补充题 19年 T9 (1) (2) (3)

20年 T8 (1) (2) (3)

19年 T9

(1) $$\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{e^{\frac{x}{2}}dx}{e^{\frac{x}{2}}\sqrt{1+e^x}}$$

$$= -2\int \frac{e^{-\frac{x}{2}}d\frac{-x}{2}}{\sqrt{e^{-x}+1}}$$

$$= -2\int \frac{de^{-\frac{x}{2}}}{\sqrt{1+(e^{-\frac{x}{2}})^2}}$$

$$= -2\,\mathrm{arcsinh}\, e^{-\frac{x}{2}} + C$$

* 出现e指数积分尝试上下同乘 $\begin{cases} e^{\frac{x}{2}} \\ e^{-\frac{x}{2}} \\ e^{x} \\ e^{-x} \end{cases}$，多半能积下去.

* 别忘记C.

* 如果不定积分忘记换回原字母就不要换元.

(2) $$\int \frac{3x+6}{(x-1)^2(x^2+x+1)}\,d$$

设 $$\frac{3x+6}{(x-1)^2(x^2+x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1}$$

通分取分子得 $3x+6 = A(x^2+x+1) + B(x-1)(x^2+x+1) + (Cx+D)(x-1)^2$ (*)

令 $x=1$ 得 $9 = 3A \Rightarrow A=3$.

(*)式化为 $3x+6-3(x^2+x+1) = B(x-1)(x^2+x+1) + (Cx+D)(x-1)^2$

$-3x^2+3 = B(x-1)(x^2+x+1) + (Cx+D)(x-1)^2$

求导得 $-6x = B(x^2+x+1) + B(x-1)(2x+1) + C(x-1)^2 + (Cx+D)2(x-1)$

令 $x=1$ 得 $-6 = 3B \Rightarrow B=-2$ 代回上式得，

$$-6x+2(x^2+x+1)+2(x-1)(2x+1)=C(x-1)^2+2(x-1)(Cx+D)$$

即 $\quad 6x(x-1)=C(x-1)^2+2(x-1)(Cx+D)$

$$6x=C(x-1)+2(Cx+D)$$

求得 $6=C+2C \Rightarrow C=2$. 代回上式得

$$6x=2x-2+4x+2D \Rightarrow 2D-2=0 \Rightarrow D=1.$$

$$原式=\int\frac{3}{(x-1)^2}dx+\int\frac{-2}{x-1}dx+\int\frac{2x+1}{x^2+x+1}dx$$

$$=-3(x-1)^{-1}-2\ln|x-1|+\int\frac{d\,x^2+x+1}{x^2+x+1}$$

$$=\frac{-3}{x-1}-2\ln|x-1|+\ln|x^2+x+1|+C$$

(3)

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\tan^2x\sec x\,dx$$

$* \int\tan x\sec x\,dx=\sec x+C$

$$=\int_{\frac{\pi}{6}}^{\pi/3}\tan x\,d\sec x$$

$$=\tan x\sec x\Big|_{\pi/6}^{\pi/3}-\int_{\pi/6}^{\pi/3}\sec x\sec^2x\,dx$$

$$=2\sqrt{3}-\frac{2}{3}-\int_{\pi/6}^{\pi/3}\sec^3x\,dx$$

$$\int_{\pi/6}^{\pi/3}\sec^3x\,dx$$

$$=\int_{\pi/6}^{\pi/3}\frac{\cos x}{\cos^3x\cdot\cos x}dx$$

$$=\int_{\pi/6}^{\pi/3}\frac{d\sin x}{\cos^4x}$$

$$=\int_{\pi/6}^{\pi/3}\frac{d\sin x}{(1-\sin^2x)^2}$$

$$=\int_{1/2}^{\sqrt{3}/2}\frac{du}{(1-u^2)^2}$$

设 $\frac{1}{(1+u)^2(1-u)^2} = \frac{A}{(1+u)^2} + \frac{B}{1+u} + \frac{C}{(1-u)^2} + \frac{D}{(1-u)}$

通分取分子得 $1 = A(1-u)^2 + B(1+u)(1-u)^2 + C(1+u)^2 + D(1-u)(1+u)^2$ (1)

令 $u=1$ 得 $1 = 4C \Rightarrow C = \frac{1}{4}$.

