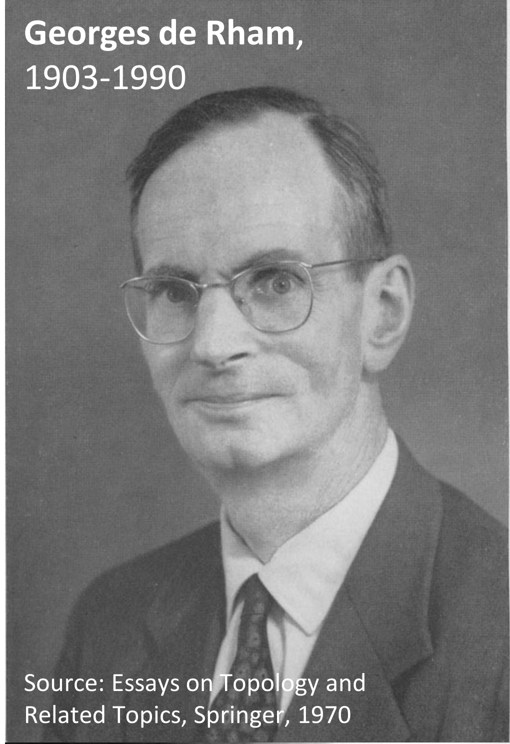
Electromagnetics and Differential Forms

Elements of Hodge theory
Outlook: Relativistic electrodynamics

In today's lecture we talk about:

- **L^2-inner product and codifferential**
- Dirac and Hodge-Laplace operators
- Maxwell's equations in space-time
- Constitutive relations in Minkowski space



$\boldsymbol{L^2}$-inner product and codifferential operator [20, 6.2.6]

Consider an oriented Riemannian manifold (M, g, ω_M). We define the $\boldsymbol{L^2}$**-inner product** for differential forms whose supports have compact intersection by integrating the pointwise inner product, $\mathcal{F}^p(M) \times \mathcal{F}^p(M) \to \mathbb{R}$,

$$(\omega, \eta) \mapsto \langle \omega, \eta \rangle := \int_M (\omega \cdot \eta)\, \omega_M = \int_M \omega \wedge \star \eta$$

We now define the **codifferential** $\boldsymbol{\delta}$ as formal L^2-adjoint of the exterior derivative, where $\omega \in \mathcal{F}^{p-1}(M), \eta \in \mathcal{F}^p(M)$,

$$\langle \mathrm{d}\omega, \eta \rangle = \langle \omega, \delta\eta \rangle$$

Intersection of supports compact in $M \backslash \partial M$ $\curvearrowright$
integrals exist,
no boundary terms

Coderivative (cont'd)

This implies for the **coderivative**

$$\delta : \mathcal{F}^p(M) \to \mathcal{F}^{p-1}(M) : \omega \mapsto (-1)^p \star^{-1} \mathrm{d} \star \omega$$

as can be seen from the product rule

By Stokes' theorem this yields a boundary term cancels out during integration

$$\begin{aligned} \mathrm{d}(\omega \wedge \star \eta) &= \mathrm{d}\omega \wedge \star \eta - \omega \wedge (-1)^p \mathrm{d} \star \eta \\ &= \mathrm{d}\omega \wedge \star \eta - \omega \wedge \star (-1)^p \star^{-1} \mathrm{d} \star \eta \\ &= \mathrm{d}\omega \wedge \star \eta - \omega \wedge \star \delta \eta \end{aligned}$$

$\omega \in \mathcal{F}^{p-1}(M)$

$\eta \in \mathcal{F}^p(M)$

- Complex property: $\delta \circ \delta = 0$
- Physical dimension: $\mathrm{pd}(\delta) = \mathrm{pd}(g)^{-1} =$ L^{-2}

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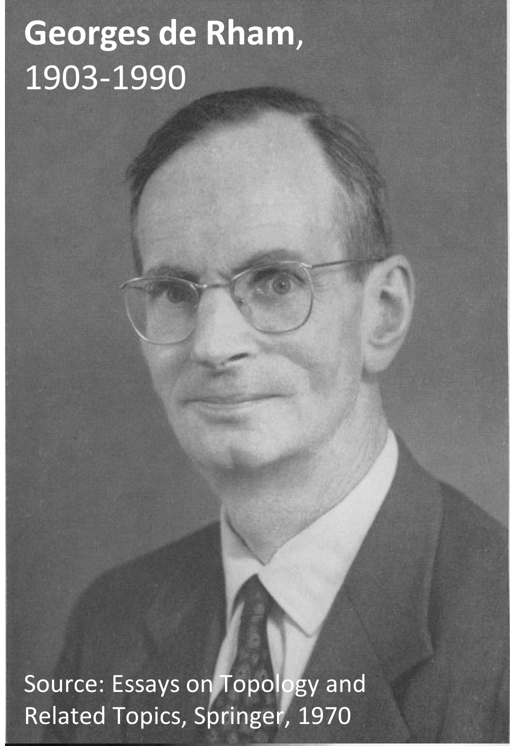


Dirac and Hodge-Laplace operators

The **Dirac operator** is defined by

$$\mathrm{D} : \mathcal{F}(M) \to \mathcal{F}(M) : \omega \mapsto (\mathrm{d} \oplus \delta)\,\omega$$

<u>Note</u>: It is of inhomogeneous degree, hence defined on the entire exterior algebra $\mathcal{F}(M) = \mathcal{F}^0(M) \oplus \cdots \oplus \mathcal{F}^n(M)$ $n = \dim M$

Direct sum over modules

The **Hodge-Laplace operator** is defined by

$$\boldsymbol{\Delta} : \mathcal{F}^p(M) \to \mathcal{F}^p(M) : \omega \mapsto \mathrm{D}^2\omega = (\mathrm{d}\delta + \delta\mathrm{d})\omega$$

Homogeneous degree

- Physical dimension: $\mathrm{pd}(\boldsymbol{\Delta}) = \mathrm{pd}(g)^{-1} = \mathrm{L}^{-2}$

Hodge-Laplace and translation isomorphisms

In Euclidean space $\left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$ it holds that

$$\begin{aligned} \boldsymbol{\Delta}^0 f &= -{}^0\Delta f \\ \boldsymbol{\Delta}^1 a &= -{}^1\Delta a \\ \boldsymbol{\Delta}^2 b &= -{}^2\Delta b \\ \boldsymbol{\Delta}^3 g &= -{}^3\Delta g \end{aligned}$$

$$\begin{aligned} &\forall f, g \in C^\infty(M) \\ &\forall a, b \in \mathcal{X}(M) \\ &M \subset \mathbb{R}^3 \quad \text{open} \end{aligned}$$

Herein, Δ denotes the ordinary scalar and vector Laplace operators from vector analysis,

$$\Delta f = \operatorname{div} \operatorname{grad} f$$

$$\Delta a = \operatorname{grad} \operatorname{div} a - \mathbf{curl}\,\mathbf{curl}\, a$$

