- Smooth homotopy.

$f, g: X \to M$ be sm maps between sm mf and the homotopy $h: X \times I \to M$ between f and g is also sm. Then h is called sm homotopy.

Facts: 1. sm homotopy is an equivalence relation on the set of all sm maps $X \to M$

2. If sm maps f and g are homotopy, they are smoothly homotopic to each other.

3. Whitney approximation theorem: $f: X \to M$ is a conti map. Then f may be approximated by a sm map homotopic to f.

- Two equivalence def of contracitible

① M is contractible iff $M \simeq pt$, i.e. $M \xrightarrow{f} pt \xrightarrow{g} M$ s.t. $gf \sim id_M$ and $\underbrace{fg \sim id_{pt}}_{\text{(always holds)}}$.

② $\exists$ pt $m \in M$ s.t. constant map $M \to m$ homotopic to id_M

($g: pt \to M$ assign a point in M)

Useful properties: TFAE

(1) M is contracitible

(2) Any $f: M \to X$ is nullhomotopic

(3) Any $f: X \to M$ is nullhomotopic

- Contractible mf can have non-contractible boundary.

example: contractible D^n has non-contractible boundary S^{n-1}.

How does this happen? $D^n = \underbrace{S^{n-1}}_{\text{non-contractible}} \cup \underbrace{D^n \setminus S^{n-1}}_{\text{contractible open ball}}$

S^{n-1} is non-contractible means $id_{S^{n-1}}$ can not homotopic to const map by homotopy only stay in S^{n-1}. But in D^n we have enough rooms. So the inclusion $i: S^{n-1} \to D^n$ can be homotopic to const map by homotopy exploring all D^n.

- Retraction $\neq$ deformation retraction.

A map $r: M \to M$ with $r(M) = X$ and $r|_X = id$ is called retraction. If r is the homotopy inverse to inclusion $i: X \to M$, then X is called a deformation retract of M and r is called a deformation retraction.

properties ① X is a deformation retraction of M, then X homotopic to M. Conversely is not true.

[Exp] $M = \mathbb{R}^3 \setminus 0$. $M \sim S^2$. S^2 enclose 0 is a defor. retra. of M. S^2 that doesn't enclose 0 is not a deform. retraction of M.

[Rmk] deformation is stronger than homotopy equivalence. defor. retra. contains location information to some extent.

② Deformation retraction $\underset{\not\Leftarrow}{\Rightarrow}$ Retraction

[Exp] Let point $p \in S^n$. $S^n \to p$ is a retraction but it's not a deformation retraction.

- Fiber bundles with contractible bundle:

Given a fiber bundle $F \to E \to B$, where E, F, B are sm mfs.
If fiber F is contractible, then total space E homotopic to base space B.

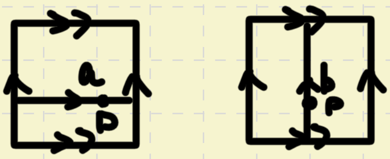
[Exp] Unbounded Mobiüs band and cylinder are homotopic, because they are total spaces with fiber $\mathbb{R}$ over S^1. So they both homotopic to S^1.

[Rmk] Oriented m.f. can homotopic to unoriented m.f.

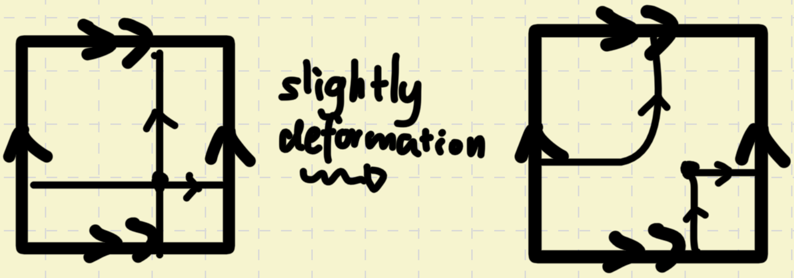
- Importance of base point of the loop in fundamental group.

Loop homotopy require fix base point. Why? Because if we do not impose any special conditions to the base point, we may loose topological information.

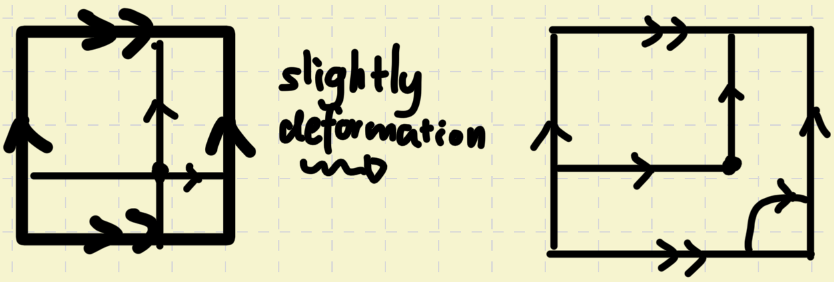
For a torus,



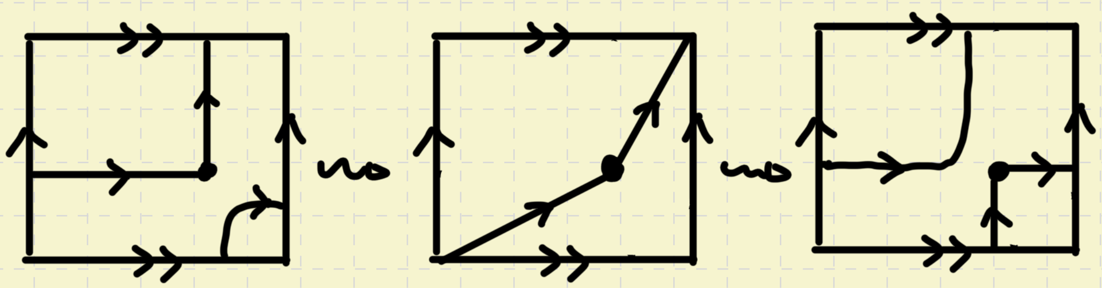
[a·b]=



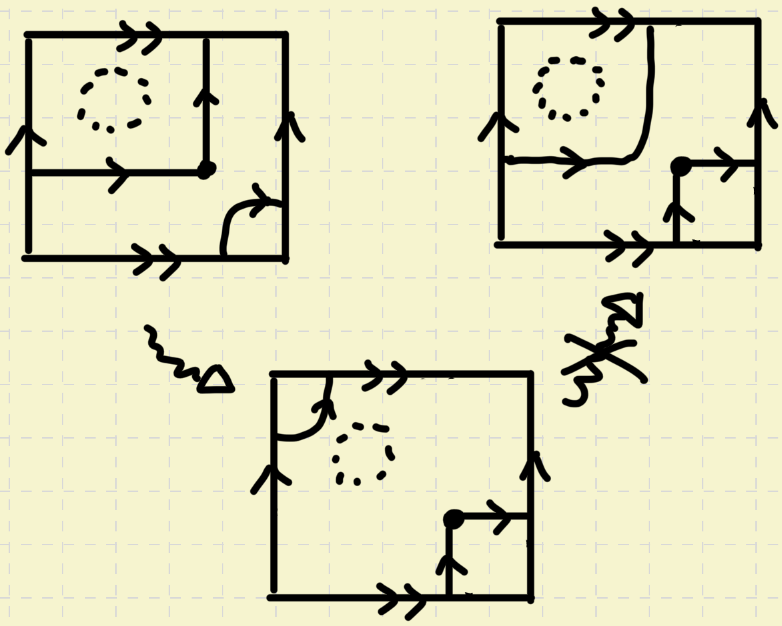
[b·a] =



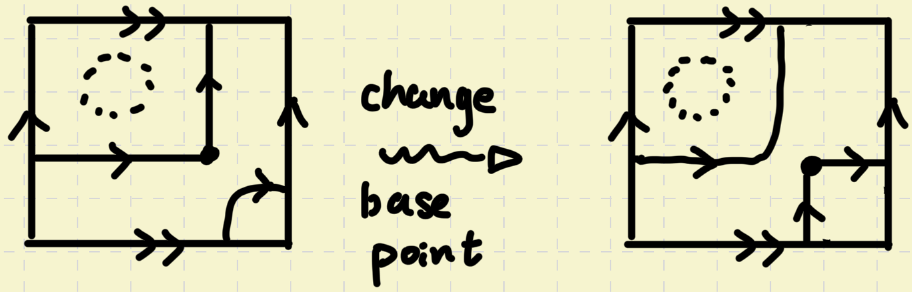
[a·b] = [b·a] because



Consider $T^2 \setminus D^2$, then



So $[a \cdot b] \not\simeq [b \cdot a]$ in sense of fix basepoint homotopy. Compare the case in T^2. This nonabilian encodes the topological information of $T^2 \setminus D^2$: $T^2 \setminus D^2$ has a hole on it. If we do not require base point be fixed. Then



So $[a \cdot b] = [b \cdot a]$, we lost information of nonabelian which encodes topo information.

- Some useful facts: 1. If M is contractible, then $\pi_k(M)=0$ for all positive k.
  2. If M is a topo m.f. then $\pi_1(M)$ is countable.
  3. $$\pi_k(S^n)=\begin{cases} 0 & k<n \\ \mathbb{Z} & k=n \\ \cdots & k>n \end{cases}$$
     $$\pi_k(S^1)=\begin{cases} \mathbb{Z} & k=n \\ 0 & k\neq n \end{cases}$$
  4. homotopy group of n-sphere

     has translation property for almostly all $n\in\mathbb{N}^*$

     $\pi_{n+1} S^n \simeq \mathbb{Z}_2$ for $n\geq 3$

     $\pi_{n+2} S^n \simeq \mathbb{Z}_2$ for $n\geq 2$

     $\pi_{n+3} S^n \simeq \mathbb{Z}_{24}$ for $n\geq 5$
  5. $\pi_k(\mathbb{R}P^n)\simeq \pi_k(S^n)$ when $k\geq 2$.

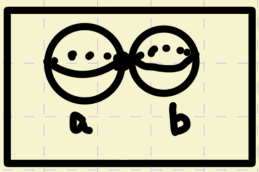
     $\pi_1(\mathbb{R}P^n)=\mathbb{Z}_2$, $\pi_1(S^n)=0$.

- Higher homotopy group is abelian. $\pi_k(X,x)$ is abelian for $k\geq 2$.

  For $a,b: S^k\to X$ be two elements in $\pi_k(X)$, we define product of a and b, $[a\cdot b]: S^k\to X$, by following steps.

  Find "equator" $S^{k-1}\subseteq S^k$ divide S^k into two parts.

  $[a\cdot b]: S^k\to X$ is squeezing S^{k-1} to the point x s.t. the two S^k can be viewed as a and b.



$a\cdot b$

By construction we find $a\cdot b=b\cdot a$.

(In Hatcher, we use $\boxed{f\,g}\simeq\boxed{\square\,\square}\simeq\boxed{\square\,\square}\simeq\boxed{g\,f}$)

- 1. X,Y: connected m.f. without boundary, then

     $\pi_k(X\times Y)\simeq \pi_k(X)\times\pi_k(Y)$
  2. Fiber bundle $F\to E\to B$ admits a section, then

$\pi_k(E) \cong \pi_k(F) \times \pi_k(B)$ for $k \geq 2$

- $X \sim Y \underset{\not\Leftarrow}{\Rightarrow} \pi_n(X) \simeq \pi_n(Y)$ for all n

[Coro] M is a sm mf with boundary, then $\pi_n(M) \simeq \pi_n(\mathring{M})$, since $M \sim \mathring{M}$.

But $\Leftarrow$ holds in some special case.

Whitehead's thm: X, Y path-connected.

If $\begin{cases} (1) \ \pi_n(X) \xrightarrow{\simeq} \pi_n(Y) \text{ with iso induced by a map } X \longrightarrow Y \\ (2) \ X, Y \text{ homeomorphic to CW complexes} \end{cases}$

Then X homotopic equivalent to Y.

[Rmk] Every sm compact mf is homeomorphic to a CW complex.

- [Def] (local homeo) A map $f: E \to M$ is called a local diffeomorphism homeomorphism if each point $p \in E$ has a n.b.h. U for which $f(U)$ is open and $f|_U$ is homeomorphism. diffeomorphism

smooth covering map

1. Every covering map is a local homeomorphism. diffeomorphism
2. Any local homeomorphism between compact, connected diffeomorphism mf is a covering map.

smooth covering map diffeomorphism

3. An injective covering map is a (global) homeomorphism.

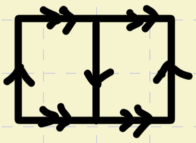
We can introduce sm structure, by red words replacement.

- Orientation covering

Every nonorientable sm mf M has an orientable two-sheeted covering E called its orientation covering or oriented double covering of M. It's unique up to diffeomorphism.

[Exp] S^n is the orientation covering of $\mathbb{RP}^n$.
oriented, two-sheeted.

[Exp] T^2 is the orientation covering of klein bottle



- M is a connected sm mf. If M is nonorientable, then $\pi_1(M)$ has a subgrp of index two. If M is simply-connected, $\pi_1(M)$ is trivial and doesn't have a subgrp of index two, so M is orientable.

- Covering spaces and homotopy grps.

  M is a mf and E is a covering of M, then $\pi_n(E)=\pi_n(M)$ for $n\geq 2$. The fundamental grp is the only homotopy grp that can distinguish between M and E.

  [Exp] Klein bottle can be covered by $\mathbb{R}^2$. So

  $$\pi_n(K)=\pi_n(\mathbb{R}^2)=0,\ n\geq 2.$$

- $\{\text{subgroups of } \pi_1(M)\} \overset{1:1}{\longleftrightarrow} \{\text{coverings over } M\}$

  [Exp] Consider $M=S^1$. $\pi_1(S^1)=\mathbb{Z}$

  subgroups of $\mathbb{Z}=\{\langle n\rangle \mid n\in\mathbb{N}\}$. So $\langle n\rangle$ with $n\geq 1$ corr. to covering $p: S^1\to S^1$ which winding n times. $p_*\pi_1(S^1)=\langle n\rangle$.
  $\langle 0\rangle$ corr. to universal covering $h:\mathbb{R}\to S^1$. $h_*\pi_1(\mathbb{R})=\langle 0\rangle$.

- Deck transformation tells you $\pi_1(\mathbb{R}P^n)=\pi_1(S^n/\mathbb{Z}_2)\simeq \mathbb{Z}_2,\ n\geq 2$

Another way to see it: S^n is simply-connected when $n\geq 2$. So S^n is the universal covering of $\mathbb{R}P^n$. Since any covering of $\mathbb{R}P^n$ can be covered by S^n and S^n is a double oriented covering, hence $\mathbb{R}P^n$ only has two covering (up to iso): $\mathbb{R}P^n$ (1-sheeted) and S^n (2-sheeted). By correspondence, $\pi_1(\mathbb{R}P^n)=\mathbb{Z}_2$ (only has two elements).

Ref: https://www.cphysics.org/article/61813.html#:~:text=The%20homotopy%20group%20with%20n%3D1%20is%20called%20the,group%20is%20only%20part%20of%20M%27s%20fundamental%20group.

