Differential geometry on bundles Ref: Gtm 65 CH III

§ Hermitian differential geometry

In this section, the term vector bundle means a differentiable $\mathbb{C}$-vec bundle over a differentiable mf $E \to X$.

We want to define lot of geometric concepts such as metric on bundles, they all need a foundation concepts — frame, which is "basis of vector bundle", analogous to basis of vector space.

[Def] $E \to X$ is a vector bundle of rank r. For $x \in X$, if there is a n.b.h. U of x and $f = (e_1, \cdots, e_r)$ with $e_i \in \mathcal{E}(U, E)$ are linearly independent at each pt of U, then we call $f = (e_1, \cdots, e_r)$ a frame at x. To illustrate its dependence of U, we sometimes denote f as f_U.

[Rmk] Note that frame is a local concepts. A bundle may not admit a global frame! Moreover, given an open set U, there may not be a frame on U. But when U sufficiently small, it admits a frame.

[Construction] (New frame by change of frame). Let f_U be a given frame and $g: U \to GL(r, \mathbb{C})$. Then we obtain a new frame fg by

$$fg = (e_1 g, e_2 g, \cdots, e_r g), \text{ i.e., } fg(x) = (e_1(x)g(x), \underbrace{e_2(x)g(x)}_{\sum_\rho e_\rho g_{\rho i}(x)}, \cdots, e_r(x)g(x))$$

Hence we call g a change of frame.

Conversely, given any two frames f, f' on U, there exists a change of frame g on U s.t. $f = f'g$ □

Local representation for sections of E:

Let $f = (e_1, \cdots, e_r)$ be frame over open set U. For a section $\xi \in \mathcal{E}(U, E)$, we have $\xi(x) = \sum \xi^\rho(f)(x) e_\rho(x)$ for $\forall x \in U$. Hence $\xi = \sum \xi^\rho(f) e_\rho$ for some $\xi^\rho \in \mathcal{E}(U) = \{\text{differentiable funs on } U\}$. This induces a mapping

$$\iota_f : \mathcal{E}(U, E) \longrightarrow \mathcal{E}(U)^r \cong \mathcal{E}(U, U \times \mathbb{C}^r)$$
$$\xi \longmapsto (\xi^\rho(f))_\rho \leftarrow \text{coefficients under basis } f = (e_\rho)_\rho.$$

Let's consider change of frame. Let $g: U \to GL(r, \mathbb{C})$ be a change of frame, what's the relationship between $\xi(f)$ and $\xi(fg)$?

$$\xi = \xi^\rho(f) e_\rho = \xi^\rho(f)(g^{-1}g)_{\rho\sigma} e_\sigma = \xi^\rho(f) g^{-1}_{\rho\iota} g_{\iota\sigma} e_\sigma = (\xi^\rho(f) g^{-1}_{\rho\iota})(g_{\iota\sigma} e_\sigma)$$

So $\xi^\iota(fg) = \xi^\rho(f) g^{-1}_{\rho\iota}$, i.e. $\xi(fg) = \xi(f) g^{-1}$.

Tell this story again in holomorphic vector bundle E, we shall have holomorphic frames, i.e., $f=(e_1, \cdots, e_r)$, $e_j \in \mathcal{O}(U, E)$ and $e_1 \wedge \cdots \wedge e_r(x) \neq 0$ for $x \in U$ (wedge nonvanishing equivalently say they are linear independent); and holomorphic changes of frame, i.e., holo mapping $g: U \to GL(r, \mathbb{C})$.

Then w.r.t. a holo frame we have vector representation

$$\mathcal{O}(U, E) \xrightarrow{\iota_f} \mathcal{O}(U)^r \text{ given by } \xi \mapsto (\xi^\rho(f))_\rho \text{ where } \xi = \sum \xi^\rho(f) e_\rho$$

The rest of this section is about metric, connection and curvature on vecter bundles.

<Metric>

[Def] Let $E \to X$ be a vector bundle. A Hermitian metric h on E is an assignment of Hermitian inner product $\langle , \rangle_x$ to each fiber E_x of E s.t. for any open $U \subseteq X$ and $\xi, \eta \in \mathcal{E}(U, E)$, the fun

$$\langle \xi, \eta \rangle : U \longrightarrow \mathbb{C}, \quad x \mapsto \langle \xi(x), \eta(x) \rangle_x \text{ is } C^\infty.$$

[Rmk] $\xi, \eta \in \mathcal{E}(U, E)$, then $\xi(x), \eta(x) \in E_x$.

[Slogan] A Hermitian metric on vec bundle is a smooth choice of Hermitian inner product at each fiber.

[Rmk] sm choice of Hermitian inner product is equivalently to say $\langle \xi(-), \eta(-) \rangle_-$ is sm dependent on x, i.e., $\langle \xi, \eta \rangle$ is C^∞.

[Def] A vector bundle E equipped with a Hermitian metric h is called a Hermitian vector bundle.

Use frame to represent Hermitian metric h: $f=(e_1, \cdots, e_r)$ is a frame for E over open U. The $r \times r$ matrix $h(f)_{\rho\sigma} = \langle e_\sigma, e_\rho \rangle$ is a local rep for Hermitian metric h. Then, for $\xi, \eta \in \mathcal{E}(U, E)$

$$\begin{aligned}\langle \xi, \eta \rangle &= \langle \xi^\rho(f) e_\rho, \eta^\sigma(f) e_\sigma \rangle \\ &= \overline{\eta^\sigma(f)}\, \xi^\rho(f) \langle e_\rho, e_\sigma \rangle \\ &= \overline{\eta^\sigma(f)}\, h(f)_{\sigma\rho}\, \xi^\rho(f)\end{aligned}$$

Hence, $\langle \xi,\eta\rangle = {}^t\overline{\eta(f)}\, h(f)\, \xi(f)$ where ${}^t\cdot$ means transpose.

Let g be a change of frame.

$$h(fg)_{\rho\sigma} = \langle (eg)_\sigma, (eg)_\rho\rangle = \langle e_\iota g_{\iota\sigma}, e_s g_{s\rho}\rangle = \overline{g_{s\rho}}\, g_{\iota\sigma} \langle e_\iota, e_s\rangle$$
$$= \overline{g_{s\rho}}\, h(f)_{s\iota}\, g_{\iota\sigma} = {}^t\overline{g}_{\rho s}\, h(f)_{s\iota}\, g_{\iota\sigma}. \text{ Hence } h(fg) = {}^t\overline{g}\, h(f)\, g$$

[Thm] Every vector bundle $E \to X$ admits a Hermitian metric.

pf: There exists a locally finite covering $\{U_\alpha\}$ of X and frames f_α defined on U_α. Define a Hermitian metric h_α on $E|_{U_\alpha}$ by

$$\langle \xi,\eta\rangle_x^\alpha = {}^t\overline{\eta(f_\alpha)}(x) \cdot \xi(f_\alpha)(x) \quad \text{for } x\in U_\alpha,\ \xi,\eta \in \mathcal{E}(U_\alpha, E).$$

To pass local to global, let $\{\rho_\alpha\}$ be C^∞ partition of unity subordinate to the covering $\{U_\alpha\}$. We define inner product on each fiber E_x as, for $a,b\in E_x$, $\langle a,b\rangle_x = \sum \rho_\alpha(x)\langle a,b\rangle_x^\alpha$ where $\langle a,b\rangle_x^\alpha = \overline{a(f_\alpha)}\cdot b(f_\alpha)$.

It's obviously a Hermitian inner product. For $\xi,\eta\in\mathcal{E}(U,E)$

$$\begin{aligned}\langle \xi,\eta\rangle : U &\longrightarrow \mathbb{C}\\ x &\longmapsto \langle \xi(x),\eta(x)\rangle_x = \sum \rho_\alpha(x)\langle \xi(x),\eta(x)\rangle_x^\alpha\\ &\qquad = \sum \rho_\alpha(x)\, {}^t\overline{\eta(f_\alpha)}(x)\cdot \xi(f_\alpha)(x)\end{aligned}$$

Here $\langle \xi,\eta\rangle \in C^\infty(U)$, complete the proof. $\square$

⟨Connection⟩

[Def] Let $\mathcal{E}^p(X,E) := \mathcal{E}(X, \Lambda^p T^*X \otimes_{\mathbb{C}} E)$ be the differential forms of degree p on X with coefficients in E.

The following property will be helpful.

[Prop] (Universal prop of sheafification) Let $\mathcal{F}$ be a presheaf with $\tilde{\mathcal{F}} = \Gamma(-,\tilde{\mathcal{F}})$ be its sheafification. Then there exists a presheaf mor $\theta:\mathcal{F}\to\tilde{\mathcal{F}}$ s.t. given any sheaf $\mathcal{G}$ together with presheaf mor $\psi:\mathcal{F}\to\mathcal{G}$, there exists a unique sheaf mor $\varphi:\tilde{\mathcal{F}}\to\mathcal{G}$, i.e., the diagram commutes:

$$\begin{tikzcd} \text{presheaf } \mathcal{F} \arrow[r, "\forall\psi"] \arrow[d, "\theta"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{G} \text{ sheaf} \\ \tilde{\mathcal{F}}_{\text{sheaf}} \arrow[ur, dotted, "\exists!\varphi"'] & \end{tikzcd}$$

(Ref: https://web.ma.utexas.edu/users/slaoui/notes/Sheaf_Cohomology_3.pdf Prop 4.11) pf: Define $\theta: \mathcal{F} \to \overline{\mathcal{F}}$ by $\theta_U(s) = \tilde{s} = [\, U \to \tilde{\mathcal{F}},\ x \mapsto s_x \,]$

- Check θ is a sheaf mor, i.e. θ_U compatible with restriction maps

$$\begin{tikzcd}
s \arrow[r, mapsto] \arrow[d, mapsto] & \theta_U(s): U \to \tilde{\mathcal{F}},\ x \mapsto s_x \arrow[d, mapsto] \\
r^U_V s \arrow[r, mapsto] & \theta_U(s)|_V : V \to \tilde{\mathcal{F}},\ x \mapsto s_x
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\theta_U"] \arrow[d, "r^U_V"'] & \overline{\mathcal{F}}(U) \arrow[d, "\overline{r}^U_V"] \\
\mathcal{F}(V) \arrow[r, "\theta_V"] & \overline{\mathcal{F}}(V)
\end{tikzcd}$$

$[\theta_V(r^U_V s): V \to \tilde{\mathcal{F}},\ x \mapsto (r^U_V s)_x$

$\Rightarrow$ Show $s_x = (r^U_V s)_x$ for $\forall x \in V$

Use direct limit

$$\begin{tikzcd}
s \arrow[rr, mapsto, bend left] \arrow[ddr, bend right] & & r^U_V s \arrow[ddl, bend left] \\
\mathcal{F}(U) \arrow[r, "r^U_V"] \arrow[dr, "r^U_x"'] & \mathcal{F}(V) \arrow[d, "r^V_x"] \arrow[d, phantom, shift right=4, "\circlearrowleft"] & \\
& \mathcal{F}_x &
\end{tikzcd}$$

$s_x = (r^U_V s)_x$

So θ is a presheaf mor.

- Given $\psi: \mathcal{F} \to \mathcal{G}$, show $\varphi: \overline{\mathcal{F}} \to \mathcal{G}$ rendering diagram commutes exists

We define $\varphi_U: \overline{\mathcal{F}}(U) \to \mathcal{G}(U)$ as following: Let $\sigma \in \overline{\mathcal{F}}(U)$. ($\overline{\mathcal{F}}(U) = \Gamma(U, \tilde{\mathcal{F}})$)

$A_i \xrightarrow{f_{ij}} A_j$, $f_i \searrow \swarrow f_j$ L $\quad$ $\forall b \in L\ \exists i$ and $a \in A_i$ s.t. $f_i a = b$.