Example: curl curl equation of magnetostatics

- From chasing the Tonti diagram we obtain in the static case

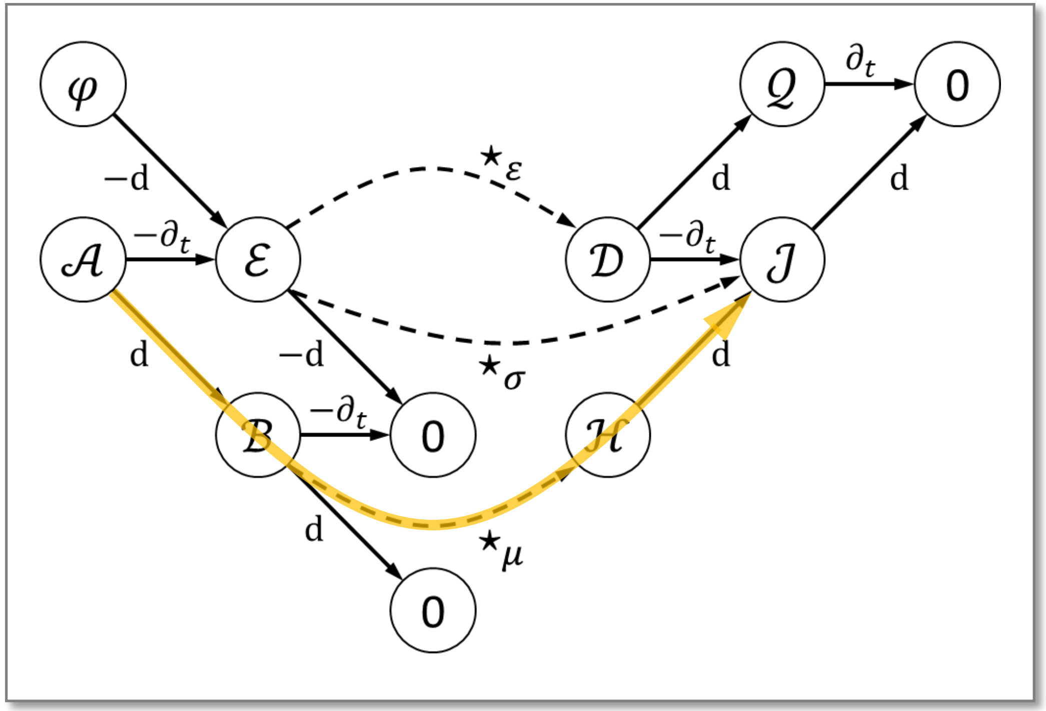
  $\curvearrowright \; \mathrm{d} \star_\mu \mathrm{d}\mathcal{A} = \mathcal{J}$ curl curl eq.

- Apply $\star_\mu$

  $\curvearrowright \; \delta_\mu \mathrm{d}\mathcal{A} = \star_\mu \mathcal{J}$

- **Coulomb gauge**: We can always find a representative (why?) $\mathcal{A}' = \mathcal{A} + \mathrm{d}\lambda$ such that $\delta_\mu \mathcal{A} = 0$

  $\curvearrowright \; \delta_\mu \mathrm{d}\mathcal{A} + \mathrm{d}\underbrace{\delta_\mu \mathcal{A}}_{=0} = \star_\mu \mathcal{J}$



$\boldsymbol{\Delta}_\mu \mathcal{A} = \star_\mu \mathcal{J}$ Vector Laplace eq.

Elements of Hodge theory

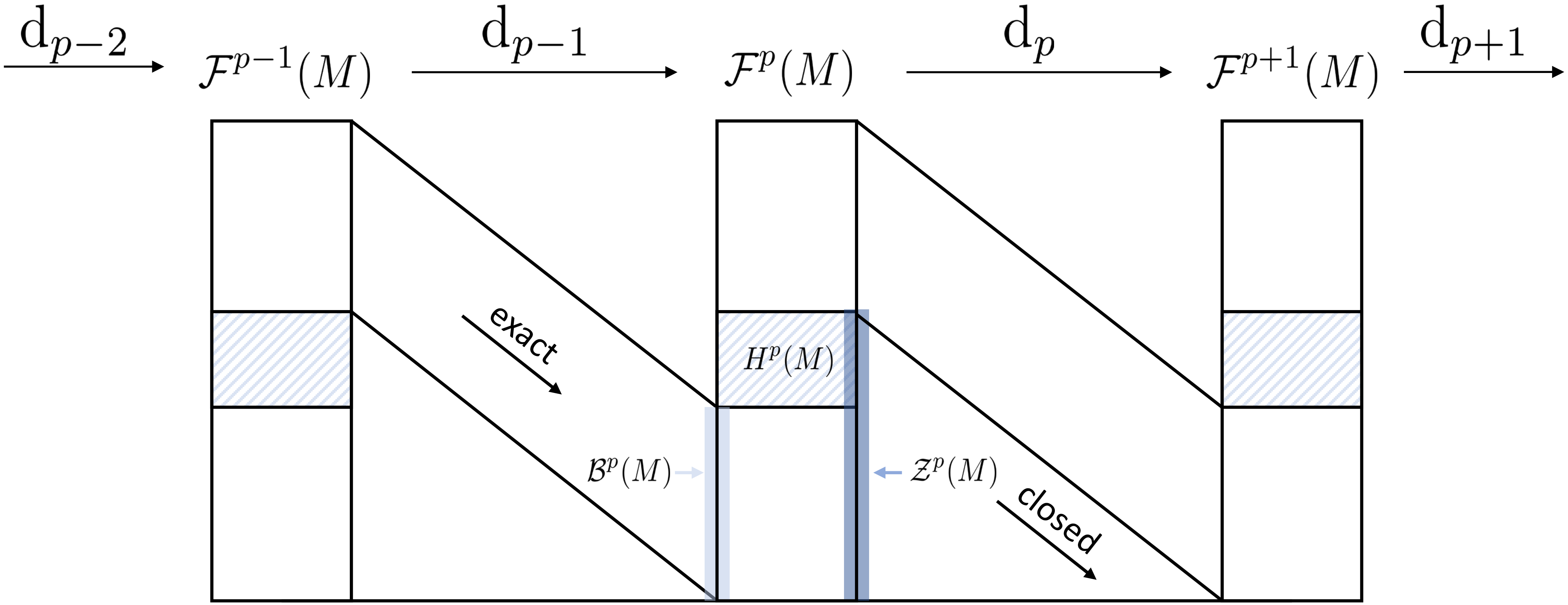
Consider an oriented Riemannian manifold (M, g, ω_M).
For the remainder of this section we assume the manifold to be **compact** and **without boundary**.

