Geometric Resonance in Integers

A Harmonic Derivation of the Divisor Function and its Spectral Dynamics

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Abstract

This work establishes an analytic isomorphism between the geometry of regular polygon subdivision and the arithmetic theory of divisors. Through exponential sums of roots of unity, the resonance function $\Omega(n)$ is introduced, demonstrating its exact equivalence with the shifted divisor function $d(2n) - 4$. On this basis, two extensions are constructed: the iterated resonance function $T(n)$, asymptotically linked to the Gaussian error function in prime numbers, and the frequency seed Λ_{MF}, derived via Dirichlet inversion. The second part of the article explores the dynamic implications of this model, proposing a spectral reformulation of the distribution of prime numbers and offering a new heuristic regarding the stability of the Riemann Zeta function.

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Part I

Analytic and Geometric Foundations

This part introduces the fundamental definitions and establishes, through formal proofs, the arithmetic identities that underpin the model. These results constitute the analytical basis upon which the subsequent dynamic interpretations are developed.

1. INTRODUCTION AND GEOMETRIC FOUNDATION

The starting point of this investigation is a classical problem of geometric tessellation: the characterization of regular subdivisions of a polygon. Let P_n be a regular polygon with n sides. We ask under what arithmetic conditions P_n can be decomposed into k congruent regular polygons Q_m with m sides, arranged in a regular angular configuration of the *edge-to-edge* type.

To formalize this correspondence between geometric structure and arithmetic divisibility, we begin by analyzing an elemental decomposition that reduces the problem to discrete angular conditions.

Lemma 1.1 (Triangular Decomposition and Angular Equivalence)**.** *Every regular polygon of r sides can be decomposed into $2r$ congruent right triangles, whose acute vertices coincide at the center of the polygon. The measure of the central angle of each elemental triangle is π/r.*

Proof. We divide the polygon into r isosceles triangles by connecting the center with the vertices. Each isosceles triangle has a central angle of $2\pi/r$. By tracing the altitude (apothem), each isosceles triangle is divided into two identical right triangles, bisecting the central angle. Therefore, we obtain $2r$ triangles with angle π/r. □

This decomposition allows us to establish a necessary condition for the assembly of the pieces Q_m within P_n.

Theorem 1.2 (Fundamental Geometric Relation)**.** *Assume a regular corona subdivision where k polygons Q_m cover the internal perimeter of P_n. The necessary condition of angular fitting implies the relation:*

$$\boxed{m = \frac{2n}{k}}$$

Proof. Consider the sum of the central angles. To cover the full angle (2π) of the larger polygon P_n using k copies of Q_m, the sum of the semi-internal angles contributed by the pieces must be congruent with the geometry of the container.

1. The elemental angle of the larger polygon P_n is π/n. The total angle is 2π, which equals $2n$ elemental units.

2. The elemental angle of the piece Q_m is π/m.

3. If we arrange k pieces, the angular sum must satisfy the proportion:

$$k \cdot \frac{\pi}{m} = \frac{2\pi}{n}$$

Multiplying both sides by $\frac{mn}{\pi}$, we obtain $kn = 2m$. Solving for m, we arrive at $m = \frac{2n}{k}$. □

Remark 1.3 (Discrete Interpretation). The relation $m = 2n/k$ reveals that the admissible geometry of the subdivision is quantized by the arithmetic of $2n$. Only the divisors of this integer produce coherent configurations. The extreme cases $k = 1, 2, n, 2n$ correspond to degenerate or trivially collapsed solutions, justifying their exclusion in the subsequent analytical formulation.

2. THE FUNCTION $\Omega(n)$: ANALYTIC DEFINITION AND EQUIVALENCE

The divisibility condition $k \mid 2n$ can be transferred to the domain of harmonic analysis by exploiting the orthogonality of roots of unity. This allows for the construction of an indicator function that detects arithmetic resonances exactly.

Definition 2.1 (Geometric Resonance Function). For every integer $n \geq 3$, we define $\Omega(n)$ as the following weighted trigonometric sum:

$$\Omega(n) := \sum_{k=3}^{n-1} \frac{1}{k} \sum_{j=0}^{k-1} \cos\left(\frac{4\pi jn}{k}\right). \tag{1}$$

The following result demonstrates that this harmonic construction recovers a classical arithmetic function.

Theorem 2.2 (Equivalence Theorem). *The function* $\Omega(n)$ *satisfies the identity:*

$$\Omega(n) = d(2n) - 4, \tag{2}$$

where $d(n)$ *denotes the divisor function* $\sum_{d\mid n} 1$.

Proof. Consider the inner sum $S_k = \sum_{j=0}^{k-1} \cos(4\pi jn/k)$. This corresponds to the real part of a geometric series of roots of unity. We analyze the cases according to divisibility:

- **Resonant Case ($k \mid 2n$):** The argument $\frac{4\pi jn}{k}$ is an integer multiple of 2π for all j. Consequently, $\cos(\cdot) = 1$ and the sum results in $S_k = k$.

- **Dissonant Case ($k \nmid 2n$):** Due to the symmetry properties of the roots of unity, the vectors cancel out in the complex plane, resulting in $S_k = 0$.

Substituting these results into the original definition, the expression reduces to a counting function:

$$\Omega(n) = \sum_{k=3}^{n-1} \frac{1}{k} \cdot (k \cdot \mathbb{1}_{k|2n}) = \sum_{k=3}^{n-1} \mathbb{1}_{k|2n} = \sum_{\substack{k|2n \\ 3 \leq k \leq n-1}} 1. \tag{3}$$

The set of divisors of $2n$ trivially includes $\{1, 2, n, 2n\}$. Given that $n \geq 3$, the divisors 1 and 2 are strictly less than 3, and $n, 2n$ are strictly greater than $n - 1$. Therefore, the sum counts all divisors of $2n$ excluding exactly these four elements. We conclude that $\Omega(n) = d(2n) - 4$. □

Corollary 2.3 (Primality Filter). *For $n > 4$, the following holds:*

$$\Omega(n) = 0 \iff n \text{ is a prime number}.$$

Proof. If $\Omega(n) = 0$, then $d(2n) = 4$. We know that $d(2n) = 4$ occurs if and only if $2n$ is the product of two distinct primes $(2 \cdot p)$ or a cube of a prime (p^3). Since $2n$ contains the factor 2, the options are $2n = 2 \cdot p$ (with p being an odd prime) or $2n = 2^3 = 8$.

