8.1

T9

$$\int \frac{1}{e^{z}+e^{-z}}\,dz$$

$$= \int \frac{e^{z}}{(e^{z})^{2}+1}\,dz$$

$$= \int \frac{de^{z}}{(e^{z})^{2}+1}$$

$$= \arctan e^{z} + C$$

T27

$$\int \frac{2\,dx}{x\sqrt{1-4\ln^{2}x}}$$

$$= \int \frac{d2\ln x}{\sqrt{1-(2\ln x)^{2}}}$$

$$= \arcsin 2\ln x + C$$

T33

$$\int_{-1}^{0} \sqrt{\frac{1+y}{1-y}}\,dy$$

$$= \int_{-1}^{0} \frac{1+y}{\sqrt{1-y^{2}}}\,dy \quad (y \geq -1)$$

$$= \int_{-1}^{0} \frac{1}{\sqrt{1-y^{2}}}\,dy + \int_{-1}^{0} \frac{y}{\sqrt{1-y^{2}}}\,dy$$

$$= \arcsin y\Big|_{-1}^{0} - \frac{1}{2}\int_{-1}^{0} \frac{d(1-y^{2})}{\sqrt{1-y^{2}}}$$

$\arcsin y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$= \left[0-\left(-\frac{\pi}{2}\right)\right] - \frac{1}{2}\,\frac{(1-y^{2})^{1/2}}{1/2}\Big|_{-1}^{0} = \frac{\pi}{2} - 1$$

T38

$$\int \frac{d\theta}{\cos\theta - 1} = \int \frac{d\theta}{-2\sin^{2}\frac{\theta}{2}}$$

$$= -\int \frac{d\,\theta/2}{\sin^{2}\theta/2}$$

令$\frac{\theta}{2}=x$

$$= -\int \frac{dx}{\sin^{2}x}$$

$$= \cot x + C$$

$$= \cot\frac{\theta}{2} + C$$

T40

$$\int \frac{\sqrt{x}}{1+x^{3}}\,dx$$

令$u=x^{3/2}$

$$= \int \frac{u^{1/3}}{1+u^{2}}\,\frac{2}{3}u^{-\frac{1}{3}}\,du$$

$$= \frac{2}{3}\arctan u + C$$

$$= \frac{2}{3}\arctan x^{3/2} + C$$

T48.

$$\int \frac{dx}{1+\sin^2 x}$$

$$= \int \frac{dx}{2-\cos^2 x}$$

$$= \int \frac{dx}{2-\frac{1}{1+\tan^2 x}}$$

令 $u=\tan x$

$$原式 = \int \frac{1}{2-\frac{1}{1+u^2}} \frac{1}{1+u^2} du$$

$$= \int \frac{1}{2(1+u^2)-1} du$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{2u^2+1} d(u\sqrt{2})$$

$$= \frac{\sqrt{2}}{2} \arctan(\sqrt{2}u) + C$$

$$= \frac{\sqrt{2}}{2} \arctan(\sqrt{2}\tan x) + C$$

8.2

T23

$$I = \int e^{2x} \cos 3x dx$$

$$= \frac{1}{2} \int \cos 3x \, de^{2x}$$

$$= \frac{1}{2}\left(\cos 3x e^{2x} + 3\int e^{2x} \sin 3x \, dx\right)$$

$$= \frac{1}{2} \cos 3x e^{2x} + \frac{3}{4} \int \sin 3x \, de^{2x}$$

$$= \frac{1}{2} \cos 3x e^{2x} + \frac{3}{4}\left(\sin 3x e^{2x} - \int e^{2x} 3\cos 3x \, dx\right)$$

$$= \frac{1}{2} \cos 3x e^{2x} + \frac{3}{4} \sin 3x e^{2x} - \frac{9}{4} I + C_1$$

解得 $\Rightarrow I = \frac{4}{13}\left(\frac{1}{2} \cos 3x e^{2x} + \frac{3}{4} \sin 3x e^{2x}\right) + C$

