# Electromagnetics and Differential Forms

Metric and
Hodge star operator

# In today's lecture we talk about:

- **Riemannian manifold**
- Metric volume form
- Hodge star operator
- Constitutive relations

# Riemannian manifold

The pair $(M, g)$ is called **Riemannian manifold**, where $M$ is a differentiable manifold and

$$g : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M)$$

the **metric tensor** field.

$$\forall v, w \in \mathcal{X}(M)$$

- Locality $\quad g(v,w)|_P = g(\tilde{v},\tilde{w})|_P \quad \left\{ \begin{array}{l} \tilde{v}, \tilde{w} \in \mathcal{X}(M) : \\ \tilde{v}_P = v_P,\ \tilde{w}_P = w_P \end{array} \right.$

smooth continuations

- Symmetry $\quad g(v,w) = g(w,v)$
- Bilinearity $\quad g(v + fv', w) = g(v,w) + fg(v',w) \quad \forall v' \in \mathcal{X}(M)$, $\forall f \in C^\infty(M)$
- Positive definiteness $\quad g(v,v) > 0\,, \quad v \neq 0$

# Metric tensor field

The metric tensor field $g$ is symmetric bilinear, not to be confused with a 2-form $\omega \in \mathcal{F}^2(M)$, which is antisymmetric bilinear.

In a chart with coordinates $(x^1, \ldots, x^n)$ we obtain the **metric coefficients**

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \in \mathcal{C}^\infty(M)$$

$$i, j = 1, \ldots, n \qquad n = \dim M$$

- Orthonormal basis fields $g_{ij} = \delta_{ij}$
- In general, it is not possible to choose coordinates such that the basis fields are orthogonal throughout. Obstruction: **curvature**
  Example: breakdown of spherical coordinates in the poles

# Euclidean space $\mathbb{R}^n$

In Euclidean space $(\mathbb{R}^n\ ,\ \cdot\ )$ we let $g(v, w) := v \cdot w$ .

Since Euclidean space is flat, there exist coordinate systems such that the basis fields are orthogonal, $g_{ij} = h_i \delta_{ij}$, where $0 < h_i \in C^\infty(\mathbb{R}^3)$ are the **scaling functions**.

| **Coordinate system** (in 3 dimensions) | **Coordinates** $(x^1, x^2, x^3)$ | **Scaling functions** $(h_1, h_2, h_3)$ |
|---|---|---|
| Cartesian coordinates | $(x\ ,\ y\ ,\ z)$ | $(1\ ,\ 1\ ,\ 1)$ |
| Cylindrical coordinates | $(r\ ,\ \varphi\ ,\ z)$ | $(1\ ,\ r\ ,\ 1)$ |
| Spherical coordinates | $(r\ ,\ \theta\ ,\ \varphi)$ | $(1\ ,\ r\ ,\ r\sin\theta)$ |

# Musical isomorphisms

The metric tensor field induces the **musical isomorphisms**

$$\begin{aligned} \flat : \quad & \mathcal{X}(M) && \to && \mathcal{F}^1(M) \\ \sharp = \flat^{-1} : \quad & \mathcal{F}^1(M) && \to && \mathcal{X}(M) \end{aligned}$$

where $\flat(v)(w) := g(v,w)\,, \quad v \in \mathcal{X}(M)\,, \quad \forall w \in \mathcal{X}(M)$

- In Euclidean space we have $\flat(v) = {}^1v$
- Isomorphisms $\flat$ and $\sharp$ amount to **lowering** and **raising** of indices:

$$v = \boxed{v^i}\frac{\partial}{\partial x^i} \quad \curvearrowright \quad \flat(v) = \underbrace{g_{ij}v^i}_{= \boxed{v_j}}\,\mathrm{d}x^j$$

contravariant components

covariant components

Checking this is an easy exercise.

RECALL THIS

# Translation isomorphisms – coordinate-free [1]

In $(\mathbb{R}^3, \cdot, \times)$ we may identify scalar/vector fields with differential forms.

