdots \in H^*(X;\mathbb{Z})$ $\quad$ ($P_i(\pi)=0$ for $i>\frac{n}{2}$)

[Thm] Let π_1, π_2 be $\mathbb{R}$-vec bdls on X. Then,

$$p(\pi_1\oplus\pi_2) = p(\pi_1)\smile p(\pi_2) \quad (\text{mod } 2\text{-torsion})$$

pf: $P_i(\pi_1\oplus\pi_2) = (-1)^i C_{2i}((\pi_1\oplus\pi_2)\otimes\mathbb{C})$

$$= (-1)^i C_{2i}((\pi_1\otimes\mathbb{C})\oplus(\pi_2\otimes\mathbb{C}))$$

$$= (-1)^i \sum_{k+l=2i} C_k(\pi_1\otimes\mathbb{C})\smile C_l(\pi_2\otimes\mathbb{C})$$

Note that when k is odd, $C_k(\pi_i\otimes\mathbb{C})$ has order two. So only even-th Chern class survives.

$$P_i(\pi_1\oplus\pi_2) = (-1)^i \sum_{a+b=i} C_{2a}(\pi_1\otimes\mathbb{C})\smile C_{2b}(\pi_2\otimes\mathbb{C}) + \{\text{elements of order } 2\}$$

$$= \sum_{a+b=i}\left((-1)^a C_{2a}(\pi_1\otimes\mathbb{C})\right)\smile\left((-1)^b C_{2b}(\pi_2\otimes\mathbb{C})\right) + \{\text{elements of order } 2\}$$

$$= \sum_{a+b=i} P_a(\pi_1)\smile P_b(\pi_2) + \{\text{elements of order } 2\}$$

So $p(\pi_1\oplus\pi_2) = p(\pi_1)\smile p(\pi_2)$

[Def] Let M be a real sm mf. Define $p(M) := p(TM)$.
Let N be a complex sm mf. Define $p(N) := p((TN)_{\mathbb{R}})$

[Thm] $c(\mathbb{C}P^n) = (1+c)^{n+1}$, $p(\mathbb{C}P^n) = (1+c^2)^{n+1}$ where $c \in H^2(\mathbb{C}P^n; \mathbb{Z})$ is a generator.

Consider $\mathbb{C}P^n \hookrightarrow \mathbb{R}^{2n+k}$. We want to find constrain of k. With embedding we have $(T\mathbb{C}P^n)_{\mathbb{R}} \oplus \nu^k \cong i^* T\mathbb{R}^{2n+k} = T\mathbb{R}^{2n+k}|_{\mathbb{C}P^n}$ which is a trivial bdl. Apply Pontryagin class to it and obtain $p((T\mathbb{C}P^n)_{\mathbb{R}}) \smile p(\nu^k) = 1$. So $p(\nu^k) = p(T\mathbb{C}P^n_{\mathbb{R}})^{-1}$. If $p_i(\nu^k) \neq 0$, we have $i \leq \frac{k}{2}$.

[Exp] Consider $\mathbb{C}P^2 \hookrightarrow \mathbb{R}^{4+k}$

$p(\mathbb{C}P^2) = (1+c^2)^3 = 1 + 3c^2$ since $c^3 \in H^6(\mathbb{C}P^2; \mathbb{Z}) = 0$

$p(\nu^k) = p(\mathbb{C}P^2)^{-1} = 1 - 3c^2$.

Hence, $p_1(\nu^k) \neq 0$, meaning that $k \geq 2$. So $\mathbb{C}P^2 \hookrightarrow \mathbb{R}^5$ is impossible.

Oriented cobordism and Pontryagin numbers

Let M be sm oriented mf and $-M$ be mf with opposite orientation.

[Def] Let M^n, N^n be sm closed oriented real mfs of dim n. We say M and N are oriented cobordant if there exists a sm compact oriented $(n+1)$-dim mf W^{n+1} s.t. $\partial W = M \sqcup (-N)$.

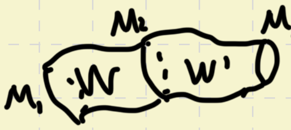
[Def] Oriented cobordism is an equivalence relation.

pf: △ $M \sim M$ since $M \sqcup (-M) = \partial(M \times [0,1])$

△ Assume $M \sim N$, then $M \sqcup (-N) = \partial W$, equivalently, $-M \sqcup N = \partial(-W)$, that is, $N \sim M$.

$= N \sqcup (-M)$

△ Assume $M_1 \sim M_2$, $M_2 \sim M_3$, then $M_1 \sqcup (-M_2) = \partial W$, $M_2 \sqcup (-M_3) = \partial W'$. We can glue W and W' along M_2,



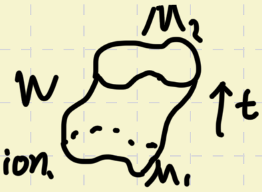
denoted by $W \sqcup_{M_2} W'$. $\partial(W \sqcup_{M_2} W') = M_1 \sqcup (-M_3)$.

