Electromagnetics and Differential Forms

Integration of Differential Forms

In today’s lecture we talk about:

- **Orientation on manifolds**
- Integration – in terms of triangulation
- Integration – over a simplex and chain
- Integration of forms – over manifolds
- Integration in coordinates

Orientation for vector spaces

Let V be a vector space over $\mathbb{R}$ and $(e_i)_{i=1}^n, (\tilde{e}_i)_{i=1}^n \subset V$ two basis of V.
They are connected by the matrix $\left(a_j^i\right)$ via $e_j = \left(a_j^i\right)\,\tilde{e}_i$.

The two bases are said to have the same orientation if

$$(e_i)_{i=1}^n \sim (\tilde{e}_j)_{j=1}^n \quad \Leftrightarrow \quad \det(a_j^i) > 0$$

This defines a equivalence relation, partitioning the set of all possible bases of V into exactly two equivalence classes.

These two classes are called **orientations** on V .

Orientation for vector spaces with multicovectors

We have two bases $(e_i)_{i=1}^n, (\tilde{e}_j)_{j=1}^n \subset V$ connected via $e_j = \left(a^i_j\right)\ \tilde{e}_i$.
We introduce their dual bases $\left(\varepsilon^i\right)_{i=1}^n, \left(\tilde{\varepsilon}^j\right)_{j=1}^n \subset V^*$ and define :

$$\eta = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n \qquad\qquad \tilde{\eta} = \tilde{\varepsilon}^1 \wedge \cdots \wedge \tilde{\varepsilon}^n$$

We know that $\exists \lambda \in \mathbb{R} : \quad \tilde{\eta} = \lambda\, \eta$

One can proof that $\lambda = \det(a^i_j)$ and therefore get an equivalent definition for equally oriented basis:

$$(e_i)_{i=1}^n \sim (\tilde{e}_j)_{j=1}^n \qquad \Leftrightarrow \qquad \exists \lambda > 0 : \ \tilde{\varepsilon}^1 \wedge \cdots \wedge \tilde{\varepsilon}^n = \lambda\ \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$$

Orientable manifolds and volume forms

Let M be a n-dimensional manifold.

A n-form $\omega \in \mathcal{F}^n(M)$ on M is called a **volume form** if $\omega_P \neq 0 \quad \forall P \in M$.

A manifold M is called **orientable** if there exists a volume form on M.

Remark:

- Not all manifolds are orientable. Example: Möbius strip
- The notion of integration can be extended to non-orientable manifolds. See for example [7].
- If there exists a metric on the manifold, a special volume form is distinct by assigning a unit-volume the value ±1. (**metric volume form**)

Orientations on manifolds

Let M be a n-dimensional orientable manifold.
Let $\omega, \tilde{\omega} \in \mathcal{F}^n(M)$ be volume forms on M.

Define an equivalence relation :

$$\omega \sim \tilde{\omega} \quad \Leftrightarrow \quad \exists\lambda > 0 \in C^\infty(M) : \omega = f\tilde{\omega}$$

Then an **orientation** on M is defined by selecting an equivalence class $[\omega]$ of this relation.

On an orientable **connected** manifold, there are exactly two orientations.

A representative ω of an orientation $[\omega]$ then induces an orientation on every tangent space of M .

Orientation preserving maps between manifolds

Let M, N be n-dimensional orientable manifolds.
Let $\omega \in \mathcal{F}^n(M)$ and $\theta \in \mathcal{F}^n(N)$ be two volume forms representing orientations on M and N.

A map $\psi : M \to N$ is called **orientation preserving** if $\mathrm{D}\psi^*\theta \in [\omega]$

$\Rightarrow$ $\mathrm{D}\psi$ sends an oriented basis of $T_P M$ on an oriented basis of $T_Q N$, where $Q = \psi(P)$.

A map $\psi : M \to N$ is called **orientation reversing** if $\mathrm{D}\psi^*\theta \in [-\omega]$

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Integration of forms - Introduction

A differential p-form ω is an object naturally to be integrated over a p-dimensional oriented integration domain K.

