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Low-energy effective description of light pseudo-scalar mesons in $SO(N)$-like dark QCD

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ABSTRACT

Recently, theories of strongly interacting massive particles (SIMP) gained a lot of attention as explanation for dark matter on the level of particle physics. This class of theories features strong self-interactions among the dark matter particles, resolving the cusp vs. core problem and offer an explanation for the stability of dark matter. In this thesis I turn towards realizations of SIMP dark matter through non-abelian gauge theories with two Dirac fermions transforming under a real representation of the gauge group. The properties of the lightest pseudo-scalar meson, which constitute stable dark matter candidates in these theories, are studied. Special focus is put on the physics of a flavor-singlet meson, called $\tilde{\eta}$, which is the analogue of the η' in QCD. A representation theoretical criterion, quantifying the phenomenological importance of the $\tilde{\eta}$-meson for dark matter phenomenology, is presented. I argue that, once the dark sector is coupled to the standard model via a dark photon mediator, the dark $\tilde{\eta}$ meson becomes relevant for the dark matter phenomenology of $SO(N)$-vector DM. Further, for the gauge theories investigated, a topological obstruction spoils the geometrical construction and classification of Wess-Zumino-Witten terms in the low energy effective description. These terms play the dominant role in the SIMP freeze-out regime and are thus of utter importance for dark matter phenomenology. I point towards an alternative construction of Wess-Zumino-Witten terms, which provides an expression useful for practical applications, even in situations where Witten's geometrical construction does not seem to be applicable.

KURZZUSAMMENFASSUNG

Theorien stark wechselwirkender massiver Teilchen (SIMP) gewannen in letzter Zeit viel Aufmerksamkeit als Erklärung für Dunkle Materie (DM) auf der Ebene elementarer Teilchenphysik. Diese Klasse von Theorien beinhaltet Dunkle Materie Teilchen, welche stark genug untereinander wechselwirken um das sogenannte "Spitz gegen Flach" (Cusp vs. Core) Problem zu lösen und bieten eine Erklärung für die augenscheinliche Stabilität Dunkler Materie. In dieser Arbeit befasse ich mich mit der Realisierung von SIMP DM durch nichtabelsche Eichtheorien mit zwei Dirac Fermionen, welche unter einer reellen Repräsentation der Eichgruppe transformieren. Ich untersuche die Eigenschaften der leichtesten pseudo-skalaren Mesonen in diesen Theorien. Diese sind die Kandidaten für die stabile DM. Ein besonderes Augenmerk wird dabei auf das $\tilde{\eta}$ Meson gelegt, welches in der trivialen Repräsentation unter allen globalen internen Quantensymmetrien transformiert. Dieses Teilchen ist das Analogon zum η' Meson in der Quantenchromodynamik (QCD). Ich präsentiere ein repräsentationstheoretisches Kriterium, welches Aussage über die phänomenologische Relevanz dieses Teilchens im Kontext von SIMP DM macht. Ich argumentiere dass dieses Teilchen in $SO(N)$-Vector Eichtheorien relevant für die DM Phänomenologie wird, wenn der dunkle Sektor durch ein dunkles Photon an das Standardmodell der Teilchen Physik (SM) gekoppelt wird. Weiters wird eine alternative Konstruktion für Wess-Zumino-Witten (WZW) Terme in der niedrig Energie Beschreibung dieser DM Theorien diskutiert. Dies ist nötig da Wittens geometrische Konstruktion aufgrund einer topologischen Komplikation nicht anwendbar zu sein scheint. Diese Terme spielen eine dominante Rolle für das Ausfrieren der DM im SIMP Regime und sind daher von großer phänomenologischer Wichtigkeit. Die alternative Konstruktion liefert einen Ausdruck der für die praktische Anwendung brauchbar ist.

Contents

1 Introduction

One of the most prominent problems of modern physics is the mystery of dark matter (DM). Despite ample observational evidence from astrophysics [1], there is yet no commonly accepted explanation on the level of particle physics. We know very little about the nature of DM, except that DM interactions with the standard model (SM) of particle physics are strongly suppressed, if not absent at all. It is mostly agreed on by physicists that DM today is cold, in the sense that DM particles are massive and move at non-relativistic speeds in the current state of the universe (in contrast to neutrinos or photons). A plethora of models have already been proposed, ranging from axion-like particles of mass $m_{\mathrm{DM}} \lesssim \mu eV$ [2][3] to primordial black holes up to masses $m_{\mathrm{DM}} \approx 10^{18}$kg [4]. The dark matter scenario of weakly interacting massive particles (WIMP) has been investigated most intensively so far [5]. The standard model of cosmology, the "Λ cold dark matter" (ΛCDM) model, assumes that DM is cold and interacts only gravitationally. Estimating the abundance of DM today, within ΛCDM, we find that it makes up about 26% of the energy budget of the universe [6]. However, the cold DM regime of the ΛCDM model fails to explain certain problems in structure formation like the cusp vs core problem [7], where the DM density profiles of dwarf galaxy halos come out more cusp in N-body simulations than observations would suggest. In 2000, Spergel and Steinhardt [8] put forward the idea that the introduction of sufficiently strong and short-ranged self-interactions among the DM particles might resolve the cusp vs. core problem. Recent investigations [9][10] put further constraints on the self-interactions of DM and suggest a velocity dependence of the self-interactions [11][12] in order to resolve the structure formation problems at various scales. As far as naturalness is concerned, composite dark matter models are favored over models with elementary particles in order to reproduce the correct self-interaction crosssections, while matching the observed relic density. Large self-interaction crosssections arise due to the underlying strong force, responsible for the bound state formation, and the spatial extent of the bound states [13]. The composite nature of DM further implements a velocity dependence of the self-interaction crosssection.

A class of particle physics models [14][15], featuring such strongly coupled composite particles, may be realized by QCD-like models. These are models of fermions, transforming under a non-abelian gauge group and exhibiting a confining phase transition, resulting in phase of spontaneously broken chiral symmetry. In these models DM is made out of the pseudo Nambu-Goldstone particles (pNGBs) of the spontaneously broken (approximate) symmetry. These particles are dubbed "dark pions", in analogy to QCD. An accidental flavor symmetry prevents these particles from decaying, offering an explanation for why we see stable DM today. Furthermore, QCD-like models feature a $3 \to 2$ cannibalization process, that may be used to set the relic density via a freeze-out process. These interactions are implemented by the so-called Wess-Zumino-Witten (WZW) term [16][17], in the low energy effective description. When coupling the dark sector to the SM via a dark photon mediator, the $3 \to 2$ process allows to reproduce the relic abundance in a region of parameter space, where the dark photon interactions to the SM are sufficiently suppressed to evade direct detection constraints, while still being able to maintain kinetic equilibrium between the dark sector and the SM. This regime of dark matter is dubbed "strongly interacting massive particles" (SIMP) [18]. On the flip side, a mediator is required to dump heat from the dark sector into the SM, since the $3 \to 2$ process necessarily injects kinetic energy into the dark sector. Such a heat up would be in conflict with certain constraints from structure formation [18] and thus motivates the existence of a mediator particle, beyond the hope for a direct detection signal.

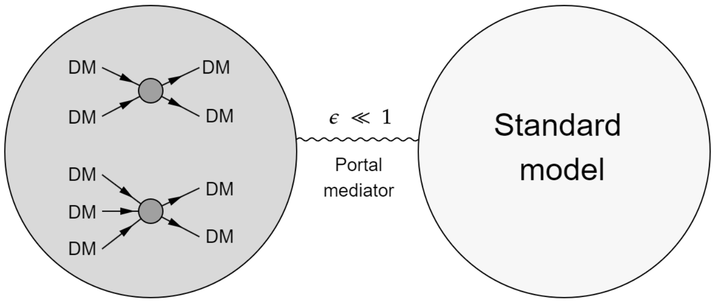


Figure 1.1: Schematic representation of the SIMP mechanism. The dark sector features $2 \to 2$ self-interaction, used to resolve the cusp vs. core problem and maintain thermal equilibrium within the dark sector. The $3 \to 2$ cannibalization process is used to set the relic density. The mediation to the SM is controlled by a parameter ε. DM-SM interactions maintain thermal equilibrium between the dark sector and the SM until freeze-out. This graphic is a copy of the one in the original SIMP paper [18].

While these models seem very tempting, the shear complexity of such a dark sector should not be underestimated. The amount of physical bound states is massive and dependent on the details of the theory. States other than the dark pions may become relevant for DM physics. Most investigations so far use effective field theory approaches and chiral perturbation theory to describe the dynamics of the relevant parts of the particle spectrum. However, it is hard to say in general, which parts of the spectrum will be relevant, if we do not know the details of the mass spectrum, which depends on the details of the underlying gauge theory. There have been novel approaches [19][20] in combining effective field theories and lattice field theory methods in the context of DM, in order to constraint or calculate the mass spectrum and low energy effective constants (LEC) for an effective description of cold DM from first principles like lattice QFT. These studies focus on symplectic gauge theories, especially $Sp(4)$, which have been very little studied so far. Most interest in symplectic gauge theories came from the composite Higgs community [21][22].
In this study we will focus on Dirac fermions transforming under a finite dimensional, unitary, real representation of a gauge group. The defining feature of such representation is that it is unitary-equivalent to its complex conjugate representation. Thus, there is no way to distinguish particles and anti-particles with respect to this gauge group on physical grounds. The prototypical theory is an $SO(N_C)$ gauge theory, with fermions transforming under the so-called vector representation of $SO(N_C)$. For this representation we take the representation space to be $\mathbb{C}^{N_C}$ and the fermions transform as a column vector, with an orthogonal matrix $U^{\text{vec}} = U^{\text{vec}\top}$ acting on the N_C gauge indices. There are plenty of other real representations e.g. the antisymmetric, traceless, rank 2 tensor (AT2T) representation of $Sp(2N_C)$ or every adjoint representation of the classical groups. As we progress with the thesis, we highlight differences between the prototypical $SO(N_C)$-vector theory and other real representation gauge theories, addressing the question of possibilities to construct SIMP dark matter other than the suggestions in [15]. Real representation gauge theories have been studied very little in the context of DM. Lattice studies for $Sp(4)$-AT2T exist in the composite Higgs community [23]. We too, have special interest in $Sp(4)$ theory, with fermions transforming in the AT2T representation, since it is well suited to study real representation theories on the lattice. The minimal gauge theory with real representations, allowing for a WZW term, would be a parity violating theory of 3 left-handed Weyl fermions. However, since we have future lattice studies on our mind and want parity to be a good symmetry, we consider Dirac fermions rather than Weyl fermions. The case of $N_F = 2$ Dirac flavors is interesting, because it is the smallest number of Dirac flavors for which a WZW term can exist. This minimal case will be considered throughout the thesis.

This thesis tries to accomplish two things. On the one hand we reinvestigate the derivations of the low energy effective description for QCD-like theories with a dark photon mediator, taking into account the presence of another light pseudo-scalar flavor-singlet meson, called $\tilde{\eta}$, which is the analog of the η' in QCD. We argue that this particle will affect the freeze-out regime due to an anomalous decay channel, once we couple the dark sector to the SM via a dark photon. Hence, it will also influence our understanding of the theories parameters space. We derive a representation theoretic criterion that characterizes for which theories the physics of the $\tilde{\eta}$ meson becomes important for DM. To the best of our knowledge, the role of this particle for DM physics was not investigated within the SIMP model so far, mostly because its QCD analog is rather heavy. However, no statements exist for general theories. On the other hand, we will lay the foundation for lattice studies of these strong dark sectors by offering classifications and construction recipes for interpolating operators of all the relevant particle states. Further, we provide some technical details on the structure of continuous and discrete symmetries of the underlying UV theory.

As another motivation for the restriction to two Dirac fermions we point out that in this case a topological obstruction renders the standard construction and classification of WZW terms by Witten, Weinberg and D'Hoker [17][24][25] inconclusive. However, since these terms are essential for the SIMP regime, we point towards a different approach, first explored by Chu, Ho and Zumino [26], in order to construct the WZW term, even if the standard approach is not available. Our work also answers a footnote remark in [27], concerning the applicability of their construction of the WZW term. The $SO(N)$-vector theory was studied to a large extend already in [28]. However, the authors do not mention the physics of the $\tilde{\eta}$ meson, which, as we argue in this thesis, is of particular significance to $SO(N_C)$-vector DM, while of subordinate importance to all other real representation gauge theories.

The structure of the thesis is as follows. In chapter 2 we will provide some introduction into well known concepts of QFT, such as anomalies and spontaneous symmetry breaking. This is mainly thought as a review and to fix convention. In section 3 we discuss the UV-description of the dark sector. We will treat the pure strong sector separately from the dark photon, in order to make the results usable for lattice field theory approaches. We discuss the UV physics relation between the $\tilde{\eta}$ meson, the axial anomaly and the theta angle in Yang-Mills theory. Special emphasis is put on the symmetry realizations in the UV, which become important when modelling the low energy effective description in section 4. There we will work out the anomalous and non-anomalous parts of the low-energy effective description separately, focusing on the physics of the $\tilde{\eta}$ meson in a 't Hooft large N_C expansion. The problem with the standard construction of the WZW term is pointed out, followed by an alternative approach for the construction. In section 5 we present an argument for the phenomenological importance of $\tilde{\eta}$ and why we think it should be taken into account for DM phenomenology. We conclude in section 6. The appendix contains several technical details, of which most are simply a refined version of things that can be found in standard textbooks.

Conventions

Indices

Throughout the thesis we maintain a consistent index convention.

Spacetime index	μ, ν, ρ, σ	$0, \ldots, 3$
Gauge / Color index	α, β, γ	$1, \ldots, \dim G$
Dirac flavor index	g, h	$1, \ldots, N_F = 2$
Weyl flavor index	i, j, k, l, m, n	$1, \ldots, N_f = 2N_F = 4$
$U(N_f)$ generator index	M, N, K	$(0), 1, \ldots, \dim SU(N_f)$
Broken generator index	a, b, c	$(0), 1, \ldots, \dim SU(N_f)/SO(N_f)$
Unbroken generator index	A, B, C	$\dim SU(N_f)/SO(N_f), \ldots, \dim SU(N_f)$

The position of the index is of importance throughout the thesis. Typically, each index belongs to some basis components of representation space. The indices indicate contravariant (upper index) or covariant (lower index) transformation behavior of components under the group, acting on the respective representation space.

Spacetime signature

We use the spacetime convention common for particle physics.

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

Summation convention

We adopt Einsteins summation convention, in the sense that summation over the whole range of an index is implicitly implied, if it appears in a product as a pair of an upper and lower index. The summation convention is only violated if explicitly indicated in the text. There is no implicit summation over an index if it appears in a pair of only lower or only upper indices.

$$\begin{aligned} \epsilon^\alpha T_\alpha &= \sum_\alpha \epsilon^\alpha T_\alpha \\ \epsilon^\alpha T^\alpha &\quad \text{no summation} \\ \epsilon_\alpha T_\alpha &\quad \text{no summation} \end{aligned}$$

Pauli matrices

For the Pauli- and γ-matrices we adopt the conventions used in appendix A of [21].

$$\overline{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \overline{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \overline{\sigma}^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \overline{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.1}$$

$$\sigma^\mu = g^{\mu\mu}\overline{\sigma}^\mu \qquad \overline{\sigma}_\mu = g_{\mu\nu}\overline{\sigma}^\nu$$

Chiral representation of γ matrices

A chiral basis for Dirac spinors consists of eigenstates of the chirality element γ^5 of the Clifford algebra [29] of $g_{\mu\nu}$. In even dimensions the representation space decomposes into left-handed (positive) and right-handed (negative) eigenspinors of the chirality element γ^5. The γ-matrices then take on a simple block form. Our definition of the charge conjugation matrix C for fermions, used in equation (3.8), differs from that in appendix A of [21], but is consistent in the sense that charge conjugation of a Dirac fermion $q_C := Cq^*$ gives the same result.

$$
\begin{aligned}
\gamma^\mu &= \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} & \gamma_5 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} & C &= \begin{pmatrix} 0 & -i\,\overline{\sigma}^2 \\ i\,\overline{\sigma}^2 & 0 \end{pmatrix} \\
\gamma_\mu &= g_{\mu\nu}\gamma^\nu & \gamma^5 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 & C &= i\gamma^2 \\
& & &= \gamma_5 & &= -i\gamma_2
\end{aligned}
\tag{1.2}
$$

If we define $E = i\,\overline{\sigma}^2$ the following useful relations hold

$$E^{-1}\,\overline{\sigma}^\mu\,E = g^{\mu\mu}(\overline{\sigma}^\mu)^\top = (\sigma^\mu)^\top \tag{1.3}$$

$$C^\dagger\gamma^0\gamma^\mu C = (\gamma^0\gamma^\mu)^\top \tag{1.4}$$

2 Preliminaries

This section is a review of well established concepts, potentially taught in more advanced courses on QFT. None of this is original. At the beginning of each section we state some references to modern standard literature for more details, while referencing some original papers in between.

2.1 Elements of representation theory

Most of the material presented in this section can be found in [30]. We summarize here only the parts we explicitly require, omitting all details on the theory of roots and weights, being the representation theoretical backbone of this section. In the following, G denotes a Lie-group and $\mathfrak{g}$ its corresponding Lie-algebra. Since we deal with gauge theories, we restrict our discussion to (semi-)simple, compact groups. These allow for finite dimensional, unitary representations $\mathcal{R} : G \to \mathrm{Aut}\,(V)$, with V the representation space. For any $g \in G$ a transformation matrix $U^{\mathcal{R}} := \mathcal{R}[g]$ of such a representation may be parametrized via the exponential map

$$U^{\mathcal{R}} = \exp\left\{i\epsilon^{\alpha}T^{\mathcal{R}}_{\alpha}\right\} \tag{2.1}$$

where $T^{\mathcal{R}}_{\alpha}$ are hermitian matrices and ϵ^{α} are some real parameters. In the case of gauge groups or local symmetries the $\epsilon^{\alpha} : \mathcal{M} \to \mathbb{R}$ constitute functions mapping spacetime $\mathcal{M}$ onto the real numbers. The generators $T^{\mathcal{R}}_{\alpha}$ span a representation $\mathcal{R}_{*}$ of the Lie-algebra. The structure constants ${C^{\gamma}}_{\alpha\beta}$ of the Lie-algebra may be obtained from the commutation relations of the generators

$$\left[T^{\mathcal{R}}_{\alpha}, T^{\mathcal{R}}_{\beta}\right] = i{C^{\gamma}}_{\alpha\beta}T^{\mathcal{R}}_{\gamma} \tag{2.2}$$

This relation holds, independent of the representation, but the values of the structure constants depend on the choice of basis of the Lie-algebra. The structure of a Lie-algebra does not induce a notion of length, additional structure has to be added. However, for compact Lie-groups a natural choice for a metric exists, the so-called invariant Killing form κ. Its components may be defined by taking the generators into the adjoint representation and trace over their product

$$\kappa_{\alpha\beta} = \lambda\,\mathrm{tr}\left\{T^{adj}_{\alpha}T^{adj}_{\beta}\right\} \tag{2.3}$$

with the scale factor λ being a matter of convention. Nevertheless, one may also choose different notions of length and geometry. For example in the defining representation of $SU(N)$, one may use the Frobenius product $\mathrm{tr}\left\{T^{F}_{\alpha}T^{F}_{\beta}\right\}$ for hermitian matrices. It is important to note that such traces of matrix products are not preserved upon a change of representation and an irreducible representation may be characterized by its so-called Dynkin index $T_{\mathcal{R}}$.

$$\mathrm{tr}\left\{T^{\mathcal{R}}_{\alpha}T^{\mathcal{R}}_{\beta}\right\} = T_{\mathcal{R}}\ \kappa_{\alpha\beta} \tag{2.4}$$

Another important quantity to characterize irreducible representations is the quadratic Casimir operator

$$C^{\mathcal{R}}_{2} := \kappa^{\alpha\beta}T^{\mathcal{R}}_{\alpha}T^{\mathcal{R}}_{\beta} \tag{2.5}$$

defined via the inverse Killing metric $\kappa^{\alpha\beta}$, satisfying $\kappa^{\alpha\beta}\kappa_{\beta\gamma} = \delta^{\alpha}_{\gamma}$. This operator commutes with all the generators $T^{\mathcal{R}}_{\alpha}$ and thus, by Schur's Lemma, must be proportional to the unity operator on the representation space V of an irreducible representation, i.e. $C^{\mathcal{R}}_{2} = c_{\mathcal{R}}\mathbb{1}$. The proportionality

Table 2.1: Dimension, Casimir number and Dynkin index of various groups and representations [31][32]. The $SO(N)$-vector representation is defined in section 1. The $Sp(2N)$-AT2T representation is explained in appendix G.

G	$\mathcal{R}$	dim$\mathcal{R}$	$T_\mathcal{R}$	$c_\mathcal{R}$	complex or real
$SU(N)$	fundamental	N	$\frac{1}{2}$	$\frac{N^2-1}{2N}$	complex
$SU(N)$	adjoint	N^2-1	N	N	real
$SO(N)$	vector	N	1	$\frac{N-1}{2}$	real
$SO(N)$	adjoint	$\frac{N(N-1)}{2}$	$N-2$	$N-2$	real
$Sp(2N)$	fundamental	$2N$	$\frac{1}{2}$	$\frac{2N-1}{4}$	pseudo real
$Sp(2N)$	adjoint	$2N^2+N$	$N+1$	$N+1$	real
$Sp(2N)$	AT2T	$N(2N-1)-1$	$N-1$	N	real

constant is called the Casimir number $c_\mathcal{R}$ of an irreducible representation. The Dynkin index and Casimir number of an irreducible representation are both independent of the choice of basis and are related via

$$c_\mathcal{R}\ d_\mathcal{R} = T_\mathcal{R}\ d_G \tag{2.6}$$

where d_G is the dimension of the Lie-group and $d_\mathcal{R}$ the dimension of the representation. It has been shown that it is always possible to fix a basis of hermitian generators such that $\kappa_{\alpha\beta} = \delta_{\alpha\beta}$. With respect to this basis, the structure constants are totally antisymmetric and one no longer has to distinguish between upper and lower indices.

$$C^\gamma{}_{\alpha\beta} = C_{\alpha\beta\gamma} = -C_{\alpha\gamma\beta} = C_{\gamma\alpha\beta} \tag{2.7}$$

While in mathematics the convention $\lambda = 1$ is often used, we adopt the most common physics convention when dealing with gauge theories $\lambda = c_{\text{adj}}^{-1}$, with c_{adj} given in table 2.1. The consequences of this convention for representations other than the adjoint are also summarized in table 2.1 for the classical groups. This choice has an advantageous feature explained in appendix D. Note that the choice of a Killing form on the Lie-algebra is a technical tool, without real physical consequences. Any choice of λ may be compensated by rescaling the coupling of the gauge theory, which is a free, running parameter anyway. However, the explicit equation for the running coupling is influenced by the choice of λ.

2.2 Spontaneous symmetry breaking in QFT

It is very well known that symmetries of the action may not be symmetries of the full quantum field theory. Examples are so-called anomalies, discussed in section 2.3 and spontaneously broken symmetries. The relevant aspects of the latter phenomena will be reviewed in this section, focusing on the breaking of global internal symmetries in relativistic quantum theories. For a more general but still pedagogical overview see [33]. For a detailed discussion of spontaneous symmetry breaking in QFTs see [34, Chpt. 19].

Consider a collection of linearly independent quantum fields $\psi^{(i)}$ together with the linear transformation $\psi^{(i)} \to \tilde{\psi}^{(i)} = \left\{U^\mathcal{R}\right\}^i_j \psi^{(j)}$. The matrices $U^\mathcal{R}$ furnish some finite dimensional, unitary representation of a group G. Furthermore, let us assume that the action of the QFT is invariant under these transformations, e.g. $S[\psi] = S[\tilde{\psi}]$. However, in order for G to be a symmetry group of the quantum theory, we require that there exist (anti-)unitary operators $U^Q : \mathcal{H} \to \mathcal{H}$ on the infinite dimensional Hilbert space $\mathcal{H}$ of the quantum theory, furnishing an infinite dimensional representation of G. In order to be a symmetry, these operators must further commute with the Hamiltonian. From the standpoint of the Lagrangian formulation, this may nicely be captured by the condition

$$U^{Q\dagger}\psi^{(i)}U^{Q} = \left\{U^{\mathcal{R}}\right\}^{i}_{\ j}\psi^{(j)} \tag{2.8}$$

In the case of a continuous group G, one may use the Noether current operators j^{μ}_{N}, corresponding to the one-parameter transformations of the respective generators $T^{\mathcal{R}}_{N}$, to construct charge operators

$$Q_N(t) := \int \mathrm{d}^3x \; j^0_N(t,\vec{x}) \tag{2.9}$$

If the fields $\psi^{(i)}$ satisfy canonical commutation rules, the charge operators obey the following algebraic relations

$$[Q_N(t), Q_M(t)] = iC^{K}_{\ \ MN}Q_K(t) \tag{2.10}$$

$$[Q_N(t), \psi^{(i)}(t,\vec{x})] = -\left\{T^{\mathcal{R}}_{N}\right\}^{i}_{\ j}\psi^{(j)}(t,\vec{x}) \tag{2.11}$$

A famous result by Coleman [35], dubbed "the invariance of the vacuum is the invariance of the world", guarantees that if $Q_N\left|0\right\rangle = 0$ holds, the operators commute with the Hamiltonian, e.g. it holds

$$\partial_t Q_N = 0 \tag{2.12}$$

However, for us the case where $Q_{N=a}$ does not annihilate the vacuum state will be of interest. In this case it can be shown that Q_a is ill-defined on $\mathcal{H}$ and thus cannot be used to generate the quantum symmetries U^Q. The consequence is a richer vacuum structure of unitary inequivalent, but stable and degenerate ground states. One of them defines the systems vacuum state and the system is said to be in a "Nambu-Goldstone phase". The symmetry group realized in the quantum theory is a subgroup $H \subset G$. Typically, such a phase is characterized by a non-zero order parameter

$$\chi_c := \left\langle 0\right| \chi \left|0\right\rangle \tag{2.13}$$

with χ some operator that transforms non-trivially under transformations $g \in G\backslash H$ and trivially under $g \in H$. A phenomenon related to this "spontaneous breakdown" of a continuous symmetry G in a relativistic theory is the appearance of massless particles, whose zero momentum states correspond to the degenerate ground states. The famous Nambu-Goldstone theorem proves the existence of these modes, gives information on their properties and relates them to a proper choice of order parameter. We state one version of this theorem below. A simple proof can be found in [36].

Theorem 2.2.1 (Nambu-Goldstone theorem)

In a non-supersymmetric, Lorentz- and translation invariant QFT, given a conserved Noether current operator j^{μ}_{a} and its corresponding charge Q_a, the following statement holds:

If there exists an operator $\mathcal{O}_a$, such that $\left\langle 0\right| [Q_a, \mathcal{O}_a] \left|0\right\rangle \neq 0$, the spectrum of the theory contains at least one massless, spin zero particle, whose zero momentum eigenstates $\left|\xi_a(0)\right\rangle$ couple to the operators $\mathcal{O}_a$ and Q_a i.e.

$$\left\langle 0\right| Q_a \left|\xi_a(0)\right\rangle \left\langle \xi_a(0)\right| \mathcal{O}_a \left|0\right\rangle \neq 0 \tag{2.14}$$

A good order parameter may then be given by $\chi_c = \left\langle 0\right| [Q_a, \mathcal{O}_a] \left|0\right\rangle$. Furthermore, we can derive the symmetry properties of the states $\left|\xi_a(k)\right\rangle$ under the remaining symmetry group H from (2.14). The number of linearly independent such Nambu-Goldstone bosons (NGBs) can be deduced by the so-called coset construction, introduced in section 2.4, and is given by

$$\#\text{NGBs} = \dim G - \dim H = \dim G/H \tag{2.15}$$

Here G/H is the space of left cosets $[g]_{G/H} = gH$. It is important to remark that theorem 2.2.1 requires the currents j_a^μ to be conserved at the quantum level i.e. the operator identity $\partial_\mu j_a^\mu = 0$ must hold. That this still holds for a spontaneously broken symmetries is a main difference to anomalous symmetry breaking. However, if this relation is violated by an operator $f(x, X)$, with X some parameter such that

$$f(x, X) \xrightarrow[X\to 0]{} 0 \tag{2.16}$$

sufficiently fast, then, also for finite X, there still exist states $|\xi_a(k)\rangle$ satisfying (2.14), but belonging to spin zero states of finite mass. The mass vanishes in the limit of $X \to 0$. These particles are called "pseudo Nambu-Goldstone bosons" (pNGBs) of a spontaneously broken approximate symmetry.

2.3 Anomalies in gauge theories

Anomalies, anomaly matching and anomaly cancellation play an important role in the construction of quantum field theories. Here we summarize the relevant properties of anomalies that we used in the thesis throughout. Although, all the material presented here can be found in the standard literature [37][38][34, Chapter 22], we try to present it in a more compact way than usual, treating global- and gauge-anomalies on an equal footing. This is done by using the gauge-principle as a technical tool to introduce appropriate source terms in order to treat global anomalies in a local formulation.

2.3.1 Ward identities and current conservation

The physical consequences of symmetries manifest themselves in Noether's current conservation laws and so called Ward-Takahashi identities (WTI) among the correlation functions. Typically, these are calculated for local symmetries. For global continuous symmetries, as a technical tool, one may promote the transformation parameter to a local function and set this function to a constant value after the calculation. An equivalent formulation is to "gauge" the global symmetry and work in a background gauge field description of the generating functional. After the derivation of the WTI's one sets the values of the background gauge fields to a physical value. We want to formulate the latter approach in a bit more detail to explain the consequences of anomalies in our theory. Suppose we have a Lagrangian $\mathcal{L}(\partial_\mu\Psi, \Psi)$, invariant under global, continuous symmetry transformations of a Lie-group G, implemented via the representation $\mathcal{R} : G \to \mathrm{Aut}\,(V)$. At every spacetime point the field satisfies $\Psi(x) \in V$, but in principle we may consider Ψ to contain scalar, fermion and vector components. By "gauging the global symmetry" we mean the assignment

$$\mathcal{L}(\partial_\mu\Psi, \Psi) \mapsto \mathcal{L}_{\text{cov.}}(\partial_\mu\Psi, \Psi, A) \tag{2.17}$$

The Lagrangian $\mathcal{L}_{\text{cov.}}$ is constructed by covariantizing $\mathcal{L}$ under the local symmetry and the corresponding action $S_{\text{cov.}}$ obtained is assumed to be unique. The gauge fields A_μ^α are organized within the connection 1-from $A = -iA_\mu^\alpha T_\alpha^{\mathcal{R}} \mathrm{d}x^\mu$, where we suppressed the label of the representation in the notation of A. In most cases the covariantization can be achieved by the replacement $\partial_\mu \mapsto \mathrm{D}_\mu^{\mathcal{R}}[A]$ of the partial derivatives with a covariant derivative. For example, the fermionic part of the Lagrangian of QCD is constructed in this manner from a variant of the free Dirac Lagrangian with additional color degrees of freedom. In order to calculate the WTI's one defines the generating functional

$$\tilde{Z}[A] = \mathrm{e}^{i\tilde{\mathrm{W}}[A]} := \int \mathcal{D}\Psi \mathrm{e}^{iS_{\text{cov.}}(\Psi, A)} \tag{2.18}$$

where the A_μ^α enter as background sources and by now have no physical meaning. However, one can promote them to dynamical gauge fields by adding an action describing their dynamics and integrating over A_μ^α in the path-integral. By doing so, one then has to deal with gauge fixing and the introduction of ghosts, which so far we could ignore. Another option is to set the physical point, such that the background gauge fields vanish, e.g. we look at the covariantized theory in the limit of $A_\mu^\alpha \to 0$. The partition function of the original "ungauged" theory is given by $Z = \tilde{Z}[0]$ and contains all the information on the symmetries, realized in the quantum theory. Under the assumption that the limit $\tilde{Z}[A] \to Z$ is sufficiently smooth, we can derive the WTI's for $\tilde{Z}[A]$ and then look at them in the decoupling limit $A \to 0$. It should be mentioned that for non-abelian groups G, $S_{\text{cov.}}[\Psi, A]$ in general does not inherit the global symmetry of $S[\Psi]$, because symmetry transformations, obtained by localizing the symmetry, involve a simultaneous transformation of the background gauge fields. The additional terms, introduced by the covariantization procedure, are not invariant if only performing a global transformation of the Ψ field alone. These terms vanish in the limit $A_\mu^\alpha \to 0$. In order to derive WTI's and conservation laws we compare the currents

$$J_\alpha^{\mathcal{R}\mu} = -i \, \frac{\partial \mathcal{L}_{\text{cov.}}}{\partial(\mathrm{D}_\mu^{\mathcal{R}}[A]\,\Psi)} \, T_\alpha^{\mathcal{R}} \psi = \frac{\partial \mathcal{L}_{\text{cov.}}}{\partial A_\mu^\alpha} \tag{2.19}$$

$$A \to 0 \quad \downarrow$$

$$j_\alpha^{\mathcal{R}\mu} = -i \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \, T_\alpha^{\mathcal{R}} \psi \tag{2.20}$$

We realize that relations for the conserved current vacuum expectation values and connected current correlation functions may be obtained from functional derivatives of $\tilde{\mathrm{W}}[A]$ with respect to background gauge fields A_μ^α.

$$\frac{\delta^n \tilde{\mathrm{W}}[A]}{\delta A_{\mu_1}^{\alpha_1}(x_1) \cdots \delta A_{\mu_n}^{\alpha_n}(x_n)} = i^{\,n-1} \left\langle \mathrm{T}\left(J_{\alpha_1}^{\mu_1}(x_1) \cdots J_{\alpha_n}^{\mu_n}(x_n)\right)\right\rangle_{\mathrm{C}} \tag{2.21}$$

We now have a look at the behavior of $\tilde{\mathrm{W}}[A]$ under a background gauge transformation. Such a transformation is parametrized by the choice of a local function $\boldsymbol{\epsilon} : \mathcal{M} \to \mathfrak{g}$, with $\mathcal{M}$ denoting spacetime. By expressing $\boldsymbol{\epsilon} = -i\epsilon^\alpha T_\alpha^{\mathcal{R}}$ in components, one defines component functions $\epsilon^\alpha : \mathcal{M} \to \mathcal{R}$. Given the matrix $U^{\mathcal{R}} = \exp\{-\boldsymbol{\epsilon}\} = \exp\{i\epsilon^\alpha T_\alpha^{\mathcal{R}}\}$ a finite background gauge transformation is given by

$$\Psi \mapsto \Psi' = U^{\mathcal{R}} \Psi \tag{2.22}$$

$$A \mapsto A' = U^{\mathcal{R}} A U^{\mathcal{R}\dagger} + U^{\mathcal{R}} \mathrm{d} U^{\mathcal{R}\dagger} \tag{2.23}$$

Here, $\mathrm{d}U^{\mathcal{R}}$ denotes the exterior derivative, defined in appendix C, of the matrix valued function $U^{\mathcal{R}}$. We may choose the component functions to be infinitesimally small. Then the transformation becomes $\Psi \mapsto \Psi' = \Psi + i\epsilon^\alpha T_\alpha^{\mathcal{R}} \Psi + \ldots$ and $A_\mu^\alpha \mapsto A_\mu'^\alpha = A_\mu^\alpha + \delta_\epsilon A_\mu^\alpha + \ldots$. The dots represent terms of order $\mathcal{O}((\epsilon^a)^2)$. The variations $\delta_\epsilon A$ are defined by

$$\delta_\epsilon A_\mu^\alpha := \partial_\mu \epsilon^\alpha + C^\alpha{}_{\beta\gamma} A_\mu^\beta \epsilon^\gamma =: \mathrm{D}_\mu^{\mathrm{adj}}[A]\,\epsilon^\alpha \tag{2.24}$$

$$\delta_\epsilon A_{\mu\alpha} := \partial_\mu \epsilon_\alpha - C^\gamma{}_{\beta\alpha} A_\mu^\beta \epsilon_\gamma =: \mathrm{D}_\mu^{\mathrm{adj}}[A]\,\epsilon_\alpha \tag{2.25}$$

Although in our convention $\delta_\epsilon A_\mu^\alpha = \delta^{\alpha\beta} \delta_\epsilon A_{\mu\beta}$, we explicitly state the result to define the covariant derivative in the adjoint representation of G, acting on lower and upper index quantities. The infinitesimal variation of $\tilde{\mathrm{W}}[A]$ is obtained by

$$\delta_\epsilon \tilde{\mathrm{W}}[A] = \int \mathrm{d}^4x \, \left(\mathrm{D}_\mu^{\mathrm{adj}}[A]\,\epsilon^\alpha\right) \frac{\delta \tilde{\mathrm{W}}[A]}{\delta A_\mu^\alpha} \overset{\mathrm{PI}}{=} -\int \mathrm{d}^4x \; \epsilon^\alpha \left(\mathrm{D}_\mu^{\mathrm{adj}}[A]\, \frac{\delta \tilde{\mathrm{W}}[A]}{\delta A_\mu^\alpha}\right) \tag{2.26}$$

We used an analog of partial integration (PI) for the covariant derivatives, that may be derived from the definition (2.24). Contracting the current expectation value (2.19) with the appropriate covariant derivative and taking the decoupling limit gives the following relation for current conservation.

$$\mathrm{D}_{\mu}^{\mathrm{adj}}\,[A]\left\langle J_{\alpha}^{\mathcal{R}\mu}\right\rangle = \mathrm{D}_{\mu}^{\mathrm{adj}}\,[A]\,\frac{\delta\tilde{\mathrm{W}}[A]}{\delta A_{\mu}^{\alpha}} \tag{2.27}$$

$$\partial_{\mu}\left\langle j_{\alpha}^{\mathcal{R}\mu}\right\rangle = \lim_{A\to 0}\mathrm{D}_{\mu}^{\mathrm{adj}}\,[A]\,\frac{\delta\tilde{\mathrm{W}}[A]}{\delta A_{\mu}^{\alpha}} \tag{2.28}$$

Thus, physically speaking, current conservation in this formulation, is related to invariance of the functional $\tilde{\mathrm{W}}[A]$ under background gauge transformations. Note that in principle a mixture of these situations is possible, because some background gauge fields may be related to dynamical fields, while some are purely auxiliary. In section 3.2 we use this to introduce the dark photon by gauging a 1-parameter subgroup of the global flavor symmetry.

