Electromagnetics and Differential Forms

Poincaré Lemmata and de Rham Cohomology

In today's lecture we talk about:

- **The Poincaré lemmata**
- De Rham cohomology
- Simplicial homology
- The theorem of de Rham

Duality paring between forms and chains

Let $\omega \in \mathcal{F}^p(M)$ be a p-form on a differentiable manifold M.

Let $c \in \mathcal{C}_p(M)$ be a p-chain on M.

$\langle \omega, c \rangle$ is a duality paring between $\mathcal{F}^p(M)$ and $\mathcal{C}_p(M)$.

The pairing $\langle \omega, c \rangle := \int_c \omega$ is

I) bilinear, $\lambda \in \mathbb{R}$

$$\langle \omega + \lambda \omega', c \rangle = \int_c \omega + \lambda \int_c \omega' \qquad \langle \omega, c + \lambda c' \rangle = \int_c \omega + \lambda \int_{c'} \omega$$

II) nondegenerate

$$\forall c \in \mathcal{C}_p(M) : \langle \omega, c \rangle = 0 \qquad \forall \omega \in \mathcal{F}^p(M) : \langle \omega, c \rangle = 0$$

$$\Rightarrow \ \omega = 0 \qquad \Rightarrow \ c = 0$$

Physical interpretation of chains and forms

Consider the duality paring

$$\big(\mathcal{F}^p(M), \mathcal{C}_p(M), \langle\cdot,\cdot\rangle\big)$$

Note that for $\mathcal{C}_p(M)$, in order to be a vector space, one also has to admit formal linear combinations of simplices with real coefficients.

From a physical perspective:

$c \in \mathcal{C}_p(M)$ models of measuring probes

$\omega \in \mathcal{F}^p(M)$ models of fields

$\langle \omega, c\rangle \longrightarrow$ measurement

Stokes' theorem revisited

We can express Stokes' theorem in terms of the duality paring:

d is the dual operator to ∂

$$\langle \mathrm{d}\omega, c \rangle = \langle \omega, \partial c \rangle$$

$\forall c \in \mathcal{C}_p(M)$
$\forall \omega \in \mathcal{F}^{p-1}(M)$

This can be used as an alternative definition of the exterior derivative:

$\mathrm{d}\omega$ is the p - form that gives the same response to any p - chain c as the $(p-1)$- form ω gives on the boundaries ∂c .

Also $\partial \circ \partial = 0 \quad \Rightarrow \quad \mathrm{d} \circ \mathrm{d} = 0$ holds as a necessity.

Classifications of forms and chains

closed p-forms

$$\mathcal{Z}^p(M) = \ker \mathrm{d}_p$$

$\forall \omega \in \mathcal{Z}^p(M) : \; \mathrm{d}\omega = 0$

p-cycles

$$\mathcal{Z}_p(M) = \ker \partial_p$$

$\forall c \in \mathcal{Z}_p(M) : \; \partial c = 0$

exact p-forms

$$\mathcal{B}^p(M) = \mathrm{Im}\ \mathrm{d}_{p-1}$$

$\forall \omega \in \mathcal{B}^p(M) : \exists \Omega \in \mathcal{F}^{p-1}(M) : \omega = \mathrm{d}\Omega$

p-boundaries

$$\mathcal{B}_p(M) = \mathrm{Im}\ \partial_{p+1}$$

$\forall c \in \mathcal{B}_p(M) : \; \exists C \in \mathcal{C}_{p+1}(M) : c = \partial C$

First lemma of Poincaré

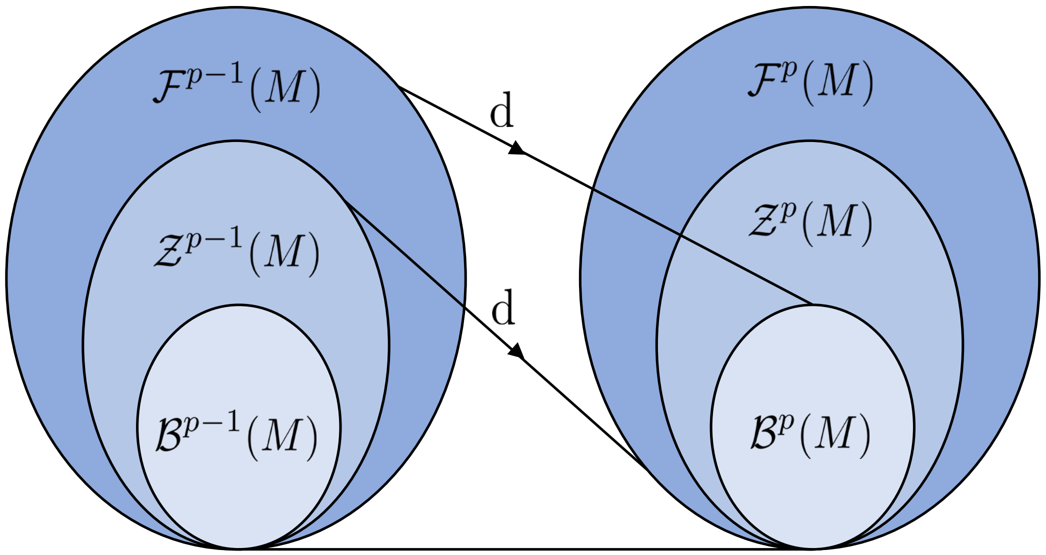
From the complex property $\mathrm{d} \circ \mathrm{d} = 0$ and $\partial \circ \partial = 0$ it follows directly

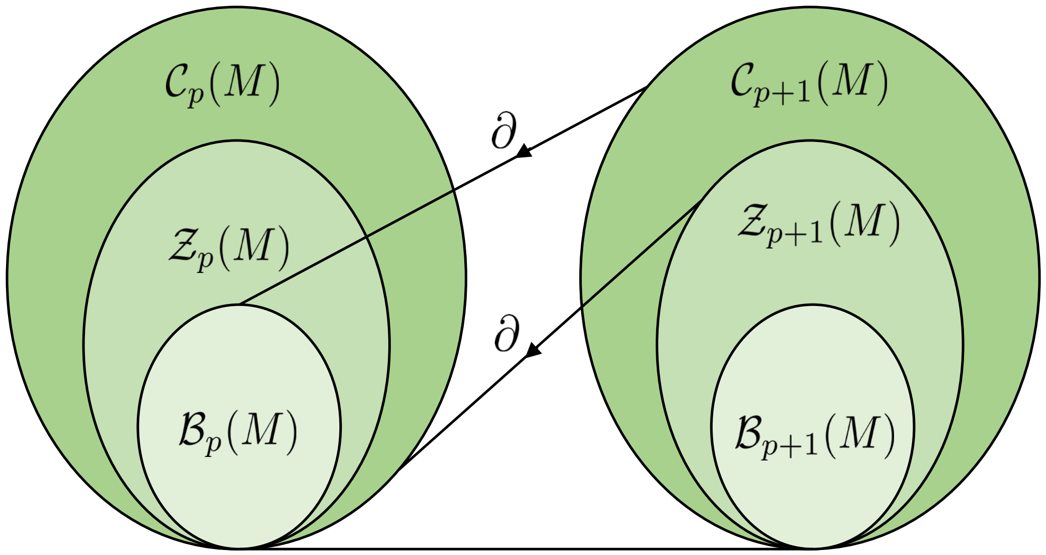
$$\mathcal{B}^p(M) \subseteq \mathcal{Z}^p M$$

Every exact form
is closed.

$$\mathcal{B}_p(M) \subseteq \mathcal{Z}_p M$$

Every boundary
is a cycle.