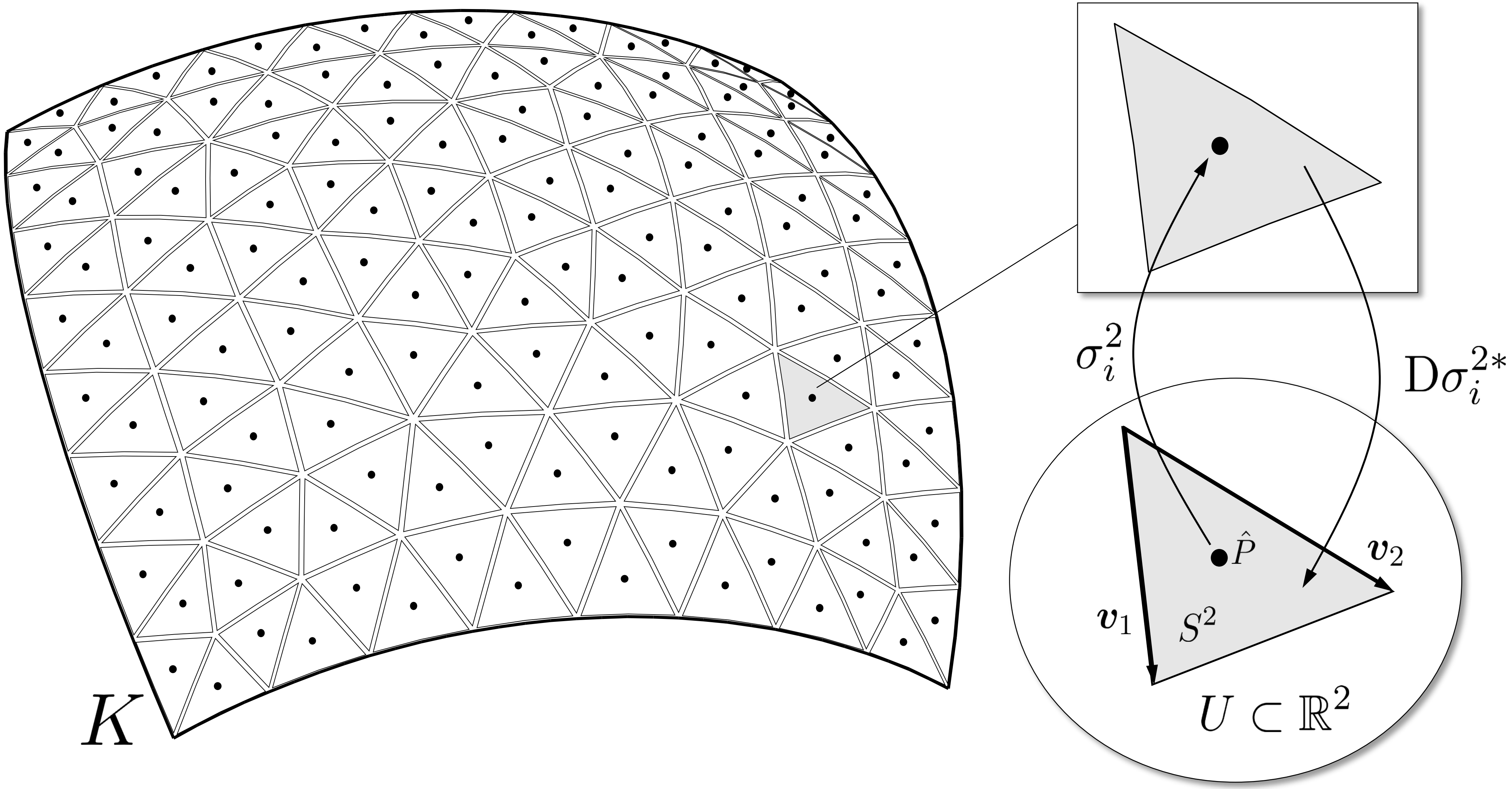
$$I(K) = \int_K \omega$$

No measures needed – each differential form comes naturally with its own.

No additional differentials needed – appear in coordinate representation.

This generalizes line, surface and volume integrals from vector analysis.

Triangulation of 2D-surface



Integration of a 2-form over a 2D-surface

The underlying concept is as follows:

1. Divide the surface into triangles and calculate the integral for every triangle
2. The i-th triangle is constructed with an orientation preserving map $\sigma_i^2 : U \subset \mathbb{R}^2 \to K$ from a reference triangle $S^2 \subset U$, which is spanned by two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$.
3. The reference triangle is equipped with an evaluation point $\hat{P}$. This defines evaluation points on the triangles in K.
4. The pullback map induced by σ_i^2 gives a differential form $\mathrm{D}\sigma_i^{2*}\,\omega$ on $U \subset \mathbb{R}^2$

Integration of a forms – The intuitive approach

5. Evaluating $\mathrm{D}\sigma_i^{2*}\,\omega$ at $\hat{P}$ gives a multicovector that maps $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ onto a real number.
6. The integral is approximated by the sum of the contributions of all triangles

$$\int_K \omega \approx \frac{1}{2} \sum_i \mathrm{D}\sigma_i^{2*}\omega_{\hat{P}}(\boldsymbol{v}_1, \boldsymbol{v}_2)$$

7. For finer and finer partitions of K this sum converges against the integral.

This is independent of the explicit triangulation.

This is independent of the choice of the reference triangle.

For more details see [8]

Integration of a 2-form over a 2D-surface

For the general integral of a p- form we use simplices instead of triangles and exchange 2 by $p!$ in the denominator.

$$\int_K \omega \approx \frac{1}{p!} \sum_i \mathrm{D}\sigma_i^{p*} \omega_{\hat{P}}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p)$$

We now have to ask:

What domains can we integrate over ?

How to calculate these integrals analytically ?

Does this integral operation have the usual properties ?

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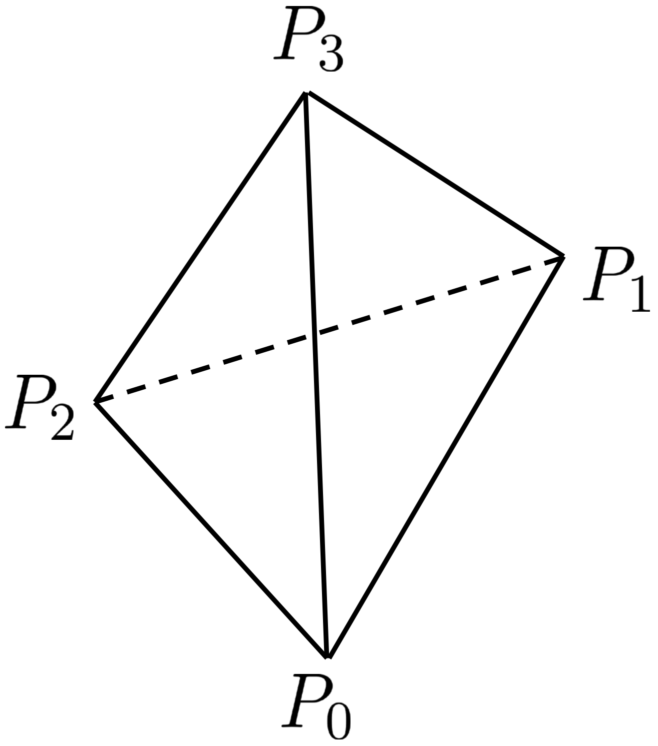
Simplices in $\mathbb{R}^n$

Let $(P_0, \dots, P_p)$ be points in $\mathbb{R}^n$ such that $\overrightarrow{P_0 P_i} \in \mathbb{R}^n$ are linearly independent vectors.