2.3.2 Properties of anomalies

Anomalies manifest as non-invariance of the quantum theory under a symmetry of the Lagrangian and are sourced by quantum effects, related to a lack of a gauge-invariant regulator for the quantum theory [39]. Especially the functional $\tilde{\mathrm{W}}$ becomes non-invariant under background gauge transformations, sourcing current non-conservation. Anomalies occur in theories with fermions coupled to dynamical gauge fields. Hence, for now, let us restrict to a single fermion field $\Psi = \psi$, furnishing a representation $\mathcal{R}$ as above. By construction, the action $S_{\text{cov.}}[\psi, A] = S_{\text{cov.}}[\psi', A']$ stays invariant under a simultaneous background gauge transformation (2.22) of matter and background gauge fields. The fermion measure in the path-integral however may not and this is the origin of the anomaly. We parametrize the non-invariance of the measure by

$$\mathcal{D}\psi'\mathcal{D}\overline{\psi'} = \mathcal{D}\psi\mathcal{D}\overline{\psi}\ \mathrm{e}^{i\mathcal{A}[\boldsymbol{\epsilon},A]} \tag{2.29}$$

with $\mathcal{A}[\boldsymbol{\epsilon}, A]$ the so-called "Anomaly functional", which is assumed to be linear in the first argument $\boldsymbol{\epsilon}$. It can be shown that $\mathcal{A}$ must be a local functional. It therefore is restricted to the form

$$\mathcal{A}[\boldsymbol{\epsilon}, A] = \int \mathrm{d}^4x\ \epsilon^{\alpha}\mathcal{A}_{\alpha}[A](x) \tag{2.30}$$

The functions $\mathcal{A}_{\alpha}$ are the so-called "Anomaly functions". These are polynomials of the background gauge fields and spatial derivatives thereof. The highest order degree of derivatives in this polynomial expansion in dimension four is restricted to two. The lecture notes of Bilal [37] give a pedagogical introduction to these claims. If we perform a finite background gauge transformation $A \to A'$ on $\tilde{\mathrm{W}}[A]$, one easily verifies that

$$\tilde{\mathrm{W}}[A'] - \tilde{\mathrm{W}}[A] = \mathcal{A}[\boldsymbol{\epsilon}, A] \tag{2.31}$$

holds. From this we can further see that $\mathcal{A}_{\alpha}$ must be a scalar under Lorentz transformations and of mass-dimension four. It is interesting to ask whether the anomaly functional itself can be reproduced by adding a non-invariant term in the Lagrangian. This would allow to remove the anomaly by adding local terms to the Lagrangian. An anomaly that can not be canceled in this way is called "essential". In more precise terms, an anomaly is essential if it cannot be written as the variation of a local functional $\mathcal{F}$.

$$\int \mathrm{d}x^4\epsilon^{\alpha}(x)\ \mathcal{A}_{\alpha}(x) \neq \delta_{\epsilon}\mathcal{F} \tag{2.32}$$

By a local functional we mean here that $\mathcal{F}$ is the spacetime integral over a polynomial in the gauge fields and finite number of derivatives of them. For further calculations the introduction of the functional derivative operator

$$\mathfrak{D}_\alpha(x) := -\mathrm{D}^{\mathrm{adj}}_\mu [A] \frac{\delta}{\delta A^\alpha_\mu(x)} = -\frac{\partial}{\partial x^\mu} \frac{\delta}{\delta A^\alpha_\mu(x)} + C^\gamma{}_{\beta\alpha} A^\beta_\mu \frac{\delta}{\delta A^\gamma_\mu(x)} \tag{2.33}$$

is useful. The commutator of this differential operator obeys a similar relation as the Lie-algebra elements themselves, which can be proven by explicit calculation.

$$[\mathfrak{D}_\alpha(x), \mathfrak{D}_\beta(y)] = \delta^4(x-y) C^\gamma{}_{\alpha\beta} \mathfrak{D}_\gamma(x) \tag{2.34}$$

Now an infinitesimal background transformation can be expressed via this differential operator via (2.26). Application of this differential operator on $\tilde{\mathrm{W}}[A]$ yields the anomaly functions.

$$\mathfrak{D}_\alpha(x)\tilde{W} = \mathcal{A}_\alpha[A](x) \tag{2.35}$$

Note that we may always add an exterior derivative $\mathrm{d}\omega[\epsilon](x)$ of a differential 4-form, depending linear on $\boldsymbol{\epsilon}$, to (2.30). This will not change the anomaly or the physics, but the appearance of the anomaly functions. Thus, (2.35) is only determined up to adding such an exterior derivative. Application of the commutator of the differential operator on $\tilde{\mathrm{W}}[A]$ results in a self-consistency relation for the anomaly functions.

$$\mathfrak{D}_\alpha(x)\mathcal{A}_\beta(y) - \mathfrak{D}_\beta(y)\mathcal{A}_\alpha(x) = C^\gamma{}_{\alpha\beta}\delta^4(x-y)\mathcal{A}_\gamma(x) \tag{2.36}$$

This condition is called the "Wess-Zumino consistency condition" and was first derived in [16]. An anomaly that satisfies (2.36) is called "consistent". This condition is so strong that a consistent anomaly is fully determined by only knowing the quadratic coefficient of the anomaly functions, i.e. the term in the polynomials $\mathcal{A}_\alpha[A]$ that only contains two gauge-fields and derivatives thereof. But even if the anomaly is completely unknown, there exist solutions that are determined uniquely up to an overall scale-factor and the addition of the gauge variation of a local functional. Thus, the solutions of the consistency condition determines the anomaly completely up to a normalization and non-essential terms. Solutions to (2.36) can be found with the help of the Stora-Zumino descent equations [37]. The exposition so far shows that every anomaly originating from a gauge symmetry, must be consistent, making the consistency condition a very powerful tool.

2.3.3 Calculation of the anomaly

The anomaly may be calculated directly from the variation of the measure by the elegant Fujikawa method [39], which works best for so-called abelian anomalies and further demonstrates nicely how the anomaly is related to the lack of a gauge-invariant regularization procedure. However, in this and the next section, we would like to review the perturbative calculation from triangle diagrams in order to put emphasis on the connection between the regularization scheme and non-essential features of the anomaly. Understanding this becomes important when modelling the consequences of anomalies in the low energy effective theory.
We consider a set of left-handed Dirac fermions q_L, transforming under a gauge group G in the representation $\mathcal{R} : G \to \mathfrak{M}$. The associated currents[1] are given by

$$J^{\mathcal{R}\mu}_\alpha = \overline{q_L}\gamma^\mu T^{\mathcal{R}}_\alpha q_L \tag{2.37}$$

[1] If following the calculation in [34, Chpt. 22.3] one may realize that the currents there have an additional factor i in front. This is because Weinberg uses a different spacetime convention. The currents are the same, the gamma matrices are different.

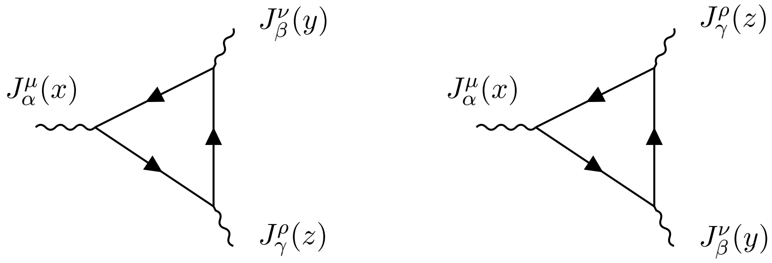


Figure 2.1: The two triangle diagrams contributing to the one loop calculation of the three point vertex.

These take on the form (3.17) if the Dirac fermions are parametrized as $q_L^{(k)\top} = (\psi^{(k)\top}, 0)$, allowing us to make contact to the left-handed formulation, used later in the bulk of this thesis. As explained in the previous section, we know the anomaly functions $\mathcal{A}_\alpha$ to be polynomials in the background gauge fields A and derivatives thereof of at most degree two. Since the mass dimension of $\mathcal{A}_\alpha$ is four, the quadratic coefficient

$$\mathcal{A}^{\ \nu\rho}_{\alpha;\beta\gamma}(x,y,z) = \frac{1}{2}\left.\frac{\delta^2 \mathcal{A}_\alpha[A](x)}{\delta A^\beta_\nu(y)\delta A^\gamma_\nu(z)}\right|_{A=0} \tag{2.38}$$

of the functional expansion of $\mathcal{A}$ in A^α_μ must contain two spacetime derivatives. From this coefficient alone, together with the properties discussed above, one can fully determine the anomaly by enforcing the Wess-Zumino consistency condition (2.36). In order to determine the quadratic coefficient we turn towards the WTI's for the current correlation functions

$$\Pi^{\mu\nu}_{\alpha\beta}(x,y) = \left.\frac{\delta^3 \tilde{\mathrm{W}}[A]}{\delta A^\alpha_\mu(x)\delta A^\beta_\nu(y)}\right|_{A=0} = i\left\langle \mathrm{T}\left(J^{\mathcal{R}\mu}_\alpha(x) J^{\mathcal{R}\nu}_\beta(y)\right)\right\rangle^{A=0}_C \tag{2.39}$$

$$\tilde{\Gamma}^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) = \left.\frac{\delta^2 \tilde{\mathrm{W}}[A]}{\delta A^\alpha_\mu(x)\delta A^\beta_\nu(y)\delta A^\gamma_\rho(z)}\right|_{A=0} = -\left\langle \mathrm{T}\left(J^{\mathcal{R}\mu}_\alpha(x) J^{\mathcal{R}\nu}_\beta(y) J^{\mathcal{R}\rho}_\gamma(z)\right)\right\rangle^{A=0}_C \tag{2.40}$$

Note that we do not denote the currents by lower case letters, because $A = 0$ must not necessarily correspond with the physical point. Functional differentiating (2.35) with respect to the background gauge fields $A^\alpha_\mu(x)$ and setting $A = 0$ gives the following anomalous ward identity

$$\frac{\partial}{\partial x^\mu}\tilde{\Gamma}^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) + 2\mathcal{A}^{\ \nu\rho}_{\alpha;\beta\gamma} = C^\delta{}_{\gamma\alpha}\delta^{(4)}(x-z)\tilde{\Pi}^{\nu\rho}_{\delta\beta}(x,y) + C^\delta{}_{\beta\alpha}\delta^{(4)}(x-z)\tilde{\Pi}^{\nu\rho}_{\delta\gamma}(x,z) \tag{2.41}$$

The part on the right-hand side of (2.41) is totally antisymmetric in the exchange of indices α, β and γ, while $\mathcal{A}^{\ \nu\rho}_{\alpha;\beta\gamma}$ is symmetric in exchanging β and γ. Thus, the anomalous contribution hides in the symmetric part of the derivative of the three-point vertex function. In order to determine this quantity we follow closely the calculation in [34, Chapter 22.3], only stating the essential steps. To one-loop order, the contribution to this vertex comes from the triangle diagrams depicted in figure 2.1. The expression for the vertex function up to this order is

$$\begin{aligned}\tilde{\Gamma}^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) = {}& \frac{i}{(2\pi)^{12}}\iint \mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\mathrm{e}^{-i(k_1+k_2)x}\mathrm{e}^{ik_1y}\mathrm{e}^{ik_2z} \quad\times \\ \times\Bigg(&\int \mathrm{d}^4p\,\mathrm{tr}\left\{\frac{\not{p}-\not{k}_1}{(p-k_1)^2-i0^+}\gamma^\nu\frac{\not{p}}{(p)^2-i0^+}\gamma^\rho\frac{\not{p}+\not{k}_2}{(p+k_2)^2-i0^+}\gamma^\mu\frac{1+\gamma_5}{2}\right\}\mathrm{tr}\left\{T^{\mathcal{R}}_\alpha T^{\mathcal{R}}_\beta T^{\mathcal{R}}_\gamma\right\} \\ +&\int \mathrm{d}^4q\,\mathrm{tr}\left\{\frac{\not{q}-\not{k}_2}{(q-k_2)^2-i0^+}\gamma^\nu\frac{\not{q}}{(q)^2-i0^+}\gamma^\rho\frac{\not{q}+\not{k}_1}{(q+k_1)^2-i0^+}\gamma^\mu\frac{1+\gamma_5}{2}\right\}\mathrm{tr}\left\{T^{\mathcal{R}}_\alpha T^{\mathcal{R}}_\beta T^{\mathcal{R}}_\gamma\right\}\Bigg)\end{aligned} \tag{2.42}$$

The expression is convergent, however its spatial derivative is not and thus requires regularization. In (2.42) we may shift the loop momenta by arbitrary, but fixed values $p \mapsto p + a$ and $q \mapsto q + b$.

This does not influence the value of $\tilde{\Gamma}^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z)$, however it changes the expression for the spatial derivative. A convenient way of regularizing the expression is to adjust the momentum shifts of the loop momenta, by choosing $a = -b$, such that divergent terms in the expression for $\partial_\mu \tilde{\Gamma}^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z)$ cancel. The result for the totally symmetric part of the divergence of the three-point function, presented in [34, Chapter 22.3] is given by

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\tilde{\Gamma}^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) = \frac{2\pi^2 D_{\alpha\beta\gamma}}{(2\pi)^{12}} \iint \mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\mathrm{e}^{-i(k_1+k_2)x}\mathrm{e}^{ik_1y}\mathrm{e}^{ik_2z}\,\epsilon^{\kappa\nu\lambda\rho}a_\kappa\{k_1+k_2\}_\lambda \\ +\ \text{antisymmetric parts}\end{aligned} \tag{2.43}$$

and still depends on the choice of a. The coefficient $D_{\alpha\beta\gamma}$ arises from the symmetrization of (2.42) and is given by

$$D^{\mathcal{R}}_{\alpha\beta\gamma} := \frac{1}{2}\,\mathrm{tr}\{T^{\mathcal{R}}_\alpha\,(T^{\mathcal{R}}_\beta T^{\mathcal{R}}_\gamma + T^{\mathcal{R}}_\gamma T^{\mathcal{R}}_\beta)\} \tag{2.44}$$

How to choose a now? If we choose $a \propto k_1 + k_2$ the anomalous contribution to the divergences of the three point vertex (2.42) with respect to x vanishes. However, as explained in [34, Chapter 22.3], in this case the anomaly appears in the other divergences with respect to y or z. It is not possible to eliminate the effects of the anomaly all together by choosing a appropriately. For now let us choose $3a = k_1 - k_2$. In this case we obtain

$$\begin{aligned}2\mathcal{A}^{\ \nu\rho}_{\alpha;\beta\gamma}(x,y,z) &= -\frac{4\pi^2 D_{\alpha\beta\gamma}}{3(2\pi)^{12}} \iint \mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\mathrm{e}^{-i(k_1+k_2)x}\mathrm{e}^{ik_1y}\mathrm{e}^{ik_2z}\,\epsilon^{\kappa\nu\lambda\rho}\{k_1\}_\kappa\{k_2\}_\lambda \\ &= \frac{D_{\alpha\beta\gamma}}{12\pi^2}\,\epsilon^{\kappa\nu\lambda\rho}\,\frac{\partial\delta^{(4)}(y-x)}{\partial y^\kappa}\,\frac{\partial\delta^{(4)}(z-x)}{\partial z^\lambda}\end{aligned} \tag{2.45}$$

The quadratic term in the polynomial expansion of $\mathcal{A}_\alpha[A]$ thus contains two spatial derivatives, confirming our expectation. To understand the appearance of the ϵ-tensor one must know that only transformation, involving a γ_5 matrix, so-called “chiral transformations”, are able to produce a phase in the fermionic path-integral measure [39]. Since the anomaly functional $\mathcal{A}$ is Lorentz-invariant, it is no wonder it involves only spacetime tensors that have the same parity features as the γ^5 matrix. Following the derivation in [34, Chapter 22.5] we can now make an appropriate ansatz involving operators of mass dimension four, built from the background gauge fields A^α_μ and at most two spatial derivatives. For brevity, we use $A_\mu = -iA^\alpha_\mu T_\alpha$

$$24\pi^2\mathcal{A}_\alpha[A] = \epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left\{T^{\mathcal{R}}_\alpha \frac{1}{2}\left(T^{R}_\alpha T^{\mathcal{R}}_\beta + T^{R}_\beta T^{\mathcal{R}}_\alpha\right)\partial_\mu A^\alpha_\nu \partial_\rho A^\beta_\sigma\right\} \tag{2.46}$$

$$- ic_1\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\{T^{\mathcal{R}}_\alpha \partial_\mu A_\nu A_\rho A_\sigma\} - ic_2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\{T^{\mathcal{R}}_\alpha A_\mu \partial_\nu A_\rho A_\sigma\} \tag{2.47}$$

$$- ic_3\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\{T^{\mathcal{R}}_\alpha A_\mu A_\nu \partial_\rho A_\sigma\} - c_4\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\{T^{\mathcal{R}}_\alpha A_\mu A_\nu A_\rho A_\sigma\} \tag{2.48}$$

In this ansatz the quadratic coefficient is fixed correctly. The other coefficients c_1 to c_4 can be determined by enforcing the consistency relation (2.36). This gives $c_1 = -c_2 = c_3 = 1/2$ and $c_4 = 0$ [34, Chapter 22.5]. The result is called the “canonical anomaly”. The anomaly functional is most conveniently formulated in terms of the connection 1-form $A = A_\mu \mathrm{d}x^\mu$.

$$\mathcal{A}[\boldsymbol{\epsilon}, A] = \frac{i}{24\pi^2}\int_{\mathcal{M}} \mathrm{tr}\left\{\boldsymbol{\epsilon}\ \mathrm{d}\left[A\mathrm{d}A - \frac{1}{2}A^3\right]\right\} \tag{2.49}$$

The result is linear in the first argument $\boldsymbol{\epsilon} = -i\epsilon^\alpha T^{\mathcal{R}}_\alpha$ and the anomaly functions $\mathcal{A}_\beta$ may be obtained back by explicitly extracting the gauge fields A^α_μ from the connection 1-form $A = -iA^\alpha_\mu T^{\mathcal{R}}_\alpha dx^\mu$. The totally antisymmetric spacetime tensor is hidden in the exterior derivatives and products of the differential forms. As is explained in [37] this anomaly is essential. It can not be removed by adding a non-invariant local functional to the Lagrangian. The whole expression (2.49) is proportional to $D^{\mathcal{R}}_{\alpha\beta\gamma}$ and thus, the presence of anomalies can be detected by looking at the D-symbol for a given representation. The D-Symbol may then be used to derive anomaly cancellation conditions. In the SM this is used to restrict the parameter space of charge assignments under the electroweak force.

2.3.4 Anomalies as source of current non-conservation

The physical connection to the abstract discussion of anomalies presented above is the non-conservation of currents, sourced by anomalies. In this section we focus on how the connection between currents and anomalies may be used to constrain the regularization parameter a. So far we could choose this parameter freely and adopted a specific choice of $3a = k_1 - k_2$ to calculate (2.49). However, this choice may not be allowed by physical restrictions like gauge-invariance. In order to establish a connection to current non-conservation we consider an arbitrary infinitesimal, background gauge transformation (2.24) and combine (2.27) with (2.35) to conclude

$$\mathrm{D}^{\mathrm{adj}}_{\mu}\left[A\right] J^{\mu}_{\alpha} = \mathrm{D}^{\mathrm{adj}}_{\mu}\left[A\right] \frac{\delta \tilde{\mathrm{W}}[A]}{\delta A^{\alpha}_{\mu}} = -\mathcal{A}_{\alpha}(x) \tag{2.50}$$

This quantifies the relation between the appearance of the anomaly and current non-conservation, already hinted in (2.27). If some gauge fields $V^{\bar{\alpha}}_{\mu} \subset A$ are dynamical we need to integrate them out in the path integral at some point. Consider for example that all background gauge fields are dynamical, then the full generating functional of the physical theory is given by

$$Z = \iint \mathcal{D}V\mathcal{D}\psi \; \mathrm{e}^{iS[\psi,V]+iS_{YM}[V]} = \int \mathcal{D}V \;\, \tilde{Z}[V]\mathrm{e}^{iS_{YM}[V]} \tag{2.51}$$

But if the currents associated to background gauge transformations of V are not conserved, then $\tilde{Z}[V]$ is not gauge-invariant and the functional Z is ill-defined. In order to guarantee that a gauge theory is well-defined the consistent anomaly (2.49) must vanish. This leads to anomaly cancellation conditions, by checking that the D-symbol (2.44) for the involved gauge groups vanishes.
However, no such constraint exists for the auxiliary background gauge fields and the related currents are allowed to be anomalous. In the physical limit, where the auxiliary fields are vanishing, the anomaly functions can only be build from the dynamical gauge fields. In order to calculate the anomaly in this case, consider a triangle diagram coupled to three currents of which two are related to dynamical gauge fields. We separate $\mathcal{L}_{\mathrm{cov.}}[\psi, A] = \mathcal{L}_{\mathrm{free}}[\psi] + \mathcal{L}_{\mathrm{Int}}[\psi, A]$ in order to obtain a free and an interactive part of the Lagrangian. The free part corresponds to $\mathcal{L}_{\mathrm{free}}[\psi] = \mathcal{L}_{\mathrm{cov.}}[\psi, 0]$. The one loop perturbative contribution of the triangle diagrams to the current divergence is thus given by

$$\begin{aligned}
\left[\left\langle \mathrm{D}^{\mathrm{adj}}_{\mu}\left[A\right] J^{\mu}_{\alpha}(x)\right\rangle_{A}\right]_{\Delta} &= \frac{i^2}{2}\iint \mathrm{d}^4y\;\mathrm{d}^4z\,\left\langle \mathrm{T}\left(\left(\mathrm{D}^{\mathrm{adj}}_{\mu}\left[A\right] J^{\mu}_{\alpha}(x)\right) J^{\mu}_{\beta}(y) J^{\mu}_{\gamma}(z)\right)\right\rangle^{A=0}_{C} A^{\beta}_{\nu}(y)A^{\gamma}_{\rho}(z) \\
&= \frac{1}{2}\iint \mathrm{d}^4y\;\mathrm{d}^4z\;\frac{\partial}{\partial x^{\mu}}\tilde{\Gamma}^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z)A^{\beta}_{\nu}(y)A^{\gamma}_{\rho}(z) \\
&= -\iint \mathrm{d}^4y\;\mathrm{d}^4z\;\mathcal{A}^{\nu\rho}_{\alpha;\beta\gamma}(x,y,z)A^{\beta}_{\nu}(y)A^{\gamma}_{\rho}(z) + \text{non-anomalous terms}
\end{aligned}$$

If the current $J^{\mu}_{\alpha}(x)$ belongs to a dynamical gauge field, the anomalous contributions must vanish. The non-anomalous contribution will sum up to zero, when considering the missing diagrams beyond triangle contributions. As we have already seen from (2.43), one may choose the regularization parameter a such that the anomaly vanishes in the divergence of one current, while contributions appear in the other two divergences. In [34, Chapter 22.3] it is explained that, choosing $a = k_1 - k_2$ in (2.43) achieves to put the anomalous contribution in the divergence of only one current, while the anomalous contributions to the other two currents vanishes. The current we choose to be anomalous may correspond to a global symmetry. Thus, the physical constraint of gauge-invariance fully fixes the regularization scheme. However, the anomaly we would now obtain, can only differ from the canonical anomaly in (2.49) by non-essential terms, since it must satisfy the consistency condition (2.36). This means that the choice of regularization scheme is reflected by non-essential terms. Further, observe that the quadratic coefficient of the resulting anomaly for $a = k_1 - k_2$ will be exactly the quadratic coefficient of the canonical consisted anomaly (2.45) scaled by a factor 3.

2.4 Phenomenological Lagrangians

The investigation of the low energy description of the relevant particles for dark matter physics in the IR is based on two principles: a "theorem" by Weinberg [40] and non-linear symmetry realizations of the global flavor symmetries. The latter results in the so-called "coset construction" and together they form the basis of what is called chiral perturbation theory (χPT). For an introduction to mesonic χPT in QCD see [36] [41] [42]. For a more general but abstract review of these methods we recommend [43].

2.4.1 Weinbergs power counting theorem

The idea raised by Weinberg is that, if we know all the symmetries of the fundamental theory, no matter how complicated that theory is, we may study certain low energy interactions of particles by writing down the most general Lagrangian for a QFT of the respective relevant fields, that is compatible with the symmetry principles of the underlying theory. Weinbergs phrased his "theorem"[2] as follows:

> "*In the context of perturbation theory: if one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles.*"[40]

Such a Lagrangian in principle contains infinitely many terms and unknown constants. In order to turn this "theorem" into a practical tool one needs to identify the terms giving the largest contribution to the process considered and drop the rest. Following the suggestion of Weinberg [40] one may look at $\mathcal{M}$-matrix elements $\mathcal{M}(p_i, m)$ under a rescaling of external momenta $p_i \mapsto tp_i$ and quark masses $m \mapsto t^2 m$, with $t > 0$.

$$\mathcal{M}(tp_i, t^2 m) = t^D \mathcal{M}(p_i, m) \tag{2.52}$$

The quantity D is called the "chiral order" of such an $\mathcal{M}$-matrix element. The chiral order may be expressed explicitly in terms of the numbers N_d and N_L. N_d is the number of terms in $\mathcal{M}(p_i, m)$ with d spacetime (or momentum) derivatives involved in the Feynmann diagram. N_L is the number of loops in the graph.

$$D = 2 + \sum_{d=2}^{\infty} N_d(d-2) + 2N_L \tag{2.53}$$

Note that due to Lorentz invariance, d must be an even number. This result tells us that for small external momenta and masses, diagrams with lower chiral order D will dominate. We thus can truncate the perturbative description at a certain chiral order. The lowest order regime, which we will adopt for dark matter, drops terms of order $D > 2$.

[2] As already remarked in [40] there exist no rigorous proof, however neither are there counterexamples known off.

2.4.2 The coset construction

Low energy theories with spontaneously broken symmetry G have been studied extensively. Among the first general studies of constructions of phenomenological Lagrangians for such theories have been the works of Coleman, Callan, Wess and Zumino [44][45]. If one wants to model a low energy effective theory for the fields ξ_a, associated to the NGBs particles $|\xi_a\rangle$, one needs to take into account the symmetry structure of the underlying theory. Clearly, due to our discussion in section 2.2 the fields ξ_a can not transform in a linear representation of G. However, an argument presented in the book of Weinberg [34, Chpt. 19], elegantly demonstrates the non-linear properties of the fields ξ_a under G transformations. We will outline the key steps of the argument. Consider some fields collected in Ψ within the QFT we are considering. These may be components of composite fields, spinor components or anything else one can think of. The group G acts linearly on these fields via a representation $\mathcal{R} : G \to \mathrm{Aut}\,(V)$ as a global symmetry. Again, for every spacetime point x it is supposed to hold that $\Psi(x) \in V$. The action is constructed to be invariant under G. If the symmetry is broken to a subgroup H, the Goldstone fields ξ_a must manifest among these fields. Now, in [34, Chpt. 19] Weinberg demonstrates that one my always decompose

$$\Psi(x) = \mathcal{R}[\gamma(x)]\tilde{\Psi}(x) \tag{2.54}$$

with $\tilde{\Psi}$ being a field that does not contain the NGB fields and $\gamma(x)$ is an element of the group. Since NGB fields are massless, the statement that $\tilde{\Psi}$ does not contain NGB fields can be expressed formally by stating that for every spacetime point $\tilde{\Psi}(x)$ lies in the orthogonal complement of the zero-eigenspace of the mass matrix of the quantum field theory. The mass matrix can be computed from the second (functional) derivatives of the quantum effective potential. The NGB fields must thus somehow reappear within the spacetime dependent function $\mathcal{R}(\gamma(x))$. Further, it is demonstrated that the choice of $\gamma(x)$ is only unique up to multiplication with elements H. Hence $\gamma(x)$ represents an element $[\gamma(x)]_{G/H}$ of the coset space G/H. For G a compact connected group one may choose a standard representation

$$\mathcal{R}[\gamma(x)] = \exp\{i\xi^a(x)T_a^R\} \tag{2.55}$$

with T_a^R the generators of the broken part of G. We may interpret the coordinates $\xi^a(x)$ of G/H as the Goldstone fields. Now, if we act with G on $\Psi(x)$, we have

$$\Psi(x) \mapsto \mathcal{R}[g]\Psi(x) = \mathcal{R}[g\,\gamma(x)]\tilde{\Psi}(x) \tag{2.56}$$

But one must be cautious, because $g\gamma(x)$ may now be a still a representative of an equivalence class $[g\gamma(x)]_{G/H}$, but it may not be in the standard form (2.55) and the connection to the NGB fields is lost. One may always return to the standard form by applying an appropriate H transformation from the right $\mathcal{R}[h^{-1}]$. This, of course, does not change the class. However, $h = h[g, \xi_a]$ depends non-linearly on $g \in G$ and the field configuration ξ_a. Thus, we have the non-linear transform behavior

$$\gamma(x) \mapsto g\,\gamma(x)\,h^{-1}[g, \xi_a] \tag{2.57}$$

Via the standard representation one can see that for $g \in H$, the group element $h[g, \xi_a]$ is uniquely determined by $h[g, \xi_a] = g \in H$. Thus, the transformation becomes linear for $g \in H$, e.g. for unbroken transformations.

The fields $\tilde{\Psi}$ must be massive and thus heavier than the NGB fields. The product structure (2.54) shows that we can always integrate out the heavier fields in order to derive a low energy effective description. On the other hand we may follow the effective field theory approach suggested by Weinbergs “theorem” discussed above and construct all terms compatible with the global symmetries. For the pions we may construct such a Lagrangian by starting with the coset representative (2.55) in the first place.

3 Short range description

The dark sector models we investigate are to some extent already discussed in [15][28]. Two Dirac fields $u = q^{(1)}$ and $d = q^{(2)}$, called dark quarks, are interacting via a new dark strong force and confine into bound states that make up the dark sector. The lightest stable states will contribute the dark matter observed today. While the isolated strong sector is a theory on its own, accessible via lattice simulation, a mediator to the standard model is required to keep DM and SM in thermal equilibrium until freeze-out. Such a mediator is modeled by a massive dark photon Z', kinetically mixing with SM hypercharge. The charge assignments of the dark quarks under $U(1)_D$ must be chosen appropriately in order not to spoil the stability of the lightest states. The mass of the dark photon is generated from an abelian Brout-Englert-Higgs (BEH) effect, triggered by the potential of a scalar field φ. Figure 3.1 tries to give an overview of the components of the dark sector, described in the following sections and how they fit together. For the gauge group representation $\mathcal{R}$, under which the dark fermions transform, we will restrict to the case of a real representation.