T29

$$I = \int \sin(\ln x) dx$$

$$= \sin(\ln x) x - \int \cos(\ln x) dx$$

$$= \sin(\ln x) x - \cos(\ln x) x - I + C_1$$

解得 $\Rightarrow I = \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) + C$

T48

$$\int_0^{\pi/2} x^3 \cos 2x dx$$

$$= \frac{1}{2} \int_0^{\pi/2} x^3 \cos 2x \, d2x$$

$$= \frac{1}{2} \int_0^{\pi/2} x^3 d\sin 2x$$

$$= \frac{1}{2}\left(x^3 \sin 2x \Big|_0^{\pi/2} - \int_0^{\pi/2} 3 \sin 2x \cdot x^2 dx\right)$$

$$= -\frac{3}{4} \int_0^{\pi/2} x^2 \sin 2x \, dx$$

$$= \frac{3}{4} \int_0^{\pi/2} x^2 d\cos 2x$$

$$= \frac{3}{4}\left(x^2 \cos 2x \Big|_0^{\pi/2} - \int_0^{\pi/2} \cos 2x \cdot 2x \, dx\right)$$

$$= -\frac{3}{4}\left(\frac{\pi}{2}\right)^2 - \frac{3}{4} \int_0^{\pi/2} 2x \cos 2x \, dx$$

$$= -\frac{3\pi^2}{16} - \frac{3}{4} \int_0^{\pi/2} x \, d\sin 2x$$

$$= -\frac{3\pi^2}{16} - \frac{3}{4}\left(x \sin 2x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin 2x \, dx\right)$$

$$= -\frac{3\pi^2}{16} + \frac{3}{4} \int_0^{\pi/2} \sin 2x dx$$

$$= -\frac{3\pi^2}{16} - \frac{3}{8} \cos 2x \Big|_0^{\pi/2}$$

$$= -\frac{3\pi^2}{16} + \frac{3}{4} = -\frac{3\pi^2}{16} + \frac{12}{16} = \frac{12-3\pi^2}{16}$$

T49

$$\int_{2/\sqrt{3}}^{2} t\sec^{-1}t\,dt$$

$$= \int_{2/\sqrt{3}}^{2} t \operatorname{arcsec} t\,dt$$

$$= \frac{1}{2}\int_{2/\sqrt{3}}^{2} \operatorname{arcsec} t\,dt^2$$

$$= \frac{1}{2}\left(t^2 \operatorname{arcsec} t \Big|_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2} t^2 \frac{1}{|t|\sqrt{t^2-1}}\,dt \right.$$

$$= \frac{1}{2}\left(4\underbrace{\operatorname{arcsec} 2}_{\cos x = \frac{1}{2}} - \frac{4}{3}\underbrace{\operatorname{arcsec}\frac{2}{\sqrt{3}}}_{\cos x = \frac{\sqrt{3}}{2}} - \frac{1}{2}\int_{2/\sqrt{3}}^{2} \frac{1}{\sqrt{t^2-1}}\,d(t^2-1) \right)$$

$$= \frac{1}{2}\left(4\,\frac{\pi}{3} - \frac{4}{3}\,\frac{\pi}{6} - \frac{1}{2}\,\frac{(t^2-1)^{1/2}}{1/2}\Big|_{2/\sqrt{3}}^{2} \right)$$

$$= \frac{1}{2}\left(\frac{4\pi}{3} - \frac{2\pi}{9} - (t^2-1)^{1/2}\Big|_{2/\sqrt{3}}^{2} \right)$$

$$= \frac{1}{2}\left(\frac{10\pi}{9} - \left(\sqrt{3} - \sqrt{\frac{1}{3}}\right) \right)$$

$$= \frac{5\pi}{9} - \frac{\sqrt{3}}{3}$$

T68

$$\int_0^{\pi/2}\cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n}\Big|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2}x\,dx.$$

因此，当n为奇数

$$\int_0^{\pi/2}\cos^n x\,dx = \frac{n-1}{\underset{\text{奇}}{n}}\cdot\frac{n-3}{n-2}\cdots\frac{2}{\underset{\text{奇}}{3}}\int_0^{\pi/2}\cos x\,dx$$

$$= \frac{2\cdot 4\cdots(n-1)}{3\cdot 5\cdots n}\left(\sin x\Big|_0^{\pi/2}\right)$$

$$= \frac{2\cdot 4\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}$$

当n为偶数.