Converse lemma ?

The following question arises :

Does the converse hold e.g. is every closed form exact and every cycle a boundary ?

The answer is given by the **second lemma of Poincaré** :

Locally – **Yes, always**! To any point there always exist a open set on the manifold where this holds.

Globally – **It depends** on the topological properties of the manifold.
$\Rightarrow$ leads to discussion of homology and cohomology classes.

Contractible manifold

A differentiable manifold M is called **contractible** if there exists a differentiable map

$$\psi : [0,1] \times M \to M$$

such that

$$\psi(1\,,\,\cdot\,) = \mathrm{id}_M$$

$$\psi(0\,,\,\cdot\,) = \mathrm{const.}$$

One can picture this as a continuous deformation to a point.

Examples:

contractible			non contractible	
n-ball	convex domains	in $\mathbb{R}^n$	torus	n-sphere
solid cylinder	star-shaped domains	in $\mathbb{R}^n$	disconnected domains	cylindrical surface

Second lemma of Poincaré

Let M be a **contractible** differentiable manifold, then it holds that

$$\mathcal{B}^p(M) = \mathcal{Z}^p(M)$$

Every closed form is exact.

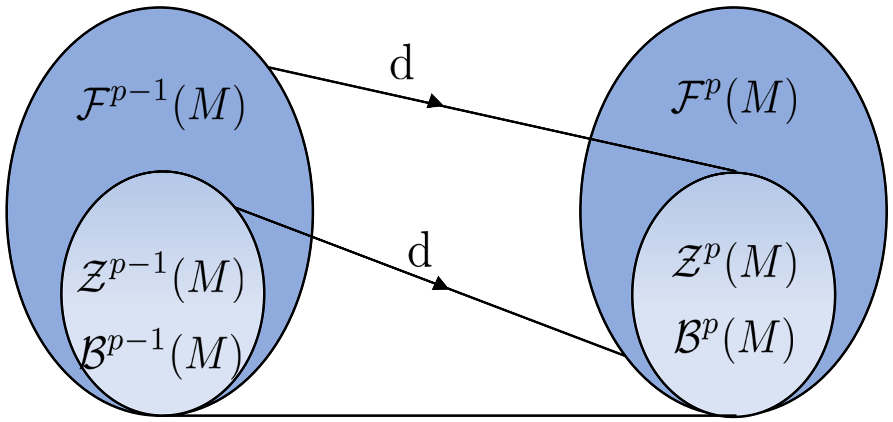
$$\mathcal{B}_p(M) = \mathcal{Z}_p(M)$$

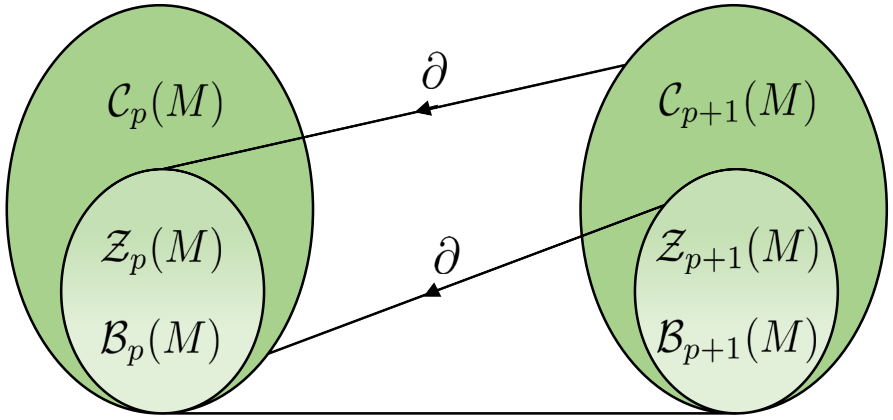
Every cycle is a boundary.

Ω is called the potential to ω.

$$\mathrm{d}\omega = 0 \Rightarrow \exists \Omega \in \mathcal{F}^{p-1}(M) : \omega = \mathrm{d}\Omega$$

$$\partial c = 0 \Rightarrow \exists C \in \mathcal{C}_{p+1}(M) : c = \partial C$$



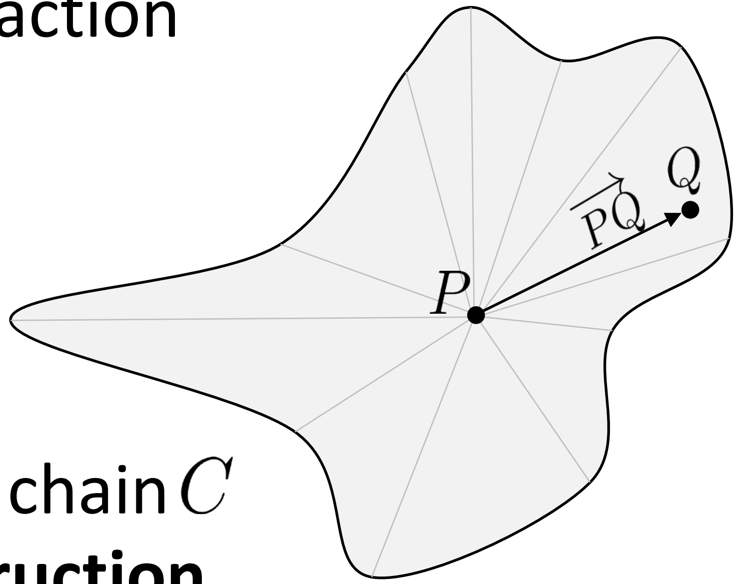


Lemma of Poincaré for star-shaped domains in $\mathbb{R}^n$

The **lemma of Poincaré** especially holds for $M \subset \mathbb{R}^n$ open and **star-shaped**.

