6. Explicit Time Integration

In contrast to an implicit solver the explicit method allows to avoid the solution of a system of equations. As explained in section 6.3 a lumped mass matrix is used as the inverse of the lumped mass matrix is represented by the fraction of the respective diagonal matrix entries. With this performance the nodal acceleration can easily be updated without the need of inverting big matrices.
This means that every solution at each time step is just a prediction of the real solution and is strongly dependent on a proper choice of time step. Different approaches are available to approximate the first and second derivative of a function. The preferred choice in this thesis is the *central difference method*, which is discussed in the next section 6.1.

6.1. Central difference formula

The basic idea of the *Central Difference method* used in this thesis is the usage of the *central difference formula* [29].

For this purpose the Taylor's series from equation (2.2) is used:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_k)}{n!} \Delta x^n \qquad with \qquad f^{(n)} = \frac{\partial^n f}{\partial x^n}. \tag{6.1}$$

If higher order terms (*n>=2*) are neglected the function values of the previous step f_{k-1},

$$f_{k-1} \approx f_k - \left(\frac{\partial f}{\partial x}\right)_k \cdot \Delta x, \tag{6.2}$$

and the function values of the next step f_{k+1} can be expressed:

$$f_{k+1} \approx f_k + \left(\frac{\partial f}{\partial x}\right)_k \cdot \Delta x. \tag{6.3}$$

These two equations described the so called *backward difference* [29],

$$\left(\frac{\partial f}{\partial x}\right)_k \approx \frac{f_k - f_{k-1}}{\Delta x}, \tag{6.4}$$

as well as the *forward difference*,

$$\left(\frac{\partial f}{\partial x}\right)_k \approx \frac{f_{k+1} - f_k}{\Delta x}. \tag{6.5}$$

To obtain the *central difference method* used in this thesis, equation (6.2) and (6.3) are assembled [29]:

$$f_{k-1} - f_{k+1} = -2 \cdot \frac{\partial f}{\partial x} \cdot \Delta x \qquad \rightarrow \qquad \left(\frac{\partial f}{\partial x}\right)_k \approx \frac{f_{k+1} - f_{k-1}}{2 \cdot \Delta x}. \tag{6.6}$$

Obviously all three differences describe an approximation of the first derivative of *f* at the current position *k*. This is visualized in the following graph.