For $x \in U$, $\sigma(x) \in \mathcal{F}_x$. By direct limit property, there exists n.b.h. V of x and $s \in \mathcal{F}(V)$ s.t. $r^V_x s = \sigma(x)$. Since $r^V_x s = \tilde{s}(x) = \theta_V(s)(x) = \sigma(x)$, we have two sections $\theta_V(s), \sigma$ agree at one point. Since section has local inverse, so two sections agree at one point will agree at a n.b.h. $W \subseteq V$, i.e. $\sigma|_W = \overline{r}^V_W \theta_V(s) = \theta_W(r^V_W s)$

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\theta_V"] \arrow[d, "r^V_W"'] \arrow[dr, phantom, "\circlearrowleft"] & \overline{\mathcal{F}}(V) \arrow[d, "\overline{r}^V_W"] \\
\mathcal{F}(W) \arrow[r, "\theta_W"] & \overline{\mathcal{F}}(W)
\end{tikzcd}$$

Hence, when x ranges over U, we have a cover $\{U_i\}$ of U and $s_i \in \mathcal{F}(U_i)$ s.t. $\sigma|_{U_i} = \theta_{U_i}(s_i)$. Consider $\psi_{U_i}(s_i) \in \mathcal{G}(U_i)$

Claim: $r_{\mathcal{G}\,U_i \cap U_j}^{\ U_i} \psi_{U_i}(s_i) = r_{\mathcal{G}\,U_i \cap U_j}^{\ U_j} \psi_{U_j}(s_j)$.

If this claim is true, then by sheaf prop there exists a unique $t \in \mathcal{G}(U)$, s.t. $r^U_{U_i} t = \psi_{U_i}(s_i)$.

pf for the claim: If $r^{U_i}_{U_i \cap U_j} s_i = r^{U_j}_{U_i \cap U_j} s_j$ (I haven't proved), by the naturality of ψ, we have diagram:

$$\begin{tikzcd}
s_i \in \mathcal{F}(U_i) \arrow[r, "\psi_{U_i}"] \arrow[d, "r^{U_i}_{\mathcal{F}\, U_i\cap U_j}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U_i) \arrow[d, "r^{U_i}_{\mathcal{G}\, U_i\cap U_j}"] \\
\mathcal{F}(U_i\cap U_j) \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U_i\cap U_j) \\
s_j \in \mathcal{F}(U_j) \arrow[u, "r^{U_j}_{\mathcal{F}\, U_i\cap U_j}"] \arrow[r, "\psi_{U_j}"] & \mathcal{G}(U_j) \arrow[u, "r^{U_j}_{\mathcal{G}\, U_i\cap U_j}"']
\end{tikzcd}$$

Hence $r_{\mathcal{G}}{}^{U_i}_{U_i\cap U_j}\psi_{U_i}(s_i) = r_{\mathcal{G}}{}^{U_j}_{U_i\cap U_j}\psi_{U_j}(s_j)$.

For the proof of $r^{U_i}_{U_i\cap U_j} s_i = r^{U_j}_{U_i\cap U_j} s_j$, it's equivalent to show $\theta_{U_i\cap U_j}$ is inj, since

$$\theta_{U_i\cap U_j}(r^{U_i}_{U_i\cap U_j} s_i) = \theta_{U_i}(s_i)|_{U_i\cap U_j} = \sigma|_{U_i\cap U_j}$$
$$= \theta_{U_j}(s_j)|_{U_i\cap U_j} = \theta_{U_i\cap U_j}(r^{U_j}_{U_i\cap U_j} s_j).$$

(But without $\mathcal{F}$ being a sheaf, $\theta_{U_i\cap U_j}$ may not be inj :(I do not know how to complete this part of proof, yet)

- Show φ is a sheaf mor

$\sigma \longmapsto t$ with $\exists$ open cover U_i, $\sigma|_{U_i} = \theta_{U_i}(s_i)$
$r^U_{U_i} t = \psi_{U_i}(s_i)$

$$\begin{tikzcd}
\bar{\mathcal{F}}(U) \arrow[r, "\varphi_U"] \arrow[d, "r^U_V"'] & \mathcal{G}(U) \arrow[d, "r_{\mathcal{G}}{}^U_V"] \\
\bar{\mathcal{F}}(V) \arrow[r, "\varphi_V"] & \mathcal{G}(V)
\end{tikzcd}$$

$\bar{r}^U_V\sigma \qquad r_{\mathcal{G}}{}^U_V$

open cover U_i with $r^{V_i}_{U_i\cap V} r^U_{U_i} t = r^U_{U_i\cap V} t = \psi_{U_i}(s_i)|_{U_i\cap V}$
$= \psi_{U_i\cap V}(r^{U_i}_{U_i\cap V} s_i)$

$\bar{r}^U_V\sigma \longmapsto t'$ with oper cover $\{U_i\cap V\}$, $\bar{r}^U_V\sigma|_{U_i\cap V} = \theta_{U_i}(s_i)|_{U_i\cap V}$
$= \theta_{U_i\cap V}(r^{U_i}_{U_i\cap V} s_i)$

$$r^U_{U_i\cap V} t' = \psi_{U_i\cap V}(r^{U_i}_{U_i\cap V} s_i)$$

Hence we have diagram commutes.

- show φ is uniqueness. Let $\varphi' : \bar{\mathcal{F}} \to \mathcal{G}$ be any sheaf mor rendering diagram commute. Since sheaf mor equality can be tested at stalk level, we have:

$$\begin{tikzcd}
\mathcal{F}_x \arrow[r, "\psi_x"] \arrow[d, "\theta_x"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}_x \\
\bar{\mathcal{F}}_x \arrow[ur, "\varphi_x,\, \varphi'_x"'] &
\end{tikzcd}$$

Since sheafification doesn't change stalk, $\theta_x = id_{\mathcal{F}_x}$. Hence $\varphi_x = \psi_x = \varphi'_x$.
Two sheaf mors agree at each stalks are identified, so $\varphi = \psi$.

$\varphi, \psi : \mathcal{F} \to \mathcal{G}$. $\varphi = \psi \Leftrightarrow \varphi_x = \psi_x, \forall x \in X$.

$\square$

[Lemma] Let E, E' be vec. bdl. over X. Then there is an iso

$$\tau: \mathcal{E}(E)\otimes_{\mathcal{E}}\mathcal{E}(E') \xrightarrow{\simeq} \mathcal{E}(E\otimes E')$$

pf: (This is special case of $\Gamma(E\otimes F)=\Gamma(E)\otimes\Gamma(F)$ where E, F are blds over x and $\Gamma(E)$ is the sheaf of sections of E. But this proof is still instructive for how to prove a sheaf map is an iso)

Let $\mathcal{H}$ denote the presheaf $U \to \mathcal{E}(U,E)\otimes_{\mathcal{E}(U)}\mathcal{E}(U,E')$.

Idea:

1.
$$\begin{tikzcd} \mathcal{H} \arrow[r, "\psi"] \arrow[d, "\theta"'] & \mathcal{E}(E\otimes E') \\ \widetilde{\mathcal{H}} \arrow[ur, dashed, "\exists!\varphi"'] & \end{tikzcd}$$
easy to define

$\rightsquigarrow$ We construct $\varphi: \widetilde{\mathcal{H}} = \mathcal{E}(E)\otimes_{\mathcal{E}}\mathcal{E}(E') \to \mathcal{E}(E\otimes E')$

2. Show φ is an iso. If ψ is an iso, then ψ_x is iso with $\theta_x = id_{\mathcal{H}_x}$, we have φ_x is iso. Try to show ψ is iso.

- Define $\psi: \mathcal{H}(U) = \mathcal{E}(U,E)\otimes_{\mathcal{E}(U)}\mathcal{E}(U,E') \to \mathcal{E}(E\otimes E')(U)$ as following

$$\begin{tikzcd} (\xi,\eta) \arrow[r, mapsto] & {[x\mapsto \xi(x)\otimes\eta(x)]} \\ \mathcal{E}(U,E)\times\mathcal{E}(U,E') \arrow[r, "\phi"] \arrow[d, "\nu"'] \arrow[dr, phantom, "?"] & \mathcal{E}(E\otimes E')(U) \\ \mathcal{E}(U,E)\otimes_{\mathcal{E}(U)}\mathcal{E}(U,E') \arrow[ur, dashed, "\exists!\psi"'] & \\ \xi\otimes\eta & \end{tikzcd}$$

basis of $E_x\otimes E'_x$

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basis of $\mathcal{E}(U,E)\otimes_{\mathcal{E}(U)}\mathcal{E}(U,E')$

For any $\gamma\in\mathcal{E}(E\otimes E')(U)$, $\gamma(x)\in E_x\otimes E'_x$. Denote $f=(e_1,\cdots,e_r)$ and $f'=(e_1',\cdots,e_r')$ be frames of E and E'. we have $\gamma(x)=\sum a_i(f)\, b_j(f')\, e_i(x)\otimes e_j(x)$. Let $\xi\in\mathcal{E}(U,E)$ with $\xi(x)=\sum_i a_i(f)\, e_i(x)$ and $\eta\in\mathcal{E}(U,E')$ with $\eta(x)=\sum_j b_j(f)\, e_j(x)$, then $\gamma=\phi(\xi,\eta)$. So ϕ is surj, i.e., ψ is surj.

Let $\psi(\xi\otimes\eta)=\psi(\xi'\otimes\eta')$, so $\psi(\xi\otimes\eta)(x)=\psi(\xi'\otimes\eta')(x)$, i.e., $\xi(x)\otimes\eta(x)=\xi'(x)\otimes\eta'(x)$. Let $\xi=\xi_i e_i$, $\eta=\eta_j e_j'$, $\xi'=\xi_i' e_i$, $\eta'=\eta_j' e_j'$.
So $\xi_i(x)\eta_j(x)\, e_i(x)\otimes e_j'(x)=\xi_i'(x)\eta_j'(x)\, e_i(x)\otimes e_j'(x)$. So $\xi_i(x)\eta_j(x)=\xi_i'(x)\eta_j'(x)$ for all $x\in U$, i.e., $\xi_i\eta_j=\xi_i'\eta_j'\in\mathcal{E}(U)$. $\xi\otimes\eta=\xi_i\eta_j\, e_i\otimes e_j'=\xi_i'\eta_j'\, e_i\otimes e_j'$, $=\xi'\otimes\eta'$. So ψ is inj and thus ψ is iso.

- Let φ be mor induced by ψ. Show φ is an iso.

$$\begin{tikzcd} \mathcal{H}(U) \arrow[r, "\psi_U"] \arrow[d, "\theta_U"'] & \mathcal{E}(E\otimes E')(U) \\ \bar{\mathcal{H}}(U) \arrow[ur, dotted, "\varphi"'] & \end{tikzcd}$$

$\xi_i \eta_j\, e_i \otimes e_j' = \xi_i' \eta_j'\, e_i \otimes e_j'$

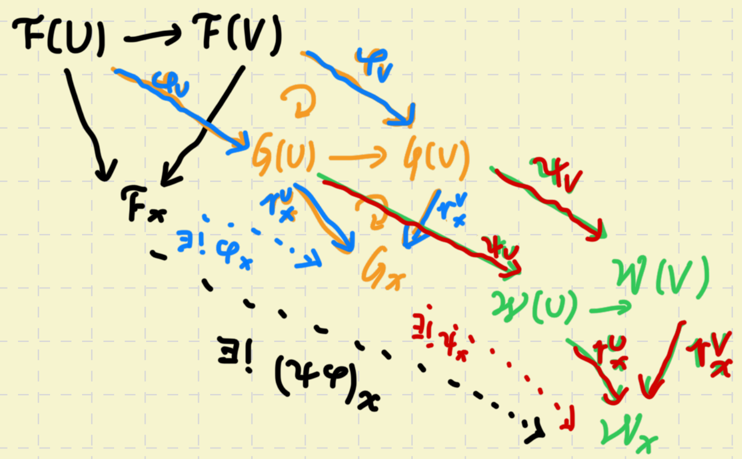
At stalk level, we have

$$\begin{tikzcd} \mathcal{H}_x \arrow[r, "\psi_x"] \arrow[d, "\mathrm{id}\,=\,\theta_x"'] & \mathcal{E}(E\otimes E')_x \\ \bar{\mathcal{H}}_x \arrow[ur, dotted, "\varphi_x"'] & \end{tikzcd}$$

ψ iso $\Leftrightarrow \psi_x$ iso, $\forall x \in X$
$\Leftrightarrow \varphi_x$ iso, $\forall x \in X$
$\Leftrightarrow \varphi$ iso.

Prop: Let $\varphi: \mathcal{F} \to \mathcal{G}$ be sheaf mor. φ is iso $\Leftrightarrow \varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is iso for $\forall x \in X$.

φ is iso $\Leftrightarrow \exists\, \varphi^{-1}$ s.t. $\varphi^{-1}\varphi = \mathrm{id}. \Leftrightarrow (\varphi^{-1}\varphi)_x = \mathrm{id}_x \quad \forall x \in X.$
$\Leftrightarrow (\varphi^{-1})_x \varphi_x = \mathrm{id}_x, \ \forall x \in X$
$\Leftrightarrow \varphi_x$ is iso, $\forall x \in X$.



