8. (12 points) Let A be a 3×3 matrix such that $\operatorname{rank}(A) = 2$ and $A^3 = 0$.

(a) Prove that $\operatorname{rank}(A^2) = 1$.

(b) Let $\alpha_1 \in \mathbb{R}^3$ be a nonzero vector such that $A\alpha_1 = 0$. Prove that there exist vectors α_2, α_3 such that $A\alpha_2 = \alpha_1$, $A^2\alpha_3 = \alpha_1$.

(c) For any vectors α_2, α_3 described as above, show that $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.

(In this problem, you are allowed to assume the statements of some questions to answer subsequent questions.)

(a) 证明 $\operatorname{rank}(A^2) = 1$.

(b) 设 $\alpha_1 \in \mathbb{R}^3$ 是满足 $A\alpha_1 = 0$ 的非零向量. 证明: 存在向量 α_2, α_3 使得 $A\alpha_2 = \alpha_1$, $A^2\alpha_3 = \alpha_1$.

(c) 证明: 对于任意满足上述条件的向量 α_2, α_3, 向量组 $\alpha_1, \alpha_2, \alpha_3$ 线性无关.

(本题中, 允许承认前面小题的结果来用于后续问题的解答.)

a) $A^3=0$ 视作 $A\cdot A^2=0$，则有 $C(A^2)\subseteq N(A)$. 故 $r(A^2)\leqslant \dim N(A)=3-r(A)=1$

若 $r(A^2)=0$. 则 $\dim N(A^2)=3$. 故 $N(A^2)=\mathbb{R}^3$. 由 $\begin{cases} \mathbb{R}^3=N(A^2)\oplus C((A^2)^T) \\ N(A^2)=\mathbb{R}^3 \end{cases}$. 得 $C((A^2)^T)=0$.

故 $A^2=0$. (其实是把 $r(M)=0 \Leftrightarrow M=0$ 证了一遍). $A^2=0$ 视作 $A\cdot A=0$. 故 $C(A)\subseteq N(A)$.

于是 $r(A)\leqslant \dim N(A)=3-2=1$ 与 $r(A)=2$ 矛盾. 因此 $r(A^2)\neq 0$、又由 $r(A^2)\leqslant 1$,

得 $r(A)=1$

b) $\forall x\in N(A)$, 有 $Ax=0$, 自然有 $A^2x=0$, 故 $x\in N(A^2)$. 因此 $N(A)\subseteq N(A^2)$.

$\dim N(A)=3-r(A)=1$. 题目任取 $\alpha_1\neq 0$ s.t. $A\alpha_1=0$，有 $\{\alpha_1\}$ 是 $N(A)$ 的基 (i.e., $N(A)=\{k\alpha_1 \mid k\in\mathbb{R}\}$). $\alpha_1\in N(A)\subseteq N(A^2)$. 计算 $\dim N(A^2)=3-r(A^2)=2$.

由扩基定理知, 存在 $v_2\in N(A^2)$, s.t. $\{\alpha_1, v_2\}$ 构成 $N(A^2)$ 的一组基.

特别地, $v_2\in N(A^2)$ 即 $A^2v_2=0$, 即 $A(Av_2)=0$. 故 $Av_2\in N(A)=\{k\alpha_1 \mid k\in\mathbb{R}\}$.

因此存在 $c\in\mathbb{R}^*$, s.t. $Av_2=c\alpha_1$. 令 $\alpha_2=v_2/c$, 则 $A\alpha_2=A\frac{v_2}{c}=\frac{c\alpha_1}{c}=\alpha_1$.

$\{\alpha_1, v_2\}$ 是 $N(A^2)$ 的一组基, 则 $\{\alpha_1, \frac{v_2}{c}=\alpha_2\}$ 也是 $N(A^2)$ 的一组基.

同理易证 $\{\alpha_1,\alpha_2\}\subseteq N(A^2)\subseteq N(A^3)$, 且 $\dim N(A^3)=3-r(A^3)=3-0=3$. 由扩基定理, 可将 $\{\alpha_1,\alpha_2\}$ 扩充成 $\{\alpha_1,\alpha_2,v_3\}$ 成为 $N(A^3)$ 的一组基.

$A^3v_3=0 \Rightarrow A^2(Av_3)=0$. 于是 $Av_3\in N(A^2)=\{k\alpha_1+l\alpha_2 \mid k,l\in\mathbb{R}\}$.

故 $\exists k_1,k_2\in\mathbb{R}$ s.t. $Av_3=k_1\alpha_1+k_2\alpha_2$. 则 $A^2v_3=k_1A\alpha_1+k_2A\alpha_2=k_2\alpha_1$
$\begin{pmatrix} A\alpha_2=\alpha_1 \\ A\alpha_1=0 \end{pmatrix}$

令 $\alpha_3=\frac{v_3}{k_2}$. 则 $A^2\alpha_3=A^2(v_3/k_2)=k_2\alpha_1/k_2=\alpha_1$.

于是找到了满足 $A^2\alpha_3=\alpha_1$, $A\alpha_2=\alpha_1$ 的 α_3, α_2.

c) $\forall c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$ (*), w.t.s. $c_1 = c_2 = c_3 = 0$.

左乘A^2，得 $c_1A^2\alpha_1 + c_2\underbrace{A^2\alpha_2}_{\because A\alpha_1=0} + c_3A^2\alpha_3 = 0$，即 $c_3\alpha_1 = 0$. 由$\alpha_1 \neq 0$知 $c_3=0$

故(*)化为 $c_1\alpha_1 + c_2\alpha_2 = 0$. (1). (1)式左乘$A$得 $c_1A\alpha_1 + c_2A\alpha_2 = 0$，即 $c_2\alpha_1 = 0$.

由$\alpha_1 \neq 0$知 $c_2 = 0$. 故(1)化为 $c_1\alpha_1 = 0$. 由$\alpha_1 \neq 0$知 $c_1 = 0$. 于是证明了 $c_1 = c_2 = c_3 = 0$.