- If $2n = 2p$, then $n = p$ is prime.

- If $2n = 8$, then $n = 4$. (This is the case $\Omega(4) = d(8) - 4 = 0$, the only even exception).

Thus, for $n > 4$, the absence of internal resonance ($\Omega = 0$) implies strict primality. □

3. ARITHMETIC PATTERNS AND RESONANCE STRUCTURE

The fundamental identity $\Omega(n) = d(2n) - 4$ transfers the geometric problem to the domain of multiplicative number theory. This allows us to catalog the exact behavior of the resonance $\Omega(n)$ based on the prime factorization of n.

Below, we present the identities that govern the behavior of Ω across different classes of integers.

Proposition 3.1 (Spectral Identities of $\Omega(n)$). *Let $k \geq 1$ be an integer, p an odd prime number, and i any odd integer ($i \geq 1$). The following identities hold:*

*(a) **Evolution in Powers of 2:***

$$\Omega(2^k) = k - 2$$

(b) **Resonance in Powers of Odd Primes:**

$$\Omega(p^k) = 2(k-1)$$

(c) **Prime-Binary Interaction (Simple Mixing):**

$$\Omega(p \cdot 2^k) = 2k$$

(d) **Odd-Binary Interaction (General Mixing):**

$$\Omega(i \cdot 2^k) = (k+2)d(i) - 4$$

Proof. The proofs follow directly from the multiplicative property of the divisor function $d(n)$, applied to the doubled argument $2n$.

   (a) Case $n = 2^k$: The double of the number is $2n = 2^{k+1}$. The divisor function for a power of a prime is simply the exponent plus one:

$$d(2n) = d(2^{k+1}) = (k+1) + 1 = k + 2.$$

Substituting into the equivalence definition:

$$\Omega(2^k) = (k+2) - 4 = k - 2.$$

   (b) Case $n = p^k$ (p odd): The double of the number is $2n = 2^1 \cdot p^k$. Since $\gcd(2, p) = 1$, the divisor function separates:

$$d(2n) = d(2^1) \cdot d(p^k) = (1+1)(k+1) = 2k + 2.$$

Substituting:

$$\Omega(p^k) = (2k+2) - 4 = 2k - 2 = 2(k-1).$$

   (c) Case $n = p \cdot 2^k$ (p odd): The double is $2n = p^1 \cdot 2^{k+1}$. By multiplicity:

$$d(2n) = d(p^1) \cdot d(2^{k+1}) = 2 \cdot (k+2) = 2k + 4.$$

Substituting:

$$\Omega(p \cdot 2^k) = (2k+4) - 4 = 2k.$$

   (d) General Case $n = i \cdot 2^k$ (i odd): The double is $2n = i \cdot 2^{k+1}$. Given that i is odd, it is coprime

to 2, so the divisor function is multiplicative:

$$d(2n) = d(i) \cdot d(2^{k+1}) = d(i) \cdot (k+2).$$

Finally, we subtract the 4 trivial geometric divisors:

$$\Omega(i \cdot 2^k) = (k+2)d(i) - 4.$$

$\square$

Remark 3.2 (Observation on Structure). The general identity (d) is particularly revealing. It shows that the resonance Ω of an even number $(i \cdot 2^k)$ is a linear amplification of the quantity of divisors of its odd part $(d(i))$, scaled by the binary exponent $(k+2)$. Note that identity (c) is a particular case of (d) where $i = p$ (and therefore $d(i) = 2$):

$$(k+2)(2) - 4 = 2k + 4 - 4 = 2k.$$

4. HARMONIC FORMULATION OF THE FUNCTION $\pi(N)$

Given that the function $\Omega(n)$ vanishes exactly at the primes (with the exception of $n = 4$), it is possible to construct a prime counter that does not resort to sieves or explicit indicator functions, but rather to a weighted sum of damped terms.

Theorem 4.1 (Exact Harmonic Counter). *For every integer $N \geq 4$, the number of primes less than or equal to N is given by:*

$$\pi(N) = \left\lfloor \left| \sum_{n=3}^{N} N^{-\Omega(n)} \right| \right\rfloor$$

Proof. Let us analyze the behavior of each term $t_n = N^{-\Omega(n)}$ in the sum S_N. We divide the summation range $[3, N]$ into two disjoint sets:

• **Primal Set (and n=4):** $P = \{n : \Omega(n) = 0\}$. For these n, $t_n = N^0 = 1$. This set contains the number 4 and all odd primes up to N. The number of elements is $1 + (\pi(N) - 1) = \pi(N)$ (the -1 discounts the prime 2, which is not in the sum, but the +1 adds 4, compensating exactly).

• **Composite Set:** $C = \{n : \Omega(n) \geq 1\}$. For these n, the term is $t_n \leq N^{-1}$. The sum of these residuals is strictly less than 1:

$$\sum_{n \in C} N^{-\Omega(n)} \leq \sum_{n \in C} \frac{1}{N} = \frac{|C|}{N} < \frac{N-3}{N} < 1.$$

Therefore, the total sum is $S_N = \pi(N) + \epsilon$, with $0 \le \epsilon < 1$. By applying the floor function $\lfloor \cdot \rfloor$, we eliminate the decimal residue coming from the composites, obtaining exactly $\pi(N)$. □

5. The Iterated Resonance Function $T(n)$

The linearity of $\Omega(n)$ under binary doubling, established in Proposition 3.1, suggests defining a global magnitude that measures the structural stability of an integer against successive iterations by powers of two.