Hence $M_1 \sim M_3$. □

equivalence class of cobordism

[Def] Let Ω_n be the set of cobordism classes of closed, oriented, sm n-mf.

[Rmk] "Cobodism classes" has physical meaning. $M_1 \sim M_2$, i.e., $M_1 \sqcup (-M_2) = \partial W$ means M_1 can evolution to M_2.



[Coro] Ω_n is an abelian grp with the disjoint union operation.

pf: △The unit of Ω_n is ϕ. $M \sqcup \phi = M$.

compact oriented

$[\phi] = \{M \in \Omega_n \mid M \sqcup \phi = \partial W\} = \{M \in \Omega_n \mid M = \partial W \text{ for some } W\}$

△ Check $[M]^{-1} = [-M]$.

$M \sqcup (-M) = \partial(M \times [0,1])$, hence $[M] \sqcup [-M] = \underset{\sim}{\overset{[\phi]}{0}}$. □

⟨Pontrjagin numbers⟩

[Def] Let M^n be closed oriented sm real n-mf, with fundamental class $\mu_M \in H_n(M;\mathbb{Z})$. Let $k = \frac{n}{4}$ and pick a partition $\alpha = (\alpha_1, \cdots, \alpha_k)$ fulfilling $\sum_{i=1}^{k} 4i\alpha_i = n$. Define Pontrjagin number of M associated to the partition α as

$$P_{(\alpha)}(M) = \langle P_1(M)^{\alpha_1} \cup P_2(M)^{\alpha_2} \cup \cdots \cup P_k(M)^{\alpha_k}, \mu_M \rangle \in \mathbb{Z}$$

[Rmk] If $\frac{n}{4} \notin \mathbb{Z}$, $P_{(\alpha)}(M) = 0$.

[Fact] $P_i(M) = P_i(-M)$, $\mu_M = -\mu_{-M}$.

[Coro] $P_{[\alpha]}(M) = -P_{[\alpha]}(-M)$

[Fact] If $[M] = 0$, i.e., $M^n = \partial W^{n+1}$, then $P_{(\alpha)}(M) = 0$ for all α.

[Fact] $n = 4k$, $P_{(\alpha)}$ defines homomorphism

$$\Omega_n \to \mathbb{Z}, \quad [M] \mapsto P_{(\alpha)}(M)$$

Oriented cobordant mfs have same pontrjagin numbers.

$\underset{\underset{0}{\|\, M_1 \sim M_2}}{P_{(\alpha)}(M_1 \sqcup (-M_2))} = P_{(\alpha)}(M_1) + P_{(\alpha)}(-M_2) \Rightarrow P_{(\alpha)}(M_1) = -P_{(\alpha)}(-M_2) = P_{(\alpha)}(M_2)$

[Exp] Show $\mathbb{CP}^{2n}$ is not an oriented boundary. $p(\mathbb{CP}^{2n}) = (1+c^2)^{2n+1}$.

$P_i(\mathbb{CP}^{2n}) = \binom{2n+1}{i} c^{2i}$, $i = 0, 1, \cdots, n = \frac{(4n)}{4}$. Let $\alpha = (0, 0, \cdots, 0, 1)$.

$\dim_{\mathbb{R}} \mathbb{CP}^{2n}$

$p_{(\alpha)}(\mathbb{C}P^{2n}) = \langle p_n(\mathbb{C}P^{2n}), \mu_{\mathbb{C}P^{2n}} \rangle = \langle \binom{2n+1}{n} c^{2n}, \mu_{\mathbb{C}P^{2n}} \rangle = \binom{2n+1}{n} \neq 0$

So $\mathbb{C}P^{2n}$ is not a boundary

[Exp] When M reverse orientation, $p_{(\alpha)}(-M) = -p_{(\alpha)}(M)$. Hence, if $p_{(\alpha)}(M) \neq 0$ for some α, M cannot diffeomorphic to orientation reversed mf. Equivalently, if M is not an oriented boundary, then M cannot diffeomorphic to orientation reversed mf. e.g., $\mathbb{C}P^{2n}$ doesn't have orientation-reversing diffeomorphism. But $\mathbb{C}P^{2n+1}$ can.

[Def] (Thom) The oriented cobordism group Ω_n is finitely generated of rank $|I|$, where I is the collection of partitions α satisfying $\sum_j 4j\alpha_j = n$. In fact, Ω_n is generated by products of even complex projective spaces modulo torsion. Moreover,

$$\bigoplus_{\alpha\in I} p_{(\alpha)} : \Omega_n \to \mathbb{Z}^{|I|}$$ is an injective homo onto a subgrp of the same rank.