Ref: 【几何拓扑短课-哔哩哔哩】https://b23.tv/JnjXxW4

- $f: X \to \mathbb{R}^n$, $g: X \to \mathbb{R}^n \Rightarrow f \sim g$

  用直线同伦 $F: X \times I \to \mathbb{R}^n$, $F(x,t) = (1-t)f(x) + tg(x)$

  (把 $\mathbb{R}^n$ 换成凸集, 也有直线同伦)

  [Exp] $\pi_n(\mathbb{R}^m, x_0) = 0$ 因为 $[S^n, \mathbb{R}^m] = *$.

- X path connected, $x_0 \in X$. If $\pi_i(X, x_0) = 0$, $\forall i \leq n$, then we call X is n-connected.

  注意. n-connected 定义的前提是 X path connected, why?

  consider 0-connected space. $\pi_0(X, x_0) = 0$ iff X is connected. 这个限制太弱了, 因此直接要求 X path-connected.

  0-connected space is path connected. (和 Hatcher 定义有出入)

  1-connected space is simply connected.

  Since $\pi_k(S^n) = 0$ for $k < n$, S^n is an $(n-1)$-connected space.

- Suspension

  Let X be a topo space. $SX = X \times I / {(x,1) \sim (y,1) \atop (x,0) \sim (y,0)}$, $\forall x, y \in X$

  picture:

  X $\qquad$ SX

For $f: X \to Y$, we define $Sf: SX \to SY$

$$(x,t) \mapsto (f(x), t)$$

Then we have a map $S: \pi_i(X, x_0) \to \pi_{i+1}(SX, x_0)$

$$[f] \longmapsto [Sf]$$

Actually, this map is a homeomorphism for $i \geq 1$.

- Fredenthal suspension thm

If X is path-connected and $(n-1)$-connected

then $S: \pi_i(X, x_0) \longrightarrow \pi_{i+1}(SX, x_0)$

is $\begin{cases} \text{iso.} & i < 2n-1 \\ \text{surj.} & i = 2n-1 \end{cases}$

[Exp] S^n is $(n-1)$-connected

$\pi_i(S^n, x_0) \cong \pi_{i+1}(S^{n+1}, x_0)$ when $i < 2n-1$

$\pi_{2n-1}(S^n, x_0) \longrightarrow \pi_{2n}(S^{n+1}, x_0)$ is surj.

1. These are helpful for computing $\pi_i(S^n)$.
2. $\pi_2(S^2) \cong \pi_3(S^3) \cong \pi_4(S^4) \cong \cdots$
3. By Fredenthal thm we cannot obtain $\pi_2(S^2) \cong \pi_1(S^1)$

- Hurewicz thm

1. Define Hurewicz map $h: \pi_n(X, x_0) \longrightarrow H_n(X, x_0)$

$$[f] \longmapsto f_* \alpha$$

where α is the generator of $H_n(S^n) \cong \mathbb{Z}$.

$f: (S^n, *) \to (X, x_0) \rightsquigarrow f_*: H_n(S^n, *) \longrightarrow H_n(X, x_0)$

2. Hurewicz thm: If X is $(n-1)$-connected, $n \geq 2$ then when $i = n$ we have h being an iso and $H_i(X) = 0$ for $i < n$.

[Exp] S^n is $(n-1)$-connected. $\pi_n(S^n) \cong H_n(S^n) = \mathbb{Z}$

- Review : fiber bundle

A morphism $p: E \to B$ is a fiber bundle with fiber F if $\forall b, \exists$ n.b.h. U of b and a homeo $h_U: p^{-1}(U) \to U \times F$ s.t.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h_U"] \arrow[dr, "p"'] & \circlearrowleft & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

$p_1 \sim$ projection

Then we call $F \hookrightarrow E \downarrow B$ a fiber bundle.

[Rmk] $p^{-1}(U)$ is "local total space". 仅有 total space iso to trivial $U \times F$ 是不够的, h_U 是 bundle iso, 用交换图刻画.

- Fiber bundle 的例子

[Exp] $\mathbb{C}P^n$.

$$S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

$$S^{2n+1} = \{ (\underbrace{x_1, y_1}_{z_1 \in \mathbb{C}}, x_2, y_2, \cdots, x_{n+1}, y_{n+1}) \in \mathbb{R}^{2n+2} \mid \sum x_i^2 + \sum y_i^2 = 1 \}$$

$$= \{ (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum |z_i|^2 = 1 \}$$

$$\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$$

$$(z_1, \dots, z_{n+1}) \longmapsto [z_1, \cdots, z_{n+1}]$$

$$\pi^{-1}(z_1, \dots, z_{n+1}) = \{ (\lambda z_1, \dots, \lambda z_{n+1}) \mid \lambda \in \mathbb{C} \} \cap S^{2n+1}$$

$$= \{ (\lambda z_1, \dots, \lambda z_{n+1}) \mid \lambda \in \mathbb{C}, |\lambda| = 1 \}$$

$$\cong S^1$$ ← Fiber is S^1.

When $n=1$ $\quad S^1 \to S^3 \to \mathbb{C}P^1 \cong S^2$ (Hopf bundle)

球极投影

When $n\to\infty$ $S^1\to S^\infty\to \mathbb{C}P^\infty$ 可用于计算同伦群.

[Exp] $\mathbb{H}P^n$, $\mathbb{H}$: 四元数

$$S^3 \hookrightarrow S^{4n+3} \xrightarrow{\pi} \mathbb{H}P^n$$

$\mathbb{H}P^n = \{$过原点四元数直线 $\subseteq \mathbb{H}^{n+1}\}$

$$S^{4n+3} = \{(a_1,b_1,c_1,d_1,\cdots,a_{n+1},b_{n+1},c_{n+1},d_{n+1})\in\mathbb{R}^{4n+4} \mid \Sigma a_i^2+b_i^2+c_i^2+d_i^2=1\}$$

$$= \{(w_1,\cdots,w_{n+1})\in\mathbb{H}^{n+1} \mid w_i\in\mathbb{H}, \Sigma |w_i|^2=1\}$$

When $n=1$ $S^3\hookrightarrow S^7\to \mathbb{H}P^1 = S^4$ Hopf bundle

When $n\to\infty$ $S^3\hookrightarrow S^\infty \to \mathbb{H}P^\infty$

[Exp] Cayley number: 八元数 $\mathbb{O}$

Similarly, we can obtain a Hopf bundle

$$S^7\hookrightarrow S^{15}\to S^8$$

[Exp] $S^2\hookrightarrow \mathbb{C}P^{2n+1} \xrightarrow{\pi} \mathbb{H}P^n$

$$[z_1,\cdots,z_{2n+2}] \longmapsto [z_1+jz_2, z_3+jz_4,\cdots, z_{2n+1}+jz_{2n+2}]$$

[Exp] Grassmann 流形 $G_n(\mathbb{R}^{n+k}) = \{V \mid V\subseteq \mathbb{R}^{n+k}, \dim V = n\}$

Stiefel 流形 $V_n(\mathbb{R}^{n+k}) = \{(\alpha_1,\cdots,\alpha_n) \mid \alpha_1,\cdots,\alpha_n$ 是 $\mathbb{R}^{n+k}$ 中规范正交向量$\}$

例如 $G_1(\mathbb{R}^{k+1}) = \mathbb{R}P^k$, $G_1(\mathbb{C}^{k+1}) = \mathbb{C}P^k$, $G_1(\mathbb{H}^{k+1}) = \mathbb{H}P^k$

$V_1(\mathbb{R}^{k+1}) = S^k$, $V_1(\mathbb{C}^{k+1}) = S^{2k+1}$, $V_1(\mathbb{H}^{k+1}) = S^{4k+3}$

Grassmannian 与 Stiefel 之间是 base space 与 total space 的关系. 有如下 fiber bundles.

$$O(n) \hookrightarrow V_n(\mathbb{R}^{n+k}) \xrightarrow{\pi} G_n(\mathbb{R}^{n+k})$$
$$(d_1, \cdots, d_n) \longmapsto \mathrm{Lin}(d_1, \cdots, d_n)$$

$$U(n) \hookrightarrow V_n(\mathbb{C}^{n+k}) \xrightarrow{\pi} G_n(\mathbb{C}^{n+k})$$
$$(d_1, \cdots, d_n) \longmapsto \mathrm{Lin}(d_1, \cdots, d_n)$$

$$Sp(n) \hookrightarrow V_n(\mathbb{H}^{n+k}) \xrightarrow{\pi} G_n(\mathbb{H}^{n+k})$$

$$V_{m-n}(\underset{\sim}{\mathbb{R}^k}) \hookrightarrow V_m(\mathbb{R}^{n+k}) \longrightarrow V_n(\mathbb{R}^{n+k})$$
$$(d_1, \cdots, d_m) \longmapsto (d_1, \cdots, d_n)$$

$(d_1, \cdots, d_n)^{\perp} \cong \mathbb{R}^k$

在垂直于 $\mathrm{Lin}(d_1, \cdots, d_n)$ 的空间选规范正交 $m-n$ 个向量.

- Fiber bundle 同伦长正合列的计算例子.

[Exp] covering space is a fiber bundle with discrete fiber.

$$\begin{array}{ccc} F & \overset{i}{\hookrightarrow} & E \\ & & \downarrow p \\ & & B \end{array}$$

$\pi_i(F, x_0) = 0$, $i \geq 1$. Let $p(x_0) = b$. We have L.E.S.

$$\cdots \to \underset{\substack{\| \\ 0}}{\pi_i(F, x_0)} \xrightarrow{i_*} \pi_i(E, x_0) \xrightarrow{p_*} \pi_i(B, b) \xrightarrow{\partial} \underset{\substack{\| \\ 0}}{\pi_{i-1}(F, x_0)}$$

$i \geq 1$ $\qquad$ $i-1 \geq 1$ & $i \geq 2$

So $\pi_i(E, x_0) \cong \pi_i(B, b)$, $i \geq 2$

[Exp] Consider $S^1 \hookrightarrow S^3 \longrightarrow S^2$

We have $\cdots \to \pi_i(S^1) \to \pi_i(S^3) \to \pi_i(S^2) \to \pi_{i-1}(S^1) \to \cdots$

$\pi_i(S^1) = \begin{cases} \mathbb{Z} & i=1 \\ 0 & i\neq 1 \end{cases}$, we have

1. $\pi_i(S^3) \cong \pi_i(S^2)$ when $i \geq 3$ $(i-1 \geq 2)$

2. When $i=3$ $\pi_3(S^2) \cong \pi_3(S^3) = \mathbb{Z} \neq 0$

不同于 homology group!

3. Consider

$$\cdots \to \underset{\overset{\shortparallel}{0}}{\pi_2(S^3)} \to \pi_2(S^2) \to \pi_1(S^1) \to \underset{\overset{\shortparallel}{0}}{\pi_1(S^3)} \to \cdots$$

So $\pi_2(S^2) \cong \pi_1(S^1)$. (By Fredenthal thm 不能得到 $\pi_2(S^2) \cong \pi_1(S^1)$，只能得到 $\pi_{i+1}(S^{i+1}) \cong \pi_i(S^i)$, $i \geq 2$.

[Exp] Stiefel 流形的性质

$$V_{n-1}(\mathbb{R}^{n+k-1}) \hookrightarrow V_n(\mathbb{R}^{n+k}) \longrightarrow V_1(\mathbb{R}^{n+k})$$

$$(\alpha_1, \cdots, \alpha_n) \longmapsto \alpha_1$$

$V_1(\mathbb{R}^{n+k}) = S^{n+k-1}$. So we have L.E.S.

$$\cdots \to \underset{\substack{\shortparallel \\ 0}}{\pi_{i+1}(S^{n+k-1})} \xrightarrow{\partial} \underset{\substack{\shortparallel \\ V_{n-1}}}{\pi_i(V_{n-1}(\mathbb{R}^{n+k-1}))} \to \underset{\substack{\shortparallel \\ V_n}}{\pi_i(V_n(\mathbb{R}^{n+k}))} \to \underset{\substack{\shortparallel \\ 0}}{\pi_i(S^{n+k-1})} \to \cdots$$

$i+1 < n+k-1$ i.e., $i \leq n+k-3$; $i < n+k-1$ i.e. $i \leq n+k-2$

Then $\pi_i(V_{n-1}) \cong \pi_i(V_n)$ when $i \leq n+k-3$

递推，有

$$\pi_i(V_n) \underset{i\leq n+k-3}{\cong} \pi_i(V_{n-1}) \underset{i \leq n+k-4}{\cong} \pi_i(V_{n-2}) \cong \cdots \cong \pi_i(V_2) \underset{i\leq k-1}{\cong} \pi_i(V_1)$$

By notation, $V_1 = V_1(\mathbb{R}^{k+1}) \cong S^k$

So $\pi_i(V_n) = \pi_i(S^k) = 0$ when $i \leq k-1$.

So $V_n = V_n(\mathbb{R}^{n+k})$ is $(k-1)$-connected.（略去π_k和的证明）

[Exp] 已知 $\pi_i(S^\infty)=0,\ i\geq 0$. We can use bundle
$S^1 \hookrightarrow S^\infty \to \mathbb{C}P^\infty$ to compute $\pi_n(\mathbb{C}P^\infty)$.

$$\underset{\substack{\| \ i+1\geq 0\\ 0}}{\pi_{i+1}(S^\infty)} \to \pi_{i+1}(\mathbb{C}P^\infty) \to \pi_i(S^1) \to \underset{\substack{\| \ i\geq 0\\ 0}}{\pi_i(S^\infty)} \to \cdots$$

$$\text{So } \pi_{i+1}(\mathbb{C}P^\infty) \cong \pi_i(S^1) = \begin{cases} \mathbb{Z} & i=1 \\ 0 & i\neq 1 \end{cases}$$

$$\text{i.e., } \pi_i(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i\neq 0 \end{cases}$$

- Spectral sequence 谱序列

Introduction: Fiber bundle $F \hookrightarrow E \to B$ 有同伦长正合列刻画$\pi_*(F)$, $\pi_*(E)$, $\pi_*(B)$ 的信息，任知两个可知剩下一个的信息. Question: $H_*(F)$, $H_*(E)$, $H_*(B)$ 之间的关系是什么？它们间的关系可由 Spectral sequence 刻画.

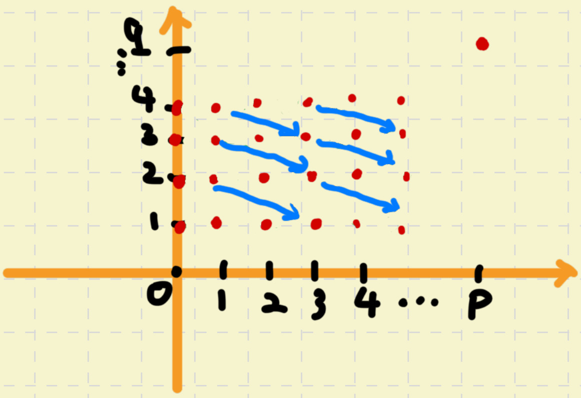
Big picture: Spectral sequence is a book.

We call the i-th page E_i-term.