Let $M \subset \mathbb{R}^3$ be an open set in $\mathbb{R}^3$ and $u, v, w \in \mathcal{X}(M)$

| | |
|---|---|
| ${}^1\cdot : \mathcal{X}(M) \to \mathcal{F}^1(M)$<br>$a \mapsto {}^1a$ | ${}^1a(v) := a \cdot v$ |
| ${}^2\cdot : \mathcal{X}(M) \to \mathcal{F}^2(M)$<br>$b \mapsto {}^2b$ | ${}^2b(u, v) := b \cdot (u \times v)$ |
| ${}^3\cdot : \mathcal{C}^\infty(M) \to \mathcal{F}^3(M)$<br>$c \mapsto {}^3c$ | ${}^3c(u, v, w) := c\, u \cdot (v \times w)$ |

# A note on the translation isomorphisms

! no summation convention !

Consider Euclidean space $\left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$ equipped with orthogonal coordinates $(x^1, x^2, x^3)$ such that $\left(e_i = \frac{1}{h_i}\frac{\partial}{\partial x^i}\right)_{i=1}^{3}$ is a direct **orthonormal basis** and $\left(\varepsilon^j = h_j \mathrm{d}x^j\right)_{j=1}^{3}$ its dual basis.

Then for the **translation isomorphisms** it holds that

$$
\begin{aligned}
{}^1e_i &= \varepsilon^i &&= h_i\,\mathrm{d}x^i && i = 1,2,3\\
{}^2e_i &= \varepsilon^{jk} &&= h_j\,\mathrm{d}x^j \wedge h_k\,\mathrm{d}x^k && ijk = 123, 231, 312\\
{}^3\,1 &= \varepsilon^{123} &&= h_1\,\mathrm{d}x^1 \wedge h_2\,\mathrm{d}x^2 \wedge h_3\,\mathrm{d}x^3 &&
\end{aligned}
$$

<u>Example</u>: cylindrical coordinates $\quad {}^1a = a_r \mathrm{d}r + a_\varphi r \mathrm{d}\varphi + a_z \mathrm{d}z$

Physical components wrt $(e_r, e_\varphi, e_z)$

# In today's lecture we talk about:

- Metric tensor

- **Metric volume form**

- Hodge star operator

- Constitutive relations

# Gram determinant

Disclaimer: In the sequel, $v$ can be either understood as tangent vector, $v \in T_P M$ , or as vector field $v \in \mathcal{X}(M)$, $\quad n = \dim M$.
In the latter case, the constructions are defined pointwise.

Consider a $p$-tupel $\;(v_1, \ldots, v_p)$, $\quad 1 \leq p \leq n$.

Define the **Gram determinant**

$$G = G(v_1, \ldots, v_p) := \det\big(g(v_i, v_k)\big)\,, \quad i, k = 1, \ldots, p$$

Properties:

$$\left.\begin{array}{l} \text{1.}\quad G \geq 0 \\ \text{2.}\quad G = 0 \quad \text{iff } (v_1, \ldots, v_p) \text{ are linearly dependent} \end{array}\right\} \text{Cauchy inequality for } p = 2$$

# Volume of a parallelotope ("Parallelflach")

Consider the parallelotope spanned by the vectors $(v_1, \ldots, v_p)$.

Define its **volume** by

$$\mathrm{Vol}(v_1, \ldots, v_p) := \sqrt{G}$$

Insensitive
to orientation

<u>Examples</u> in Euclidean space $\left(\mathbb{R}^3 \,,\, \cdot \,,\, \times\right)$:

1. Single vector: $\mathrm{Vol}(v) = |v|$
2. Parallelogramm: $\mathrm{Vol}(v_1, v_2) = |v_1 \times v_2|$
3. Parallelepiped ("Spat"): $\mathrm{Vol}(v_1, v_2, v_3) = \big|v_1 \cdot (v_2 \times v_3)\big|$

# Orientable manifolds and volume forms

RECALL THIS

Let $M$ be a $n$-dimensional manifold.

A $n$-form $\omega \in \mathcal{F}^n(M)$ on $M$ is called a **volume form** if $\omega_P \neq 0 \quad \forall P \in M$ .

A manifold $M$ is called **orientable** if there exists a volume form on $M$.

Remark:

- Not all manifolds are orientable. Example: Möbius strip
- The notion of integration can be extended to non-orientable manifolds. See for example [7].
- If there exists a metric on the manifold, a special volume form is distinct by assigning a unit-volume the value ±1. ( **metric volume form** )

# Metric volume form: definition

Start from an orientable Riemannian manifold $(M, g)$ and orient it, by picking a volume form $\omega \in \mathcal{F}^n(M)$.