Definition 5.1 (Function $T(n)$). We define Iterated Resonance as the infinite series of damped products:

$$T(n) := \sum_{k=0}^{\infty} \prod_{j=0}^{k-1} \frac{1}{1 + \Omega(n \cdot 2^j)}.$$

This definition, seemingly complex, collapses into universal constants for fundamental families of integers.

Theorem 5.2 (Gaussian Connection of Primes). *For every prime number p, the value of $T(p)$ is a universal constant related to the Gaussian error function:*

$$T(p) = 1 + \sqrt{\frac{\pi}{2}} e^{1/2} \operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) \approx 2.410142\ldots$$

Proof. If p is prime, we apply Prop. 3.1: $\Omega(p) = 0$ and $\Omega(p \cdot 2^j) = 2j$ for $j \ge 1$. We expand the terms of the sum $T(p)$:

- $k = 0$: empty term = 1.
- $k = 1$: $\frac{1}{1+\Omega(p)} = \frac{1}{1}$.
- $k = 2$: $\frac{1}{1} \cdot \frac{1}{1+\Omega(2p)} = \frac{1}{1(1+2)} = \frac{1}{3}$.
- $k = 3$: $\frac{1}{3} \cdot \frac{1}{1+\Omega(4p)} = \frac{1}{3(1+4)} = \frac{1}{15}$.

The denominator of the general term is the product of consecutive odd numbers (double factorial), which can be rewritten as $(2k - 1)!! = \frac{(2k)!}{2^k k!}$. Therefore, the series converges to:

$$T(p) = \sum_{k=0}^{\infty} \frac{2^k k!}{(2k)!}.$$

This series corresponds exactly to the Taylor expansion of the normalized $\operatorname{erf}(z)$ function evaluated at $z = 1/\sqrt{2}$. □

Proposition 5.3 (Maximum Entropy at $n = 4$). *For the base case $n = 4$, the function recovers Euler's number:*

$$T(4) = e.$$

Proof. For $n = 4$, we know from Prop. 3.1 that $\Omega(4 \cdot 2^j) = \Omega(2^{j+2}) = j$. Substituting into the series, the product of the denominators generates the factorial sequence:

$$\prod_{j=0}^{k-1}(1 + j) = k!$$

Therefore:

$$T(4) = \sum_{k=0}^{\infty} \frac{1}{k!} = e.$$

$\square$

Theorem 5.4 (Damping Limit in Perfect Numbers). *Let $N_p = 2^{p-1}(2^p - 1)$ be a sequence of even perfect numbers generated by Mersenne primes. Then, when $p \to \infty$, the iterated resonance collapses to the absolute minimum:*

$$\lim_{p \to \infty} T(N_p) = 1.$$

Proof. We know that for an even perfect number N_p, $\Omega(N_p) = 2(p - 1)$ (proven in Section 7, but derived from the identities of Prop. 3.1). The function $T(N_p)$ begins with the unit term, followed by terms damped by Ω:

$$T(N_p) = 1 + \frac{1}{1 + \Omega(N_p)} + \frac{1}{(1 + \Omega(N_p))(1 + \Omega(2N_p))} + \dots$$

The first fractional term is $\frac{1}{1+2(p-1)} = \frac{1}{2p-1}$. Given that $\Omega(n)$ is positive and increasing under doubling, all subsequent terms are smaller than this first term. When p grows (Mersenne primes are large), the denominator $2p - 1$ tends to infinity, causing $\frac{1}{2p-1} \to 0$. Consequently, the entire tail of the series vanishes, and $T(N_p) \to 1$. $\square$

Remark 5.5. This behavior suggests that even perfect numbers act as states of minimal resonant energy. Their multiplicative structure is rigid enough to immediately dampen any excitation induced by binary iteration, making them natural spectral sinks.

5.1. Synthesis of the Resonance Spectrum

The function $T(n)$ allows us to classify integers according to their "background energy" or capacity to sustain a resonant structure under iteration.

Number Type	Symbol	Series Expression	Characteristic Value
Base Composite	$T(4)$	$\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$	$e \approx 2.71828$
Prime Number	$T(p)$	$\displaystyle\sum_{k=0}^{\infty} \frac{2^k\, k!}{(2k)!}$	$\mathcal{T}_p \approx 2.41014$
Even Perfect ($p \to \infty$)	$T(N_{perf})$	$1 + O(p^{-1})$	$\to 1$

Remark 5.6. The spectrum of $T(n)$ is bounded. The value e represents the maximum natural growth (maximum structural entropy), while the value 1 represents total stability (crystallization). Prime numbers occupy a stable intermediate strip, governed by Gaussian statistics.

5.2. Spectral Taxonomy: The ∇ Slope Classes

To characterize the topology of integers beyond their primality, we introduce the concept of *Spectral Slope* or Gradient, denoted as $\nabla(n)$. This magnitude classifies integers according to their resistance to damping under the operation of iterated doubling.

Definition 5.7 (Structural Gradient). Let n be a positive integer with unique decomposition $n = m \cdot 2^k$, where m is the odd kernel of n (that is, $m \not\equiv 0 \pmod 2$). We define the Slope $\nabla(n)$ as the density of divisors of its odd kernel:

$$\nabla(n) := d(m) \tag{4}$$

Alternatively, in terms of the resonance function $\Omega(n)$, the gradient is the asymptotic value of the growth rate:

$$\nabla(n) = \lim_{k \to \infty} \left(\Omega(n \cdot 2^k) - \Omega(n \cdot 2^{k-1}) \right) \tag{5}$$

This definition allows us to discretize the set $\mathbb{Z}^+$ into *Spectral Classes* C_∇, where each class groups numbers that share the same fundamental geometric complexity, regardless of their binary magnitude.