令 $u=-1$ 得 $1 = 4A \Rightarrow A = \frac{1}{4}$

将A与C代回(1)式得. $1 - \frac{1}{4}(1-u)^2 - \frac{1}{4}(1+u)^2 = B(1+u)(1-u)^2 + D(1-u)(1+u)^2$

$1 - \frac{1}{2} - \frac{1}{2}u^2 = B(1+u)(1-u)^2 + D(1-u)(1+u)^2$

求导得 $-u = B(1-u)^2 - B(1+u)2(1-u) - D(1+u)^2 + D(1-u)2(1+u)$

令 $u=1$ 得 $-1 = -4D \Rightarrow D = \frac{1}{4}$

令 $u=-1$ 得 $1 = 4B \Rightarrow B = \frac{1}{4}$

因此 $\frac{1}{(u^2-1)^2} = \frac{1}{4}\left[\frac{1}{(u+1)^2} + \frac{1}{u+1} + \frac{1}{(1-u)^2} + \frac{1}{1-u}\right]$

$$\int \frac{1}{(u^2-1)^2}\,du = \frac{1}{4}\left[\int \frac{du}{(u+1)^2} + \int \frac{du}{u+1} + \int \frac{du}{(1-u)^2} + \int \frac{du}{1-u}\right]$$

$$= \frac{1}{4}\left[\frac{(u+1)^{-1}}{-1} + \ln|u+1| - \frac{(1-u)^{-1}}{-1} - \ln|1-u|\right] + C$$

$$= \frac{1}{4}\left[-\frac{1}{u+1} + \ln|u+1| + (1-u)^{-1} - \ln|1-u|\right] + C$$

$$= \frac{1}{4}\left[\ln\left|\frac{1+u}{1-u}\right| + \frac{2u}{1-u^2}\right] + C$$

故原式 $= 2\sqrt{3} - \frac{2}{3} - \int_{\pi/6}^{\pi/3} \sec^3 x\,dx$

$$= 2\sqrt{3} - \frac{2}{3} - \int_{1/2}^{\sqrt{3}/2} \frac{du}{(1-u^2)^2}$$

$$= 2\sqrt{3} - \frac{2}{3} - \frac{1}{4}\left[\ln\left|\frac{1+u}{1-u}\right| + \frac{2u}{1-u^2}\right]\Bigg|_{1/2}^{\sqrt{3}/2}$$

$$= 2\sqrt{3} - \frac{2}{3} - \frac{1}{4}\left[\ln\left|\frac{1+\sqrt{3}/2}{1-\sqrt{3}/2}\right| + \frac{\sqrt{3}}{1-3/4} - \ln\left|\frac{3/2}{1/2}\right| - \frac{1}{1-1/4}\right]$$

$$= 2\sqrt{3} - \frac{2}{3} - \frac{1}{4}\ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right| - \sqrt{3} + \frac{1}{4}\ln 3 + \frac{1}{3} = \sqrt{3} - \frac{1}{3} + \frac{1}{4}\ln 3 - \frac{1}{2}\ln(2+\sqrt{3})$$

T8

(1) $\int_{\pi/6}^{\pi/3} \csc^3 x \, dx$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{\sin^3 x} dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin x}{\sin^4 x} dx$$

$$= -\int_{\pi/6}^{\pi/3} \frac{d\cos x}{(1-\cos^2 x)^2}$$

$$= \int_{1/2}^{\sqrt{3}/2} \frac{du}{(1-u^2)^2}$$

$$= \frac{1}{4}\left[\ln\left|\frac{1+u}{1-u}\right| + \frac{2u}{1-u^2}\right]\Bigg|_{1/2}^{\sqrt{3}/2}$$

$$= \frac{1}{4}\left[\ln\left|\frac{1+\sqrt{3}/2}{1-\sqrt{3}/2}\right| + \frac{\sqrt{3}}{1-3/4} - \ln\left|\frac{3/2}{1/2}\right| - \frac{1}{3/4}\right]$$