$\Rightarrow (\psi\varphi)_x = \varphi_x \circ \psi_x$.

[Coro] Let E be a bundle over X. Then $\mathcal{E}^p \otimes_{\mathcal{E}} \mathcal{E}(E) \cong \mathcal{E}^p(E)$.

pf: Note that $\mathcal{E}^p = \mathcal{E}(\wedge^p T^*M)$. So

$$\mathcal{E}^p \otimes_{\mathcal{E}} \mathcal{E}(E) = \mathcal{E}(\wedge^p T^*M) \otimes_{\mathcal{E}} \mathcal{E}(E) \overset{\text{By above prop}}{=} \mathcal{E}(\wedge^p T^*M \otimes E) \overset{\text{By def}}{=:} \mathcal{E}^p(E)$$

[Rmk] Differential form of degree p valued in E $\quad \mathcal{E}^p(E) = \mathcal{E}(\wedge^p T^*M \otimes E)$
is $\underbrace{\mathcal{E}^p}_{\text{diff. form. of degree } p} \otimes_{\mathcal{E}} \underbrace{\mathcal{E}(E)}_{\text{valued in } E}$, as it sounds.

Local representation of diff. form of degree p valued in E:

For $\xi \in \mathcal{E}(\Lambda^p T^*M \otimes E)(U)$. On U, there is a frame $(\omega_1, \cdots, \omega_s)$ of $\Lambda^p T^*M$ and frame $(e_1, \cdots, e_r)$ of E.

1. View $\xi \in \mathcal{E}(U, \Lambda^p T^*M \otimes E)$

$\xi(x) \in \Lambda^p T_x^*M \otimes E_x$, so $\xi(x) = \sum a_{ij}(x) b_i(x) \otimes c_j(x)$ where $b_i \in \mathcal{E}(U, T_x^*M)$, $c_j \in \mathcal{E}(U, E)$. $b_i = \sum f_{ik} \omega_k$ and $c_j = \sum g_{jl} e_l$. So

$$\xi(x) = \sum a_{ij}(x) f_{ik}(x) g_{jl}(x) \omega_k(x) \otimes e_l(x) =: \sum \xi_{kl}(x) \omega_k(x) \otimes e_l(x).$$

$\xi_{kl} = \sum a_{ij} f_{ik} g_{jl}$, so ξ is a section, i.e., $\xi_{kl} \in \mathcal{E}(U)$

Hence $\xi = \sum \xi_{kl} w_k \otimes e_l$.

2. View $\xi \in \mathcal{E}(\Lambda^p T^*M) \otimes \mathcal{E}(E) \cong \mathcal{E}^p(E)$.

we've prove $\mathcal{E}(U, \Lambda^p T^*M) \otimes \mathcal{E}(U, E) \cong \mathcal{E}^p(U, E)$

$$a \otimes b \longmapsto a \cdot b = [x \mapsto a(x) \otimes b(x)]$$

$\xi \in \mathcal{E}^p(U, E)$ means $\xi = \xi_{ij} a_i \cdot b_j$

$$= \xi_{ij} a_i b_{jk} \cdot e_k.$$

$$= f_k \cdot e_k., \quad f_k \in \mathcal{E}(U, \Lambda^p T^*M) = \mathcal{E}^p(U).$$

So we have an iso
$$\mathcal{E}^p(U, E) \longrightarrow [\mathcal{E}^p(U)]^r$$
$$\xi \longmapsto \begin{bmatrix} \xi^1(f) \\ \vdots \\ \xi^r(f) \end{bmatrix} \text{ with } \xi = \sum \underbrace{\xi^\rho(f)}_{\mathcal{E}^p(U)} \cdot \underbrace{e_\rho}_{\text{frame}}$$

[Def] Let $E \to X$ be a vect. bdl. Then a connection D on $E \to X$ is a $\mathbb{C}$-linear mapping $D: \mathcal{E}(X, E) \longrightarrow \mathcal{E}^1(X, E)$ satisfies
$$D(\varphi\xi) = \underbrace{d\varphi}_{\mathcal{E}^1(X)} \cdot \underbrace{\xi}_{\mathcal{E}(X,E)} + \underbrace{\varphi}_{\mathcal{E}(X)} \underbrace{D\xi}_{\mathcal{E}^1(X,E)}$$
where $\varphi \in \mathcal{E}(X)$, $\xi \in \mathcal{E}(X, E)$

[Rmk] $D(\varphi\xi) = d\varphi \cdot \xi + \varphi D\xi$ implies D is a first-order diff. operator. mapping $\mathcal{E}(X, E)$ to $\mathcal{E}^1(X, E) = \mathcal{E}(X, \Lambda^1 T^*X \otimes E)$.

[Exp] When $E = X \times \mathbb{C}$ is the trivial line bdl, we can take ordinary exterior differentiation as a connection on E.

$$\mathcal{E}^1(X, E) = \mathcal{E}(X, \Lambda^1 T^*X \otimes (X \times \mathbb{C})) \overset{\text{trivial bdl has global frame}}{\cong} \mathcal{E}^1(X) \otimes \mathcal{E}(X \times \mathbb{C})$$

$= \mathcal{E}'(X) \otimes \mathcal{E}(X) = \mathcal{E}'(X)$. So $D: \mathcal{E}(X) \longrightarrow \mathcal{E}'(X)$ with $D(\varphi\xi) = d\varphi \cdot \xi + \varphi D\xi$ can be chosen as d.

[Slogan] Connection is a generalization of exterior diff. d.

Local description of a connection: Let $f = (e_1, \dots, e_r)$ be a frame over U for vec bdl $E \to X$ equipped with a connection D.

$$D : \mathcal{E}(U, E) \longrightarrow \mathcal{E}'(U, E)$$

element has form $a_i(f) e_i$, $a_i(f) \in \mathcal{E}(U)$

element has form $b_i(f) \cdot e_i$ with $b_i(f) \in \mathcal{E}'(U)$

This motivate us to focus on De_σ.

Define as $\theta_{\rho\sigma}(D,f)$ not $\theta_{\sigma\rho}(D,f)$, the reason is to obtain $D\xi(f) = (d+\theta(f))\xi(f)$

$$De_\sigma = \theta_{\rho\sigma}(D,f) \cdot e_\rho \quad \text{where } \theta_{\rho\sigma}(D,f) \in \mathcal{E}'(U)$$

means it's dependent on D and f

So we define the connection matrix $\theta(D,f) = [\theta_{\rho\sigma}(D,f)]$, when D fixed we denote it by $\theta(f)$.

For $\xi \in \mathcal{E}(U, E)$, $D\xi = D(\xi^\rho(f) e_\rho) = \cancel{\xi^\rho(f) De_\rho}$

$$= d\xi^\rho(f) \cdot e_\rho + \xi^\rho(f) De_\rho$$

$$= d\xi^\rho(f) \cdot e_\rho + \xi^\rho(f)\, \theta_{\sigma\rho}(f) e_\sigma$$

$$= \left[d\xi^\sigma(f) + \xi^\rho(f)\, \theta_{\sigma\rho}(f) \right] \cdot e_\sigma$$

Let $d\xi(f)$ be the matrix $\begin{bmatrix} d\xi^1(f) \\ \vdots \\ d\xi^r(f) \end{bmatrix}$, $\xi(f) = \begin{bmatrix} \xi^1(f) \\ \vdots \\ \xi^r(f) \end{bmatrix}$ and $\theta(f) = [\theta_{\rho\sigma}(f)]$,

we have $D\xi(f) = d\xi(f) + \theta(f)\xi(f) = (d + \theta(f))\xi(f)$

In conclusion, we have $D = d + \theta(f)$ acting on $\xi(f) \in \mathcal{E}(U, E)$.

<Curvature>

[Slogan] connection D induces in a natural manner an element $\Theta_E(D) \in \mathcal{E}^2(X, \mathrm{Hom}(E,E))$ called curvature tensor.

Local description of $\mathcal{E}^p(X, Hom(E,E))$: $\chi \in \mathcal{E}^p(X, Hom(E,E))$ and $f = (e_1, \dots, e_r)$ be a frame for E over U in X.

$$\mathcal{E}^p(U, Hom(E,E)) \cong \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{E}(U, Hom(E,E)) .$$

$$\mathcal{E}(U, Hom(E,E)) = \mathfrak{M}_r(U) = \underbrace{\mathfrak{M}_r \otimes_{\mathbb{C}} \mathcal{E}(U)}_{\wr} \quad \text{where } \mathfrak{M}_r = M_{r\times r}(\mathbb{C})$$

$x \mapsto Hom(E_x, E_x)$
|| frame f
$\mathfrak{M}_r \in M_{r\times r}(\mathbb{C})$

an $r\times r$-matrix (a_{ij}) with $a_{ij} \in \mathcal{E}(U)$.

So $\mathcal{E}(U, Hom(E,E))$ is an $\mathcal{E}(U)$-module.

[Rmk] $\mathfrak{M}_r \otimes_{\mathbb{C}} \mathcal{E}(U) = \delta_{11} \otimes a_{11} + \delta_{12} \otimes a_{12} + \cdots + \delta_{ij} \otimes a_{ij} + \cdots$
where δ_{ij} is the matrix with only ij-entry equals to 1 and other entries vanishing. e.g. $\begin{bmatrix} x & x^2 \\ x^3 & x^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes x^2$
$+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x^3 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x^4 = \begin{bmatrix} 1\cdot x & 0\cdot x \\ 0\cdot x & 0\cdot x \end{bmatrix} + \cdots = \begin{bmatrix} x & x^2 \\ x^3 & x^4 \end{bmatrix}$

Hence
$$\begin{aligned} \mathcal{E}^p(U, Hom(E,E)) &\cong \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} (\mathfrak{M}_r \otimes_{\mathbb{C}} \mathcal{E}(U)) \\ &\cong \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} (\mathcal{E}(U) \otimes_{\mathbb{C}} \mathfrak{M}_r) \\ &\cong (\mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{E}(U)) \otimes_{\mathbb{C}} \mathfrak{M}_r \\ &\cong \mathcal{E}^p(U) \otimes_{\mathbb{C}} \mathfrak{M}_r . \end{aligned}$$

Hence, for any $\chi \in \mathcal{E}^p(U, Hom(E,E))$, under frame f we denote χ as an $r\times r$ matrix $\chi(f) = [\chi(f)_{\rho\sigma}]$, $\chi(f)_{\rho\sigma} \in \mathcal{E}^p(U)$.

Actually, χ determines a global homomorphism

$$\begin{aligned} \chi : \mathcal{E}(X, E) &\longrightarrow \mathcal{E}^p(X, E) \\ \xi &\longmapsto e_i \chi_{ij}(f) \xi(f)_j \end{aligned}$$

at x, $\exists U \ni x$ and frame f on U s.t. $\xi = e_i \xi_i$ where $\xi_i(f) \in \mathcal{E}^0(U)$

$\chi(f)_{ij} \in \mathcal{E}^p(U)$,
so $\chi(f)_{ij} \xi(f)_j \in \mathcal{E}^p(U) \Rightarrow \chi_{ji}(f) \xi_j(f) \in \mathcal{E}^p(X, E)$,

$\Rightarrow \chi$ is well-defined.

The following diagram commutes :

$$\begin{array}{ccc} \mathcal{E}(U,E) & \xrightarrow{\chi} & \mathcal{E}^p(U,E) \\ \wr\Vert & & \Vert\wr \\ [\mathcal{E}(U)]^r & \xrightarrow{\chi(f)} & [\mathcal{E}^p(U)]^r \\ \xi(f)=(\xi_i(f)) & \longmapsto & \end{array}$$

Change of frame : $eg\chi(fg)\xi(fg) = e\,\chi(f)\xi(f)$

$$\overset{\shortparallel}{eg\chi(fg)g^{-1}\xi(f)} \Rightarrow g\chi(fg)g^{-1} = \chi(f)$$

$$\Rightarrow \chi(fg) = g^{-1}\chi(f)g$$

$\Big(e_i g_{ij}\xi(fg)_j = e_i\xi(f)_i \Rightarrow g_{ij}\xi(fg)_j = \xi(f)_i \Rightarrow g\xi(fg) = \xi(f)$
$\Rightarrow \xi(fg) = g^{-1}\xi(f). \Big)$

Conversely, any matrix of p-forms $\chi(f)$ to given a frame which is defined for all frames and satisfies $\chi(fg) = g^{-1}\chi(f)g$ defines an element $\chi \in \mathcal{E}^p(\underline{\underline{X}}, \mathrm{Hom}(E,E))$.
global section

[Construction] (Curvature) Let $E \to X$ be a vect bdl with a connection D and let $\theta(f) = \theta(D,f)$ be associated connection matrix.
We define $\Theta(D,f) = d\theta(f) + \theta(f)\wedge\theta(f) \in \mathfrak{M}_r \otimes \mathcal{E}^2(U)$, an $r\times r$ matrix of 2-forms.