A p- simplex in $\mathbb{R}^n$, for $p \leq n$, is then defined as the closed convex hull of the point set $(P_0, \dots, P_p)$.

$$S^p := \langle P_0, \dots, P_p \rangle$$

$$= \left\{ \sum_{i=0}^{p} t_i \, P_i \;\middle|\; \sum_{i=0}^{p} t_i = 1 \;,\; t_i \in [0,1] \right\}$$



The $(t_0, \dots, t_p)$ are called **barycentric coordinates**.

Euclidian volume of simplices in $\mathbb{R}^n$

The space $\mathbb{R}^n$ has a canonical chart $\mathrm{id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$.

This chart induces a canonical volume form

$$\mathrm{Vol} := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

If we consider a maximal simplex $S^n := \langle P_0, \ldots, P_n \rangle$ it can be assigned the **Euclidean volume**

$$\mathrm{Vol}\left[S^n\right] := \frac{1}{n!} \mathrm{Vol}_{\hat{P}} \left(\overrightarrow{P_0 P_1}, \ldots, \overrightarrow{P_0 P_n} \right) \quad \text{with } \hat{P} \in S^n \text{ arbitrary}$$

Remark:

The volume form Vol is distinguished by the Euclidean metric on $\mathbb{R}^n$ since it assigns a unit volume spanned by orthonormal vectors the value ±1. (See later in lecture on metric)

Canonical orientation of a simplex in $\mathbb{R}^n$

The space $\mathbb{R}^n$ has a canonical orientation fixed by the order of the canonical basis vectors.

This induces an orientation to a simplex $S^n := \langle P_0, \ldots, P_n \rangle$.

The simplex is said to have the same orientation as $\mathbb{R}^n$ if

$$\mathrm{Vol}\left[S^n\right] \propto \quad \mathrm{d}x^1|_{\hat{P}} \wedge \cdots \wedge \mathrm{d}x^n|_{\hat{P}} \left(\overrightarrow{P_0P_1}, \ldots, \overrightarrow{P_0P_n}\right) > 0$$

Otherwise, since there are only two orientations, it is said to have the opposite orientation.

Integration of n-forms on $\mathbb{R}^n$ over a n-simplex

We consider the canonical chart $\mathrm{id}_{\mathbb{R}^n}$, inducing the coordinates $(x^1, \ldots, x^n)$.

Any n - form $\omega \in \mathcal{F}^n(\mathbb{R}^n)$ can be written as

$$\omega = \omega_{1\ldots n}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$
$$= \omega_{1\ldots n}\, \mathrm{Vol} \quad ,$$

where

$$w_{1\ldots n} = \omega\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) \in \mathcal{C}^\infty(\mathbb{R}^n)$$

Riemann sum for n-forms on $\mathbb{R}^n$ over a n-simplex

Every n-simplex S^n in $\mathbb{R}^n$ can be decomposed into sub-simplices.
This simplicial partition can be made arbitrarily fine.

We define the Riemann sum

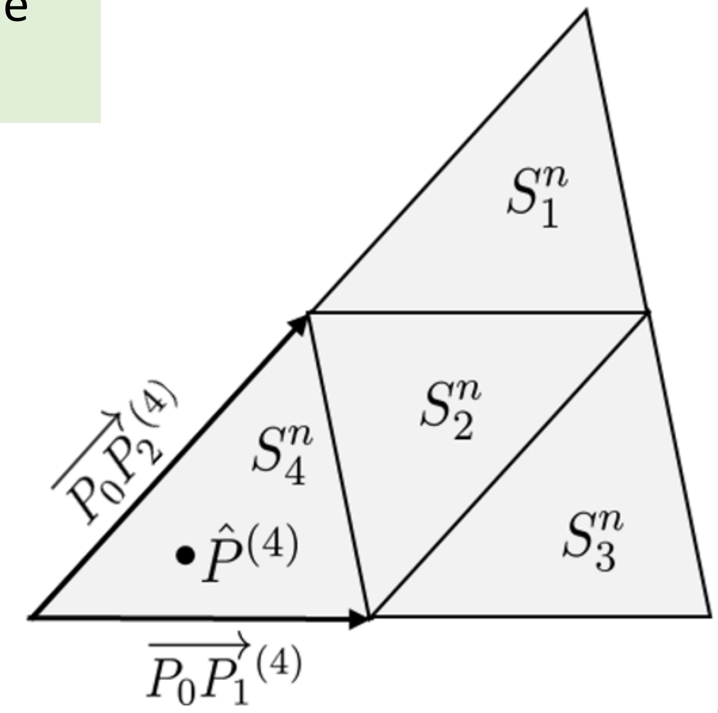
multicovector assigned by differential form at the point $\hat{P}^{(i)}$

$$\mathcal{R}[\omega] := \frac{1}{n!} \sum_i \omega_{\hat{P}^{(i)}} \left(\overrightarrow{P_0 P_1}^{(i)}, \ldots, \overrightarrow{P_0 P_n}^{(i)} \right)$$

factor due to partition in simplices

vectors spanning the i-th sub-simplex S_i^n.

evaluation point within the i-th sub-simplex S_i^n.