$$= \frac{1}{4}\ln\frac{2+\sqrt{3}}{2-\sqrt{3}} + \sqrt{3} - \frac{1}{4}\ln 3 - \frac{1}{3}$$

$$= \frac{1}{2}\ln(2+\sqrt{3}) + \sqrt{3} - \frac{1}{4}\ln 3 - \frac{1}{3}$$

(2) $\int \sqrt{\frac{x}{x-2}}\,dx$

令 $t=\sqrt{\frac{x}{x-2}}$

$$=\int t\cdot\left(\frac{-4t}{(t^2-1)^2}\right)dt$$

$$=-4\int\frac{t^2-1+1}{(t^2-1)^2}\,dt$$

$$=-4\int\frac{1}{t^2-1}+\frac{1}{(t^2-1)^2}\,dt$$

$$=-4\left[\int\frac{1/2}{t-1}-\frac{1/2}{t+1}\,dt+\int\frac{1}{(t^2-1)^2}dt\right]$$

$$=-4\left[\frac{1}{2}\ln|t-1|-\frac{1}{2}\ln|t+1|\right]-4\times\frac{1}{4}\left[\ln|1-t^2|+\frac{2}{1-t^2}\right]+C$$

$$=-2\ln|t-1|+2\ln|t+1|-\ln|1-t^2|-\frac{2}{1-t^2}+C$$

$$=\ln\frac{(t+1)^2}{(t-1)^2(1-t^2)}-\frac{2}{1-t^2}+C$$

$$=\ln\left|\frac{1+t}{(t-1)^3}\right|-\frac{2}{1-t^2}+C$$

再把 t 写成 $\sqrt{\frac{x}{x-2}}$

$$=\ln\left|\frac{1+\sqrt{\frac{x}{x-2}}}{\left(\sqrt{\frac{x}{x-2}}-1\right)^3}\right|+x+C$$

(3) $\int_1^e \ln^3 x \, dx$

$= x\ln^3 x\Big|_1^e - \int_1^e x \; 3\ln^2 x \cdot \frac{1}{x} dx$

$= e + 3\int_e^1 \ln^2 x \, dx$

$= e + 3\left(\ln^2 x \cdot x\Big|_e^1 - \int_e^1 x \cdot 2\ln x \cdot \frac{1}{x} dx\right)$

＊分部积分前通过换上下限把积分前符号变正 不易出错.

$= e - 3e + 3\int_1^e 2\ln x \, dx$

$= -2e + 6\left(\ln x \cdot x\Big|_1^e - \int_1^e x \cdot \frac{1}{x} dx\right)$

$= -2e + 6e - 6\int_1^e 1 \, dx$

$= 6 - 2e$

8.5

T20

$$\int \frac{x^2 dx}{(x-1)(x^2+2x+1)}$$

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+1}$$

通分取分子得

$x^2 = A(x^2+2x+1) + (Bx+C)(x-1)$

令 $x=1$ 得 $1=4A \Rightarrow A=\frac{1}{4}$

故上式化为 $\frac{3}{4}x^2 - \frac{1}{2}x - \frac{1}{4} = (Bx+C)(x-1)$

两边求导得 $\frac{3}{2}x - \frac{1}{2} = B(x-1) + Bx + C = 2Bx + C - B$

故 $2B = \frac{3}{2} \Rightarrow B = 3/4$, $C - B = -\frac{1}{2} \Rightarrow C = \frac{1}{4}$

原式 $= \frac{1}{4}\int \frac{dx}{x-1} + \int \frac{\frac{3}{4}x + \frac{1}{4}}{x^2+2x+1} dx$

$= \frac{1}{4}\ln|x-1| + \int \frac{\frac{3}{4}(x+1) - \frac{2}{4}}{(x+1)^2} dx$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{(x+1)^2}$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2}\frac{1}{x+1} + C$$