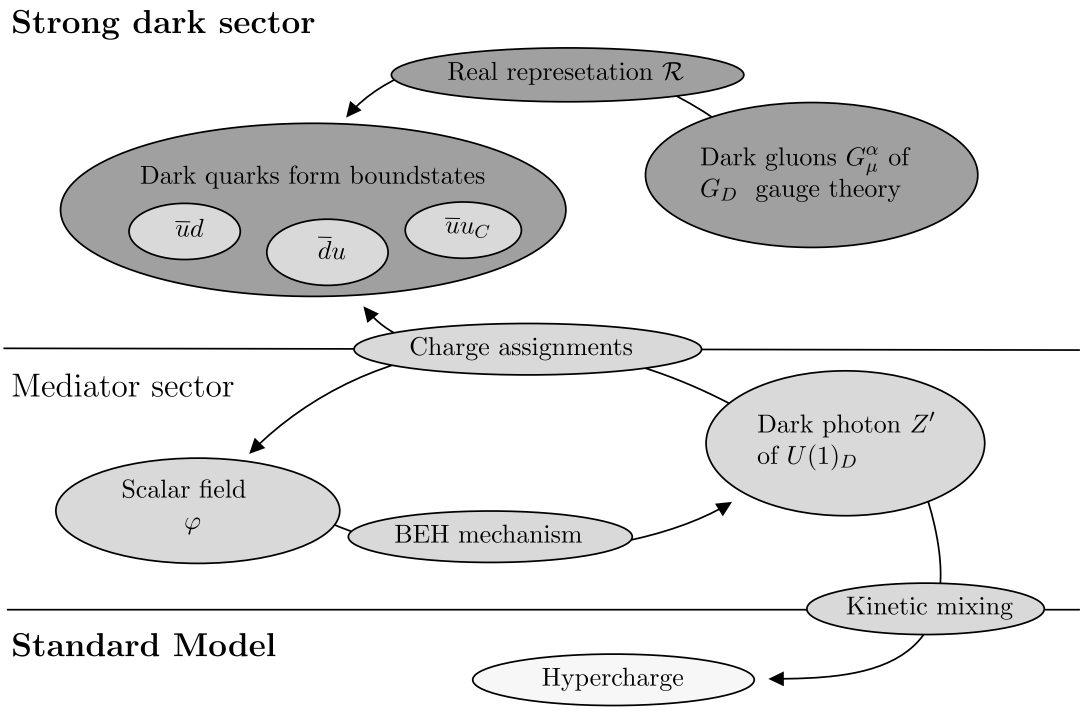


Figure 3.1: Schematic overview of the components of the dark sector. The strong dark sector consists of dark quarks, charged under a real representation of a gauge group G_D. DM is made of the quark bound states. The mediator sector between the dark sector and the SM is realized by a massive dark photon Z', corresponding to an additional $U(1)_D$ gauge symmetry. The quarks are charged under $U(1)_D$ and Z' kinetically mixes with SM Hypercharge. The dark photon obtains its mass from a scalar field φ under an abelian Brout-Englert-Higgs (BEH) mechanism.

3.1 The strong dark sector

The new dark strong force in this model is carried by massless vector bosons G^α_μ, called dark gluons, related to a non-abelian dark gauge group G_D. Here μ is a spacetime index and α a color index, related to the adjoint representation of the Lie-algebra $\mathfrak{g}_D$. The kinematics of these gluons are described by a Yang-Mills-Lagrangian

$$\mathcal{L}^{UV}_{YM} = -\frac{1}{4} G^\alpha_{\mu\nu} G^{\mu\nu}_\alpha \tag{3.1}$$

with the field strength tensor

$$G^\alpha_{\mu\nu} = \partial_\mu G^\alpha_\nu - \partial_\nu G^\alpha_\mu + g\, C^\alpha{}_{\beta\gamma} G^\beta_\mu G^\gamma_\nu \tag{3.2}$$

For the matter content of the theory we consider two species of Dirac fermions $q^{(g)}$, called dark quarks, furnishing a Dirac spinor representation and are charged under a color representation $\mathcal{R} : G_D \to \mathrm{Aut}\,(V_C)$ of the gauge group. The Lagrangian is constructed by virtue of the gauge principle

$$\mathcal{L}^{UV}_{\mathrm{q}} = \sum_{g=1}^{2} \overline{q}^{(g)} i\gamma^\mu \mathrm{D}^{\mathcal{R}}_\mu\,[G]\, q^{(g)} - m_f\, \overline{q}^{(g)} q^{(g)} \tag{3.3}$$

with $\overline{q}^{(g)}$ the Dirac adjoint spinor and the covariant derivative

$$\mathrm{D}^{\mathcal{R}}_\mu\,[G]\, q := \partial_\mu q - ig\, G^\alpha_\mu T^{\mathcal{R}}_\alpha q \tag{3.4}$$

The use of Dirac fermions allows parity to be a good symmetry of the dark sector, as long as we do not include explicit breaking terms. If the representation $\mathcal{R}$ is real, particles and anti-particles, with respect to the gauge group, are physically indistinguishable. Mathematically, this may be expressed by the fact the $\mathcal{R}$ is unitary equivalent to its complex conjugate representation $\overline{\mathcal{R}}$. The linear transformation, relating the two representations, may be given by a matrix[1] S, satisfying

$$S = S^\top \qquad\qquad SS^* = \mathbb{1} \tag{3.5}$$

and the equivariance condition

$$SU^{\mathcal{R}}S^{-1} = (U^{\mathcal{R}})^* \cong U^{\overline{\mathcal{R}}} \qquad \Leftrightarrow \qquad ST^{\mathcal{R}}S^{-1} = -(T^{\mathcal{R}})^\top \cong T^{\overline{\mathcal{R}}} \tag{3.6}$$

Here * denotes the complex conjugation of the entries of the matrix, with respect to a fixed basis in color space V_C. As a consequence, the fundamental degrees of freedom are not the Dirac fermions, but rather four Majorana fermions $q^{(k)}_M$. This can be seen explicitly by adopting a chiral basis for the Dirac spinors, such that the γ-matrices are in the chiral representation (1.2) and the charge conjugation matrix C takes on the form

$$C = \begin{pmatrix} 0 & E^{-1} \\ E & 0 \end{pmatrix} \qquad \text{with} \qquad E := i\,\overline{\sigma}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.7}$$

Here E maps a left-handed onto a right-handed Weyl spinor. The notation is borrowed from [46]. Due to (3.5) the matrix S is unitary. We may augment the charge conjugation operation by the corresponding linear transformation

$$\mathcal{C} : \quad q \longmapsto \mathcal{C}q := S\, C\, q^* \tag{3.8}$$

This augmentation with the linear map S makes explicit the physical fact that fermions and anti-fermions transform in the same representation of the gauge group. Majorana fermions $q^{(k)}_M$

[1] Details on the definition and properties of a real representation are given in Appendix B

may then be defined as positive eigenstates of $\mathcal{C}$ i.e. $\mathcal{C}q_M^{(k)} = q_M^{(k)}$. In a chiral basis they can be explicitly parametrized as

$$q_M^{(k)} = q_M[\phi^{(k)}] = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi^{(k)} \\ E\,S\,\phi^{(k)*} \end{pmatrix} \tag{3.9}$$

with $\phi^{(k)}$ a left-handed Weyl fermion. Due to the reality of the overall fermion representation, the Dirac fermions may be decomposed into two Majorana fermions $q^{(k)} = q_M^{(2k-1)} + iq_M^{(2k)}$. In foresight of the inclusion of a dark photon it will be more convenient to use a different basis than Majorana fermions. Via the relation

$$q^{(k)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi^{(2k-1)} + i\phi^{(2k)} \\ E\,S\,\phi^{(2k-1)*} + iE\,S\,\phi^{(2k)*} \end{pmatrix} = \begin{pmatrix} \psi^{(2k-1)} \\ E\,S\,\psi^{(2k)*} \end{pmatrix} \tag{3.10}$$

we introduce a Nambu-Gorkov [47] basis of left-handed Weyl fermions. The ψ-fields are related to the ϕ-fields by a unitary transformation. As in QCD, using the chiral projection operators $P_{L/R}$, the kinetic term may be split in two terms, each containing Dirac spinors $q_{L/R} := P_{L/R}\,q$ of distinct chirality. The right-handed part

$$\overline{q}_R\gamma^\mu \mathrm{D}_\mu^{\mathcal{R}}\,[G]\,q_R = \overline{Cq_R^*}\gamma^\mu \mathrm{D}_\mu^{\overline{\mathcal{R}}}\,[G]\,Cq_R^* = \overline{SCq_R^*}\gamma^\mu \mathrm{D}_\mu^{\mathcal{R}}\,[G]\,SCq_R^* = \overline{P_L q_{\mathcal{C}}}\gamma^\mu \mathrm{D}_\mu^{\mathcal{R}}\,[G]\,P_L q_{\mathcal{C}} \tag{3.11}$$

may be rewritten in terms of the left-handed projection $P_L q_{\mathcal{C}} = SCq_R^*$ of a charge conjugate Dirac fermion $q_{\mathcal{C}} := \mathcal{C}q$. Hence, the kinetic term can be formulated entirely in terms of left-handed fermions and anti-fermions, all transforming in the same representation $\mathcal{R}$ of the gauge group. Making use of the Nambu-Gorkov parametrization (3.10), the kinetic term reads

$$\sum_{g=1}^{2} \overline{q}_L^{(g)}\gamma^\mu \mathrm{D}_\mu^{\mathcal{R}}\,[G]\,q_L^{(g)} + \overline{q}_R^{(g)}\gamma^\mu \mathrm{D}_\mu^{\mathcal{R}}\,[G]\,q_R^{(g)} = \sum_{k=1}^{4} \psi^{(k)\dagger}\overline{\sigma}^\mu \mathrm{D}_\mu^{\mathcal{R}}\,[G]\,\psi^{(k)} \tag{3.12}$$

In order to rewrite the Dirac mass term, we look at expressions like $\overline{q}^{(g)}q^{(h)}$ and apply the Nambu-Gorkov formalism one more time. Taking into account the anti-commutativity of the fermions one obtains

$$\begin{aligned} \overline{q}^{(g)}q^{(h)} &= \psi^{(2g-1)\dagger}SE\psi^{(2h)*} + \psi^{(2g)\top}S^\dagger E^\dagger\psi^{(2h-1)} \\ &= \psi^{(2g-1)\dagger}SE\psi^{(2h)*} - \psi^{(2h-1)\top}S^* E^*\psi^{(2g)} \end{aligned} \tag{3.13}$$

Due to (3.5) one may further realize that the expression $\psi^{(k)\dagger}SE\psi^{(j)*} = \psi^{(j)\dagger}SE\psi^{(k)*}$ is symmetric upon exchange of Weyl fermions. This allows to reformulate the mass term with a symmetric mass matrix M

$$\sum_{g=1}^{2} m_{(g)}\overline{q}^{(g)}q^{(g)} = \frac{1}{2}M_{kj}\left(\psi^{(k)\dagger}ES\psi^{(j)*} - \psi^{(k)\top}E^*S^*\psi^{(j)}\right) \tag{3.14}$$

In the mass degenerate case $m_{(1)} = m_{(2)} = m$, the mass matrix $M = m\omega$ defines a symmetric, covariant rank 2 tensor ω_{ij}. This tensor will be realized as the invariant structure of the flavor symmetry of the theory.

$$M = \begin{pmatrix} 0 & m_{(1)} & 0 & 0 \\ m_{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{(2)} \\ 0 & 0 & m_{(2)} & 0 \end{pmatrix} \qquad \omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.15}$$

Collecting all the Weyl fermions in a column $\Psi^\top = (\psi^{(1)\top}, \dots, \psi^{(4)\top})$, the action of charge conjugation may be conveniently written as $\mathcal{C} : \Psi \mapsto \Psi_{\mathcal{C}} = \omega\Psi$ and the Lagrangian of the dark quarks becomes

$$\mathcal{L}_{\mathrm{q}}^{UV} = \Psi^\dagger \overline{\sigma}^\mu \mathrm{D}_\mu^{\mathcal{R}} [G] \Psi - \frac{1}{2}\left(\Psi^\dagger ESM\Psi^* - \Psi^\top E^* S^* M\Psi\right) \tag{3.16}$$

In this representation the classical symmetries of the Lagrangian become manifest. Besides parity and charge conjugation, the kinetic term is invariant under global $U(4)$ rotations of the four Weyl flavors $\psi^{(k)}$, while the mass term provides a source of explicit symmetry breaking for this global flavor symmetry.

3.1.1 Anomalous flavor symmetries in the chiral limit

From (3.16) its evident that, in the chiral limit $M \to 0$, the Lagrangian possesses $U(4)$ symmetry, implemented via the defining representation $\mathcal{F} : U(4) \to \mathbb{C}^{4\times 4}$. The symmetry is generated by the 15 generators $T_N^{\mathcal{F}}$, given in Appendix A and $T_0^{\mathcal{F}} = \mathbb{1}/\sqrt{8}$. We denote the representation matrices, acting only on the flavor indices, by $U^{\mathcal{F}}$. The corresponding conserved currents are given as

$$j_N^\mu = \delta_{lp} \left\{T_N^{\mathcal{F}}\right\}^p{}_m \psi^{(l)\dagger}\overline{\sigma}^\mu \psi^{(m)} = \Psi^\dagger \overline{\sigma}^\mu T_N^{\mathcal{F}} \Psi \tag{3.17}$$

However, in the full quantum theory, only a Lie-subgroup of the $U(4)$ symmetry is realized, due to so-called anomalous symmetry breaking by quantum effects. This is analogous to the axial anomaly in QCD. In fact, symmetries generated by $T_0^{\mathcal{F}}$, if translated back into the Dirac formulation, correspond exactly to what is called a $U(1)_A$-transformation in QCD. However, as we show next, a slight difference to QCD arises, concerning the anomalous breaking pattern. We dwell on the details. Using the means of section 2.3 we can derive the anomalous contributions to the triangle diagram depicted in figure 3.2. If we note that each generator $T_N^{\mathcal{F}}$ of a flavor symmetry, associated with a current (3.17), generates a $U(1)$ symmetry, commuting with the gauge-symmetries, we can use the result for the general abelian anomaly [34, Chpt. 22.3] and obtain

$$\begin{aligned}\langle \partial_\mu j_N^\mu \rangle_G &= -g^2 D_{N\alpha\beta} \frac{\epsilon^{\mu\nu\rho\sigma}\delta_{\alpha\beta}}{32\pi^2} G_{\mu\nu}^\alpha G_{\rho\sigma}^\beta & (3.18)\\ &= -2g^2 T_{\mathcal{R}} \operatorname{tr}\left\{T_N^{\mathcal{F}}\right\} \frac{\epsilon^{\mu\nu\rho\sigma}\delta_{\alpha\beta}}{64\pi^2} G_{\mu\nu}^\alpha G_{\rho\sigma}^\beta & (3.19)\end{aligned}$$

for the current conservation law (2.28) in the physical limit. For the calculation of the D-symbol $2D_{N\alpha\beta} = \operatorname{tr}\left\{T_N^{\mathcal{F}}\left(T_\alpha^{\mathcal{R}} T_\beta^{\mathcal{R}} + T_\beta^{\mathcal{R}} T_\alpha^{\mathcal{R}}\right)\right\}$ we used that $T_N^{\mathcal{F}}$ acts on flavor indices and $T_\alpha^{\mathcal{R}}$ on gauge indices only. This allows us to factor the trace. Since, all the matrices $T_N^{\mathcal{F}}$ are traceless, except for $T_0^{\mathcal{F}}$, only the current j_0^μ becomes non-conserved. The non-conservation is sourced by the dark gluon fields.

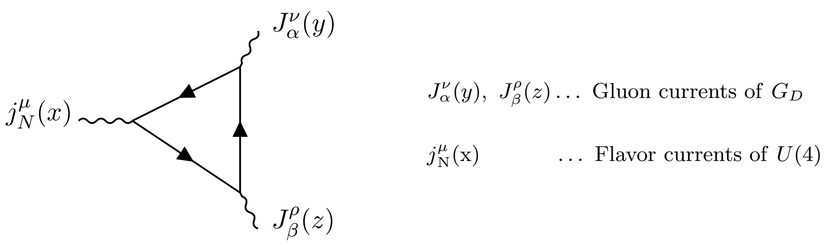


Figure 3.2: Triangle diagram contributing to the axial anomaly. The axial anomaly leads to non-conservation of the singlet flavor current j_0^μ, sourced by the dark gluons.

By virtue of (2.50), we may read off the anomaly functional for a general flavor transformation matrix $U^{\mathcal{F}} = \exp(-\boldsymbol{\epsilon})$ directly form the current non-conservation law.

$$\mathcal{A}_{\text{axial}}[\boldsymbol{\epsilon}, G] = 2ig^2 T_{\mathcal{R}} \operatorname{tr}\{\boldsymbol{\epsilon}\} \frac{\epsilon^{\mu\nu\rho\sigma}\delta_{\alpha\beta}}{64\pi^2} \int_{\mathcal{M}} \mathrm{d}^4x \; G^{\alpha}_{\mu\nu} G^{\beta}_{\rho\sigma} \tag{3.20}$$

$$= -2iT_{\mathcal{R}} \operatorname{tr}\{\boldsymbol{\epsilon}\} \mathcal{Q}_{\text{Topo}}[G] \tag{3.21}$$

Here $\mathcal{Q}_{\text{Topo}}[G]$ denotes the topological charge of a fixed gauge-field configuration G^{α}_{μ} for a simple gauge-group G_D. Equation (3.21) agrees with the result presented in [37], derived via Fujikawas method [39]. It is well known and further demonstrated in Appendix D, that $\mathcal{Q}_{\text{Topo}}[G]$ can only take on integer values. The existence of topological non-trivial gauge field configurations of unit topological charge has first been proven in [48] for $SU(2)$ and later for other simple groups in [49]. Since all gauge field configurations must be considered in the path-integral, the topological charge may be non-zero and current-conservation can be violated in the quantum theory, if the trace does not vanish. But this also means that the corresponding continuous symmetries cannot be fully realized in the quantum theory. From (2.29) we may explicitly determine which symmetries can be realized. A transformation $U^{\mathcal{F}}$ leaves the path-integral measure invariant if $\mathcal{A}_{\text{axial}}[\epsilon, G] = 2\pi\mathbb{Z}$. This translates into $\operatorname{tr}\{\boldsymbol{\epsilon}\} = \frac{2i\pi}{K}\mathbb{Z}$ with $K = 2T_{\mathcal{R}}$. By using $\log\left(\det\left(U^{\mathcal{F}}\right)\right) = \operatorname{tr}\left\{\log\left(U^{\mathcal{F}}\right)\right\}$ we see that transformations with

$$\det\left(U^{\mathcal{F}}\right) = \exp\left(-\frac{2ki\pi}{K}\right) \qquad k = 0, \dots, K-1 \tag{3.22}$$

may be realized in the quantum theory[2]. The remaining symmetry group has the structure of a closed Lie-subgroup and may be written as a semidirect product $SU(4) \rtimes \mathbb{Z}_K$, where $SU(4)$ is a normal subgroup of $SU(4) \rtimes \mathbb{Z}_K$. From a physics perspective such a semidirect product structure may be seen as the fact that every symmetry transformation may be decomposed (in a non-unique) product of two non-commuting matrices, each representing one element of the factor groups $SU(4)$ and $\mathbb{Z}_K$. A quantity is invariant if and only it is invariant under independent transformations of both group factors. To make this more concrete we consider a concrete way of representing the matrices of $\mathbb{Z}_K$. Inspired by the Lie-group homomorphism from the universal covering group of U(4) to the unitary group itself, presented in [50], we may represent $n \in \mathbb{Z}_K$ by the matrices

$$\mathcal{F}[n] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \exp\left(-\frac{2ki\pi}{K}\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.23}$$

Then every matrix in $U^{\mathcal{F}} \in G_F = \mathbb{Z}_2 \ltimes SU(4)$ may be decomposed as a product[3] $U^{\mathcal{F}} = \mathcal{F}[n]\mathcal{F}[g]$ with $g \in SU(4)$ and $n \in \mathbb{Z}_K$. The matrices $\mathcal{F}[n]$ and $\mathcal{F}[g]$ do not commute in general. The decomposition however is not unique and one may choose to represent $\mathbb{Z}_K$ differently. For all real representations in table 2.1 the Dynkin index $T_{\mathcal{R}}$ is an integer and thus $K \in 2\mathbb{N}$. This means that the Lie-subgroup with $\det\left(U^{\mathcal{F}}\right) = \pm 1$ is always realized in the quantum theory. Especially, for the case of $SO(N_C)$-vector representations $K = 2$. Hence, the symmetry, realized in the quantum theory in this case, is $G_F = \mathbb{Z}_2 \ltimes SU(4)$. The physical interpretation of $\mathbb{Z}_2$ is related to charge-conjugation of only the first dark quark, while keeping the other quark unchanged. This interpretation can be made explicit by representing $\mathbb{Z}_2$ different than in (3.23). Details will be discussed in section (3.1.4). For the general case, we are not aware of a clear physical interpretation of the $\mathbb{Z}_K$ semi-factor.

[2]This argument depends on $T_{\mathcal{R}} \in \mathbb{N}$. If $T_R = \frac{1}{2}$ then $\det\left(U^{\mathcal{F}}\right) = 1$ must hold. This is the situation in QCD. This argument may seem weird, because $T_{\mathcal{R}}$ to a large part is an arbitrary definition. However, for the convention discussed in section 2.1 this result is valid, and additional factors appear in different conventions, but the physics stay the same. See Appendix D for more details.

[3]The order of the product-decomposition with n being the first factor is convention. But one must stay consistent.

3.1.2 Explicit symmetry breaking by mass term

As in QCD, the mass term introduces a source of explicit symmetry breaking. Let us investigate the details, by looking at (3.14). In order to avoid explicit CP-violating terms, we define the mass matrix M_{ij} to be real. The invariance condition of the mass term reads

$$\left\{U^{\mathcal{F}}\right\}_l^k M_{kj} \left\{U^{\mathcal{F}}\right\}_n^j = M_{ln} \quad \Leftrightarrow \quad U^{\mathcal{F}\top} M U^{\mathcal{F}} = M \tag{3.24}$$

Let us first consider the mass degenerate case $M = m\omega$. As explained in Appendix A, the Lie-algebra associated to the Lie-subgroup satisfying the invariance condition, is given by $\mathfrak{so}(4)$. The invariance condition, on the level of the Lie-algebra, may be written as

$$\text{Broken } U(4) \text{ generators} \qquad T_a^{\mathcal{F}\top}\omega - \omega T_a^{\mathcal{F}} = 0 \qquad a = (0), 1, \ldots, 9 \tag{3.25}$$

$$\text{Unbroken } U(4) \text{ generators} \qquad T_A^{\mathcal{F}\top}\omega + \omega T_A^{\mathcal{F}} = 0 \qquad A = 10, \ldots, 15 \tag{3.26}$$

and the broken generators will determine the corresponding subgroup. Although $U(4)$ is compact, by exponentiation of the generators we may only reach every group transformation that is connected to unity. These, being generated from the $\mathfrak{so}(4)$-subalgebra, satisfy $\det(U^{\mathcal{F}}) = +1$ and span an $SO(4)$-subgroup. However, taking the determinant of (3.24) we obtain the necessary condition $\det\left(U^{\mathcal{F}}\right)^2 = 1$. This lets us conclude that actually $O(4)$ is the full symmetry group. Indeed, in section 3.1.4 we find that the charge conjugation supplements a discrete symmetry that is not an element of $SO(4)$ itself and relates between the two connected components of $O(4)$. Then $O(4)$ is the relevant flavor symmetry in the quantum theory for the mass-degenerate case, since transformations satisfying $\det\left(U^{\mathcal{F}}\right) = \pm 1$ are not broken by the axial anomaly. The classification of particles under this symmetry will be discussed in section 3.3. In order to investigate the analog of the PCAC-relations in QCD, we will introduce the following pseudo-scalar operators

$$\mathcal{O}_a^{\mathrm{PS}} := \Psi^\top E S \omega T_a^{\mathcal{F}} \Psi + \Psi^\dagger E S \omega T_a^{\mathcal{F}} \Psi^* \tag{3.27}$$

An infinitesimal $U(4)$ symmetry-variation of the Lagrangian then gives the following partially conserved broken current (PCBC) relations

$$\partial_\mu j_A^\mu = 0 \tag{3.28}$$

$$\partial_\mu j_a^\mu = -i\, m\, \mathcal{O}_a^{\mathrm{PS}} \tag{3.29}$$

$$\partial_\mu j_0^\mu = -i\, m\, \mathcal{O}_0^{\mathrm{PS}} - g^2 T_{\mathcal{R}} \frac{\epsilon^{\mu\nu\rho\sigma}\delta_{\alpha\beta}}{32\sqrt{2}\pi^2} G_{\mu\nu}^\alpha G_{\rho\sigma}^\beta \tag{3.30}$$

where $A = 10, \ldots, 15$ is the index of the unbroken generators and $a = 1, \ldots, 9$ is the index of the traceless broken generators. The operators $\mathcal{O}_a^{\mathrm{PS}}$ may be interpreted as the analog of the pseudo-scalar interpolating operators in QCD.
Let us now consider the mass non-degenerate case, where $m_{(1)} \neq m_{(2)}$ and the mass matrix is M given as in (3.15). In this case, the remaining symmetry is given by $O(2) \times O(2)$, by they same argument as before, with the difference that the only unbroken generators are the commuting matrices $T_{12}^{\mathcal{F}}$ and $T_{15}^{\mathcal{F}}$. This case will be of little interest for dark matter phenomenology due to the existence of flavor singlet pNGB states. Further details are explained in section 3.3. We could consider other symmetry breaking terms, but all of them have the same problem as in the mass non-degenerate case. We comment on such possibilities in appendix A.

3.1.3 Spontaneous symmetry breaking

The dark matter candidates in the theories proposed in [14][15][28] are made of the pNGBs of a spontaneously broken symmetry scenario. Similar to QCD we may define an order parameter, called the chiral condensate

$$\chi_c := \langle 0|\, \Psi^\dagger S E \omega \Psi^* \,|0\rangle - \langle 0|\, \Psi^\top S^* E^* \omega^* \Psi \,|0\rangle = 2\delta_{gh} \langle 0|\, \overline{q}^{(g)} q^{(h)} \,|0\rangle \tag{3.31}$$

This parameter has the same structure as the mass term. Consider now again the chiral limit. If the theory is in a chiral broken phase, indicated by a non-zero vacuum expectation value $\chi_c \neq 0$, the symmetry group $G_F = \mathbb{Z}_K \ltimes SU(4)$ of the quantum theory is broken down to its $H_F = O(4)$ subgroup. Thus, by virtue of the Nambu-Goldstone theorem, we expect massless excitations, corresponding to each continuous degree of freedom of the coset space G_F/H_F. In the case of $G_D = SO(N_C)$, with fermions transforming in the vector representation, we have $G_F = \mathbb{Z}_2 \ltimes SU(4)$ and $H_F = O(4) = \mathbb{Z}_2 \ltimes SO(4)$. Hence

$$G_F/H_F \cong SU(4)/SO(4) \tag{3.32}$$

where $\cong$ denotes a diffeomorphism of manifolds. For the number of Nambu-Goldstone modes it thus holds

$$\#\text{NGBs} = \text{dim}G_F - \text{dim}H_F = 15 - 6 = 9 \tag{3.33}$$

Thus we expect nine massless single particle states in the chiral limit, if the flavor symmetry is spontaneously broken. The mass term may be reintroduced as a small perturbation, driving the system into the desired scenario of spontaneously broken symmetry. This argument allows us to choose the spontaneous breaking scenario from a model building perspective, since the chiral condensate is aligned with the mass term. However, introduction of the mass term also breaks the flavor symmetry G_F explicitly, leaving us with a scenario of spontaneously broken approximate symmetry. As a consequence, the Nambu-Goldstone modes will no longer be massless, but very light particles, with vanishing mass close to the chiral limit. Since these particles are necessarily very light, they will be our stable dark matter candidates.

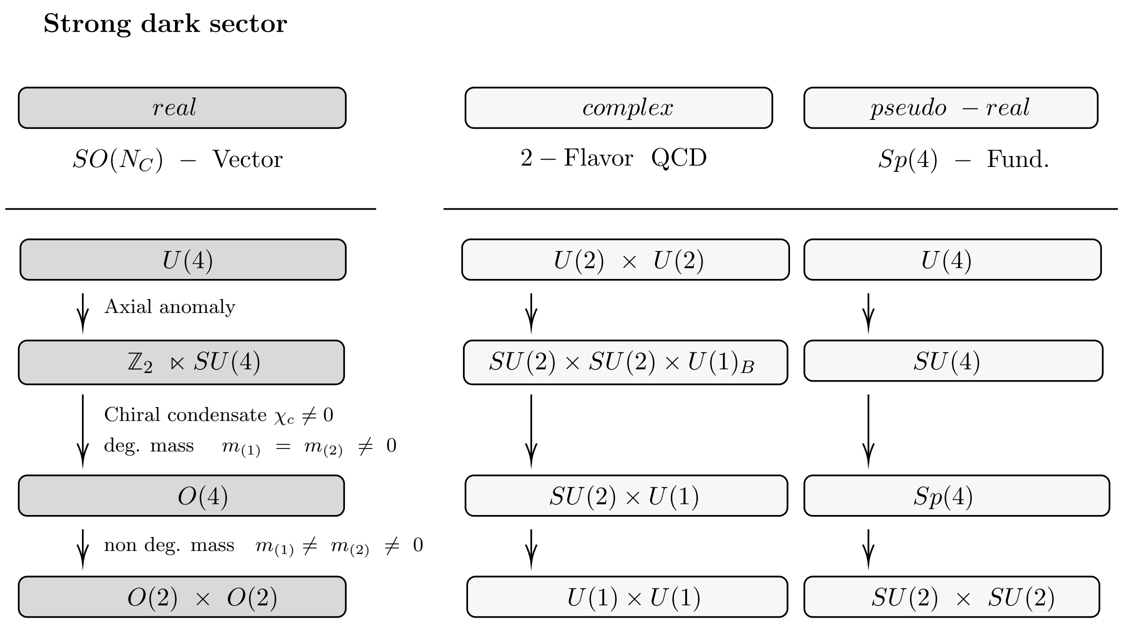


Figure 3.3: Comparison of symmetry breaking patterns in QCD-like theories with two Dirac fermions. The main features of the patterns are determined by the gauge group representation being real, pseudo-real or complex. On the right: The breaking pattern for the dark sector considered here for fermions gauged under $SO(N_C)$-vector representation. In the middle: 2-flavor QCD. On the left: A dark $Sp(4)$ theory with two fermions discussed in [19].

For other gauge theories than $SO(N_C)$-vector, all statements hold true, except that there may be additional discrete broken symmetries. Some part of the $\mathbb{Z}_K$ semi-factor will be broken, since the chiral condensate is only invariant under $\det\left(U^{\mathcal{F}}\right) = \pm 1$ transformations. The number of continuous Nambu-Goldstone modes remains however the same.

3.1.4 Discrete Symmetries

The Lagrangian of the strong dark sector is, by construction, invariant under spatial parity $\boldsymbol{P}$ and charge conjugation $\mathcal{C}$. For Dirac fermions the spatial parity transformation may be represented by $\boldsymbol{P} : q(t,\vec{x}) \mapsto \eta_P \gamma_0 q(t,-\vec{x})$, with η_p an arbitrary complex phase [51]. In QCD one typically adopts the choice $\eta_p = +1$. However, we would like the parity transformations to commute with the unbroken flavor symmetries in order to classify particles in the spectrum by their parity and flavor quantum numbers. This is not possible when adopting the QCD convention for parity [52][19]. If choosing $\eta_p = -i$, parity transformations in they Nambu-Gorkov basis manifest as[4]

$$\boldsymbol{P} : \quad \Psi(t,\vec{x}) \longmapsto i\,\omega E S \Psi^*(t,-\vec{x}) \tag{3.34}$$

Suppressing the spacetime argument and using the invariance condition (3.26) we can easily conclude that $\boldsymbol{P} U^{\mathcal{F}} \boldsymbol{P} \Psi = i\,\omega E S \left(U^{\mathcal{F}}\, i\,\omega E S \Psi^*\right)^* = \omega U^{\mathcal{F}*} \omega \Psi = U^{\mathcal{F}} \Psi$ for an arbitrary flavor transformation $U^{\mathcal{F}} \in O(4)$. Hence, the unbroken flavor transformations commute with this version of parity. Similarly, we can conclude that

$$\begin{aligned}\Psi^\dagger \overline{\sigma}^\mu T_N^{\mathcal{F}} \Psi \;&\xrightarrow{\boldsymbol{P}}\; \Psi^\top E^\dagger \overline{\sigma}^\mu E\, \omega T_N^{\mathcal{F}} \omega\, \Psi^* \\ &= -\,\Psi^\dagger \left(E^\dagger \overline{\sigma}^\mu E\right)^\top \left(\omega T_N^{\mathcal{F}} \omega\right)^\top \Psi = \begin{cases} -g^{\mu\mu}\; \Psi^\dagger \overline{\sigma}^\mu T_a^{\mathcal{F}} \Psi & \text{for } a = 0,\dots,9 \\ \;\; g^{\mu\mu}\; \Psi^\dagger \overline{\sigma}^\mu T_A^{\mathcal{F}} \Psi & \text{for } A = 10,\dots,15 \end{cases}\end{aligned} \tag{3.35}$$

Hence, the currents of the broken generators behave like axial vectors, while the unbroken ones are proper vectors. Parity is a good symmetry of this quantum theory, meaning it is neither anomalous nor spontaneously broken.
In the Nambu-Gorkov formulation of the theory, simultaneous charge conjugation of both Dirac fermions may conveniently be formulated by a matrix multiplication

$$\mathcal{C} : \Psi(t,\vec{x}) \longmapsto \omega \Psi(t,\vec{x}) \tag{3.36}$$

Since ω is unitary, satisfies (3.24) and $\det(\omega) = 1$ applies, charge conjugation actually manifests as $SO(4)$ flavor symmetry. This illustrates the fact that we can not distinguish particles and anti-particles with respect to a real gauge theory. Additionally, charge conjugation gives rise to another symmetry. Consider charge conjugating the first Dirac fermion only e.g. $q^{(1)} \to \mathcal{C} q^{(1)}$, while keeping $q^{(2)}$ unchanged. By the use of the Nambu-Gorkov formalism, this may be implemented with the following matrix.

$$C_u = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.37}$$

This matrix is self-inverse i.e. $C_u^2 = \mathbb{1}$ and satisfies (3.24). Further, it holds that $\det(C_u) = -1$. The matrix does not commute with the $SO(4)$ flavor transformations. The set $\{\mathbb{1}, C_u\}$ corresponds to a representation of $\mathbb{Z}_2$. We may thus conclude that this transformation relates the two disconnected components of $O(4) = \mathbb{Z}_2 \ltimes SO(4)$ and substitutes the $\mathbb{Z}_2$ subgroup of the flavor symmetry[5], which is left unbroken by the anomaly. The action of this $\mathbb{Z}_2$ symmetry has another interesting aspect. As explained in appendix A.2, $SO(4) \cong SU(2)_\rightarrow \times SU(2)_\leftarrow$. The $SU(2)_\rightarrow$ transformations are generated by $T_{10}^{\mathcal{F}}, T_{11}^{\mathcal{F}}, T_{12}^{\mathcal{F}}$ and $SU(2)_\leftarrow$ is generated by $T_{13}^{\mathcal{F}}, T_{14}^{\mathcal{F}}, T_{15}^{\mathcal{F}}$. Now using the conjugation $C_u^{-1} U^{\mathcal{F}} C_u = \exp\{-i\epsilon^N C_u T_N^{\mathcal{F}} C_u\}$ one can calculate that the action of C_u interchanges the role of $SU(2)_\rightarrow$ and $SU(2)_\leftarrow$ transformations [53]. A $SU(2)_\rightarrow$ transformation can always be expressed by conjugation of an appropriate $SU(2)_\leftarrow$ matrix with C_u. The discussion of this $\mathbb{Z}_2$ semi-factor is also similar to parts of the exposition in [54]

[4]Here $\omega E S$ should be read as $\omega \otimes E \otimes S \in \mathrm{Aut}\left(\mathbb{C}^4 \otimes V_{Weyl} \otimes V_C\right)$, with $\omega^i_j := \delta^{ik}\omega_{kj}$ a transformation acting on flavor space $\mathbb{C}^4$. Here V_{Weyl} is the space of the left-handed Weyl spinor representation.