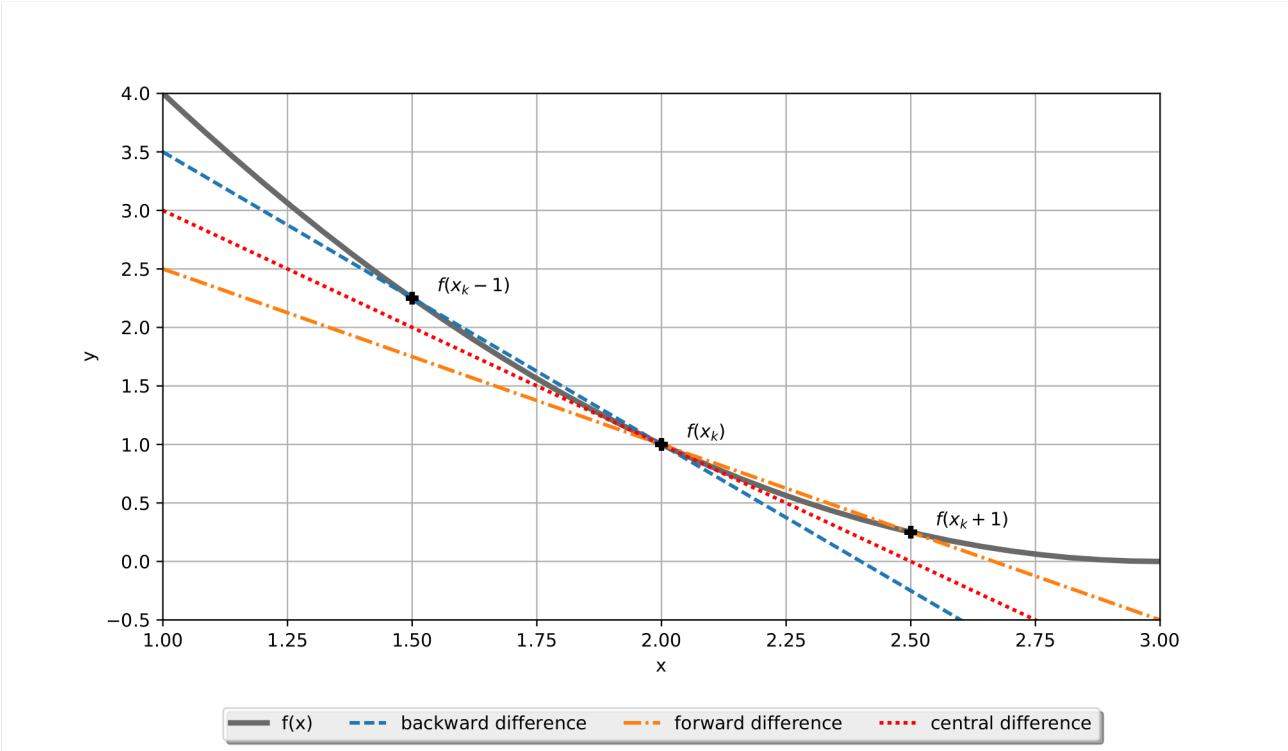

Figure 92 Finite Differencing

6.2. Derivation of system properties

In the following derivation a constant time increment is assumed for simplicity for every step *n*. In case variable time steps are needed this derivation has to be slightly adopted:

$$\Delta t = \Delta t^n = \Delta t^{n+0.5} = t^{n+0.5} - t^{n-0.5} = t^{n+1} - t^n = t^n - t^{n-1}. \tag{6.7}$$

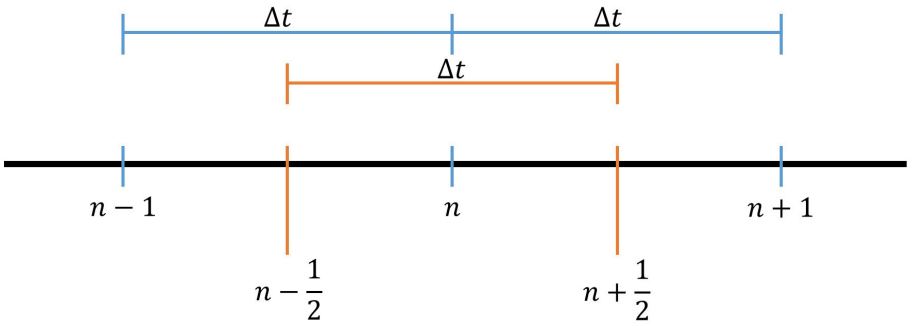

Figure 93 Time steps

Using the *central difference formula* (6.6) the velocity **v** (time derivative of the displacement **d**) is described at the midpoint of the time interval ([16] p.330):

$$\dot{\mathbf{d}}^{n+0.5} = \left(\frac{\partial \mathbf{d}}{\partial t}\right)^{n+0.5} \equiv \mathbf{v}^{n+0.5} = \frac{\mathbf{d}^{n+1} - \mathbf{d}^n}{\Delta t}, \tag{6.8}$$

$$\dot{\mathbf{d}}^{n-0.5} = \left(\frac{\partial \mathbf{d}}{\partial t}\right)^{n-0.5} \equiv \mathbf{v}^{n-0.5} = \frac{\mathbf{d}^{n} - \mathbf{d}^{n-1}}{\Delta t}. \tag{6.9}$$

This allows us to describe the displacement vector at the next time step *n+1*:

$$\mathbf{d}^{n+1} = \mathbf{v}^{n+0.5} \cdot \Delta t + \mathbf{d}^n. \tag{6.10}$$

Also the acceleration **a** (time derivative of the velocity **v**) is approximated by the central difference formula:

$$\ddot{\mathbf{d}}^{n} = \left(\frac{\partial^2 \mathbf{d}}{\partial t^2}\right)^{n} \equiv \mathbf{a}^{n} = \frac{\mathbf{v}^{n+0.5} - \mathbf{v}^{n-0.5}}{\Delta t}. \tag{6.11}$$

By rearranging equation (6.11) the midpoint velocity can be predicted:

$$\mathbf{v}^{n+0.5} = \mathbf{a}^n \cdot \Delta t + \mathbf{v}^{n-0.5}. \tag{6.12}$$

If the midpoint velocities obtained in equation (6.8) and (6.9) are put into equation (6.11) the current acceleration can be expressed with the help of the displacement vector:

$$\mathbf{a}^n = \frac{\frac{\mathbf{d}^{n+1} - \mathbf{d}^n}{\Delta t} - \frac{\mathbf{d}^n - \mathbf{d}^{n-1}}{\Delta t}}{\Delta t} = \frac{\mathbf{d}^{n+1} - 2 \cdot \mathbf{d}^n + \mathbf{d}^{n-1}}{(\Delta t)^2}. \tag{6.13}$$