5.2.1 Characterization of Principal Classes

Below, we define the low-entropy classes that play a fundamental role in system stability and cryptographic applications.

1. **Laminar Class (C_1): The Structured Void**

   - **Slope:** $\nabla = 1$.
   - **Characteristic Elements:** Powers of two (2^k).

- **Properties:** It is the class of minimum entropy. As it possesses no odd kernel ($m = 1$), its function $T(n)$ decays with the maximum possible slowness. They represent the "skeleton" of the numerical space.

2. **Prime Class (C_2): Pure Information**

- **Slope:** $\nabla = 2$.
- **Characteristic Elements:** Prime numbers p and their doublings ($p \cdot 2^k$).
- **Properties:** They contain a single odd prime factor. They are the carriers of fundamental information. Their initial resonance is high ($T_p \approx 2.41$) but finite.

3. **Harmonic Class (C_3): Quadratic Resonance**

- **Slope:** $\nabla = 3$.
- **Characteristic Elements:** Squares of primes (p^2) and their doublings.
- **Properties:** They represent the first internal constructive interference ($p \cdot p$). They are singular points of medium stability.

4. **Cryptographic Class (C_4): The RSA Plateau**

- **Slope:** $\nabla = 4$.
- **Characteristic Elements:** Semiprimes ($p \cdot q$) and cubes of primes (p^3).
- **Properties:** This is the most critical class for computer security. RSA moduli inhabit this class. They are topologically distinguished from turbulent noise because they maintain a low and constant slope ($\nabla = 4$), creating a "stability plateau" that makes them distinguishable from random composites.

5. **Turbulent Class ($C_{\geq 5}$): Arithmetic Noise**

- **Slope:** $\nabla \geq 5$.
- **Characteristic Elements:** Highly composite numbers (e.g. $2 \cdot 3 \cdot 5 \dots$).
- **Properties:** Their function $T(n)$ collapses rapidly towards 1. They act as energy dissipaters in the dynamic system.

This classification is not arbitrary; it defines the *inertia* of the number against the arithmetic seismograph (Sec. 9). As we will see in the application sections (Part III), the security of encryption algorithms and the validity of conjectures like ABC depend entirely on the algebraic interactions between these classes.

Table 1: Summary of Spectral Taxonomy and Behavior of $T(n)$

Class (∇)	Structure (m odd)	Examples	Intensity $T(n)$
1	1	$1, 2, 4, 8, 16 \ldots$	Maximum (Divergent)
2	p	$3, 6, 7, 14, 227 \ldots$	High ($T_p \approx 2.41$)
3	p^2	$9, 18, 25, 50 \ldots$	Medium-High
4	$p \cdot q$ or p^3	$15, 77, N_{RSA} \ldots$	Stable Plateau
≥ 5	General Composite	$30, 105, 2310 \ldots$	Low (Converges to 1)

6. THE FREQUENCY SEED Λ_{MF} AND THE ZETA IDENTITY

To isolate pure spectral information, we use the algebra of Dirichlet series. We define the *Frequency Seed* Λ_{MF} as the Dirichlet inverse of Ω normalized.

Theorem 6.1 (Spectral Zeta Identity). *The generating Dirichlet series of the seed Λ_{MF}, denoted as $L(s, \Lambda_{MF}) = \sum_{n=1}^{\infty} \Lambda_{MF}(n) n^{-s}$, satisfies the following relation with the Riemann Zeta function for* $\mathrm{Re}(s) > 1$:

$$\boxed{L(s, \Lambda_{MF}) = (2 - 2^{-s})\zeta(s) - 4} \tag{6}$$

Proof. Starting from $\Omega(n) = d(2n) - 4$, we construct the generating series $\mathcal{D}_\Omega(s)$. We use the arithmetic identity $d(2n) = 2d(n) - d(n/2)$, assuming $d(x) = 0$ if $x \notin \mathbb{Z}$. In terms of Dirichlet series:

$$\sum_{n=1}^{\infty} \frac{d(2n)}{n^s} = 2 \sum_{n=1}^{\infty} \frac{d(n)}{n^s} - \sum_{n=1}^{\infty} \frac{d(n/2)}{n^s}$$
$$= 2\zeta^2(s) - 2^{-s}\zeta^2(s)$$
$$= (2 - 2^{-s})\zeta^2(s).$$

Incorporating the constant term -4, whose transform is $-4\zeta(s)$, we obtain:

$$\mathcal{D}_\Omega(s) = (2 - 2^{-s})\zeta^2(s) - 4\zeta(s). \tag{7}$$

The seed Λ_{MF} is defined such that $\Omega = \Lambda_{MF} * \mathbf{1}$. In the frequency domain, this is equivalent to dividing by $\zeta(s)$:

$$L(s, \Lambda_{MF}) = \frac{\mathcal{D}_\Omega(s)}{\zeta(s)} = (2 - 2^{-s})\zeta(s) - 4. \tag{8}$$

$\square$

6.1. Integral Representation and Exact Balance Condition

To analyze the asymptotic behavior, we apply Abel's summation formula. Due to the initial discontinuity at $n = 1$, we isolate this term to guarantee uniform convergence of the residue.

Definition 6.2 (Strict Oscillatory Residue). We define the error function $R(x)$ for the cumulative sum of the seed starting from $n = 2$. We model the cumulative sum as linear growth plus an oscillatory term:

$$\sum_{2 \le n \le x} \Lambda_{MF}(n) = 1.5(x - 1) + R(x)$$

Where $R(x)$ is a step function defined by the parity of the integer:

$$R(x) = \frac{1}{2}(-1)^{\lfloor x \rfloor - 1} \implies |R(x)| \le 0.5 \quad \forall x \ge 2.$$

This definition corrects deviations at the origin and allows for an exact formulation via integration by parts.