[5]Indeed, the matrix C_u is related to (3.23) via $C_u = V^\dagger \mathcal{F}\left[n[k=1, K=2]\right] V$. The matrix V is given in (A.7).

3.1.5 Remark on additional discrete symmetries

Let us conclude our discussion with a remark on the appearance of additional symmetries. We implicitly assume that the list of symmetries discussed so far is exhaustive. However, due to the generality of the investigation we would like to point out a few remarks. For certain gauge groups there may exist symmetries that do not commute with the generators of the gauge symmetry. These, by definition, can not be flavor symmetries and are referred to as "outer automorphism of the Lie-algebra". A full list is also cited in [55]. For $G_D = SO(N_C = 2k+1)$, $Sp(2N_C)$ no such additional symmetries occurs. For $G_D = SO(N_C = 4k+2)$, there exist such an outer automorphism, but it may be shown to be equivalent to charge conjugation up to gauge transformation. In the case of $G_D = SU(N_C)$ with $N_C > 2$ these outer automorphism coincide with charge conjugation [55]. Thus, up to gauge transformation, these symmetries are already considered in our discussion. The physical bound states considered in the thesis are gauge-invariant and are thus not affected by these additional symmetries. Hence, for these cases, we may conclude that our discussion is exhaustive. Included in this list is also the $Sp(4) \cong SO(5)$ theory presented in appendix G.
The situation for $SO(N_C = 4k)$ gauge theories is different, since there exists an additional outer automorphism of the gauge groups Lie-algebra. This symmetry behaves like a parity on color space and may thus be termed "color-parity" [55]. The implications of this theory for DM phenomenology (if there are any at all) are left for future investigations and are ignored in this thesis.

3.1.6 QCD-like theories and conformal window considerations

So far the discussed theory only depends on two structural features, the reality of the representation and the spontaneous symmetry breaking pattern by the formation of the chiral condensate χ_c in (3.31). Although, there are plenty of real representations, we may already reduce the parameter space by excluding those for which we do not expect a chiral condensate to form. For this we take a look at the so-called conformal window in the (N_C, N_F)-theory space of a fixed representation $\mathcal{R}$. Here N_F is the number of Dirac flavors and N_C is the number of colors for the gauge groups $G_D = SU(N_C), Sp(N_C), SO(N_C)$. For the definition of the conformal window we need to understand the features of the β-function for Yang-Mills theory of a group G_D, coupled to N_F massless fermions transforming in the representation $\mathcal{R}$. Up to two-loop level, the beta function is given [56] by

$$\beta(g) = -\frac{\beta_0}{(4\pi)^2} g^3 - \frac{\beta_1}{(4\pi)^5} g^5 + \dots \tag{3.38}$$

The coefficients β_0 and β_1 are independent of the renormalization scheme and can be expressed in terms of the Casimir number $c_\mathcal{R}$ and Dynkin index $T_\mathcal{R}$ for a representation of the group G_D. These were introduced in section 2.1.

$$\beta_0 = \frac{11}{3} c_{\text{adj}} - \frac{4}{3} T_\mathcal{R} N_F \tag{3.39}$$

$$\beta_1 = \frac{34}{3} c_{\text{adj}}^2 - \frac{20}{3} c_{\text{adj}} T_\mathcal{R} N_F - 4 c_\mathcal{R} T_\mathcal{R} N_F \tag{3.40}$$

If one fixes N_C, the coefficients become functions of N_F. Now a QCD-like theory must at least be asymptotically free, meaning it as a trivial UV fixed point. For $g \approx 0$ this can be determined by the condition $\beta_0 > 0$, marking the upper-bound of the conformal window. For N_F above the conformal window, the theory is asymptotically unfree [56], showing a QED like or asymptotic safe behavior. If $\beta_1 < 0$ the beta function has another fixed point in the IR, called the Banks–Zaks fixed point, and its existence predicts a scale invariant IR description of the theory. However, if we lower N_F further, β_1 may become positive too, as can be seen from (3.40). However, higher order coefficients of the beta function may still produce an IR fixed point for growing g.

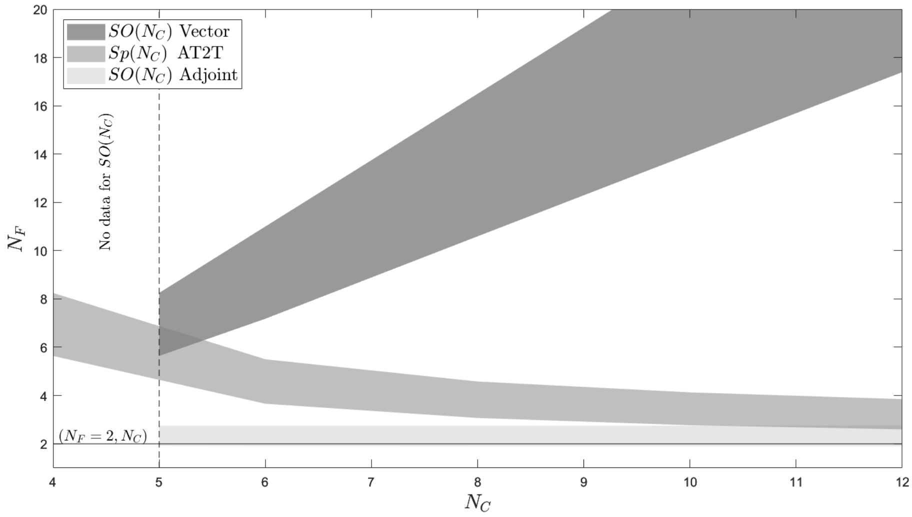


Figure 3.4: Conformal window for $SO(N_C)$-vector representation, $SO(N_C)$-adjoint and $Sp(N_C)$ -AT2T representation. The data points are taken from [56] and linearly interpolated in order to improve visualization. For $Sp(N_C)$ only data points with N_C even make sense. There is no data for $SO(4) \cong SU(2) \times SU(2)$, because this Lie-group is not simple. The black solid line marks the subspace in which the theories, considered in this thesis, live.

The number of flavors N_F^*, at which the beta function lacks an IR fixed point rendering the IR theory scale invariant marks the lower bound of the conformal window. The formation of a chiral condensate introduces a scale Λ_c in the IR theory, breaking the scale invariance. This means, in order to realistically expect[6] the formation of a chiral condensate, the theory must lie below the conformal window in (N_C, N_F)-theory space. Unfortunately, the lower bound of the conformal theory for various theories is a subject of current research and not exactly know [31][56]. In figure 3.4, an estimate of the conformal window from [56] for three real gauge-theories are reviewed. The black line marks the subspace of theories for $N_F = 2$ Dirac flavors, as considered in this work. For $SO(N)$-vector representations we do not penetrate the conformal window, even if $N_C \to \infty$. This will become important in section 3.4. For $SO(N)$-adjoint the situation does not look promising, since for any number of colors the black line cuts into the conformal window. $Sp(4)$-AT2T on the other hand looks like a promising theory. We can further easily exclude a lot of representations already by the criterion $\beta_0 \leq 0$, stating the loss of asymptotic freedom. For example, this makes higher rank tensor representations less favorable.

3.2 The dark photon

The dark photon [57] will take the role of a mediator between the strong dark sector and the standard model and is given by a $U(1)_D$ gauge field Z'. The dynamics of the dark photon field Z'_μ will be given by a Yang-Mills term. Contact to the standard model is introduced by a kinetic mixing term [58] with the hypercharge field B.

$$\mathcal{L}_{Z'}^{UV} = -\frac{1}{4} F'^{\mu\nu} F'_{\mu\nu} + \frac{\varepsilon}{\cos(\Omega_W)} F'^{\mu\nu} B_{\mu\nu} \tag{3.41}$$

with $F'_{\mu\nu} = \partial_\mu Z'_\nu - \partial_\nu Z'_\mu$ and ε the kinetic mixing parameter. Furthermore, $B_{\mu\nu}$ is the hypercharge field strength tensor and Ω_W the Weinberg mixing angle.

[6]This is however only a necessary, but not a sufficient condition.

Charge assignment $\mathcal{Q}$	Remaining symmetry H_F'	Pion multiplets
$a_1 T_{10}^{\mathcal{F}} + a_2 T_{11}^{\mathcal{F}} + a_3 T_{12}^{\mathcal{F}}$	$SU(2)_{\rightarrow} \times U(1)_{\leftarrow}$	$\mathbf{3} \oplus \mathbf{3} \oplus \mathbf{3}$
$b_1 T_{13}^{\mathcal{F}} + b_2 T_{14}^{\mathcal{F}} + b_3 T_{15}^{\mathcal{F}}$	$U(1)_{\rightarrow} \times SU(2)_{\leftarrow}$	$\mathbf{3} \oplus \mathbf{3} \oplus \mathbf{3}$
Others	$U(1)_{\rightarrow} \times U(1)_{\leftarrow}$	$\mathbf{1} \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{2}$

Table 3.1: Charge assignment for dark photon. The parameters a_i and b_i are real in order to make $\mathcal{Q}$ hermitian. The pion multiplets are discussed in more detail in section 3.3. Here it is only important that no dark pion singlets occurs for the first two assignments.

3.2.1 Charge assignments

The dark photon may be coupled to the matter of the strong dark sector by gauging a one-parameter subgroup of the flavor symmetry. One way to achieve this was already discussed in [28]. Gauging a subgroup will introduce an additional explicit symmetry breaking term

$$\mathcal{L}_{qZ'}^{UV} = -i\, e_D\, \Psi^\dagger \overline{\sigma}^\mu \mathcal{Q} \Psi Z'_\mu \tag{3.42}$$

and one must be careful in order not to spoil the stability of the dark matter. Here $\mathcal{Q}$ is a 4×4 matrix. Since we introduced the mass term by hand as an external symmetry breaking term (in contrast to the SM, where it arises from Yukawa couplings to the Higgs field), we may only consider gauging the unbroken flavor symmetries, otherwise the Lagrangian will not be gauge-invariant and thus ill-defined. Because $O(4)$ is a real symmetry group, we do not have a problem with $U(1)_D - U(1)_D - U(1)_D$-triangle anomalies and the $U(1)_D$-gauge theory is well-defined. If we pick a charge assignment $\mathcal{Q}$, the remaining symmetry can be determined by the condition

$$\delta\mathcal{L}_{qZ'}^{UV} = e_D \psi^\dagger \overline{\sigma}^\mu [T_A^{\mathcal{F}}, \mathcal{Q}] \Psi Z'_\mu \stackrel{!}{=} 0 \tag{3.43}$$

since all other terms in the Lagrangian are already $O(4)$ invariant. In order to guarantee the stability of the pion dark matter we require that, after gauging the one-parameter subgroup, the dark pions must form multiplets under the remaining flavor symmetry group $H_F' \subset H_F = O(4)$. In order to investigate all the possible charge assignments $\mathcal{Q}$, achieving this, we may look for all possible subgroups of the form $\tilde{H} \times U(1) \subset SO(4)$. As explained in section 3.3, the maximally broken symmetry group $U(1) \times U(1)$ always allows for a flavor singlet dark pion state. Hence, we want to avoid this scenario. Using that $SO(4) \cong SU(2)_{\rightarrow} \times SU(2)_{\leftarrow}$, we already see that we only have $H' \in \{SU(2)_{\rightarrow}, SU(2)_{\leftarrow}, SU(2)_D\}$ left as options. Here $SU(2)_D$ denotes the diagonal subgroup, consisting of transformations $(U_{\rightarrow}, U_{\leftarrow} = U_{\rightarrow}) \in SU(2)_{\rightarrow} \times SU(2)_{\leftarrow}$. As explained in appendix A.2, the $SU(2)_{\rightarrow}$ transformations are generated by $T_{10}^{\mathcal{F}}, T_{11}^{\mathcal{F}}, T_{12}^{\mathcal{F}}$ and $SU(2)_{\leftarrow}$ is generated by $T_{13}^{\mathcal{F}}, T_{14}^{\mathcal{F}}, T_{15}^{\mathcal{F}}$. Gauging either the left or the right $SU(2)$ subgroup is one possibility to couple the dark photon. What about gauging any one-parameter subgroup[7] within $SU(2)_{\rightarrow} \times SU(2)_{\leftarrow}/SU(2)_D$? It turns out that no such one-parameter subgroup, commuting with all $SU(2)_D$ transformations, exists. Therefore, we have two distinct ways to choose the charge assignment, summarized in tabular 3.1.

In the section 3.1.4 it is demonstrated that charge conjugating only the first Dirac dark quark $q^{(1)}$ results in an exchange of the roles of $SU(2)_{\rightarrow}$ and $SU(2)_{\leftarrow}$. Thus, from the strong sector perspective, they are physically not distinguishable because this is the action of the $\mathbb{Z}_2$ action

[7] The coset space is again a group since $SU(2)_D$ is a normal subgroup. However, in general actions of the coset space may not commute with $SU(2)_D$ action.

in $O(4)$. Hence, there is only one physically relevant class of charge assignments. All the transformation in table 3.1 can be related by a basis transformation. The choice of charge assignment explicitly breaks the symmetry and favors a direction in field space. For explicit calculations we will thus choose

$$\mathcal{Q} = \sqrt{8}T^{\mathcal{F}}_{15} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.44}$$

which corresponds to gauging the σ_3 subgroup of $SU(2)_{\leftarrow}$. The Nambu-Gorkov fields are then already the charge eigenbasis. As a sanity check, we need to consider that no anomalous contributions to broken and unbroken flavor currents occur, due to the coupling to the $U(1)_D$ gauge-field. In the case of an unbroken current, this would spoil the symmetry, as in the $U(4)$ case. In the case of broken currents, additional anomalous terms in the low-energy effective Lagrangian must be added, describing analogous effects such as the $\pi_0 \to \gamma\gamma$ decay in QCD. Nevertheless, for $\mathcal{Q} = a_1 T^{\mathcal{F}}_{10} + a_2 T^{\mathcal{F}}_{11} + a_3 T^{\mathcal{F}}_{12}$ and $\mathcal{Q} = b_1 T^{\mathcal{F}}_{13} + b_2 T^{\mathcal{F}}_{14} + b_3 T^{\mathcal{F}}_{15}$, it holds that

$$\mathcal{Q}^2 \propto \mathbb{1} \tag{3.45}$$

This condition, leads to cancelation of the triangle $SU(4) - U(1)_D - U(1)_D$ anomaly, because the D-symbol

$$D_{NZ'Z'} \propto \mathrm{tr}\{T^{\mathcal{F}}_N Q^2\} = 0 \tag{3.46}$$

vanishes due to $T^{\mathcal{F}}_N$ being traceless for $N = 1, \ldots, 15$. This was to be expected, since all the pions transform as multiplets and thus, since Z' is a flavor singlet, can not decay into a dark photon. Furthermore, $\mathrm{tr}\{\mathcal{Q}\} = 0$ holds true, rendering the $U(1)_D$ gauge theory well-defined from the viewpoint of gravitational anomalies [34, Chpt. 22.4].

3.2.2 Making the dark photon massive

The argument of Spergel and Steinhardt [8], in order to resolve the cusp vs. core problem, requires the absence of long range forces. Also, there are constraints, ruling out a massless dark photon [59]. Therefore, we would like to introduce a mass to the dark photon. In order to maintain the gauge structure we use the abeliean version of the Brout-Englert-Higgs mechanism. In order to do so we need to add a complex, proper scalar field φ, transforming under $U(1)_D$, to the Lagrangian.

$$\mathcal{L}^{UV}_{\varphi} = (\partial_\mu \varphi^\dagger - ie_\varphi Z'_\mu \varphi^\dagger)(\partial^\mu \varphi + ie_\varphi Z'^\mu \varphi) - V_\varphi[\varphi^\dagger \varphi] \tag{3.47}$$

The potential $V_\varphi[\varphi^\dagger\varphi] = \mu^2_\varphi \varphi^\dagger \varphi + \frac{\lambda_\varphi}{3!}(\varphi^\dagger\varphi)^2$ takes on a non-trivial minimum if we choose $\mu^2_\varphi = -|\mu^2_\varphi| < 0$ and $\lambda_\varphi > 0$. The potential then triggers the Brout-Englert-Higgs effect. Working in unitary gauge, the tree-level propagator $D^{Z'}_{\mu\nu}$ of the dark photon Z'_μ [60, Chpt. 3.5]

$$D^{Z'}_{\mu\nu}(p) = \frac{1}{p^2 - 3e_\varphi \frac{|\mu^2_\varphi|}{\lambda_\varphi}} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{3e_\varphi \frac{|\mu^2_\varphi|}{\lambda_\varphi}} \right) \tag{3.48}$$

behaves like for a vector field of mass $m^2_Z = 3e_\varphi \frac{|\mu^2_\varphi|}{\lambda_\varphi}$. There also appears a scalar field $\tilde{\varphi}$ of mass $|\mu_\varphi|$ in the theory. For all phenomenology considerations we may vary the parameters $|\mu_\varphi|$ and λ_φ separately and thus adjust the mass of Z' and $\tilde{\varphi}$ independently. Henceforth, since there are no vertices coupling the scalar φ to the fermions $q^{(g)}$, the massive scalar mode $\tilde{\varphi}$ can be ignored in the low energy effective description, assuming it is sufficiently heavy and can be integrated out.

3.3 Classification of dark matter particles

In QCD we talk of mesons and baryons, which are defined as integer and half-integer spin particles, coupling to the strong force. Further, there exists a global symmetry, called $U(1)_B$ under which all the baryons get multiplied by a complex phase. Due to the fact that the fundamental representations of $SU(3)$ is complex, the mesons in QCD are necessarily made up of an equal number of quarks and anti-quarks.
In the dark theory we are investigating, the situation differs because of the reality of the gauge group representation $\mathcal{R}$. Mesons can not only be $\overline{q}q$ bound states, but also $\overline{q}q_\mathcal{C}$ bound states. This explains the enlarged amount of nine instead of three pNGBs in the theory. Moreover, there exists also an analog of the baryon number symmetry. However, also $\overline{q}q_\mathcal{C}$ transform with a non-trivial complex phase under this symmetry. Hence, the notion of a meson is not a priory clear. In the following we will refer to physical integer spin particles, coupling to the strong dark force, as dark mesons.
In this section we investigate which particles are the lightest states in the theory, relevant for dark matter phenomenology, as well as their transformation behavior under the symmetries discussed in the sections above. Besides the pNGBs, resulting from the spontaneously broken approximate flavor symmetry, there is another meson called $\tilde{\eta}$, related to a non-zero expectation value of the chiral condensate (3.31). This particle is a flavor singlet and the analog of the η' meson in QCD.

3.3.1 Spin zero dark mesons

In appendix F we construct all two quark scalar operators allowed by gauge invariance in this theory. A complete, linearly independent basis is given by

$$\mathcal{O}_a^S = \Psi^\top ES\omega T_a^\mathcal{F}\Psi - \Psi^\dagger ES\omega T_a^\mathcal{F}\Psi^* \tag{3.49}$$
$$\mathcal{O}_a^{PS} = \Psi^\top ES\omega T_a^\mathcal{F}\Psi + \Psi^\dagger ES\omega T_a^\mathcal{F}\Psi^* \tag{3.50}$$

with $a = 0, \ldots, 9$. The operators $\mathcal{O}_a^{PS}$ transform as pseudo-scalars, while $\mathcal{O}_a^S$ behave like proper scalars. The commutator of the pseudo-scalar operators with the charges Q_a, corresponding to the currents j_a^μ in (3.17), is proportional to the chiral condensate χ_c.

$$\langle 0| \left[Q_a(t'), \mathcal{O}_a^{PS}(x)\right] |0\rangle = \frac{\delta_{gh}}{2} \langle 0| \overline{q}^{(g)}(x) q^{(h)}(x) |0\rangle = 4\chi_c(x) \tag{3.51}$$

This can be proven by explicit calculation and the use of relation (2.11). By virtue of the Nambu-Goldstone theorem, the $\mathcal{O}_a^{PS}$ couple to the nine pNGB states in the theory. We may use them to derive the symmetry properties of the dark pion states $|\pi^a(p)\rangle$. However, we see that (3.51) also holds true for $a = 0$. The first part of the proof of the Nambu-Goldstone theorem only uses (3.51) and $\chi_c \neq 0$. Borrowing from the proof presented in [36, Chpt. 3] one thus may conclude that there exists a pseudo-scalar particle $\tilde{\eta}$, coupling to Q_0 and $\mathcal{O}_0^{PS}$ at low momentum.

$$\langle 0| Q_0 |\tilde{\eta}(0)\rangle \langle\tilde{\eta}(0)| \mathcal{O}_0^{PS} |0\rangle \neq 0 \tag{3.52}$$

Such a state would be the analog of the η' meson in QCD. However, even in the chiral limit, the symmetry corresponding to Q_0 is not realized in the quantum theory, due to the axial anomaly, and $\tilde{\eta}$ behaves more like a pNGB of finite mass. It will thus also differ in mass from the pions, due to a contribution from the anomaly, as is also indicated by the PCBC relation (3.30). In QCD this is considered the solution of the $U(1)_A$ [34, Chpt. 19.10], explaining why η' is so much heavier than the pions. However, we cannot make a general statement on how much heavier this particle may be in general theories. In $SU(N)$ QCD it is believed that for $N \to \infty$ the η' becomes mass-degenerate to the pions in the so-called 't Hooft limit. Also, with growing fermion mass m, the effects of the anomaly become comparatively smaller. In QCD the contribution of the heave strange quark to the mass η' meson must also be considered. Hence, the relevance of $\tilde{\eta}$

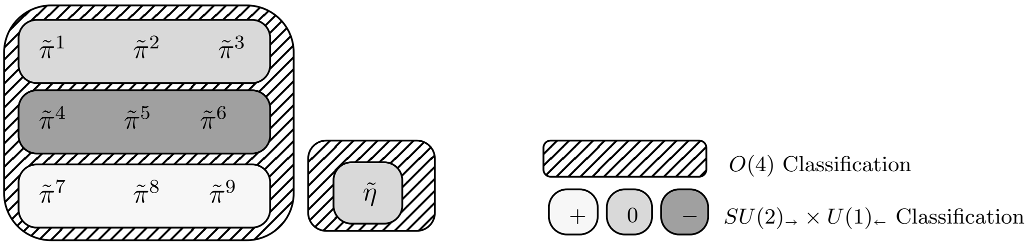


Figure 3.5: Classification of pseudo scalar meson states under the $O(4)$ flavor symmetry and its $SU(2)_\rightarrow \times U(1)_\leftarrow$ subgroup. The triplets are charged differently under $U(1)_D$. The grayscale indicates the charge of the particles.

depend on the details of the specific theory. In [61] it was pointed out that for an $Sp(4)$ theory with two fermions in the fundamental representation, the mass of the $\tilde{\eta}$-mass becomes close to the pions in the relevant parts of parameter space for DM phenomenology.
Equation (3.51) does not hold for the scalar states. Thus, the existence of light scalar states with a large overlap with these operators is not guaranteed. Our experience from QCD tells us that the scalar states are rather heavy, hence we will not consider them in this study. In order to study the multiplet structure of the π and $\tilde{\eta}$ states, we may look at the multiplet structure formed by the operators $\mathcal{O}_A^{PS}$ under $O(4)$ flavor transformation. When charging the fermions under $U(1)_D$, we used the fact that $SO(4) = SU(2)_\rightarrow \times SU(2)_\leftarrow$. The charge assignment breaks the flavor symmetry $O(4)$ down to $SU(2)_\rightarrow \times U(1)_\leftarrow$. The remaining symmetry allows for physical an interpretation by comparison with 2 flavor QCD. In appendix A.2 it is demonstrated that the $SU(2)_\rightarrow$ symmetry subgroup may be interpreted as the analog of Isospin in QCD. The symmetry $U(1)_\leftarrow$, being the one-parameter subgroup of $SU(2)_\leftarrow$ generated by $T^{\mathcal{F}}_{15}$, is the analog of the $U(1)_B$ symmetry in QCD. This idea is originally from [53]. Since the dark quarks transform in the same[8] manner under $U(1)_\leftarrow$ as by global $U(1)_D$ gauge transformation, we may use $U(1)_\leftarrow$ to classify the charge-eigenstates of the dark quarks. It is instructive to find a basis $|\tilde{\pi}^a(p)\rangle$ of pion states, which are simultaneous eigenstates of the quantum analog of the z-isospin generator in $SU(2)_\rightarrow$

$$I_z := \frac{1}{2} T^{\mathcal{F}}_{12} \tag{3.53}$$

and the charge assignment matrix $\mathcal{Q}$, defined in (3.44). The pion states $|\pi^a(p)\rangle$ are related to these eigenstates $|\tilde{\pi}^a(p)\rangle$ via a unitary transformation.

$$\begin{pmatrix} |\tilde{\pi}^1(p)\rangle \\ |\tilde{\pi}^2(p)\rangle \\ |\tilde{\pi}^3(p)\rangle \\ |\tilde{\pi}^4(p)\rangle \\ |\tilde{\pi}^5(p)\rangle \\ |\tilde{\pi}^6(p)\rangle \\ |\tilde{\pi}^7(p)\rangle \\ |\tilde{\pi}^8(p)\rangle \\ |\tilde{\pi}^9(p)\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} |\pi^1(p)\rangle \\ |\pi^2(p)\rangle \\ |\pi^3(p)\rangle \\ |\pi^4(p)\rangle \\ |\pi^5(p)\rangle \\ |\pi^6(p)\rangle \\ |\pi^7(p)\rangle \\ |\pi^8(p)\rangle \\ |\pi^9(p)\rangle \end{pmatrix} \tag{3.54}$$

Using the $\mathcal{O}_a^{PS}$ we deduce the multiplet structure of the $|\tilde{\pi}^a\rangle$ states under flavor symmetries. The result is summarized in figure 3.5. Under the full $O(4)$ symmetry, the pions form a nineplet, while the $\tilde{\eta}$ is a flavor singlet. Under $SU(2)_\rightarrow \times U(1)_\leftarrow$ the pions split into multiples under $SU(2)_\rightarrow$.

[8] It is important that global $U(1)_D$ gauge transformations are not identical with $U(1)_\leftarrow$ transformations, since the first also include a simultaneous transformation of the gauge fields, while the latter does not.

All states within such an Isospin multiplet are eigenstates of $\mathcal{Q}$, but of different charges $+1, 1$ and 0.
As a closing remark for this section, let us consider how the pion states would transform if the symmetry had been broken down further to $O(2) \times O(2)$ in the mass non-degenerate scenario. In this case, there always appears a singlet pNGB state. The breaking patterns of the mass degenerate and non-degenerate case are compared in figure 3.6. The appearance of flavor singlet pNGB is troublesome for the dark matter scenario considered here, since it spoils the stability argument of the DM. The pion state π^3 may spontaneously decay into the vacuum, lacking any symmetries forbidding such a decay, while simultaneously coupling to the rest of the pions by strong force interactions. This allows the other pion states to easily scatter into the unstable singlet, if the mass-differences are not too large. Hence, the dark matter is at best metastable or becomes unstable. In principle, $\tilde{\eta}$ has the same problem, but its increased mass may render the dark matter sufficiently meta-stable more easily. Hence, $\tilde{\eta}$ may in principle significantly impact the freeze out regime, but does not spoil the fact that we observe dark matter today. Details will be discussed in section 5.

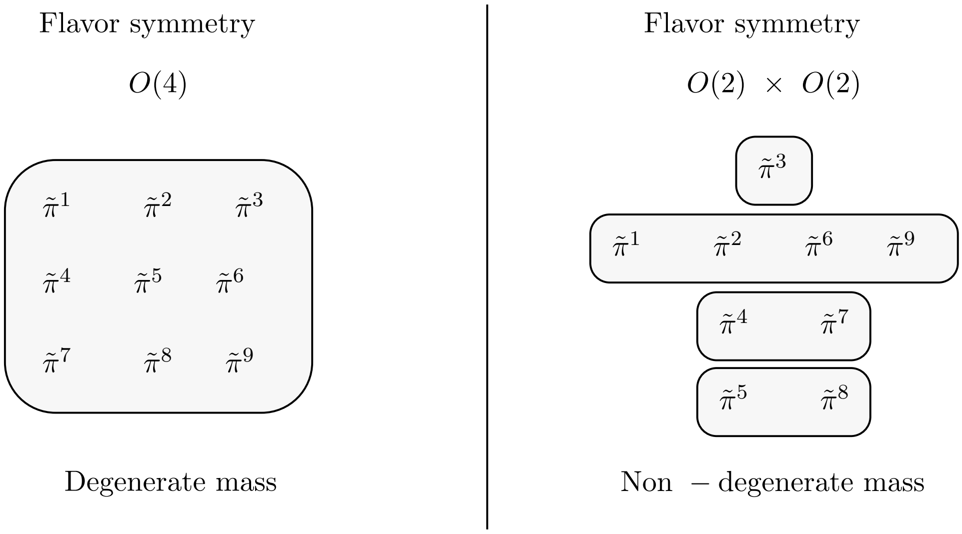


Figure 3.6: Comparison of pion flavor multiplet structure in the mass degenerate and mass non-degenerate case. In the latter, $\tilde{\pi}^3$ is a flavor singlet.

3.4 Large N_C limit

Below the conformal window of a strong, confining gauge theory there typically exists some scale Λ_C, called the confinement scale. In some sense the masses of the gauge-invariant composite states will depend on this scale. However, defining this scale precisely is a delicate task and typically the definition is ambiguous, differing between for example perturbative, functional and lattice approaches to QFT.
In the following we want to compare meson masses among gauge theories, whose gauge structure differs only by the value N_C. Especially we want to see what happens if N_C becomes arbitrarily large. But if we want to do this consistently we must ensure that the scale Λ_C for the theories we compare is approximately the same. In order to get an intuition on how to do this, we may look at a functional approach at one-loop level. We may define Λ_C as the spatial momentum scale at which the solution for the running gauge coupling g diverges at one loop, if approached from momentum scales of the asymptotic free regime of the theory.

$$\Lambda_C^2 = \Lambda_{UV}^2 \exp\left\{ \frac{(4\pi)^2}{\beta_0 g^2(\Lambda_{UV}^2)} \right\} \tag{3.55}$$

Here β_0 is the one-loop coefficient (3.39) of the β-function (3.38) and $\Lambda_{UV}^2 > \Lambda_C^2$ is some large UV cutoff. The value $g^2(\Lambda_{UV}^2)$ is the running coupling at Λ_{UV}^2. Equation (3.55) suggest that we may fix Λ_C for various theories if we demand $\beta_0 g^2(\Lambda_{UV}^2) = \text{fixed}$. This allows us to compare quantities among series of gauge theories with $G_D(N_C) = SU(N_C), Sp(2N_C), SO(N_C)$ and representations $\mathcal{R} : G_D(N_C) \to \mathrm{Aut}\,(V_C(N_C))$ in consistent manner. As it turns out, doing this for $SU(N_C)$ theories with N_F fundamental fermions, one may apply perturbative arguments for an expansion in powers of $1/N_C$ in the limit of $N_C \to \infty$ [62]. This is done by a geometrical argument. Looking at the color structure of Feynman diagrams one can assign each diagram an oriented surface such as a sphere, a torus or a more complicated structure like a pretzel. The contributions of each diagram are weighted by a factor N_C^χ, with χ a topological invariant called the "Euler character" of that surface. The dominant contribution for large N_C comes from diagrams which correspond to spheres, called planar diagrams [62]. Since arbitrarily large Feynman diagrams contribute all at the same order if they have the same geometry in color space, it was argued that such a large N_C expansion may capture effects lost in a perturbative coupling expansion. Further, it may even be applicable in the low energy regime where g^2 becomes very large. Features in meson phenomenology such as the Zweig rule can be understood in such a large N_C expansion [63]. Witten realized that in the 't Hooft large N_C limit [64] the η' meson may behave like an exact NGB in massless QCD, since the contribution of the anomaly in the PCAC relation is suppressed by a factor $g^2 \to 0$. In the following we want to see if we can also apply such arguments in order to investigate the physics of the $\tilde{\eta}$ meson in the dark matter theory considered in this thesis.