What's in E_i-term?

$$E_i\text{-term contains } \{\underbrace{E_r^{p,q}}_{\text{Abel grp}},\ p,q\in\mathbb{Z};\ \underbrace{d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}}_{d_r\circ d_r=0}\}$$

E_2-term



1. 每一个红点 • 代表一个 ab grp $E_r^{p,q}$

2. → 表示$d_2: E_r^{p,q} \to E_r^{p+2,q-1}$

3. $d_r^2=0$ 意味着每一条连起来的线是 chain complexes

E_i-term 与 E_{i+1}-term 之间有关联. "$E_r \Rightarrow E_{r+1}$"

$$\cdots \to E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1} \to \cdots$$

$$E_{r+1}^{p,q} = \frac{\ker d_r}{\operatorname{Im} d_r}$$

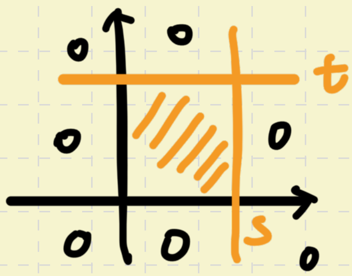
Hence from $E_r^{p,q}, d_r$, we can compute $E_{r+1}^{p,q}$

但并没有统一的办法计算 d_{r+1} 要具体情形具体分析.

- $E_2 \to E_3 \to E_4 \to \cdots$ 一直计算下去是否会稳定?

在具体应用时, 总存在 $r, s, t \in \mathbb{N}^*$ s.t. $p \geq s$ or $q \geq t$,

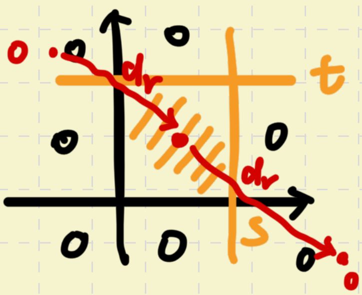
we have $E_r^{p,q} = 0$.



非0部分,

外部全是0

当 r 充分大时, d_r 的头或尾是0, 如此 $E_{r+1}^{p,q} = E_r^{p,q}$



于是有 $E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots =: E_\infty^{p,q}$

Δ 若 $\exists R$ s.t. $r \geq R$ 时 $d_r = 0$ ($\forall p, q$)

Then $E_\infty^{p,q} := E_R^{p,q} = E_{R+1}^{p,q} = \cdots$ 稳定.

则称谱序列 $\{E_r^{p,q}, d_r\}$ 在 E_R-term 是 degenerate (collapse)

Δ $E_\infty^{p,q}$ 有什么用?

$H^n(X)$ 是我们关心的未知待求上同调.

$H^n(X)$ 存在 filtration $\quad H^n(X)\supset F^0\supset F^1\supset\cdots\supset F^n\supset 0$.

若有 $E_\infty^{p,q}=F^p/F^{p+1}$ 则我们可以求 $H^n(X)$，方法如下:

$E_\infty^{n,0}=F^n/0=F^n$ 因此 F^n 得到了

$E_\infty^{n-1,1}=F^{n-1}/F^n$，于是有短正合列

$$0\to \underset{\text{已知}}{F^n}\to \underset{\text{可以解出 }F^{n-1}}{F^{n-1}}\to \underset{\text{已知}}{E_\infty^{n-1,1}}\to 0$$

⋮

$E_\infty^{0,n}=F^0/F^1$ 有短正合列

$$0\to \underset{\text{已知}}{F^1}\to \underset{\text{可以解出 }H^n(X)}{\overset{H^n(X)}{\overset{\parallel}{F^0}}}\to \underset{\text{已知}}{E_\infty^{0,n}}\to 0$$

• 谱序列的构造 (filtered differential graded (Abelian group) chain complex)

1. A is a chain complex, i.e., $\boxed{\cdots\to A^n\to A^{n+1}\to A^{n+2}\to\cdots}=:A$

chain complex A has a filtration

$$A\supset\cdots\supset \underbrace{F^pA}_{\text{子链复形}}\supset F^{p+1}A\supset\cdots$$

具体写出来如下，称为 filtered chain complex 滤化链复形

$$\begin{array}{ccccccc}
\vdots & & & & & & \\
\uparrow & & & & & & \\
A^{n+1} & \supset\cdots\supset & F^pA^{n+1} & \supset & F^{p+1}A^{n+1} & \supset\cdots & \text{每一行是 filtration} \\
\uparrow & & & & & & \\
A^n & \supset\cdots\supset & F^pA^n & \supset & F^{p+1}A^n & \supset\cdots & \\
\uparrow & & & & & & \\
A^{n-1} & \supset\cdots\supset & F^pA^{n-1} & \supset & F^{p+1}A^n & \supset\cdots & \\
\uparrow & & & & & & \\
\vdots & & & & & & \\
A & \supset\cdots & \supset F^pA & \supset & F^{p+1}A & \supset\cdots &
\end{array}$$

每一列是 chain complex

2. $H(A,d)$: cohomology of chain complex $(\underset{\text{differential}}{A,d})$

$H(A,d)$ has a filtered chain complex

$$H(A,d) \supset \cdots \supset F^{p-1}H(A,d) \supset F^{p}H(A,d) \supset \cdots$$

where $F^pH(A,d) := \mathrm{Im}\left(H(F^pA,d) \to H(A,d)\right)$

3. Assume filtration of A is bounded.

$\forall n$, $\exists s(n), t(n) \in \mathbb{Z}$ s.t.

$$A^n = F^tA^n \supset F^{t-1}A^n \supset \cdots \supset F^sA^n = 0$$

Thm: Let (A,d,F^*) be a filtered differential graded Abelian group chain complex and $\forall n, \exists s(n), t(n)$ s.t. $A^n = F^tA^n \supset \cdots \supset F^sA^n = 0$

Then (A,d,F^*) determines a spectral sequence $\{E_r^{p,q}, d_r\}_{r\geq 1}$ where $d_r : E_r^{p,q} \to E_r^{p+r,\,p-r+1}$

and $\cdot$ $E_1^{p,q} \cong H^{p+q}(F^pA/F^{p-1}A)$

$\cdot$ $\{E_r^{p,q}, d_r\} \underset{\text{collapse}}{\Longrightarrow} H(A,d)$,

i.e., $E_\infty^{p,q} = F^pH^{p+q}(A,d)/F^{p+1}H^{p+q}(A,d)$

$$\left(\begin{array}{l} F^pH^{p+q}(A,d) := \mathrm{Im}\left(H^{p+q}(F^pA,d) \to H^{p+q}(A,d)\right) \\ H(A,d) \supset \cdots \supset F^{p-1}H(A,d) \supset F^pH(A,d) \supset \cdots \end{array}\right)$$

[Rmk] 另一种构造谱序列的办法是利用exact couple.

给一个 exact couple 就能造一个谱序列.

- Serre spectral sequence (or Leray spectral sequence)

* Serre spectral sequence 联系了 $F \hookrightarrow X \to B$ 的上同调.

Thm: 若 $F \hookrightarrow X \to B$ 是一个 fiber bundle

- B path-connected
- $\pi_1(B)$ 在 $H^*(F;G)$ 上作用平凡 (例如 $\pi_1(B)$ 平凡，即 B 单连通)

则一定存在一个 spectral sequence 收敛到 $H^*(X;G)$

其中 G 为 Abelian grp.

即 $\{E_r^{p,q}, d_r\}$

$\cdot\ E_\infty^{p,n-p} \cong F_p^n / F_{p+1}^n$，其中 $F_p^n = \ker(H^n(X;G) \to H^n(X_{p-1}))$

且 $E_2^{p,q} \cong H^p(B; H^q(F;G))$

[Rmk] $\underline{E_2}^{p,q} \cong H^p(\underline{B}; H^q(\underline{F}; G))$

spectral seq collapse to total space cohomo $H^*(X;G)$

base space

fiber

- 谱序列上有乘法结构

$\cup: E_r^{p,q} \times E_r^{s,t} \longrightarrow E_r^{p+s,q+t}$ 满足

$$d_r(x \cup y) = d_r(x) \cup y + (-1)^{p+q} x \cup d_r(y)$$

when $r=2$

$$\begin{array}{ccccc} \cup: & E_2^{p,q} \times E_2^{s,t} & \longrightarrow & E_2^{p+s,q+t} \\ & \wr\wr \quad\quad \wr\wr & & \wr\wr \\ & H^p(B;H^q(F;G)) \times H^s(B;H^t(F;G)) & \longrightarrow & H^{p+s}(B;H^{q+t}(F;G)) \\ & (x,y) & \longmapsto & (-1)^{qs} x \cup y \end{array}$$

一般的 cup 积.

Application1: 计算上同调环 $H^*(\mathbb{C}P^n;\mathbb{Z})$

$*$: $H^*(\mathbb{C}P^n;\mathbb{Z})$ 的群结构是容易的，即

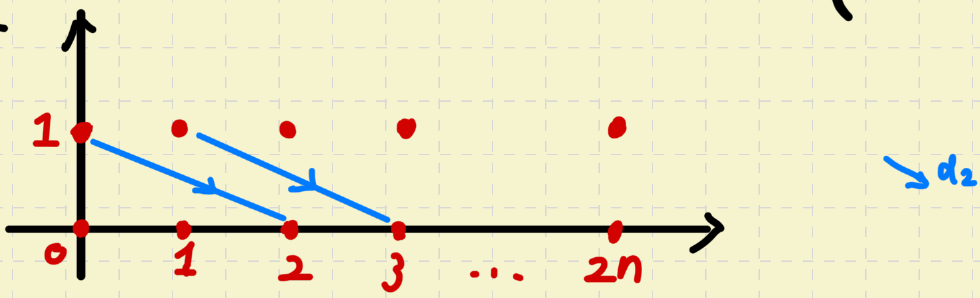
$$H^m(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & m = 0, 2, \cdots, 2n \\ 0 & \text{o/w} \end{cases}$$

* 对 fiber bundle $S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$ 用 Serre spectral bundle. $\mathbb{C}P^n$ path-connected and $\pi_1(\mathbb{C}P^n)=0$, hence there is a spectral sequence $\{E_r^{p,q}, d_r\} \Rightarrow H^*(S^{2n+1};\mathbb{Z})$ 考虑$\mathbb{Z}$系数

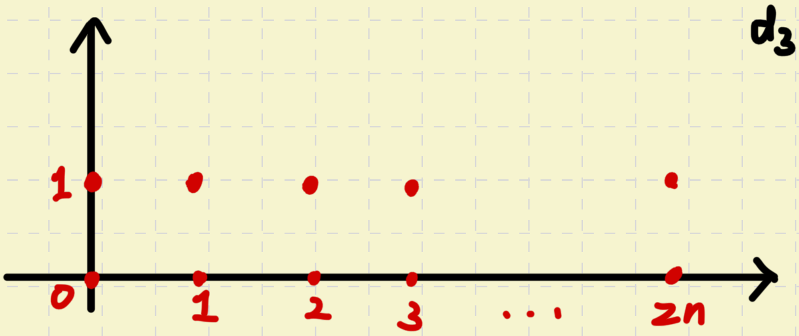
$$E_2^{p,q} = H^p(\mathbb{C}P^n; H^q(S^1;\mathbb{Z}))$$

$E_2^{p,q}$ 最多只有如下 nontrivial grp $\left(\begin{array}{l} H^p(\mathbb{C}P^n;\mathbb{Z})=0,\ p>2n \\ H^q(S^1;\mathbb{Z})=0 \quad q\neq 0,1 \end{array} \right)$

E_2



Consider E_3



$d_3: E_3^{p,q} \longrightarrow E_3^{p+3,\,q-2}$ 往下走两格必是 0

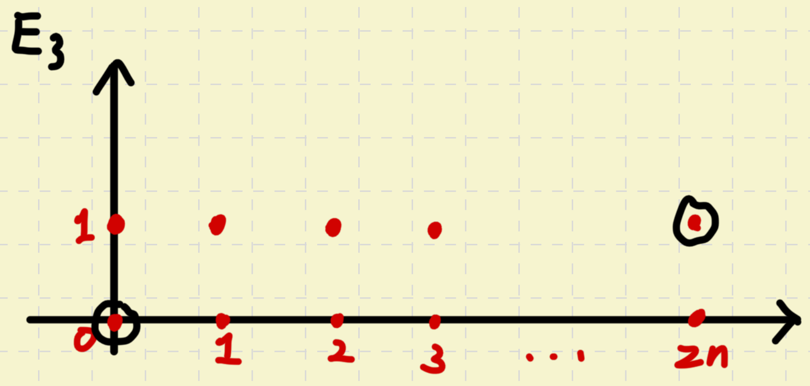
因此 $d_r=0$, $r\geqslant 3$. $E_\infty^{p,q}=E_3^{p,q}$

$$H^m(S^{2n+1};\mathbb{Z}) = F_0^m \supset F_1^m \supset F_2^m \supset \cdots \supset F_t^m \supset 0$$

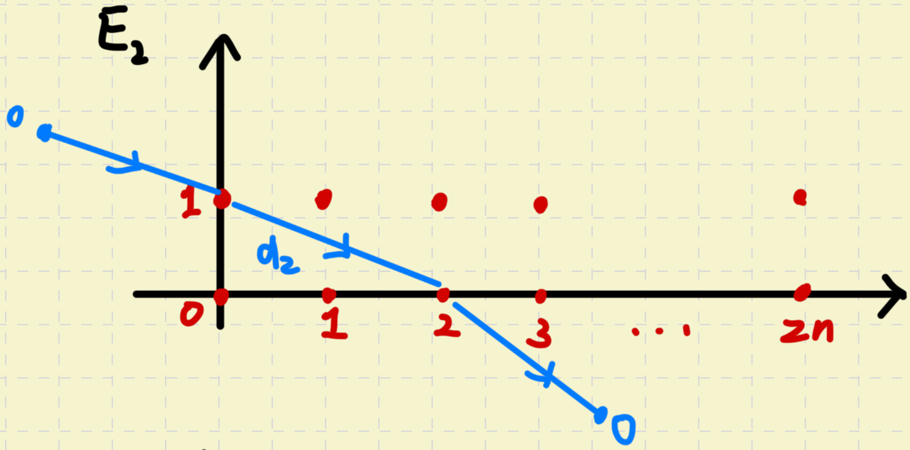
$E_\infty^{p,m-p} = F_p^m / F_{p+1}^m$. Luckly, $H^m(S^{2n+1};\mathbb{Z})=0$ when $m\neq 0, 2n+1$. So $F_n^m \equiv 0$, i.e., $E_\infty^{p,m-p}=0$ when $m\neq 0, 2n+1$.

即两个 index 之和非 $0, 2n+1$ 时 $E_\infty^{p,q}=0$.

$E_3^{p,q} = E_\infty^{p,q} = 0$ 从 E_3 反推 E_2.