Assertion: There exists a unique **metric volume form** $\omega_M \in [\omega]$ that replies in each point to a direct basis $(v_1, \ldots, v_n)$ with the volume of the associated parallelotope,

$$\omega_M(v_1, \ldots, v_n) = +\mathrm{Vol}(v_1, \ldots, v_n)$$

Hence we write for an **oriented Riemannian manifold**

$$\boxed{(M, g, \omega_M)}$$

Equivalently, one may ask that $\omega_M$ assigns each direct orthonormal basis the value +1

# Metric volume form in local coordinates

Consider an oriented Riemannian manifold $(M, g, \omega_M)$ and a chart $\phi : U \to \mathbb{R}^n$, with local coordinates $(x^1, \ldots, x^n)$, such that $\omega_M \sim \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$. Hence $\exists \lambda > 0 \in C^\infty(U)$ such that

$$\omega_M = \lambda \, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n.$$

Assertion: It can be shown that

$$\omega_M = \sqrt{\det(g_{ij})} \, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$$

**Invariant volume element** in vector analysis

In orthogonal coordinates:

$$\omega_M = h_1 \mathrm{d}x^1 \wedge \ldots \wedge h_n \mathrm{d}x^n$$

Example:
spherical coordinates
$\mathrm{d}\tau = r^2 \sin\theta \; \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi$

# Metric volume form in local coordinates: proof

Proof of the assertions.

Consider $\mathrm{Vol}(v_1, \ldots, v_n) = \sqrt{\det\big(g(v_k, v_\ell)\big)}$

$$
\begin{aligned}
&= \sqrt{\det(g_{ij}\, v_k^i\, v_\ell^j)} \\
&= \sqrt{\det(g_{ij})}\, |\det(v_k^i)| \\
&= \sqrt{\det(g_{ij})}\, \big|\det\big(\mathrm{d}x^i(v_k)\big)\big| \\
&= \underbrace{\sqrt{\det(g_{ij})}}_{:= \lambda}\ \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n (v_1, \ldots, v_n) \\
&\qquad\qquad\qquad\qquad = \omega_M(v_1, \ldots, v_n) > 0
\end{aligned}
$$

# In today’s lecture we talk about:

- Riemannian manifold
- Metric volume form
- **Hodge star operator**
- Constitutive relations

# Hodge star operator: basic idea

Consider an oriented Riemannian manifold $(M, g, \omega_M)$. The spaces of $p$-covectors and $(n-p)$-covectors share the same dimension,

$$\dim \bigwedge^p T_P^* M = \binom{n}{p} = \binom{n}{n-p} = \dim \bigwedge^{n-p} T_P^* M$$

↷ The spaces $\mathcal{F}^p(M)$ and $\mathcal{F}^{n-p}(M)$ are isomorphic, but not canonically isomorphic.

On an oriented Riemannian manifold, a specific isomorphism is defined by the **Hodge star operator**:

$$\star : \mathcal{F}^p(M) \xrightarrow{\sim} \mathcal{F}^{n-p}(M) \ : \omega \mapsto \star\omega$$

# Hodge star operator: definition

The Hodge star operator is defined pointwise as follows, where all test vectors $\xi \in T_P M$. This extends to differential forms.

$$\star\omega_P\underbrace{(\xi_1,\ldots,\xi_q)}_{\text{Given } q\text{-tupel of linearly independent test vectors}} := \omega_P\underbrace{(\xi_1^\perp,\ldots,\xi_p^\perp)}_{p\text{-tupel constructed from test vectors}}$$

$$p+q=n$$
$$n=\dim M$$

↷ The definition boils down to defining the map $(\xi_1,\ldots,\xi_q) \mapsto (\xi_1^\perp,\ldots,\xi_p^\perp)$

# Hodge star operator: definition (cont‘d)

The map $(\xi_1, \ldots, \xi_q) \mapsto (\xi_1^\perp, \ldots, \xi_p^\perp)$ is characterized by <u>three conditions</u>:

1. $(\xi_1^\perp, \ldots, \xi_p^\perp)$ span the orthogonal complement of $(\xi_1, \ldots, \xi_q)$:

$$\mathrm{span}(\xi_1^\perp, \ldots, \xi_p^\perp) = \mathrm{span}(\xi_1, \ldots, \xi_q)^\perp$$ i.e. $$g(\xi_j^\perp, \xi_i) = 0$$

$$j = 1, \ldots, q$$
$$i = 1, \ldots, p$$

2. $(\xi_1^\perp, \ldots, \xi_p^\perp)$, $(\xi_1, \ldots, \xi_q)$ span the same volume:

$$\mathrm{Vol}(\xi_1^\perp, \ldots, \xi_p^\perp) = \mathrm{Vol}(\xi_1, \ldots, \xi_q)$$

3. $(\xi_1^\perp, \ldots, \xi_p^\perp)$, $(\xi_1, \ldots, \xi_q)$ form a direct basis of $T_P M$:

$$\omega_M(\xi_1^\perp, \ldots, \xi_p^\perp, \xi_1, \ldots, \xi_q) > 0$$

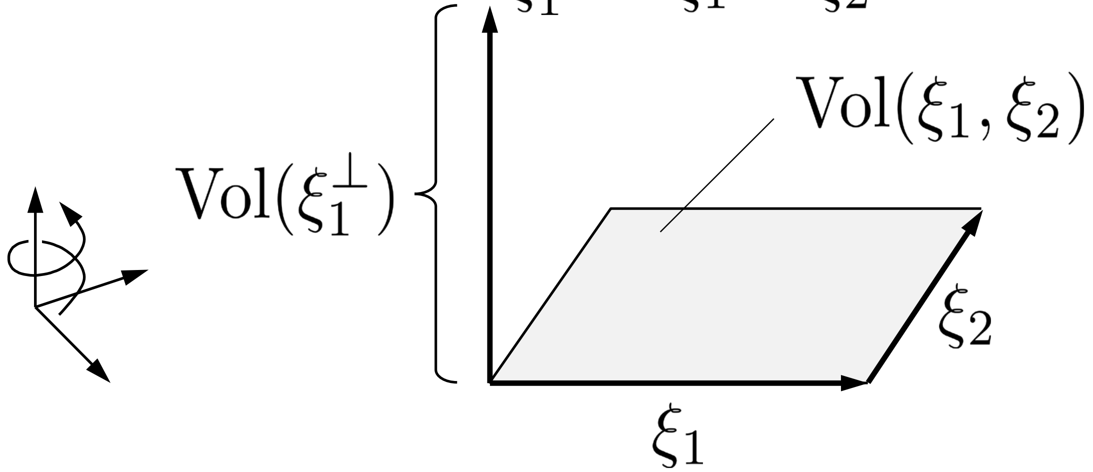
# Hodge star operator: definition (cont'd)

Conditions 1. – 3. fix $(\xi_1^\perp, \dots, \xi_p^\perp)$ up to a transformation from the special linear group[1] $\curvearrowright$ $\omega_P(\xi_1^\perp, \dots, \xi_p^\perp)$ uniquely defined

Example in Euclidean space $\left(\mathbb{R}^3\,,\ \cdot\,,\ \times\ \right)$ $\curvearrowright$ $\omega_{\mathbb{R}^3} = {}^3 1$

$p = 1, q = 2$

$\xi_1^\perp = \xi_1 \times \xi_2$




1. $\xi_1^\perp$ perpendicular to $\xi_1, \xi_2$
2. Length of arrow $\xi_1^\perp$ equals area of parallelogram $\xi_1, \xi_2$
3. $\xi_1^\perp, \xi_1, \xi_2$ right-handed system

[1] A linear transformation whose matrix has determinant +1

# Hodge star operator: properties

For proofs and background information see [1, 5.8], [2, 12.3], [4, 4.2]

- Well-definedness. $\star\omega_P$ <u>does</u> define a multicovector (not obvious[1])
- $C^\infty(M)$- linearity

$$\star(\omega + f\eta) = \star\omega + f\star\eta$$

$$\forall \omega, \eta \in \mathcal{F}^p(M)$$
$$\forall f \in C^\infty(M)$$

- Self-inverse property

$$\star\star\,\omega = (-1)^{pq}\omega$$

in odd dimensions

$$\star^{-1} = \star$$

- Symmetry

$$\omega \wedge \star\eta = \eta \wedge \star\omega$$

plus positive definiteness

$$\omega \wedge \star\omega \sim \omega_M\,, \quad \omega \neq 0$$

[1] See material on the learning platform about an equivalent reformulation of the definition.