Integration of n-forms on $\mathbb{R}^n$ over a n-simplex

In the limit of an arbitrary fine partition, the Riemann sum can be evaluated as a Riemann integral.

$$\begin{aligned}
\mathcal{R}[\omega] &= \sum_i \frac{1}{n!}\,\omega_{\hat{P}^{(i)}}\left(\overrightarrow{P_0P_1}^{(i)},\ldots,\overrightarrow{P_0P_n}^{(i)}\right)\\
&= \sum_i \omega_{1\ldots n}\left(\hat{P}^{(i)}\right)\,\frac{1}{n!}\,\mathrm{Vol}_{\hat{P}^{(i)}}\left(\overrightarrow{P_0P_1}^{(i)},\ldots,\overrightarrow{P_0P_n}^{(i)}\right)\\
&= \sum_i \omega_{1\ldots n}(\hat{P}^{(i)})\,\mathrm{Vol}[S_i^n]\\
&\xrightarrow[\max(\mathrm{Vol}[S_i^n])\to 0]{} \int_{S^n} \omega_{1\ldots n}(x^1,\ldots,x^n)\,\mathrm{d}x^1\cdots\mathrm{d}x^n
\end{aligned}$$

Integration of n-forms on $\mathbb{R}^n$ over a n-simplex

We then can define the integration of $\omega \in \mathcal{F}^n(\mathbb{R}^n)$ over a n-simplex S^n as the demonstrated limiting process

$$\int_{S^n} \omega := \lim_{\max(\mathrm{Vol}[S_i^n]) \to 0} \mathcal{R}[\omega] = \underbrace{\int_{S^n} \omega_{1\ldots n}(x^1, \ldots, x^n)\, \mathrm{d}x^1 \cdots \mathrm{d}x^n}_{\text{❗ ordinary Riemannian integral ❗}}$$

If ω - i.e. $\omega_{1\ldots n}$ - is sufficiently smooth, this limiting process does not depend on the partition series, as known from Riemannian integration theory.

Simplices on manifolds

Let M be a n-dimensional manifold and $p \leq n$.

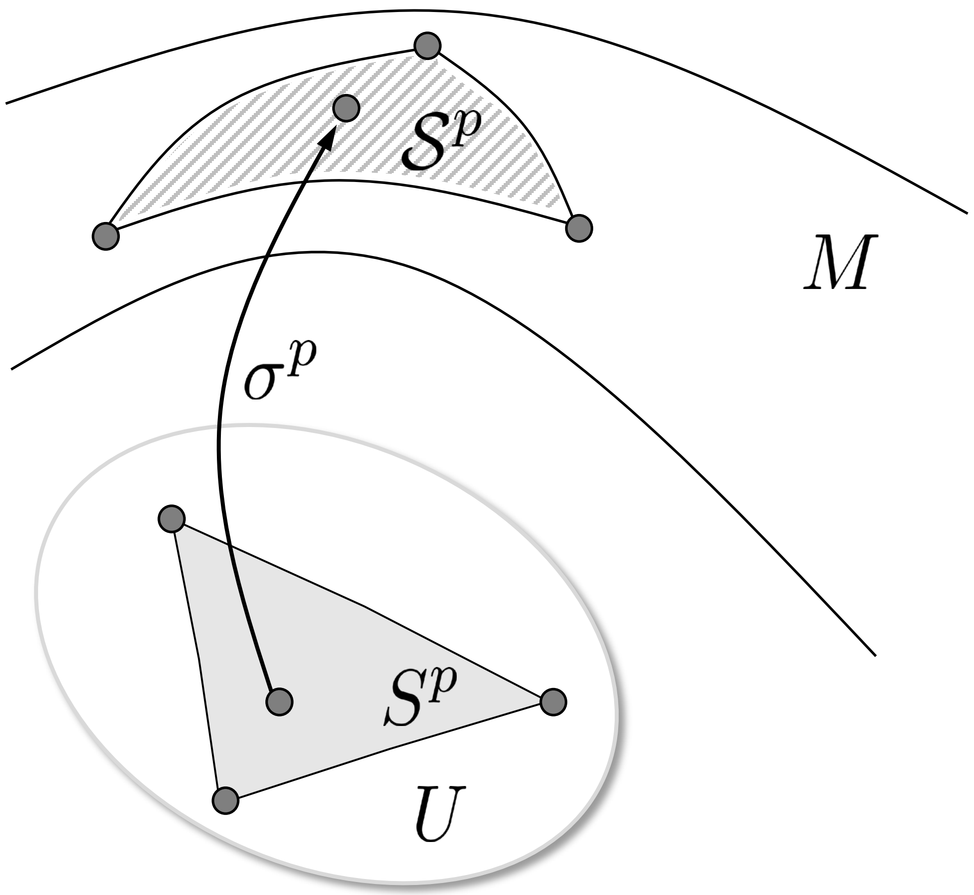
A p-simplex $\mathcal{S}^p$ on M is represented by a triple (S^p, U, σ^p), where

$S^p = \langle P_0, \ldots, P_p \rangle$ is a p-simplex in $\mathbb{R}^p$.

$U \subset \mathbb{R}^p$ is an open set such that $S^p \subset U$.

$\sigma^p : U \to M$ is a differentiable, injective, orientation preserving map between manifolds.

S^p is called a **reference simplex** to $\mathcal{S}^p$.