$$\int_0^{\pi/2}\cos^n x\,dx = \frac{n-1}{n}\cdots\frac{1}{2}\int_0^{\pi/2}\cos^0 x\,dx$$

$$= \frac{1\cdot 3\cdots(n-1)}{2\cdot 4\cdots n}\left(\frac{\pi}{2}\right)$$

$$\int_0^{\pi/2}\sin^n x\,dx = -\int_0^{\pi/2}\sin^{n-1}x\,d\cos x$$

$$= -\left(\sin^{n-1}x\cos x\Big|_0^{\pi/2}\right.$$

$$- \int_0^{\pi/2}\cos^2 x\,(n-1)\sin^{n-2}x\,dx$$

$$= \int_0^{\pi/2}(n-1)(1-\sin^2 x)\sin^{n-2}x\,dx$$

$$= (n-1)\int_0^{\pi/2}\sin^{n-2}x\,dx$$

$$-(n-1)\int_0^{\pi/2}\sin^n x\,dx$$

$$\Rightarrow \int_0^{\pi/2}\sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^n x\,dx$$

同理可得

$$\int_0^{\pi/2}\sin^n x\,dx = \begin{cases} \frac{1\cdot 3\cdots(n-1)}{2\cdot 4\cdots n}\,\frac{\pi}{2} & n\text{为偶} \\ \frac{2\cdot 4\cdots(n-1)}{1\cdot 3\cdot 5\cdots n} & n\text{为奇} \end{cases}$$

T69

证明 $\int_a^b \int_x^b f(t)\,dt\,dx = \int_a^b (x-a)f(x)\,dx$

令 $\varphi(x) = \int_x^b f(t)\,dt$, $\varphi'(x) = -f(x)$

$\psi(x) = x$, $\psi'(x) = 1$.

$$\int_a^b \int_x^b f(t)\,dt\,dx$$

$$= \int_a^b \varphi(x)\,d\psi(x)$$

$$= \varphi(x)\psi(x)\Big|_a^b - \int_a^b \psi(x)\varphi'(x)\,dx$$

$$= \varphi(b)\psi(b) - \varphi(a)\psi(a) + \int_a^b x f(x)\,dx$$

$$= -a\int_a^b f(t)\,dt + \int_a^b x f(x)\,dx$$

$$= \int_a^b (x-a) f(x)\,dx$$

T72

用公式 $\int f^{-1}(x)\,dx = x f^{-1}(x) - \int f(y)\,dy$

$$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \tan y\,dy$$

$$= x\tan^{-1}(x) - \int \frac{\sin y}{\cos y}\,dy$$

$$= x\tan^{-1}(x) + \int \frac{d\cos y}{\cos y}$$

$$= x\tan^{-1}(x) + \ln|\cos y| + C$$

$$= x\tan^{-1}(x) + \ln|\cos \tan^{-1}(x)| + C$$

补充题.

T11 (4)

$$y' = -\frac{1}{\sqrt{1-e^{-2t}}} \cdot (-e^{-t})$$

$$= \frac{e^{-t}}{\sqrt{1-e^{-2t}}}$$

(5)

$$y = e^{\arctan x \ln x}$$

$$y' = e^{\arctan x \ln x}\left(\frac{\ln x}{1+x^2} + \frac{\arctan x}{x}\right)$$

$$= x^{\arctan x}\left(\frac{\ln x}{1+x^2} + \frac{\arctan x}{x}\right)$$

T15

(1)

要证 $2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1} = \frac{\pi}{2} \quad (x \geqslant 0)$

只需证 $2\arctan\sqrt{x} - \frac{\pi}{2} = \arcsin\frac{x-1}{x+1}$

只需证 $\sin\left(2\arctan\sqrt{x} - \frac{\pi}{2}\right) = \frac{x-1}{x+1}$

只需证 $\cos(2\arctan\sqrt{x}) = \frac{2}{x+1} - 1$

只需证 $\cos^2\arctan\sqrt{x} = \frac{1}{x+1}$ 用二倍角公式

只需证 $\frac{1}{\cos^2\arctan\sqrt{x}} = x+1$

只需证 $\tan^2(\arctan\sqrt{x}) = x$

即 $x = x$ 成立.