# Hodge star operator: properties (cont'd)

In coordinates:
$\omega \cdot \eta = \omega_I \eta_J \det(g^{IJ})$

- Pointwise inner product. By symmetry and positive definiteness we define

$$\mathcal{F}^p(M) \times \mathcal{F}^p(M) \to C^\infty(M) : (\omega, \eta) \mapsto \omega \cdot \eta :$$
$$(\omega \cdot \eta)\, \omega_M = \omega \wedge \star \eta$$

Note also:

$$\star 1 = \omega_M$$
$$\star \omega_M = 1$$

- Hodge and contraction

$$\mathbf{i}_v \star \omega = \star\big(\omega \wedge \flat(v)\big)$$

$$\forall \omega \in \mathcal{F}^p(M)$$
$$\forall v \in \mathcal{X}(M)$$

- Scaling of metric tensor $(M, g, \omega_M) \mapsto (M, \lambda g, \lambda^{n/2} \omega_M)$

$$\star \omega \mapsto \lambda^{n/2 - p} \star \omega$$

$$0 < \lambda \in C^\infty(M)$$

Interesting case: $n$ even, $p = n/2$

# Hodge star operator: coordinate expression

Consider orthogonal coordinates $(x^1, x^2, x^3)$ in Euclidean space $(\mathbb{R}^3\,,\,\cdot\,,\,\times\,)$, with metric coefficients $(g_{ij}) = \mathrm{diag}(h_1^2\,,\,h_2^2\,,\,h_3^2)$.

Then it holds

$$\star : \begin{cases} 1 & \mapsto h_1 \mathrm{d}x^1 \wedge h_2 \mathrm{d}x^2 \wedge h_3 \mathrm{d}x^3 \mapsto 1 \\ h_i \mathrm{d}x^i & \mapsto h_j \mathrm{d}x^j \wedge h_k \mathrm{d}x^k \qquad\qquad\quad \mapsto h_i \mathrm{d}x^i \end{cases}$$

$$ijk = 123,\; 231, 312$$

<u>Proof</u>: easy consequence of

$$\star 1 = \omega_{\mathbb{R}^3}\,, \qquad \star\star = 1\,,$$

$$\mathbf{i}_{(\frac{1}{h_i}\frac{\partial}{\partial x^i})} \star 1 = \star\Big(1 \wedge \flat\big(\frac{1}{h_i}\frac{\partial}{\partial x^i}\big)\Big) = \star\, h_i \mathrm{d}x^i$$

# Hodge star operator and translation isomorphisms

In Euclidean space $\left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$ it holds that

$$\begin{aligned} \star^0 f &= {}^3 f \\ \star^1 a &= {}^2 a \\ \star^2 b &= {}^1 b \\ \star^3 g &= {}^0 g \end{aligned}$$

$\forall f, g \in C^\infty(M)$

$\forall a, b \in \mathcal{X}(M)$

$M \subset \mathbb{R}^3$ open

<u>Proof</u>: From the definitions of Hodge star and the translation isomorphisms it follows that

$$(\star^0 f)(u,v,w) = {}^0 f\, u \cdot (v \times w) = f\, u \cdot (v \times w) = {}^3 f(u,v,w)$$

$$(\star^1 a)(u,v) = {}^1 a(u \times v) = a \cdot (u \times v) = {}^2 a(u,v)$$

$u, v, w \in \mathcal{X}(M)$

# In today's lecture we talk about:

- Riemannian manifold
- Metric volume form
- Hodge star operator
- **Constitutive relations**

# Constitutive relations for simple media

Consider Euclidean space. **Simple media** are linear and local
tensorial constitutive relations. Tensors shall be time-independent, symmetric positive definite [1], no electric-magnetic coupling [2].

