5.5

T15

a. $\int \csc^2 2\theta \cot 2\theta \, d\theta.$

$= \frac{1}{2}\int \csc^2 2\theta \cot 2\theta \, d2\theta$

$= -\frac{1}{2}\int \cot 2\theta \, d\cot 2\theta$

$= -\frac{1}{2}\int u\,du = -\frac{1}{2}\cdot\frac{u^2}{2} + C$

$= -\frac{1}{4}\cot^2 2\theta + C$

b. $\int \csc^2 2\theta \cot 2\theta \, d\theta$

令 $u = \csc 2\theta$. $du = -\frac{2\cos 2\theta}{\sin^2 2\theta} d\theta.$

$d\theta = -\frac{\sin^2 2\theta}{2\cos 2\theta} du$

(这些写草稿纸上. 因为写出来不规范)

$= -\int \csc^2 2\theta \cot 2\theta \, \frac{\sin^2 2\theta}{2\cos 2\theta} du.$

$= -\int \frac{1}{\sin^2 2\theta} \, \frac{\cos 2\theta}{\sin 2\theta} \, \frac{\sin^2 2\theta}{2\cos 2\theta} du$

$= -\frac{1}{2}\int \mathrm{css}\, 2\theta \, du$

$= -\frac{1}{2}\int u\,du = -\frac{1}{2}\cdot\frac{u^2}{2} = -\frac{u^2}{4}$

$= -\frac{1}{4}\csc^2 2\theta + C$

* 不定积分换元后要记得换回去.

* 记得加 C.

T21

$\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$

令 $u = 1+\sqrt{x}$. $du = \frac{1}{2\sqrt{x}} dx = \frac{dx}{2(u-1)}$

$\Rightarrow 2(u-1)du = dx$

原式 $= \int \frac{2(u-1)du}{(u-1)\cdot u^2}$

$= \int \frac{2}{u^2} du = \frac{2}{-1} u^{-1} + C = -\frac{2}{u} + C$

$= -\frac{2}{1+\sqrt{x}} + C$

T26

$\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} dx$

$= 2\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} d\frac{x}{2}$

$= 2\int \tan^6 \frac{x}{2} \sec \frac{x}{2} d\sec\frac{x}{2}$

$= 2\int (\sec^2 \frac{x}{2} - 1)^3 \sec\frac{x}{2} d\sec\frac{x}{2}$

$= \int (\sec^2\frac{x}{2} - 1)^3 d(\sec^2\frac{x}{2} - 1)$

$= \frac{1}{4}(\sec^2\frac{x}{2} - 1)^4 + C$

$= \frac{1}{4}\tan^8\frac{x}{2} + C$

T40

$$\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$$

$$=\int \sqrt{\frac{x^2-1}{x^2}}\,d\left(\frac{x^{-2}}{-2}\right)$$

$$=\frac{1}{-2}\int \sqrt{1-x^{-2}}\,dx^{-2}$$

$$=\frac{1}{2}\int \sqrt{1-x^{-2}}\,d(1-x^{-2})$$

$$=\frac{1}{2}\,\frac{(1-x^{-2})^{3/2}}{3/2}$$

$$=\frac{1}{3}(1-x^{-2})^{3/2}+C$$

T48

$$\int 3x^5\sqrt{x^3+1}\,dx$$

$$=\int 3x^3\cdot x^2\sqrt{x^3+1}\,dx$$

$$=\int x^3\sqrt{x^3+1}\,dx^3$$

令 $\sqrt{x^3+1}=t$ ＊令根号为新元 $\left(\begin{array}{l} x^3+1=t^2 \\ dx^3=2t\,dt \end{array}\right)$

$$=\int (t^2-1)\,t\cdot 2t\,dt$$

$$=\frac{2}{5}t^5-\frac{2}{3}t^3+C=\frac{2}{5}\left(\sqrt{x^3+1}\right)^5-\frac{2}{3}\left(\sqrt{x^3+1}\right)^3+C$$

T54

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta$$

令 $\sqrt{\theta}=t$ ＊令根号为新元.

$$d\sqrt{\theta}=dt \rightsquigarrow \frac{d\theta}{2\sqrt{\theta}}=dt \rightsquigarrow d\theta=2\sqrt{\theta}\,dt=2t\,dt$$

$$=\int \frac{\sin t}{t\sqrt{\cos^3 t}}\,2t\,dt$$

$$=\int \frac{2\sin t}{\sqrt{\cos^3 t}}\,dt$$

$$=-2\int \frac{1}{\sqrt{\cos^3 t}}\,d\cos t$$

$$=-2\,\frac{\cos^{-1/2}t}{-1/2}+C$$

$$=4\frac{1}{\sqrt{\cos\sqrt{\theta}}}+C$$

T60

$$\frac{d^2y}{dx^2} = 4\sec^2 2x \tan 2x$$

$$\int 4\sec^2 2x \tan 2x \, dx$$

$$= 2\int \sec^2 2x \tan 2x \, d2x$$

$$= 2\int \sec 2x \, d\sec 2x$$

$$= 2\,\frac{\sec^2 2x}{2} + C_1 = \sec^2 2x + C_1$$

由 $y'(0)=4$ 得 $1 + C_1 = 4 \Rightarrow C_1 = 3$

$$\int \sec^2 2x + 3 \, dx$$

$$= \int \sec^2 2x \, dx + \int 3 \, dx$$

$$= \frac{1}{2}\int \sec^2 2x \, d2x + \int 3 \, dx$$

$$= \frac{1}{2}\tan 2x + 3x + C_2$$

由 $y(0) = -1$ 得 $C_2 = -1$

故 $y = \frac{1}{2}\tan 2x + 3x - 1$

5.6

T18

$$\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right) \sec^2\left(\frac{\theta}{6}\right) d\theta$$

$$= 6\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right) \sec^2\left(\frac{\theta}{6}\right) d\,\frac{\theta}{6}$$

$$= 6\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right) d\tan\frac{\theta}{6}$$

$$= \frac{6}{-4}\tan^{-4}\frac{\theta}{6}\Big|_{\pi}^{3\pi/2} = 12$$

T23

$$\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$$

令 $\theta^{3/2} = t$ * 定积分换元要变上下限.