- Due to compactness, the scalar product $\langle\omega, \eta\rangle$ is defined **for all** $\omega, \eta \in \mathcal{F}^p(M)$
- Since $\partial M = \emptyset$ the duality formula $\langle \mathrm{d}\omega, \eta\rangle = \langle\omega, \delta\eta\rangle$ for the codifferential holds **for all** $\omega \in \mathcal{F}^{p-1}(M), \eta \in \mathcal{F}^p(M)$

The assumptions can be relaxed in several ways [18]. In particular, for manifolds with boundary, certain **boundary conditions** in terms of trace and Hodge star apply.

De Rham complex

Recall this



Classifications of forms revisited

closed p-forms

$$\mathcal{Z}^p(M) = \ker \mathrm{d}_p$$

$\forall \omega \in \mathcal{Z}^p(M) : \mathrm{d}\omega = 0$

co-closed p-forms

$$\mathcal{Z}^{*,p}(M) = \ker \delta_p$$

$\forall \omega \in \mathcal{Z}^{*,p}(M) : \delta\omega = 0$

exact p-forms

$$\mathcal{B}^p(M) = \mathrm{Im}\ \mathrm{d}_{p-1}$$

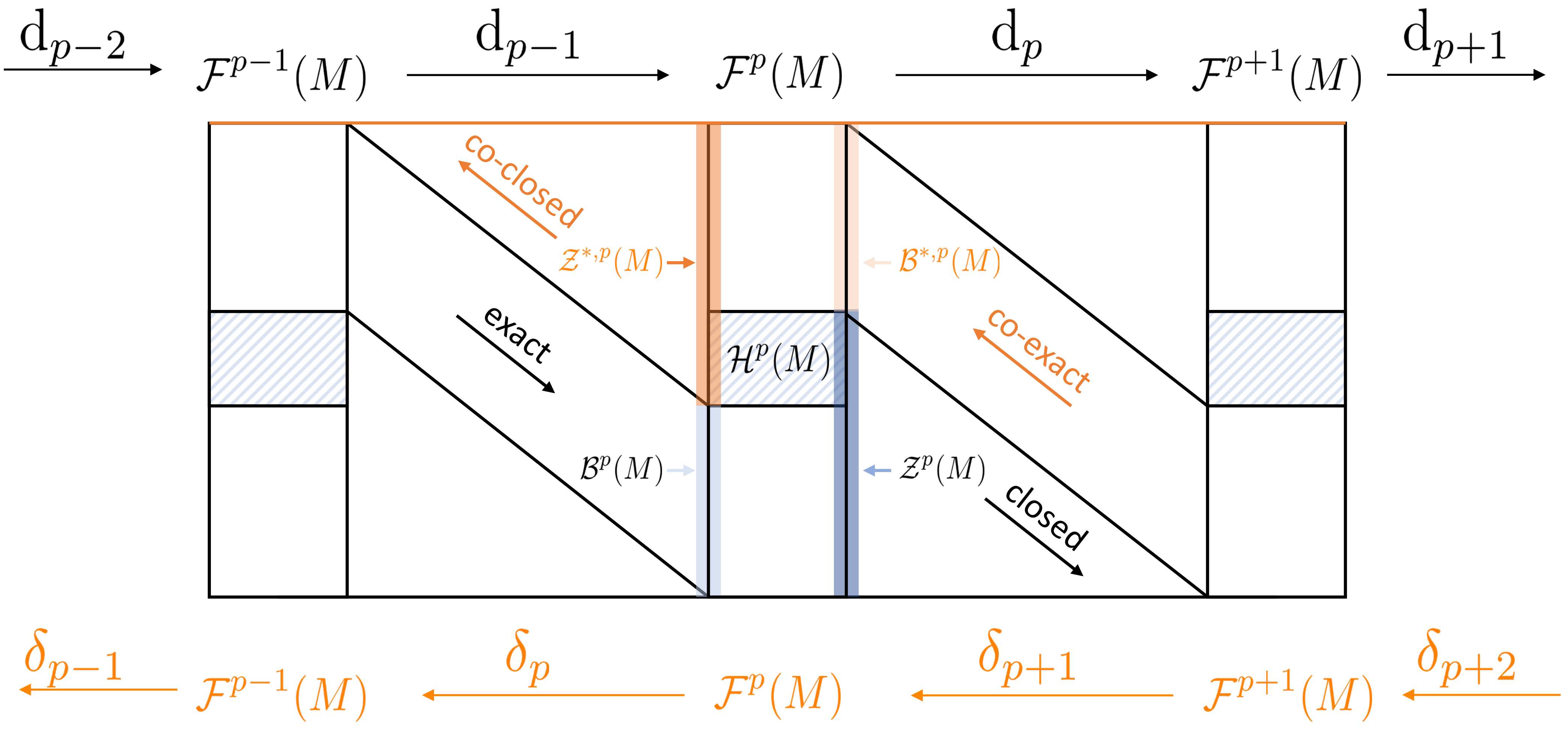
$\forall \omega \in \mathcal{B}^p(M) : \exists \Omega \in \mathcal{F}^{p-1}(M) : \omega = \mathrm{d}\Omega$

co-exact p-forms

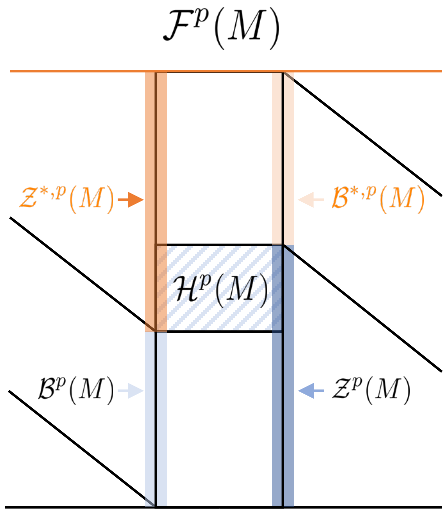
$$\mathcal{B}^{*,p}(M) = \mathrm{Im}\ \delta_{p+1}$$

$\forall \omega \in \mathcal{B}^{*,p}(M) : \exists \Omega \in \mathcal{F}^{p+1}(M) : \omega = \delta\Omega$

Hodge decomposition



Harmonic forms $\mathcal{H}^p(M)$



The linear space of **harmonic forms** is defined by

$$\mathcal{H}^p(M) = \ker \boldsymbol{\Delta} = \{\, \omega \in \mathcal{F}^p(M) \,|\, \boldsymbol{\Delta}\omega = 0 \,\}$$