⊙ 可以非 0



$E_3^{0,1} = E_3^{2,0} = 0 \Rightarrow$ 上面的蓝色 sequence 在 $E_2^{0,1}$ 与 $E_2^{2,0}$ 处正合

$\Rightarrow d_2$ 是同构 $\Rightarrow E_2^{0,1} \cong E_2^{2,0}$.

$$E_2^{0,1} = H^0(\mathbb{C}P^n; H^1(S^1;\mathbb{Z}))$$
$$= H^0(\mathbb{C}P^n;\mathbb{Z}) \otimes H^1(S^1;\mathbb{Z}) = \mathbb{Z}x$$

where x is the generator of $H^1(S^1;\mathbb{Z})$

Then $E_2^{2,0} = \mathbb{Z}(d_2 x) =: \mathbb{Z}a$ where $a := d_2 x$

Similarly, $\underset{\sim\sim}{E_2^{2,1}} \cong E_2^{4,0}$

用 universal coefficient thm 计算

$$E_2^{2,1} = H^2(\mathbb{C}P^n; H^1(S^1;\mathbb{Z}))$$
$$\cong H^2(\mathbb{C}P^n;\mathbb{Z}) \otimes H^1(S^1;\mathbb{Z})$$

generator $\quad a \quad \otimes \quad x$

$$= \mathbb{Z}(a\otimes x)$$

$$E_2^{2,1} \xrightarrow{d_2} E_2^{4,0}$$
$$ax \longmapsto d_2(ax) = \overset{0}{d_2(a)}x + a\,d_2(x) = a\,d_2x = a^2$$
$$(d_2 a = d_2^2 x = 0)$$

故 $H^4(\mathbb{C}P^n;\mathbb{Z}) = E_2^{4,0}$ 由 a^2 生成

$$E_2^{4,1} = H^4(\mathbb{C}P^n; H^1(S^1;\mathbb{Z}))$$
$$= H^4(\mathbb{C}P^n;\mathbb{Z}) \otimes H^1(S^1;\mathbb{Z})$$
$$= \mathbb{Z}(a^2\cdot x)$$

So $E_2^{4,1} \xrightarrow[\sim]{d_2} E_2^{6,1}$

$$a^2x \longmapsto d_2(a^2)x + a^2 d_2 x = a^3.$$

$\vdots$

We have $E_2^{2n,1} = \mathbb{Z}a^n$

$$E_2^{2n,1} = H^{2n}(\mathbb{C}P^n; \mathbb{Z}x)$$
$$= \mathbb{Z}(a^n\cdot x)$$

$$E^{2n,1} \xrightarrow{d_2} E^{2n+2,0} = 0$$
$$a^n\cdot x \longmapsto a^{n+1}$$

so $a^{n+1} = 0$.

Hence $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[a]/a^{n+1}$.

Application 2：计算 $H^*(\Omega S^n)$ 的群结构

$\Omega(S^n) = \{\alpha: I \to S^n \mid \alpha(0)=\alpha(1)=*\}$（所有闭路）

寻找 fiber bundle 出现 ΩS^n：

$P := P(S^n) = \{\alpha: I \to S^n \mid \alpha(0) = *\}$ where $*$ is the base point of S^n.

$$\Omega(S^n) \hookrightarrow P(S^n) \longrightarrow S^n$$
$$\alpha \longmapsto \alpha(1)$$

Note that P is contractible, i.e., $P \simeq *$

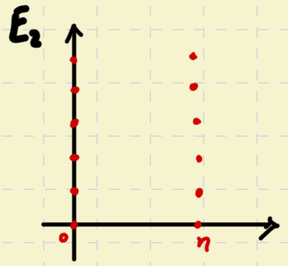
$\pi_1(S^n)=0$. S^n path-connected，可以用 serre spectral seq.

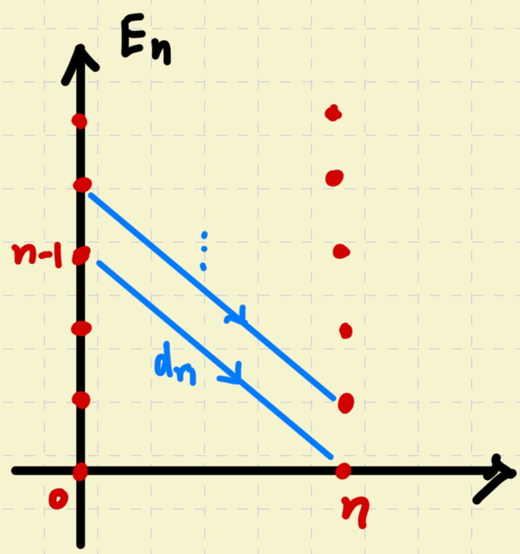
$$E_2^{p,q} = H^p(S^n; H^q(\Omega S^n;\mathbb{Z}))$$
$$= H^p(S^n) \otimes H^q(\Omega S^n)$$

$\Rightarrow$ 1. 仅当 $p=0$ or n 时 $E_2^{p,q}$ 才可能非0.

2. $H^0(S^n) = H^p(S^n) = \mathbb{Z}$，同一行两个群同构.

3. $q=0$ 与 $q=n$ 上的群正是待求的 $H^*(\Omega S^n)$





$d_r: E_r^{p,q} \longrightarrow E_r^{p+r,\,q-r+1}$

只有 $r=n$ 时 d_r 才可能非0.

$d_r = 0$ when $r \neq n$.

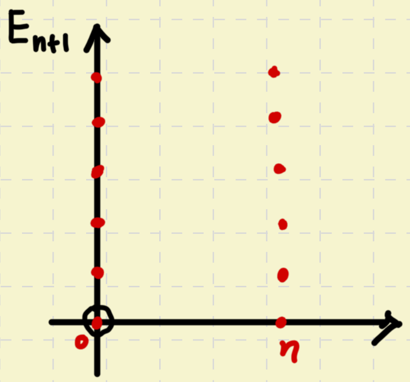
即 $E_2^{p,q} = E_3^{p,q} = \cdots = E_n^{p,q}$

$E_{n+1}^{p,q} = E_{n+2}^{p,q} = \cdots = E_\infty^{p,q}$

$$E_r^{p,q} \Rightarrow H^*(P), \text{ i.e., } \begin{cases} H^m(P) = F_0^m \supset F_1^m \supset \cdots \supset F_t^m \\ E_\infty^{p,m-p} = F_p^m / F_{p+1}^m \end{cases}$$

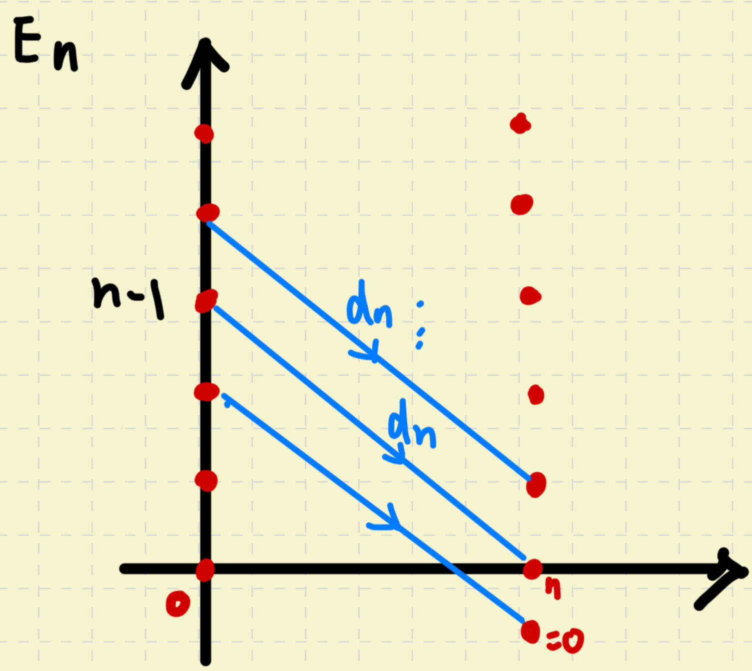
$H^m(P) = H^m(*) = 0$ when $m \neq 0$.

So $E_{n+1}^{p,m-p} = E_\infty^{p,m-p} = 0$ when $m \neq 0$



只有$E_{n+1}^{0,0}$ 可能非0.
其余都是0.

$\Downarrow$ E_n中有很多同构

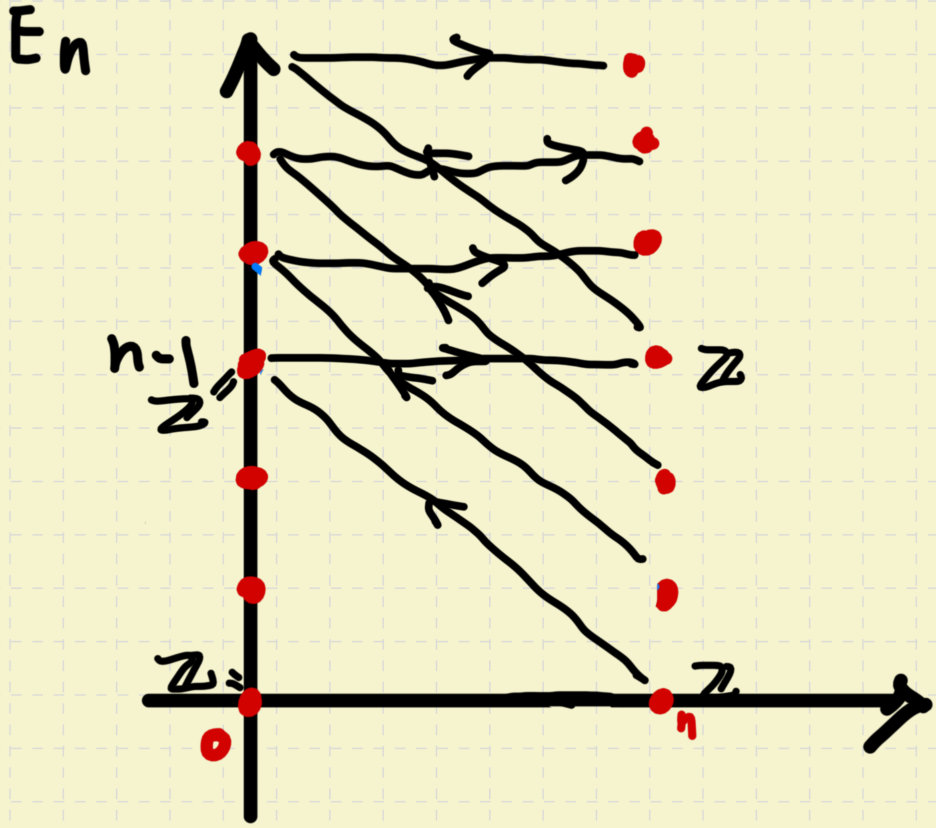


$E_n^{0,q} \cong 0$, $1 \le q < n-1$, i.e.,

$H^q(\Omega S^n) = E_2^{0,q} = E_n^{0,q} = 0$, $1 \le q < n-1$

$E_n^{0,n-1} = H^{n-1}(\Omega S^n) \cong H^n(S^n) \cong \mathbb{Z}$

So $E_n^{0,0} \cong E_n^{n,0} \cong E_n^{0,n-1} \cong \mathbb{Z}$



以  的路径可以把上同调群解出来

$$H^*(\Omega S^n) = \begin{cases} \mathbb{Z} & * = k(n-1) \quad k \geq 0 \\ 0 & o/w \end{cases}$$

amazing! ΩS^n 在任意高阶上同调群都会出现非平凡群.

<u>Application3</u> 计算 $\pi_{n+1}(S^n) = ?$

* Serre spectral seq 可以帮助计算同位群.

由 Suspension, $\pi_{n+1}(S^n) \cong \pi_n(S^{n-1}) \cong \cdots \underset{\substack{\uparrow \\ 4 < 2\times 3 - 1 \text{ 满足}}}{\cong} \pi_4(S^3)$.

Hence we only need to compute $\pi_4(S^3)$.

需要把$\pi_4(S^3)$同构到另一个容易计算的群.

Consider Eilenberg-Maclane space $K(\mathbb{Z}, n)$,

i.e., $\pi_i(K(\mathbb{Z},n)) = \begin{cases} \mathbb{Z} & i=n \\ 0 & o/w \end{cases}$

有 fibration $F \hookrightarrow S^3 \xrightarrow{f} K(\mathbb{Z},3)$ (可以不用管 Fiber F 是什么)

1. Fact: F is 3-connected.

By Hurewicz thm, $\pi_4(F) \cong H_4(F)$.

2.

$$0 \longrightarrow \underset{\substack{\parallel \\ 0}}{\pi_5(K(\mathbb{Z},3))} \longrightarrow \pi_4(F) \longrightarrow \pi_4(S^3) \longrightarrow \underset{\substack{\parallel \\ 0}}{\pi_4(K(\mathbb{Z},3))} \longrightarrow 0$$

So $\pi_4(F) \cong \pi_4(S^3)$

$\Rightarrow$ Therefore, $\pi_4(S^3) \cong H_4(F)$

Ideal: Compute $H^5(F)$ and $H^4(F)$, then by universal coefficient thm, we can compute $H_4(F)$.

(把计算同伦群化规到求上同调的问题)

$F \hookrightarrow S^3 \longrightarrow K(\mathbb{Z},3) \rightsquigarrow$

$$\begin{array}{c} K(\mathbb{Z},2) \hookrightarrow F \\ \downarrow \\ S^3 \end{array}$$

$K(\mathbb{Z},2) = \mathbb{C}P^\infty$

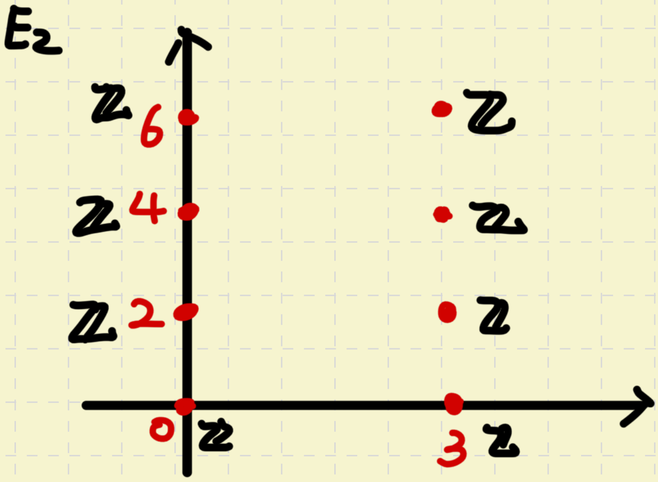
we obtain a bundle $\mathbb{C}P^\infty \to F \to S^3$

$\pi_1(F)=0$, S^3 path-connected. 故可用 Serre spectral seq.

$$E_2^{p,q} = H^p(S^3; H^q(\mathbb{C}P^\infty)) = H^p(S^3) \otimes H^q(\mathbb{C}P^\infty)$$

$$= \begin{cases} \mathbb{Z} & (0,2k),\ (3,2k),\ k \in \mathbb{N} \\ 0 & o/w \end{cases}$$

E_2



F is 3-connected, so by Hurewicz thm we have $H_2(F)=0$. By universal coefficient thm, we have $H^2(F)=\mathrm{Hom}(H_2(F),\mathbb{Z})=0$.

$0=H^2(F)=F_0^2\supset F_1^2\supset\cdots\supset F_t^2=0.$

$E_\infty^{p,m-p}=F_p^m/F_{p+1}^m=0$ when $m=2$.

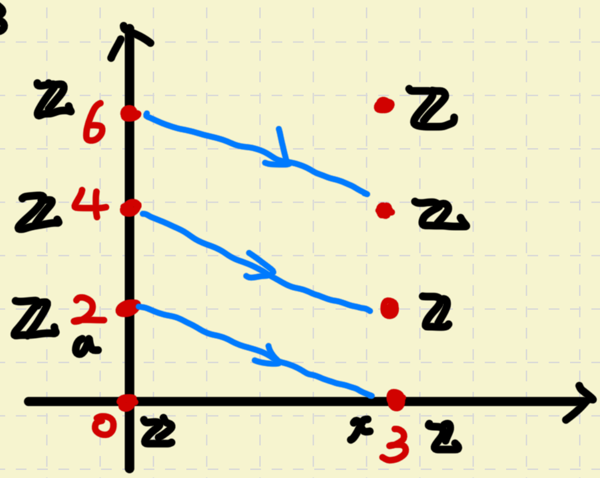
So $E_\infty^{2,0}=E_\infty^{0,2}=E_\infty^{1,1}=0$

Consider d_r. $d_r=\begin{cases}0 & o/w\\ \neq 0 & r=3\end{cases}$

So $E_4^{p,q}=E_\infty^{p,q} \rightsquigarrow E_4^{2,0}=E_4^{0,2}=E_\infty^{1,1}=0$

We need to focus on E_3

E_3



$E_4^{0,2}=0 \rightsquigarrow (E_3, d_r)$ exact at $E_3^{0,2}$

So $E_3^{0,2}\xrightarrow{d_3}E_3^{3,0}$ is

generator: $a\longmapsto x\neq 0$

injective.