This can also directly be obtained by using a Taylor's expansion to describe the displacement and truncating the series for every power bigger than two:

$$\mathbf{d}^{n+1} \approx \mathbf{d}^n + \mathbf{v}^n \cdot \Delta t + \frac{\mathbf{a}^n}{2} \cdot (\Delta t)^2 \; + \; .... \, , \tag{6.14}$$

$$\mathbf{d}^{n-1} \approx \mathbf{d}^n - \mathbf{v}^n \cdot \Delta t + \frac{\mathbf{a}^n}{2} \cdot (\Delta t)^2 \; + \; .... \, . \tag{6.15}$$

By adding these two functions up the same result as in equation (6.13) is obtained:

$$\mathbf{d}^{n+1} + \mathbf{d}^{n-1} = 2 \cdot \mathbf{d}^n + \mathbf{a}^n \cdot (\Delta t)^2 \qquad \rightarrow \qquad \mathbf{a}^n = \frac{\mathbf{d}^{n+1} - 2 \cdot \mathbf{d}^n + \mathbf{d}^{n-1}}{(\Delta t)^2}. \tag{6.16}$$

This corresponds to the approximation of second-order derivatives with the help of the central difference method as described in [29].

6.3. Solving the equation of motion

With respect to [17] as well as [16] the equation of motion reads as follows. **f** represents the force residual:

$$\mathbf{M} \cdot \mathbf{a} + \mathbf{C} \cdot \mathbf{v} + \mathbf{K} \cdot \mathbf{d} = \mathbf{f}_{ext} \qquad \rightarrow \qquad \mathbf{M} \cdot \mathbf{a} + \mathbf{C} \cdot \mathbf{v} = \mathbf{f}_{ext} - \mathbf{f}_{int} = \mathbf{f}. \tag{6.17}$$

This equation is used by the explicit time integration scheme to update the current acceleration [16] p.333:

$$\mathbf{a}^n = \mathbf{M}^{-1} \cdot \left(\mathbf{f}^n - \mathbf{C} \cdot \mathbf{v}^{n-0.5} \right). \tag{6.18}$$

Here **M** describes the mass matrix whereas **C** represents the damping matrix. Equation (6.18) contains the inverse mass matrix. This is the reason only *lumped* mass matrices are used in explicit integration schemes.

With,

$$M^{ij}_{lumped} = 0 \qquad and \qquad \left(M^{ii}_{lumped} \right)^{-1} = \frac{1}{M^{ii}_{lumped}}, \tag{6.19}$$

no system of equations has to be solved to obtain the inverse of a lumped mass matrix. In other words this means that every entry of the residual force vector in equation (6.18) is divided by the respective nodal mass, as described by Newtons second law of motion.

6.3.1. Solution Process

The total solution process and the respective loop are visualized in a flow chart.