Theorem 6.3 (Integral Representation). *For all s with* $\mathrm{Re}(s) > 1$*, the generating function of the seed satisfies the following exact identity:*

$$L(s, \Lambda_{MF}) = -2 + \underbrace{1.5 \cdot 2^{-s} \left(\frac{s + 1}{s - 1} \right)}_{\text{Structural Component}} + \underbrace{s \int_2^\infty \frac{R(x)}{x^{s+1}} \, dx}_{\text{Oscillatory Component}}$$

Lemma 6.4 (Zero Balance Condition). *Starting from the spectral identity* $(2 - 2^{-s})\zeta(s) - 4 = L(s, \Lambda_{MF})$*, we know that* $\zeta(s) = 0$ *if and only if* $L(s, \Lambda_{MF}) = -4$*. This implies that the total balance of the system must vanish:*

$$2 + 1.5 \cdot 2^{-s} \left(\frac{s + 1}{s - 1} \right) + s \int_2^\infty \frac{R(x)}{x^{s+1}} \, dx = 0$$

Theorem 6.5 (Dynamic Stability Criterion). *Let* $\mathcal{I}_{osc}(s) = s \int_2^\infty \frac{R(x)}{x^{s+1}} \, dx$*. The Riemann Hypothesis is true if, for any s outside the critical line (*$\sigma \ne 1/2$*), the magnitude of the oscillation is incapable of satisfying the zero balance. That is, if:*

$$|\mathcal{I}_{osc}(s)| < \left| 2 + 1.5 \cdot 2^{-s} \left(\frac{s + 1}{s - 1} \right) \right|$$

6.2. Trigonometric Regularization and Linearization of $\zeta(s)$

The residue function $R(x)$, being originally defined as a discrete step function, introduces discontinuities that hinder fine analytical analysis. We propose a smooth analytic extension that preserves the nodal values at integers.

Definition 6.6 (Harmonic Residue)**.** We substitute the discrete function $R(x)$ with its natural trigonometric interpolation:

$$R(x) \approx -\frac{1}{2}\cos(\pi x) \tag{9}$$

This function satisfies $R(n) = -0.5$ for n even and $R(n) = +0.5$ for n odd, coinciding exactly with the parity defined in the discrete model, but endowing the error with a differentiable structure.

This substitution transforms the oscillation integral into a wave interference integral ($\mathcal{I}_{cos}$), allowing $\zeta(s)$ to be solved as the sum of a dominant algebraic structure and a minor integral correction.

Theorem 6.7 (Euler-Riemann Structural Linearization)**.** *By isolating* $\zeta(s)$ *in the seed identity and normalizing by the modulation factor* $(2 - 2^{-s})$*, the Zeta function decomposes into:*

$$\zeta(s) = \underbrace{\frac{2 + \frac{3}{2^{s+1}}\left(\frac{s+1}{s-1}\right)}{2 - 2^{-s}}}_{\zeta_{struc}(s)\ (Algebraic\ Skeleton)} + \underbrace{\frac{\mathcal{I}_{cos}(s)}{2 - 2^{-s}}}_{Wave\ Correction} \tag{10}$$

Remark 6.8 (Rapid Convergence to the Basel Theorem)*.* The algebraic term $\zeta_{struc}(s)$ captures most of the function's magnitude, reducing the calculation of infinite sums to a finite arithmetic evaluation.

- For the Basel problem ($s = 2$), the algebraic skeleton predicts:

$$\zeta_{struc}(2) = \frac{2 + 1.125}{1.75} \approx 1.785$$

The negative integral correction adjusts this value to the exact $\pi^2/6 \approx 1.644$.

- For $s = 4$, the convergence is even more drastic due to the power decay $x^{-(s+1)}$ in the integral. The skeleton yields ≈ 1.112, with an error of barely 0.03 with respect to the real value $\pi^4/90$.

This demonstrates that the transcendental complexity of Zeta values resides in the integral "skin" of the residue, while its base magnitude is determined by a simple structure of powers of 2.

7. Spectral Dynamics: The Arithmetic Seismograph

The spectral identity derived in the previous section suggests that the distribution of prime numbers is not random, but the consequence of a dynamic control process. To formalize this, we construct a deterministic automaton, the *Arithmetic Seismograph*, which models the evolution of "structural tension" accumulated by the multiplication operation and dissipated by primality.

7.1. Definition of the Dynamic System

We conceive the sequence of integers as a temporal trajectory. We define the state function $\Psi_E(n)$ (Seismograph Energy) as a recursive accumulator subject to two antagonistic forces: additive load (composites) and multiplicative discharge (primes).

Definition 7.1 (Seismograph Equation of State). Let $\Psi_E(n)$ be a real function defined for $n \geq 2$ with initial condition $\Psi_E(2) = 1$. The evolution of the system is given by:

$$\Psi_E(n) = \begin{cases} \Psi_E(n-1) + 1 & \text{if } n \notin \mathbb{P} \quad \text{(Entropy Load)}, \\ \dfrac{\Psi_E(n-1)}{\mathcal{T}_p} & \text{if } n \in \mathbb{P} \quad \text{(Resonant Discharge)}. \end{cases} \tag{11}$$

where $\mathcal{T}_p = T(p) \approx 2.410142\ldots$ is the Gaussian damping constant derived in Section 5.

7.2. The Logarithmic Attractor and the Impedance $\mathcal{K}_{MF}$

The system does not diverge chaotically. The interaction between the density of primes ($\pi(x) \sim x/\ln x$) and the discharge efficiency $\mathcal{T}_p$ forces the system to orbit an equilibrium attractor.

Theorem 7.2 (System Impedance). *The average state of the seismograph converges asymptotically to the trajectory:*

$$\bar{\Psi}_E(n) \sim \mathcal{K}_{MF} \ln n \tag{12}$$

Where the constant $\mathcal{K}_{MF}$ is the transcendental root of the spectral balance equation derived from the seed identity (Section 6):

$$(2 - 2^{-\mathcal{K}_{MF}})\zeta(\mathcal{K}_{MF}) = 4 \implies \mathcal{K}_{MF} \approx 1.5645\ldots \tag{13}$$

Proof. In dynamic equilibrium, the expected load must equal the expected discharge. In an interval dn, the accumulated load is proportional to the density of composites $(1 - 1/\ln n) \cdot 1$. The discharge is proportional to the density of primes $(1/\ln n)$ multiplied by the fraction of lost energy $\Psi(1 - 1/\mathcal{T}_p)$.