By Hurewicz thm, $H^3(F)=0$.

So $E_\infty^{p,3-p}=F_p^3/F_{p+1}^3=0$.

So $\underbrace{E_4^{0,3} = E_4^{1,2} = E_4^{2,1}}_{\text{useless}} = \underbrace{E_4^{3,0}}_{\text{useful}} = 0$

$E_4^{3,0} = 0 \Longrightarrow d_3 : E_3^{0,2} \longrightarrow E_3^{3,0}$ is surjective.

$\Rightarrow$ So $d_3 : E_3^{0,2} \rightarrow E_3^{3,0}$ is an iso.

Since $d_2 = 0$, we have $E_3^{p,q} = E_2^{p,q}$.

$E_3^{0,2k} = E_2^{0,2k} = H^{2k}(\mathbb{C}P^\infty) = \mathbb{Z}\, a^k$

$$d_3 : E_3^{0,4} \longrightarrow E_3^{3,2} = x$$

gen: $a^2 \longmapsto d_3(a^2) = \underbrace{d_3(a)}\cdot a + a \cdot d_3(a) = 2ax$

相当于 $d_3 : \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$

$$E_3^{0,4} \xrightarrow{d_3} E_3^{3,2} \longrightarrow 0, \quad E_3^{3,2} = \mathbb{Z}$$

$$E_4^{3,2} = E_3^{3,2} / \mathrm{Im}\, d_3 = \mathbb{Z}/2$$

The $E_\infty^{3,2} = \mathbb{Z}/2$. $\rightsquigarrow H^5(F) = \mathbb{Z}/2$ (略去步骤)

同理算得 $H^4(F) = 0$. 于是 $H_4(F) = \mathbb{Z}/2$. $\rightsquigarrow \pi_4(S^3) = \mathbb{Z}/2$.

- For absolute homotopy grp $\pi_n(X, x_0) = \begin{cases} \text{abelian grp} & n \geq 2 \\ \text{grp} & n = 1 \\ \text{may not be a grp.} & n = 0 \end{cases}$

[Rmk] $\pi_0(X, x_0)$ = a set with card being path connecting components.

Any element in $\pi_0(X, x_0)$ is a pair (x_0, x) where $x \in X$. $(x_0, x_1) \sim (x_0, x_2) \Leftrightarrow x_1$ path connected to x_2

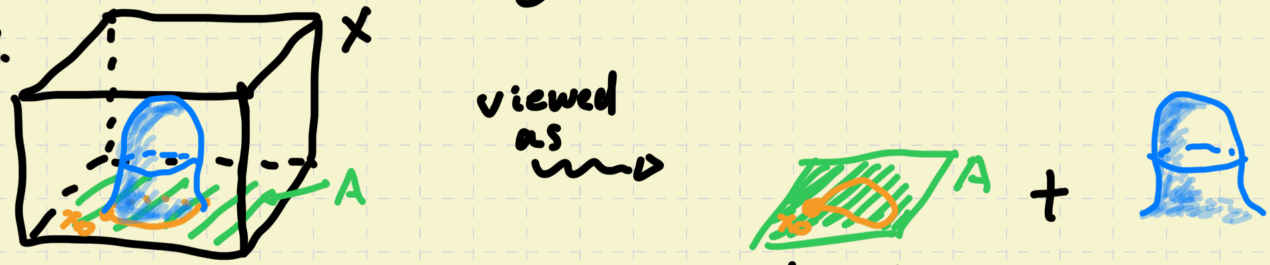
For relative homotopy grp $\pi_n(X,A,x_0) = \begin{cases} \text{abelian grp} & n\geq 3 \\ \text{grp} & n=2 \\ \text{may not be a grp} & n=1 \end{cases}$

Why the conclusions are so similar?

Reason:

1. 观点：我们可把 $\pi_n(X,A,x_0)$ 中的元素视作 $\pi_{n-1}(A,x_0)$ 的元素并以之为 boundary 覆上 n-dim 的 bulk.

e.g.



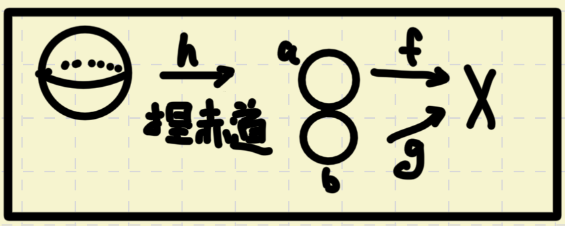
an element in $\pi_2(X,A,x_0)$

an element in $\pi_1(A,x_0)$

事实上这正是同伦长正合列中的 $\underline{\partial}: \pi_n(X,A,x_0) \longrightarrow \pi_{n-1}(A,x_0)$ 作 restriction.

($\pi_0(X,A,x_0)$ 没有好的定义，可以用 $\pi_{-1}(A,x_0)$ 不好定义解释)

2. 群乘法结构从哪里来?



Hatcher上的图说清楚：

$f\vee g: S^2 \vee S^2 \longrightarrow X$

$f\vee g|_{\text{sphere } a} = f$, $f\vee g|_{\text{sphere } b} = g$

$f\cdot g = (f\vee g)\circ h : S^2 \longrightarrow X$.

乘法结构最关键的一步是捏赤道. 只要捏赤道能裂成两个自己就有乘法.

3. 群乘法的 abelian 性质从哪里来?

捏赤道后分裂的两个东西继承原来的定向 定向有差别则不 abelian；反之 abelian.

4. 最终的解释.

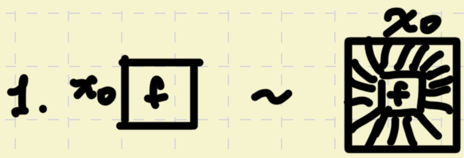
$\pi_n(X,A,x_0)$ 主要被 $a\in\pi_{n-1}(A,x_0)$ 部分的性质决定，因为在 a 上覆盖的 n-dim bulk 由于维数比 $\dim a=n-1$ 大所以性质一定比 a 好(观察 n-sphere 得来) 因此

$\pi_0(A,x_0)$ is not a grp $\rightsquigarrow$ $\pi_1(X,A,x_0)$ is not a grp.

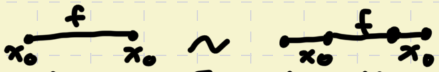
$\pi_1(A,x_0)$ is a grp $\rightsquigarrow$ $\pi_2(X,A,x_0)$ is a grp

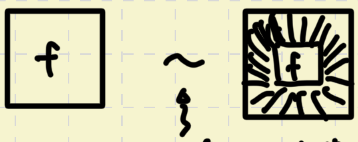
$n\geq 2$, $\pi_n(A,x_0)$ is ab grp $\rightsquigarrow$ $n\geq 3$, $\pi_n(X,A,x_0)$ is ab grp.

- 关于图像

1.  是什么意思?

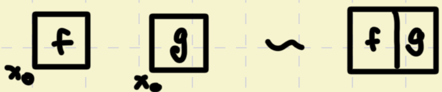
表示 $g: I^2\to X$，$g|_{中间方块}=f$，$g|_{shadow}=x_0$

注意当考虑 I^1 时， 可以用时间(或运动速度)来解释二者的同伦. 但当 $n\geq 2$ 时，时间不再是刻画同伦的好的量，应用密度(或想象成橡皮膜各点处的张力)来描述.



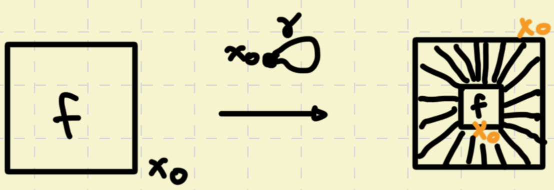
密度变化. (一定不能用速度、时间解释 ……)

把周围橡皮膜卷成一个点，只利用中间部分捏一个和原来 Im f 一样的球.

用球模型也是类似的 

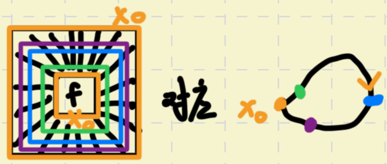
同样大小的橡皮膜(I^2) 现在要捏两个球，张力更大.

2. 考虑 $\pi_1(X,x_0)$ 作用在 $\pi_n(X,x_0)$ 时的图像



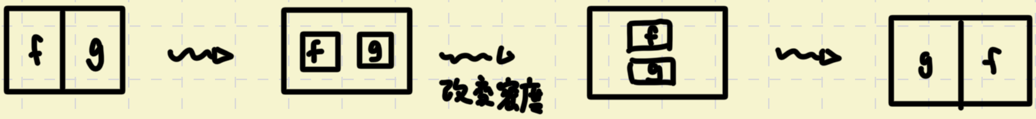
这幅图的含义是：按相同颜色捏.

方块图的 path 对应像里的 path.



对应

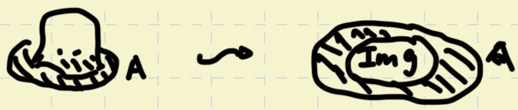
3. 从方块模型看 abelian 的性质.



4. Compression criterion for relative homotopy grp.

$[f]\in\pi_n(X,A,x_0)$. $[f]=0 \Leftrightarrow \exists g$ s.t. $f\overset{H}{\sim}g$ rel S^{n-1} and $\mathrm{Img}\subseteq A$

pf: $\Leftarrow$ key: $\mathrm{Img}\sim D^n$.



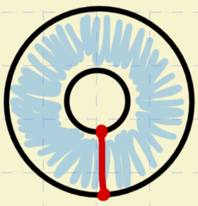
Img is contractible. Let $F:\mathrm{Img}\times I\to X$ be homotopy, $F(\cdot,0)=id$, $F(\cdot,1)=x_0$

Then $f\overset{F\circ H}{\sim}x_0$ so $[f]=0$.

$\Rightarrow$ Suppose $[f]=0$ via homotopy $F:D^n\times I\to X$, $f\sim g$ where g null homotopy

$D^n\times I$

key: Find what's $S^{n-1}\times I$.

e.g.  is 

同伦柱图中的连线代表一点的运动轨迹.

因此考虑 a family of maps $\{H:=F|_{D^n\times\{t\}\cup S^{n-1}\times[0,t]}\}_{t\in[0,1]}$

$H(0)=F|_{D^n\times\{0\}}\sim f$, $H(1)=F|_{D^n\times\{1\}\cup S^{n-1}\times[0,1]}$

$g\sim F|_{D^n\times\{1\}}\sim F|_{D^n\times\{1\}\cup S^{n-1}\times[0,1]}\sim f$ 故 $f\overset{H_t}{\sim}g$ 且 H_t fix S^{n-1}.

$F|_{S^{n-1}\times[0,1]}$ is null homotopy.

图像上看是

$D^n\times\{t\}$ (F map 过去后是 t时刻 D^n 的像)

$S^{n-1}\times I$ 被 F map过去之后是 $F(D^n\times\{t\})$ 边缘每一点运动回 0 时刻 S^{n-1} 的像

这种瓶盖同伦可以固定边缘点.

- $p:(\tilde{X},\tilde{x}_0) \longrightarrow (X, x_0)$ is a covering space. Then,

  $p_*: \pi_n(\tilde{X},\tilde{x}_0) \longrightarrow \pi_n(X,x_0)$ is an iso for $n \geq 2$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ S^n \arrow[ur, "\exists \tilde{f}"] \arrow[r, "f"'] \arrow[ur, phantom, "\Omega"] & X \end{tikzcd}$$

For any $[f] \in \pi_n(X, x_0)$,
$f_* \pi_1(S^n) = 0 \subseteq p_* \pi_1(\tilde{X})$, so we can lift it to $\tilde{f}: S^n \to \tilde{X}$
which lies in $\pi_n(\tilde{X})$. Since $p \circ \tilde{f} = f$, $\tilde{f}$是f在p下的原像.

suppose $f = p\tilde{f} \sim g$ where g is a const map.
w.t.s. $\tilde{f}$ is null homotopic. Suppose $f \overset{H}{\simeq} g$. Then we can lift H to $\tilde{H}$ s.t. $\tilde{f} \overset{\tilde{H}}{\sim} \tilde{g} = \text{const}$, so $\tilde{f}$ is null homotopic.

出现同伦将同伦写出来会有帮助.

- Good example for $\pi_n(X) = \pi_n(Y)$, $\forall n$ but $X \not\simeq Y$.
  $X = S^2 \times \mathbb{R}P^3$, $Y = \mathbb{R}P^2 \times S^3$.

$\pi_n(X) = \pi_n(S^2) \times \pi_n(\mathbb{R}P^3)$, $\pi_n(Y) = \pi_n(\mathbb{R}P^2) \times \pi_n(S^3)$.

When $n \geq 2$, $\pi_n(X) = \pi_n(S^2) \times \pi_n(S^3)$ (Since $S^3 \to \mathbb{R}P^3$ is a covering)

$\pi_n(Y) = \pi_n(S^2) \times \pi_n(S^3)$ (Since $S^2 \to \mathbb{R}P^2$ is a covering)

when $n=1$, $\pi_1(X) = \pi_1(\mathbb{R}P^3) = \pi_1(S^3/\mathbb{Z}_2) \cong \mathbb{Z}_2$.

$\pi_1(Y) = \pi_1(\mathbb{R}P^2) = \pi_1(S^2/\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Hence $\pi_n(X) = \pi_n(Y)$ for all n.
However, $\mathbb{R}P^3$ is orientable while $\mathbb{R}P^2$ not. So $X \not\simeq Y$. $\square$

- $\pi_1(X)$ acts on $\pi_n(X)$.

  $\underline{\underline{\pi_1(A)}}$ acts on $\pi_n(X,A)$

  NOT $\pi_1(X,A)$!

If A is path-connected, then any path γ with $\gamma(0)=x_0$, $\gamma(1)=x_1$ induces an iso

$\beta_\gamma: \pi_n(X,A,x_0) \to \pi_n(X,A,x_1)$

by

When $x_0 = x_1$, we have $\pi_1(A)$ acting on $\pi_n(X,A,x_0)$ $\square$

- Hatcher P346.

TFAE: (1) Every map $S^i \to X$ is null homotopic

(2) Every map extends to a map $D^{i+1} \to X$

(3) $\pi_i(X, x) = 0$ for all $x \in X$.

* X is n-connected if one of these conditions holds for $\forall i \leq n$.

Pf: (1)$\Leftrightarrow$(3), obviously.

(1)$\Rightarrow$(2) Assume $f: S^i \to X$ is null homotopic. Then there exists $\phi: S^i \times I \to X$ satisfying $\phi_0 = f$, $\phi_1 = C_x$ (where C_x means const map)

Since $\phi_1 = C_x$, we have induced map $\bar{\phi}: S^i \times I / S^i \times \{1\} = CS^i \to X$.