$$\left(\begin{aligned} &\frac{3}{2}\theta^{\frac{1}{2}}d\theta = dt \\ &\Rightarrow d\theta = \frac{2}{3}\theta^{-1/2}dt \\ &\qquad\;\; = \frac{2}{3}t^{-1/3}dt \end{aligned}\right)$$

$$原式 = \int_0^{\pi} t^{1/3}(\cos^2 t)\frac{2}{3}t^{-1/3}\,dt$$

$$= \frac{2}{3}\int_0^{\pi}\cos^2 t\,dt$$

$$= \frac{2}{3}\int_0^{\pi}\frac{1+\cos 2t}{2}\,dt$$

$$= \frac{\pi}{3} + \frac{1}{3}\int_0^{\pi}\cos 2t\,dt = \frac{\pi}{3}$$

TST

$x=y^2-1 \quad x=|y|\sqrt{1-y^2}$

$y^2-1=|y|\sqrt{1-y^2}$

$y=\pm1$ 是一个解.

$y\neq\pm1$ 时 $-\sqrt{1-y^2}=|y|$.

$$1-y^2=y^2 \Rightarrow y=\pm\frac{1}{\sqrt{2}}$$

由表达式 $x=|y|\sqrt{1-y^2}$ 知 $x>0$.

但 $y=\pm\frac{1}{\sqrt{2}}$ 时 $x<0$, 故 $y=\pm\frac{1}{\sqrt{2}}$ 不是解.

因此 $y=\pm1$.

$$\int_{-1}^{1} y^2-1-|y|\sqrt{1-y^2}\,dy$$

$$=\int_{-1}^{0} y^2-1+y\sqrt{1-y^2}\,dy+\int_0^1 y^2-1-y\sqrt{1-y^2}\,dy$$

$$=\int_{-1}^{0} y^2-1\,dy+\int_{-1}^{0} y\sqrt{1-y^2}\,dy+\int_0^1 y^2-1\,dy-\int_0^1 y\sqrt{1-y^2}\,dy$$

$$=\left(\frac{y^3}{3}-y\right)\Big|_{-1}^{0}-\frac{1}{2}\int_{-1}^{0}\sqrt{1-y^2}\,d(1-y^2)+\frac{y^3}{3}-y\Big|_0^1+\frac{1}{2}\int_0^1\sqrt{1-y^2}\,d(1-y^2)$$

$$=2$$

T69

$$\int_0^{\pi/2} 3\sin y\sqrt{\cos y}\, dy$$

$$= -\int_0^{\pi/2} 3\sqrt{\cos y}\, d\cos y$$

$$= -3\left.\frac{\cos^{3/2} y}{3/2}\right|_0^{\pi/2}$$

$$= 2$$

T70

$$S_1 = 2\int_0^a (a^2 - x^2)\, dx$$

$$= 2\left(a^2 x - \frac{1}{3}x^3\right)\Big|_0^a$$

$$= 4a^3/3$$

$$S_2 = \frac{1}{2}(2a)(a^2) = a^3.$$

$$\lim_{a\to 0^+} \frac{a^3}{4a^3/3} = \frac{3}{4}.$$

T87

$$I = \int_0^a \frac{f(x)\, dx}{f(x) + f(a-x)}$$

令 $u = a - x$ $\quad du = -dx$

$$I = \int_a^0 \frac{-f(a-u)\, du}{f(a-u) + f(u)}$$

$$= \int_0^a \frac{f(a-x)\, dx}{f(a-x) + f(x)}.$$

$$2I = \int_0^a \frac{f(x) + f(a-x)}{f(a-x) + f(x)}\, dx$$

$$= \int_0^a 1\, dx = a$$

$$\Rightarrow I = a/2$$

6.1

T14

solid和cone有相同高度，且同高度处截面积相同。

由 Cavalieri's Principle

solid与cone有相同体积。

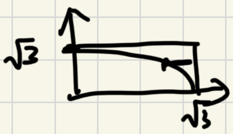
T29

$$V=\int_0^{\pi/2}\pi\left(\sqrt{2\sin 2y}\right)^2 dy$$

$$=\pi\int_0^{\pi/2}\sin 2y \, d2y$$

$$=\pi\cos 2y\Big|_{\pi/2}^{0}$$

$$=2\pi$$

T44 

$R(y)=\sqrt{3}$ $\quad r(y)=\sqrt{3-y^2}$

$$V=\int_0^{\sqrt{3}}\pi\left[R(y)^2-r(y)^2\right]dy$$

$$=\pi\int_0^{\sqrt{3}}3-(3-y^2)\,dy$$

$$=\pi\frac{y^3}{3}\Big|_0^{\sqrt{3}}=\pi\sqrt{3}$$

T57

$R(y)=\sqrt{256-y^2}$

$$V=\int_{-16}^{-7}\pi R(y)^2 dy$$

$$=\pi\int_{-16}^{-7}256-y^2\,dy$$

$$=\pi 256y-\pi\frac{y^3}{3}\Big|_{-16}^{-7}$$

$$=1053\pi\ cm^3$$