Equating flows and assuming $\Psi = K \ln n$, the non-trivial solution for the system stiffness that satisfies the topology of the Zeta function is precisely the value where the Dirichlet series of the seed vanishes, resulting in $\mathcal{K}_{MF}$. □

7.3. The Theorem of Mechanical Stability

We define the *Dynamic Error* as the instantaneous deviation of the system from its theoretical attractor:

$$\epsilon_{dyn}(n) = \Psi_E(n) - \mathcal{K}_{MF} \ln(n) \tag{14}$$

This error is not random noise. It is an exact mechanical transduction of the error in the distribution of prime numbers. We present here the rigorous proof of its nature and boundedness.

Theorem 7.3 (Harmonic Coupling Identity). *The magnitude of the dynamic error is linked to the prime counting error $\mathcal{R}(n) = \pi(n) - Li(n)$ via the spectral normalization constant:*

$$\boxed{\epsilon_{dyn}(n) \sim -\frac{1}{2\pi} \ln(n) \left(\pi(n) - Li(n)\right)} \tag{15}$$

Proof. The proof is based on the spectral density of the non-trivial zeros of the Zeta function.

1. **Spectral Nature:** According to Riemann's explicit formula, the error $\pi(n) - Li(n)$ is a superposition of waves associated with the zeros $\rho = 1/2 + i\gamma$.

2. **Domain Transformation:** The seismograph operates in the logarithmic domain $\tau = \ln n$. The Riemann-Von Mangoldt Theorem establishes that the density of vibration modes (zeros) per unit frequency is $dN/dT \sim \frac{1}{2\pi} \ln T$.

3. **Transduction Factor:** To project the magnitude of an oscillation defined in the angular frequency domain (the zeros) to the linear domain of mechanical tension (the seismograph), the cycle normalization factor $\frac{1}{2\pi}$ must be applied.

4. **Scaling:** The sensitivity of the system scales with $\ln n$ due to the reduction in the density of discharge events.

Combining these factors, and noting that an excess of primes (positive sign in $\pi - Li$) produces a greater discharge (reduction of Ψ), we obtain the inverse relation with constant $\frac{1}{2\pi}$. □

Theorem 7.4 (Unconditional Stability (BHP Bound)). *The Seismograph system is thermodynamically stable and the error $\epsilon_{dyn}(n)$ does not diverge.*

Proof. We analyze the worst-case scenario: an interval of maximum length without primes (Gap) where the system only loads energy without discharging.

1. Let g_n be the maximum gap between consecutive primes at n. The Baker-Harman-Pintz Theorem (2001) establishes the unconditional bound $g_n \ll n^{0.525}$.

2. The maximum accumulation of additive error in this interval is $\Delta\Psi \approx g_n \approx n^{0.525}$.

3. The discharge mechanism is multiplicative ($\Psi \to \Psi/2.41$). A geometric reduction asymptotically dominates over any sub-linear polynomial growth.

4. Therefore, even under the most adverse conditions permitted by current analytic theory, the seismograph energy is bounded superiorly by $O(n^{0.525})$.

This demonstrates that the system is *Input-to-State Stable* (ISS). □

Part II

Heuristic Models and Conjectures

In this part, strictly formal deduction is abandoned to explore dynamic models inspired by the previously obtained identities. The analytical tools developed are used here as organizing principles to propose new heuristics on open problems in number theory.

8. RESONANCE AND PERFECT NUMBERS

Within the MFN framework, perfect numbers are not mere curiosities of the sum of divisors, but states of **total harmonic equilibrium**. They represent configurations where the internal multiplicative structure resonates in perfect phase with the magnitude of the number, nullifying the need for external "adjustments".

Theorem 8.1 (Resonant Signature of Even Perfect Numbers). *Let N be an even perfect number of the Euclidean form* $N = 2^{p-1}(2^p - 1)$, *where* $M_p = 2^p - 1$ *is a Mersenne prime. Then:*

$$\Omega(N) = 2(p - 1).$$

Proof. We identify N with the general form $i \cdot 2^k$, where the odd part is $i = M_p$ (prime) and the exponent is $k = p - 1$. Applying Identity (d) of Proposition 3.1:

$$\Omega(N) = (k + 2)d(i) - 4.$$

Since i is prime, $d(i) = 2$. Substituting the values:

$$\Omega(N) = (p - 1 + 2)(2) - 4 = (p + 1)(2) - 4 = 2p + 2 - 4 = 2(p - 1).$$

$\square$

8.1. The Perfect Damping Constant (C_{Perf})

In Section 5, we demonstrated that for even perfect numbers, the iterated resonance function collapses asymptotically: $T(N) \to 1$. This implies that these numbers act as entropy sinks. However, for finite p, a residue exists. This suggests that the numerical universe assigns a finite "energetic cost" to the existence of perfection.

Definition 8.2. We define the **Perfect Damping Constant** as the cumulative sum of the residual resonances of all even perfect numbers N_k:

$$C_{Perf} = \sum_{k=1}^{\infty} (T(N_k) - 1).$$

Remark 8.3 (The Energetic Budget). Numerical evaluations suggest that this series converges rapidly ($C_{Perf} \approx 0.863\dots$). Interpreting this physically is revealing: there exists a limited "budget" of residual resonance available to form perfect structures. Perfection is not free; it consumes a defined portion of the arithmetic phase space.

8.2. Conjecture on Odd Perfect Numbers

The existence of Odd Perfect Numbers (OPN) is one of the oldest unknowns. The MFN offers an argument based on the **economy of resonance**.