- A p-form is harmonic iff it is closed and co-closed (non-trivial),

$$\boldsymbol{\Delta}\omega = 0 \quad \Longleftrightarrow \quad \mathrm{d}\omega = 0\,,\ \ \delta\omega = 0$$

$$\mathcal{H}^p(M) = \mathcal{Z}^p(M) \cap \mathcal{Z}^{*,p}(M)$$

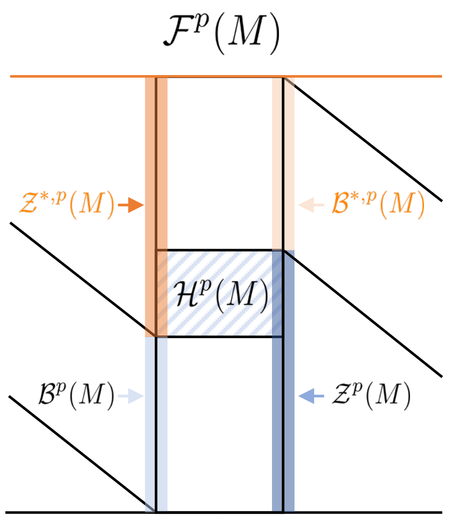
- **Hodge Theorem** (highly non-trivial): each de Rham cohomology class contains exactly one harmonic representative

$$\mathcal{H}^p(M) \xrightarrow{\sim} H^p(M) \ \ : \ \ \omega \mapsto [\omega]$$

Canonical map is an isomorphism

$$\dim \mathcal{H}^p(M) = \dim H^p(M) = b_p$$

Hodge decomposition



For an oriented compact Riemannian manifold without boundary it holds that

$$\boxed{\mathcal{F}^p(M) = \overbrace{\mathcal{B}^p(M) \oplus \mathrlap{\underbrace{\phantom{\mathcal{H}^p(M) \oplus \mathcal{B}^{*,p}(M)}}_{\mathcal{Z}^{*,p}(M)}}\mathcal{H}^p(M)}^{\mathcal{Z}^p(M)} \oplus \mathcal{B}^{*,p}(M)}$$

is a direct **orthogonal** sum with respect to $\langle \omega, \eta \rangle$.

- That is, any p-form can be written **globally** as

$$\boxed{\omega = \mathrm{d}\alpha + \delta\beta + \gamma}$$

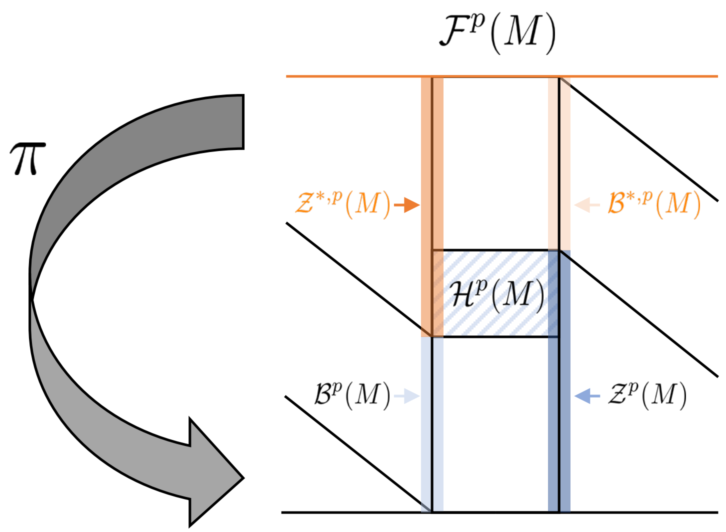
$\alpha \in \mathcal{F}^{p-1}(M)$

$\beta \in \mathcal{F}^{p+1}(M)$

$\gamma \in \mathcal{H}^p(M)$

- Generalization of **Helmholtz decomposition**
  (d = generalized curl, δ = generalized divergence)

Poincaré duality



Hodge star renders the following spaces isomorphic

$$\star : \left\{ \begin{array}{ccc} \mathcal{B}^p(M) & \xrightarrow{\sim} & \mathcal{B}^{*,q}(M) \\ \mathcal{Z}^p(M) & \xrightarrow{\sim} & \mathcal{Z}^{*,q}(M) \\ \mathcal{H}^p(M) & \xrightarrow{\sim} & \mathcal{H}^{*,q}(M) \end{array} \right.$$

$$p + q = n = \dim M$$

By Hodge theorem this carries over to

POINCARÉ : $H^p(M) \xrightarrow{\sim} H^q(M)$, hence $\boxed{b_p = b_q}$

<u>Examples</u>: n-sphere $b_0 = b_n = 1\,,\ b_k = 0$ otherwise

n-torus $b_k = \binom{n}{k}$

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From 3D space to 4D space-time (i)

So far, we considered time-dependent p-forms α in space $X \ni P$,
time $T = \mathbb{R} \ni t$ as universal parameter.

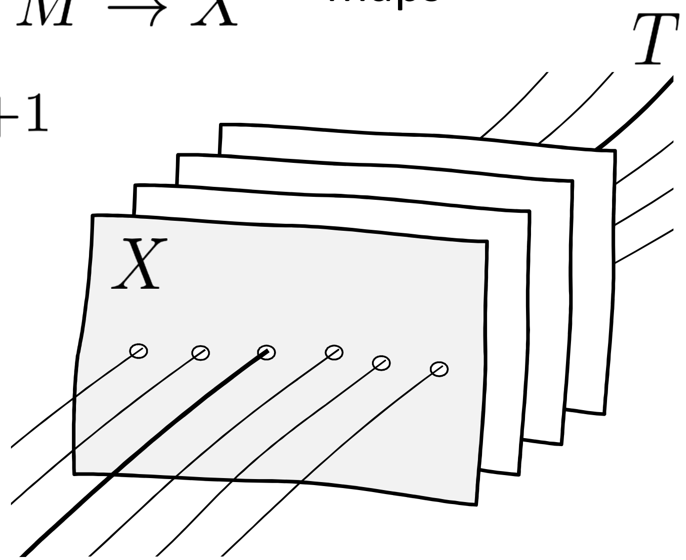
Notation: $\alpha \in \mathcal{F}^p(X;T)$ Chart: $\phi_X : U_X \to \mathbb{R}^n$ $\curvearrowright$ $\alpha = \alpha_I(t,P)\,\mathrm{d}x^I$

Now let's introduce **space-time**

$$\boxed{M = T \times X} \qquad \begin{cases} \Pi_T : M \to T \\ \Pi_X : M \to X \end{cases}$$

Projection maps

$\curvearrowright$ Adapted chart $\phi = (\Pi_T\,,\,\phi_X \circ \Pi_x)\ :\ U \to \mathbb{R}^{n+1}$
with coordinates $(t, x^1, \ldots, x^n)$



From 3D space to 4D space-time (ii)

Consider the space of **horizontal p-forms** (“*spatial forms*”)

$$\underline{\mathcal{F}}^p(M) = \left\{ \omega \in \mathcal{F}^p(M) \,\middle|\, \mathbf{i}_{\ker(\mathrm{D}\Pi_X)}\omega = 0 \right\}$$

In plain words: contains no $\mathrm{d}t$

Insights:

1. Time-dependent p-forms in space and horizontal p-forms can be canonically identified, $\mathcal{F}^p(X;T) \cong \underline{\mathcal{F}}^p(M)$