Integration over simplices on manifolds

We define the integral of a p -form on a p -simplex $\mathcal{S}^p$ on a manifold, represented by the triple (S^p, U, σ^p) , by pulling it back onto a p-form on $U \subset \mathbb{R}^p$ with $\mathrm{D}\sigma^{p*}$,

$$\int_{\mathcal{S}^p} \omega := \int_{S^p} \mathrm{D}\sigma^{p*}\omega$$

This definition corresponds to the original, intuitive definition with a Riemann sum.

$$\int_{\mathcal{S}^p} \omega = \lim_{\max(\mathrm{Vol}[S_i^p]) \to 0} \mathcal{R}\left[\mathrm{D}\sigma^{p*}\omega\right]$$

Independence of reference simplex

For more details see [6, Ch. 3.7 & 5.4]

The definition of the integral does not depend on the reference simplex.

To see this, go to the coordinate representation. A change of reference simplex corresponds to change in coordinates $(x^1, \ldots, x^n) \to (y^1, \ldots, y^n)$.

$$\int_{S^p} \omega = \int_{S^p} \left(\mathrm{D}\sigma^{p*}\omega\right)_{1\ldots p}\left(x^1, \ldots, x^p\right) \mathrm{d}x^1 \cdots \mathrm{d}x^p$$

coordinate transformation of the component function gives a factor

$$\det\left(\frac{\partial y^k}{\partial x^j}\right)$$

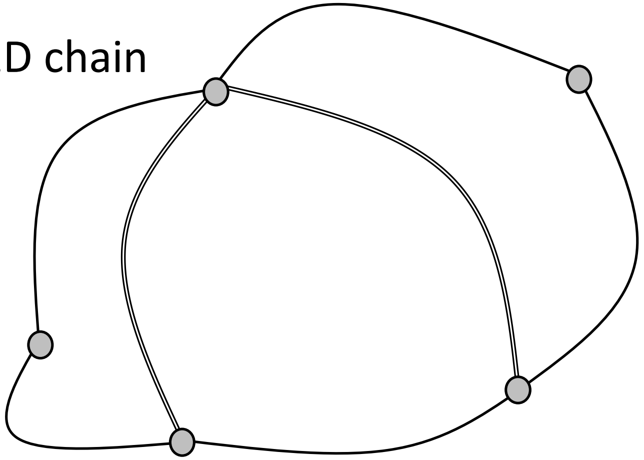
transformation formula for integrals gives a factor

$$\left|\det\left(\frac{\partial x^k}{\partial y^j}\right)\right|$$

The factors cancel out. The expression is invariant under change of coordinates!

It is a good exercise to proof this with rigor. A lot of things we discussed in the lecture will be used here.

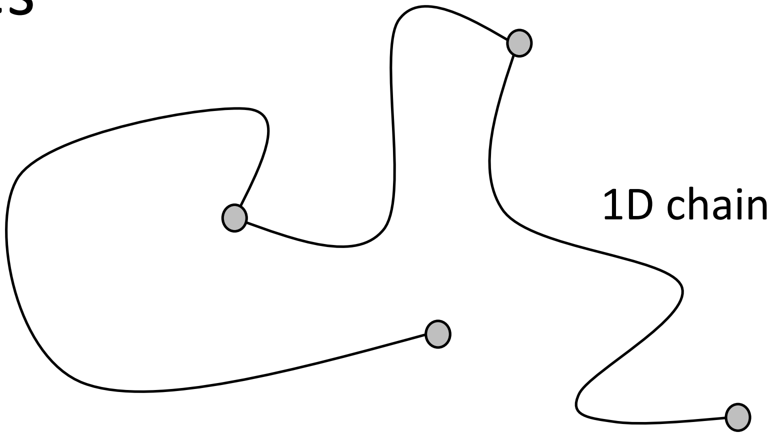
Chains



A **simplicial chain** is a compound of various simplices of the same dimension, such that [6]

1) they are disjoint or
2) the only intersection is along their boundaries

Formally one writes for a p-dimensional chain



$$K = \sum_i \lambda_i \mathcal{S}_i^p \quad \text{where} \quad \lambda_i \in \mathbb{Z}$$

The coefficients allow for example to take the same simplex twice. Negative coefficients denote a reversal of orientation.

The corresponding ∞-dimensional space is denoted $\mathcal{C}_p(M)$

Integration over chains

The integration of a p - form ω over a p - chain K is defined by the integration over the chain's simplices.

$$\int_K \omega := \sum_i \lambda_i \int_{\mathcal{S}_i^p} \omega$$

Note that convergence is not guaranteed if K is an infinite chain.

No problems occur if ω has a compact support and the individual integrals exist.

Triangulation of manifolds

We now formalized integration under the assumption that we can homomorphically represent the integration domain by a simplicial chain on a manifold.

For manifolds with dimension

$p \leq 3$ this is always possible

$p = 4$ there are some manifolds where this is not possible

$p > 4$ the situation is unknown

Can this integration concept be generalized such that it always works ?

The representation of an integration domain by a simplicial chain is not unique. How do we know the integral is always the same ?

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Measurable partition of a manifold

Let M be a n-dimensional manifold.

We call a family $(A_i, U_i, \phi_i)_{i\in\mathbb{N}}$ where

$U_i \subset M$ is an open set

$\phi_i : U_i \to \mathbb{R}^n$ is a chart

$A_i \subset U_i$

a **measurable partition** of M if

$(A_i)_{i\in\mathbb{N}}$ is a partition of M

$\phi_i(A_i)$ is a Lebesgue measurable set in $\mathbb{R}^n$ for all $i \in \mathbb{N}$.