If – in addition – the media are homgeneous and isotropic they can be characterized by constant parameters.

| Symbol | Parameter | Physical dimension |
|---|---|---|
| $\varepsilon$ | permittivity | $U^{-1}\, I\, T\, L^{-1}$ |
| $\mu$ | permeability | $U\, I^{-1}\, T\, L^{-1}$ |
| $\sigma$ | conductivity | $U^{-1}\, I\, L^{-1}$ |

[1] Positive semidefinite in case of conductivity  [2] Excludes in particular moving media and bi-isotropic media [Link]

# Constitutive relations for simple media (cont'd)

Under these assumptions, the constitutive relations read

$$\left.\begin{aligned} D &= \varepsilon\, E \\ B &= \mu\, H \\ J &= \sigma\, E \end{aligned}\right\} \quad \Longleftrightarrow \quad \left\{\begin{aligned} \mathcal{D} &= \varepsilon \star \mathcal{E} \\ \mathcal{B} &= \mu \star \mathcal{H} \\ \mathcal{J} &= \sigma \star \mathcal{E} \end{aligned}\right.$$

Ohm's law

This is a simple consequence of the translation isomorphisms and Hodge star

<u>Note</u>: We defined the metric tensor and Hodge star dimensionless.
In physics, the metric tensor acquires the **physical dimension** $\mathrm{pd}(g) = \mathsf{L}^2$. From the scaling law it follows on $p$-forms in $n$ dimensions

$$\mathrm{pd}(\star) = \mathsf{L}^{n-2p}$$

$$\underbrace{\mathcal{D}}_{\mathsf{I\,T}} = \underbrace{\varepsilon}_{\mathsf{U^{-1}\,I\,T\,L^{-1}}}\ \underbrace{\star}_{\mathsf{L}}\ \underbrace{\mathcal{E}}_{\mathsf{U}}$$

# Constitutive relations for simple media (cont'd)

The material properties can be absorbed in the metric tensor with the help of the scaling law, by choosing $\lambda$ successively as $\varepsilon^2, \mu^2, \sigma^2$.

For 1-forms $\omega$ in 3 dimensions this yields $\star_\lambda \omega = \lambda \star \omega$, hence:

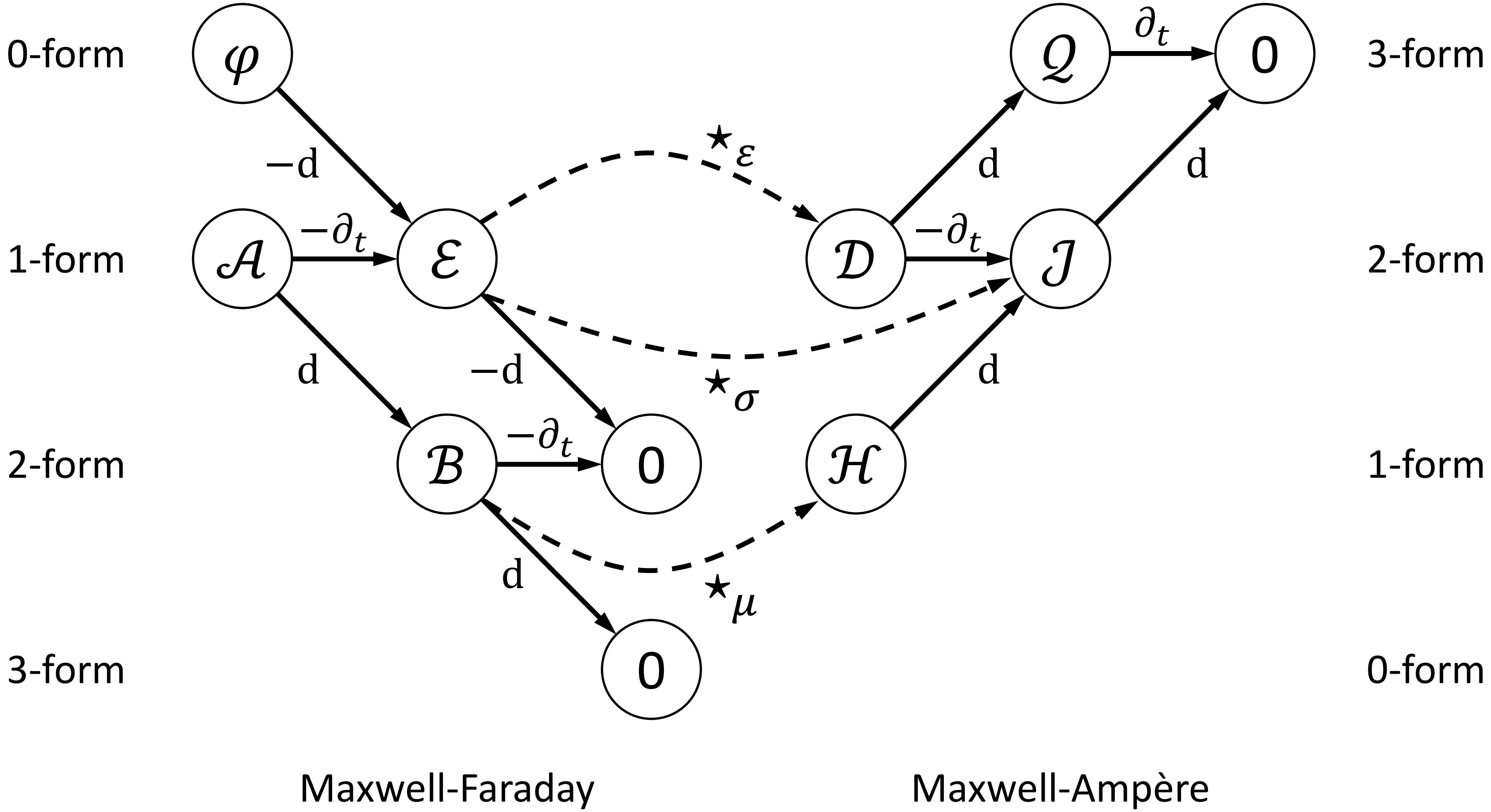
$$\mathcal{D} = \star_\varepsilon \mathcal{E}$$
$$\mathcal{B} = \star_\mu \mathcal{H}$$
$$\mathcal{J} = \star_\sigma \mathcal{E}$$