Since CS^i homotopy to D^{i+1} fixed S^i, denoted by $g: D^{i+1} \to CS^i$ with $g(S^i) \subseteq S^i \times \{0\}$. Then we define $\tilde{f}: D^{i+1} \to X$ by setting $x \mapsto \bar{\phi} \circ g(x)$. $\tilde{f}|_{S^i} = \bar{\phi} g(S^i) = \bar{\phi}(S^i \times \{0\}) = \phi(S^i \times \{0\}) = \phi_0 = f$. Hence $\tilde{f}: D^{i+1} \to X$ is an extension of $f: S^i \to X$

(2)$\Rightarrow$(1) Assume $f: S^i \to X$ can extends to $\tilde{f}: D^{i+1} \to X$.

Since $\tilde{f}: D^{i+1} \to X$ is null homotopic, i.e., there exists ϕ_t s.t. $\phi_0 = \tilde{f}$, $\phi_1 = C_x$. Let $\psi = \phi|_{S^i}: S^i \times I \to X$ be restriction. $\psi_0 = \phi_0|_{S^i} = f$, $\psi_1 = C_x$. Hence $f \overset{\psi}{\simeq} C_x$, meaning that f is null homotopic. □

[Rmk] VERY USEFUL. 在证明 S^n上 map可延拓到中间e^{n+1}上只要证map零伦.

[Rmk] ("Kill $[\alpha]$") Let $\alpha: S^n \to X$ corresponding to an $[\alpha] \in \pi_n(X, x_0)$. From above properties, we find that:

$[\alpha] = 0 \Leftrightarrow \alpha$ can extend to $e^{n+1} \to X$.

If we enlarge X with an $(n+1)$-cell e^{n+1} by attaching map S^n, we obtain $X' = X \cup e^{n+1}$. For inclusion $j: X \to X'$, we have $[j_*\alpha] = 0 \in \pi_n(X')$. We say $[\alpha]$ has been killed.

Because $j_*\alpha$ can extend to $e^{n+1} \to X'$ in X'

• 从一个 property 学习 CW complex 的 cell structure 如何运用.

Homotopy extension property (HEP) 

can extend to 

$X\times\{0\}\cup A\times I \longrightarrow Y$ $\qquad$ $X\times I \longrightarrow Y$

CW pair (X, A) has HEP, because CW pairs consists of D^i's, which can deformation to a point! Sketch for proving CW pair has HEP: It suffices to show CW pair (X,A) has deformation retraction $r: X\times I \longrightarrow X\times\{0\}\cup A\times I$, if r exists, then any $f: X\times\{0\}\cup A\times I \longrightarrow Y$ extends to $fr: X\times I \longrightarrow Y$.

We construct $r_n: X_n\times I \longrightarrow X_n\times\{0\}\cup(A_n\cup X_{n-1})\times I$ and concatenates those r_n by performing r_n over $[1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}]$.

Construct r_n: $\quad X_n\times I = [X_n\times\{0\}\cup(A_n\cup X_{n-1})\times I] \cup_\alpha [D^n_\alpha\times I]$

where D^n are n-cells in $X_n\setminus A_n$ Deformation $D^n\times I \to (S^{n-1}\times I)\cup D^n\times\{0\}$

(Since we should fix $X_{n-1}\times I$)

• Although there is no excition thm of homotopy group for general space, we have excition thm for CW-complex!

[Thm] (Excition) $X = A\cup B$ are CW-complexes. Let $C = A\cap B$.

If $\begin{cases}(A, C) \text{ is } m\text{-connected} \\ (B, C) \text{ is } n\text{-connected} \\ C \text{ path-connected}\end{cases}$ $m, n\geq 1$, then

$\pi_i(A, C)\longrightarrow \pi_i(X, B)$ induced by inclusion is

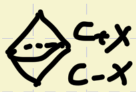
① iso. when $i < n+m$ ② surj. when $i = n+m$.

We use Excition to prove Freudenthal suspension thm.

[Thm] (Freudenthal suspension thm) Let X be $(n-1)$-connected CW complex. $\pi_i(X)\longrightarrow \pi_{i+1}(\Sigma X)$ by $[f]\mapsto[\Sigma f: \Sigma S^i = S^{i+1}\to\Sigma X]$

is ① iso. when $i < 2n-1$ ② surj. when $i = 2n-1$.

pf: (a good exercise for L.E.S.)

Consider $\Sigma X = C_+X\cup C_-X$ 

with $C_+X\cap C_-X = X$.

Apply L.E.S. to $(C_\pm X, X)$, we obtain $\pi_i(X)\cong\pi_{i+1}(C_\pm X, X)$

Apply L.E.S. to $(\Sigma X, C_\pm X)$, we obtain $\pi_i(\Sigma X)\cong\pi_i(\Sigma X, C_\pm X)$

Consider $h_i : \pi_i(\underset{A}{C_+X}, \underset{C}{X}) \to \pi_i(\underset{A\cup B}{\Sigma X}, \underset{B}{C_-X})$.

$(C_\pm X, X)$ is n-connected: $\pi_{i+1}(C_\pm X, X) = \pi_i(X) \overset{X \text{ is } (n-1)\text{-connected}}{=} 0$ when $i \leq n-1$

So h is ① iso when $i < 2n$. ② surj when $i = 2n$

$g : \pi_i(X) \to \pi_{i+1}(\Sigma X)$ is a composition:

$$\pi_i(X) \cong \pi_{i+1}(C_\pm X, X) \xrightarrow{h_{i+1}} \pi_{i+1}(\Sigma X, C_-X) \cong \pi_{i+1}(\Sigma X)$$

So g is ① iso. when $i+1 < 2n$, i.e., $i < 2n-1$

② surj. when $i+1 = 2n$, i.e., $i = 2n-1$. □

• Cellular approximation

[Lemma] $f : X \cup e^n \to Y \cup e^k$ be a map of CW complexes.
If $f|_X : X \to f(X) \subseteq Y$ is cellular and $n < k$, then $f \sim f'$ (rel X) s.t. $\operatorname{Im} f' \subseteq Y$.

[Rmk] It's a crucial lemma. The spirit of this lemma is we can always homotop such f misses higher dimension cells, i.e., $f(e^n) \subseteq Y_n$.
"spirit" of cellular

[Thm] (Cellular approximation) Any map $f : X \to Y$ between CW-complexes has a cellular approximation $f' : X \to Y$ (i.e., $f \sim f'$, f' cellular) Moreover, if $\exists A \subseteq X$ with $f|_A$ cellular, we can take $f'|_A = f|_A$.

pf: We prove by induction. Suppose $f : X \to Y$ satisfying $f|_{X_{n-1}}$ cellular.
For any cell e_n, since $\bar{e}_n$ compact, $f(\bar{e}_n)$ meets finite open cells in Y and so does $f(e_n)$. Let e_k be the highest dimension cell in Y meets $f(e_n)$.
If $k \leq n$, $f(e_n) \subseteq Y_n$. If $k > n$, consider $g := f|_{X_{n-1} \cup e_n} : X_{n-1} \cup e_n \to (Y_n - e^k) \cup e^k$
$g|_{X_{n-1}} = f|_{X_{n-1}}$ is cellular, hence we can homotop g to $g' : X_{n-1} \cup e_n \to Y_n - e^k$ rel X_{n-1}. Do finite steps we obtain $\tilde{f} : X_{n-1} \cup e_n \to Y_n - e^{k_1} - e^{k_2} - \dots - e^{k_s}$ fullfilling $f(e_n)$ misses all cells with dimension $> n$. Then we homotop f to a cellular map $\tilde{f} : X_{n-1} \cup e_n \to Y$, i.e., a homotopy:
$X \times \{0\} \cup [(X_{n-1} \cup e_n) \times I] \to Y$. By HEP we extend it to $\phi : X \times I \to Y$ for which $\phi|_{X_{n-1} \cup e_n}$ can homotop $f|_{X_{n-1} \cup e_n}$ to $\tilde{f} : X_{n-1} \cup e_n \to Y$ (cellular).
We perform those for all n-cells in X and fix n-cells in A (on which f is already cellular), and thus obtain a homotopy $\Psi : X \times I \to Y$ for

which $\varphi|_{X_n}$ can homotop $f|_{X_n}$ to a cellular map on X_n.
If n is infinite, we perform homotop in time interval $[1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}]$
$\square$.

[Rmk] The proof of props of CW-complexes are always prove for one cell. "One cell by one cells" (step by steps) we complete the whole proof.

[Thm] (Relative version cellular approximation) Any map between CW pairs $f:(X,A)\to(Y,B)$ has cellular approximation

pf: Idea: above thm makes sure for homotoping f to a cellular map $f':X\to Y$ with $f(A)$ may not lie in B. Construct a homotopy fixing something can obtained by HEP.

We can homotop $f|_A: A\to B$ to a cellular map $f': A\to B$ by H.
By HEP, we extend $\underset{f}{X\times\{0\}}\cup \underset{H}{A\times I}\longrightarrow Y$ to $\varphi: X\times I\to Y$.
Since $\varphi_0=f$, $\varphi_0|_A=f|_A$, $\varphi_1|_A=f'$, we homotop f to a map φ_1 cellular on A.
By cellular apporximation, we can homotop φ_1 to f'' (rel A), where $f'':X\to Y$ is cellular. $f''|_A=\varphi_1|_A$ is cellular. $\square$

- $[f]\in\pi_n(X,A,x_0)$. $[f]=0 \Leftrightarrow f\sim g$ with $\mathrm{Im}\,g\subseteq A$

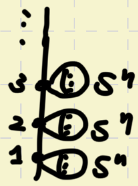
- Homotopy group $\pi_i(S^n)$ when $i\leq n$.

Δ $\pi_i(S^n)=0$ when $i<n$. $f:(S^i,s_0)\to(S^n,s_1)$ corresponds $[f]\in\pi_i(S^n)$.
Consider standard cell structure of S^i and S^n. f has cellular approximation f' with $f'(S^i)\subseteq(S^n)_i=pt$. So $[f]=[f']=0$.

Δ $\pi_n(S^n)=\mathbb{Z}$. Fact: $\pi_1(S^1)=\mathbb{Z}$.
State suspension thm for S^i: $\pi_i(S^n)\to\pi_{i+1}(S^{n+1})$ is
① iso $i<2n-1$ ② surj $i=2n-1$. when $n>1$, $n<2n-1$.
So we have $\mathbb{Z}\cong\pi_1(S^1)\overset{surj}{\twoheadrightarrow}\pi_2(S^2)\overset{\sim}{\to}\pi_3(S^3)\cong\cdots\cong\pi_n(S^n)\cong\cdots$
Consider Hopf fiberation $S^1\hookrightarrow S^3\to S^2$, we have L.E.S.
$\pi_i(S^1)\to\pi_i(S^3)\to\pi_i(S^2)\to\pi_{i-1}(S^1)\to\cdots\to\underset{\overset{\shortparallel}{0}}{\pi_2(S^3)}\to\pi_2(S^2)\to\pi_1(S^1)$
$\to\underset{\overset{\shortparallel}{0}}{\pi_1(S^3)}\to\cdots$ so $\pi_2(S^2)\cong\pi_1(S^1)=\mathbb{Z}$ and thus $\pi_n(S^n)\cong\mathbb{Z}$

- Homotopy grps of finite CW-complex can be infinitely generated.

Show $\pi_n(S^1 \vee S^n) = \mathbb{Z}[t, t^{-1}]$.

$\widetilde{S^1 \vee S^n}$, the universal covering, is 

attach each $z \in \mathbb{Z}$ an S^n!

Clearly, $\widetilde{S^1 \vee S^n} \sim \bigvee_{\alpha \in \mathbb{Z}} S^n_\alpha$. Hence $\pi_n(S^1 \vee S^n) = \pi_n(\widetilde{S^1 \vee S^n}) = \pi_n(\vee S^n_\alpha)$

[Lemma] $\pi_n(\vee_\alpha S^n_\alpha) = \langle f_\alpha : S^n \to S^n_\alpha \hookrightarrow \vee S^n_\alpha \mid \alpha \in \mathbb{Z} \rangle$ (Free abelian group generated by $f_\alpha : S^n \to S^n_\alpha \hookrightarrow \vee S^n_\alpha$)

pf: Suppose there are finite $\{S^n_\alpha\}$. Regard $\vee S^n_\alpha$ as n-skeleton of $\prod_\alpha S^n_\alpha$ (e.g., $S^1 \vee S^1$ is 1-skeleton of $S^1 \times S^1$).

Since $S^n_\alpha = e^0_\alpha \cup e^n_\alpha$, $\prod_\alpha S^n_\alpha$ has cells $\begin{cases} \text{0-cell } \prod_\alpha e^0_\alpha \\ \text{n-cells } \left(\prod_{\alpha \neq \beta} e^0_\alpha\right) \prod e^n_\beta \\ \vdots \end{cases}$ $\Big]$ n-skeleton of $\prod_\alpha S^n_\alpha = \bigvee_\alpha S^n_\alpha$

Hence $\prod_\alpha S^n_\alpha \setminus (\vee_\alpha S^n_\alpha)$ has dimension $\geq 2n$. So $\pi_i(\prod_\alpha S^n_\alpha, \vee_\alpha S^n_\alpha) = 0$ when $i \leq 2n-1$.

Idea: homotopy grps $\pi_i(\prod_\alpha S^n_\alpha, \vee_\alpha S^n_\alpha)$, $\pi_i(\prod_\alpha S^n_\alpha)$, $\pi_i(\vee_\alpha S^n_\alpha)$ have relations! (L.E.S.)

By L.E.S., $\pi_i(\vee_\alpha S^n_\alpha) \cong \pi_i(\prod_\alpha S^n_\alpha) = \prod_\alpha \pi_i(S^n_\alpha) \overset{\text{finite}\{S^n_\alpha\}}{=} \bigoplus \pi_i(S^n_\alpha) = \bigoplus \mathbb{Z}$.

Suppose there are infinite $\{S^n_\alpha\}$. Consider homomorphism

$$\Phi : \bigoplus_\alpha \pi_n(S^n_\alpha) \longrightarrow \pi_n(\vee_\alpha S^n_\alpha)$$
$$\{f_\alpha : S^n \to S^n_\alpha\}_\alpha \longmapsto \begin{bmatrix} f : S^n \to \vee_\alpha S^n_\alpha \\ \text{s.t. } f|_{S^n_\alpha} = f_\alpha \end{bmatrix}$$

Any $f : S^n \to \vee_\alpha S^n_\alpha$ has compact image which contained in finite S^n_α's of $\vee_\alpha S^n_\alpha$. By finite case Φ is onto. Similarly, a null homotopic $g \in \bigoplus_\alpha \pi_n(S_\alpha)$ also contained in finite S^n_α's, Φ is inj by finite case. □

$\pi_n(\vee_\alpha S^n_\alpha) = \langle f_\alpha : S^n \to S^n_\alpha \to \vee_\alpha S^n_\alpha \mid \alpha \in \mathbb{Z} \rangle$ is a $\mathbb{Z}[\pi_1(\vee_\alpha S^n_\alpha)]$-module.