For any manifold there exists at least one such partition, due to the second countability property. See [2, Ch. 5]

The decomposition in simplices, if possible, would be such a partition.

Integration over entire manifolds

A n-form ω on a n-dimensional manifold M is called integrable, if for any measurable partition $(A_i, U_i, \phi_i)_{i\in\mathbb{N}}$ of M all of the functions

$$f_{\omega,\phi_i} = \omega_{1\ldots n} \circ \phi_i^{-1} \qquad \text{with} \quad \omega_{1\ldots n} = \omega\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$$

are Lebesgue-integrable over $\phi_i(A_i)$, and if

$$\sum_{i=1}^{\infty} \int_{\phi_i(A_i)} \left| f_{\omega,\phi_i}(x^1, \ldots, x^n) \right| \, \mathrm{d}x^1 \cdots \mathrm{d}x^n < \infty$$

Then we define

$$\boxed{\int_M \omega := \sum_{i=1}^{\infty} \underbrace{\int_{\phi_i(A_i)} f_{\omega,\phi_i}(x^1, \ldots, x^n) \, \mathrm{d}x^1 \cdots \mathrm{d}x^n}_{\text{Ordinary Lebesgue integral}}}$$

Integrability of forms on manifolds

1) One can show that if the integral of ω over M exists for a single measurable partition $(A_i, U_i, \phi_i)_{i\in\mathbb{N}}$ of M it exists for all partitions and it **always has the same value**. See [2, Ch. 5]

2) If ω is a differential form with **compact support** then it holds that :

ω is integrable over M	$\Leftrightarrow$	The functions $f_{\omega,\phi}$ are integrable over $\phi(U)$ for **every chart** $\phi : U \to \mathbb{R}^n$ in an **arbitrary** atlas.

! 3) To integrate p-forms on M over p-dimensional submanifolds $N \subset M$, use the pullback map $\mathrm{D}\psi^*$ induced by the embedding $\psi : N \hookrightarrow M$. !

Integration over compact domains

Let N be a p-dimensional manifold and ω a p-form on N.

Consider a compact integration domain $K \subset N$ that allows a simplicial partition.

Integration over more general, so called "regular" domains, can be defined in an analogous way.
See [4, p. 281]

Then one can formally define

$$\int_K \omega := \int_N \mathbb{1}_K \cdot \omega \ ,$$

where $\mathbb{1}_K$ denotes the characteristic function to $K \subset N$.

Properties of the integral

Consider integration domains K, K' with $K \cap K' = \emptyset$.

Let ω, η be p - forms and $\alpha, \beta \in \mathbb{R}$.

$$\int_{K \cup K'} \omega = \int_K \omega + \int_{K'} \omega$$

Additivity

$$\int_K a\,\omega + b\,\eta = a \int_K \omega + b \int_K \eta$$

Linearity

Transformation formula [2, Ch. 5.5]

Let K, K' be p-dimensional integration domains and $\psi : K' \to K$ an orientation-preserving diffeomorphism.

Let ω be a p-form on a n- dimensional manifold $M \supset K$, $p \leq n$.

Then it holds:

ω is integrable on K $\Leftrightarrow$ $\mathrm{D}\psi^*\omega$ is integrable on K'

and

$$\int_K \omega = \int_{K'} \mathrm{D}\psi^*\omega$$

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Line integral : electric voltage along a curve

Let $\mathcal{E}$ be a 1-form on a 3-dimensional differentiable manifold M and C a curve on M. W.l.o.g. we assume:

I

$C = \psi([0,1])$ where $\psi : \mathbb{R} \supset [0,1] \to M$

$\exists \phi : V \subset M \to \mathbb{R}^3$ chart, such that $C \subset V$

Domain of chart, avoid confusion with U

Then we can perform the integration with

$$U(C) := \int_C \mathcal{E} = \int_{\psi([0,1])} \mathcal{E} = \int_{[0,1]} \mathrm{D}\psi^* \mathcal{E}$$

We now have to compute the pullback into the parameter space.

Line integral evaluation – parameter space

II Due to our assumption $C \subset V$ we can write

$$\mathcal{E} = \mathcal{E}_1 \mathrm{d}x^1 + \mathcal{E}_2 \mathrm{d}x^2 + \mathcal{E}_3 \mathrm{d}x^3 \qquad \text{and} \qquad \begin{pmatrix} C^1(t) \\ C^2(t) \\ C^3(t) \end{pmatrix} = \phi \circ \psi(t)$$

Coordinate representation of the curve

III With this we can perform the pullback

1) $$\mathcal{E}_i(P) \xrightarrow{\psi} \tilde{\mathcal{E}}_i(t) := \mathcal{E}_i(\psi(t))$$

2) $$\mathrm{D}\psi^* \mathrm{d}x^i = \frac{\partial C^i(t)}{\partial t} \mathrm{d}t$$

Component function of pullback to parameter t.