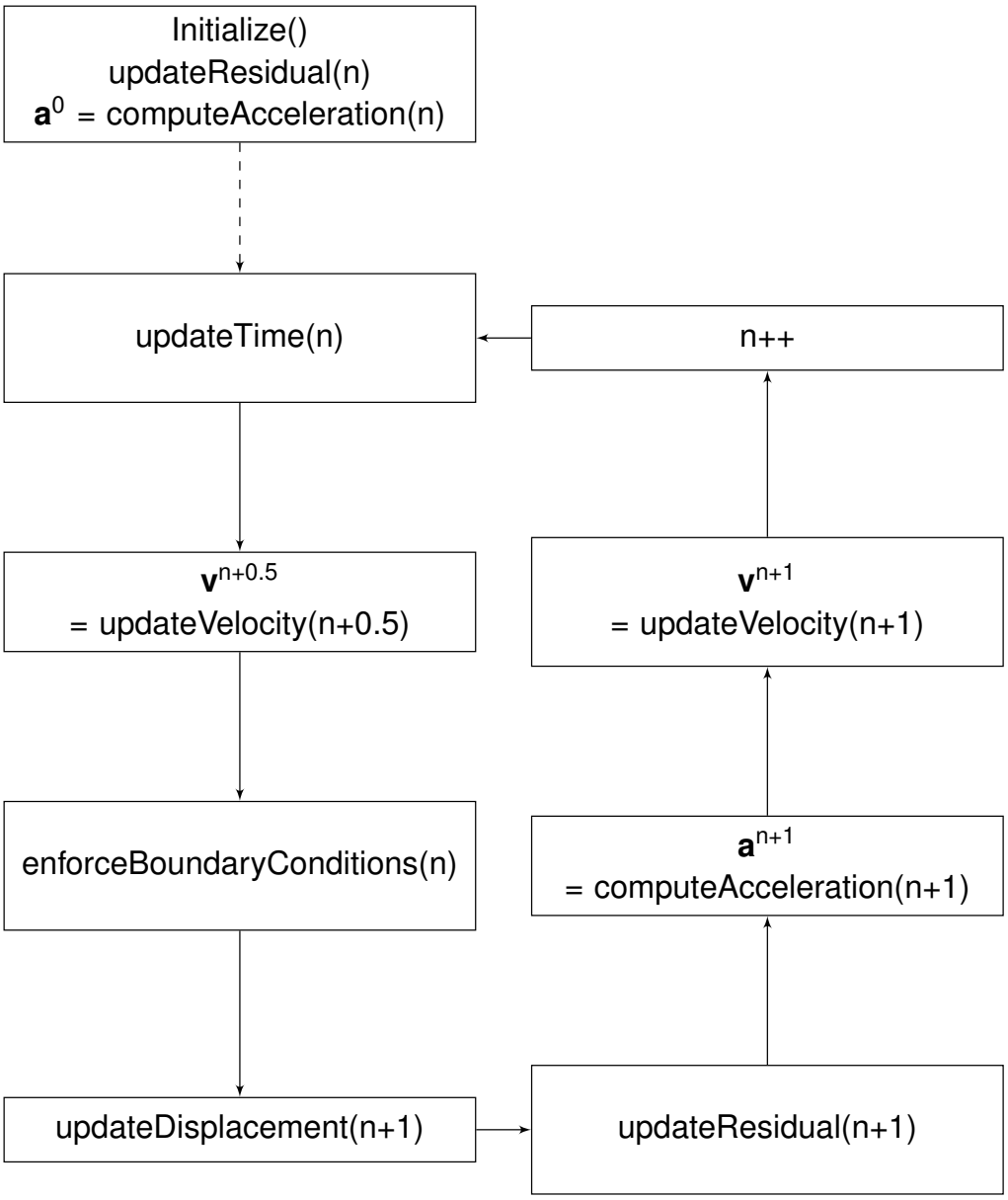

Figure 94 Solution Process

The respective functions are given on the next page (see [16]).

Algorithm 2 Functions used in figure 94

define $Initialize()$
    apply $\mathbf{v}^0$ and pre-stresses
    compute **M**
    $\mathbf{d}^0 = \mathbf{0}$
    n = 0, $t^0 = 0$
define $computeAcceleration(n)$
    $\mathbf{a}^n = inv\left(\mathbf{M}\right) \cdot \left(\mathbf{f}^n - \mathbf{C} \cdot \mathbf{v}^{n-0.5}\right)$
define $updateTime(n)$
    $t^{n+1} = t^n + \Delta t$
    $t^{n+0.5} = \left(t^n + t^{n+1}\right) \cdot 0.5$
define $updateVelocity(n)$
    $\mathbf{v}^n = \mathbf{v}^{n-0.5} + \mathbf{a}^{n-0.5} \cdot \Delta t \cdot 0.5$
define $enforceBoundaryConditions(n)$
    enforce velocity boundary conditions on Γ_v
define $updateDisplacement(n)$
    $\mathbf{d}^n = \mathbf{d}^{n-1} + \mathbf{v}^{n-0.5} \cdot \Delta t$
define $updateResidual(n)$
    get $\mathbf{f}_{\text{int}}(\mathbf{d}^n, t^n)$ from the element
    $\mathbf{f}^n = \mathbf{f}_{ext}\left(\mathbf{d}^n, t^n\right) - \mathbf{f}_{int}\left(\mathbf{d}^n, t^n\right)$

6.3.2. Stable time step

As already mentioned in the introduction of this section, the choice of a proper time step is of great importance for the explicit time integration scheme.