Since all S^n_α's are identified under π_1 actions, hence $\pi_n(\vee_\alpha S^n_\alpha)$ is a free $\mathbb{Z}[\pi_1]$-module of rank 1, i.e.,

$$\pi_n(\vee_\alpha S^n_\alpha) = \mathbb{Z}[\pi_1(\vee_\alpha S^n_\alpha)] \cong \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$$

$$\begin{aligned} 1 &\longmapsto t \\ -1 &\longmapsto t^{-1} \\ n &\longmapsto t^n \end{aligned}$$

. □

- Homotopy group equals zero, what does this mean?

[Lemma] (Compression lemma) Let (X,A) be a CW pair, (Y,B) be any pair with $B \neq \phi$. For each n $X \setminus A$ has cells of dimension n, assume that $\pi_n(Y,B,y_0)=0$, $\forall y_0 \in B$. Then every map $f:(X,A) \to (Y,B)$ is homotopic rel A to a map $X \to B$.

Sketch of proof: Assume inductively $f(X_{k-1} \cup A) \subseteq B$. ($f(A) \subseteq B$) is obvious) $e^k \subseteq X \setminus A$ is a k-cell with characteristic map $d:(D^k, S^{k-1}) \to (X, X_{k-1} \cup A)$. View $[d] \in \pi_k(X, X_{k-1} \cup A)$. Since $f(X_{k-1} \cup A) \subseteq B$, rewrite $f:(X, X_{k-1} \cup A) \to (Y,B)$. $f_*[d]=0$ since $\pi_k(Y,B)=0$ (regardless base points). $f_*[d]=0$ means we have homotopy $H:(D^k, S^{k-1}) \times I \to (Y,B)$ s.t. $H_0 = f_*[d] = f \circ d$, $Im(H_1) \subseteq B$, i.e., Use HEP, we obtain a homotopy on X which homotop f to f' with $f'(A \cup X_{k-1} \cup e_k) \subseteq B$. Do above steps for all k-cells in A we obtain a map $g \sim f$ with $g(A \cup X_k) \subseteq B$. □

[Rmk] From proof of compression lemma, we find homotopy grp of target space vanishes can be used to compress cells of X a to B. □

Application of compression lemma:

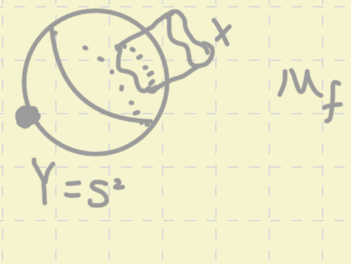
[Thm] (Whitehead thm) X, Y are CW pairs. If there exists $f: X \to Y$ s.t. X, Y are weak homotopy equiv then X, Y are homotopy equiv.

Sketch of proof: Case I: X can inclusion to Y. Then $\pi_n(X) \cong \pi_n(Y)$, in L.E.S. means $\pi_n(Y,X)=0$, $\forall n$. Consider $id:(Y,X) \to (Y,X)$, use compression lemma, we get retraction $r: Y \to X$.

Case II: Use mapping cylinder $M_f = X \times I \amalg Y / (x,1) \sim f(x)$. $M_f \sim Y$, so it suffices to show $M_f \sim X$. Homotop f to a cellular map, making X a subcomplex of M_f, and then (M_f, X) is a CW pair.

(When f cellular, n-skeleton of X are taken to n-skeleton of Y)

(When f is not cellular, M_f may not be a CW pair. e.g.



[Rmk] 'f induced iso of homotopy grps' 这个条件证明中哪里用到?

Case II reduce to Case I, we need to show $\pi_n(X, M_f)=0$, which follows from $\pi_n(X) \overset{i_X^*}{\cong} \pi_n(M_f)$ (iso induced by inclusion) Since

$\pi_n(X) \boxed{\overset{f_*}{\cong}} \pi_n(Y) \overset{i_Y^*}{\cong} \pi_n(M_f)$. ($i_X^* = i_Y^* f_*$, since $f: X \xrightarrow{i} M \xrightarrow{r} Y$, $X\times\{0\}$ retraction.

it induce
$$\begin{tikzcd} \pi_n(X) \arrow[r, "i_X^*"] \arrow[dr, "f_*"'] & \pi_n(M) \arrow[d, shift left, "r_*"] \\ & \pi_n(Y) \arrow[u, shift left, "i_Y^*"] \end{tikzcd}$$
Since $Y \underset{r}{\overset{i_Y}{\rightleftarrows}} M$ are homotopy equivalence, $r_*^{-1} = i_Y^*$.

Hence $i_Y^* f_* = i_X^*$)

把这个 iso 换掉就用不了 L.E.S. 得到 $\pi_n(M_f, X)=0$ 了.

所以 f induced iso 是由 L.E.S. 中 $\pi_n(X) \xrightarrow{i_*} \pi_n(Y)$ is induced by inclusion 所限制得来.

- Show $\pi_i(S^\infty)=0$. $S^\infty = \bigcup_{i=0}^{\infty} S^n$. Let $[\alpha] \in \pi_i(S^\infty)$, where $\alpha: S^i \to S^\infty$. Homotop α to a cellular map β s.t. $\beta: S^i \to S^n$ for $i<n$. Since $\pi_i(S^n)=0$, $[\alpha]=[\beta]=0$.
  Besides S^∞ is a covering of $\mathbb{R}P^\infty$.

- n-connected CW model and CW approximation.

[Def] (X, A) : X any space, $\emptyset \neq A$ CW complex.
An n-connected CW model is a CW pair (Z, A) with $f: Z \to X$ satisfying ① $f|_A = id_A$ ② $f_*: \pi_i(Z) \to \pi_i(X)$ is $\begin{cases} \text{iso} & i>n \\ \text{inj} & i=n \end{cases}$ □

△ n-connected CW model 是一种截断近似.

(X, A) CW: $\pi_0(X)$, $\pi_1(X)$, $\pi_2(X)$, $\vdots$, $\pi_n(X)$, $\pi_{n+1}(X)$, $\vdots$

Find CW pair (Z, A), $f: Z \to X$

$\pi_0(Z)=0$, $\pi_1(Z)=0$, $\pi_2(Z)=0$, $\vdots$

$f^*: \pi_n(Z) \xrightarrow{\text{inj}} \pi_n(X)$

iso: $\pi_{n+1}(Z) \overset{f^*}{\cong} \pi_{n+1}(X)$, $\vdots$

△ Pick $A = pt$, $n=0$ we obtain CW approximation of X.

- $[\alpha] \in \pi_i(X)$ ① can be regarded as $\alpha: S^i \to X$
  ② can be regarded as a characteristic map if X is a CW complex.

- 如何用 CW complex 近似任何一个空间 X？从下面定理的证明可以看到具体的构造.

[Thm] Any pair (X,A), $\Phi \neq A$ is a CW complex, there exists n-connected models (Z,A) of (X,A).

Sketch of proof: Idea: build Z by attach k-cells with A, $k > n$.

$$\begin{tikzcd} A: Z_n \arrow[d, "f_n"'] \arrow[r, phantom, "\subseteq"] & Z_{n+1} \arrow[dl, "f_{n+1}"] \arrow[r, phantom, "\subseteq \cdots \subseteq"] & Z_k \arrow[dll, "f_k"] \arrow[r, phantom, "\subseteq"] & \cdots \\ X & & \cdots & \end{tikzcd}$$

Assume we've constructed $f_k: Z_k \to X$ s.t. ① $f_k|_A = id_A$ ② $f_{k*}: \pi_i(Z_k) \to \pi_i(X)$ is
$$\begin{cases} \text{inj} & i = n \\ \text{iso} & n < i < k \\ \text{sur} & i = k \end{cases}$$

We only show the construction here and omit inj, iso, surj check.

$Z_{k+1} = Z_k \cup \underbrace{e_\alpha^{k+1}}_{\text{part I}} \cup \underbrace{S_\beta^{k+1}}_{\text{part II}}$.

part I: pick $\underline{\{\phi_\alpha\}_\alpha = \ker(f_{k*})}$

$\phi_\alpha \in \ker f_{k*} \subseteq \pi_k(Z_k)$, so $\phi_\alpha: S^k \to Z^k$ can be extended to $D^{k+1} \to Z^k$. Hence f_k can be extended to $g: Y_{k+1} \to X$, where $Y_{k+1} = Z_k \cup e_\alpha^{k+1}$.

Part II: Let $\underline{\{\gamma_\beta: S^{k+1} \to X\}}$ be generators of $\pi_{k+1}(X)$.

Let $Z_{k+1} = Y_{k+1} \vee S_\beta^{k+1}$. Then $g: Y_{k+1} \to X$ can be extended to $f_{k+1}: Z_{k+1} \to X$ by $f_{k+1}|_{S_\beta^{k+1}} = \gamma_k$.

[Rmk] ker 与 generators 近似刻画 X 的 CW 结构.

- Tower obtained from CW-approximation.

[Prop] (Functority of CW models) $f: (Z, A) \to (X, A)$ is an n-connected CW model; $f': (Z', A') \to (X', A')$ is an n'-connected CW model; $n \geq n'$. Then for $g: (X,A) \to (X',A')$, there exists unique (up to homotopy) $h: (Z, A) \to (Z', A')$ with $h|_A = g|_A$ and diagram commutes up to homotopy equivalence:

$$\begin{tikzcd} (X,A) \arrow[r, "g"] & (X',A') \\ (Z,A) \arrow[u, "f"] \arrow[r, "h"'] & (Z',A') \arrow[u, "f'"'] \end{tikzcd}$$

△ Tower : $\mathrm{id}: (X,A) \to (X,A)$ induces maps

$$\begin{tikzcd}
\boxed{(X,A)} \arrow[r, "\mathrm{id}"] & \boxed{(X,A)} \\
(Z^n,A) \arrow[u, "f^n"] \arrow[r, "g_n"] & (Z^{n-1},A) \arrow[u, "f^{n-1}"]
\end{tikzcd}$$

n-connected CW model   (n-1)-connected CW model

Then we have a tower

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & Z^2 \arrow[d, "g_2"] \arrow[ddr, "f^2"] & \\
 & Z^1 \arrow[d, "g_1"] \arrow[dr, "f^1"] & \\
A \arrow[r, hook] \arrow[ur] \arrow[uur] & Z^0 \arrow[r, "f^0"'] & X
\end{tikzcd}$$

Right triangles commute up to homotopy equivalence ;

Left triangles commutes.

Postnikov tower is the tower of (CX, X) , we denote $X^i := Z^i$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & X^1 \arrow[d] \arrow[dr] & \\
X \arrow[r] \arrow[ur] \arrow[uur] & X^0 \arrow[r] & CX
\end{tikzcd}$$

$\pi_i(X^n) \cong \pi_i(CX) = 0, \quad i > n$

$\pi_i(X^n, X) = 0 \quad i \le n$

n-connected

i.e., $\pi_i(X^n) \cong \pi_i(X)$, $i \le n$.

Whitehead tower is the tower of (X, pt)

$$\begin{tikzcd}
\vdots \arrow[d] & \\
Z^1 \arrow[d] \arrow[dr] & \\
Z^0 \arrow[r] & X
\end{tikzcd}$$

$\pi_i(Z^n, *) = 0, \quad i \le n \Rightarrow \pi_i(Z^n) = 0$

$\pi_i(Z^n) \cong \pi_i(X)$, $i > n$

- An example for using cone CA : See following prop (similar to $\tilde{H}(X,A) \cong \tilde{H}(X/A)$)

[Prop] CW-pair (X,A) is r-connected $(r \ge 1)$ and A is s-connected $(s \ge 0)$. $q: X \to X/A$ be projection. Then the induced $q_*: \pi_i(X,A) \to \pi_i(X/A)$ is ① iso , $i \le r+s$ ② onto , $i = r+s+1$

pf: Let $Y = X \cup_A CA$. Since CA is contractible, $Y \to Y/CA \cong X/A$ is homotopy equiv , hence $\pi_i(X/A) \cong \pi_i(Y)$.

Factor q_* as following:

$q_*: \pi_i(X,A) \xrightarrow{i_*} \pi_i(Y, CA) \xleftarrow{\simeq} \pi_i(Y) \cong \pi_i(X/A)$

iso because L.E.S.

(X,A) is r-connected ; (CA, A) is $s+1$-connected (L.E.S.).

Therefore by excition, i_* is ① iso, $i \le r+s+1$ ② onto, $i = r+s+1$ and so does q_*.

- Topo space and group

[Prop] Assume $n \ge 2$. If $X = \bigvee_\alpha S^n_\alpha \cup \bigcup_\beta e^{n+1}_\beta$ is obtained from $\bigvee_\alpha S^n_\alpha$ by attaching $(n+1)$-cells e^{n+1}_β via basepoint-preserving maps $\varphi_\beta : S^n_\beta \to \bigvee_\alpha S^n_\alpha$, then

$$\pi_n(X) = \pi_n(\bigvee_\alpha S^n_\alpha)/\langle \varphi_\beta \rangle = (\bigoplus_\alpha \mathbb{Z})/\langle \varphi_\beta \rangle$$

pf: Consider L.E.S.

cellular appr.

$$\cdots \to \pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \xrightarrow{\partial} \pi_n(\bigvee_\alpha S^n_\alpha) \to \pi_n(X) \to \pi_n(X, \bigvee_\alpha S^n_\alpha) = 0$$

Hence $\pi_n(X) \cong \pi_n(\bigvee_\alpha S^n_\alpha)/\mathrm{Im}\,\partial$

Since $X \setminus \bigvee_\alpha S^n_\alpha \cong \bigvee_\beta S^{n+1}_\beta$,

$$\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \cong \pi_{n+1}(X/\bigvee_\alpha S^n_\alpha) \cong \pi_{n+1}(\bigvee_\beta S^{n+1}_\beta)$$

$n+1 < n+n-1$ by above prop

generated by

$$\left[\Phi_\beta : S^{n+1} \to S^{n+1}_\beta \hookrightarrow \bigvee_\beta S^{n+1}_\beta\right]$$

Φ_β corresponds to φ_β in $\pi_n(\bigvee_\alpha S^n_\alpha)$, since Φ_β  S^{n+1} is obtained by $e^{n+1}_\beta / S^n_\alpha$.

[Exp] For any abelian group $G = \langle g_\alpha \mid r_\beta \rangle$. Let $X = \bigvee_\alpha S^n_\alpha \cup \bigcup_\beta e^{n+1}_\beta$. s.t. generator of $\pi_n(\bigvee_\alpha S^n_\alpha)$ are $g_\alpha : S^n \to S^n_\alpha \hookrightarrow \bigvee_\alpha S^n_\alpha$.

$\varphi_\beta : S^n_\beta \to \bigvee_\alpha S^n_\alpha$ be characteristic maps s.t. $\varphi_\beta = r_\beta$.

e.g. $\langle a, b, c \mid a+b \rangle$

$\rightsquigarrow$ attach 2-cell along $S^1 \to$ loop $a+b$.

[Thm] For $\begin{cases} n \ge 1 \\ \text{any } G \end{cases}$ or $\begin{cases} n \ge 2 \\ \text{ab } G \end{cases}$ there exists $K(G, n)$

pf (sketch): Construct X_{n+1} s.t. $\pi_n(X_{n+1}) = G$, $\pi_i(X_{n+1}) = 0$, $i < n$. We want to construct X_{n+2} s.t. $\pi_i(X_{n+2}) = \pi_i(X_{n+1})$, $i \le n$, $\pi_{n+1}(X_{n+2}) = 0$.