3.4.1 $\tilde{\eta}$ meson in the 't Hooft large N_C limit

In order to implement the large N_C limit more formally, for the theories considered in this thesis, we must start by assuming that the theories, with $N_F = 2$ fixed, are confining and stay below the conformal window as $N_C \to \infty$. The first assumption is plausible while the second may be checked by looking at conformal window investigations as in section 3.1.6. If these assumptions apply, it holds that $\beta_0 > 0$ and $\beta_1 > 0$ as N_C becomes arbitrarily large. By virtue of (3.39) and (3.40) this suggests

$$\beta_0 \underset{N_C\to\infty}{\approx} \mathcal{O}(c_{\text{adj}}) \qquad \beta_1 \underset{N_C\to\infty}{\approx} \mathcal{O}(c_{\text{adj}}^2) \tag{3.56}$$

In order to fix the confinement scale for large N_C, because we want to make the theories comparable, one may, instead of $\beta_0 g^2 = \text{fixed}$, demand that

$$\overline{g}(\Lambda_{UV}) := g^2(\Lambda_{UV}) c_{\text{adj}} = \text{fixed} \tag{3.57}$$

The β-function (3.38) then implies a renormalization group equation for $\overline{g}$.

$$\overline{\beta}(\overline{g}) = -\frac{2}{(4\pi)^2}\frac{\beta_0}{C_{\text{adj}}}\overline{g}^2 - \frac{2}{(4\pi)^5}\frac{\beta_1}{C_{\text{adj}}^2}\overline{g}^3 + \dots \tag{3.58}$$

Due to (3.56) this equation does not depend on N_C as $N_C \to \infty$. This implies that Λ_C, independent of its precise definition, may become comparable between theories. Further, since we are below the conformal window, we do not expect the appearance of any infrared fixed points for $\overline{g}$. Hence, it is plausible that the behavior of $\overline{\beta}(\overline{g})$ holds beyond two loop order. Now let us have a look at the PCBC relation (3.30) for the current associated with the $\tilde{\eta}$ meson.

$$\partial_\mu j_0^\mu = -i\, m\, \mathcal{O}_0^{\text{PS}} - \frac{T_{\mathcal{R}}}{c_{\text{adj}}}\,\overline{g}\,\frac{\epsilon^{\mu\nu\rho\sigma}\delta_{\alpha\beta}}{32\sqrt{2}\pi^2} G^\alpha_{\mu\nu} G^\beta_{\rho\sigma} \tag{3.59}$$

This result suggests that the anomalous contribution to the PCBC relation vanishes, if $T_R/c_{\text{adj}} \to 0$ for $N_C \to \infty$. This might suggest that the effects of the axial chiral anomaly vanish in this limit and the full $U(4)$ symmetry gets restored in the unbroken chiral phase of the theory. The $\tilde{\eta}$ meson would then behave as an additional NGB. However, as it is evident from equation (3.21), the effect of the anomaly in this case does not really vanish in the large N_C limit, because the phase shift in the action is always of the same finite order. Since this interpretation of η' as a NGB in the large N_C works quite well in QCD, let us however try to argue why this may also be a good way to think about $\tilde{\eta}$. Clearly the criterion

$$\frac{T_{\mathcal{R}}}{c_{\text{adj}}} \xrightarrow[N_C\to\infty]{} 0 \tag{3.60}$$

is physical in the sense that it does not depend on the arbitrary choice of Killing metric and field normalization. Changing the convention, discussed in section 2.1, would lead to a rescaling of the Dynkin index and Casimir number, but the ratio stays the same. Restricting ourselves to $G_D(N_C) = SU(N_C), Sp(2N_C), SO(N_C)$, the only theories for which the criterion (3.60) holds, are the fundamental representations of $SU(N_C)$ and $Sp(2N_C)$, as well as the vector representation of $SO(N_C)$. The last being the only real representation. For $SU(N_C)$ the behavior at large N_C is well explored and worked out in very much detail [63]. For $SO(N_C)$ the gauge fields satisfy an additional reality condition which leads to the fact there are additional surfaces which we can assign to Feynman diagrams in colors space. These correspond to nonorientable surfaces. However, these surfaces contribute only at higher orders in the $1/N_C$ expansion, due to their topological properties. Thus, at lowest order, we expect the large N_C behavior to be the same as in $SU(N_C)$-QCD [65]. It might thus be fair to interpret the suggestion of (3.59) as $\tilde{\eta}$ behaving like a pNGB of a spontaneously broken approximate symmetry. Its existence and transformation behavior under flavor transformations were already established in section (3.3). The symmetry breaking effects of the chiral anomaly for $\tilde{\eta}$, as suggested by (3.59), get suppressed for large N_C, compared to the explicit symmetry breaking contributions of the mass term.

4 Long range description

We derive phenomenological Lagrangians describing the dynamics of the dark pions and the $\tilde{\eta}$ at lowest chiral order $D = 2$. We focus separately on non-anomalous and anomalous processes, the latter leading to a discussion of the construction of Wess-Zumino terms in the theory. The dark photon is included by coupling it as an external source to the flavor currents of $H_F = O(4)$. Thus, most results hold for the pure dark strong sector. In this chapter we restrict to $SO(N_c)$-vector dark sectors.

4.1 Chiral Lagrangian for dark pions

In order to derive a low energy effective description for the dark pions we follow Weinbergs approach, explained in section 2.4. The relevant symmetry structure to be taken into account is the chiral symmetry $G_F = \mathbb{Z}_2 \ltimes SU(4)$, spontaneously broken to $H_F = O(4)$. A similar Lagrangian for the pions fields was already derived for QCD-like theories [66], however in a different context than dark matter. Also, these constructions did not mention the discrete $\mathbb{Z}_2$ symmetry. The approach presented here follows closely the derivation in QCD found in [41].
In order to account for the non-linear symmetry structure of the pNGBs, dictated by the spontaneous break of the symmetry, we start with the coset representative (2.55) of $G_F/H_F \cong SU(4)/SO(4)$ in the standard form (2.55)

$$\mathcal{F}\left[\gamma(x)\right] = \exp\left\{i\xi^a(x)T_a^{\mathcal{F}}\right\} \tag{4.1}$$

with $T_a^{\mathcal{F}}$ the nine broken generators of $SU(4)$ and $a = 1, \ldots, 9$. $\mathcal{F}$ denotes the fundamental representation. As explained in appendix A.2, the coset space $SU(4)/SO(4)$ is a symmetric space. This allows us to construct a coset representative that transforms linear [45] under G_F.

$$\Sigma_\pi = \mathcal{F}[\gamma]\,\omega\,(\mathcal{F}[\gamma])^\top = \mathcal{F}[\gamma]^2\omega \tag{4.2}$$

We used here the property $\omega(\mathcal{F}[\gamma])^\top = \mathcal{F}[\gamma]\omega$, related to the fact that the coset space is symmetric. The matrix ω was introduced in (3.15). Under a G_F transformation we then get that

$$\Sigma_\pi \mapsto \mathcal{F}[g\gamma h^{-1}]\,\omega\,(\mathcal{F}[g\gamma h^{-1}])^\top = U^{\mathcal{F}}\Sigma_\pi U^{\mathcal{F}\top} \tag{4.3}$$

where $U^{\mathcal{F}} = \mathcal{F}[g]$ is the representation matrix of g under $\mathcal{F}$ and $\mathcal{F}[h^{-1}]\,\omega\,(\mathcal{F}[h^{-1}])^\top = \omega$ due to (3.24). The group element $h = h[g, \xi^a] \in H_F$ depends non-linear on g and ξ^a. However, this non-linearity is hidden in Σ_π. We can interpret the dark pions as the local parametrization ξ^a of the coset space times some constant F_π and collect them in a matrix valued 0-form π.

$$\pi^a = F_\pi \xi^a \tag{4.4}$$

$$\pi = \pi^a T_a^{\mathcal{F}} \tag{4.5}$$

In order to later incorporate explicit symmetry breaking by the mass term we apply the method of external sources. For this we introduce an external, matrix-valued, scalar source X in the chiral UV Lagrangian.

$$\mathcal{L}^{UV}[X] = -\frac{1}{4g^2}G^\alpha_{\mu\nu}G^{\mu\nu}_\alpha + \Psi^\dagger\overline{\Sigma_\pi}^\mu \mathrm{D}^{\mathcal{R}}_\mu\left[G\right]\Psi - \frac{1}{2}\left(\Psi^\dagger ESX^*\Psi^* - \Psi^\top E^* S^* X\Psi\right) \tag{4.6}$$

The Lagrangian $\mathcal{L}_q^{UV}$ in (3.16) is obtained by setting $X = m\omega$. If we perform a G_F transformation of the fields $\Psi \mapsto U^{\mathcal{F}}\Psi$, which is explicitly broken by X, we may compensate the explicit breaking contribution by changing the source to a different value $X \to U^{\mathcal{F}} X U^{\mathcal{F}\top}$. For the purpose of DM phenomenology, contributions of lowest chiral order $D = 2$ are sufficient, since dark matter will be cold and external momenta in scattering processes low compared to relativistic scales. Since X controls the mass we count this external field as $\mathcal{O}(X) = \mathcal{O}(p^2)$. We list all lowest order terms, invariant under G_F.

$$\mathcal{O}(p^0) : \mathrm{tr}\Big\{\Sigma_\pi^\dagger \Sigma_\pi\Big\} = \mathrm{tr}\{\mathbb{1}\} = \text{const.} \tag{4.7}$$

$$\mathcal{O}(p^1) : \mathrm{tr}\Big\{\Sigma_\pi^\dagger \partial_\mu \Sigma_\pi\Big\} = -\,\mathrm{tr}\Big\{\Sigma_\pi \partial_\mu \Sigma_\pi^\dagger\Big\} = 0 \tag{4.8}$$

$$\mathcal{O}(p^2) : \mathrm{tr}\Big\{\partial_\mu \Sigma_\pi^\dagger \partial_\mu \Sigma_\pi\Big\} \overset{PI}{=} \mathrm{tr}\Big\{\Sigma_\pi^\dagger \partial_\mu \partial_\mu \Sigma_\pi\Big\} \neq 0 \tag{4.9}$$

$$\mathrm{tr}\Big\{X^\dagger \Sigma_\pi\Big\} \neq 0 \tag{4.10}$$

$$\mathrm{tr}\Big\{X \Sigma_\pi^\dagger\Big\} \neq 0 \tag{4.11}$$

The constant term is of little interest, since we may always remove a constant from the action. In (4.9) we have used partial integration (PI) under the assumption of appropriate boundary conditions. The vanishing of the $\mathcal{O}(p^1)$ term can be seen by the introduction of the so-called Maurer-Cartan 1-form, a concept which will prove useful later on.

$$\Omega_\mu := \exp\{i\xi^a T_a^{\mathcal{F}}\}\partial_\mu \exp\{-i\xi^a T_a^{\mathcal{F}}\} = \frac{1}{2}\gamma^2 \partial_\mu \gamma^{\dagger 2} \overset{F_\pi \xi^a = \pi^a}{=} \frac{1}{2}\Sigma_\pi \partial_\mu \Sigma_\pi^\dagger \tag{4.12}$$

The Maurer-Cartan form $\Omega = \Omega_\mu \mathrm{d}x^m u$ is a map into the Lie-algebra $\mathfrak{g}_F$. However, all elements of $\mathfrak{g}_F = \mathfrak{su}(4)$ are represented by traceless matrices and thus (4.8) vanishes. The most general local Lagrangian, compatible with the symmetry breaking structure $G_F \to H_F$, Lorentz invariance and parity is given by

$$\mathcal{L}_{(D=2)}^{IR} = C_1\,\mathrm{tr}\Big\{\partial_\mu \Sigma_\pi^\dagger \partial^\mu \Sigma_\pi\Big\} + C_2\,\mathrm{tr}\Big\{X^\dagger \Sigma_\pi + X \Sigma_\pi^\dagger\Big\} \tag{4.13}$$

Fixing $X = m\omega$ we obtain a manifestly $O(4)$ invariant Lagrangian incorporating the effects of explicit symmetry breaking by the degenerate mass term m. An expansion of the exponential function in $\Sigma_\pi = \exp\Big\{i\frac{2\pi}{F_\pi}\Big\}\omega$ further allows to fix the interpretation of the constants C_1 and C_2.

$$\Sigma_\pi \omega = \mathbb{1} + i\frac{2\pi}{F_\pi} - \frac{2\pi^2}{F_\pi^2} - i\frac{4}{3}\frac{\pi^3}{F_\pi^3} + \frac{2}{3}\frac{\pi^4}{F_\pi^4} + \mathcal{O}\left(\frac{\pi^5}{F_\pi^5}\right) \tag{4.14}$$

For the kinetic term we can fix C_1 by demanding that the pion fields are canonically normalized. Due to the normalization of the generators $\mathrm{tr}\{T_a^{\mathcal{F}} T_b^{\mathcal{F}}\} = \frac{1}{2}\delta_{ab}$ and $\pi^\dagger = \pi$ we see that

$$C_1\,\mathrm{tr}\Big\{\partial_\mu \Sigma_\pi^\dagger \partial^\mu \Sigma_\pi\Big\} = \sum_{a=1}^{9} \frac{4C_1}{F_\pi^2}\,\mathrm{tr}\{\partial_\mu \pi^a \partial^\mu \pi^a\} + \mathcal{O}\left(\frac{\pi^4}{F_\pi^4}\right) \tag{4.15}$$

and thus $C_1 = F_\pi^2/4$. The constant C_2 can be related to the value of the chiral condensate χ_C in (3.31). First, we want to fix C_2 such that the expectation value $\Sigma_c(x) := \langle 0|\,\Sigma(x)\,|0\rangle$ is aligned with the mass term e.g. $\Sigma_c = \omega$. The "classical" field Σ_c can be used as the argument of a quantum effective action description. Since the vacuum is supposed to be translation invariant $\Sigma_c(x) = \Sigma_c(0)$. Hence, we can determine Σ_c as the minimum of the quantum effective potential $V[\Sigma_c]$. At tree-level we have

$$V[\Sigma_c] = -mC_2\left(\mathrm{tr}\{\omega\Sigma_c\} + (\mathrm{tr}\{\omega\Sigma_c\})^*\right) \tag{4.16}$$

Observing that $\mathrm{Re}(\mathrm{tr}\{\omega^\dagger \Sigma_c\})$ is a positive definite inner product [66] on the space of unitary matrices we can conclude that $V[\Sigma_c]$ is extreme if $\Sigma_c = \pm\omega$. If $C_2 > 0$ then $\Sigma_c = \omega$ is the minimum. Due to $X = m\omega$ we may actually also interpret m as an external source field. In the domain of validity of the low energy effective action it should hold that

$$\frac{\delta Z^{UV}}{\delta m} \approx \frac{\delta Z^{IR}}{\delta m} \tag{4.17}$$

where the generating functional in the UV and IR are defined as path-integrals over the respective action.

$$Z^{UV}[m] := \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi}\mathcal{D}G \exp\{i\, S^{UV}[\Psi, G, m]\} \tag{4.18}$$

$$Z^{IR}[m] := \int \mathcal{D}\pi \exp\{i\, S^{UV}[\pi, m]\} \tag{4.19}$$

The functional differentiation gives

$$\frac{\delta Z^{UV}}{\delta m} = \frac{i}{2} \langle 0| \Psi^\dagger S E \omega \Psi^* |0\rangle - \langle 0| \Psi^\top S^* E^* \omega^* \Psi |0\rangle = \frac{i}{2}\chi_c \tag{4.20}$$

$$\frac{\delta Z^{IR}}{\delta m} = iC_2 \langle 0| \mathrm{tr}\{\omega\Sigma\} + (\mathrm{tr}\{\omega\Sigma\})^* |0\rangle = i2C_2\, \mathrm{tr}\{\omega^2\} = i8C_2 \tag{4.21}$$

and hence $16C_2 = \chi_c$. Expanding the exponential in the mass term to quadratic order one can read off a mass term for the pion fields. This is the analog of the Gell-Mann-Oakes-Renner (GMOR) relation in QCD.

$$m_\pi^2 = \frac{m\chi_c}{4F_\pi^2} \tag{4.22}$$

The result agrees with the relation presented in [66]. Further, the alignment of the vacuum expectation value $\Sigma_C = \omega$ allows to interpret the field Σ as fluctuations around the vacuum configurations. This becomes especially clear of we look at the definition (4.2). Further, as can be read of from (4.3), the broken, non-linearly realized transformation relate different, degenerate vacuum configurations. This fits with the discussion we had in section 2.2.

4.2 Inclusion of the dark $\tilde{\eta}$-meson

In order to include the $\tilde{\eta}$ into the low energy effective description we first introduce another external source field $\theta(x)$ into the theory. This field $\theta(x)$ couples to the topological charge density and will be used to organize the symmetry breaking contribution of the axial anomaly systematically. We follow closely the ideas of [67] where the external source method is used to derive the next to leading order, large N_C, effective Lagrangian for QCD, describing the physics of the QCD pions and the η' meson. The systematic application of a combined expansion in powers of $\frac{1}{N_C}$ and external momenta is discussed in [68] [69]. As in [67] the source term

$$\mathcal{L}^{UV}[X, \theta] = \mathcal{L}^{UV}[X] + \theta \frac{g^2 \epsilon^{\mu\nu\rho\sigma} \delta_{ab}}{64\pi^2} G^a_{\mu\nu} G^b_{\rho\sigma} \tag{4.23}$$

is added to the UV the Lagrangian in (4.6). In order for θ to cancel the contributions of the anomaly (3.20) it must transform accordingly under a $U(4)$ flavor transformation. For $U^F = \exp\{-\boldsymbol{\epsilon}\}$ we demand that

$$\theta \mapsto \theta - 2iT_{\mathcal{R}}\, \mathrm{tr}\{\boldsymbol{\epsilon}\} \tag{4.24}$$

With this the Lagrangian may be non-invariant under $U(4)$ symmetries, however the quantum theory is invariant and the explicit breaking contribution to the conservation of the current operator j_0^μ cancels exactly the anomalous contribution. In this theory the $U(4)$ symmetry is a good symmetry and may be broken explicitly by fixing θ to a constant value. For our dark sector we need to set $\theta = 0$ and $X = m\omega$, with $m > 0$. This is analogous to the argument used before with X in the Lagrangian (3.20). This further means that the $U(4)$ symmetry can be spontaneously broken to $O(4)$ and $\tilde{\eta}$ becomes an exact NGB in the chiral limit $X \to 0$. Since $U(4)/O(4)$ is also a symmetric space, we may construct the same coset representative as in (4.1) and (4.2), but summing over $a = 0, \ldots, 9$. Now we interpret

$$\tilde{\eta}^0 := F_{\tilde{\eta}}\, \xi^0 \tag{4.25}$$

$$\tilde{\eta} := \tilde{\eta}^0 T_0^{\mathcal{F}} \tag{4.26}$$

where $\sqrt{8}T_0^{\mathcal{F}} = \mathbb{1}$. As before in (4.4), the other local coset coordinates $\xi^{a>0}$ correspond to the dark pions. The $\tilde{\eta}$ enters then as a phase shift of the linear representative (4.2).

$$\Sigma = \exp\left\{2i\frac{\tilde{\eta}}{F_{\tilde{\eta}}}\right\}\Sigma_\pi \tag{4.27}$$

$$\sqrt{2}\tilde{\eta}^0 = -iF_{\tilde{\eta}} \ln\left(\det\left(\Sigma\right)\right) \tag{4.28}$$

With this, plus the linear transformation behavior $\Sigma \mapsto U^{\mathcal{F}}\Sigma U^{\mathcal{F}\top}$, one can deduce that $\tilde{\eta}$ gets shifted by the anomalous $U(4)$ symmetries. For $U^F = \exp\{-\boldsymbol{\epsilon}\}$ we obtain

$$\frac{\tilde{\eta}^0}{F_{\tilde{\eta}}} \mapsto \frac{\tilde{\eta}^0}{F_{\tilde{\eta}}} + i\sqrt{2}\,\mathrm{tr}\{\boldsymbol{\epsilon}\} \tag{4.29}$$

Before constructing all the $U(4)$ invariant quantities we can construct the Lagrangian, let us comment on θ under parity transformation. In order not to break parity explicitly by introduction of the source term, θ must be a pseudo-scalar field. This has further consequences. The combined field

$$Y := \frac{\eta_0}{F_{\tilde{\eta}}} + \frac{\theta}{\sqrt{2}T_{\mathcal{R}}} \tag{4.30}$$

is not only a pseudo-scalar, it is also flavor singlet under $U(4)$. Since $\mathcal{O}(\tilde{\eta}) = \mathcal{O}(p^0)$ by Weinberg power counting, we can construct Lorentz- and $U(4)$-invariant quantities of order $\mathcal{O}(p^0)$ by using any analytical function

$$W_k(Y) = W_k(0) + \sum_{l=1}^{\infty} C_k^l Y^l =: W_k(0) \tag{4.31}$$

Therefore, every term we can construct from Σ is multiplied by Y-dependent amplitudes, introducing an infinite amount of low energy effective constants (LEC). Of course, for a specific scattering process, we may reduce the system to a finite number of LEC's, due to loop suppression in the Weinberg counting scheme. As demonstrated below, adopting large N_C arguments, the relevant LEC's can be reduced even further. The complete list of $U(4)$ invariant terms of order $\mathcal{O}(p^2)$ is given by

$$\mathcal{O}(p^0) : \mathrm{tr}\left\{\Sigma^\dagger\Sigma\right\} = \mathrm{tr}\{\mathbb{1}\} = \text{const.} \tag{4.32}$$

$$W_k(Y) \tag{4.33}$$

$$\mathcal{O}(p^1) : \mathrm{tr}\left\{\Sigma^\dagger\partial_\mu\Sigma\right\} = i\sqrt{2}F_{\tilde{\eta}}^{-1}\partial^\mu\tilde{\eta}^0 \tag{4.34}$$

$$\partial^\mu\theta \tag{4.35}$$

$$\mathcal{O}(p^2) : \mathrm{tr}\left\{\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right\} \neq 0 \tag{4.36}$$

$$\mathrm{tr}\left\{X^\dagger\Sigma\right\} \neq 0 \tag{4.37}$$

$$\mathrm{tr}\left\{X\Sigma^\dagger\right\} \neq 0 \tag{4.38}$$

The non-vanishing of (4.34) can be traced back to the fact that $\mathfrak{u}(4)$-matrices are not all traceless. The $\mathcal{O}(p^1)$ terms can not appear alone in the Lagrangian, due to Lorentz-invariance. The most general local Lagrangian, compatible with the symmetry breaking structure $U(4) \to O(4)$, Lorentz-invariance and parity is given by

$$\begin{aligned} \mathcal{L}^{IR}_{(D=2)}[X,Y] = &W_0(Y) + W_1(Y)\,\mathrm{tr}\Big\{\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\Big\} + W_2(Y)\,\mathrm{tr}\Big\{\Sigma^\dagger \partial_\mu \Sigma\Big\}\,\mathrm{tr}\Big\{\Sigma \partial^\mu \Sigma^\dagger\Big\} \\ &+ W_3(Y)\,\mathrm{tr}\Big\{X^\dagger \Sigma + X\Sigma^\dagger\Big\} + iW_4(Y)\,\mathrm{tr}\Big\{X^\dagger\Sigma - X\Sigma^\dagger\Big\} \\ &+ W_5(Y)\,\mathrm{tr}\Big\{\Sigma^\dagger\partial_\mu\Sigma\Big\}\partial^\mu\theta + W_6(Y)\partial_\mu\theta\partial^\mu\theta \end{aligned} \tag{4.39}$$

The first line gives the terms that remain if all external sources are switched off, while the second line gives the explicit symmetry breaking contributions incorporated by the quark mass term. In order to obtain back the Lagrangian of the theory we fix the external sources $X = m\omega$ and $\theta = 0$. Then we obtain

$$\begin{aligned} \mathcal{L}^{IR}_{(D=2)} = &W_0\left(\frac{\tilde{\eta}^0}{F_{\tilde{\eta}}}\right) + W_1\left(\frac{\tilde{\eta}^0}{F_{\tilde{\eta}}}\right)\mathrm{tr}\Big\{\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\Big\} + W_2\left(\frac{\tilde{\eta}^0}{F_{\tilde{\eta}}}\right)\left(\mathrm{tr}\Big\{\Sigma^\dagger\partial_\mu\Sigma\Big\}\right)^2 \\ &+ W_3\left(\frac{\tilde{\eta}^0}{F_{\tilde{\eta}}}\right)\mathrm{tr}\Big\{X^\dagger\Sigma + X\Sigma^\dagger\Big\} + iW_4\left(\frac{\tilde{\eta}^0}{F_{\tilde{\eta}}}\right)\mathrm{tr}\Big\{X^\dagger\Sigma - X\Sigma^\dagger\Big\} \end{aligned} \tag{4.40}$$

The symmetry breaking effects of the axial anomaly are captured by the functions W_k due to the affine shift (4.29) of $\tilde{\eta}$ under anomalous $U(4)$ transformations. The explicit breaking effects of the mass term are incorporated by the second line. The Lagrangian obtained is manifestly invariant under $O(4)$ transformations. In order to get the low energy effective constants better under control let us employ a phenomenological large N_C argument, similar to [67], based on the assumption that $\tilde{\eta}$ behaves like a pNGB if $N_C \to \infty$. The quantity $W_0(0)$ is a constant term in the Lagrangian that can be ommitted. For the interpretation of $W_1(0)$ and $W_2(0)$ we use

$$\mathrm{tr}\Big\{\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\Big\} = \mathrm{tr}\Big\{\partial_\mu\Sigma_\pi^\dagger\partial^\mu\Sigma_\pi\Big\} + \frac{4}{F_{\tilde{\eta}}^2}\,\mathrm{tr}\{\partial_\mu\tilde{\eta}\partial^\mu\tilde{\eta}\} \tag{4.41}$$

$$\mathrm{tr}\Big\{\Sigma^\dagger\partial_\mu\Sigma\Big\}\,\mathrm{tr}\Big\{\Sigma\partial^\mu\Sigma^\dagger\Big\} = \frac{4}{F_{\tilde{\eta}}^2}\,\mathrm{tr}\{\partial_\mu\tilde{\eta}\partial^\mu\tilde{\eta}\} \tag{4.42}$$

Hence, the freedom in choosing $W_1(0)$ and $W_2(0)$ independently simply reflects the fact that the decay constants F_π and $F_{\tilde{\eta}}$ must not be equal. This is in agreement with the fact that, under the physical $O(4)$ flavor symmetry, these two particles furnish different sectors of a reducible representation of $O(4)$. As shown in [45], the non-linear realization does not mix these subspaces, even for non-linear realized G_F transformations. The only way to introduce interactions between these sectors is by breaking the G_F symmetry explicitly, for example via mass term. By choosing

$$W_1(0) = \frac{F_\pi^2}{4} \qquad\qquad W_2(0) = \frac{F_{\tilde{\eta}}^2 - F_\pi^2}{4} \tag{4.43}$$

we achieve a canonical normalization of the fields π and $\tilde{\eta}$. The constant $W_3(0)$ scales the mass term of the pions, determining the GMOR relation. Due to parity, W_4 must be an odd function. Hence, $W_4(0) = 0$. If it holds that $\tilde{\eta}$, in the limit of $N_C \to \infty$, behaves like a PNGB, it must hold that all the functions W_k become constant in this limit. Or rather, the coefficients C_k^l must all vanish as N_C becomes infinitely large. For finite but large N_C we can thus conclude that

$$\mathcal{O}(C_k^l) = \mathcal{O}(\Delta) \qquad \text{and} \qquad \Delta \xrightarrow[N_C\to\infty]{} 0 \tag{4.44}$$

From the lowest, non-constant term in W_2 we obtain a contribution to the mass of $\tilde{\eta}$, which thus differs from the pion mass at the order $\mathcal{O}(\Delta)$. If we hence estimate that $\mathcal{O}(\Delta) = \mathcal{O}(p^2) = \mathcal{O}(m_\pi)$, we can combine this with Weinberg's power counting rules to determine the important terms for a phenomenological description of cold dark matter with sufficiently large N_C. The lowest order terms correspond to order $\mathcal{O}(p^2) = \mathcal{O}(\Delta^1)$. However, this still does not restrict the coefficient function W_0, since this term is of order $\mathcal{O}(p^0)$ and our simplistic argument does not distinguish between different coefficients C_0^l. In QCD we would expect that terms for larger indices l are stronger suppressed in the large N_C limit. For DM we can make an alternative argument why we drop these terms. For DM we want to look into processes $\tilde{\eta}\pi \leftrightarrow \pi\pi$ or $\tilde{\eta}\tilde{\eta} \leftrightarrow \pi\pi$. Vertices with powers of η^l and $l \geq 4$ only occur in loops in the corresponding $\mathcal{M}$-matrix elements. Thus, they are stronger suppressed at lower momenta. Hence, we can neglect them. The lowest order Lagrangian, capturing the essential features of $\tilde{\eta}$ for dark matter with sufficiently large N_C is

$$\begin{aligned}\mathcal{L}^{IR}_{(\Delta)} = \frac{F_\pi^2}{4}\operatorname{tr}\Big\{\partial_\mu \Sigma_\pi^\dagger \partial^\mu \Sigma_\pi\Big\} + \operatorname{tr}\{\partial_\mu\tilde{\eta}\partial^\mu\tilde{\eta}\} \\ + W_3(0)\operatorname{tr}\Big\{X^\dagger\Sigma + X\Sigma^\dagger\Big\} - \frac{\Delta m_{\tilde{\eta}}^2}{2F_{\tilde{\eta}}^2}\tilde{\eta}^0\tilde{\eta}^0\end{aligned} \tag{4.45}$$

The interpretation of $16W_3(0) = \chi_c$ works as in the section before. The only thing that is different is that we must argue that for the ground state $\langle\Sigma_C\rangle = \omega$ holds. To show this with the effective potential method at tree-level one must require $\Delta m_{\tilde{\eta}}^2 > 0$ and use (4.28) in order to relate $\tilde{\eta}$ in the breaking term with Σ. Thus, for the pions the GMOR relation (4.22) remains valid, while for the $\tilde{\eta}$ mass it holds

$$m_{\tilde{\eta}}^2 = \frac{m\chi_c}{4F_{\tilde{\eta}}^2} + \frac{\Delta m_{\tilde{\eta}}^2}{F_{\tilde{\eta}}^2} \tag{4.46}$$

A few remarks are in order. Our result is in agreement with what we would expect from QCD [69]. Our derivation however gives little information on the N_C-scaling of the coefficients we dropped. This is not as elegant as the method Leutwyler [68] applied in QCD. However, our argument is plausible enough to convince the reader of the validity of the low energy effective Lagrangian in light of the phenomenological purposes we derived it for, without the necessity to go to deep into large N_C arguments of $SO(N_C)$-vector theories. Furthermore, at the moment we did not check if the large N_C counting rules are applicable for $SO(N_C)$-vector in the same way as in QCD. Although the argument, given in section 3.4, suggests that this may be a plausible assumption. If we naively apply the methods known from QCD, the result for the Lagrangian is the same. Additionally, we would obtain two rather useful results, which we miss in our line of arguments. First, applying the large N_C rules of Kaiser and Leutwyler [69], we would see that $W_2(0)$ vanishes faster than $W_1(0)$ in the large N_C limit, letting us conclude that $F_{\tilde{\eta}} \to F_\pi$ for $N_C \to \infty$. This is known as a result of the Zweig rule in QCD. Further, we would see that $\Delta m_{\tilde{\eta}}^2/m_\pi^2 = \mathcal{O}(1/N_C)$. Our method incorporates the phenomenological argument that the result holds if $\tilde{\eta}$ is comparatively close in mass to the dark pions. For future applications it would be interesting to make a decisive statement on the applicability of the QCD rules or their possible modifications. Especially, in order to relate the mass of $\tilde{\eta}$ meson with the scaling of the Wess-Zumino terms, explained in the next section, via the common parameter N_C.

4.3 Wess-Zumino terms

So far all the terms we described in the low energy effective Lagrangian only describe scattering of an even number of particles. For dark matter applications a $3 \to 2$ cannibalization process is essential for phenomenology [18]. The description so far can not contain such a vertex due to the pseudo-scalar nature of the particles π and $\tilde{\eta}$. Further, we observe that the Lagrangians (4.13) and (4.45) possess a symmetry

$$\xi^a(t,\vec{x}) \mapsto -\xi^a(t,\vec{x}) \tag{4.47}$$

we will refer to as "naive parity". Compared to actual spatial parity $\boldsymbol{P}$, the conjugation $\vec{x} \mapsto -\vec{x}$ is missing. Although useful for formal statements on the vertices, this naive parity has no analog in the UV and is thus nonphysical and superfluous. In QCD the lowest order term introducing a $3 \to 2$ process is the so-called Wess-Zumino-Witten (WZW) term, first derived in [16] by Wess and Zumino. This term also removes the superfluous symmetry from the low energy effective description. Witten provided an elegant geometrical construction [17], which later on was generalized by D'Hoker and Weinberg to a large class of theories of fields $\Sigma : S^4 \to G_F/H_F$, where G_F/H_F is a connected coset space that satisfies the topological condition $\boldsymbol{\pi}_4(G_F/H_F) = 0$. Here $\boldsymbol{\pi}_4$ denotes the so-called "fourth homotopy group" of a topological space. This criterion enters because the construction uses that for all maps Σ, there exists a continuous deformation that allows to deform Σ into a constant function on S^4, mapping all points onto a fixed origin of G_F/H_F. This means that all the maps Σ, considered in the path-integral, must lie in the same equivalence class under continuous deformations. These equivalence classes form the group $\boldsymbol{\pi}_4\left(G_F/H_F\right)$. However, since the path-integral takes into account every representative Σ of all such equivalence classes, it must hold that $\boldsymbol{\pi}_4\left(G_F/H_F\right)$ can contain only one element, since all Σ are supposed to be within the same class. $\boldsymbol{\pi}_4\left(G_F/H_F\right)$ must thus be trivial. The connection from the sphere S^4 to a topologically trivial spacetime like Minkowski space $\mathcal{M}^4$ may be established by appropriate boundary conditions. If all the fields on $\mathcal{M}^4$ vanish in the infinite, it is equivalent to a compactification of spacetime to a four dimensional sphere. Another topological condition, namely $\boldsymbol{\pi}_5\left(G_F/H_F\right) = \mathbb{Z}$, allows one to conclude that the coefficient of the WZW term in the low energy effective theory must be quantized e.g. it can only take integer values. In the case of QCD this integer is the number if colors [17]. If one wants to promote a subgroup of the flavor symmetry to a gauge symmetry, the geometrical construction also allows to covariantize these terms in QCD-like theories by use of the iterative Noether method [17]. A more systematic construction of these "gauged" WZW terms was worked out by D'Hoker [25]. Recently these gauged WZW terms gained some attention in [27], where the authors provide a more ready-to-use version of the D'Hoker construction and provide explicit expressions for different use cases.
By virtue of Botts periodicity theorem [70] the precondition $\boldsymbol{\pi}_4\left(G_F/H_F\right) = 0$ holds for almost all theories. However, as demonstrated in appendix A.3, for the case of $SO(N_C)$ QCD with two Dirac flavors, this condition does not hold. This is troublesome because the geometrical construction of WZW terms, gauged or not, is not available. There have been less restrictive approaches, trying to classify all topological terms of Wess-Zumino type in non-linear sigma models for homogenous spaces G_F/H_F over a general spacetime [71]. As demonstrated in [71], their classification and construction can also be seen as a generalization of the ungauged WZW terms discussed before. However, the abstract formula for the Wess-Zumino terms, provided in [71], is of limited use for concrete applications. Although applicable for some use cases in composite Higgs theories [72], their formula does not provide a useful expression for the cases of the symmetric coset space $SU(4)/SO(4)$ we are dealing with.
We thus retreat to the original argument of Wess and Zumino [16] in order to derive the WZW term. In this approach the WZW term emerges as a solution of an anomalous Ward identity, calculated in the UV theory and transferred to the IR via an anomaly matching argument by the 't Hoof. In contrast to the previous methods, where the WZW is constructed only from the knowledge of the symmetries of the effective theory, this approach directly connects the properties of the underlying UV theory and effective theory in the IR. This approach is not new and was mostly derived in a work by Chu, Ho and Zumino [26]. They, as stated in their abstract, did this with having in mind the condition $\boldsymbol{\pi}_4\left(G_F/H_F\right) = 0$ to be fulfilled. However, going through the derivations in their paper, there exists a useful preliminary result, valid even if this condition is not satisfied. This result, although more complicated than the expression given by Wittens construction, is still very convenient. Further, this method fixes the coefficient of the WZW term, without the need to separately inspect anomalous decays of the dark pions to a dark photon. In what following we explain the details.