2. The projection map Π_T induces an isomorphism

In plain words: use $\mathrm{d}t$ as add’l basis differential

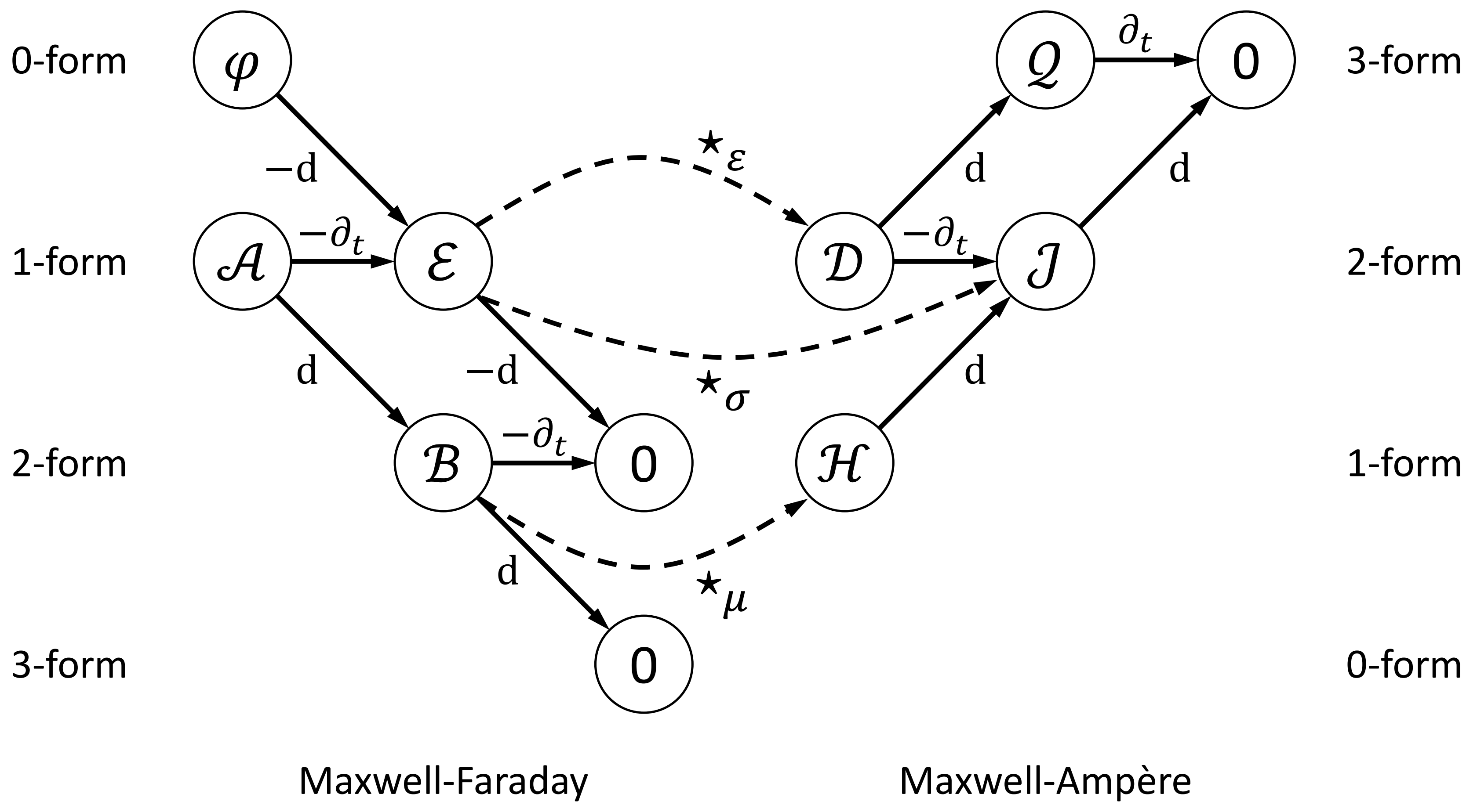
$$\underline{\mathcal{F}}^p(M) \times \underline{\mathcal{F}}^{p-1}(M) \xrightarrow{\sim} \mathcal{F}^p(M) \;:\; (\alpha, \beta) \mapsto \omega = \alpha + \mathrm{d}t \wedge \beta$$

Any p-form in space-time can be uniquely decomposed into a pair of horizontal $(p, p-1)$-forms

$$\beta = \mathbf{i}_{\frac{\partial}{\partial t}}\omega$$

Tonti-diagram in 3D [16, §10.8][17, Table III]

RECALL THIS



Electromagnetic fields and potentials in space-time

Tonti diagram $\curvearrowright$ compose pairs of quantities that are vertically adjacent

Differential form	Description	Degree	Physical dimension
$\mathfrak{A} = \mathcal{A} - \mathrm{d}t \wedge \varphi$	Electromagnetic potential	1-form	U T
$\mathfrak{F} = \mathcal{B} - \mathrm{d}t \wedge \mathcal{E}$	Electromagnetic field	2-form	U T
$\mathfrak{G} = \mathcal{D} + \mathrm{d}t \wedge \mathcal{H}$	Electromagnetic excitation	2-form	I T
$\mathfrak{J} = \mathcal{Q} - \mathrm{d}t \wedge \mathcal{J}$	Electric charge-current	3-form	I T

Note:

- All fundamental electromagnetic quantities can be expressed in terms of two physical dimensions, namely **flux** (UT) and **charge** (IT)
- Correct signs result from the decomposition of Maxwell‘s equations

Decomposition of the exterior derivative

We define derivative operators $(\underline{\mathrm{d}}, \partial_t)$ on horizontal p-forms by a commutative diagram

$$\begin{array}{ccc} \mathcal{F}^p(X;T) & \xrightarrow{\sim} & \underline{\mathcal{F}}^p(M) \\ \mathrm{d}\Big\downarrow & & \Big\downarrow\underline{\mathrm{d}} \\ \mathcal{F}^{p+1}(X;T) & \xrightarrow{\sim} & \underline{\mathcal{F}}^{p+1}(M) \end{array}$$

$$\mathcal{F}^p(X;T) \xrightarrow{\sim} \underline{\mathcal{F}}^p(M), \quad \mathrm{d} \,\Big\downarrow\, \partial_t, \quad \underline{\mathrm{d}} \,\Big\downarrow\, \partial_t, \quad \mathcal{F}^{p+1}(X;T) \xrightarrow{\sim} \underline{\mathcal{F}}^{p+1}(M)$$

$$\partial_t = \frac{\partial}{\partial t}$$

For the decomposition of the exterior derivative we obtain

$$\boxed{\mathrm{d}\,\omega \cong \begin{pmatrix} \underline{\mathrm{d}} & \\ \frac{\partial}{\partial t} & -\underline{\mathrm{d}} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}}$$