T26

$$\int\frac{s^4+81}{s(s^2+9)^2}\,ds$$

设 $\frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{(s^2+9)^2} + \frac{Ds+E}{s^2+9}$

$$A(s^2+9)^2 + (Bs+C)s + (Ds+E)s(s^2+9) = s^4+81$$

令 $s=0$ 得 $81A = 81 \Rightarrow A=1$

上式化为 $(Bs+C)s + (Ds+E)s(s^2+9) = s^4+81-(s^2+9)^2 = -18s^2$

即 $Bs^2 + Cs + \underline{Ds^4} + \underline{9Ds^2} + \underline{Es^3} + \underline{9Es} = -18s^2$

$$\Rightarrow D=0,\ E=0,\ B=-18,\ C=0$$

原式 $= \int \frac{1}{s} + \frac{-18s}{(s^2+9)^2}\,ds$

$$= \ln|s| - 9\int\frac{ds^2}{(s^2+9)^2}$$

$$= \ln|s| + 9(s^2+9)^{-1} + C$$

T37

$$\int\frac{y^4+y^2-1}{y^3+y}\,dy$$

$$= \int\frac{y(y^3+y)-1}{y^3+y}\,dy$$

$$= \int y\,dy - \int \frac{1}{y^3+y}\,dy$$

$$= \frac{y^2}{2} - \int \frac{1}{y(y^2+1)}\,dy$$

$$\frac{A}{y} + \frac{By+C}{y^2+1} = \frac{1}{y(y^2+1)} \xRightarrow{\text{通分}} A(y^2+1)+(By+C)y = 1$$

令 $y=0$ 得 $A=1$. 故 $(By+C)y = -y^2$ 即 $By^2+Cy=-y^2$

$\Rightarrow B=-1, C=0.$

原式 $= \frac{y^2}{2} - \int \frac{1}{y} + \frac{-y}{y^2+1}\,dy$

$$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\int \frac{dy^2}{y^2+1}$$

$$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln|y^2+1| + C$$

T46

$$\int \frac{1}{(x^{1/3}-1)\sqrt{x}}\,dx$$

令 $x^{1/6} = u$.

原式 $= \int \frac{1}{(u^2-1)u^3}\,6u^5\,du = \int \frac{6u^2}{u^2-1}\,du = \int \frac{6u^2-6+6}{u^2-1}\,du = \int 6 + \frac{6}{u^2-1}\,du$

$$= 6u + 6\int \frac{1}{(u+1)(u-1)}\,du = 6u + 3\int \frac{1}{u-1} - \frac{1}{u+1}\,du = 6u + 3\ln|u-1| - 3\ln|u+1| + C$$

$$= 6x^{1/6} + 3\ln\left|\frac{x^{1/6}-1}{x^{1/6}+1}\right| + C$$

T4) $\int \frac{\sqrt{x+1}}{x}\,dx$

令 $x+1=u^2$

原式 $= \int \frac{u}{u^2-1}\,2u\,du$

$=\int \frac{2u^2}{u^2-1}\,du$

$=\int \frac{2u^2-2+2}{u^2-1}\,du$

$=\int 2+\frac{2}{(u-1)(u+1)}\,du$

$=2u+\int \frac{1}{u-1}-\frac{1}{u+1}\,du$

$=2u+\ln\left|\frac{u-1}{u+1}\right|+C$

$=2\sqrt{x+1}+\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$

8.7

T8 $\int_2^4 \frac{1}{(s-1)^2}\,ds$

I. (a) $n=4,\ \Delta x=\frac{4-2}{4}=\frac{1}{2}$

$T=\frac{\Delta x}{2}(S_0+2S_1+2S_2+2S_3+S_4)=0.70500.$

$f(s)=(s-1)^{-2}\Rightarrow f''(s)=\frac{6}{(s-1)^4}\Rightarrow M=6$

$\Rightarrow |E_T|\le \frac{4-2}{12}\left(\frac{1}{2}\right)^6(6)=\frac{1}{4}$

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds=\frac{-1}{s-1}\Big|_2^4=\frac{2}{3}$

$\Rightarrow E_T=\frac{2}{3}-0.705\approx-0.03833$

$\Rightarrow |E_T|\approx 0.03833$

(c) $\frac{|E_T|}{\text{True value}}\times 100=\frac{0.03833}{2/3}\times 100\approx 6\%$