4.3.1 't Hooft anomaly matching argument

There exists an argument, originally made by 't Hooft [73], that proposes that the anomalous WTIs of a theory in the UV should carry over to the low energy effective description in the IR. This has far-reaching consequences for the modelling of the low energy effective theory. In QCD the argument predicts that the low energy effective action of the theory can not be written as spacetime integral over a Lagrange density, if one wants the capture the correct structure of the WTIs. In section 2.3 we already introduced the generating functional $\tilde{\mathrm{W}}$ in (2.18). The WTIs can be obtained by appropriate functional derivatives of $\tilde{\mathrm{W}}$. We will now rephrase 't Hoofts argument, tailored towards our theory, in terms of $\tilde{\mathrm{W}}$. Essentially it is a four-step argument:
Step 1: Consider the strongly interacting gauge theory, described in section 3.1, in the chiral limit $m = 0$. By covariantization, we may promote the global flavor symmetry to a local symmetry. This introduces fictitious background gauge fields A^N_μ for all the generators of the global symmetry. For now let us restrict to the $SU(4)$ generators only. We also have the dynamical gauge fields G^α_μ for the dark gluons. A functional $\tilde{\mathrm{W}}[A, G]$ can be obtained as in (2.18) by path-integrating the exponential of the covariantized action $S^{UV}_{\text{cov.}}[q, A, G]$ over the dark quarks. An arbitrary background gauge variation of $\tilde{\mathrm{W}}[A, G]$ will produce an anomaly, proportional to the D - symbol, defined in (2.44).

$$D_{\alpha\beta\gamma} = \frac{1}{2}\,\mathrm{tr}\big\{T^{\mathcal{R}}_\alpha\left(T^{\mathcal{R}}_\beta T^{\mathcal{R}}_\gamma + T^{\mathcal{R}}_\gamma T^{\mathcal{R}}_\beta\right)\big\} = 0 \tag{4.48}$$

$$D_{N\alpha\beta} = \frac{1}{2}\,\mathrm{tr}\big\{T^{\mathcal{F}}_N\big\}\,\mathrm{tr}\big\{T^{\mathcal{R}}_\alpha T^{\mathcal{R}}_\beta + T^{\mathcal{R}}_\beta T^{\mathcal{R}}_\gamma\big\} = 0 \tag{4.49}$$

$$D_{\alpha NM} = \frac{1}{2}\,\mathrm{tr}\big\{T^{\mathcal{R}}_\alpha\big\}\,\mathrm{tr}\big\{T^{\mathcal{F}}_N T^{\mathcal{F}}_M + T^{\mathcal{F}}_M T^{\mathcal{F}}_N\big\} = 0 \tag{4.50}$$

$$D_{NMK} = \frac{1}{2}\,\mathrm{tr}\big\{T^{\mathcal{F}}_N\left(T^{\mathcal{F}}_M T^{\mathcal{F}}_K + T^{\mathcal{F}}_K T^{\mathcal{F}}_M\right)\big\} \neq 0 \tag{4.51}$$

The first equation vanishes because $\mathcal{R}$ is a real representation. The second and third equation vanish because the generators of G_D and $SU(4)$ are traceless. The last D-Symbol does not vanish, as can be shown by explicit calculation. The first three lines indicate that a gauge variation, only concerning the Gluon fields, is not anomalous. These field can thus be considered as dynamical and can be integrated out in the path-integral, together with the Yang-Mills term, describing their free dynamics. The functional $\tilde{\mathrm{W}}[A]$, obtained by integrating out the gluons, depends only on the fictitious gauge fields A^N_μ. A background gauge variation of this functional, with respect to the fields A^N_μ, produces an anomaly proportional to the D-symbol in (4.51). In the decoupling limit $A^N_\mu \to 0$ we obtain back the partition function of the strong dark sector $\tilde{\mathrm{W}}[0] = -i \ln Z$.
Step 2: We add a Lagrangian of four massless fictitious Weyl spectator fermions, transforming under the complex conjugate representation $\overline{\mathcal{F}}$ of the fundamental representation of $SU(4)$, to the theory. If the $SU(4)$ symmetry is now promoted to a local symmetry and if we follow the same steps as before to obtain a functional $\tilde{\mathrm{W}}[A]$, the theory is now free of anomalies. The anomaly of the dark sector is canceled by the anomaly produced by the fictitious spectator fermions.
Step 3: If the external gauge fields are considered to be coupled arbitrarily weak to the fermions, they will not affect the strong sector dynamics. Hence, in the low energy regime, we may approximate the action of the dark sector by an effective action. The fictitious gauge fields are massless and interact very weakly. We may consider this description further valid in the IR. Since the theory was free of anomalies, it should be also in the IR.
Step 4: Since the description of the fermions did not change in the IR, they produce exactly the same anomaly. However, this means that the action of the low energy effective description must also produce the same anomaly in the UV, in order to cancel the anomaly of the spectator fermions. However, if there are no fermions in the IR description of the strong sector, the anomaly cannot arise from the path-integral measure.

From this argument we can conclude that there must be a term in the low energy effective theory, that reproduces the anomaly under a gauge variation, if the global symmetry is promoted to a local one. Formally, if we derive the functional $\tilde{\mathrm{W}}^{IR}[A]$ form the low energy effective action in the IR, it must hold

$$\delta_{\boldsymbol{\epsilon}}\tilde{\mathrm{W}}^{UV}[A] = \mathcal{A}[\boldsymbol{\epsilon}, A] \qquad \Rightarrow \qquad \delta_{\boldsymbol{\epsilon}}\tilde{\mathrm{W}}^{IR}[A] = \mathcal{A}[\boldsymbol{\epsilon}, A] \tag{4.52}$$

Here $\boldsymbol{\epsilon} : \mathcal{M} \to \mathfrak{su}(4)$ is any local function parametrizing the infinitesimal background gauge variation as in section 2.3.1. In the next subsection we will derive the WZW term as a functional that must be added to the low energy effective action in order for $\tilde{\mathrm{W}}^{IR}[A]$ to satisfy the anomaly equation (4.52).

4.3.2 $SU(4)/SO(4)$ Wess-Zumino term à la Chu, Ho and Zumino

For the following consider a functional $S_{WZ}[\xi, A]$, that depends on the external gauge fields A_μ^N, associated with the gauged flavor symmetry, and the NGB fields ξ^a. The NGB obey the non-linear transformation law (2.57) under global flavor transformations. For the localized symmetry, such a transformation may be used to define a functional derivative operator $\Xi_N(x)$, acting only on the NGBs ξ^a. Given a local flavor symmetry transformation matrix $\mathcal{F}[g(x)] = \exp\{-\boldsymbol{\epsilon}(x)\} = \exp\{i\epsilon^N(x)T_N^{\mathcal{F}}\}$, we define

$$\exp\left\{\int \mathrm{d}^4y\ \epsilon^N(y)\Xi_N(y)\right\} \mathrm{e}^{i\xi^a T_a^{\mathcal{F}}} = \mathcal{F}[g]\,\mathrm{e}^{i\xi^a T_a^{\mathcal{F}}}\mathcal{F}[h^{-1}] \tag{4.53}$$

where $h = h[g, \xi] \in H_F$ is given by the H_F transformation in (2.57) depending non-linear on g and ξ^a. A background gauge variation of $S_{WZ}[\xi, A]$, parametrized by a local function $\boldsymbol{\epsilon} : \mathcal{M} \to \mathfrak{su}(4)$ is given by

$$\delta_{\boldsymbol{\epsilon}} S_{WZ}[\xi, A] := \int \mathrm{d}^4x\ \left(\epsilon^N(x)\mathfrak{D}_N(x)S_{WZ}[\xi, A] + \epsilon^N(x)\Xi_N(x)S_{WZ}[\xi, A]\right) \tag{4.54}$$

Here $\mathfrak{D}_N(x)$ is the functional derivative operator, introduced in (2.33), acting on the background gauge fields. It was pointed out by Wess and Zumino [16] that, if one can construct a functional $S_{WZ}[\xi, A]$ for the action of the IR description, such that

$$\delta_{\boldsymbol{\epsilon}} S_{WZ}[\xi, A] = \mathcal{A}[\boldsymbol{\epsilon}, A] \tag{4.55}$$

the IR theory will satisfy the same anomalous WTIs as the underlying UV theory. To see why this works one has to make two observations. First the anomaly equation (4.55) behaves like an inhomogeneous linear differential equation. This means, we may add any homogenous solution S_0, with $\delta_{\boldsymbol{\epsilon}} S_0 = 0$, to a particular solution of (4.55) to obtain a new solution. Second, the functional $\tilde{\mathrm{W}}[A]$ constructed from the action $S^{IR} = S_0 + S_{WZ}$ satisfies the 't Hooft anomaly matching condition (4.52) because

$$\delta_{\boldsymbol{\epsilon}}\tilde{\mathrm{W}}[A] = \frac{-i}{\tilde{Z}[A]}\int \mathcal{D}\xi\ \left(i\delta_{\boldsymbol{\epsilon}} S^{IR}\right)\ e^{iS^{IR}[\xi,A]} = \mathcal{A}[\boldsymbol{\epsilon}, A]\frac{1}{\tilde{Z}[A]}\int \mathcal{D}\xi\ e^{iS^{IR}[\xi,A]} = \mathcal{A}[\boldsymbol{\epsilon}, A]$$

Now, the only task left is to determine the anomaly in the UV and then solve the anomaly equation (4.55) in the IR. First let us determine the anomaly in the UV. The consistency condition constrains the general form of the anomaly up to an overall scale and non-essential terms. The non-essential terms may be determined by enforcing an additional condition that fixes the regularization scheme. For the background gauge transformation of the fictitious fields A_μ^N, there is no physical constraint by gauge-invariance. However, since the D-symbol vanishes for triangle diagrams of three unbroken currents, due to the reality of $SO(4)$, we may adopt a regularization scheme such that the anomaly shows up only in the divergence of the broken currents. In terms of the anomaly this can be phrased as the following condition.

$$\forall \boldsymbol{\epsilon} \in \mathfrak{h} = \mathfrak{so}(4) \ : \ \mathcal{A}[\boldsymbol{\epsilon}, A] = 0 \tag{4.56}$$

Now, there exists a further constraint on the anomaly by parity. We may decompose the Lie-algebra $\mathfrak{su}(4) = \mathfrak{h} \oplus \mathbf{k}$ into a direct sum, where $\mathfrak{h} = \mathfrak{so}(4)$ and $\mathbf{k}$ is a linear space, spanned by the broken generators $T_a^{\mathcal{F}}$. Remember these are determined by the condition (3.25). The algebra $\mathfrak{so}(4)$ is spanned by the unbroken generators of the flavor symmetry. Every $\mathfrak{su}(4)$-valued 1-form $A = -iA_\mu^N T_N^{\mathcal{F}} \mathrm{d}x^\mu$ may be decomposed into an $\mathfrak{h}$-valued 1-form A_h and a $\mathbf{k}$-valued 1-form A_k.

$$A = A_h + A_k \tag{4.57}$$

Similarly, the curvature 2-form $F = \mathrm{d}A + A^2 = -\frac{i}{2} F_{\mu\nu}^N T_N^{\mathcal{F}}\, dx^\mu \wedge dx^\nu$ may be split as $F = F_h + F_k$ with $F_h = \mathrm{d}A_h + A_h^2 + A_k^2$ and $F_k = \mathrm{d}A_k + A_h A_k + A_k A_h$. Now, we may define a linear automorphism[1] $\hat{\sigma} : \mathfrak{su}(4) \to \mathfrak{su}(4)$ by

$$\hat{\sigma}(A) = \hat{\sigma}(A_h + A_k) := A_h - A_k \tag{4.58}$$

If we require that external fields valued in $\mathfrak{h}$ transform as vectors under Lorentz transformations and $\mathbf{k}$-valued fields transform as pseudo-vector fields, we obtain the following transformation rule under parity.

$$A_\mu^A(t, \vec{x}) \overset{\boldsymbol{P}}{\longmapsto} g_{\mu\mu} A_\mu^A(t, -\vec{x}) \tag{4.59}$$

$$A_\mu^a(t, \vec{x}) \overset{\boldsymbol{P}}{\longmapsto} -g_{\mu\mu} A_\mu^a(t, -\vec{x}) \tag{4.60}$$

or formulated in matrix valued forms

$$A_h(t, \vec{x}) \overset{\boldsymbol{P}}{\longmapsto} A_h(t, -\vec{x}) \tag{4.61}$$

$$A_k(t, \vec{x}) \overset{\boldsymbol{P}}{\longmapsto} -A_k(t, -\vec{x}) \tag{4.62}$$

$$A(t, \vec{x}) \overset{\boldsymbol{P}}{\longmapsto} \hat{\sigma}(A(t, -\vec{x})) \tag{4.63}$$

Using the relations for broken and unbroken generators, one may show $\omega A_h \omega = -A_h^\top$ and $\omega A_k \omega = A_k^\top$. Equipped with these definitions and recalling the definition of spatial parity (3.34), one can prove, by explicit calculation, that the flavor-gauged version of the Lagrangian L_{q}^{UV} in (3.16)

$$L_{\mathrm{q,\,cov.}}^{UV}[\Psi, G, A] = \Psi^\dagger \bar{\sigma}^\mu (\partial_\mu + gG_\mu + A_\mu)\Psi \tag{4.64}$$

behaves such that the action $S_{\mathrm{q,\,cov.}}^{UV}[\Psi, G, A] = S_{\mathrm{q,\,cov.}}^{UV}[\boldsymbol{P}\Psi, \boldsymbol{P}G, \boldsymbol{P}A]$ is invariant under parity. Since parity was a good symmetry and left the path-integral measure invariant, we get $\tilde{\mathrm{W}}[A] = \tilde{\mathrm{W}}[\boldsymbol{P}A]$. From (2.29) we can conclude that the anomaly functional must have the same symmetry under parity, e.g. $\mathcal{A}[\boldsymbol{\epsilon}, A] = \mathcal{A}[\boldsymbol{\epsilon}, \boldsymbol{P}A]$. Up to an overall scale, Chu, Ho and Zumino [26] provide an explicit expression for an anomaly functional, satisfying the parity constraint and (4.56), by counterterm shifting the canonical anomaly with a gauge variation $\delta_{\boldsymbol{\epsilon}} B$ of some local functional B, followed by projecting out the relevant positive parity part. Using their result, we get

$$\mathcal{A}[\boldsymbol{\epsilon}, A] \propto \mathrm{tr}\big\{\boldsymbol{\epsilon}_k \left(3F_h^2 + F_k^2 - 4(A_k^2 F_h + A_k F_h A_k + F_h A_k^2) + 8A_k^4\right)\big\} \tag{4.65}$$

with $2\boldsymbol{\epsilon}_k = \boldsymbol{\epsilon} + \omega \boldsymbol{\epsilon}^\top \omega$. The overall scale can be determined by going back to the perturbative calculation of the canonical anomaly. If we rescale all expression of the canonical anomaly in

[1] The fact that we can decompose $\mathfrak{su}(4)$ into a direct sum and the automorphism $\hat{\sigma}$ are closely related to the fact that the coset space $SU(4)/SO(4)$ of the spontaneous symmetry breaking pattern is symmetric. For more details see appendix A.2.

[26] to match our result obtained in (2.49), we can fix the correct scale. This gives an overall factor of $\frac{i}{24\pi^2}$. This result (2.49) was obtained under the implicit assumption that at every spacetime point, the fermions span the representation space of the flavor symmetry (since, after localization, this is the relevant gauge symmetry producing the anomaly). However, in our theory there are also color-indices, which have nothing to do with the flavor indices. Thus, if $d_{\mathcal{R}}$ is the dimensionality of the gauge group representation $\mathcal{R}$, there exist $d_{\mathcal{R}}$ copies of the flavor representation space, one for each color index. Thus, the anomaly is produced $d_{\mathcal{R}}$ times under a local flavor transformation. A factor d_R must be multiplied to the overall scale. In total, we obtain

$$\mathcal{A}[\boldsymbol{\epsilon}, A] = \frac{id_{\mathcal{R}}}{24\pi^2}\operatorname{tr}\left\{\boldsymbol{\epsilon}_k\left(3F_h^2 + F_k^2 - 4(A_k^2F_h + A_kF_hA_k + F_hA_k^2) + 8A_k^4\right)\right\} \tag{4.66}$$

Now the only task left to solve is the solution of the anomaly equation (4.55). However, thanks to Chu, Ho and Zumino [26], this task is already done. We define the parametrized gauge field

$$\begin{aligned} A_\tau :=& \exp\left\{-\tau\int \mathrm{d}^4y\ \xi^a(y)\mathfrak{D}_a(y)\right\}A \\ =& \exp\left\{-i\tau\xi^aT_a^{\mathcal{F}}\right\}A\exp\left\{i\tau\xi^aT_a^{\mathcal{F}}\right\} + \exp\left\{-i\tau\xi^aT_a^{\mathcal{F}}\right\}\mathrm{d}\exp\left\{i\tau\xi^aT_a^{\mathcal{F}}\right\} \end{aligned} \tag{4.67}$$

with $\tau \in [0,1]$ and d the exterior derivative of matrix valued forms. It was proven in [26] that, if the anomaly satisfies the condition (4.56), the following functional provides a particular solution to anomaly equation (4.55).

$$S_{WZW}[\xi, A] = \int_0^1 \mathrm{d}\tau \int \xi^a \mathcal{A}[-iT_a^{\mathcal{F}}, A_\tau] \tag{4.68}$$

Note that the NGB fields ξ^a enter not only through the explicit factor in front of the anomaly functional but also through A_t withing the expression of the anomaly functional. The expression (4.68) is of great phenomenological use. To see this, we first must set the physical point. As pointed out in [26] the functional $S_{WZ}[\xi, A]$, although not invariant under global G_F transformations, is invariant under local H_F transformations. This means we can assign dynamics to the fields A_h in a well-defined manner. We will use this to introduce the dark photon in a later section. However, for now we set $A = 0$ to recover the theory with the global flavor symmetry G_F. As we can see then from (4.67), the parametrized gauge field

$$A_\tau = \Omega_\tau = \exp\left\{-i\tau\xi^aT_a^{\mathcal{F}}\right\}\mathrm{d}\exp\left\{i\tau\xi^aT_a^{\mathcal{F}}\right\} \tag{4.69}$$

takes on the form of a pure gauge for every value of t. This means that $F_\tau = \mathrm{d}A_\tau + A_\tau^2 = 0$ and hence also $(F_\tau)_h = (F_\tau)_k = 0$. The Wess-Zumino action then reduces to

$$S_{WZW}[\xi] = \frac{d_{\mathcal{R}}}{3\pi^2}\int_0^1 \mathrm{d}\tau \int \xi^a \operatorname{tr}\left\{T_a^{\mathcal{F}}(\Omega_\tau)_k(\Omega_\tau)_k(\Omega_\tau)_k(\Omega_\tau)_k\right\} \tag{4.70}$$

This expression is invariant under global $H_F = \mathbb{Z}_2 \ltimes SO(4)$ transformations, since $T_a^{\mathcal{F}} \mapsto U^{\mathcal{F}\dagger}T_a^{\mathcal{F}}U^{\mathcal{F}}$ and $\Omega_\tau \mapsto U^{\mathcal{F}\dagger}\Omega_\tau U^{\mathcal{F}}$ for global $U^{\mathcal{F}} \in H_F$. According to [26], the ungauged WZW-action is also invariant under global G_F transformations. For phenomenological applications, one only needs to expand Ω_τ in a Taylor series and then integrate out t term by term. In the lowest order expansion we have $\Omega_\tau \approx i\tau\mathrm{d}\xi^aT_a^{\mathcal{F}}$. To this order $\Omega_\tau = (\Omega_\tau)_k$, because the lowest term is a $\mathbf{k}$-valued function. Then the expression for the WZW term becomes

$$S_{WZ}[\xi] \approx \frac{(i)^4d_{\mathcal{R}}}{3\pi^2}\int_0^1 \mathrm{d}\tau\ \tau^4 \int \xi^a\mathrm{d}\xi^b \wedge \mathrm{d}\xi^c \wedge \mathrm{d}\xi^c \wedge \mathrm{d}\xi^e \operatorname{tr}\left\{T_a^{\mathcal{F}}T_b^{\mathcal{F}}T_c^{\mathcal{F}}T_d^{\mathcal{F}}T_e^{\mathcal{F}}\right\} \tag{4.71}$$

$$= \frac{d_{\mathcal{R}}}{15\pi^2}\operatorname{tr}\left\{T_a^{\mathcal{F}}T_b^{\mathcal{F}}T_c^{\mathcal{F}}T_d^{\mathcal{F}}T_e^{\mathcal{F}}\right\}\ \epsilon^{\mu\nu\rho\sigma}\int d^4x\ \xi^a\partial_\mu\xi^b\partial_\nu\xi^c\partial_\rho\xi^c\partial_\sigma\xi^e \tag{4.72}$$

This gives exactly the NGB five point vertices we require in order to implement a $3 \to 2$ freeze out in the dark sector. Now a few comments are in order. First, note that the low energy effective Lagrangian (4.13) in the chiral limit can be made gauge-invariant under gauged flavor transformation by simply replacing the derivatives with a covariant one. This means that the action vanishes under an infinitesimal flavor background gauge transformation and hence is a solution of the homogenous anomaly equation. Thus, the WZW term may be added to the massless pion Lagrangian in order to model a low energy effective theory, also taking into account the anomalous WTIs. Further, note that the S_{WZ} action is parity invariant, but not invariant under the naive parity transformation, where we do not transform $(t, \vec{x})$ [26]. Thus, the WZW term also removes the superfluous symmetry. In the case of massive pions, we cannot gauge the full symmetry if the mass term is fixed. However, if we introduce the mass term via the external source X, and impose the correct transformation behavior for X, the derivation works exactly the same. We fix X only when we set the external fields A to the physical point. The result for the WZW term remains the same.
The lowest order expression for the WZW term may be rewritten further. We introduce the following antisymmetrized trace

$$\mathrm{tr}\Big\{T_a^{\mathcal{F}} T_{[b_1}^{\mathcal{F}} T_{b_2}^{\mathcal{F}} T_{b_3}^{\mathcal{F}} T_{b_4]}^{\mathcal{F}}\Big\} := \sum_{\sigma \in S_4} \mathrm{sign}(\sigma)\, \mathrm{tr}\Big\{T_a^{\mathcal{F}} T_{b_{\sigma(1)}}^{\mathcal{F}} T_{b_{\sigma(2)}}^{\mathcal{F}} T_{b_{\sigma(3)}}^{\mathcal{F}} T_{b_{\sigma(4)}}^{\mathcal{F}}\Big\} \tag{4.73}$$

where S_4 is the group of all permutations of four elements. Then, due to the antisymmetry of the wedge product of 1-forms, we get (no summation convention)

$$\mathcal{L}_{D=4}^{WZ} = \frac{d_{\mathcal{R}}}{15\pi^2} \sum_{a=1}^{9} \sum_{1 \le b < c < d < e \le 9} \cdots \sum \mathrm{tr}\Big\{T_a^{\mathcal{F}} T_{[b}^{\mathcal{F}} T_c^{\mathcal{F}} T_d^{\mathcal{F}} T_{e]}^{\mathcal{F}}\Big\}\, \epsilon^{\mu\nu\rho\sigma}\, \xi^a \partial_\mu \xi^b \partial_\nu \xi^c \partial_\rho \xi^d \partial_\sigma \xi^e \tag{4.74}$$

4.3.3 WZ-term and the $\tilde{\eta}$-meson

So far we have taken into account only the $SU(4)$ flavor symmetry. Now we try to extend the discussion above to the $U(4)$ symmetry of the Lagrangian. Consider again the Lagrangian $\mathcal{L}_{\mathrm{q}}^{UV}$ and we covariantize the full $U(4)$ symmetry. The first problem arises in the anomaly matching argument when we try to integrate out the dynamical gluon fields in order to construct the functional $\tilde{\mathrm{W}}[A]$, because (4.49) does not vanish anymore. This is because the generator $T_0^{\mathcal{F}}$ is not traceless. However, since the D-symbol (4.48), involving only gauge generators $T_\alpha^{\mathcal{R}}$, vanishes due to the reality of $\mathcal{R}$, we can choose the regularization scheme such that the anomaly shows up only in the divergence of the flavor currents. Since the D-symbol, involving only unbroken currents, also vanishes due to the reality of $\mathfrak{so}(4)$, we may even put the anomaly only in the divergence of the broken flavor currents. This allows us to integrate out the gauge fields of the gluons, while simultaneously satisfying the condition (4.56). The anomaly matching argument now works without a problem. We may also extend the splitting of the Lie-algebra to $\mathfrak{u}(4) = \mathfrak{so}(4) \oplus \mathbf{k}$. Here $\mathbf{k}$ is again generated by the broken generators, now also involving $T_0^{\mathcal{F}}$. The discussion of the parity constraint and the derivation of the anomaly stay exactly the same as before. Hence, the rest of the argument follows along the same lines as above. Formally, we obtain the same result for the WZ-action (4.70) as before. The only difference now is that we have one more ξ^0 field, which we interpret as the $\tilde{\eta}$-meson according to (4.25). However, the $\tilde{\eta}$-field enters the parametrized Maurer-Cartan form Ω_τ in a very simply way, due to its flavor-singlet nature.

$$\Omega_\tau = \exp\Big\{-i\tau \frac{\tilde{\eta}}{f_{\tilde{\eta}}}\Big\} \mathrm{d} \exp\Big\{i\tau \frac{\tilde{\eta}}{f_{\tilde{\eta}}}\Big\} + \exp\Big\{-i\tau \frac{\pi}{f_\pi}\Big\} \mathrm{d} \exp\Big\{i\tau \frac{\pi}{f_\pi}\Big\} = \Omega_\tau^{\tilde{\eta}} + \Omega_\tau^{\pi} \tag{4.75}$$

Further it holds $F_{\tilde{\eta}} \Omega_t^{\eta} = it\mathrm{d}\tilde{\eta}$ and $\tilde{\eta}$ is a $\mathbf{k}$-valued field. Thus, $(\Omega_\tau)_k = \Omega_\tau^{\tilde{\eta}} + (\Omega_\tau^{\pi})_k$. However, because $T_0^{\mathcal{F}}$ commutes with all the other generators, including itself, we obtain $(\Omega_\tau^{\tilde{\eta}})^2 = -(\Omega_\tau^{\tilde{\eta}})^2$

and thus $(\Omega_\tau^{\tilde{\eta}})^2 = 0$. Further, we may conclude that $(\Omega_\tau^\pi)_k \Omega_\tau^{\tilde{\eta}} = -\Omega_\tau^{\tilde{\eta}}(\Omega_\tau^\pi)_k$. Hence, we get

$$(\Omega_\tau)_k^2 = (\Omega_\tau^\pi)_k^2 + (\Omega_\tau^\eta)^2 + (\Omega_\tau^\pi)_k \Omega_\tau^{\tilde{\eta}} + \Omega_\tau^{\tilde{\eta}}(\Omega_\tau^\pi)_k = (\Omega_\tau^\pi)_k^2 \tag{4.76}$$

and the $\tilde{\eta}$-meson can only enter via ξ^a in front of the trace of (4.70). Thus, by virtue of (4.74), a vertex from the WZW-term including $\tilde{\eta}$ must be proportional to

$$\mathrm{tr}\Big\{ \mathbb{1}\, T^{\mathcal{F}}_{[b} T^{\mathcal{F}}_c T^{\mathcal{F}}_d T^{\mathcal{F}}_{e]} \Big\} = 0 \tag{4.77}$$

That the totally anti-symmetrized trace of four $SU(4)$ generators vanishes can be checked by explicit calculation or the use of the identities found in [74]. Hence, the $\tilde{\eta}$ mesons drops out of the WZW-term completely. Therefore, we get no new interactions by extending the discussion to $U(4)$ flavor symmetry anomalies. More importantly, the $\tilde{\eta}$ meson will not contribute to the $3 \to 2$ freeze-out process. The situation changes if we couple the dark photon to the theory.

4.4 Inclusion of dark photon

So far the discussion describes the strong dark sector in isolation. A dark photon mediator may be added by gauging a one-parameter subgroup of the flavor symmetry. Mass is added to the dark photon via an abelian BEH-mechanism. The involved scalar mode $\tilde{\varphi}$ can be made arbitrarily heavy and is thus considered to be integrated out in the low energy effective theory. To describe the dynamics of Z' we add the following term to the low energy effective Lagrangian.

$$\mathcal{L}^{IR}_{Z'} = -\frac{1}{4} F'^{\mu\nu} F'_{\mu\nu} - \frac{m_Z^2}{2} Z'_\mu Z'^\mu + \frac{\varepsilon}{\cos(\omega_W)} F'_{\mu\nu} B^{\mu\nu} \tag{4.78}$$

Here again ε is the kinetic mixing parameter, $B_{\mu\nu}$ is the SM hypercharge field strength tensor and Ω_W the Weinberg mixing angle. These are the same parameter as in the UV. The mass m_Z^2 is a free parameter of the effective theory, related to the UV via (3.48). The running of the $U(1)_D$ coupling behaves like in QED. For low momenta the $U(1)_D$ coupling will be small compared to the strong coupling g of G_D. Hence, for modelling the low energy effective theory, we consider the field Z'_μ as a small perturbation to the strong dark sector. The effective Lagrangian may be obtained by gauging the correct flavor subgroup in the Lagrangian derived so far. This will introduce explicit breaking terms, further reducing the global flavor symmetry. In the case relevant for dark matter, with charge assignment $\mathcal{Q} = \sqrt{8} T^{\mathcal{F}}_{15}$, the remaining symmetry is $SU(2)_\rightarrow \times U(1)_\leftarrow$. This symmetry is generated by $T^{\mathcal{F}}_{10}, T^{\mathcal{F}}_{11}, T^{\mathcal{F}}_{12}$ and $T^{\mathcal{F}}_{10}$. In the following we will discuss anomalous and non-anomalous parts of the action separately.