$\omega \in \mathcal{F}^p(M)$

$\alpha \in \underline{\mathcal{F}}^p(M)$

$\beta \in \underline{\mathcal{F}}^{p-1}(M)$

Decomposition of the exterior derivative (details)

For horizontal forms α it follows from the evaluation of the exterior derivative in adapted coordinates that

$$\mathrm{d}\alpha = \underline{\mathrm{d}}\alpha + \mathrm{d}t \wedge \frac{\partial \alpha}{\partial t}$$

$$\omega \in \mathcal{F}^p(M)$$

$$\alpha \in \underline{\mathcal{F}}^p(M)$$

$$\beta \in \underline{\mathcal{F}}^{p-1}(M)$$

if α is time-dependent the result is not purely horizontal

For the decomposition of the exterior derivative we obtain

$$\begin{aligned}
\mathrm{d}\omega = \mathrm{d}(\alpha + \mathrm{d}t \wedge \beta) &= \mathrm{d}\alpha - \mathrm{d}t \wedge \mathrm{d}\beta \\
&= \underline{\mathrm{d}}\alpha + \mathrm{d}t \wedge \frac{\partial \alpha}{\partial t} - \mathrm{d}t \wedge \underline{\mathrm{d}}\beta \\
&= \underline{\mathrm{d}}\alpha + \mathrm{d}t \wedge \left(\frac{\partial \alpha}{\partial t} - \underline{\mathrm{d}}\beta\right)
\end{aligned}$$

$$\mathrm{d}\,\omega \cong \begin{pmatrix} \underline{\mathrm{d}} & \\ \frac{\partial}{\partial t} & -\underline{\mathrm{d}} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Maxwell's equations in space-time [19, B.4]

$$\left.\begin{aligned} \underline{\mathrm{d}}\,\mathcal{E} &= -\frac{\partial \mathcal{B}}{\partial t} \\ \underline{\mathrm{d}}\,\mathcal{B} &= 0 \end{aligned}\right\} \quad \Longleftrightarrow \quad \mathrm{d}\,\mathfrak{F} = 0$$

Flux conservation

$$\left.\begin{aligned} \underline{\mathrm{d}}\,\mathcal{H} &= \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t} \\ \underline{\mathrm{d}}\,\mathcal{D} &= \mathcal{Q} \end{aligned}\right\} \quad \Longleftrightarrow \quad \mathrm{d}\,\mathfrak{G} = \mathfrak{J}$$

Charge conservation

- Independent from the specific product manifold structure from which it was derived ("*observer-independent*")
- Metric-independent, no notion of curvature on this level

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Minkowski space (i)

We consider 4D oriented **Minkowski space** $(\mathbb{R}^4, \eta, \omega_{\mathbb{R}^4})$ with coordinates $(t\,, x^1, x^2, x^3)$.

The **Minkowski metric** is defined by

$v, w \in \mathcal{X}(\mathbb{R}^4)$

$$\begin{aligned}\eta(v,w) &:= c\,\mathrm{d}t(v)\,c\,\mathrm{d}t(w) - \underline{\eta}(v,w) \\ &= (c\,\mathrm{d}t \otimes c\,\mathrm{d}t - \underline{\eta})(v,w)\end{aligned}$$

c speed of light in empty space

$\otimes$ tensor product

Its horizontal piece is the Euclidean metric,

$$\underline{\eta}(v,w) := \underline{v} \cdot \underline{w}$$

$\underline{v}, \underline{w}$ are the horizontal projections of v, w; hence isomorphic to time-dependent vector fields in Euclidean space $\left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$

Minkowski space (ii)

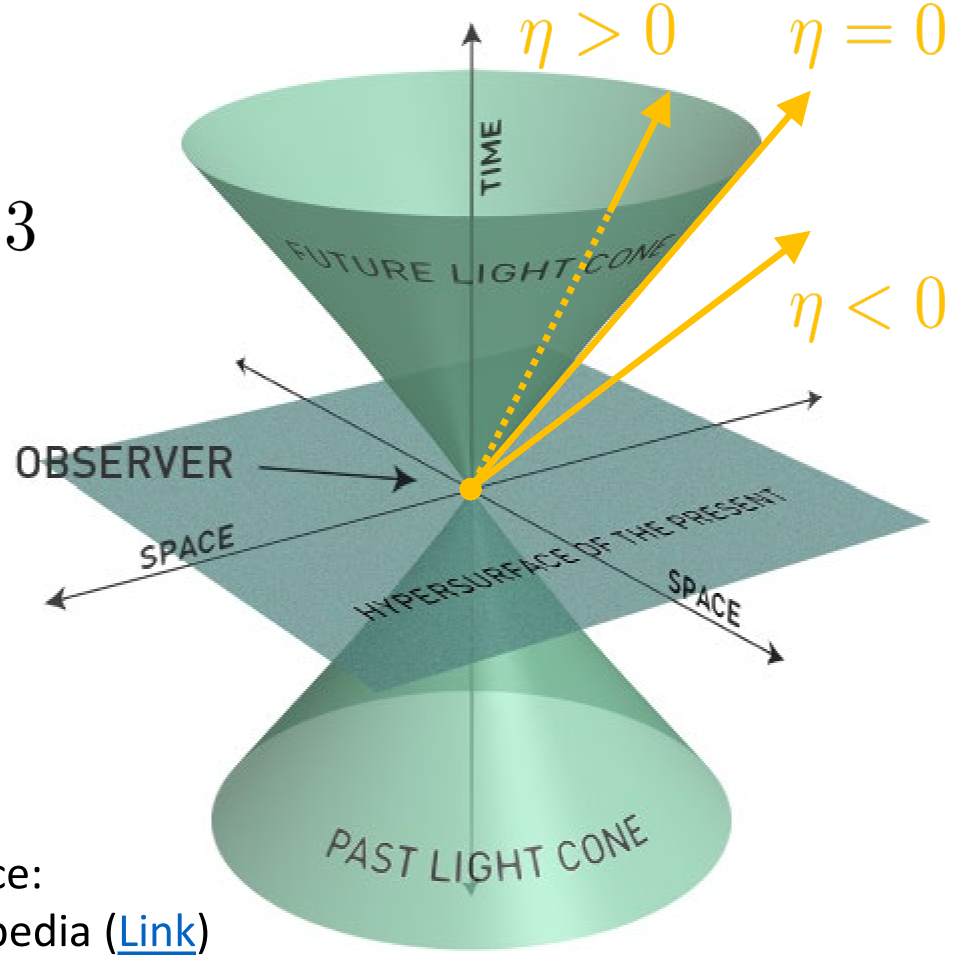
For orthogonal coordinates (x^1, x^2, x^3) the metric coefficients take the form

$$(\eta_{\mu\nu}) = \mathrm{diag}(c^2\,, -h_1^2\,, \,-h_2^2\,, \,-h_3^2)$$

$$\mu, \nu = 0, 1, 2, 3$$

The metric tensor is not positive definite, its sign provides additional information:

$$\eta(v,v) = \begin{cases} > 0 & \text{time-like} \\ = 0 & \text{light-like} \\ < 0 & \text{space-like} \end{cases}$$