(2)

令 $\varphi(u) = u$，$\varphi'(u) = 1$

$\psi(u) = \int_0^u f(t)dt$，$\psi'(u) = f(u)$

$$\int_0^x \int_0^u f(t)dt\,du$$

$$= \int_0^x \psi(u)\varphi'(u)\,du$$

$$= \int_0^x \psi(u)\,d\varphi(u)$$

$$= \psi(u)\varphi(u)\Big|_0^x - \int_0^x \varphi(u)\psi'(u)\,du$$

$$= x\int_0^x f(t)dt - \int_0^x u f(u)du$$

$$= x\int_0^x f(u)du - \int_0^x u f(u)du$$

$$= \int_0^x (x-u)f(u)du$$

T18

$\int_0^x t f(2x-t)dt = \frac{1}{2}\arctan x^2.$

令 $2x-t=u$

$$-\int_{2x}^{x}(2x-u)f(u)\,du = \frac{1}{2}\arctan x^2$$

$$\int_x^{2x}(2x-u)f(u)du = \frac{1}{2}\arctan x^2$$

$$2x\int_x^{2x}f(u)du - \int_x^{2x}uf(u)du = \frac{1}{2}\arctan x^2$$

两边同时对x求导得

$$2\int_x^{2x}f(u)du + 2x\left[f(2x)\cdot 2 - f(x)\right] - \left[2xf(2x)\cdot 2 - xf(x)\right] = \frac{1}{2}\cdot\frac{2x}{1+x^4}$$

上式取 $x=1$ 得 $2\int_1^2 f(u)du + 4f(2) - 2f(1) - 4f(2) + f(1) = \frac{1}{2}$

代入 $f(1)=1$ 得 $\int_1^2 f(u)du = \frac{3}{4}$

T19.

(1) $\lim\limits_{x\to0^+}\dfrac{e^{ax}+x^2-ax-1}{x\sin\frac{x}{4}}$

$\overset{洛}{=}\lim\limits_{x\to0^+}\dfrac{ae^{ax}+2x-a}{\sin\frac{x}{4}+\frac{x}{4}\cos\frac{x}{4}}$

$\overset{洛}{=}\lim\limits_{x\to0^+}\dfrac{a^2e^{ax}+2}{\frac{1}{4}\cos\frac{x}{4}+\frac{1}{4}\cos\frac{x}{4}-\frac{x}{16}\sin\frac{x}{4}}.$

$=\dfrac{a^2+2}{\frac{1}{2}}=2a^2+4=6\Rightarrow a^2=1$

$\lim\limits_{x\to0^-}\dfrac{\ln(1+ax^3)}{x-\arcsin x}$

$\overset{洛}{=}\lim\limits_{x\to0^-}\dfrac{\frac{3ax^2}{1+ax^3}}{1-\frac{1}{\sqrt{1-x^2}}}$

$=\lim\limits_{x\to0^-}\dfrac{3ax^2}{1+ax^3-\frac{1+ax^3}{\sqrt{1-x^2}}}$

$\overset{洛}{=}\lim\limits_{x\to0^-}\dfrac{6ax}{3ax^2-\frac{3ax^2\sqrt{1-x^2}-(1+ax^3)\frac{-2x}{2\sqrt{1-x^2}}}{1-x^2}}$

$=\lim\limits_{x\to0^-}\dfrac{6ax(1-x^2)}{3ax^2(1-x^2)-3ax^2\sqrt{1-x^2}-\frac{(1+ax^3)x}{\sqrt{1-x^2}}}$

$\overset{洛}{=}\lim\limits_{x\to0^-}\dfrac{6a(1-x^2)+6ax(-2x)}{6ax(1-x^2)+3ax^2(-2x)-6ax\sqrt{1-x^2}-3ax^2\frac{-2x}{2\sqrt{1-x^2}}-\frac{(4ax^3+1)\sqrt{1-x^2}-(x+ax^4)\frac{-2x}{2\sqrt{1-x^2}}}{1-x^2}}$

$=\dfrac{6a}{-1}=-6a=6\quad\Rightarrow a=-1.$

(2)

$-6a=2a^2+4\quad\Rightarrow a=-1$ 或 $-2.$

由于 $-6a\neq6\Rightarrow a=-2.$