A star-shaped domain contains all lines $\overline{PQ}$ from the star point $P \in M$ to a point $Q \in M$. We can construct an explicit contraction

$$\psi(t, Q) = P + t\,\overrightarrow{PQ}$$



In this special case:

1) One can explicitly construct, to a boundary c, a chain C such that $c = \partial C$ holds. $\Rightarrow$ **Poincaré cone construction**

2) One can give an explicit formula for the potential Ω to a p-form ω.

The Poincaré cone construction in $\mathbb{R}^3$

The cone is not unique since the top point can be chosen arbitrarily !

Consider a 1-cycle $c \in \mathcal{C}_1(M)$, $M \subset \mathbb{R}^3$

A cone is constructed by the family

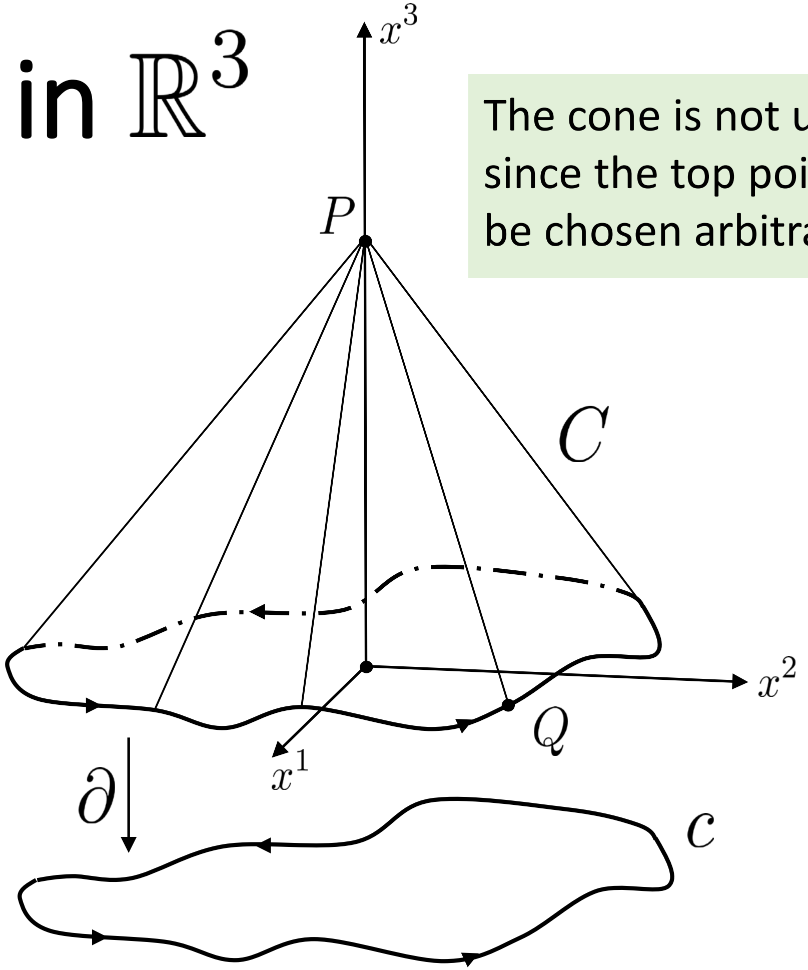
$$\psi_t := \psi(t, \cdot) : M \to M$$

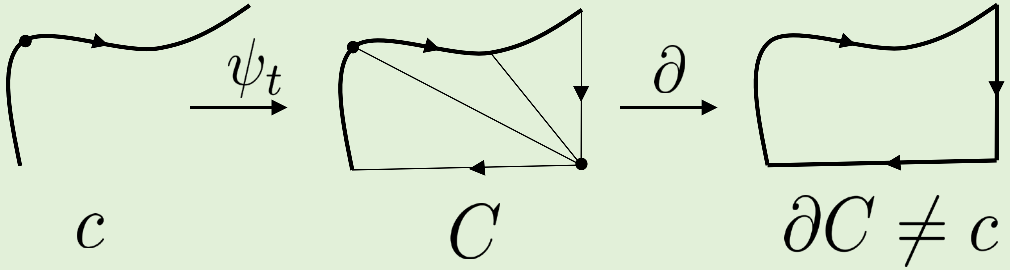
as the set

$$C = \{\psi_t(Q) \mid t \in [0,1], Q \in c\}$$

with properties

$$C \in \mathcal{C}_2(M) \quad \text{and} \quad \partial C = c$$





If c is not a cycle, this construction is in general not possible !

The Poincaré potential construction [2, Ch. 11.5]

Consider the position vector field $x : M \subset \mathbb{R}^n \to T_P M \cong \mathbb{R}^n, \; Q \mapsto \overrightarrow{PQ}$
and the contraction maps $\psi_t := \psi(t, \cdot) : M \to M$

Define the **cone integral operator**

$$\begin{aligned} \mathrm{I} : \mathcal{F}^p(M) &\to \mathcal{F}^{p-1}(M) \\ \omega &\mapsto \int_0^1 \frac{1}{\tau}\, \mathbf{i}_x \mathrm{D}\psi_\tau^* \omega \, \mathrm{d}\tau \end{aligned}$$

It can be shown that: $\mathrm{d} \circ \mathrm{I} + \mathrm{I} \circ \mathrm{d} = \mathrm{id}_M$

Therefore: $\mathrm{d}\omega = 0 \quad \Rightarrow \quad \omega = \mathrm{d}\,\mathrm{I}\omega = \mathrm{d}\Omega \quad \Rightarrow \quad \boxed{\Omega = \mathrm{I}\,\omega}$

This gives an explicit formula to calculate the potential Ω on star-shaped domains

Cone integral operator in coordinates

Use a chart ϕ such that for the star point $\phi(P) = 0$.

$$\mathrm{I}\,\omega = \int_0^1 \frac{1}{\tau}\,\mathbf{i}_x \mathrm{D}\psi_\tau^* \omega \,\mathrm{d}\tau \;= \int_0^1 \frac{1}{\tau}\,\mathbf{i}_x \left(\omega_{i_1 \ldots i_p} \circ \psi_\tau \; \mathrm{D}\psi_\tau^* \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{D}\psi_\tau^* \mathrm{d}x^{i_p}\right) \mathrm{d}\tau$$

$$= \int_0^1 \frac{1}{\tau} \omega_{i_1 \ldots i_p}(tx)\,\mathbf{i}_x \left(\,\tau \mathrm{d}x^{i_1} \wedge \cdots \wedge \tau \mathrm{d}x^{i_p}\right) \mathrm{d}\tau$$