4.4.1 Non-anomalous part of the Lagrangian

The use of the linear representative Σ in (4.2) of the coset space makes the gauging of a one-parameter subgroup convenient. Under a local $U(4)$ transformation we have

$$\Sigma_\pi \mapsto U^{\mathcal{F}} \Sigma_\pi U^{\mathcal{F}T} \tag{4.79}$$

Thus, in order for the Lagrangian (4.13) and (4.45) to be invariant under a local transformation generated by $\mathcal{Q}$, we must replace every partial derivative $\partial_\mu \Sigma_\pi$ with a covariant one.

$$\mathrm{D}_\mu \left[Z'\right] \Sigma_\pi = \partial_\mu \Sigma_\pi - i e_D Z'_\mu \left(\mathcal{Q} \Sigma_\pi + \Sigma_\pi \mathcal{Q}^\top \right) \tag{4.80}$$

Here e_D is the coupling of the $U(1)_D$, representing the bare dark charge unit. The $\tilde{\eta}$-meson is a singlet under $SO(4)$ transformations. Thus, we do not need to introduce a covariant derivative for it. The only term in the lowest order effective theory that gets modified is the pion kinetic term.

$$\operatorname{tr}\Big\{(\mathrm{D}_\mu\Sigma_\pi)^\dagger\,\mathrm{D}^\mu\Sigma_\pi\Big\} = \operatorname{tr}\Big\{(\partial_\mu\Sigma_\pi)^\dagger\,\partial^\mu\Sigma_\pi\Big\} \tag{4.81}$$

$$- 2i\,e_D Z'_\mu\ \left(\operatorname{tr}\Big\{\Sigma_\pi\partial^\mu\Sigma_\pi^\dagger\mathcal{Q}\Big\} - \operatorname{tr}\Big\{\Sigma_\pi^\dagger\partial^\mu\Sigma_\pi\mathcal{Q}^\top\Big\}\right) \tag{4.82}$$

$$+ 2\,e_D^2 Z'_\mu Z'^\mu\ \left(\operatorname{tr}\Big\{\Sigma_\pi\mathcal{Q}^\top\Sigma_\pi^\dagger\mathcal{Q}\Big\} + \operatorname{tr}\big\{\mathcal{Q}^2\big\}\right) \tag{4.83}$$

The last term in this expression seems a bit off, because there is a term that would add to the mass of the dark photon, dependent on the charge assignment. However, this is of course not the case and the expression simply looks misleading. The expression becomes more instructive if we also expand Σ_π in a Taylor series. Using (4.14) we obtain

$$(\Sigma\omega)^\dagger\,\partial^\mu\,(\Sigma_\pi\omega) = i\frac{2}{F_\pi}\partial^\mu\pi + \frac{2}{F_\pi^2}\left(\pi\partial^\mu\pi - (\partial^\mu\pi)\pi\right) + \mathcal{O}(\pi^3) \tag{4.84}$$

$$(\Sigma\omega)\,\partial^\mu\,(\Sigma_\pi\omega)^\dagger = -i\frac{2}{F_\pi}\partial^\mu\pi + \frac{2}{F_\pi^2}\left(\pi\partial^\mu\pi - (\partial^\mu\pi)\pi\right) + \mathcal{O}(\pi^3) \tag{4.85}$$

Using this expansion, together with the relations $\omega\mathcal{Q}\omega = -\mathcal{Q}^\top$ and $\mathcal{Q} = \mathcal{Q}^\dagger$ allows to conveniently expand the part of the Lagrangian that describes interactions between the dark photon and the dark pions.

$$\mathcal{L}^{IR}_{\pi-Z'} = 2\,e_D^2\,Z'_\mu Z'^\mu\operatorname{tr}\{\pi^2\mathcal{Q}^2 - \pi\mathcal{Q}\pi\mathcal{Q}\} \tag{4.86}$$

$$- 2i\,e_D\,Z'_\mu\operatorname{tr}\{(\pi\partial^\mu\pi - (\partial^\mu\pi)\pi)\,\mathcal{Q}\} + \mathcal{O}(\pi^4) \tag{4.87}$$

This gives $Z'Z'\pi\pi$ and $Z'\pi\pi$ interactions at lowest order χPT . Since Z'_μ is introduced via the covariant derivative we must count it as $\mathcal{O}(Z'_\mu) = \mathcal{O}(p^1)$ in Weinberg's power counting rules. Using the charge assignment (3.44) and the corresponding pion eigenbasis $\tilde{\pi}^a$ we obtain

$$\begin{aligned}\mathcal{L}^{IR}_{\pi-Z'} &= 4\,e_D^2\,Z'_\mu Z'^\mu\left(\tilde{\pi}^4\tilde{\pi}^7 + \tilde{\pi}^5\tilde{\pi}^8 + \tilde{\pi}^6\tilde{\pi}^9\right)\\ &\quad - 2i\,e_D\,Z'_\mu\left(\tilde{\pi}^4\partial_\mu\tilde{\pi}^7 - \tilde{\pi}^7\partial_\mu\tilde{\pi}^4 + \tilde{\pi}^5\partial_\mu\tilde{\pi}^8 - \tilde{\pi}^5\partial_\mu\tilde{\pi}^8 + \tilde{\pi}^6\partial_\mu\tilde{\pi}^9 - \tilde{\pi}^6\partial_\mu\tilde{\pi}^9\right)\end{aligned}$$

As can be seen, by comparison with figure 3.5, interactions take place between dark pions of opposite charge. Only particle anti-particle pairs are involved in the vertices. For example $\tilde{\pi}^4$ is the antiparticle to $\tilde{\pi}^7$. This can also be seen by the explicit action of the charge conjugation operation (3.36) on the interpolating operators (3.50). On the level of pion fields, charge conjugation of a single quark or both quarks acts as

$$\left(q^{(1)}, q^{(2)}\right) \mapsto \left(q_C^{(1)}, q^{(2)}\right) \quad \Rightarrow \quad \pi \mapsto C_u\pi C_u \tag{4.88}$$

$$\left(q^{(1)}, q^{(2)}\right) \mapsto \left(q_C^{(1)}, q_C^{(2)}\right) \quad \Rightarrow \quad \pi \mapsto \omega\pi\omega = \pi^* \tag{4.89}$$

The matrices C_u and ω where introduced in (3.37) and (3.15). We see that the $\mathbb{Z}_2$ semifactor of the flavor symmetry $O(4) = \mathbb{Z}_2 \ltimes SO(4)$, related to charge conjugating only one Dirac fermion, is now explicitly broken by the interaction terms between the dark pions and the dark photon. This is also not surprising from a representation theoretic point of view. The existence of the $\mathbb{Z}_2$ symmetry was due to the reality of the gauge group representation under which each Weyl fermion transforms. However, after gauging a one parameter subgroup of the flavor symmetry, each Weyl fermion transforms under a complex representation of the overall gauge group $G_D \times U(1)_D$. The Lagrangian also describes kinetic interactions between the pions, mediated by Z'. This becomes manifest if we us (4.89) to rewrite the Lagrangian.

$$\begin{aligned}\mathcal{L}^{IR}_{\pi-Z'} &= 2\,e_D^2\,Z'_\mu Z'^\mu\left(\tilde{\pi}^4\tilde{\pi}^{4*} + \tilde{\pi}^7\tilde{\pi}^{7*} + \tilde{\pi}^5\tilde{\pi}^{5*} + \tilde{\pi}^8\tilde{\pi}^{8*} + \tilde{\pi}^6\tilde{\pi}^{6*} + \tilde{\pi}^9\tilde{\pi}^{9*}\right)\\ &\quad - i\,e_D\,Z'_\mu\left(\tilde{\pi}^4\partial_\mu\tilde{\pi}^{4*} - \tilde{\pi}^{4*}\partial_\mu\tilde{\pi}^4 + \tilde{\pi}^5\partial_\mu\tilde{\pi}^{5*} - \tilde{\pi}^{5*}\partial_\mu\tilde{\pi}^5 + \tilde{\pi}^6\partial_\mu\tilde{\pi}^{6*} - \tilde{\pi}^{6*}\partial_\mu\tilde{\pi}^6\right)\\ &\quad + i\,e_D\,Z'_\mu\left(\tilde{\pi}^7\partial_\mu\tilde{\pi}^{7*} - \tilde{\pi}^{7*}\partial_\mu\tilde{\pi}^7 + \tilde{\pi}^8\partial_\mu\tilde{\pi}^{8*} - \tilde{\pi}^{8*}\partial_\mu\tilde{\pi}^8 + \tilde{\pi}^9\partial_\mu\tilde{\pi}^{9*} - \tilde{\pi}^{9*}\partial_\mu\tilde{\pi}^9\right)\end{aligned}$$

4.4.2 Anomalous part of the Lagrangian

In order to derive anomalous contributions to the Lagrangian we again solve the anomaly equation (4.55). The general solution is already given by (4.68). However, now the physical point is set to $A_\mu = -ie_D Z_\mu \mathcal{Q}$. Then we have for the parametrized gauge field

$$A_\tau = -ie_D Z'_\mu \, \mathrm{e}^{-i\tau\xi^a T_a^{\mathcal{F}}} \, \mathcal{Q} \, \mathrm{e}^{i\tau\xi^a T_a^{\mathcal{F}}} \, \mathrm{d}x^\mu + \Omega_\tau$$

In general now $(F_\tau)_h \neq 0$ and$(F_\tau)_k \neq 0$. The expression for the gauged WZW term thus becomes much more complicated, but contains all the information on the anomalous processes involving the dark photon. In order to access the information systematically, a perturbative expansion is required. The lowest order expansion, up to order $\mathcal{O}(\tau)$, is given by

$$A_\tau = -i \left(e_D Z'_\mu \mathcal{Q} - \tau \partial_\mu \xi^a T_a^{\mathcal{F}} + i\tau e_D Z'_\mu \xi^a [\mathcal{Q}, T_a^{\mathcal{F}}] \right) \mathrm{d}x^\mu + \mathcal{O}(\tau^2) \tag{4.90}$$

From the perturbative expansion one can easily read off the splitting $A_\tau = (A_\tau)_h + (A_\tau)_k$. Here we have $\mathcal{Q} \in \mathfrak{h}$, $T_a^{\mathcal{F}} \in \mathbf{k}$ and $[\mathcal{Q}, T_a^{\mathcal{F}}] \in \mathbf{k}$.

$$(A_\tau)_h = -ie_D Z'_\mu \mathcal{Q} \mathrm{d}x^\mu + \mathcal{O}(\tau^2) \tag{4.91}$$

$$(A_\tau)_k = i\tau \partial_\mu \xi^a T_a^{\mathcal{F}} \mathrm{d}x^\mu + \tau e_D Z'_\mu \xi^a [\mathcal{Q}, T_a^{\mathcal{F}}] \mathrm{d}x^\mu + \mathcal{O}(\tau^3) \tag{4.92}$$

The expression for $(A_\tau)_h$ and $(A_\tau)_k$ must then be plugged in (4.68). The expression (4.71) is of order $\mathcal{O}(\tau^4)$, so it seems that it cannot be recovered in this expansion. However, the way $(A_\tau)_h$ and $(A_\tau)_k$ are organized in (4.66) allows to retrieve this term in consistent manner. All terms that involve $(F_\tau)_h$ or $(F_\tau)_k$ vanish in the limit $A \to 0$. This means that terms of the type $\tau^4 \mathrm{d}\xi\mathrm{d}\xi\mathrm{d}\xi\mathrm{d}\xi$ can only occur in $(A_\tau)_k^4$. However, since $(A_\tau)_k \propto \tau$, the term $\tau^4 \mathrm{d}\xi\mathrm{d}\xi\mathrm{d}\xi\mathrm{d}\xi$ is the lowest order contribution to $(A_\tau)_k^4$. This term thus reproduces the expression (4.71), if we do not drop the lowest contribution. Since the rest of the terms do not contribute to (4.71), we use (4.91) and (4.92) for these terms and drop all terms of order $\mathcal{O}(\tau^2)$ or higher. When doing so we find a $\xi^a Z' Z'$ vertex. This vertex describes an anomalous decay process of a pion or the $\tilde{\eta}$-meson into two dark photons Z'.

$$\mathcal{L}^{IR}_{\xi Z' Z'} = -\epsilon^{\mu\nu\rho\sigma} \partial_\mu Z'_\nu \partial_\rho Z'_\sigma \frac{3 d_{\mathcal{R}} e_D^2}{24\pi^2} \xi^a \, \mathrm{tr}\{T_a^F \mathcal{Q}\mathcal{Q}\} \tag{4.93}$$

Comparing with the discussion in section 2.3.4, this vertex looks like the anomalous current divergence contribution to j_a^μ from a $j_a - U(1)_D - U(1)_D$ triangle diagram calculation, contracted to a point. A graphical depiction of this process in the UV and IR is given in figure 4.1. The gauge fields Z'_μ, associated with $U(1)_D$, are dynamical. Thus, in order to compare the vertex with the triangle diagram calculation in section 2.3.4, the regularization scheme must be chosen such that the corresponding currents are free of anomalies, putting the anomaly solely in the divergence of j_a^μ. Further, we rediscover the anomaly cancellation condition (3.45), already discussed for the charge assignments in section (3.2). For the pion currents, this vertex vanishes because $\mathcal{Q}^2 \propto \mathbb{1}$ and $T_a^{\mathcal{F}}$ is traceless for $a = 1, \ldots, 9$.

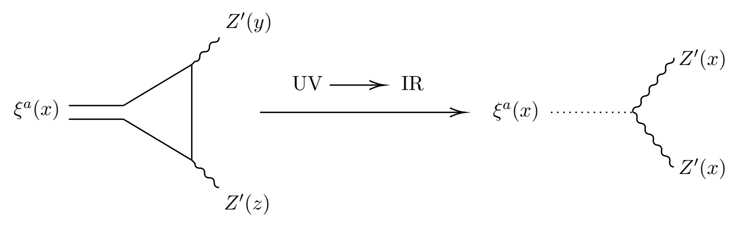


Figure 4.1: Diagrammatic depiction of the relation between the anomalous decay $\xi^a \to Z'Z'$ in the UV and IR.

As discussed in section 4.3.3, we may extend the derivation of the WZW term such that we can include the $\tilde{\eta}$ meson. If the dark photon is absent, the result is the same. However, in the presence of the dark photon, an anomalous decay channel for $\tilde{\eta}$ opens. The flavor-singlet meson $\tilde{\eta}$, although neutral under $U(1)_D$, decays via the process (4.93), since $T_0^{\mathcal{F}}$ is not traceless. The phenomenological implications of this process will be discussed in section 5.
In general the gauged WZW term describes interaction involving any odd number off pNGBs and up to four dark photons. If such a vertex is allowed by the theory one may find a contribution to it when expanding the WZW to high enough order. For example, all terms contributing to $Z'\xi\xi\xi$ interactions can be obtained by an expansion up to order $\mathcal{O}(\tau^2)$. In general, all contributions to a vertex containing n fields ξ^a are contained in an expansion up to order $\mathcal{O}(\tau^{n-1})$. For the phenomenological discussion we have in mind, we will not require higher order terms and details are left to future projects.

5 Phenomenological outlook

After discussing in detail the properties and formulation of the low energy effective description of the dark matter theory, we now wish to give an outlook on how to use the derived description for DM phenomenology. Especially we want to point out why we think the inclusion of the $\tilde{\eta}$-meson in the description is relevant for DM phenomenology. The following will be a qualitative description. Quantitative calculations will be the subject of future projects, based on the theory discussed in this thesis. We again only discuss $SO(N_C)$-vector theory, since the $\tilde{\eta}$ meson is not expected to be light in other theories with fermions transforming under a real representation.

For SIMP dark matter, we are especially interested in $2 \to 2$ and $3 \to 2$ processes within the dark sector. In order to only deal with the relevant contributions, we again expand the Σ-field (4.27) in the low energy Lagrangian $\mathcal{L}^{IR}_{(\Delta)}$, given in (4.45). If we drop all terms that contribute in these processes only at one-loop level or higher, we end up with the following phenomenological Lagrangian for SIMP dark matter, describing the strong dark sector in isolation and for large but finite N_C.

$$
\begin{aligned}
\mathcal{L}^{IR}_{SIMP} &= \operatorname{tr}\{\partial_\mu \pi \partial^\mu \pi\} - m_\pi^2 \operatorname{tr}\{\pi^2\} + \operatorname{tr}\{\partial_\mu \tilde{\eta} \partial^\mu \tilde{\eta}\} - m_{\tilde{\eta}}^2 \operatorname{tr}\{\tilde{\eta}^2\} \\
&+ \frac{2}{3F_\pi^2} \operatorname{tr}\{\partial^\mu \pi \pi \partial_\mu \pi \pi - \pi^2 \partial_\mu \pi \partial^\mu \pi\} + \frac{m_\pi^2}{3F_\pi^2} \operatorname{tr}\{\pi^4\} \\
&+ \frac{2m_\pi^2}{F_{\tilde{\eta}}^2} \operatorname{tr}\{\tilde{\eta}^2 \pi^2\} + \frac{4m_\pi^2}{3F_{\tilde{\eta}} F_\pi} \operatorname{tr}\{\tilde{\eta} \pi^3\} + \frac{m_\pi^2 F_\pi^2}{3F_{\tilde{\eta}}^4} \operatorname{tr}\{\tilde{\eta}^4\} \\
&+ \frac{d_{\mathcal{R}}}{15\pi^2 F_\pi^5} \epsilon^{\mu\nu\rho\sigma} \operatorname{tr}\{\pi \partial_\mu \pi \partial_\nu \pi \partial_\rho \pi \partial_\sigma \pi\}
\end{aligned} \tag{5.1}
$$

We remind the reader that $\pi = \pi^a T_a^{\mathcal{F}}$ and $\sqrt{8}\tilde{\eta} = \tilde{\eta}^0 \mathbb{1}$. The masses m_π^2 and $m_{\tilde{\eta}}^2$ are defined by the GMOR (4.22) and (4.46). The dimensionality $d_{\mathcal{R}} = N_C$ of the gauge group representation is given by the number of colors. The first line in (5.1) defines the propagators of the dark pions and the $\tilde{\eta}$-meson. The second line gives the self-interactions of the dark pions. The stable dark matter is made up of the stable pions. Hence, the second line in (5.1) determines the phenomenologically relevant self-interactions of dark matter in the present state of the universe. This is also the term that determines the self-interactions, necessary to resolve the cusp vs. core problem. We see an explicit momentum dependence of the crosssection. However, it is an open question if this momentum dependence suffices to resolve the cusp vs. core problem at different scales. It was already considered that a linear sigma model, taking into account also scalar mesons, might do a pretty good job [75]. However, the situation for the model considered here is unknown. The third line in (5.1) describes interactions between the dark pions and $\tilde{\eta}$. These are all momentum independent. We dropped all terms that would contribute to a momentum dependence in the discussion in section (4.2), since such terms are suppressed if N_C is large. The last line describes the five point vertex, responsible for the dark pion $3 \to 2$ cannibalization process. In principle, this term is of order $\mathcal{O}(p^4)$ and thus of higher order than the other processes. Nevertheless, it is the lowest order vertex in the theory that can contribute to a number changing process like $3 \to 2$. That is why we include it in the theory.
Let us review the SIMP freeze-out regime [18]. The $2 \to 2$ processes within the dark sector maintain kinetic equilibrium within the dark sector, such that it has an overall temperature T_{DM}. This temperature may in principle be independent of the SM temperature T_{SM}, dictated

by the SM photon bath. However, for the relic density to be set by a freeze-out scenario, the dark sector and the SM must be in thermal equilibrium before the time of freeze-out. If the kinetic mixing parameter ε is large enough, the dark photon mediator can achieve this equilibrium. Nevertheless, if the kinetic mixing parameter ε is too large, $2 \to 2$ annihilation processes between the dark sector and SM will dominate over $3 \to 2$ scattering in the freeze-out. Hence, for a $3 \to 2$ driven freeze-out, only a certain domain of ε is valid. The range of validity can be estimated[1] roughly as $10^{-9} \lesssim \varepsilon \lesssim 3 \times 10^{-6}$[18].

If we couple the dark photon, by gauging the correct flavor subgroup, we add vertices between Z' and the charged pions to the Lagrangian. The relevant vertices are captured in the interaction Lagrangian $\mathcal{L}^{IR}_{\pi Z'}$ in (4.86). Further, from the WZW-term, we obtain anomalous processes, involving an odd number of fields $\xi^a = \tilde{\eta}, \pi^a$ and a dark photon. At lowest order, the gauged WZW term contributes the anomalous decay vertex (4.93), allowing for (off shell) decays $\xi \to Z'Z'$. Due to the anomaly cancelation condition (3.45) the pions do not decay into two dark photons and remain stable. However, the situation is different for $\tilde{\eta}$. Let us further assume a regime with $m_{Z'} \ll 91\text{Gev}$. For the SIMP regime it additionally holds that the kinetic mixing parameter $\varepsilon \ll 1$ is small. In this case, the gauge eigenbasis and the mass eigenbasis of Z' will not differ significantly. Further, after electroweak symmetry breaking, Z' mixes mostly with the SM photon γ, allowing us to neglect mixing to the heavy SM Z-boson [58]. The anomalous vertex (4.93) then allows $\tilde{\eta}$ to decay into the standard model via the processes depicted in figure 5.1.

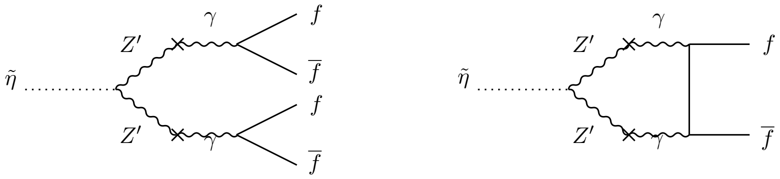


Figure 5.1: Diagrammatic depiction of the decay processes $\tilde{\eta} \to f\overline{f}f\overline{f}$ and $\tilde{\eta} \to f\overline{f}$ of the dark $\tilde{\eta}$ meson into standard model fermions f. The little $\times$ on the photon line denotes the kinetic mixing between the dark photon Z' and SM photon γ.

Depending on the lifetime $\tau_{\tilde{\eta}}$ and mass $m_{\tilde{\eta}}$ of the $\tilde{\eta}$-meson, the presence of these decays may significantly affect the dark matter parameter space. To elaborate further on this statement, first notice that, due to radiative corrections through the dark photon, there will be a splitting in mass between the neutral and the charged dark pions, typically rendering the charged particles heavier. After the $2 \to 2$ annihilation processes between DM and SM become subdominant, due to the expansion of the universe, the charged particles will scatter into the lighter neutral states via the strong interaction processes. Besides the strong $3 \to 2$ we have four different strong $2 \to 2$ processes within the dark sector, depicted in figure 5.2. Of special interests are the vertices, allowing for scattering of dark pions into $\tilde{\eta}$-states.

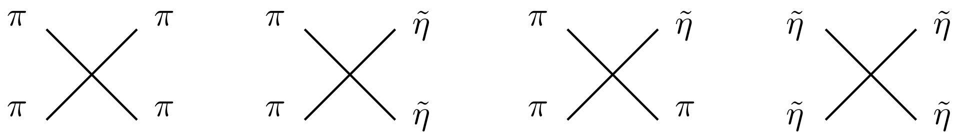


Figure 5.2: Diagrammatic depiction of the strong $2 \to 2$ interactions within the dark sector. Especially the $\tilde{\eta}\pi\pi\pi$ vertex is of interest. This vertex does not occur for $Sp(2N_C)$-fundamental SIMP dark matter [19].

[1]It is possible to make the lower bound of ε even smaller than estimated in [18]. However, then kinetic equilibrium may not be maintained during freeze-out and the dark sector temperature evolves independent of the SM photon bath. This makes the computation challenging.

Especially the $\tilde{\eta}\pi\pi\pi$ vertex is of significance[2] for $SO(N_C)$-vector dark matter, because the related scattering processes are kinematically favored over the one related to the $\tilde{\eta}\tilde{\eta}\pi\pi$ vertex at low momenta. If $\tilde{\eta}$ is sufficiently light, pions may scatter regularly into unstable $\tilde{\eta}$ states, which then can decay into the SM. However, since $2 \to 2$ DM-SM annihilation processes are already frozen out, such decays to the SM can not be compensated by annihilation of SM particles into the dark sector. Hence, the scattering of pions into $\tilde{\eta}$ states, followed by decay of the $\tilde{\eta}$ state, may lead to additional depletion of dark matter.
A more detailed analysis of the flavor structure of the vertices shows that the $\tilde{\eta}\pi\pi\pi$ vertex is only non-vanishing, if two oppositely charged pion states are involved. Further the charged pions are typically heavier than the neutral ones, due to the radiative corrections. Thus, the depletion of DM, via the anomalous decay channel, becomes more inefficient with decreasing temperature and a reducing number density of charged pion states. Eventually the $\tilde{\eta}$ production ceases, after sufficient down scattering of charged pion states. Dependent on the lifetime $\tau_{\tilde{\eta}}$ and mass $m_{\tilde{\eta}}$ of the $\tilde{\eta}$-meson, the depletion via the anomalous decay channel may affect the relic density calculation. To give an extreme example, if the $\tilde{\eta}$ decays instantly, this dark matter theory may not even be able to reproduce the correct relic density. If it is further mass degenerate with the pions, the dark matter becomes unstable and will deplete completely. The relevant points in parameter space for dark matter phenomenology is determined by a combination of relic-density calculation and an estimation of self-interaction crosssections. Hence, the physics of $\tilde{\eta}$ may significantly influence our understanding of the parameters of these dark matter theories.
So far our arguments were based on the assumption that $\tilde{\eta}$ might be sufficiently close in mass to the dark pions. However, our experience from QCD tells us that its analog particle, the η'-meson, is rather heavy. By now it is very well understood that the mass difference is not only due to the contribution of the axial anomaly, but partly originates from the large mass splitting between the light u- and d-quarks and the heavier s-quark. The situation thus quantitatively differs in mass degenerate theories [76]. Looking at relation (4.46) for the mass of $\tilde{\eta}$,

$$m_{\tilde{\eta}}^2 = \frac{F_\pi^2}{F_{\tilde{\eta}}^2} m_\pi^2 + \frac{\Delta m_{\tilde{\eta}}^2}{F_{\tilde{\eta}}^2} \tag{5.2}$$

we see that there exists at least two phenomenological limits where the mass of the dark pions and $\tilde{\eta}$ become similar. On the one hand, for $N_C \to \infty$ it holds $\Delta m_{\tilde{\eta}}^2 \to 0$. Further $F_{\tilde{\eta}} \to F_\pi$ is plausible to hold. Hence, for large N_C, the pions and $\tilde{\eta}$ should be similar in mass. Further, at N_C fixed, the anomalous contribution $\Delta m_{\tilde{\eta}}^2$ becomes relatively suppressed if m_π^2 gets large. However, m_π^2 cannot be made arbitrary large, because we treated the quark mass term as a small perturbation to the system in the chiral limit. The estimates in [15] show that, in the SIMP regime, the dark pion mass is approximately $m_\pi^2 \approx \mathcal{O}(0.2\text{GeV})$.
For future projects it will be interesting to investigate the role of the $\tilde{\eta}$-meson in a more quantitative way by estimating its lifetime in dependence of the parameters $\varepsilon, m_{\tilde{\eta}}, N_C, e_D, f_\eta$ and compare it to the rate of $\pi\pi \leftrightarrow \tilde{\eta}\pi$ and $\pi\pi \leftrightarrow \tilde{\eta}\tilde{\eta}$ scattering. For the latter two processes, an estimate of the mass hierarchy will be required. The radiative corrections can be calculated via a perturbative calculation. For crude phenomenological considerations an ansatz $m_{\tilde{\eta}} = \alpha m_\pi$ with $\alpha \approx 1.2$ could be used. The actual mass hierarchy within the isolated strong dark sector may be calculated from first principles via lattice field theory.

[2]In $Sp(2N_C)$ dark matter theories the $\tilde{\eta}\pi\pi\pi$ vertex is absent[19], while the $\tilde{\eta}\tilde{\eta}\pi\pi$ vertex is not.

6 Conclusion

In this thesis we mostly studied possible model realizations of dark matter via QCD-like $SO(N_C)$ gauge theories with two Dirac fermions transforming in the vector representation. The isolated strongly interacting dark sector can be coupled to the SM via a massive dark photon mediator. In section 3 we concerned ourselves with the UV description of the dark sector. The discussion put special focus on the symmetry structure, pointing out a convenient implementation of spatial parity and the presence of a discrete $\mathbb{Z}_2$ semi-factor in the global flavor symmetry $\mathbb{Z}_2 \ltimes SU(4)$. The physical interpretation of this $\mathbb{Z}_2$ symmetry was charge conjugation of one dark quark while the other dark quark remains unchanged. The $\mathbb{Z}_2$ symmetry is broken explicitly by the charge assignments of the dark photon. The formation of a chiral condensate breaks the flavor symmetry according to $\mathbb{Z}_2 \ltimes SU(4) \to O(4)$. Spatial parity and the global $O(4)$ flavor symmetry are used to classify the physical states of the theory. The dark matter candidates are the nine mass-degenerate dark pions, resulting from the spontaneously broken flavor symmetry. Moreover, we demonstrated the existence of another pseudo-scalar flavor-singlet state, called $\tilde{\eta}$. We discussed two limits, the 't Hooft large N_C limit and the limit of large quark mass, in which these particles become close in mass to the dark pions. For the required mediator sector, we discussed all possible charge assignments in section 3.2. We could conclude that there is only one physically distinct way to charge the dark quarks such that all the pions remain stable. The stability of the dark pions is required to explain the observation of dark matter in the current state of the universe. However, once we couple the isolated strong dark sector in this way to the SM, an anomalous decay channel for the $\tilde{\eta}$ to the SM opens. The strong interactions among the dark particles allow pions to scatter into $\tilde{\eta}$ states, which may subsequently decay. This might have destabilizing effects on the dark matter, depending on the details of the theory. As was argued in section 5, within the parameter range of the mediator sector for the SIMP regime, the presence of the $\tilde{\eta}$ may alter our understanding of the SIMP freeze-out. Since the parameters of the strong dark sector are estimated by a combination of relic density calculation and an estimation for the self-scattering crosssection [18], the physics of $\tilde{\eta}$ can also affect the estimation of the dark pion masses. We intend to study the physics of the $\tilde{\eta}$ meson more quantitatively in the future.
The low energy effective description, derived in chapter 4, was based on a non-linear Σ-model over the coset space $SU(4)/SO(4)$. The pion part of the Lagrangian agreed with earlier studies presented in [15][66]. However, the precense of the discrete, unbroken $\mathbb{Z}_2$ semi-factor was not mentioned in these previous works. The investigation of the $\tilde{\eta}$ physics in the IR was based on chiral perturbation theory and large N_C arguments. The resulting Lagrangian, describing all phenomena relevant for SIMP dark matter, was given in (5.1).
In section 4.3 we dealt with the WZW-term of the theory, which implements the $3 \to 2$ processes for the SIMP freeze-out. We explained why the non-vanishing of the fourth homotopy group of the relevant coset space, i.e. $\boldsymbol{\pi}\,(SU(4)/SO(4)) \neq 0$, renders the geometrical construction of the WZW term by Witten [17] inconclusive. Subsequently, we presented an alternative construction, based on the work of Chu, Ho and Zumino [26], which results in the expression (4.68). Thus, SIMP dark matter can still be realized with the theories presented. The alternative formula (4.68) for the WZW-term is more cumbersome than the standard expression but still very well suited for practical applications. Further, the presented construction automatically fixes the coefficient of the WZW-term in the low energy effective theory. We consider this an advantageous feature of the construction. Moreover, we showed that the $\tilde{\eta}$ does not take part in the $3 \to 2$ process. Hence, the presence of $\tilde{\eta}$ enters the SIMP phenomenology only via its anomalous decay.

Finally, let us discuss various generalizations of the $SO(N_C)$-vector DM model with two Dirac fermions. The formulas presented in this thesis where kept quite general, allowing for a generalization to an arbitrary number N_F of Dirac flavors. While the coset space generalizes to $SU(2N_F)/SO(2N_F)$ and the number of pions increases, there still exists at least one charge assignment that keeps the pions stable. These assignments also give rise to an anomalous decay of $\tilde{\eta}$. However, these assignments must not be unique and other possibilities may exist.
In the case of $N_F > 2$ Dirac flavors the topological condition $\boldsymbol{\pi}\,(SU(2N_F)/SO(2N_F)) = 0$ holds and hence the standard classification and construction methods for the WZW discussed in [24][25][27] are applicable.
Further, one may consider real representation gauge theories other than $SO(N_C)$-vector. Our considerations where kept general enough in order to be applicable to this case. The reality condition of the gauge group representation is strong enough to qualitatively restrict most features of these theories, relevant for the low energy effective description and dark matter phenomenology. The only difference that arises is the presence of additional discrete symmetries, arising from the anomalous breaking $U(4) \to \mathbb{Z}_K \ltimes SU(4)$ due to the axial anomaly. We are currently not aware of a simple physical interpretation of these discrete semi-factors in the flavor symmetry. Further, some discrete symmetries become spontaneously broken once a chiral condensate forms. We conclude that the low energy effective description for DM, derived in chapter 4, still remains valid as a perturbative description in this generalized case. However, it is clear that some non-perturbative features, mostly related to the presence of the spontaneously broken discrete symmetries, will be missed. Furthermore, we presented a representation theoretical criteria, stated in (3.60), which qualifies the importance of $\tilde{\eta}$ in the generalized case. We conclude that, except for $SO(N)$-vector theories, this particle will in general be of little importance for dark matter phenomenology. When dealing with the generalized case, one further should take into account the conformal window, in order to check if the theory is asymptotically free and if the formation of a chiral condensate is plausible. These considerations can be used to heavily restrict which theories can be used for a generalization of the presented DM theory.
The exposition in this thesis was structured such that the results on the isolated dark sector can be used independently, with the dark photon mediator introduced as an external perturbation. This allows to study properties of the strongly interacting dark sector via lattice field theory. In the future we intend to use the results in this thesis to set up a study of a $Sp(4)$-AT2T dark sector with two Dirac fermions via lattice field theory. This will allow a quantitative calculation of particle masses and low energy effective constants. We also intend to further study the physics of $\tilde{\eta}$ in the SIMP regime in order to grasp a more quantitative understanding of its phenomenological relevance for strongly interacting dark matter.

Appendix A

Group theory

A.1 Generators of $SU(4)$

For reference we fix a basis of generators of $SU(4)$ in the fundamental representation. The generators given in (A.1) are chosen to be hermitian and obey a block form. Each block is either zero or given by a Pauli matrix times some power of i. The normalization is fixed according to $\mathrm{tr}\left\{T_N^{\mathcal{F}} T_M^{\mathcal{F}}\right\} = \frac{1}{2}\delta_{NM}$. Since $Sp(4)$ and $SO(4)$ are both subgroups of $SU(4)$, this also fixes a generator basis for these groups. The respective generator subsets are stated in corresponding sections below.

$$
\begin{aligned}
T_1^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} &
T_2^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} &
T_3^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
T_4^{\mathcal{F}} &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
T_5^{\mathcal{F}} &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} &
T_6^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
T_7^{\mathcal{F}} &= \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
T_8^{\mathcal{F}} &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} &
T_9^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\
T_{10}^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} &
T_{11}^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} &
T_{12}^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
T_{13}^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} &
T_{14}^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} &
T_{15}^{\mathcal{F}} &= \frac{1}{\sqrt{8}}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
\tag{A.1}
$$

The supperscript $\mathcal{F}$ indicates that this basis generates the flavor symmetry of the Lagrangian with respect to the Nambu-Gorkov basis Ψ, implemented by the fundamental representation $\mathcal{F} : SU(4) \to \mathbb{C}^{4\times 4}$.

A.2 The orthogonal subgroup $SO(4)$

The mass term (3.14) acts as a source of explicit symmetry breaking and determines the breaking pattern via the invariance condition 3.24 of the mass tensor. In the mass degenerate case, the mass tensor is proportional to the covariant 2-tensor ω_{ij}. We distinguish between broken and unbroken generators by the following conditions.

$$\begin{array}{c} \text{Broken } SU(4) \text{ generators} \\ SU(4)/SO(4) \text{ generators} \end{array} \qquad T_a^{\mathcal{F}\top}\omega - \omega T_a^{\mathcal{F}} = 0 \qquad a = 1, \dots, 9 \tag{A.2}$$

$$\begin{array}{c} \text{Unbroken } SU(4) \text{ generators} \\ SO(4) \text{ generators} \end{array} \qquad T_A^{\mathcal{F}\top}\omega + \omega T_A^{\mathcal{F}} = 0 \qquad A = 10, \dots, 15 \tag{A.3}$$

Due to these relations we may define an involutive, Lie-algebra automorphism $\hat{\sigma} : \mathfrak{su}(4) \to \mathfrak{su}(4)$ given by

$$\hat{\sigma}A = -\omega^{-1}A^\top\omega \tag{A.4}$$

The Lie algebra decomposes into a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathbf{k}$, where $\mathfrak{h}$ is the space of -1 eigenstates of $\hat{\sigma}$, spanned by the unbroken generators. The broken generators span the eigenspace $\mathbf{k}$ of $+1$ eigenstates of $\hat{\sigma}$. For $h, h' \in \mathfrak{h}$ and $k, k' \in \mathbf{k}$ it further holds

$$[h, h'] \in \mathfrak{h} \qquad [h, k] \in \mathbf{k} \qquad [k, k'] \in \mathfrak{h} \tag{A.5}$$

The map $\hat{\sigma}$ renders the coset space $SU(4)/SO(4)$ a symmetric space [77]. For physical applications, $\hat{\sigma}$ may be used to define an implementation of spatial parity for fields $A_\mu(x) \in \mathfrak{su}(4)$ valued in the flavor Lie-algebra [26]. Examples are the dark photon Z' or the ρ-mesons of the theory.

$$\boldsymbol{P} : A_\mu(t, \vec{x}) \mapsto \hat{\sigma}A_\mu(t, -\vec{x}) \tag{A.6}$$

The perk of this parity implementation is that it commutes with the flavor symmetry. In the case of particles like the pion, this could also be done and coincides with the action of spatial parity (3.34) on the operators $\mathcal{O}_a^{PS}$. The symmetry transformations produced by the unbroken generators span the Lie-algebra of the subgroup under which the degenerate mass tensor is invariant. To see that these are equivalent to the vector representation of $\mathfrak{so}(4)$ we perform a change of basis with the matrix V.

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \tag{A.7}$$

This transformation diagonalizes the covariant 2-tensor $\omega_{ij} \mapsto V^l{}_i V^n{}_j \omega_{ln} = \delta_{ij}$, while the generators transform as (1,1)-tensors $T_n^{\mathcal{F}} \mapsto V^{-1}T_n^{\mathcal{F}}V =: T_n^{\phi}$. The new representation matrices of the generators now satisfy $T_n^{\phi\top} = \pm T_n^{\phi}$, where the plus sign holds for the broken generators. Being antisymmetric matrices, the unbroken generators span the algebra $\mathfrak{so}(4)$ and the implementation via the generators $T_n^{\mathcal{F}}$ correspond to a complex rotated version of the vector representation of $SO(4)$. The basis transformation V also relates the Nambu-Gorkov spinors to the Majorana spinors via $\psi^{(l)} = V^l{}_n\phi^{(n)}$, as can be read off from (3.10).