3) $$\mathrm{D}\psi^* \mathcal{E} = \left(\tilde{\mathcal{E}}_1(t) \frac{\partial C^1(t)}{\partial t} + \tilde{\mathcal{E}}_2(t) \frac{\partial C^2(t)}{\partial t} + \tilde{\mathcal{E}}_3(t) \frac{\partial C^3(t)}{\partial t} \right) \mathrm{d}t = (\mathrm{D}\psi^* \mathcal{E})_t \, \mathrm{d}t$$

Line integral evaluation – explicit integral

After the pullback use the definition of an integral of a 1-form over $\mathbb{R}$.

$$U(C) := \int_C \mathcal{E} = \int_{\psi([0,1])} \mathcal{E} = \int_{[0,1]} \mathrm{D}\psi^* \mathcal{E} = \int_0^1 (\mathrm{D}\psi^* \mathcal{E})_t \, \mathrm{d}t$$

The integral is then given as

IV

$$\boxed{U(C) = \int_C \mathcal{E} = \int_0^1 \underbrace{\left(\tilde{\mathcal{E}}_i(t) \frac{\partial C^i(t)}{\partial t} \right) \mathrm{d}t}_{\text{Classical vector analysis:}}}$$

Classical vector analysis:

$$E \cdot \frac{\partial \vec{r}}{\partial t} \mathrm{d}t = E \cdot \mathrm{d}s \,, \quad \mathcal{E} = {}^1 E$$

This easily generalizes to n-dimensional manifolds

Surface integral : magnetic flux across a surface

I

Let $\mathcal{B}$ be a 2-form on a 3-dimensional differentiable manifold M and A a surface on M. We assume:

$A = \psi(\alpha)$ where $\psi : \mathbb{R}^2 \supset \alpha \to M$

$\exists \phi : U \subset M \to \mathbb{R}^3$ chart, such that $A \subset U$.

The we can perform the integration with

$$\Phi(A) := \int_A \mathcal{B} = \int_{\psi(\alpha)} \mathcal{B} = \int_\alpha \mathrm{D}\psi^* \mathcal{B}$$

We now have to compute the pullback into the parameter space.

Surface integral evaluation – parameter space

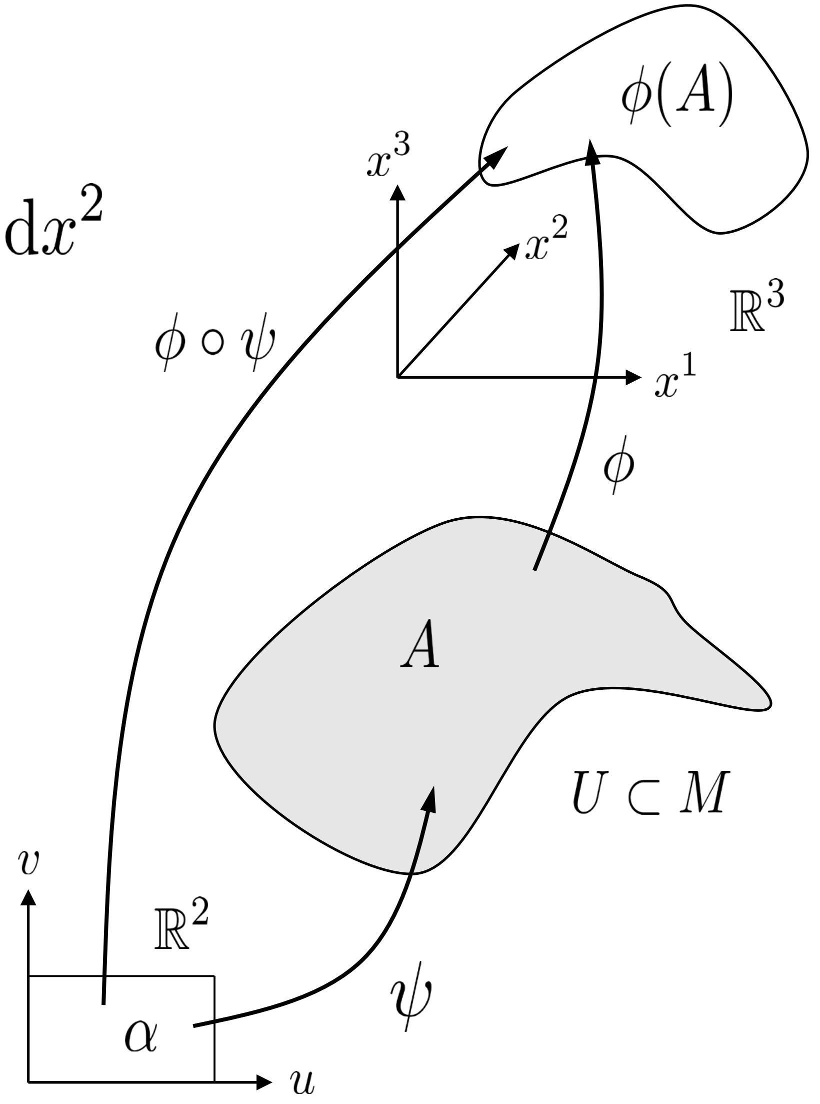
Due to our assumption $A \subset U$ we can write

$$\mathcal{B} = \mathcal{B}_{23}\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \mathcal{B}_{31}\, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + \mathcal{B}_{12}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2$$

and for the coordinate representation of the surface we have

$$\begin{pmatrix} A^1(u,v) \\ A^2(u,v) \\ A^3(u,v) \end{pmatrix} = \phi \circ \psi(u,v)$$

$\alpha \subset \mathbb{R}^2$ is the parameter space of the surface.