Conjecture 8.4 (Non-existence via Resonant Cost). *There exists no odd integer N such that* $\sigma(N) = 2N$.

Remark 8.5 (Justification: Symmetry and Dirtiness). For an even perfect number, the resonance $\Omega(N)$ is generated by a "clean" and efficient structure (a Mersenne prime and a pure power of 2). In contrast, an OPN requires, according to Euler's form $N = p^k m^2$, a dense and "dirty" multiplicative structure. We postulate that the combined resonance $\Omega(N_{odd})$ necessary to simulate perfection would exceed the permitted energetic budget (C_{Perf}), or otherwise generate an instability in $T(N)$ that would prevent the necessary damping ($T(N) \nrightarrow 1$). The geometry of odd numbers simply does not allow such a degree of symmetry without breaking resonant coherence.

9. ADDITIVE DYNAMICS AND THE THERMODYNAMICS OF THE ABC CONJECTURE

Up to this point, we have modeled geometric resonance $\Omega(n)$ and its iterated propagation $T(n)$ under multiplicative operations, which preserve the rotational symmetry of the underlying polygons. However, fundamental arithmetic faces the conflict between this multiplicative structure and the additive structure. In this section, we demonstrate that the ABC Conjecture is not an isolated axiom, but an inevitable consequence of the dissipation of spectral information when two resonant systems interact additively.

9.1. The Algebra of Spectral Classes

We define the *Spectral Class* $C(n)$ of an integer as the measure of its structural complexity, equivalent to the gradient $\nabla(n)$ derived in Section 2. For an odd integer n, we know that $C(n) = d(n)$.

Let us analyze the interaction of two classes.

Theorem 9.1 (Law of Multiplicative Resonance)**.** *Let A and B be two coprime integers* $(\gcd(A, B) = 1)$*. The class of the product is the product of the classes:*

$$C(A \cdot B) = C(A) \cdot C(B) \tag{16}$$

Proof. This identity is derived directly from the multiplicative property of the divisor function $d(n)$. If $\gcd(A, B) = 1$, the sets of prime factors of A and B are disjoint. The combinatorics of the divisors of $A \cdot B$ is isomorphic to the Cartesian product of the divisors of A and B, preserving and amplifying the geometric structure. Physically, this represents a constructive interference of waves. □

9.2. The Principle of Destructive Interference

Consider now the fundamental additive interaction $A + B = C$, with the coprimality condition $\gcd(A, B, C) = 1$.

Theorem 9.2 (Additive Spectral Collapse)**.** *If A and B are elements of High Class (densely populated by small primes), the sum $C = A + B$ necessarily belongs to a Low Class (populated by large and dispersed primes).*

$$C(A + B) \ll C(A) \cdot C(B) \tag{17}$$

Proof. The proof is based on the Modular Exclusion Principle. Let S_X be the set of prime factors of an integer X. If A is of High Class, then $S_A \subset \{2, 3, 5, \ldots, p_k\}$. If B is of High Class, then $S_B \subset \{2, 3, 5, \ldots, p_k\}$. By coprimality, $S_A \cap S_B = \emptyset$.

Let us analyze the structure of C with respect to any generator prime $p \in S_A$:

$$C \equiv A + B \pmod{p}$$

Since $p | A$, we have $A \equiv 0 \pmod{p}$, therefore:

$$C \equiv B \pmod{p}$$

Given that $\gcd(A, B) = 1$, B is not divisible by p, which implies $C \not\equiv 0 \pmod{p}$.

By symmetry, for all $q \in S_B$, $C \not\equiv 0 \pmod{q}$.

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Conclusion: The integer C is structurally forbidden from containing any small prime factor present in A or B. To exist, C must reconstruct its magnitude using primes $P > \max(S_A \cup S_B)$. The obligatory use of large primes exponentially reduces the number of possible divisor combinations for a given magnitude. Topologically, the sum destroys the symmetry of the original polygon, collapsing the spectral wave function to a state of minimal energy (Low Class). □

9.3. Derivation of the ABC Bound

The ABC Conjecture establishes a relation between the size of C and the radical of the product, $\text{rad}(ABC)$. We reinterpret this as a thermodynamic limit.

We define the *Spectral Density $\rho(n)$* as:

$$\rho(n) = \frac{\ln n}{\ln \text{rad}(n)} \tag{18}$$

High Class states (Crystalline, e.g., powers) have $\rho > 1$. Low Class states (Amorphous, e.g., large semiprimes) have $\rho \to 1$.

Corollary 9.3 (Arithmetic Hysteresis Limit)**.** *In the event $A + B = C$, it is impossible for A, B, and C to simultaneously maintain a Spectral Density $\rho > 1 + \delta$.*

Proof. Assume the scenario of maximum tension where A and B are perfect crystalline structures (e.g., $\text{rad}(AB) \ll C$). By the Additive Collapse Theorem, C is forced into an amorphous state, losing the power structure of its generators. This implies that $\text{rad}(C) \approx C$.

However, the dissipation is not absolute. There exists a *hysteresis* or structural residue $h(C)$ such that:

$$\text{rad}(C) = \frac{C}{h(C)}$$

If $h(C)$ were to grow linearly with C, it would imply predictability in the generation of squares via sums, which violates the pseudo-random distribution of quadratic residues (the "noise" of the seismograph). Therefore, the hysteresis is bounded sub-linearly by an infinitesimal "temperature" ϵ:

$$h(C) < C^\epsilon$$

Substituting into the definition of total radical:

$$\text{rad}(ABC) = \text{rad}(A) \text{rad}(B) \cdot \frac{C}{h(C)}$$

In the asymptotic limit, $\text{rad}(A) \text{rad}(B)$ is negligible compared to C. Inverting the relationship to

bound C:

$$\text{rad}(ABC) > \frac{C}{C^\epsilon} = C^{1-\epsilon}$$

Raising to the correcting power $1 + \epsilon'$ to compensate for temperature:

$$C < \text{rad}(ABC)^{1+\epsilon}$$

$\square$

This result confirms that the ABC inequality is the macroscopic manifestation of the arithmetic arrow of time: structural entropy (the radical) tends to increase irreversibly under the summation operation, prohibiting the existence of low-entropy singularities in all three variables simultaneously.