Let $\{\varphi_\gamma\}$ be generators of $\pi_{n+1}(X_{n+1})$. Let $X_{n+2} = X_{n+1} \bigcup_{\varphi_\gamma} e^{n+2}_\gamma$ satisfying the condition. Iterate this step and pick the union.

				π_n	π_{n+1}		
X_{n+1}	0	$\cdots$	0	G	?	?	$\cdots$
X_{n+2}	0	$\cdots$	0	G	0	?	$\cdots$
π_{n+3}	0	$\cdots$	0	G	0	0	$\cdots$

$\downarrow$ the limit (union) is X.

[Coro] For an ab grp seq $\{G_n\}_{n\geq 2}$, there exists X s.t. $\pi_n(X)\cong G_n$.

pf: Let $X^n = K(G,n)$. Let $X=\prod_n X^n$.

$\pi_0 X = (G_0, 0, 0, \cdots)$

$\pi_1 X = (0, G_1, 0, \cdots)$

$\pi_2 X = (0, 0, G_2, \cdots)$

$\pi_i(K(G_0,0))$ $\pi_i(K(G_1,1))$

- The homotopy type of a <u>CW complex</u> $K(G,n)$ is determined uniquely by G and n.

- Hurewicz thm: 已知 homotopy n-conn.信息, 计算 homology grp.

[Thm] (Hurewicz) X is $(n-1)$-connected, $n\geq 2$.

then $\tilde{H}_i(X) = \begin{cases} 0 & i<n \\ \pi_n(X) & i=n \end{cases}$

(X,A) is $(n-1)$-connected with $n\geq 2$ and $\pi_1(A)=0$

$H_i(X,A) = \begin{cases} 0 & i<n \\ \pi_n(X,A) & i=n \end{cases}$

[Rmk] We can not know any inf. about relationship between $\pi_i(X)$ and $H_i(X)$ when $i>n$. Here are two good examples:

① Trivial homology, nontrivial homotopy: S^n. $\pi_i(S^n)$ may nontrivial when $i>n$, but $H_i(S^n)=0$ when $i>n$

② Nontrivial homology, trivial homotopy: $\mathbb{C}P^\infty = K(\mathbb{Z},2)$. $\pi_i(\mathbb{C}P^\infty)=0$ when $i>2$. $H_i(\mathbb{C}P^\infty)$ may nontrivial when $i>2$ (Computed by spectral seq)

pf:

[prop] $f: X\to Y$ is weak homotopy equivalence. Then

$f_*: H_i(X;G)\to H_i(Y;G)$ and $f^*: H^i(X;G)\to H^i(Y;G)$ are iso.

(Prop 4.21 in Hatcher)

This prop allows us consider CW complexes in the proof.

e.g., in proof of Hurewicz thm, we only consider CW cases

$\left(\underset{CW}{f: Z} \xrightarrow{w.h.e} X. \quad \tilde{H}_i(X)\cong H_i(Z) = \begin{cases} 0 & i<n \\ \pi_n(Z)=\pi_n(X) & i=n \end{cases} \right)$

Assume X and (X,A) are CW-complexes. Reduce relative case to absolute case: $\pi_i(X,A) \overset{i\le n-1+1=n}{\cong} \pi_i(X/A) \cong \tilde{H}_i(X/A) \cong H_i(X,A)$.

(X,A) $(n-1)$-connected, A 1-connected

So it suffices to show absolute case.

Recall prop "Any n-connected (X,A) homotopy equiv to (Z,A) rel A where Z is built from A by attaching k-cells, $k>n$." (Coro in CW appro)

Fix $x_0 \in X$. X is $(n-1)$-connected and so is (X,x_0). We can assume X is built from x_0 by attaching k-cells, $k>n-1$, i.e., X has cell structure: $X_{n-1}=X_{n-2}=\cdots=X_0=x_0$, $X_n=?$, $X_{n+1}=?$ ...

Thus $\tilde{H}_i(X)=0$, $i<n$. Disregard S^{n+1}_α and e^k $k\geq n+2$ in X has no effect on π_n or H_n. Hence we assume $X=\bigvee_{x_0,\alpha} S^n_\alpha \cup e^{n+1}_\beta$.

$\pi_n(X) = \bigoplus \mathbb{Z}/\langle\varphi_\beta\rangle \overset{\text{By computation}}{=} H_n(X)$.

[Rmk] In a very special case $X=\bigvee S^n_\alpha \cup e^{n+1}_\beta$ we have $\pi_i(X)\cong\tilde{H}_i(X)$, any general case will reduction to this special case. That's key point of proving Hurewicz thm.

[Rmk] $X=S^2\times\mathbb{R}P^3$, $Y=\mathbb{R}P^2\times S^3$. Since $H_*(X)\ncong H_*(Y)$, there is no weak homotopy equiv $f:X\to Y$. It's an application for "w.h.e induces isos on π_i, H_i, H^i".

- Fibers of fibrations are homotopy equivalence while fibers of fiber bundles are homeomorphic.
  (by def $p^{-1}(U_b) \overset{h}{\cong} U_b\times F$, both over U_b)
  prop 4.65 in Hatcher

- [Thm] (Hurewicz) Fiber bundles over paracompact spaces are fibrations.

- Fiber bundle $S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$

$(z_0,\cdots,z_n)\mapsto[z_0:\cdots:z_n]$

$S^1 \hookrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$ By cellular approximation S^∞ is contracitible.

Hence L.E.S.

$$\cdots \longrightarrow \pi_i(S^\infty) \longrightarrow \pi_i(\mathbb{C}P^\infty) \longrightarrow \pi_{i-1}(S^1) \longrightarrow \pi_{i-1}(S^\infty) \longrightarrow \cdots$$

implies $\pi_i(\mathbb{C}P^\infty)\cong\pi_{i-1}(S^1)$. Hence $\pi_i(\mathbb{C}P^\infty)=\begin{cases}\mathbb{Z} & i=2\\ 0 & \text{o/w}\end{cases}$

Therefore $\mathbb{C}P^\infty = K(\mathbb{Z},2)$

- Any map can homotopic to a fibration.

此构造与universal covering的构造是类似的. Let $f: A \to B$.

Δ Define $E_f = \{(a, \gamma) \mid a \in A,\ \gamma: [0,1] \to B \text{ with } \gamma(0) = f(a)\}$

It's a topo space with compact open topo.

Δ Define inclusion $i: A \to E_f$ $\quad a \mapsto (a, C_{f(a)})$ where $C_{f(a)}$ is const loop.

i is a homotopy equivalence

($\pi: E_f \to a$ be projection. $i \circ \pi = id_A$.

$H: E_f \times I \longrightarrow E_f$ $\quad H(0) = \pi \circ i$

$((a,\gamma), t) \mapsto (a, I \to B, s \mapsto \gamma(st))$ $\quad H(1) = id_{E_f}$ $\quad$ so $\pi \circ i \sim id_{E_f}$.)

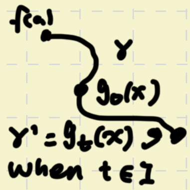
Δ Define projection $p: E_f \longrightarrow B$, $(a, \gamma) \mapsto \gamma(1)$

p is a fibration: The following shows p satisfying H.L.P.

$$\begin{tikzcd} X \arrow[r, "\tilde{g}_0"] \arrow[d, "id"'] & E_f \arrow[d, "p"] \\ X \arrow[r, "g_t"'] \arrow[ur, dashed, "\tilde{g}_t"] & B \end{tikzcd}$$

Construction for $\tilde{g}_t$ for any $x \in X$.

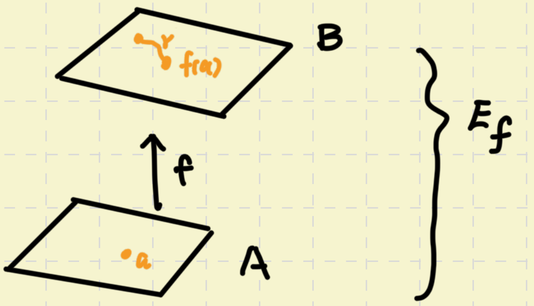
$\tilde{g}_0(x) = (a, \gamma)$ with $\gamma(0) = f(a)$, $\gamma(1) = g_0(x)$.

$\tilde{g}_t(x) = (a,$  $)$

Δ f factor through E_f,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i", "\text{h.e}"'] & B \\ E_f \arrow[ur, "p"'] & \end{tikzcd}$$ fibration. hence f homotopic to a fibration.

[Rmk] Picture for E_f.



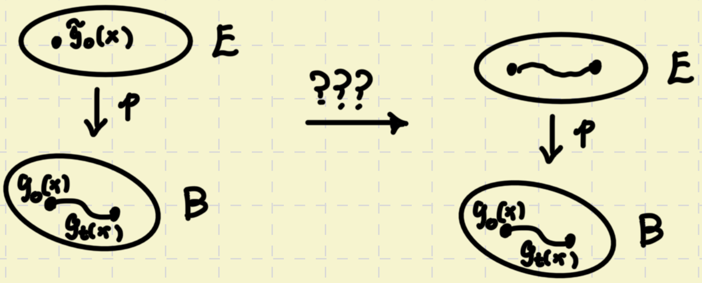
is an element in E_f.

[Rmk] How does this construction work?

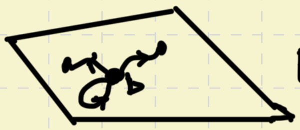
picture of HLP:

$\forall x \in X$ $$\begin{tikzcd} X \arrow[r, "\tilde{g}_0"] \arrow[d, equal] & E \arrow[d, "p"] \\ X \arrow[r, "g_t"'] \arrow[ur, "\tilde{g}_t"] & B \end{tikzcd}$$

given a path $x \frown g_t(x)$ in B and $\tilde{g}_0(x)$ in E can we lift it to a path in E? (If the answer is yes, we say it satisfies HLP w.r.t. X)



- Let $f: A = \{b\} \hookrightarrow B$. $E_f =: PB$ is called the path space of B



PB

The above fibration is $\Omega B \hookrightarrow PB \to B$ (with $\Omega B = p^{-1}(b)$, h.e.)
where ΩB is the space of all loops in B base at b and $PB \to B$, $(b, \gamma) \mapsto \gamma(1)$

Since PB is contractible, by L.E.S. $\pi_i(B) \cong \pi_{i-1}(\Omega B)$.

A useful tool for computations.

[prop] (Puppé seq) Given a fibration $F \hookrightarrow E \to B$, there is a sequence of maps

$$\cdots \to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

with any two consecutive maps forming a fibration.

- Stable homotopy grp $\pi_i(O(n))$

Consider fiber bundle $V_{n-m}(\mathbb{R}^{k-m}) \to V_n(\mathbb{R}^k) \to V_m(\mathbb{R}^k)$

pick $k = n$, $m = 1$, $V_{n-1}(\mathbb{R}^{n-1}) \cong O(n-1)$ $(V_n(\mathbb{R}^n) = O(n))$

$V_n(\mathbb{R}^n) \cong O(n)$, $V_1(\mathbb{R}^n) \cong S^{n-1}$. Hence we obtain a fiber bundle

$O(n-1) \longrightarrow O(n) \to S^{n-1}$. Since S^∞ is contractible,

hence $\pi_i(O(n))$ is independent of n when n large by L.E.S.

- [Thm] Let X be a path connected space. Then the abelianization of $\pi_1(X, x_0)$ is iso. to $H_1(X)$.

- 完整版 Serre's thm (刻画了 fibration $F \hookrightarrow E \to B$ 三者同调群关系.)

Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration with $\pi_1(B) = \pi_0(E) = 0$. Then there is a first quadrant spectral seq with E^2-page

$$E^2_{p,q} = H_p(B; H_q(F)) \text{ which converges to } H_*(E),$$

i.e., there exists a fibration $H_n(E) = D_{n,0} \supseteq D_{n-1,1} \supseteq \cdots \supseteq D_{-1,n+1} = 0$

s.t. $E^\infty_{p,q} = D_{p,q} / D_{p-1,q+1}$

(a) We have commutative diagram

$$\begin{tikzcd}
H_p(B) = E^2_{p,0} \supseteq \ker d^2_{p,0} = E^3_{p,0} \supseteq \ker d^3_{p,0} = E^4_{p,0} \supseteq \cdots \supseteq \ker d^p_{p,0} = E^{p+1}_{p,0} \\
\vdots \arrow[u, "="] \\
E^\infty_{p,0} \arrow[u, "="] \\
H_p(E)/D_{p-1,1} \arrow[u, "="] \\
H_p(E) \arrow[u, two heads] \arrow[uuuu, bend left, "\pi_*"]
\end{tikzcd}$$

i.e., the composition $H_p(E) \twoheadrightarrow E^\infty_{p,0} \subseteq E^2_{p,0} = H_p(B)$, called edge homomorphism, coincides with $\pi_* : H_p(E) \to H_p(B)$.

(b) We have commutative diagram

$$\begin{tikzcd}
H_q(F) = E^2_{0,q} \arrow[r, two heads] \arrow[rrrddddd, "i_*"] & E^3_{0,q} = H_q(F)/\mathrm{Im}\, d^2 \arrow[r, two heads] & \cdots \arrow[r, two heads] & E^{q+2}_{0,q} \arrow[d, "="] \\
 & & & \vdots \arrow[d, "="] \\
 & & & E^\infty_{0,q} \arrow[d, "="] \\
 & & & D_{0,q} \arrow[d, hook] \\
 & & & H_q(E)
\end{tikzcd}$$

(c) Image of the Hurewicz map $h^n_B : \pi_n(B) \to H_n(B)$ is contained in $E^n_{n,0}$, which is called the grp of transgression elements.

We have commutative diagram:

$$\begin{tikzcd}
\pi_n(B) \arrow[r, "h^n_B"] \arrow[d, "\partial"', "\text{L.E.S.}"] & H_n(B) = E^2_{n,0} \supseteq \cdots \supseteq E^n_{n,0} \arrow[d, "d^n"] \\
\pi_{n-1}(F) \arrow[r, "h^{n-1}_F"] & H_{n-1}(F) = E^2_{0,n-1} \twoheadrightarrow \cdots \twoheadrightarrow E^n_{0,n-1}
\end{tikzcd}$$

[Rmk] $H_*(F) \xrightarrow{i_*} H_*(E) \xrightarrow{p_*} H_*(B)$ has decomposition:

$$
\begin{tikzcd}
H_n(F)=E^2_{0,n} \arrow[r, two heads] \arrow[dddrr, "i_*"'] & \cdots \arrow[r, two heads] & E^{n+2}_{0,n} \arrow[d, "="] & & & \\
& & E^{\infty}_{0,n} \arrow[d, "="] & & & \\
& & D_{0,n} \arrow[d, hook] & & & \\
& & H_n(E) \arrow[r, two heads] \arrow[ddddrrr, "\pi_*"'] & H_n(E)/D_{n-1,1} = E^{\infty}_{n,0} \arrow[r, "="'] & \cdots \arrow[r, "="'] & E^{n+1}_{n,0} \arrow[d, phantom, sloped, "\subseteq"] \\
& & & & & E^{n}_{n,0} \arrow[d, phantom, sloped, "\subseteq"] \\
& & & & & \vdots \arrow[d, phantom, sloped, "\subseteq"] \\
& & & & & E^{2}_{n,0} \arrow[d, phantom, sloped, "="] \\
& & & & & H_n(B)
\end{tikzcd}
$$