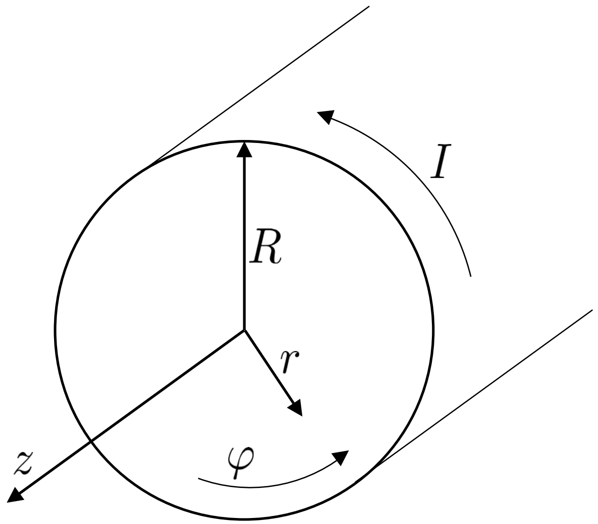
$$= \left(\int_0^1 \tau^{p-1} \omega_{i_1 \ldots i_p}(tx) \mathrm{d}\tau\right)\; \mathbf{i}_x \left(\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}\right)$$

Follows by evaluation of the contracted form with the Laplace theorem.

$$= \sum_{j=1}^{p} (-1)^{j-1} \left(\int_0^1 \tau^{p-1} \omega_{i_1 \ldots i_p}(tx) \mathrm{d}\tau\right)\; x^{i_j}\; \mathrm{d}x^{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x^{i_j}} \wedge \cdots \wedge \mathrm{d}x^{i_p}$$

Vector potential of a long, densely wound round coil

Describe the problem in polar coordinates:



R ... Radius of coil
I ... Current
n ... Winding density of coil

$$B = \mu_0 I n \begin{cases} \frac{\partial}{\partial z} & \text{if} \quad r < R \\ 0 & \text{if} \quad r > R \end{cases}$$

$$\Rightarrow \quad \mathcal{B} = {}^2 B = \mu_0 I n \begin{cases} \mathrm{d}r \wedge r\,\mathrm{d}\varphi & \text{if} \quad r < R \\ 0 & \text{if} \quad r > R \end{cases}$$

Vector potential of a long, densely wound round coil

Cone integral operator in 2D cross-sectional area:

polar coordinate representation : $\tilde{\psi}_\tau : (r,\varphi) \mapsto (\tau r, \varphi) \quad \Rightarrow \quad \begin{aligned} \mathrm{D}\psi_\tau^* \mathrm{d}r &= \mathrm{d}(\tau r) = \tau \mathrm{d}r \\ \mathrm{D}\psi_\tau^* \mathrm{d}\varphi &= \mathrm{d}\varphi \end{aligned}$

$$\mathrm{D}\psi_\tau^* \mathcal{B} = \mu_0 I n \begin{cases} \tau^2 \mathrm{d}r \wedge r\,\mathrm{d}\varphi & \text{if} \quad \tau r < R \\ 0 & \text{if} \quad \tau r > R \end{cases}$$

$$\Rightarrow \quad \int_0^1 \frac{1}{\tau} \mathrm{D}\psi_\tau^* \mathcal{B}\,\mathrm{d}\tau = \mu_0 I n \int_0^{\min\left(1,\frac{R}{r}\right)} \tau\,\mathrm{d}\tau\;\mathrm{d}r \wedge r\,\mathrm{d}\varphi$$

$$= \mu_0 I n \frac{r}{2} \min\left(1, \frac{R^2}{r^2}\right) \mathrm{d}r \wedge \mathrm{d}\varphi$$

In a last step we need to contract this 2-form with the position vector field x .

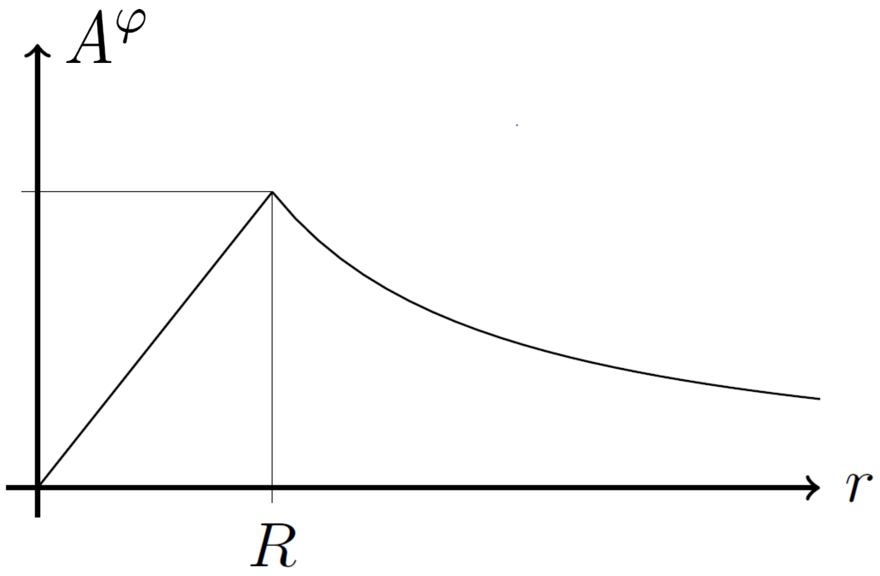
Vector potential of a long, densely wound round coil

Perform the contraction $\mathbf{i}_x$ with position vector field $x \cong r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}$

$$\mathbf{i}_x(\mathrm{d}r \wedge \mathrm{d}\varphi)(\xi) = (\mathrm{d}r \wedge \mathrm{d}\varphi)(r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}, \xi) = (\mathrm{d}r \wedge \mathrm{d}\varphi)(r\frac{\partial}{\partial r}, \xi) \qquad \forall \xi \in \mathcal{X}(\mathbb{R}^3)$$

$$= \begin{vmatrix} r & 0 \\ \mathrm{d}r(\xi) & \mathrm{d}\varphi(\xi) \end{vmatrix} = r\,\mathrm{d}\varphi(\xi)$$

$$\Rightarrow \quad \mathcal{A} = \mathrm{I}\,\mathcal{B} = \frac{\mu_0 I n}{2} \underbrace{r\mathrm{d}\varphi}_{= {}^1\frac{\partial}{r\partial\varphi} = {}^1 e_\varphi} \begin{cases} r & \text{if} \quad r \leq R \\ \frac{R^2}{r} & \text{if} \quad r \geq R \end{cases}$$



Existence of potentials and gauge transformations

The 2nd lemma of Poincaré guarantees the existence of potential forms if the domain of the form is topologically trivial.

In other words:

$\Omega \in \mathcal{F}^{p-1}(M): \quad \omega = \mathrm{d}\Omega \quad \omega \in \mathcal{F}^{p}(M)$ is solvable if M is contractible.