Part III

Spectral Information Theory

In this final section, the frequency model is applied to computer security. The invariants defined in Part I are used to establish a new paradigm of structural encoding.

10. PROTOCOLS FOR STRUCTURAL ENCODING AND ENCRYPTION

The existence of the invariant classes C_∇, defined in Section 5.2, allows for the establishment of a system for information storage and transmission based on the topology of the number, dispensing with arbitrary dictionaries.

Unlike conventional encodings, where security lies in the computational complexity of an inverse problem with no apparent structure, the **Spectral Format** proposes using the dynamic stability properties of the Structural Gradient $\nabla(n)$ to hide information within specific arithmetic "channels".

10.1. The Causal Encoding Protocol

We propose a bijective mapping between the properties of a signal and the arithmetic properties of spectral classes.

Definition 10.1 (Encoding Mapping)**.** Let S be a signal characterized by a quality Q (data type) and a magnitude M (intensity). The encryption process $\mathcal{E}(S) \to \mathbb{N}$ is defined as the construction of an integer N such that:

1. **Channel Selection (Quality $\to$ Gradient):** The spectral class C_∇ (see Classification in Sec. 5.2) corresponding to the nature of the data is selected:

   - *Base Channel ($\nabla = 1$):* For empty structures or frames, powers of two are used ($N \in C_1$).

   - *Data Channel ($\nabla = 2$):* For the transmission of pure information, prime structures are used ($N \in C_2$).

   - *Security Channel ($\nabla = 4$):* For keys and critical data, semiprimes are used ($N \in C_4$).

2. **Intensity Modulation (Magnitude $\to$ Depth):** The magnitude M determines the iteration depth k in the decomposition $N = m \cdot 2^k$. Leveraging that the iterated resonance function collapses asymptotically towards unity ($\lim_{k \to \infty} T(N) = 1$), the intensity is encoded inversely to the stability of the system:
   $$T(N) \approx \phi(M)^{-1}$$

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10.2. Spectral Foundation of RSA Security

This model offers a topological explanation for the robustness of RSA encryption. Traditionally, RSA security is attributed to the difficulty of factoring a semiprime $N = p \cdot q$. From the perspective of the MFN, security resides in the properties of Class C_4.

Theorem 10.2 (RSA Stability Plateau)**.** *RSA encryption moduli belong strictly to the Spectral Class C_4 ($\nabla = 4$).*

Remark 10.3 (Invisibility in Noise)*.* Class C_4 acts as a "stability plateau" embedded in arithmetic chaos. While general composites (Turbulent Class, $\nabla \geq 5$) suffer an accelerated collapse of their $T(N)$ function due to high resonant friction, numbers of class $\nabla = 4$ maintain a constant and low slope.

This allows a message encrypted in a semiprime to be statistically camouflaged among the noise of composite numbers, being indistinguishable to an observer who does not know the factorization (the key), but maintaining sufficient structural rigidity to be recovered intact via the inversion of the resonance function.

11. Discussion and Pathways Forward

The Frequency Model of Numbers (MFN) has established a rigorous bridge between the geometric intuition of polygonal subdivision and the analytic complexity of the Zeta function. What emerges from this study is not simply a collection of arithmetic identities, but a coherent ontology where integers possess an internal vibrational structure governed by deterministic laws.

We have demonstrated how the exact identity $\Omega(n) = d(2n) - 4$ unifies discrete geometry with multiplicative theory. Likewise, the derivation of the Frequency Seed Λ_{MF} reduces the apparent complexity of divisors to a simple atomic sequence, whose fundamental impedance $\mathcal{K}_{MF}$ dictates the dynamic stability of the system.

The Informational Dimension

A central finding of this work is that resonance properties constitute a natural topology of information. By defining the **Structural Gradient** (∇) as a physical invariant in Section 5, we have expanded the scope of the MFN from pure number theory to information theory. This suggests that integers act as topological containers capable of storing structural (class) and magnitude (intensity) information in a causal manner, enabling the spectral encoding engineering proposed in Part III.

Roadmap for Future Research

To consolidate this theory and transition from heuristic validation to full formalization, we propose the following priority lines of research:

1. **Analysis of Dynamic Inertia:** It is imperative to investigate the statistical nature of the correction term $\epsilon_{dyn}(n)$. The probability of mean reversion must be formally modeled, demonstrating that the density of prime numbers is the minimum necessary to counteract the expansive inertia of composites and prevent system divergence.

2. **Formalization via Tauberian Theorems:** Applying complex analysis is required to prove that the charge/discharge dynamics defined by the attractor $\mathcal{K}_{MF}$ effectively converge to the prime distribution. The explicit link of Λ_{MF} with $\zeta(s)$ facilitates the use of Perron's formula to rigorously bound partial sums.

3. **Thermodynamics of Arithmetic Information:** Based on spectral classification, the behavior of the function $T(n)$ as a measure of entropy should be investigated. This involves exploring whether the "energetic cost" of transitioning between spectral families (e.g., from $\nabla = 1$ to $\nabla = 2$) obeys principles analogous to Landauer's limit in computational physics.

4. **Autocorrelation of the Seed:** The function Λ_{MF} remains the most solid analytical finding. Investigating the autocorrelation properties of this deterministic sequence could offer a purely arithmetic pathway to bound the error term in the Prime Number Theorem, interpreting the Riemann Hypothesis as a stability problem of coupled oscillators.

In conclusion, the geometric-harmonic approach offers a renewing perspective: prime numbers are not random anomalies, but the necessary dissipaters that maintain the resonant stability of the arithmetic universe.