II. (a) $n=4,\ \Delta x=\frac{b-a}{4}=\frac{1}{2}$

$S=\frac{\Delta x}{3}(S_0+4S_1+2S_2+4S_3+S_4)\approx 0.67148$

$f^{(4)}(s)=\frac{120}{(s-1)^6}\Rightarrow M=120$

$|E_S|\le \frac{4-2}{180}\left(\frac{1}{2}\right)^4(120)=\frac{1}{12}\approx 0.08333$

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds=\frac{2}{3}$

$\Rightarrow E_S=\int_2^4 \frac{1}{(s-1)^2}\,ds-S\approx -0.00481$

$\Rightarrow |E_S|\approx 0.00481$

(c) $\frac{|E_S|}{\text{True value}}\times 100=\frac{0.00481}{2/3}\times 100=1\%$

T20

(a) $f(x)=\frac{1}{\sqrt{x+1}}\Rightarrow f''(x)=\frac{3}{4(x+1)^{-5/2}}$

$\Rightarrow M=3/4.\quad \Delta x=\frac{3}{n}\Rightarrow |E_T|\le \frac{3}{12}\left(\frac{3}{n}\right)^2\left(\frac{3}{4}\right)$

$<10^{-4}\Rightarrow n>\sqrt{\frac{3^4(10^4)}{48}}\Rightarrow n>129.9$

令 $n=130$.

(b) $f^{(4)}(x)=\frac{105}{16}(x+1)^{-9/2}$

$\Rightarrow M=\frac{105}{16}.$

$\Delta x=\frac{3}{n}\Rightarrow |E_S|\le \frac{3}{180}\left(\frac{3}{n}\right)^4\left(\frac{105}{16}\right)<10^{-4}$

$\Rightarrow n>\sqrt[4]{\frac{3^5\cdot 105\cdot 10^4}{16\cdot 180}}\Rightarrow n>17.25$

令 $n=18$.

8.8

T20

$$\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}dx$$

$$=\int_0^\infty 16\tan^{-1}x\, d\tan^{-1}x$$

$$=8(\tan^{-1}x)^2\Big|_0^\infty$$

$$=8\left(\frac{\pi}{2}\right)^2=2\pi^2$$

T32

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}}$$

$$=\int_0^1 \frac{dx}{\sqrt{1-x}}+\int_1^2\frac{dx}{\sqrt{x-1}}$$

$$=-\frac{(1-x)^{1/2}}{1/2}\Big|_0^1+\frac{(x-1)^{1/2}}{1/2}\Big|_1^2$$

$$=4$$

T41

$$\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$$

$$<\int_0^\pi\frac{dt}{\sqrt{t}}$$ 收敛

T43

$$\int_0^2\frac{dx}{1-x^2}$$

$$=\int_0^1\frac{dx}{1-x^2}+\underbrace{\int_1^2\frac{dx}{1-x^2}}_{<0}$$

$\frac{1}{1-x^2}>0$，当 $x\in(0,1)$

故 $\int_0^2\frac{dx}{1-x^2}>\int_1^2\frac{dx}{1-x^2}$

$$=\frac{1}{2}\int_1^2\frac{1}{1-x}+\frac{1}{1+x}dx$$

$$=\left(-\frac{1}{2}\ln|1-x|+\frac{1}{2}\ln|1+x|\right)\Big|_1^2$$

$$=\left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_1^2=-\infty$$

因此原积分发散.

T65

a. $$\int_1^2\frac{dx}{x(\ln x)^p}$$

$$=\int_1^2\frac{d\ln x}{(\ln x)^p}$$

$$=\int_0^{\ln 2}\frac{du}{u^p}$$

当 $p<1$ 时收敛.

b. $$\int_2^\infty\frac{dx}{x(\ln x)^p}$$

$$=\int_2^\infty\frac{d\ln x}{(\ln x)^p}$$

$$=\int_{\ln 2}^\infty\frac{du}{u^p}$$

当 $p>1$ 时收敛.