As already used in the analytical solution of the dynamic test cases for the co-rotational beam as well as the ones for the truss (equation (4.236) and (3.78)) the homogeneous vibrating response function can be obtained by superposition of the respective eigen-modes of of each eigen-frequency multiplied with a weighting factor [17]:

$$\mathbf{w}\left(t\right) = \sum_i \boldsymbol{\phi}_i \cdot Y\left(t\right)_i \qquad and \qquad \ddot{\mathbf{w}}\left(t\right) = \sum_i \boldsymbol{\phi}_i \cdot \ddot{Y}\left(t\right)_i . \tag{6.20}$$

Following the derivation from [30] p.111 the well known homogeneous equation of motion is expressed with the help of the eigen-mode vector,

$$\mathbf{M} \cdot \boldsymbol{\phi} \cdot \ddot{\mathbf{Y}} + \mathbf{K} \cdot \boldsymbol{\phi} \cdot \mathbf{Y} = 0, \tag{6.21}$$

which can be simplified [17]. Where ω is the natural eigen-frequency in rad/s:

$$\ddot{\mathbf{Y}}_i + \omega_i^2 \cdot \mathbf{Y} = \mathbf{0}. \tag{6.22}$$

Using the equation for the current acceleration, derived in equation (6.16),

$$\ddot{\mathbf{Y}}_i^n = \frac{\mathbf{Y}_i^{n+1} - 2 \cdot \mathbf{Y}_i^n + \mathbf{Y}_i^{n-1}}{(\Delta t)^2}, \tag{6.23}$$

the displacement vector for the next solution step *n+1* is predicted by assembling equation (6.22) and (6.23) [12] p.13:

$$\mathbf{Y}_i^{n+1} = \left(2 - \omega_i^2 \cdot \Delta t^2\right) \cdot \mathbf{Y}_i^n - \mathbf{Y}_i^{n-1}. \tag{6.24}$$

With respect to [12] the following matrix is built:

$$\begin{pmatrix} \mathbf{Y}_i^{n+1} \\ \mathbf{Y}_i^n \end{pmatrix} = \begin{pmatrix} 2 - \omega_i^2 \cdot \Delta t^2 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{Y}_i^n \\ \mathbf{Y}_i^{n-1} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_i^{n+1} \\ \mathbf{Y}_i^n \end{pmatrix} = \mathbf{A} \cdot \begin{pmatrix} \mathbf{Y}_i^n \\ \mathbf{Y}_i^{n-1} \end{pmatrix}. \tag{6.25}$$

This matrix is then used to solve the eigen-wert problem $det\left(\mathbf{A} - \lambda \cdot \mathbf{I}\right) = 0$
with $|\lambda| \leq 1$ [12] restricting the spectral radius of **A** and thus the maximum absolute eigenwert of the matrix:

$$\lambda^2 - \lambda \cdot \left(2 - \omega_i^2 \cdot \Delta t^2\right) + 1 = 0 \rightarrow \left| \frac{2 - \omega_i^2 \cdot \Delta t^2}{2} \pm \sqrt{\left(\frac{2 - \omega_i^2 \cdot \Delta t^2}{2}\right)^2 - 1} \right| \leq 1, \tag{6.26}$$

$$\pm \sqrt{\left(\frac{2 - \omega_i^2 \cdot \Delta t^2}{2}\right)^2 - 1} \qquad \leq \qquad \frac{\omega_i^2 \cdot \Delta t^2}{2}, \tag{6.27}$$

$$\omega_i^2 \cdot \Delta t^2 \qquad \geq \frac{\omega_i^4 \cdot \Delta t^4}{4}. \tag{6.28}$$

Finally an upper boundary is obtained for the time step value (see [12] as well as [16]):

$$\Delta t \leq \frac{2}{\omega_{max}}. \tag{6.29}$$

It follows that, the maximum eigen-frequency relates to the eigen-frequency of single elements and not the total structure.

Thus either one knows the maximum eigen-frequency of each element or the wave-speed in each element is calculated to derive the eigen-frequency.

First an elastic parameter *K* is calculated:

$$K = \begin{cases} Young's\ Modulus & E & for\ 1D - structures \\ Bulk\ Modulus & \dfrac{E}{3\,(1 - 2 \cdot \nu)} & for\ solids \end{cases}. \tag{6.30}$$

This parameter is then used to calculate the wave-speed *v* for the respective element,

$$v = \sqrt{\frac{K}{\rho}}, \tag{6.31}$$

the frequency ω,

$$\omega = \frac{2 \cdot v}{L}, \tag{6.32}$$

and finally the maximum time step Δt (see [12] p.14 as well as [16] p.335):

$$\Delta t_{max} = \frac{2}{\omega} = \frac{L}{v} = L \cdot \sqrt{\frac{\rho}{K}}. \tag{6.33}$$