Source: Wikipedia (Link)

Decomposition of the Hodge star operator

We define the metric volume form by $\omega_{\mathbb{R}^4} = c\,\mathrm{d}t \wedge \underline{\star}1$

Lack of positive definiteness of the Minkowski metric $\curvearrowright$ adaptations in the definition and properties of Hodge star [12, 7.9]

For the decomposition of the Hodge star operator we obtain

$$\boxed{\star\omega \cong \begin{pmatrix} & (-1)^{p-1}c^{-1} \\ c & \end{pmatrix} \underline{\star} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}}$$

$\omega \in \mathcal{F}^p(M)$

$\alpha \in \underline{\mathcal{F}}^p(M)$

$\beta \in \underline{\mathcal{F}}^{p-1}(M)$

$\underline{\star} : \underline{\mathcal{F}}^p(M) \to \underline{\mathcal{F}}^{q-1}(M)$ denotes the horizontal Hodge star

$M \subset \mathbb{R}^4,\ p + q = 4$

Decomposition of the Hodge star operator (details)

For horizontal forms α it can be shown from the definition of Hodge star that [13, 8.8.3]

$$\star\,\alpha = c\,\mathrm{d}t \wedge \underline{\star}\,\alpha$$

$$\omega \in \mathcal{F}^p(M)$$

$$\alpha \in \underline{\mathcal{F}}^p(M)$$

$$\beta \in \underline{\mathcal{F}}^{p-1}(M)$$

For the decomposition of the Hodge star operator we obtain

$$\begin{aligned}
\star\,\omega &= \star\,(\alpha + \mathrm{d}t \wedge \beta) = \star\,\alpha + (-1)^{p-1} \star (\beta \wedge \mathrm{d}t) \\
&= \star\,\alpha + (-1)^{p-1} c^{-2}\, \mathbf{i}_{\frac{\partial}{\partial t}} \star\beta \\
&= c\,\mathrm{d}t \wedge \underline{\star}\,\alpha + (-1)^{p-1} c^{-2}\, \mathbf{i}_{\frac{\partial}{\partial t}} (c\,\mathrm{d}t \wedge \underline{\star}\,\beta) \\
&= \mathrm{d}t \wedge c\,\underline{\star}\,\alpha + (-1)^{p-1} c^{-1} \underline{\star}\,\beta
\end{aligned}$$

$$\star\,\omega \cong \begin{pmatrix} & (-1)^{p-1}c^{-1} \\ c & \end{pmatrix} \underline{\star} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Constitutive relations in space-time

We restrict ourself to empty space, i.e. $\varepsilon = \varepsilon_0\,,\;\; \mu = \mu_0\,,\;\; \sigma = 0.$
Then

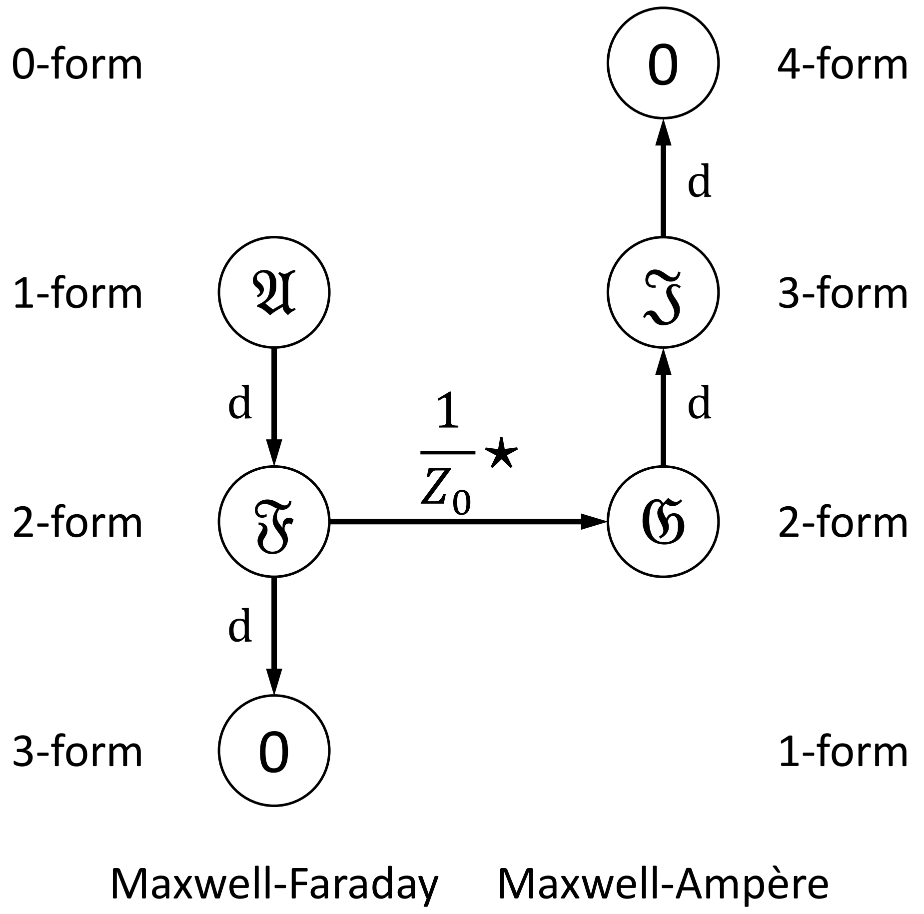
$$\left.\begin{aligned} \mathcal{D} &= \varepsilon_0 \,\underline{\star}\, \mathcal{E} \\ \mathcal{B} &= \mu_0 \,\underline{\star}\, \mathcal{H} \end{aligned}\right\} \quad\Longleftrightarrow\quad \mathfrak{G} = \frac{1}{Z_0} \star \mathfrak{F}$$

Herein, Z_0 is the **wave impedance** of empty space,

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}\;\;;\;\; \text{note that}\;\; c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