Surface integral evaluation – pullback to param. space

III

1) $$\mathcal{B}_I(P) \xrightarrow{\psi} \tilde{\mathcal{B}}_I(u,v) := \mathcal{B}_I(\psi(u,v))$$

2) $$\mathrm{D}\psi^*\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 = \left(\frac{\partial A^2}{\partial u}\mathrm{d}u + \frac{\partial A^2}{\partial v}\mathrm{d}v\right) \wedge \left(\frac{\partial A^3}{\partial u}\mathrm{d}u + \frac{\partial A^3}{\partial v}\mathrm{d}v\right)$$

$$= \left(\frac{\partial A^2}{\partial u}\frac{\partial A^3}{\partial v} - \frac{\partial A^2}{\partial v}\frac{\partial A^3}{\partial u}\right)\mathrm{d}u \wedge \mathrm{d}v \qquad = \frac{\partial(A^2, A^3)}{\partial(u,v)}\mathrm{d}u \wedge \mathrm{d}v$$

3) $$\mathrm{D}\psi^*\mathcal{B} = \left(\tilde{\mathcal{B}}_{23}\frac{\partial(A^2, A^3)}{\partial(u,v)} + \tilde{\mathcal{B}}_{31}\frac{\partial(A^3, A^1)}{\partial(u,v)} + \tilde{\mathcal{B}}_{12}\frac{\partial(A^1, A^2)}{\partial(u,v)}\right)\mathrm{d}u \wedge \mathrm{d}v$$

$$= \left(\mathrm{D}\psi^*\mathcal{B}\right)_{uv}\mathrm{d}u \wedge \mathrm{d}v$$

Component function of pullback to parameters u, v.

Surface integral evaluation – explicit integral

After the pullback use the definition of an integral of a 2-form over $\mathbb{R}^2$:

$$\Phi(A) := \int_A \mathcal{B} = \int_{\psi(\alpha)} \mathcal{B} = \int_\alpha \mathrm{D}\psi^* \mathcal{B} = \int_u \int_v (\mathrm{D}\psi^* \mathcal{B})_{uv} \, \mathrm{d}v \, \mathrm{d}u$$

IV

$$= \int_u \int_v \underbrace{\left(\tilde{\mathcal{B}}_{23} \frac{\partial(A^2, A^3)}{\partial(u,v)} + \tilde{\mathcal{B}}_{31} \frac{\partial(A^3, A^1)}{\partial(u,v)} + \tilde{\mathcal{B}}_{12} \frac{\partial(A^1, A^2)}{\partial(u,v)} \right) \mathrm{d}u \wedge \mathrm{d}v}$$

Classical vector analysis:

$$B \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) \mathrm{d}u \, \mathrm{d}v = B \cdot \mathrm{d}A \,, \quad \mathcal{B} = {}^2 B$$

Volume integral : electric charge in a volume

Let $\mathcal{Q}$ be a 3-form on a 3-dimensional differentiable manifold M and V a volume in M. We assume:

I $\quad V = \psi(\nu) \qquad$ where $\quad \psi : \mathbb{R}^3 \supset \nu \to M$

II $\quad \exists \phi : U \subset M \to \mathbb{R}^3$ chart, such that $V \subset U$. We can write

III $$\mathcal{Q} = \mathcal{Q}_{123}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \quad \Rightarrow \quad \mathrm{D}\psi^* \mathcal{Q} = \tilde{\mathcal{Q}}_{123}(u,v,w)\, \frac{\partial(x^1,x^2,x^3)}{\partial(u,v,w)} \mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w$$

and we get for the integral

IV $$Q(V) := \int_{V=\psi(\nu)} \mathcal{Q} = \int_{\nu} \mathrm{D}\psi^* \mathcal{Q} = \int_u \int_v \int_w \underbrace{\tilde{\mathcal{Q}}_{123}(u,v,w) \frac{\partial(x^1,x^2,x^3)}{\partial(u,v,w)}\, \mathrm{d}w\, \mathrm{d}v\, \mathrm{d}u}$$

Classical vector analysis: $\rho\, \mathrm{d}\tau\,, \quad \mathcal{Q} = {}^3\rho$

Wrap up: Explicit integration in coordinates

The explicit integration of p-forms on a p-dimensional domain of an n-dimensional manifold boils down to four simple steps:

I Split integration domain into subdomains, such that every subdomain can be covered by a chart and can be parametrized.

II Find a parametrisation and coordinate representation of your integration domains.

III Perform a pullback into the parameter domain.

IV Calculate the p-fold integral of the pulled back component function over the parameter domain.

Relation to integrals from vector analysis in $(\mathbb{R}^3, \cdot, \times)$

By inspection of the coordinate expressions we obtained for the integrals over lines, surfaces and volumes we can recognize the well-known integrals from vector analysis in $(\mathbb{R}^3, \cdot, \times)$.

$$\int_C \mathcal{E} = \int_C {}^1E = \int_C E \cdot \mathrm{d}s$$

Line integral

$E \in \mathcal{X}(\mathbb{R}^3)$
$\mathrm{d}s$ line element

$$\int_A \mathcal{B} = \int_A {}^2B = \int_A B \cdot \mathrm{d}A$$

Surface integral

$B \in \mathcal{X}(\mathbb{R}^3)$
$\mathrm{d}A$ surface element

$$\int_V \mathcal{Q} = \int_V {}^3\rho = \int_V \rho \, \mathrm{d}\tau$$

Volume integral

$\rho \in \mathcal{C}^\infty(\mathbb{R}^3)$
$\mathrm{d}\tau$ volume element

Wrap up: Integral Quantities

		physical dim.
$U(C) = \int_C \mathcal{E}$	electrical voltage along C	U
$V(C) = \int_C \mathcal{H}$	magnetic voltage along C	I
$\Phi(A) = \int_A \mathcal{B}$	magnetic flux across A	UT
$\Omega(A) = \int_A \mathcal{D}$	electric flux across A	IT
$I(A) = \int_A \mathcal{J}$	electric current across A	I
$Q(V) = \int_V \mathcal{Q}$	electric charge in V	IT

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