[Rmk] $\Theta_{\rho\sigma} = d\theta_{\rho\sigma} + \sum \theta_{\rho k}\wedge\theta_{k\sigma}$

[Prop] $[d+\theta(f)][d+\theta(f)]\xi(f) = \Theta(f)\xi(f)$

pf:

$$(d+\theta)(d+\theta)\xi = \underbrace{d^2\xi}_{0} + d(\theta\cdot\xi) + \theta\cdot d\xi + \theta\wedge\theta\cdot\xi$$

$$= d\theta\cdot\xi - \theta\cdot d\xi + \theta\cdot d\xi + \theta\wedge\theta\cdot\xi$$

$$= d\theta\cdot\xi + \theta\wedge\theta\cdot\xi = \Theta\cdot\xi$$

[Prop] Let g be a change of frame.

(a) $dg + \theta(f)g = g\theta(fg)$

(b) $\Theta(fg) = g^{-1}\Theta(f)g$

pf: (a) Let $e' = eg$.

$D(e'_\sigma) = (eg)_m \theta(fg)_{m\sigma} = e_m g_{ml} \theta(fg)_{l\sigma}$

$D(e'_\sigma) = D(e_l g_{l\sigma}) = D(g_{l\sigma} e_l) = dg_{l\sigma} \cdot e_l + g_{l\sigma} De_l$

$= dg_{l\sigma} \cdot e_l + g_{l\sigma} e_m \theta(f)_{ml}$

$= (dg_{m\sigma} + g_{l\sigma} \theta(f)_{ml}) \cdot e_m$

Hence $g_{ml} \theta(fg)_{l\sigma} = dg_{m\sigma} + g_{l\sigma} \theta(f)_{ml}$, i.e.,

$g\theta(fg) = dg + \theta(f) g \quad (*)$

(b) Take exterior derivative of $(*)$, we obtain

$dg \cdot \theta(fg) + g\, d\theta(fg) = d\theta(f) \cdot g - \theta(f) dg \quad (☆)$ ($\theta(f)$ is 1-form) (g is 0-form)

By $(*)$ we have $\theta(fg) = g^{-1} dg + g^{-1} \theta(f) g \quad (\Delta)$

Take (Δ) to ☆ we have $g[d\theta(fg) + \theta(fg) \wedge \theta(fg)]$

Then we can show

$g(d\theta(fg) + \theta(fg) \wedge \theta(fg)) = (d\theta(f) + \theta(f) \wedge \theta(f)) g$ $\square$

[Def] Let D be a connection in a vect. bdl $E \to X$. The curvature $\Theta_E(D)$ is defined to be $\Theta \in \mathcal{E}^2(X, Hom(E,E))$ s.t. the $\mathbb{C}$-linear mapping $\Theta : \mathcal{E}(X,E) \longrightarrow \mathcal{E}^2(X,E)$ has the representation w.r.t. a frame

$\Theta(D,f) = d\theta(f) + \theta(f) \wedge \theta(f)$

[Rmk] We've discussed that any matrix of p-form defined for any frames and satisfying $g\chi(fg)g^{-1} = \chi(f)$ is an element in $\mathcal{E}^p(X, Hom(E,E))$, so Θ can be defined globally.

<Generalizing> Goal: define the action of D on higher-order differential forms.

[Def] We define the covariant differentiation

$$D: \mathcal{E}^p(X,E) \longrightarrow \mathcal{E}^{p+1}(X,E)$$

satisfying $D\xi(f) = d\xi(f) + \theta(f)\wedge\xi(f)$ (θ is still 1-form)

[Rmk] We need to show whether $\theta\xi(f) \in \mathcal{E}^{p+1}(X,E)$, i.e, we need to check it's invariant under frame changing, which is to show

$$g\big(d\xi(fg) + \theta(fg)\xi(fg)\big) = d\xi(f) + \theta(f)\wedge\xi(f)$$

$$\text{L.H.S.} = d(\underbrace{g\cdot\xi(fg)}_{\xi(f)}) - dg\cdot\xi(fg) + \underbrace{g\theta(fg)}_{g\theta(fg) = dg+\theta(f)g}\wedge\xi(fg)$$

$$= d\xi(f) \underbrace{- dg\cdot\xi(fg) + dg\cdot\xi(fg)}_{0} + \theta(f)\wedge\underbrace{g\,\xi(fg)}_{\xi(f)}$$

$$= d\xi(f) + \theta(f)\wedge\xi(f) = \text{R.H.S.}$$

we've proved $D^2 = \Theta$ for $p=0$. Actually, we have generalization:

[Def] $\Theta = D^2 : \mathcal{E}^p(X,E) \longrightarrow \mathcal{E}^{p+2}(X,E)$

[Rmk] still need to show "$g\Theta(fg)g^{-1} = \Theta(f)$", making sure $\Theta \in \mathcal{E}^2(X, \mathrm{Hom}(E,E))$. The fact is $g\Theta(fg)g^{-1} = \Theta(f)$ still holds for $\Theta = D^2$, $p>0$.

[Rmk] Curvature Θ is the obstruction to $D^2=0$, i.e., the obstruction to forming complex:

$$\mathcal{E}^0(X,E) \xrightarrow{D} \mathcal{E}^1(X,E) \xrightarrow{D} \mathcal{E}^2(X,E) \longrightarrow \cdots$$

⟨Bianchi identity⟩

Let $\mathcal{E}^*(X, \mathrm{Hom}(E,E)) = \sum_p \mathcal{E}^p(X, \mathrm{Hom}(E,E))$.

Goal: define a Lie product on $\mathcal{E}^*(X, \mathrm{Hom}(E,E))$, making it into a Lie alg.

[Construction] Let $\chi \in \mathcal{E}^p(X, \mathrm{Hom}(E,E))$ and f a frame for E over open U. We've proved that $\chi(f) \in \mathfrak{M}_r \otimes \mathcal{E}^p(U)$.

Let $\psi \in \mathcal{E}^q(X, \mathrm{Hom}(E,E))$, $\psi(f) \in \mathfrak{M}_r \otimes \mathcal{E}^q(U)$. We define

$$[\chi(f), \psi(f)] = \underline{\chi(f) \wedge \psi(f) - (-1)^{pq} \psi(f) \wedge \chi(f)}$$ matrix multiplication

Change of frame:

With $\begin{cases} \chi(fg) = g^{-1}\chi(f) g \\ \psi(fg) = g^{-1}\psi(f) g \end{cases}$, we have

$$[\chi(fg), \psi(fg)] = g^{-1}\chi(f) \wedge \psi(f) g - (-1)^{pq} g^{-1}\psi(f) \wedge \chi(f) g$$
$$= g^{-1}[\chi(f), \psi(f)] g$$

i.e., $[\chi, \psi](fg) = g^{-1}[\chi, \psi](f) g$, so

$$[\chi, \psi] \in \mathcal{E}^{p+q}(X, \mathrm{Hom}(E,E)),$$

making Lie bracket is well-defined. It satisfies Jacobian identity: $[\chi,[\psi,\varphi]] + [\psi,[\varphi,\chi]] + [\varphi,[\chi,\psi]] = 0$ making $\mathcal{E}^*(X, \mathrm{Hom}(E,E))$ into a Lie algebra.

More details in Lie alg, see Ref: http://staff.ustc.edu.cn/~wangzuoq/Courses/23F-Manifolds/Notes/Lec15.pdf

[Prop] $d\Theta(f) = [\Theta(f), \theta(f)]$

pf: Since $\Theta = d\theta + \theta\wedge\theta$, we have

$$d\Theta = d^2\theta + d\theta\wedge\theta - \theta\wedge d\theta = d\theta\wedge\theta - \theta\wedge d\theta \qquad (\theta \text{ is } 1\text{-form})$$

$$[\Theta, \theta] = [d\theta + \theta\wedge\theta, \theta] = [d\theta, \theta] + [\theta\wedge\theta, \theta]$$
$$= d\theta\wedge\theta - (-1)^{1\cdot 2}\theta\wedge d\theta + \underbrace{\theta\wedge\theta\wedge\theta - (-1)^{1\cdot 2}\theta\wedge\theta\wedge\theta}_{=0}$$
$$= d\theta\wedge\theta - \theta\wedge d\theta$$

< Does vec. bdl. admits a connection compatible with Hermitian metric ? >

Recall Hermitian metric on Hermitian vec bdl is a sm choice of Hermitian inner product at each fiber $\langle a,b\rangle_x \in \mathbb{C}$ where $a,b \in E_x$. The following we'll generalize this inner product to E-valued covectors, i.e., define a mapping $h: \mathcal{E}^p(X,E) \otimes \mathcal{E}^q(X,E) \to \mathcal{E}^{p+q}(X)$

[Construction] $\mathcal{E}^p(X,E) = \mathcal{E}(X, \wedge^p T^*M \otimes E) = \mathcal{E}(X, \wedge^p T^*X) \otimes \mathcal{E}(X,E)$.

A basis of it has the form $\omega\otimes\xi$ where $\omega \in \mathcal{E}(X, \wedge^p T^*X)$

and $\xi \in \mathcal{E}(X, E)$. Let $h(\omega\otimes\xi, \omega'\otimes\xi') = \omega\wedge\overline{\omega'}\,\langle \xi, \xi'\rangle$, this extends to a mapping

$$h: \mathcal{E}^p(X,E)\otimes\mathcal{E}^q(X,E)\longrightarrow \mathcal{E}^{p+q}(X)$$
$$(\omega\otimes\xi)\otimes(\omega'\otimes\xi')\longmapsto(\omega\wedge\overline{\omega'})\otimes\langle\xi,\xi'\rangle$$

Similary, we define a matrix $h(f)$ for a frame U on E. with $h(f)_{\rho\sigma} = \langle e_\sigma, e_\rho\rangle$.

[Def] A connection on D is said to be compatible with a Hermitian metric h on E if $d\langle\xi,\eta\rangle = \langle D\xi,\eta\rangle + \langle\xi, D\eta\rangle$

[Prop] D compatible with h if and only if $dh = h\theta + {}^t\overline{\theta}h$.

pf: $\Rightarrow$ Suppose D compatible with h.

$$\begin{aligned} dh_{\rho\sigma} = d\langle e_\sigma, e_\rho\rangle &= \langle De_\sigma, e_\rho\rangle + \langle e_\sigma, De_\rho\rangle \\ &= \langle e_\tau\theta_{\tau\sigma}, e_\rho\rangle + \langle e_\sigma, e_\mu\theta_{\mu\rho}\rangle \\ &= \theta_{\tau\sigma}\langle e_\tau, e_\rho\rangle + \overline{\theta}_{\mu\rho}\langle e_\sigma, e_\mu\rangle \\ &= \theta_{\tau\sigma}h_{\rho\tau} + \overline{\theta}_{\mu\rho}h_{\mu\sigma} \\ &= (h\theta)_{\rho\sigma} + ({}^t\overline{\theta}h)_{\rho\sigma} \quad \Rightarrow dh = h\theta + {}^t\overline{\theta}h. \end{aligned}$$

$\Leftarrow$ Suppose $dh = h\theta + {}^t\overline{\theta}h$ satisfied for all frames.

$$\langle\xi,\eta\rangle = \langle e_\rho\xi_\rho, e_\sigma\eta_\sigma\rangle = \xi_\rho\overline{\eta}_\sigma h_{\sigma\rho} = \overline{\eta}_\sigma h_{\sigma\rho}\xi_\rho$$

Hence $\langle\xi,\eta\rangle = {}^t\overline{\eta}h\xi$. So $d\langle\xi,\eta\rangle = d({}^t\overline{\eta}h\xi) =$

$${}^t(d\overline{\eta})h\xi + {}^t\overline{\eta}\,\underbrace{dh}_{h\theta+{}^t\overline{\theta}h}\,\xi + {}^t\overline{\eta}h\,d\xi$$
$$= \underline{{}^t(d\overline{\eta})h\xi} + {}^t\overline{\eta}h\theta\xi + \underline{{}^t\overline{\eta}\,{}^t\overline{\theta}h\xi} + {}^t\overline{\eta}h\,d\xi$$
$$= \langle\xi, d\eta + \theta\eta\rangle + \langle\theta\xi + d\xi, \eta\rangle$$
$$= \langle\xi, D\eta\rangle + \langle D\xi, \eta\rangle$$

[Prop] $E\to X$ is a Hermitian vec. bdl.. Then there exists a connection D on E compatible with the Hermitian metric on E.

pf: We can use Schmidt orthogonalization to find orthonormal basis for E_x at all pts x near x_0, hence we can find a locally finite covering U_α and unitary frames f_α U_α s.t. $h(f_\alpha) = I$. Then D compatible with h on frame $\{f_\alpha\}$ $\Leftrightarrow$ $0 = \theta(f_\alpha) + {}^t\bar{\theta}(f_\alpha)$ $(dh = h\theta + {}^t\bar{\theta}h)$. The following are constructions of D. Pick $\theta(f_\alpha) =: \theta_\alpha = 0$. Then $\underbrace{\theta(f_\alpha g)}_{\overset{\#}{0}\ (\theta(f_\alpha)=0)} = g^{-1}dg + 0$ $(\theta(fg) = g^{-1}dg + g^{-1}\theta(f)g)$

$f_\alpha g$ represents any frame. So we define $\theta(f_\alpha g) = g^{-1}dg$.