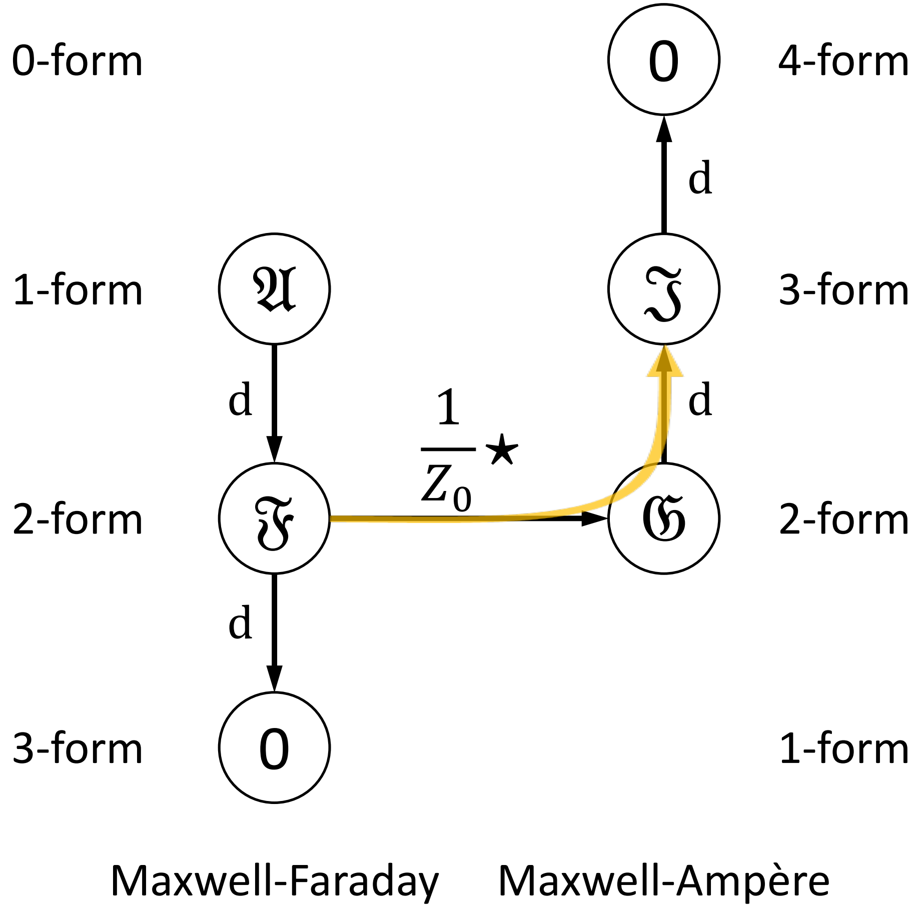
Given the decomposition of the Hodge star operator, the proof of the equivalence is an easy exercise.

Tonti-diagram in 4D



Vast generalization possible: replace Minkowski space by 4D Lorentzian manifold!

Tonti-diagram in 4D



From chasing the Tonti diagram we obtain

$$\mathrm{d}\,\frac{1}{Z_0}\star\mathfrak{F}=\mathfrak{J}$$

$$\underbrace{\star\,\mathrm{d}\,\star}_{\delta}\,\mathfrak{F}=Z_0\star\mathfrak{J}$$

Vast generalization possible: replace Minkowski space by 4D Lorentzian manifold!

Can we even be more concise?

Utilize full exterior algebra!

$$\left.\begin{aligned} \mathrm{d}\,\mathfrak{F} &= 0 \\ \delta\,\mathfrak{F} &= z_0 \star \mathfrak{J} \end{aligned}\right\} \oplus \quad \Longleftrightarrow \quad \mathrm{D}\,\mathfrak{F} = z_0 \star \mathfrak{J}$$

The Maxwell equation

$$\left.\begin{aligned} \mathrm{d}\,\mathfrak{A} &= \mathfrak{F} \\ \delta\,\mathfrak{A} &= 0 \end{aligned}\right\} \oplus \quad \Longleftrightarrow \quad \mathrm{D}\,\mathfrak{A} = \mathfrak{F}$$

Definition of potential

Lorenz gauge
$\delta\mathcal{A} + \partial_\tau\varphi = 0$

$$\partial_\tau := \frac{1}{c^2}\frac{\partial}{\partial t}$$

$$\boldsymbol{\Delta}\,\mathfrak{A} = z_0 \star \mathfrak{J}$$

4D wave equation

$\boldsymbol{\Delta} = \mathrm{D}^2$
Hodge-Laplace of space-time metric

Sommerfeld’s voice

Ich wünsche meinen Zuhörern den Eindruck zu verschaffen, dass die wahre mathematische Form dieser Gebilde erst jetzt hervortreten wird, wie bei einer Gebirgslandschaft, wenn der Nebel zerreißt.

A. Sommerfeld. Elektrodynamik §26. Harri Deutsch, 1988.

$$\mathrm{D}\,\mathfrak{F} = z_0 \star \mathfrak{J}$$

$$\mathrm{D}\,\mathfrak{A} = \mathfrak{F}$$

$$\boldsymbol{\Delta}\,\mathfrak{A} = z_0 \star \mathfrak{J}$$

The End.

Literature

[1] M. Fecko. *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, 2006.

[2] K. Jänich. *Vector Analysis*. Springer, 2001. Link

[3] J. Nestruev. *Smooth manifolds and observables*. Springer, 2003.

[4] C. von Westenholz. *Differential forms in mathematical physics*. North-Holland, 1981.

[5] W. H. Greub. *Multilinear Algebra*. Springer, 1967.

[6] H. J. Dirschmid. *Tensoren und Felder*. Springer, 1996.

[7] G. de Rham. *Differentiable Manifolds.* Springer, 1984.

[8] A. Bossavit. On the geometry of electromagnetism. (3): Faraday's law. J. Japan Soc. Appl. Electromagn. & Mech., Vol. 6, p. 233-240, 1998.

Literature (cont'd)

[9] S. Morita. *Geometry of Differential Forms*. Iwanami Series in Modern Mathematics, 2001.

[10] R.E. Terrell. *The fundamental theorem of calculus and the Poincaré lemma*. Tech Report, Cornell University, 2009. Link

[11] B.-Y. Hou and B.-Y. Hou. *Differential Geometry for Physicists*. World Scientific, 1997.

[12] M. Nakahara. *Geometry, Topology and Physics*, 2nd ed. Taylor & Francis, 2003.

[13] G. Lehner, S. Kurz. *Elektromagnetische Feldtheorie für Ingenieure und Physiker*, 9. Auflage. Springer Vieweg, 2021.

[14] M. Pellikka et al. Homology and Cohomology Computation in Finite Element Modeling. SIAM J Sci Comput, 35(5), 2013. Link

Literature (cont'd)

[15] A. Bossavit. *On the notion of anisotropy of constitutive laws: Some implications of the "Hodge implies metric" result.* COMPEL, 20(1), 2001.

[16] E. Tonti. *Mathematical Structure of Classical and Relativistic Physics*. Springer, 2013.

[17] G.A. Deschamps. *Electromagnetics and Differential Forms*. Proc. IEEE, 69(6), 1981.

[18] G. Schwarz. *Hodge Decomposition - A Method for Solving Boundary Value Problems*. Springer, 1995.

[19] F. Hehl and Y. Obukhov. *Foundations of Classical Electrodynamics*. Birkhäuser, 2003.

[20] D.N. Arnold. *Finite Element Exterior Calculus*. SIAM, 2018.