The solution is **not unique** !

$$\forall \lambda \in \mathcal{F}^{p-2}(M) \qquad \mathrm{d}(\Omega + \mathrm{d}\lambda) = \mathrm{d}\Omega + \mathrm{d} \circ \mathrm{d}\lambda = \mathrm{d}\Omega = \omega$$

$\Rightarrow \quad \mathrm{d}\Omega \to \mathrm{d}(\Omega + \mathrm{d}\lambda)$ **gauge transformation**

The choice of a different apex in the Poincaré cone construction leads also to a different gauge.

2^{nd} lemma of Poincaré in classical vector analysis [10]

If the open set $U \subset \mathbb{R}^3$ is contractible then for sufficiently smooth vector fields and functions it holds that:

1) To every vector field E on U with $\mathbf{curl}\, E = 0$ there exists a function Φ such that $E = -\mathrm{grad}\, \Phi$.

2) To every vector field B on U with $\mathbf{div} B = 0$ there exists a vector field A such that $B = \mathbf{curl}\, A$.

3) To every function f there exists a vector field F such that $f = \mathrm{div}\ F$.

Try to prove this with the 2^{nd} lemma of Poincaré and the translation isomorphisms

In today's lecture we talk about:

- The Poincaré lemmata
- **De Rham cohomology**
- Simplicial homology
- The theorem of de Rham

Cohomologous forms

Two closed p-forms $\omega_1, \omega_2 \in \mathcal{Z}^p(M)$ are called **cohomologous** if they only differ by an exact p-form $\mathrm{d}\Omega$.

$$\omega_1 \sim \omega_2 \qquad \Leftrightarrow \qquad \exists \Omega \in \mathcal{F}^{p-1}(M) :\ \omega_1 - \omega_2 = \mathrm{d}\Omega$$

This defines an equivalence relation.

$$\Rightarrow \quad \mathcal{B}^p(M) = \mathrm{Im}\, \mathrm{d}_{p-1}$$

$$= [0]$$

is the equivalence class of all closed p-forms cohomologous to 0.

Cohomology groups

The quotient space

$$H^p(M) := {\mathcal{Z}^p(M)}\big/{\mathcal{B}^p(M)}$$

is called the p-th **de Rham cohomology group/space**.

It is the set of all equivalence classes $[\omega]$ of the relation of last slide.

$H^p(M)$ can be given the structure of an Abelian group or vector space over $\mathbb{R}$ by :

$$[\omega] + [\omega'] := [\omega + \omega'] \qquad \forall \omega, \omega' \in \mathcal{Z}^p(M)$$

$$\lambda[\omega] := [\lambda\omega] \qquad \forall \lambda \in \mathbb{R}$$

Homotopy invariance of cohomology [2, Ch. 11.2]

Let M, N be differentiable manifolds.
Two differentiable maps $f, g : M \to N$ are called **differentiably homotopic** if there exists a differentiable map

$$h : [0,1] \times M \to N \text{ such that } h(0,\cdot) = f \text{ and } h(1,\cdot) = g$$

If $f, g : M \to N$ are differentiably homotopic maps, then

$$\mathrm{D}f^* = \mathrm{D}g^* : H^p(N) \to H^p(M)$$

holds for all p, e.g.

$$[\mathrm{D}f^*\omega] = [\mathrm{D}g^*\omega] \qquad \forall \omega \in \mathcal{Z}^p(N)$$

Betti numbers (cohomology)

The dimension of the p-th de Rham cohomology space

$$b^p := \dim H^p(M)$$

is called the p-th **Betti number** of M.

- b^p gives the number of **essentially different closed** p**- forms** on M.
- These numbers are topological invariants, they depend only on the topology of the manifold.
- $b^p < \infty$ holds for finite dimensional, compact manifolds.

Examples

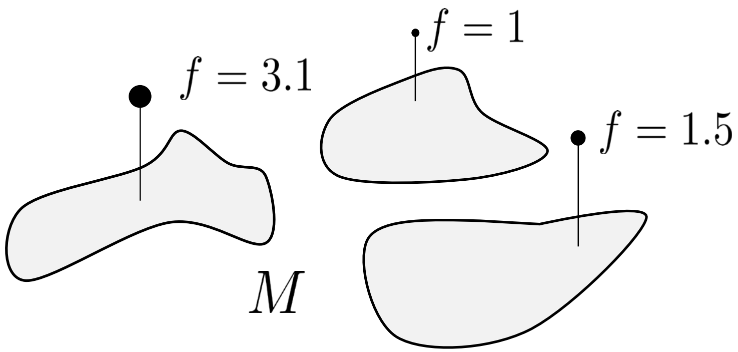
There are no forms of negative degree

$\omega = 0$ for $p > n$

$$H^p(M) = 0 \quad \text{for } p < 0 \text{ and } p > n = \dim M\,.$$

$$H^0(M) := \mathcal{Z}^0(M) = \{f \in \mathcal{C}^\infty(M) \mid \mathrm{d}f = 0\} \cong \mathbb{R}^c$$

Piecewise constant functions on M.



Here c is the number of connected components of M.