We want to show $dh(f_\alpha g) = h(f_\alpha g)\theta(f_\alpha g) + {}^t\bar{\theta}(f_\alpha g)h(f_\alpha g)$, i.e., D compatible with h.

Since $h(f_\alpha g) = {}^t\bar{g}\underbrace{h(f)}_{= I}g = {}^t\bar{g}g$, $L.H.S. = d({}^t\bar{g}\cdot g) = d\,{}^t\bar{g}\cdot g + {}^t\bar{g}\,dg$

$$\left.\begin{array}{l} {}^t\bar{\eta}(fg)h(f_\alpha g)\xi(fg) = {}^t\bar{\eta}(f)h(f)\xi(f) \\ \qquad \| \\ {}^t\overline{g\eta(f)}\,h(f_\alpha g)\,g\xi(f) = {}^t\bar{\eta}(f){}^t\bar{g}\,h(f_\alpha g)g\xi(f) \end{array}\right\} \Rightarrow {}^t\bar{g}\,h(f_\alpha g)g = h(f)$$

$R.H.S. = {}^t\bar{g}g\,g^{-1}dg + d\,{}^t\bar{g}\cdot\underbrace{{}^t\bar{g}^{-1}}_{\#} = {}^t\bar{g}\,dg + d\,{}^t\bar{g}\cdot g$

g is a change of frame, which is unitary

${}^t\bar{g}g = I \Rightarrow g^{-1} = {}^t\bar{g}$

The rest is about passing local to global, clearly, we need partition of unity. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$ and let D_α be the connection in $E|_{U_\alpha}$ defined by $(D_\alpha\xi)(f_\alpha) = d\xi(f_\alpha)$. $(D\xi = d\xi + \theta\xi = d\xi$, since $\theta = 0)$

$(D_\alpha\xi)(f_\alpha g) = d\xi(f_\alpha g) + \theta(f_\alpha g)\xi(f_\alpha g) = d(g\xi(f_\alpha)) + g^{-1}dg\cdot g\xi(f_\alpha)$.

D_α is compatible with the Hermitian metric on $E|_{U_\alpha}$ by construction.

Let $D = \sum_\alpha \varphi_\alpha D_\alpha$ which is $D: \mathcal{E}(X,E) \to \mathcal{E}^1(X,E)$.

It suffices to show D is compatible with metric h on E.

$$\langle D\xi, \eta\rangle + \langle \xi, D\eta\rangle = \sum \varphi_\alpha[\langle D_\alpha\xi, \eta\rangle + \langle \xi, D_\alpha\eta\rangle]$$
$$= \sum \varphi_\alpha\, d\langle \xi, \eta\rangle = d\langle \xi, \eta\rangle. \qquad \square$$

[Rmk] From the proof, we find connection compatible with metric is not unique. But for holomorphic vec. bdl., we can obtain a unique connection satisfying an additional restriction on type of θ

§ Special cases: Hermitian holomorphic vec. bdl's

holo^m vec. bdl. equipped with a differentiable Hermitian metric

<Connection>

- X is a complex m.f. and $E \to X$ a holo^m vec. bdl.

1. $\mathcal{E}^*(E) = \sum_r \mathcal{E}^r(E) = \sum_{p,q} \mathcal{E}^{p,q}(E)$ $\left(\sum_r \sum_{p+q=r} \mathcal{E}^{p,q}(E)\right)$

where $\mathcal{E}^{p,q}(E) = \mathcal{E}_X^{p,q} \otimes_{\mathcal{E}_X} \mathcal{E}(E)$

2. Suppose we have a connection on E

$$D: \mathcal{E}(X,E) \longrightarrow \mathcal{E}'(X,E) = \mathcal{E}^{0,1}(X,E) \oplus \mathcal{E}^{1,0}(X,E)$$

Then D splits into $D = D' + D''$ where

$$D': \mathcal{E}(X,E) \longrightarrow \mathcal{E}^{1,0}(X,E)$$
$$D'': \mathcal{E}(X,E) \longrightarrow \mathcal{E}^{0,1}(X,E)$$

[Thm] Let h be a Hermitian metric on a holo^m vec. bdl. $E \to X$, then h induces canonically a connection $D(h)$ on E, satisfying:

(a) D is compatible with metric h, i.e., for $\forall\, W \overset{\text{open}}{\subseteq} X$, $\xi, \eta \in \mathcal{E}(W,E)$,

$$d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle$$

(b) If $\xi \in \underset{\text{holo}^m\text{ section}}{\underline{\mathcal{O}}(W,E)}$, then $D''\xi = 0$, where $D'': \mathcal{E}(X,E) \to \mathcal{E}^{0,1}(X,E)$

Pf: For $\xi \in \mathcal{O}(W,E)$ and f a holo^m frame,

$$D\xi(f) = (d + \theta(f))\,\xi(f)$$
$$= (\partial + \bar\partial + \theta^{(1,0)}(f) + \theta^{(0,1)}(f))\,\xi(f).$$

So $D'\xi(f) = \underbrace{\partial\xi(f)}_{\in \mathcal{E}^{1,0}(W)} + \theta^{(1,0)}(f)\xi(f)$, $D''\xi(f) = \bar\partial\xi(f) + \theta^{(0,1)}(f)\,\xi(f)$.

ξ and f are holo^m, so $\bar\partial\xi(f) = 0$. Thus $D''\xi(f) = \theta^{(0,1)}(f)\,\xi(f)$.

Condition (b) $\Leftrightarrow D''\xi(f) = \theta^{(0,1)}(f)\xi(f) = 0, \forall \xi \in \mathcal{O}(W,E) \Leftrightarrow \theta^{(0,1)}(f) = 0$

$$\Leftrightarrow \theta(f) = \theta^{(1,0)}(f)$$

Suppose a connection D satisfying (a) and (b).

Condition (a) $\Leftrightarrow dh = h\theta + {}^t\bar\theta h \Leftrightarrow \partial h + \bar\partial h = h\theta + {}^t\bar\theta h$

In addition condition (b) $\Leftrightarrow \theta(f) = \theta^{(1,0)}(f)$ $\Downarrow$

$$\partial h + \bar\partial h = h\theta^{(1,0)} + {}^t\bar\theta^{(1,0)} h \Rightarrow \begin{cases} \partial h = h\theta^{(1,0)} = h\theta \\ \bar\partial h = {}^t\bar\theta^{(1,0)} h = {}^t\bar\theta h \end{cases}$$

So $\theta = h^{-1}\partial h$, and we can define θ by $h^{-1}\partial h$. Such θ satisfies condition (a) and condition (b).

h is Hermitian ${}^t\bar{h} = h$

$\Big(h\theta + {}^t\bar{\theta}h = hh^{-1}\partial h + {}^t\overline{h^{-1}\partial h}\, h = \partial h + {}^t\overline{\partial h}\, {}^t\bar{h}^{-1} h = \partial h + \bar{\partial} h$ so θ satisfies condition (a). Clearly $\theta^{(0,1)} = 0$, satisfying condition (b) $\Big)$

It suffices to show $g\theta(fg) = \theta(f)g + dg$, basis transformation. for D becoming a global section.

$$\text{L.H.S.} = g\,\theta(fg) = g\,h^{-1}(fg)\,\partial h(fg) = g g^{-1} h^{-1}(f) ({}^t\bar{g})^{-1}\, \partial({}^t\bar{g}\, h(f) g)$$

$h(fg) = {}^t\bar{g}\, h(f) g$

$h^{-1}(fg) = g^{-1} h^{-1}(f)\, {}^t\bar{g}^{-1}$

$$= h^{-1}(f) ({}^t\bar{g})^{-1} [{}^t\bar{g}\, \partial h(f) \cdot g + {}^t\bar{g}\, h(f)\, \partial g] = h^{-1}(f)\, \partial h(f) \cdot g + dg$$

$\partial {}^t\bar{g} = \overline{\bar{\partial}\, {}^t g} = 0$, $\partial g = dg$

$$= \theta(f) g + dg = \text{R.H.S.} \qquad \square$$

[Conclution] 1. canonical connection of holo^m vec bdl induced by h is $\theta(f) = h(f)^{-1}\partial h(f)$ for f holo^m. D

2. $D = D' + D''$ with $\begin{cases} D' = \partial + \theta(f) \\ D'' = \bar{\partial} \end{cases}$

Because of those simple formula, we have good formulas for curvature Θ.

<Curvature>

[Prop] Let D be the canonical connection of a Hermitian holo^m vec. bdl $E \to X$ with Hermitian metric h. Let $\theta(f)$, $\Theta(f)$ be connection and curvature matrices defined by D w.r.t. a holo^m frame f. Then

(a) $\theta(f)$ is of type $(1,0)$, and $\partial\theta(f) = -\theta(f) \wedge \theta(f)$.

(b) $\Theta(f) = \bar{\partial}\theta(f)$, and $\Theta(f)$ is of type $(1,1)$

(c) $\bar{\partial}\Theta(f) = 0$, and $\partial\Theta(f) = [\Theta(f), \theta(f)]$

pf: (a) Since $dh = h\theta + {}^t\bar{\theta}h$, we have $\partial h = h\theta$. (h holo^m)

$0 = \partial I = \partial(hh^{-1}) = \partial h \cdot h^{-1} + h \cdot \partial h^{-1} = h\theta h^{-1} + h\partial h^{-1}$.

Hence $\partial h^{-1} = -\theta h^{-1} = -h^{-1}\partial h \cdot h^{-1}$.

Then, $\partial\theta = \partial(h^{-1}\partial h) = \partial h^{-1} \wedge \partial h + h^{-1} \underbrace{\partial^2 h}_{=0}$

$$= -h^{-1}\partial h \wedge h^{-1}\partial h = -(h^{-1}\partial h) \wedge (h^{-1}\partial h) = -\theta \wedge \theta$$

(b) $\Theta = d\theta + \theta\wedge\theta = \underbrace{\partial\theta + \theta\wedge\theta}_{=0 \text{ by (a)}} + \bar\partial\theta = \bar\partial\theta$

(c) $\bar\partial\Theta = \bar\partial^2\theta = 0$. We've proved that $d\Theta = [\Theta, \theta]$,

so $\partial\Theta(f) = [\Theta(f), \theta(f)]$ $\square$

Let $f = (e_1, \cdots, e_r)$ be a frame for E defined near a pt $p \in X$. Choose local coord. $z = (z_1, \cdots z_n)$ near p, so that p is given by $z=0$. Then we have

1. $f(z) = (e_1(z), \cdots, e_r(z))$ ($z \in \mathbb{C}^n$ in Euclidean sp)
2. $h(z) = h(f(z))$
3. $\Theta(z) = \Theta(f(z))$

The following property allows us to compute curvature Θ at a point without having to compute the inverse of the $h(z)$.

[Prop] There exists a holo^m frame f s.t.

(a) $h(z) = I + O(|z|^2)$

(b) $\Theta(0) = \bar\partial\partial h(0)$

<Examples> A computable example see Exp 2.4 in Gtm 65.

§ Chern classes of differentiable vect. bdls.

⟨Multilinear algebra⟩

$\mathfrak{M}_r$ denote the set of $r \times r$ matrices with complex entries.