$$\underbrace{\mathcal{D}}_{\mathrm{I\,T}} = \underbrace{\star_\varepsilon}_{\mathrm{U^{-1}\,I\,T}} \underbrace{\mathcal{E}}_{\mathrm{U}}$$

! Length is irrelevant !

<u>Note</u>: It can be shown that the requirements of homogeneity and isotropy can be dropped. <u>Simple media</u> can always be represented by adapted metrics and Hodge star operators [15].

# Tonti-diagram in 3D [16, §10.8][17, Table III]



# Literature


[1] M. Fecko. *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, 2006.

[2] K. Jänich. *Vector Analysis*. Springer, 2001. Link

[3] J. Nestruev. *Smooth manifolds and observables*. Springer, 2003.

[4] C. von Westenholz. *Differential forms in mathematical physics*. North-Holland, 1981.

[5] W. H. Greub. *Multilinear Algebra*. Springer, 1967.

[6] H. J. Dirschmid. *Tensoren und Felder*. Springer, 1996.

[7] G. de Rham. *Differentiable Manifolds.* Springer, 1984.

[8] A. Bossavit. On the geometry of electromagnetism. (3): Faraday's law. J. Japan Soc. Appl. Electromagn. & Mech., Vol. 6, p. 233-240, 1998.

# Literature (cont'd)


[9] S. Morita. *Geometry of Differential Forms*. Iwanami Series in Modern Mathematics, 2001.

[10] R.E. Terrell. *The fundamental theorem of calculus and the Poincaré lemma*. Tech Report, Cornell University, 2009. Link

[11] B.-Y. Hou and B.-Y. Hou. *Differential Geometry for Physicists*. World Scientific, 1997.

[12] M. Nakahara. *Geometry, Topology and Physics*, 2nd ed. Taylor & Francis, 2003.

[13] G. Lehner, S. Kurz. *Elektromagnetische Feldtheorie für Ingenieure und Physiker*, 9. Auflage. Springer Vieweg, 2021.

[14] M. Pellikka et al. Homology and Cohomology Computation in Finite Element Modeling. SIAM J Sci Comput, 35(5), 2013. Link

# Literature (cont'd)


[15] A. Bossavit. *On the notion of anisotropy of constitutive laws: Some implications of the "Hodge implies metric" result.* COMPEL, 20(1), 2001.

[16] E. Tonti. *Mathematical Structure of Classical and Relativistic Physics*. Springer, 2013.

[17] G.A. Deschamps. *Electromagnetics and Differential Forms*. Proc. IEEE, 69(6), 1981.