$$b^0 := \dim H^0(M) = c$$

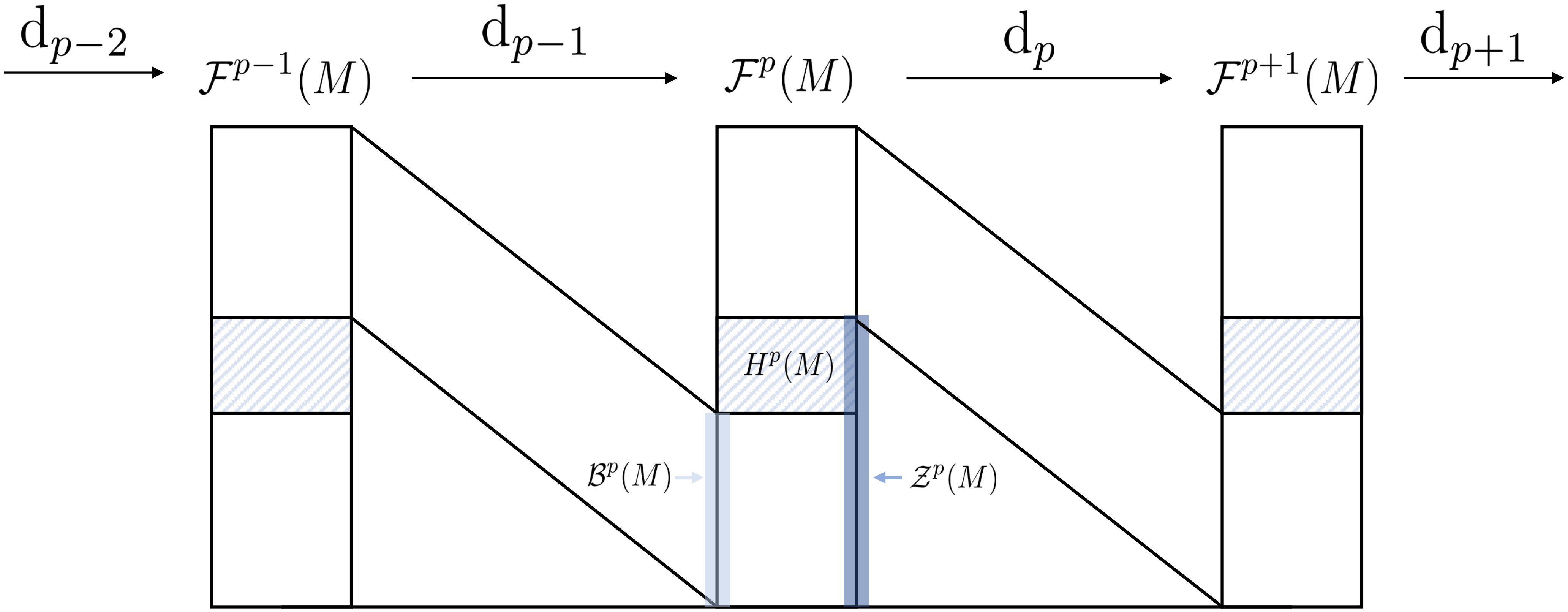
$$H^0(\mathbb{R}^n) \cong \mathbb{R}$$

$$H^p(\mathbb{R}^n) = 0 \quad \text{for } p > 0$$

This follows directly from the 2nd lemma of Poincaré.

The de Rham complex of M

The sequence $(\mathcal{F}^p(M), \mathrm{d}_p)$ is called the **de Rham complex** of M.



The 2^{nd} lemma of Poincaré revisited

The 2^{nd} lemma of Poincaré can also be stated in terms of de Rham cohomology groups:

M is a contractible manifold.	$\Leftrightarrow$	$H^0(M) \cong \mathbb{R}$ $H^p(M) = 0$ for $p > 0$

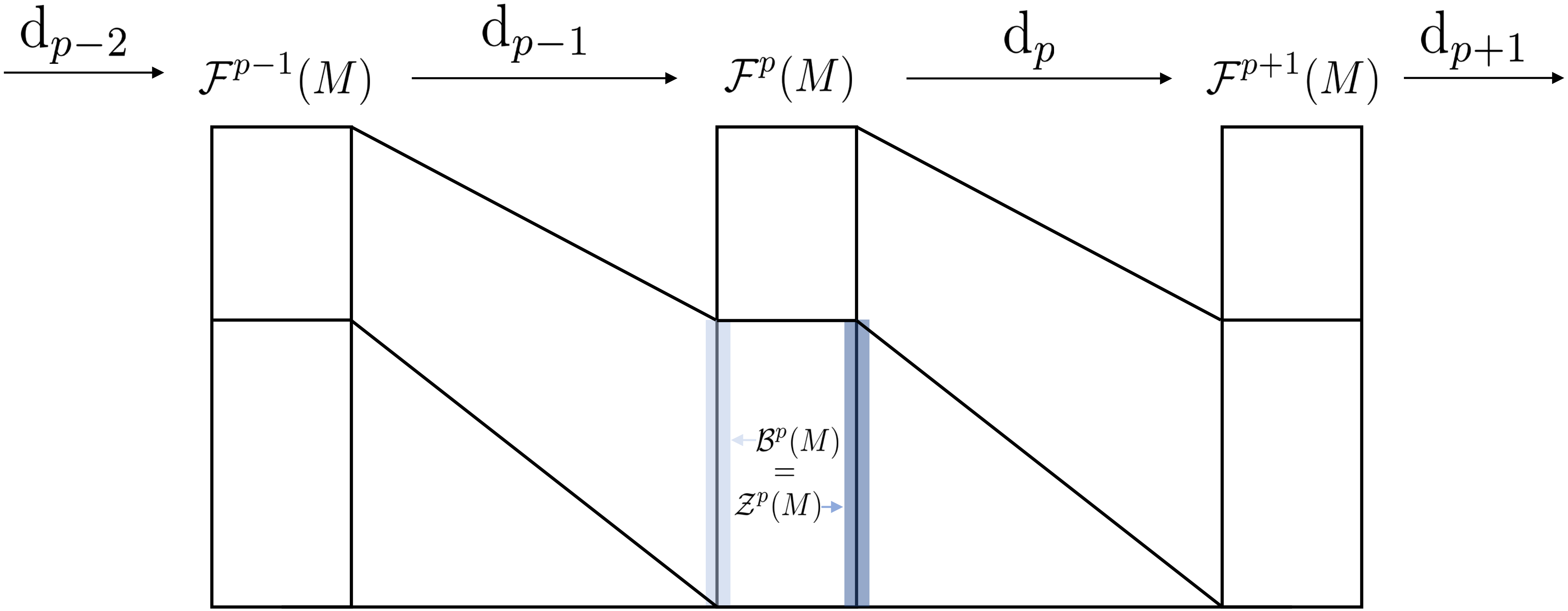
Manifolds with the Betti numbers

$$b^0 = 1 \qquad \text{and} \qquad b^p = 0 \quad \text{for} \quad p > 0$$

are called **topologically trivial**.

De Rham complex for topologically trivial manifolds

The sequence $(\mathcal{F}^p(M), \mathrm{d}_p)$ is called **exact** in the case of topologically trivial manifolds.



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Homologous chains

Two p-cycles $c_1, c_2 \in \mathcal{Z}_p(M)$ are called **homologous** if they only differ by a boundary ∂C_{p+1}.

$$c_1 \sim c_2 \qquad \Leftrightarrow \qquad \exists C \in \mathcal{C}_{p+1}(M) :\ c_1 - c_2 = \partial C$$

This defines an equivalence relation.

$$\begin{aligned} \Rightarrow\ \ \mathcal{B}_p(M) &= \mathrm{Im}\, \partial_{p+1} \\ &= [0] \end{aligned}$$

Boundaries are said to be "homologous to zero".

Homology groups

The quotient space

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is called the p-th **simplicial homology group/space.**

It is the set of all equivalence classes $[c]$ of the relation of last slide.