[Def] A k-linear form $\tilde{\varphi} : \mathfrak{M}_r \times \cdots \times \mathfrak{M}_r \longrightarrow \mathbb{C}$ is said to be invariant if $\tilde{\varphi}(gA_1g^{-1}, \cdots, gA_kg^{-1}) = \tilde{\varphi}(A_1, \cdots, A_k)$ for $\forall g \in GL(r, \mathbb{C})$, $A_i \in \mathfrak{M}_r$. Let $\tilde{I}_k(\mathfrak{M}_r)$ be the $\mathbb{C}$-vec. sp. of all invariant k-linear forms on $\mathfrak{M}_r$.

[Construction] Let $\tilde{\varphi} \in \tilde{I}_k(\mathfrak{M}_r)$. $\tilde{\varphi}$ induces $\varphi : \mathfrak{M}_r \longrightarrow \mathbb{C}$ with $\varphi(A) = \tilde{\varphi}(A, \cdots, A)$. Since $\tilde{\varphi} \in \tilde{I}_k(\mathfrak{M}_r)$, $\varphi(gAg^{-1}) = \tilde{\varphi}(gAg^{-1}, \cdots, gAg^{-1}) = \tilde{\varphi}(A, \cdots, A) = \varphi(A)$. So φ is inv.

$\varphi(\lambda A) = \tilde{\varphi}(\lambda A, \cdots, \lambda A) \underset{\text{linear}}{\overset{\text{multi-}}{=}} \lambda^k \tilde{\varphi}(A, \cdots, A) = \lambda^k A$. It holds for every A and every λ, so φ is a homogeneous poly. of degree k in the entries of A. Let $I_k(\mathfrak{M}_r)$ be the set of inv. homogeneous polys of degree k.
So $\varphi \in I_k(\mathfrak{M}_r)$.

Besides, from φ we can obtain $\tilde{\varphi}$ by polarization:

$$\tilde{\varphi}(A_1, \cdots, A_k) = \frac{(-1)^k}{k!} \sum_{j=1}^{k} \sum_{i_1 < \cdots < i_j} (-1)^j \varphi(A_{i_1} + \cdots + A_{i_j})$$

[Construction] (Action of φ on $\mathfrak{M}_r(U) \otimes_{\mathcal{E}} \mathcal{E}^p(U)$)

Let U open in X, consider φ act on $\mathfrak{M}_r(U) \otimes_{\mathcal{E}(U)} \mathcal{E}^p(U)$. For $\underbrace{A_i \cdot w_i}_{\text{Basis}} \in \mathfrak{M}_r(U) \otimes_{\mathcal{E}(U)} \mathcal{E}^p(U)$, we set $\underbrace{\varphi_U(A_1 \cdot w_1, \cdots, A_k \cdot w_k)}_{\text{defined on basis}} = w_1 \wedge \cdots \wedge w_k \, \varphi(A_1, \cdots, A_k)$

By linearity, φ extends to act on $\mathfrak{M}_r(U) \otimes_{\mathcal{E}(U)} \mathcal{E}^p(U)$.

[Construction] (Action of φ on $\mathcal{E}^p(U, \mathrm{Hom}(E,E))$)

Let $\chi_j \in \mathcal{E}^p(U, \mathrm{Hom}(E,E))$. For frame f on U, we set

$$\varphi_U(\chi_1, \cdots, \chi_k) = \varphi_U(\chi_1(f), \cdots, \chi_k(f))$$

Change of frame: $\varphi_U(\chi_1(fg), \cdots, \chi_k(fg)) = \varphi_U(g^{-1}\chi_1(f)g, \cdots, g^{-1}\chi_k(f)g)$
$= \varphi_U(\chi_1(f), \cdots, \chi_k(f))$

Hence this definition is independent of the choice of frame. Thus we get an extension of φ to all of X

$$\varphi_X : \mathcal{E}^p(X, \mathrm{Hom}(E,E)) \times \cdots \times \mathcal{E}^p(X, \mathrm{Hom}(E,E)) \longrightarrow \mathcal{E}^{pk}(X).$$

(Note that $\mathcal{E}^p(X, \mathrm{Hom}(E,E)) = \mathcal{E}^p(X) \otimes \mathfrak{M}_r$, φ acts on $\mathcal{E}^p(U) \otimes \mathfrak{M}_r$ has defined before) When we restrict to diagonal, we have

$$\varphi_X : \mathcal{E}^p(X, \mathrm{Hom}(E,E)) \longrightarrow \mathcal{E}^{pk}(X).$$

$\square$

Let $D : \mathcal{E}(X,E) \to \mathcal{E}^1(X,E)$ be a connection and $\Theta_E(D)$ be the curvature depending on bdl $E \to X$ and the connection. Since $\Theta_E(D)$ is a 2-form over X, $\varphi_X(\Theta_E(D))$ is a global $2k$-form on X.

[Thm] Let $E \to X$ be a differentiable $\mathbb{C}$-vect. bdl., let D be a connection on E and suppose $\varphi \in I_k(\mathfrak{M}_r)$. Then

(a) $\underbrace{\varphi_X(\Theta_X(D))}_{2k\text{-form}}$ is closed, i.e., $d\varphi_X(\Theta_X(D)) = 0$

(b) The image of $\varphi_X(\Theta_E(D))$ in the de Rham group $H^{2k}(X, \mathbb{C})$ is independent of the connection D.

proof can be find Thm 3.2 in Gtm 65.

<Chern classes of a differentiable vec. bdl.>

[Def] Define $\Phi_k \in I_k(\mathfrak{M}_r)$ by matrices satisfying

$$\det(I + A) = \sum_k \Phi_k(A) \quad , A \in \mathfrak{M}_r.$$

[Def] (Chern classes) Let $E \to X$ be a differentiable vec. bdl. equipped with a connection D. Then the k-th Chern form of E relative to the connection D is

$$C_k(E,D) := (\Phi_k)_X\left(\frac{i}{2\pi}\Theta_E(D)\right) \in \mathcal{E}^{2k}(X)$$

The (total) Chern form of E relative to D is defined to be

$$C(E,D) := \sum_{k=0}^{r} C_k(E,D) \quad , \quad r = \operatorname{rank} E.$$

The kth Chern class of the vec. bdl. E, denoted by $C_k(E)$, is the cohomology class of $C_k(E,D)$ in the de Rham group $H^{2k}(X, \mathbb{C})$ and the total Chern class of E, denoted by $c(E)$, is the cohomology class of $c(E,D)$ in $H^*(X, \mathbb{C})$, i.e.,

$c(E) = \sum_{k=0}^{r} C_k(E)$.

[Rmk] From above thm, we've known $dC_k(E,D) = dC(E,D) = 0$, hence $\underset{C_k(E)}{\overset{\shortparallel}{[C_k(E,D)]}}, \underset{C(E)}{\overset{\shortparallel}{[C(E,D)]}} \in H^{2k}(X, \mathbb{C})$. Besides, $C_k(E), C(E)$ are independent of connection D.

We'll see Chern classes are obstructions to finding global frames.

[Rmk] $C(E)=\sum_k C_k(E,D)=\sum_k (\Phi_k)_X(\frac{i}{2\pi}\Theta_E(D))=\det(I+\frac{i}{2\pi}\Theta_E(D))$

[Slogan] Chern classes are cohomology classes in the base space of the vect bdl.

[prop] Let D be a connection on a Hermitian vect. bdl. E compatible with the Hermitian metric h. Then the Chern form $c(E,D)$ is a REAL differential form and it follows that $c(E)\in H^*(X,\mathbb{R})$ under the Canonical inclusion $H^*(X,\mathbb{R})\subset H^*(X,\mathbb{C})$

pf: Since Φ_k is independent on choice of frame, we only need to show the Chern form is a real differential form for a local frame, then we choose unitary frame f, i.e., choose a orthonormal basis by Schimit orthonomalization, s.t. ${}^t\overline{h(f)}\,h(f)=I$, $h(f)$ unitary.

$$D \text{ compatible with } h \Leftrightarrow dh=h\theta+{}^t\bar\theta h \quad (\Delta)$$

$$\xrightarrow{d} 0=dh\wedge\theta+hd\theta+d{}^t\bar\theta\cdot h-{}^t\bar\theta\wedge dh \quad (\star)$$

Substituting dh in $(\star)$ by (Δ),

$$0=(h\theta+{}^t\bar\theta h)\wedge\theta+hd\theta+d{}^t\bar\theta\cdot h-{}^t\bar\theta\wedge(h\theta+{}^t\bar\theta h)$$

$$=h\theta\wedge\theta+\underline{{}^t\bar\theta h\wedge\theta}+hd\theta+d{}^t\bar\theta\cdot h-\underline{{}^t\bar\theta\wedge(h\theta)}-({}^t\bar\theta\wedge{}^t\bar\theta)h$$

${}^t\bar\theta h\wedge\theta = {}^t\bar\theta\wedge h\theta$

$\Theta=d\theta+\theta\wedge\theta$ $\quad =h\Theta+{}^t\bar\Theta h$

$$C(E)=\sum_k C_k(E,D,f)=\sum(\Phi_k)_X(\frac{i}{2\pi}\Theta_E(D))$$
$$=\det(I+\frac{i}{2\pi}\Theta_E(D))$$

It suffices to show $c=\bar c$. In the choosing frame (${}^t\bar h h=I$)

$$\det(I+\frac{i}{2\pi}\Theta)=\det(h+\frac{i}{2\pi}\Theta h)=\det(I+\frac{i}{2\pi}\Theta)\det h$$

$\Vert\ \Theta h=-h{}^t\bar\Theta$

$$\det(h-\frac{i}{2\pi}h{}^t\bar\Theta)=\det(h)\cdot\det(I-\frac{i}{2\pi}{}^t\bar\Theta)$$

Since $h(f)$ unitary, $\det(h)=1$, so we have

$$\det(I+\frac{i}{2\pi}\Theta)=\det(I-\frac{i}{2\pi}{}^t\bar\Theta).$$

Then $c=\det(I+\frac{i}{2\pi}\Theta)=\det(I-\frac{i}{2\pi}{}^t\bar\Theta)=\det(I-\frac{i}{2\pi}\bar\Theta)$

$$\bar c=\det(I-\frac{i}{2\pi}\bar\Theta)=c$$

${}^t(I+A)={}^tI+{}^tA$.

So $\det(I+A)=\det({}^t(I+A))=\det({}^tI+{}^tA)$
$=\det(I+{}^tA)$

$\square$

Cup product on de Rham grp $H^*(X,\mathbb{R})$: $c, c' \in H^*(X,\mathbb{R})$, $c=[\varphi]$, $c' := [\varphi']$, then $c \cdot c' := [\varphi \wedge \varphi']$. This is a representation of cup product in alg. topo.

Next we focus on computation formula for Chern classes.

[Thm] Suppose E and E' are differentiable $\mathbb{C}$-vect. bdls over a differentiable m.f. X. Then

(a) If $\varphi: Y \to X$ is a differentiable mapping where Y is a differentiable mf, then $c(\varphi^* E) = \varphi^* c(E)$ where $\varphi^* E$ is the pull back vect. bdl and $\varphi^* c(E)$ is the pullback of $c(E)$.

(b) $c(E \oplus E') = c(E) \cdot c(E')$

(c) $c(E)$ depends only on the isomorphism class of the vect bdl E.

(d) If E^* is the dual vect. bdl. to E, then $c_j(E^*) = (-1)^j c_j(E)$.

Pf:

(a) It suffices to find connection D^* on $\varphi^* E$ with
$$\Theta(D^*) = \varphi^*(\Theta(D)) \text{ where}$$

1. $\Theta(D^*) = d\theta(D^*) + \theta(D^*) \wedge \theta(D^*)$

2. $\varphi^*(\Theta(D)) \in \mathcal{E}^2(\varphi^{-1}U, \mathrm{Hom}(\varphi^* E, \varphi^* E)) = \mathcal{E}^2(\varphi^{-1}U) \otimes \mathfrak{M}_r$.

$\Theta(D) \in \mathcal{E}^2(X) \otimes \mathfrak{M}_r$, i.e., $\Theta(D) = \underbrace{(A_{ij})}_{\text{a matrix}}$ with $A_{ij} \in \mathcal{E}^2(\varphi^{-1}(U))$

then $\varphi^*(\Theta(D)) = (\varphi^* A_{ij})$ where φ^* are induced map on 2-forms.

If we find such D^*, we have
$$c(\varphi^* E) = \sum \Phi_k\left(\tfrac{i}{2\pi}\Theta(D^*)\right)$$
$$= \sum \Phi_k\left(\tfrac{i}{2\pi}\varphi^*(\Theta(D))\right)$$

$$\varphi^* c(E) = \sum \varphi^* \Phi_k\left(\tfrac{i}{2\pi}\Theta(D)\right)$$
$$= \sum \Phi_k\left(\tfrac{i}{2\pi}\Theta(D)\right)$$
they are equal by right diagram.

$\tilde{\Phi}_k$ is a k-linear form, so Φ_k has functority, i.e., diagram commutes

$$\begin{tikzcd}
\underset{(A_{ij})}{\mathcal{E}(U) \otimes \mathfrak{M}_r} \arrow[r, "\varphi^*"] \arrow[d, "\Phi_k"] & \underset{(\varphi^* A_{ij})}{\mathcal{E}^2(\varphi^{-1}(U)) \otimes \mathfrak{M}_r} \arrow[d, "\Phi_k"] \\
\underset{\Phi_k(A_{ij})}{\mathcal{E}^{2k}(U)} \arrow[r, "\varphi^*"] & \underset{\varphi^* \Phi_k(A_{ij})}{\mathcal{E}^{2k}(\varphi^{-1}(U))}
\end{tikzcd}$$

Let $f = (e_1, \dots, e_r)$ be a frame on U and $f^* = (e_1^*, \dots, e_r^*) = (e_1\varphi, e_2\varphi, \dots, e_r\varphi)$ be a frame on $\varphi^{-1}(U)$, and the frames of form f^* covers Y.

If $g: U \to GL(r, \mathbb{C})$ is a change of frame, then

$g^* = g \circ \varphi : \varphi^{-1}(U) \to GL(r, \mathbb{C})$ is a change of frame over $\varphi^* E$ on $\varphi^{-1}(U)$.

Define $\theta^*(f^*) := \varphi^* \theta(f) = [\varphi^* \theta_{\rho\sigma}]$ $\qquad \theta_{\rho\sigma} \in \mathcal{E}^1(U)$, $\varphi^* \theta_{\rho\sigma} \in \mathcal{E}^1(\varphi^{-1}(U))$

where φ^* is induced map on forms.

$$g^* \theta^*(f^* g^*) = g^* \theta^*((fg)^*) = g^* \varphi^* \theta(fg)$$

$f^* g^* = (f \cdot g) \circ \varphi = (fg)^*$

$$\begin{aligned}
&= g^* \varphi^* (g^{-1} \theta(f) g + g^{-1} dg) \\
&= g^* (g^*)^{-1} \theta^*(f^*) g^* + g^* (g^*)^{-1} \varphi^* dg \\
&= \theta^*(f^*) g^* + dg^*
\end{aligned}$$

$\varphi^* d = d \varphi^*$ since φ^* is homo of graded alg $\wedge^* T^* X \to \wedge^* T^* Y$

(Not very sure)

Then $\Theta(D^*, f^*) = d\theta^*(f^*) + \theta^*(f^*) \wedge \theta^*(f^*)$

$$\begin{aligned}
&= d(\varphi^* \theta(f)) + \varphi^* \theta(f) \wedge \varphi^* \theta(f) \\
&= \varphi^* d\theta(f) + \varphi^* (\theta(f) \wedge \theta(f)) \\
&= \varphi^* \Theta(D, f),
\end{aligned}$$

φ^* is homo of graded alg.

complete the proof.

(b) Suppose D and D' are connections on E and E'.

Goal: Find a connection $D^{\oplus}$ on $E \oplus E'$ s.t.

$c(E \oplus E', D^{\oplus}) = c(E, D) \wedge c(E', D')$.

It suffices to consider a local argument. Let $\theta^{\oplus} = \begin{bmatrix} \theta & 0 \\ 0 & \theta' \end{bmatrix}$.

Let f and f' be frames of E and E' over U. Then (f, f') is a frame of $E \oplus E'$ over U. Let g and g' be changing frames of f and f', then $g \oplus g'$ is a frame changing of (f, f'). To show $\theta^{\oplus}$ is defining a global connection on $E \oplus E'$, we need to prove

$(g \oplus g') \cdot \theta^{\oplus}(fg, f'g') = \theta^{\oplus}(f, f')(g \oplus g') + d(g \oplus g')$. Indeed,

$$(g \oplus g') \cdot \theta^{\oplus}(fg, f'g') = \begin{bmatrix} g & \\ & g' \end{bmatrix} \begin{bmatrix} \theta(fg) & \\ & \theta'(f'g') \end{bmatrix} = \begin{bmatrix} g\theta(fg) & 0 \\ 0 & g'\theta'(f'g') \end{bmatrix}$$

$$= \begin{bmatrix} \theta(f)g + dg & \\ & \theta'(f')g' + dg' \end{bmatrix} = \theta^{\oplus}(f, f')(g \oplus g') + d(g \oplus g')$$

The associate curvature is $\Theta^{\oplus} = \begin{bmatrix} \Theta & \\ & \Theta' \end{bmatrix}$.

Then $c(E\oplus E', D^{\oplus})|_U = \det\left(I + \frac{i}{2\pi}\begin{bmatrix}\Theta & \\ & \Theta'\end{bmatrix}\right)$

$$= \det\begin{bmatrix} I+\frac{i}{2\pi}\Theta & \\ & I+\frac{i}{2\pi}\Theta' \end{bmatrix}$$

$$= \det(I+\tfrac{i}{2\pi}\Theta)\det(I+\tfrac{i}{2\pi}\Theta')$$

$$= c(E,D)|_U \wedge c(E',D')|_U$$

(c) Let $\alpha: E' \to E$ be bdl iso, Goal: show $c(E) = c(E')$.

Let θ, f be connections and frames on E. Then we define $f' = \alpha f = (\alpha e_1, \dots, \alpha e_r)$. Let $\theta'(f') = \theta(f)$. Let $f \to fg$, it induces $f' = \alpha f \to \alpha fg = f'g =: f'g'$, so $g = g'$

Then $g'\theta'(f'g') = g\theta'(f'g) = g\theta'((fg)') = g\,\theta(fg) = \theta(f)g + dg = \theta'(f')g' + dg'$

$(fg)' = \alpha fg = \alpha f\cdot g = f'\cdot g$ $\quad$ $g=g'$, $\theta(f) = \theta'(f')$

Then $\Theta'(f') = \Theta(f)$. So $c(E') = c(E)$.

(d) Suppose the duality between E and E^* is represented by $\langle , \rangle$, i.e., we have $\langle e_\rho, e^*_\sigma\rangle = \delta_{\rho\sigma}$ where $f = (e_\rho)$, $f^* = (e^*_\sigma)$ be frames over U.

Define a connection $\theta^* := \theta(D^*, f^*) := -{}^t\theta(D, f)$.

when $f \to fg$, we have inner product changing

$$e_\rho^t(e^*_\sigma) = \delta_{\rho\sigma} \longrightarrow e_\rho g^t(e^*_\sigma g^*) = \delta_{\rho\sigma}$$

So $g^* = {}^tg^{-1}$. Claim: $(fg)^* = f^*g^*$. pf: $(fg)^* = (e_\rho g)^* = e^*_\sigma\, {}^tg^{-1} = e^*_\sigma g^*$

check $e_\rho g^t(e^*_\sigma\,{}^tg^{-1}) = \delta_{\rho\sigma}$

$= f^*g^*$ Bingo :) !

$g^*\theta^*(f^*g^*) = g^*\theta^*((fg)^*) = -g^*({}^t\theta(fg)) = -g^*({}^tg){}^t\theta(f)({}^tg^{-1}) - g^*(d\,{}^tg)\,{}^tg^{-1}$

$\theta(fg) = g^{-1}\theta(f)g + g^{-1}dg$ $\qquad = -{}^t\theta(f)\cdot{}^tg^{-1} - {}^tg^{-1}d\,{}^tg\cdot{}^tg^{-1}$ $(*)$

$\theta^*(f^*)g^* + dg^* = -{}^t\theta(f)\,{}^tg^{-1} + d({}^tg^{-1})$ (Δ) To show $(*) = (\Delta)$, it suffices to show $d\,{}^tg^{-1} = -{}^tg^{-1}d\,{}^tg\cdot{}^tg^{-1}$, i.e., $dg^{-1} = -g^{-1}dg\,g^{-1}$

$0 = dI = d(g\cdot g^{-1}) = dg\cdot g^{-1} + g\,dg^{-1}$. So $dg^{-1} = -g^{-1}\cdot dg\cdot g^{-1}$, completing the proof of $* = \Delta$, meaning that $\theta(f)$ is a connection (global defined).

$$\Theta^* = d\theta^*(f^*) + \theta^*(f^*)\wedge\theta^*(f^*) = -d\,{}^t\theta(f) + {}^t\theta(f)\wedge{}^t\theta(f)$$

$$= -d\,{}^t\theta(f) - {}^t(\theta(f)\wedge\theta(f)) \quad ({}^t(\alpha\wedge\beta) = -{}^t\beta\wedge{}^t\alpha)$$

$$= -{}^t(d\theta(f) + \theta(f)\wedge\theta(f)) = -{}^t\Theta$$

Thus, Chern forms related to V are

$$c_k(E^*, D^*) = \Phi_k\left(\frac{i}{2\pi}\Theta^*\right) = \Phi_k\left(-\frac{i}{2\pi}{}^t\Theta\right) = \Phi_k\left(-\frac{i}{2\pi}\Theta\right)$$

$\det(I+A) = \det(I+{}^tA)$
$\to$ so $\Phi_k(A) = \Phi_k({}^tA)$

$$= (-1)^k \Phi_k\left(\frac{i}{2\pi}\Theta\right) = (-1)^k c_k(E, D)$$ $\square$

<Obstruction>

Then we'll derive obstruction-theoretic props of Chern classes, i.e., the obstruction of finding global sections.

[Thm] Let $E \to X$ be a differentiable vec. bdl. of rank r.

(a) $c_0(E) = 1$

(b) If $E \cong X \times \mathbb{C}^r$ is trivial, then $c_j(E) = 0$, $j = 1, \cdots, r$; i.e., $c(E) = 1$.

(c) If $E \cong E' \oplus T_s$ where T_s is a trivial vec. bdl. of rank s, then
$$c_j(E) = 0,\ j = r-s+1, \cdots, r$$

pf:

(a) $$\det(I+A) = \sum_{k=0}^{n} \operatorname{tr}(\wedge^k A) = 1 + \operatorname{tr}(A) + \det(A)$$

where $$\operatorname{tr}(\wedge^k A) = \frac{1}{k!}\begin{vmatrix} \operatorname{tr}A & k-1 & 0 & \cdots & 0 \\ \operatorname{tr}A^2 & \operatorname{tr}A & k-2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \operatorname{tr}A^{k-1} & \operatorname{tr}A^{k-2} & & \cdots & 1 \\ \operatorname{tr}A^k & \operatorname{tr}A^{k-1} & & \cdots & \operatorname{tr}A \end{vmatrix}$$

Hence $\Phi_0 = 1$. So $c_0(E) = \Phi_0\left(\frac{i}{2\pi}\Theta\right) = 1$.

(b) Let $E = X \times \mathbb{C}^r$. Since there is a global frame, we have $\mathcal{E}(X, E) \cong (\mathcal{E}(X))^r$. We've seen in an example that connection of trivial bdl can be exterior diffrentiation d.

(Chern class is independent of connection, so we only need to pick an appropriate connection)

$$D : \mathcal{E}(X, E) \longrightarrow \mathcal{E}^1(X, E)$$

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_r \end{bmatrix} \longmapsto D\xi = d\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_r \end{bmatrix}, \quad \xi_i \in \mathcal{E}(X)$$

$\mathcal{E}(X,E) \cong \mathcal{E}(X)^r$

θ is defined by $De_\rho = e_\sigma \cdot \theta_{\sigma\rho}$.

$$De_\rho = d\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = 0, \text{ so } \theta_{\sigma\rho} = 0, \forall \sigma, \rho.$$

Then $\Theta = d\theta + \theta\wedge\theta = 0$. Hence $c(E) = \det(I+0) = 1$.