$H_p(M)$ can be given the structure of an Abelian group or vector space

$$[\omega] + [\omega'] := [\omega + \omega'] \qquad \forall \omega, \omega' \in \mathcal{Z}_p(M)$$

$$\lambda[\omega] := [\lambda\omega] \qquad \forall\lambda \begin{cases} \in \mathbb{Z} \;\Rightarrow\; \text{group structure only} \;\Rightarrow\; H_p(M,\mathbb{Z}) \\ \in \mathbb{R} \;\Rightarrow\; \text{vector space over } \mathbb{R} \;\Rightarrow\; H_p(M,\mathbb{R}) \end{cases}$$

Betti numbers (homology)

We define the p-th **Betti number** of M in terms of homology as

$$b_p := \dim H_p(M, \mathbb{R}) \qquad\qquad b_p := \operatorname{rank} H_p(M, \mathbb{Z})$$ [1)]

$$=$$

follows by
universal coefficient theorem [11, Sect. 7.3]

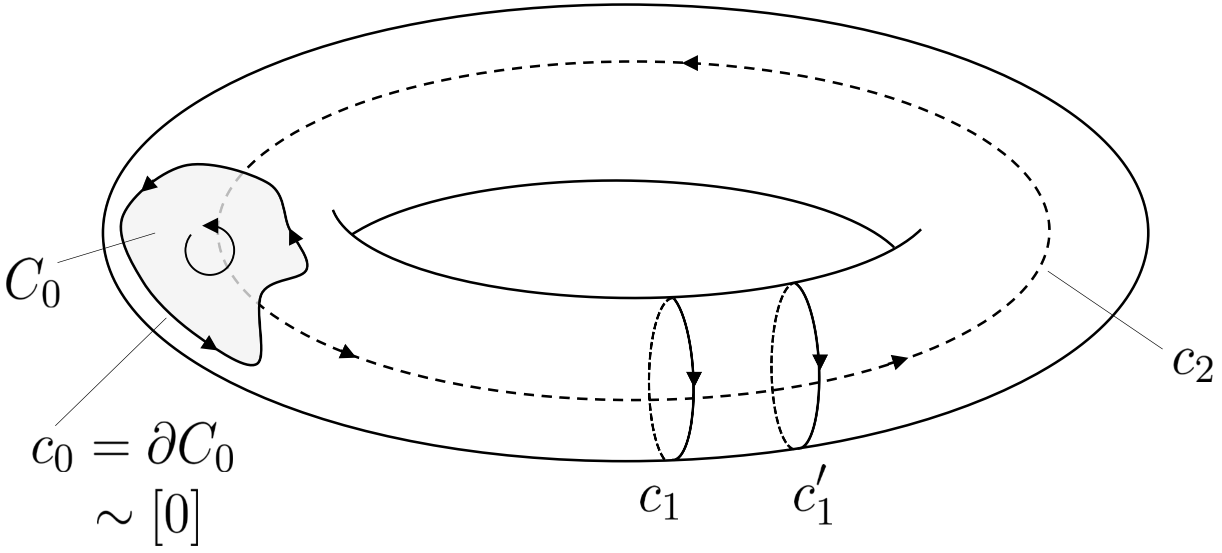
Their intuitive meaning can be better understood in terms of groups.

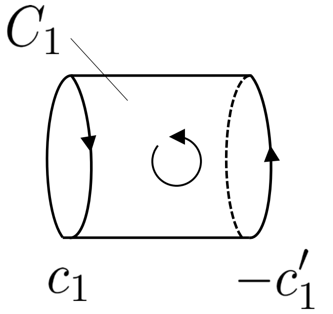
For de Rham's theorem we need $H_p(M)$ to have a vector space structure.

[1)] Precisely: The rank of the free Abelian part, neglecting the torsion subgroup [12, Sect. 3.4.3]

Example: Homology of a torus

Let M be a torus, $\dim M = 2$.





$$c_1 - c_1' = \partial C_1$$

Meridians Längenkreise Parallels Breitenkreise

$$H^1 = \{n[c_1] + m[c_2] \mid m, n \in \mathbb{Z}\}$$

$3[c_1] + 2[c_2]$ can be interpreted as closed curves that are wound around the torus

3 times like c_1

2 times like c_2

Betti numbers

$b_0 = 1$ manifold is connected

$b_1 = 2$ number of “ loops ”

$b_2 = 1$ number of “ cavities ”

Example: Betti numbers

For practical computations see [14]

Topological trivial manifolds: $b_0 = 1$, $b_k = 0 \;\; \forall k > 0$

n- sphere: $b_0 = b_n = 1$, $b_k = 0 \;\; \forall k \notin \{1, n\}$

n- torus: $b_k = \begin{pmatrix} n \\ k \end{pmatrix}$

Hollow n- ball: $b_0 = b_{n-1} = 1$, $b_k = 0 \;\; \forall k \notin \{1, n-1\}$

Homeomorphic to $(n-1)$-sphere $\times$ line segment

For topological spaces X, Y it holds

$$b_k(X \times Y) = \sum_{\lambda+\mu=k} b_\lambda(X) b_\mu(Y)$$

→ Künneth formula [12, eq. (6.45)]

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Consequences of Stokes's theorem (1)

The integral of an **exact form** $\omega \in \mathcal{B}^p(M)$ over a **cycle** $c \in \mathcal{Z}_p(M)$ vanishes.

Proof:

$$\int_c \omega = \int_c \mathrm{d}\Omega = \int_{\partial c} \Omega = 0$$

Consequences of Stokes's theorem (2)

The integral of a **closed form** $\omega \in \mathcal{Z}^p(M)$ over a **boundary** $c \in \mathcal{B}_p(M)$ vanishes.

Proof:

$$\int_c \omega = \int_{\partial C} \omega = \int_C \mathrm{d}\omega = 0$$

Consequences of Stokes's theorem (3)

The integral of a **closed form** $\omega \in \mathcal{Z}^p(M)$ over a **cycle** $c \in \mathcal{Z}_p(M)$ depends only on the homology class of the cycle.

Proof:

Let $c \sim c'$, then $\exists C \in \mathcal{C}_{p+1}(M) : \ c - c' = \partial C$

$$\Rightarrow \quad \int_c \omega - \int_{c'} \omega = \int_{c-c'} \omega = \int_{\partial C} \omega = \int_C \mathrm{d}\omega = 0$$

Consequences of Stokes's theorem (4)

The integral of a **closed form** $\omega \in \mathcal{Z}^p(M)$ over a **cycle** $c \in \mathcal{Z}_p(M)$ depends only on the cohomology class of the form.

Proof:

Let $\omega \sim \omega'$, then $\exists \Omega \in \mathcal{F}^{p-1}(M): \; \omega - \omega' = \mathrm{d}\Omega$

$$\Rightarrow \quad \int_c \omega - \int_c \omega' = \int_c \omega - \omega' = \int_c \mathrm{d}\Omega = \int_{\partial c} \Omega = 0$$

The theorem of de Rham [4, Thm. 3.6]

Follows from the previous four consequences of Stokes' theorem.