$1 = c(E) = 1 + \sum_{j=2}^{r} c_j(E,D) \Rightarrow c_j(E,D) = 0$ (Use $H^*(X;\mathbb{Z}) = \underbrace{\bigoplus}_{\text{direct sum}} H^k(X;\mathbb{Z})$)

(c) $c(E) = c(E'\oplus T_s) = c(E')\cdot c(T_s) \overset{c(T_s)=1 \text{ by (b)}}{=} c(E')$

$$c(E) = 1 + \underset{H^2(X;\mathbb{Z})}{\underset{\uparrow}{c_1(E)}} + \underset{H^4(X;\mathbb{Z})}{\underset{\uparrow}{c_2(E)}} + \cdots + \underset{H^{2r}(X;\mathbb{Z})}{\underset{\uparrow}{c_r(E)}} = 1 + \underset{H^2(X;\mathbb{Z})}{\underset{\uparrow}{c_1(E')}} + c_2(E') + \cdots + \underset{H^{2r-2s}(X;\mathbb{Z})}{\underset{\uparrow}{c_{r-s}(E')}}.$$

By direct sum, $c_j(E) = 0$ for $j = r-s+1, \cdots, r$ $\square$

[Rmk] There're some computations for Chern classes, see Example 3.7, Example 3.8 in Gtm 65.

§ Complex line bundles

* Complex line bdls : diff. or holo^m $\mathbb{C}$-vec. bdl. of rank 1.

Intro : There are many ways to define Chern classes. The first approach is the differential-geometric definition in the above section. Another approach is the classifying space approach (Detailed in Ch III.4) A third approach is to define Chern classes only for line bdls, called sheaf-theoretic definition of Chern class. (Application of sheaf theory)

Notation : Consider holo^m line bdls over a mf X. $\mathcal{O}$ is the structure sheaf of X and $\mathcal{O}^*$ be the sheaf of nonvanishing holo^m funs on X.

[Lemma] There is a one-to-one correspondence between the equivalence classes of holo^m line bdls on X and the elements of the cohomology grp $H^1(X,\mathcal{O}^*)$.

Pf: $\Rightarrow$ Given a line bdl $E\to X$, let's construct an element in $H^1(X,\mathcal{O}^*)$. The trivialization of bdl E forming an open covering $\{U_\alpha\} =: \mathfrak{U}$ and transition map $g_{\alpha\beta} : U_\alpha\cap U_\beta \to GL(1,\mathbb{C}) = \mathbb{C}^*$ which is holo^m (E is holo^m line bdl).

(Recall that a q-cochain of $\mathcal{U}$ with coefficients in $\mathcal{F}$ is a mapping g associate q-simplex σ a $f(\sigma) \in \mathcal{F}(|\sigma|)$ where

1. q-simplex is an ordered set of $\mathcal{U}$, $(U_0, \cdots, U_q)$, with $\bigcap_{i=0}^{q} U_i \neq \emptyset$.
2. $|\sigma| = \bigcap_{i=0}^{q} U_i$, called surpport of σ.

)

1\. Construct $g \in C^1(\mathcal{U}, \mathcal{O}^*)$,

(Recall that

1. $C^q(\mathcal{U}, \mathcal{O}^*)$ consists of all q-simplex.
2. $\delta : C^q(\mathcal{U}, \mathcal{O}^*) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{O}^*)$ by $\delta f(\sigma) = \sum_{i=0}^{q+1} (-1)^i r_{\sigma}^{|\sigma_i|} f(\sigma_i)$ where $f \in C^q(\mathcal{U}, F)$, $\sigma_i = (U_0, \cdots, \widehat{U_i}, \cdots, U_{q+1})$.
3. There is a cochain complex

$$C^1(\mathcal{U}, \mathcal{O}^*) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{O}^*) \xrightarrow{\delta} C^3(\mathcal{U}, \mathcal{O}^*) \xrightarrow{\delta} \cdots$$

$Z^q(\mathcal{U}, \mathcal{O}^*) = \ker \delta$, $B^q(\mathcal{U}, \mathcal{O}^*) = \operatorname{Im} \delta$, $H^q(\mathcal{U}, \mathcal{O}^*) = Z(\mathcal{U}, \mathcal{O}^*) / B(\mathcal{U}, \mathcal{O}^*)$)

i.e., a map g associate (U_0, U_1), $U_0, U_1 \in \mathcal{U}$ an element $g(U_0, U_1) \in \mathcal{O}^*(U_0 \cap U_1)$ = nonvanishing holo^m funs on $U_0 \cap U_1$.
We find transition map $g_{01} : U_0 \cap U_1 \to GL(1, \mathbb{C}) = \mathbb{C}^*$ lies in $\mathcal{O}^*(U_0 \cap U_1)$
so we define $g(U_0, U_1) = g_{01}$

(Note that $\underset{\text{coord in } U_0 \to U_1}{g_{01} : U_0 \cap U_1 \to \mathbb{C}^*}$ is not equal to $\underset{\text{coord in } U_1 \to U_0}{g_{10} : U_1 \cap U_0 \to \mathbb{C}^*}$)

2\. Show $g \in Z^1(\mathcal{U}, \mathcal{O}^*)$, i.e., show $\delta g = 1$ ($\mathcal{O}^*$ sheaf is ab. grp with multiplication)

$$\delta g(U_0, U_1, U_2) = r_{|\sigma|}^{|\sigma_0|} g(U_1, U_2) \cdot \left(r_{|\sigma|}^{|\sigma_1|} g(U_0, U_2)\right)^{-1} \cdot r_{|\sigma|}^{|\sigma_2|} g(U_0, U_1)$$

$$= r_{|\sigma|}^{|\sigma_0|} g_{12} \left(r_{|\sigma|}^{|\sigma_1|} g_{02}\right)^{-1} \cdot r_{|\sigma|}^{|\sigma_2|} g_{01} \underset{\substack{\text{when focusing on a} \\ \text{pt in } U_0 \cap U_1 \cap U_2}}{=} g_{12} \cdot g_{20} \cdot g_{01} = 1$$

3\. Since $g \in Z^1(\mathcal{U}, \mathcal{O}^*)$, we have $[g] \in H^1(\mathcal{U}, \mathcal{O}^*)$.

(Recall if $\underline{m}$ is a refinement of $\mathcal{U}$, there is a natural grp homo

$$\mu_{\underline{m}}^{\mathcal{U}} : H^q(\mathcal{U}, \mathcal{S}) \to H^q(\underline{m}, \mathcal{S}) \text{ and } \varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{S}) \cong H^q(X, \mathcal{S}) \;)$$

We pass $[g] \in H^1(\mathcal{U}, \mathcal{O}^*)$ to its image in direct limit $H^q(X, \mathcal{S})$. So we construct an element in $H^q(X, \mathcal{S})$.

4. Show any line bdl $E' \to X$ which is iso to $E \to X$ will correspond to same class in $H^1(X, \mathcal{O}^*)$.

E' has transition funs $\{g'_{\alpha\beta}\}$ on $\{U'_\alpha\}$, E has transition funs $\{g_{\alpha\beta}\}$ on $\{U_\alpha\}$. Let $\varphi: E' \to E$ be iso. From $g'_{\alpha\beta}: U'_\alpha \cap U'_\beta \to \mathbb{C}$ there is $g'_{\alpha\beta} \circ \varphi : \varphi^{-1}(U'_\alpha) \cap \varphi^{-1}(U'_\beta) \to \mathbb{C}$ is also transition maps on covering $\{\varphi^{-1}(U_\alpha)\}$. So it suffices to show $g' \in H^1(\{\varphi^{-1}U'_\alpha\}, \mathcal{O}^*)$ with $g'(\varphi^{-1}(U'_\alpha), \varphi^{-1}(U'_\beta)) = g'_{\alpha\beta} \circ \varphi$ and $g \in H^1(\{U_\alpha\}, \mathcal{O}^*)$ with $g(U_\alpha, U_\beta) = g_{\alpha\beta}$ passing to same class in $\varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{O}^*) = H^1(X, \mathcal{O}^*)$. Since they have same refinement, their image in $H^1(X, \mathcal{O}^*)$ are the same.

$\Leftarrow$ Given cohomology class $\xi \in H^1(X, \mathcal{O}^*)$, let's construct a holo^m line bdl. By prop of direct limit, there exists a covering $\mathcal{U}$ and $g = \{g_{\alpha\beta}\}$ ($g(U_\alpha, U_\beta) = g_{\alpha\beta}$) s.t. $[g] \in H^1(\mathcal{U}, \mathcal{O}^*)$.

$$\begin{tikzcd} A_i \arrow[rr, "f_{ij}"] \arrow[dr, "f_i"'] & & A_j \arrow[dl, "f_j"] \\ & L & \end{tikzcd}$$

$\forall b \in L$, $\exists i$ and $a \in A_i$ s.t. $f_i a = b$

Let $\tilde{E} = \coprod U_\alpha \times \mathbb{C} / \sim$ where $\underset{\in U_\alpha \times \mathbb{C}}{(x, z)} \sim \underset{\in U_\beta \times \mathbb{C}}{(y, w)} \Leftrightarrow x = y$ and $z = g_{\alpha\beta} w$

We obtain a holo^m vec bdl.

We omit the proof of one-to-one correspondence (By common refinement arguement) $\square$

[Construct] Consider exact seq of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

This induces a cohomology seq

$$\begin{tikzcd} H^1(X, \mathcal{O}) \arrow[r] & H^1(X, \mathcal{O}^*) \arrow[r, "\delta"] & H^2(X, \mathbb{Z}) \arrow[r] \arrow[d, "j"] & H^2(X, \mathcal{O}) \\ & & H^2(X, \mathbb{R}) & \end{tikzcd}$$

where 1. j induced by inclusion $\mathbb{Z} \to \mathbb{R}$

2. δ is the Bockstein operator.

We construct third mor $c_1 : H^1(X, \mathcal{O}^*) \longrightarrow H^2(X; \mathbb{R})$

$g \longmapsto$ g corresponds to a holo^m line bdl E (by above thm), let $c_1(g) = c_1(E) \in H^2(X; \mathbb{R})$

Turns out we obtain a commutative triangle.

[Thm] The diagram
$$\begin{tikzcd} H^1(X, \mathcal{O}^*) \arrow[r, "\delta"] \arrow[dr, "c_1"'] & H^2(X;\mathbb{Z}) \arrow[d, "j"] \\ & H^2(X;\mathbb{R}) \end{tikzcd}$$
is commutative.

[Rmk] $H^1(X, \mathcal{O}^*) = \{\text{holo}^m \text{ line bdl } E \text{ over } X\}$. This commutative diagram means we can compute first Chern class of line bdls by using the Bockstein operator. Chern class $c(E) = c_0(E) + c_1(E) = 1 + c_1(E)$

⟨Divisors⟩

Consider the exact seq of multiplicative sheaves
$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$$
where $\mathcal{M}^*$ is the sheaf of nontrivial meromorphic funs on X, i.e., the stalk $\mathcal{M}^*_x$ is the grp of non-zero elements of the quotient field of the integral domain $\mathcal{O}_x$ at a point $x \in X$.

($\mathcal{O}_x$ is a Noetherian local ring, $\mathcal{O}_x$ is an integral domain, with unique factorization)

[Def] Let $\mathcal{D} = \mathcal{M}^*/\mathcal{O}^*$, this is called the sheaf of divisors on X. A section of $\mathcal{D}$ is called a divisor. □

Let $D \in H^0(X, \mathcal{D}) = \Gamma(X, \mathcal{D})$. Then there is a covering $\mathcal{U} = \{U_\alpha\}$ and meromorphic funs (sections of $\mathcal{M}^*$) f_α defined in U_α s.t.
$$\frac{f_\beta}{f_\alpha} = g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$
with $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

$H^0(\{U_\alpha\}, \mathcal{D})$ g with $g(U_\alpha, U_\beta) = g_{\alpha\beta}$, $\delta g = 0$.
$$\downarrow$$
$H^0(X, \mathcal{D})$

Then $E = \coprod U_\alpha \times \mathbb{C}/\sim$ $\quad \underset{U_\alpha\times\mathbb{C}}{(x, z)} \sim \underset{U_\beta\times\mathbb{C}}{(y, w)} \Leftrightarrow x = y$, $z = g_{\alpha\beta} w$. is a line bdl.

[Conclusion] A divisor gives rise to an equivalence class of line bdls represented by the cocycle $\{g_{\alpha\beta}\}$.

Brief discussion: 1. A divisor determines an equivalence class of holo^m line bdls AND that two different divisors give the same class if they "differ by" (multiplicatively) a global meromorphic fun (called linear equivalence in alg. geo.)

2. Divisors occur very often as the divisor determined by a subvariety $V \subset X$ of codim 1.

V can be defined by the following data: a covering $\{U_\alpha\}$ of X, holo^m funs f_α in U_α and $f_\beta / f_\alpha = g_{\alpha\beta}$ nonvanishing and holo^m on $U_\alpha \cap U_\beta$. V is defined to be the zeros of the funs f_α in U_α.