The following bilinear mapping is well defined and **non-degenerate**. !!!

$$\begin{aligned} \langle \cdot, \cdot \rangle : H^p(M) \times H_p(M, \mathbb{R}) &\to \mathbb{R} \\ ([\omega], [c]) &\mapsto \int_c \omega \end{aligned}$$

Important consequences :

$$H^p(M) \cong H_p(M, \mathbb{R})^* \; \forall p \geq 0$$

$$b_p = \dim H^p(M) = \dim H_p(M, \mathbb{R})$$

Corollary: Existence of potentials

The **periods of a p-form** $\omega \in \mathcal{F}^p(M)$ are the values of the pairing

$$\langle \omega, c \rangle = \int_c \omega \quad \text{, where } c \in \mathcal{Z}_p(M)$$

From the theorem of de Rham it follows:

$\omega \in \mathcal{B}^p(M)$	$\Leftrightarrow$	$\langle \omega, c \rangle = 0 \ \ \forall c \in \mathcal{Z}_p(M)$
ω is an exact form e.g. there exist a potential form Ω	$\Leftrightarrow$	All periods of ω vanish.

Proof: Existence of potentials corollary

i) Consider $\omega \in \mathcal{Z}^p(M)$

$$\omega \in \mathcal{B}^p(M)$$

(class definition)
$$\Leftrightarrow \quad [\omega] = 0$$

(non-degeneracy)
$$\Leftrightarrow \quad \langle [\omega], [c] \rangle = 0 \qquad \forall [c] \in H_p(M)$$

(well definedness)
$$\Leftrightarrow \quad \langle \omega, c \rangle = 0 \qquad \forall c \in \mathcal{Z}_p(M)$$

ii) Consider $\omega \in \mathcal{F}^p(M)$

$$\langle \omega, c \rangle = 0 \qquad \forall c \in \mathcal{Z}_p(M)$$
$$\Rightarrow \quad \langle \omega, c \rangle = 0 \qquad \forall c \in \mathcal{B}_p(M)$$
$$\Rightarrow \quad \langle \omega, \partial C \rangle = 0 \qquad \forall C \in \mathcal{C}_{p+1}(M)$$
$$\Rightarrow \quad \langle \mathrm{d}\omega, C \rangle = 0 \qquad \forall C \in \mathcal{C}_{p+1}(M)$$
$$\Rightarrow \quad \omega \in \mathcal{Z}^p(M)$$

iii) $\omega \in \mathcal{B}^p(M) \subset \mathcal{Z}^p(M) \overset{\text{(i)}}{\Rightarrow} \langle \omega, c \rangle = 0 \ \forall c \in \mathcal{Z}_p(M)$

iv) $\omega \in \mathcal{F}^p(M)$ and $\langle \omega, c \rangle = 0 \ \forall c \in \mathcal{Z}_p(M) \overset{\text{(ii + i)}}{\Rightarrow} \omega \in \mathcal{B}^p(M)$

Integrability condition and gauge freedom

Consider the equation

$$\omega = \mathrm{d}\Omega \qquad \omega \in \mathcal{F}^p(M)$$

The **integrability condition**

$$\boxed{\langle \omega, c \rangle = 0 \ \ \forall c \in \mathcal{Z}_p(M)}$$

The integrability condition is necessary **and** sufficient

$$\Downarrow$$

$$\mathrm{d}\omega = 0$$

only necessary

guarantees the existence of a particular solution $\Omega' \in \mathcal{F}^{p-1}(M)$.

General solution:

$$\Omega = \Omega' + \underbrace{\mathrm{d}\lambda + \Gamma}_{\text{gauge freedom}} \ \text{ with } \ \lambda \in \mathcal{F}^{p-2}(M),\ \Gamma \in [\Omega - \Omega'] \in H^{p-1}(M)$$

Example: Magnetic vector potential

Let $\mathcal{B} \in \mathcal{F}^2(M)$ be the magnetic flux density 2-form.

From Maxwell's equation in integral form it follows

2-boundaries

$$\langle \mathcal{B}, \partial V \rangle = \int_{\partial V} \mathcal{B} = 0 \qquad \Rightarrow \qquad \langle \mathcal{B}, A \rangle = 0 \quad \forall A \in \mathcal{B}_2(M)$$

We can generalize this to: $$\underbrace{\langle \mathcal{B}, A \rangle = 0 \quad \forall A \in \mathcal{Z}_2(M)}_{\textit{No magnetic monopoles}}$$

Then the theorem of de Rham implies

$$\exists \mathcal{A} \in \mathcal{F}^1(M) : \; \mathcal{B} = \mathrm{d}\mathcal{A}$$

Maxwell's equation in **integral form** can be formulated to guarantee the existence of potentials, **even for topological non trivial manifolds**.

Example: Magnetic vector potential, hollow sphere

Consider $M = B_R(0) \setminus B_\varepsilon(0)$ and spherical coordinates (r, θ, φ) with the orthonormal basis fields $\frac{\partial}{\partial r}$, $\frac{1}{r}\frac{\partial}{\partial \theta}$, $\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}$ and dual forms $\mathrm{d}r$, $r\mathrm{d}\theta$, $r\sin\theta\,\mathrm{d}\varphi$.

$b_2(M) = 1$

Not every closed 2-form is exact !

Consider the magnetic flux density of a “hidden monopole”

$$B = \frac{c}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}, \qquad R > r > \varepsilon > 0, \; c \neq 0 .$$

This flux density satisfies Gauss’s law for magnetism in differential form

$$B = \frac{c}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r} \longrightarrow \mathcal{B} = {}^2B = \frac{c}{4\pi}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \longrightarrow \boxed{\mathrm{d}\mathcal{B} = 0}$$

$$\mathrm{div} B = 0$$

Example: Magnetic vector potential, hollow sphere

Consider now a sphere $S_\rho(0)$ with $R > \rho > \varepsilon$. Then

$$\langle \mathcal{B}, S_\rho(0) \rangle = \int_{S_\rho(0)} \mathcal{B} \quad = \int_{[0,\pi]\times[0,2\pi]} \frac{c}{4\pi} \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\varphi \quad = \int_0^{2\pi} \int_0^{\pi} \frac{c}{4\pi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi = c \neq 0$$

But then $\exists \mathcal{A} \in \mathcal{F}^1(M) : \mathcal{B} = \mathrm{d}\mathcal{A}$ is **impossible**, despite $\mathrm{d}\mathcal{B} = 0$, since

$$0 \neq c = \langle \mathcal{B}, S_\rho(0) \rangle = \langle \mathrm{d}\mathcal{A}, S_\rho(0) \rangle = \langle \mathcal{A}, \partial S_\rho(0) \rangle = 0$$

The vanishing of periods requires $\langle \mathcal{B}, S_R(0) \rangle = 0$ and rules out this situation !

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