A.2.1 Double covering $SO(4)$ with $SU(2) \times SU(2)$

As outlined in [53][78] the group $SU(2)_\rightarrow \times SU(2)_\leftarrow$ is a double cover of $SO(4)$. The covering map can be obtained by relating an element $x \in \mathbb{R}^4$ of the representation space of the vector representation with a complex 2×2 matrix via $\Phi : \mathbb{R}^4 \to H = \Phi(\mathbb{R}^4) \subset \mathbb{C}^{2\times 2}$

$$\left(x^1, x^2, x^3, x^4\right)^\top \xmapsto{\Phi} \begin{pmatrix} x^1 + ix^2 & -x^3 + ix^4 \\ x^3 + ix^4 & x^1 - ix^2 \end{pmatrix} \tag{A.8}$$

As outlined in [53] this map is a Lie-group homeomorphism. It follows that for every $O \in SO(4)$ there exist two matrices $U_\rightarrow, U_\leftarrow \in SU(2)$ such that

$$\Phi\left(Ox\right) = U_\rightarrow\, \Phi(x)\, U_\leftarrow \tag{A.9}$$

Following [78], reversely every $O \in SO(4)$ may be decomposed into two orthogonal, commuting matrices $O = O_\rightarrow O_\leftarrow$, each corresponding to a left transformation $U_\rightarrow$ or right transformation $U_\leftarrow$. The matrices $O_\rightarrow$ are generated by $T^\phi_{10}, T^\phi_{11}, T^\phi_{12}$ and the transformations $O_\leftarrow$ are generated by $T^\phi_{13}, T^\phi_{14}, T^\phi_{15}$. Remember that these are related via $T^\phi_n = V^{-1} T^{\mathcal{F}}_n V$ to the flavor symmetry implementation in the Nambu-Gorkov basis. The map Φ physically corresponds exactly to the relation $q^{(k)} = q^{2k-1}_M + i q^{2k}_M$ between Majorana and Dirac fermions, with the Majorana fermions as the real degrees of freedom. The connection may be given by arranging the fermions in a square matrix

$$\Phi(q_M) = \begin{pmatrix} q^{(1)}_M + iq^{(2)}_M & -q^{(3)}_M + iq^{(4)}_M \\ q^{(3)}_M + iq^{(4)}_M & q^{(1)}_M - iq^{(2)}_M \end{pmatrix} = \begin{pmatrix} q^{(1)} & -q^{(2)}_{\mathcal{C}} \\ q^{(2)} & q^{(1)}_{\mathcal{C}} \end{pmatrix} \tag{A.10}$$

where $q^{(g)}_{\mathcal{C}} = \mathcal{C} q^{(g)}$ denotes the charge conjugate Dirac spinor (3.8). Due to the parametrization (3.9) we may use the Majorana fermions to relate symmetries in the Dirac and Weyl formulation.

$$\{T^\phi_N\}^k{}_j\, q^{(j)}_M = \{T^\phi_N\}^k{}_j\, q_M[\phi^{(j)}] = q_M[\{T^\phi_N\}^k{}_j \phi^{(j)}] \tag{A.11}$$

The double covering property can be seen by the fact that the kernel of the group homeomorphism $SU(2)_\rightarrow \times SU(2)_\leftarrow \mapsto SO(4)$ is a group of two elements, given by $(\mathbb{1}_2, \mathbb{1}_2)$ and $(-\mathbb{1}_2, -\mathbb{1}_2)$, implementing a representation of $\mathbb{Z}_2$.
The arrangement (A.10) also allows to identify the action of $SU(2)_\rightarrow$ as the analog of QCD isospin transformations. Less obvious is the interpretation of the action of the one-parameter subgroup of $SU(2)_\leftarrow$, generated by $T^{\mathcal{F}}_{15}$. By exponentiation $U^{\mathcal{F}}[\alpha] = \exp\left\{-i\alpha\sqrt{8}T^{\mathcal{F}}_{15}\right\}$ in the Nambu-Gorkov representation and then translating the transformation back to the Dirac formalism, we obtain

$$\begin{pmatrix} u \\ d \end{pmatrix} \xmapsto{U^{\mathcal{F}}} \mathrm{e}^{-i\alpha} \begin{pmatrix} u \\ d \end{pmatrix} \tag{A.12}$$

Hence, the symmetry generated by $T^{\mathcal{F}}_{15}$ may be interpreted as the analog of the baryon number symmetry, if $T^{\mathcal{F}}_{15}$ is appropriately normalized. This was already pointed out in [53].

A.2.2 Alternative breaking terms to realize $SO(4)$ subgroups

There are other $SO(4)$ subgroups in $SU(4)$ and one may think of writing down different symmetry breaking terms. For example, the generators $T^{\mathcal{F}}_2, T^{\mathcal{F}}_7, T^{\mathcal{F}}_8, T^{\mathcal{F}}_9, T^{\mathcal{F}}_{11}, T^{\mathcal{F}}_{14}$ are antisymmetric and thus span an $\mathfrak{so}(4)$ algebra. In fact, the resulting coset space $SU(4)/SO(4)$ is also symmetric. The mass tensor required to achieve this breaking pattern would be a mass-degenerate Majorana mass term. Due to the relation $\psi^{(l)} = V^l{}_n \phi^{(n)}$ and the fact that the Majorana fermions are the fundamental degrees of freedom anyhow, the physical theory emerging is equivalent to the one we are considering.
Another $SO(4)$ subgroup is given by the generators $T^{\mathcal{F}}_4, T^{\mathcal{F}}_5, T^{\mathcal{F}}_7, T^{\mathcal{F}}_8, T^{\mathcal{F}}_{12}, T^{\mathcal{F}}_{15}$. These span the

Lie-algebra $su(2) \oplus su(2) \cong so(4)$. They would correspond to the unbroken generators of a mass tensor

$$M_{su(2)\oplus su(2)} = \begin{pmatrix} 0 & 0 & 0 & m_{(1)} \\ 0 & 0 & -m_{(1)} & 0 \\ 0 & m_{(2)} & 0 & 0 \\ -m_{(2)} & 0 & 0 & 0 \end{pmatrix} \tag{A.13}$$

with two free parameters. This symmetry breaking scenario is often considered in Composite Higgs theories [72] in order to realize a symmetry breaking scenario with a flavor singlet pNGB. However, it is only realizable within symplectic gauge theories [19] but not for real ones. Anyhow, symmetry breaking scenarios with a flavor singlet pNGB are not favorable for a dark matter, as explained in section 3.3. For that reason we do not need to consider this scenario anyway. The resulting $SU(4)/SO(4)$ coset space differs further by the fact that it is not symmetric.

A.3 Homotopy groups of $SU(4)/SO(4)$

Wittens construction of the Wess-Zumino term in QCD [17], as well as several generalizations of it [24][25][27] have the preliminary assumption that the forth homotopy group[1] $\boldsymbol{\pi}_4(G/H)$ of the corresponding coset space is trivial. We will show that in the case of our theory, where $G/H = SU(4)/SO(4)$, this preliminary assumption is violated. Hence, the geometrical construction by Witten is not applicable in this case. Unfortunately, $SU(4)/SO(4)$ is out of the range of Bott's periodicity theorem [70], thus we have to work a little. We know the homotopy groups of $SO(4)$ and $SU(4)$, summarized in table A.1. In the case of coset we are interested in, $SO(4)$

Table A.1: Homotopy groups of $SO(4)$ and $SU(4)$ [80, Appendix A, Table 6.VII]

	$\boldsymbol{\pi}_3$	$\boldsymbol{\pi}_4$	$\boldsymbol{\pi}_5$
$SO(4)$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$
$SU(4)$	$\mathbb{Z}$	0	$\mathbb{Z}$

is an embedded Lie-subgroup of $SU(4)$. Results from differential geometry and Lie-theory [29] tell us that $SU(4)/SO(4)$ then has a uniquely defined manifold structure and the projection map $\Pi : SU(4) \to SU(4)/SO(4)$ defines a fiber bundle $SO(4) \to SU(4) \xrightarrow{\Pi} SU(4)/SO(4)$. In Algebraic Topology [81] such a fibration gives rise to a long exact sequence

$$\begin{array}{ccccccccc} \to & \boldsymbol{\pi}_4(SU(4)) & \xrightarrow{h_1} & \boldsymbol{\pi}_4(SU(4)/SO(4)) & \xrightarrow{h_2} & \boldsymbol{\pi}_3(SO(4)) & \xrightarrow{h_3} & \boldsymbol{\pi}_3(SU(4)) & \to \\ \to & 0 & \xrightarrow{h_1} & ? & \xrightarrow{h_2} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{h_3} & \mathbb{Z} & \to \end{array}$$

of group homomorphism between the homotopy groups. From this sequence we can extract a lot of information. First we observe that $\{0\} = h_1(\{0\}) = \text{Img } h_1 = \text{Ker } h_2$. This lets us conclude that h_2 is injective. Henceforth, $\boldsymbol{\pi}_4(SU(4)/SO(4)) \cong \text{Img } h_2 = \text{Ker } h_3$. But h_3 is a group homomorphism mapping the rank two group $\mathbb{Z}\oplus\mathbb{Z}$ on the smaller rank one group $\mathbb{Z}$. This means that Ker h_3 cannot be trivial and conclusively $\boldsymbol{\pi}_4(SU(4)/SO(4))$ is non-trivial. This answers a footnote remark in [27], concerning the applicability of their methods to this coset space: They are never applicable, independent of how $SO(4)$ sits inside $SU(4)$.

[1] A good explanation of what homotopy groups are can be found in [79]. Also, don't confuse the symbol of the homotopy group with the pion field. These are completely different things.

Appendix B

Real representations

We define a finite dimensional, unitary representation $\mathcal{R} : G \to \mathrm{Aut}\,(V)$ to be (pseudo) real, if it is equivalent to its complex conjugate representation $\overline{\mathcal{R}} : G \to \mathrm{Aut}\,(\overline{V})$. Here V denotes a complex vector space and $\overline{V}$ the complex conjugate space of V. Introducing the same basis on V and $\overline{V}$ this may be expressed in components by the existence of a unitary matrix S that satisfies the equivariance condition (3.6), e.g. for all matrices representing group elements it holds

$$SU^{\mathcal{R}}S^{-1} = (U^{\mathcal{R}})^* \tag{B.1}$$

As explicitly done in [30], one can show that the matrix S is either symmetric are antisymmetric. To see this we transpose (B.1) and take the inverse on both sides to get $U^{\mathcal{R}} = S^{-1\top}(U^{\mathcal{R}})^* S^\top$. Plugging in (B.1) again shows that the matrix $S^{-1}S^\top$ commutes with all representation matrices of $\mathcal{R}$. By Schur's lemma, on each irreducible subspace, it must hold $S^{-1}S^\top = \eta_S \mathbb{1}$ or equivalently

$$S^\top = \eta_S S \tag{B.2}$$

From transposing S twice we can infer that $\eta_S^2 = 1$ and thus $\eta_S = \pm 1$. The equations (B.2) and (B.1) hold up to an arbitrary rescaling $S \to \lambda S$. It can be shown that, up to this rescaling, S is unique [30]. Using the arguments in [54] one can further show that the scale may be fixed by an unitarity condition for S. To do so, we introduce an anti-linear map $\hat{J} : V \to V$, whose action, with respect to the above basis, is defined as $\{\hat{J}(x)\}^a = S^a_{\ b} x^{b*}$. With this we can express the equivariance condition (B.1) as $JU^{\mathcal{R}} = U^{\mathcal{R}}J$. We see that J^2 is a non-zero operator, again commuting with all representation matrices of $\mathcal{R}$ and by Schur's lemma, J^2 it is proportional to the unity operator.

$$J^2 = \eta_J \mathrm{id}_V \qquad \Leftrightarrow \qquad SS^* = \eta_J \mathbb{1} \tag{B.3}$$

We may choose λ now such that $|\eta_J| = 1$ holds. Using (B.2) we further see that $SS^\dagger = \eta_J \eta_S \mathbb{1}$. Because $SS^\dagger$ is a positive definite, hermitian matrix $\eta_J \eta_S = 1$ must hold and hence $\eta_J = \eta_S$. We define $\mathcal{R}$ to be real if $\eta_J = 1$ holds.

Definition B.0.1 (Real representation)
A finite dimensional, unitary representation $\mathcal{R} : G \to \mathrm{Aut}\,(V)$ of a group G is called *real* if there exists an anti-linear, conjugation $\hat{J} : V \to V$ that is $\mathcal{R}$-equivariant, e.g. it holds

$$\hat{J}U = U\hat{J}$$

In the pseudo-real case $\eta_J = -1$ holds and $\hat{J}$ is a skew-conjugation. For non-unitary real representation $\hat{J}$ is only an anti-linear involution [29] instead of a conjugation. The difference manifest in S not being unitary. We now further show that the existence of S is tied closely to the existence of a complex bilinear product on V, that is invariant under the representation $\mathcal{R}$. More precisely, the following theorem holds.

Theorem B.0.1 (Bilinear invariants and conjugations)
Let $\mathcal{R} : G \to \mathrm{Aut}(V)$ be a finite dimensional, unitary representation of a group G. The following two statements are equivalent:

i) The representation is real.

ii) There exists a $\mathcal{R}$-invariant, complex bilinear, non-degenerate, symmetric inner product $(.,.) : V \times V \to \mathbb{C}$.

Proof. i)$\Rightarrow$ ii) Since $\mathcal{R}$ is real there exists the unitary matrix S, satisfying (B.1). Due to (B.2) and unitarity we see that $S = S^\top$ is symmetric. The complex bilinear inner product defined by

$$\forall X, Y \in V : \qquad (X, Y) := X^a \delta_{ab} S^b_{\ c} Y^c \tag{B.4}$$

is symmetric and non-degenerate, due to the properties of S. It remains to proof its invariance. The representation $\mathcal{R}$ acts as

$$X^\alpha \xmapsto{\mathcal{R}} \{\mathcal{R}X\}^a = \left\{U^{\mathcal{R}}\right\}^a_{\ b} X^b \tag{B.5}$$

$$(X, Y) \xmapsto{\mathcal{R}} (\mathcal{R}X, \mathcal{R}Y) = X^d \left\{U^{\mathcal{R}}\right\}^a_{\ d} \delta_{ab} S^b_{\ c} \left\{U^{\mathcal{R}}\right\}^c_{\ e} Y^e \tag{B.6}$$

Using the unitarity of $\mathcal{R}$ and (B.1), we obtain $\mathcal{R}$-invariance because

$$\left\{U^{\mathcal{R}}\right\}^a_{\ d} \delta_{ab} S^b_{\ c} \left\{U^{\mathcal{R}}\right\}^c_{\ e} = \left\{U^{\mathcal{R}\top} S U^{\mathcal{R}}\right\}_{de} = \left\{U^{\mathcal{R}\top} U^{\mathcal{R}*} S\right\}_{de} = \delta_{da} S^a_e \tag{B.7}$$

i)$\Leftarrow$ ii) Now assume that $(.,.) : V \times V \to \mathbb{C}$ is a $\mathcal{R}$-invariant, complex bilinear, symmetric, non-degenerate inner-product and $(e_a)_{a=1}^{\dim V}$ is a basis of V. The component matrix is defined as $S_{ab} := (e_a, e_b)$ and analogous to (B.5) we have due

$$S_{ab} = (\mathcal{R}e_a, \mathcal{R}e_b) = \left\{U^\top S U\right\}_{ab} \tag{B.8}$$

Pulling up the first index with δ^{ab} and multiplying this matrix equation from the left with U^* gives back the equivariance condition (B.1). Then, in the same manner as at the beginning of the section, we may argue that $SS^* = \eta_J \mathbb{1}$ and $S^\top = \eta_S S$. Because S is symmetric $\eta_J = \eta_S = 1$ must hold, which concludes the proof. $\square$

A similar theorem holds for the pseudo-real case and is proven in [30], where the role of the symmetric inner product is taken by a complex symplectic form, e.g. an alternating, complex, non-degenerate, covariant two-tensor. This theorem nicely illustrates the connections between the additional structures on V, introduced for a real representation.
Note that the matrix representation of the complex bilinear form $(.,.)$ can always be brought to a canonical form due to Sylvester's law of inertia. Thus, as a corollary of this proof, we see that a representation $\mathcal{R}$ of finite dimension $d_\mathcal{R}$ always provides a group homomorphisms $G \to SO(d_\mathcal{R})$, since the introduction of an appropriate basis on V tells us that the representation matrices leave invariant the canonical, symmetric, bilinear form on the representation space. However, this group homomorphism must neither be injective nor surjective. If $\dim SO(d_\mathcal{R}) = \dim G$ the groups are at least locally isomorphic. An example of such a local isomorphism would be given by $Sp(4)$-AT2T and $SO(5)$-Vector.
Ultimately the fact that we have the additional bilinear structure on color space, allows us to build gauge invariant combinations of Majorana fermions and explains the additional pNGB states that we find in this theory. As concrete non-fundamental examples of real representation we have $Sp(4)$-AT2T, explained in appendix G or every adjoint representation of a gauge group. In the latter case the invariant, bilinear product is given by the Killing metric.

Appendix C

Matrix valued differential forms

When dealing with anomalies, it is useful to introduce the concept of matrix-valued differential forms. Let $\mathfrak{M}$ be a vector space of matrices over a field $\mathbb{k} = \{\mathbb{C}, \mathbb{R}\}$. A matrix valued p-form is defined by

$$A(x) = a^\alpha(x) \otimes T_\alpha \tag{C.1}$$

where a^α are $\mathbb{K}$-vaued differential p-form over spacetime $\mathcal{M}$ and T_α are some matrices forming a basis of $\mathfrak{M}$. The space of all such p-forms is denoted by $\Omega^p(\mathcal{M}, \mathfrak{M})$. Using the operations for $\mathbb{K}$-vaued differential forms and assuming that the corresponding matrix operations are closed on $\mathfrak{M}$, we may introduce the following operations on $A \in \Omega^p(\mathcal{M}, \mathfrak{M}), B \in \Omega^q(\mathcal{M}, \mathfrak{M})$

$$\mathrm{d}A := (\mathrm{d}a^\alpha) \otimes T_\alpha \qquad \in \Omega^{p+1}(\mathcal{M}, \mathfrak{M}) \tag{C.2}$$

$$AB := \Big(a^\alpha \wedge b^\beta(x)\Big) \otimes (T_\alpha T_\beta) \qquad \in \Omega^{p+q}(\mathcal{M}, \mathfrak{M}) \tag{C.3}$$

$$A^\top := a^\alpha \otimes T_\alpha^\top \qquad \in \Omega^p(\mathcal{M}, \mathfrak{M}) \tag{C.4}$$

$$A^* := a^{\alpha *} \otimes T_\alpha^* \qquad \in \Omega^p(\mathcal{M}, \mathfrak{M}) \tag{C.5}$$

$$\mathrm{tr}\{A\} := \mathrm{tr}\{T_\alpha\}\, a^\alpha \qquad \in \Omega^p(\mathcal{M}, \mathbb{K}) \tag{C.6}$$

Further $A^\dagger := A^{*\top}$. First we want to mention that the product (C.3) of two such forms does neither commute nor anti-commute in general. For these operations the rules (C.7)-(C.13) hold. Note that these are mostly different to what one may be used from ordinary differential forms or matrices.

$$(AB)^\top = (-1)^{pq} B^\top A^\top \tag{C.7}$$

$$\mathrm{tr}\{AB\} = (-1)^{pq}\, \mathrm{tr}\{BA\} \tag{C.8}$$

$$\mathrm{tr}\{AB\} = \mathrm{tr}\Big\{(AB)^\top\Big\} = \mathrm{tr}\Big\{A^\top B^\top\Big\} \tag{C.9}$$

$$\mathrm{d}(\mathrm{d}A) = 0 \tag{C.10}$$

$$\mathrm{d}(AB) = (\mathrm{d}A)B + (-1)^p A(\mathrm{d}B) \tag{C.11}$$

$$\mathrm{d}A^\top = (\mathrm{d}A)^\top \tag{C.12}$$

$$\mathrm{d}\,\mathrm{tr}\{A\} = \mathrm{tr}\{\mathrm{d}A\} \tag{C.13}$$

One has to be especially careful when dealing with powers of the same form. Consider for example a 1-form A. Then it holds

$$A^2 = \frac{a^\alpha \wedge a^\beta}{2} \otimes [T_\alpha, T_\beta] \neq 0 \tag{C.14}$$

$$(A^2)^\top = -(A^\top)^2 \tag{C.15}$$

For anomalies it is useful to organize the gauge fields A_μ^α of a gauge group G in a matrix valued differential 1-form

$$A^\mathcal{R} = -iA_\mu^\alpha\, \mathrm{d}x^\mu \otimes T_\alpha^\mathcal{R} \tag{C.16}$$

commonly called a connection 1-form. Here $\mathcal{R} : G \to \mathfrak{M}$ may be any representation of the gauge group. In section 2.3.1 this 1-form was introduced, but the tensor product and representation label are suppressed in the notation.

Appendix D

Topological charge and Instantons

In section 3.1 we used an instanton argument to proof that only a certain subgroup of $U(4)$ is realized in the quantum theory, due to anomalous symmetry breaking. In this appendix we explain the details of the argument further. For two pedagogical reviews on the topic of instanton calculus, see [82] [83].
We start with considerations in euclidean spacetime $\mathbb{R}^4$. Instantons are gauge-field configurations A_μ^α of finite action $S_{YM}[A] < \infty$ that satisfy the classical equation of motion. A finite action demands that $r^2 F_{\mu\nu}^\alpha$ vanishes in the limit $r \to \infty$. This is equivalent to demand that the gauge fields approach a pure gauge for every point on an infinitely large 3-sphere $S^3(r)$, e.g.

$$A^{\mathcal{R}} \xrightarrow[r\to\infty]{} U^{-1}(\hat{x})\mathrm{d}U(\hat{x}) + \mathcal{O}\left(r^{-1}\right) \tag{D.1}$$

where $\hat{x}$ is the unit vector, specifying points on S^3 and $U : S^3 \to \mathcal{R}(G)$ maps spacetime on a representation matrix of G in some faithful representation $\mathcal{R}$. Above, d denotes the exterior derivative of matrix valued forms. We suppress the representation label, since the gauge-fields A_μ^α do not depend on the choice of $\mathcal{R}$, only on the choice of basis of the Lie-algebra $\mathfrak{g}$. By picking a reference direction $\hat{x}_0$ and by insertion of unity $\mathbb{1} = U(\hat{x}_0)U^{-1}(\hat{x}_0)$, we see that each pure gauge-field configuration is in one-to-one correspondence with maps $U : S^3 \to \mathcal{R}(G)$ that map $\hat{x}_0$ to the identity. These maps fall into equivalence classes of $\pi_3(G)$, the so-called third homotopy group of G. For the classical groups $SU(N), Sp(2N), SO(N)$ it holds that $\pi_3(G) = \mathbb{Z}$. Therefore, every instanton can be assigned a unique integer ν, called the instanton number or winding-number. The calculation of the instanton number with the help of the so-called Pontrjagin index is explained in [34, Chpt. 23], and we follow the discussion there. We define the Integral

$$I[U] = \int_{S^3} tr_{\mathcal{R}} \left\{ \left(U^{-1}\mathrm{d}U\right)^3 \right\} \tag{D.2}$$

As discussed in [34, Chpt 23.4], this integral is only sensitive to the homotopy class $[U]_{\pi_3}$ of U. To evaluate it, one may use that any map $U : S^3 \to G$ may be continuously deformed to a map $\tilde{U} : S^3 \to \mathrm{Std}(SU(2)) \subset G$, where $\mathrm{Std} : SU(2) \to G$ denotes a standard embedding of $SU(2)$ into G. This result was first obtained by Bott [84]. The standard embedding $\mathrm{Std} : SU(2) \to G$ may be defined for the classical groups [85] as follows:

$SU(N)$: $N \geq 3$	$\mathrm{Std}(SU(2))$ is corresponds to the $SU(2)$ subgroup of $SU(N)$ acting only on the first two components in the defining representation.
$Sp(2N)$: $N \geq 2$	$\mathrm{Std}(SU(2))$ corresponds to the $Sp(2) \cong SU(2)$ subgroup of $Sp(2N)$, acting only on the first two components in the defining representation.
$SO(N)$: $N \geq 5$	Using that $SO(4) \cong SU(2) \times SU(2)$, $\mathrm{Std}(SU(2))$ corresponds to either $SU(2)$ subgroup in the $SO(4)$ subgroup of $SO(N)$, acting on the first four components in the defining vector representation.

The derivation in [34, Chpt 23.4] now uses that U and the deformation $\tilde{U}$ are within the same homotopy class and the integral can be evaluated generally for $\tilde{U}$, by using the standard embedding. The result for $I[U]$ may be expressed in terms of two constants $\lambda > 0$ and $\tilde{T}_{\mathcal{R}}$, tied to the properties of the three generators $T^{\mathcal{R}}_{\alpha}$ of G, generating the standard subgroup Std($SU(2)$).

$$[T^{\mathcal{R}}_{\alpha}, T^{\mathcal{R}}_{\beta}] = \sqrt{\lambda}\ \epsilon_{\alpha\beta\gamma}\delta^{\gamma\gamma'}T^{\mathcal{R}}_{\gamma'} \tag{D.3}$$

$$tr_{\mathcal{R}}\left\{T^{\mathcal{R}}_{\alpha}T^{\mathcal{R}}_{\beta}\right\} = \lambda\tilde{T}_{\mathcal{R}}\ \delta_{\alpha\beta} \tag{D.4}$$

Note that the relation (D.3), fixing λ, is only dependent on the choice of basis, but not on the metric on $\mathfrak{g}$ or on the representation $\mathcal{R}$. The relation (D.3) on the other hand depends on both, metric and representation. If ν is now the unique integer, assigned to the homotopy class of $[U]_{\pi_3}$, the result is

$$I[U] = 48\pi^2\ \tilde{T}_R\ \nu \tag{D.5}$$

The result does not depend on λ. This can be realized as the fact that the result is not depending on the choice of metric on $\mathfrak{g}$ and confirms the expectation that topological features should be independent of the geometry induced by the metric. We may use $I[U]$ to evaluate the following integral over Minkowski space

$$\mathcal{Q}_{\text{Topo}} = \frac{\epsilon^{\mu\nu\rho\sigma}}{\mathcal{N}}\int d^4x\ tr_{\mathcal{R}}\left\{F^{\mathcal{R}}_{\mu\nu}F^{\mathcal{R}}_{\rho\sigma}\right\} = \frac{4}{\mathcal{N}}\int_{\mathcal{M}} tr_{\mathcal{R}}\left\{F^{\mathcal{R}}F^{\mathcal{R}}\right\} \tag{D.6}$$

Here $F^{\mathcal{R}} = \mathrm{d}A^{\mathcal{R}} + A^{\mathcal{R}}A^{\mathcal{R}}$ is the matrix valued 2-form[1], corresponding to the field-strength tensor and $A^{\mathcal{R}}$ is the connection 1-form, defined in (C.16). $\mathcal{N}$ is a normalization factor, to be specified later. The integral on the left, formulated in terms of differential forms, is insensitive to the choice of coordinates or the metric. On purely topological grounds, a compactified version of Minkowski space is equivalent to an arbitrary large 4-ball $B^4(r)$ of radius r and compactified euclidean space. Using the so-called Chern-Simons class

$$G^{\mathcal{R}} = A^{\mathcal{R}}\mathrm{d}A^{\mathcal{R}} + \frac{2}{3}A^{\mathcal{R}}A^{\mathcal{R}}A^{\mathcal{R}} \tag{D.7}$$

$$tr_{\mathcal{R}}\left\{\mathrm{d}G^{\mathcal{R}}\right\} = tr_{\mathcal{R}}\left\{F^{\mathcal{R}}F^{\mathcal{R}}\right\} \tag{D.8}$$

and Stokes theorem, we may evaluate the integral over the boundary of $B^4(r)$, given by $S^3(r)$. In the limit $r \to \infty$ the gauge fields approach a pure gauge, given by (D.1). It then holds $\mathrm{d}A = -A^2$. Thus, we obtain

$$\mathcal{Q}_{\text{Topo}} = \lim_{R\to\infty}\frac{4}{\mathcal{N}}\int_{S^3(r)} tr_{\mathcal{R}}\left\{G^3\right\} = \lim_{R\to\infty}\frac{-4}{3\mathcal{N}}\int_{S^3(r)} tr_{\mathcal{R}}\left\{A^3\right\} \tag{D.9}$$

However since any pure gauge field is characterized by a unique $U : S^3 \to G$, we obtain $-3\mathcal{N}\mathcal{Q}_{\text{Topo}} = 4I[U]$. Choosing $\mathcal{N} = 64\pi^2\tilde{T}^{\mathcal{R}}$ we get $\mathcal{Q} = \nu$.

We finish this appendix with a closing remark on a convenient choice of metric $\kappa : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ on $\mathfrak{g}$, that allows to determine the quantities λ and $\tilde{T}_{\mathcal{R}}$ easily. The results obtained for I and $\mathcal{Q}_{\text{Topo}}$, as well as the definition of $\mathcal{Q}_{\text{Topo}}$, depend on the choice of representation, but not on the choice of metric on $\mathfrak{g}$ or the basis of $\mathfrak{g}$. However, the choice of metric, explained in appendix 2.1 and the corresponding basis of generators giving totally antisymmetric structure constants for this metric is most convenient. In this convention one obtains that $\lambda = 1$, by explicit evaluation of the relation (D.3) for the accordingly normalized $SU(2)$-subgroup generators of $SU(2)$, $Sp(2)$ and $SO(4)$ in the defining representation $D : G \to \mathrm{GL}(N, \mathbb{C})$. As a consequence, since (D.3) is independent of the choice of representation, we see that $\tilde{T}_R = T_{\mathcal{R}}$, with $T_{\mathcal{R}}$ the corresponding Dynkin index for this choice of metric on $\mathfrak{g}$. The topological charge in this convention thus reads

$$\mathcal{Q}_{\text{Topo}} = -\frac{\epsilon^{\mu\nu\rho\sigma}}{64\pi^2 T_{\mathcal{R}}}\int d^4x\ tr_{\mathcal{R}}\left\{F^{\mathcal{R}}_{\mu\nu}F^{\mathcal{R}}_{\rho\sigma}\right\} \tag{D.10}$$

[1] Note that in equation (D.6) the integrand is the trace of a product of matrix valued differential forms. The topological charge must not be confused with the Yang-Mills action i.e. $tr_{\mathcal{R}}\left\{T^{\mathcal{R}}_{\alpha}T^{\mathcal{R}}_{\beta}\right\}F^{\alpha}_{\mu\nu}F^{\mu\nu\beta} \neq tr_{\mathcal{R}}\left\{F^{\mathcal{R}}F^{\mathcal{R}}\right\}$

and only takes on integer values ν. Note that the result in this form only holds for the specified choice of metric. In gauge theories, with a kinetic term

$$L_{YM} = -\frac{1}{4g^2} F^\alpha_{\mu\nu} F^{\mu\nu}_\alpha \tag{D.11}$$

for the gauge fields, one may actually fix the metric and basis of the Lie-algebra [34, Chpt. 15]. The scale of the gauge-fields enters as a free parameter g. However, often a field redefinition $\tilde{A} = \frac{1}{g} A$ with a simultaneous redefinition of the generators $\tilde{T}^{\mathcal{R}}_\alpha = g T^{\mathcal{R}}_\alpha$ is performed, in order to write the Lagrangian as in (3.1). To be consistent, the structure constants, in order to stay totally antisymmetric, must also be rescaled $\tilde{C}_{\alpha,\beta,\gamma} = g C_{\alpha,\beta,\gamma}$. However, this means that in (D.3) and (D.4) it holds that $\sqrt{\lambda} = g$ and nothing changes on the fact that $\tilde{T}_{\mathcal{R}} = T_{\mathcal{R}}$. However, evaluating the trace in (D.10) gives a different result. Since conventions may case a great deal of confusion we summarize how to rescale the formula (D.10), if performing the field redefinition $A \mapsto \tilde{A} = \frac{1}{g} A$ consistently.

$$L_{YM} = -\frac{1}{4\lambda} F^\alpha_{\mu\nu} F^{\mu\nu}_\alpha \qquad \longmapsto \qquad \tilde{L}_{YM} = -\frac{1}{4} \tilde{F}^\alpha_{\mu\nu} \tilde{F}^{\mu\nu}_\alpha \tag{D.12}$$

$$\mathcal{Q}_{\text{Topo}} = -\frac{\epsilon^{\mu\nu\rho\sigma}\delta_{\alpha\beta}}{64\pi^2} \int d^4x \; F^\alpha_{\mu\nu} F^\beta_{\rho\sigma} \longmapsto \tilde{\mathcal{Q}}_{\text{Topo}} = -g^2 \frac{\epsilon^{\mu\nu\rho\sigma}\delta_{\alpha\beta}}{64\pi^2} \int d^4x \; \tilde{F}^\alpha_{\mu\nu} \tilde{F}^\beta_{\rho\sigma} \tag{D.13}$$

Appendix E

Feynman diagrams

We now derive the Feynman rules for this theory. For this we will take all external momenta π^a of the external states as incoming. Four-momentum and energy conservation are captured by the equations

$$p_1 + p_2 + p_3 + \ldots = 0 \tag{E.1}$$

$$(p_1)^2 + (p_2)^2 + (p_3)^2 + \ldots = m_1^2 + m_2^2 + m_3^2 + \ldots \tag{E.2}$$

For $2 \to 2$ scattering we define the Mandelstamm variables

$$s := (p_1 + p_2)^2 = (p_1 + p_b)^2 \tag{E.3}$$

$$t := (p_1 + p_3)^2 = (p_2 + p_4)^2 \tag{E.4}$$

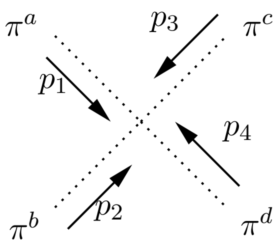
$$u := (p_1 + p_4)^2 = (p_2 + p_3)^2 \tag{E.5}$$

Within the calculation we meet pion fields with derivative fields. In momentum space these become (no summation convention)

$$\partial^\mu \pi_a(p) = i p^\mu \pi_a \tag{E.6}$$

E.1 4-pion scattering

We derive the Feynman rules for $\pi_a(p_1)\pi_b(p_2) \to \pi_c(p_c)\pi_d(p_d)$ scattering, depicted in figure E.1. For the derivation of this rule we consider all contributions from the Lagrangian (5.1)



$$
\begin{aligned}
&= \; -i\; M^{4\pi}_{abcd}(p_1, p_2; p_3, p_4) \\
&= \; \tfrac{2}{F_\pi^2} \left(\mathrm{tr}\left\{T_a^{\mathcal{F}} T_b^{\mathcal{F}} T_c^{\mathcal{F}} T_d^{\mathcal{F}}\right\} \; + \; \mathrm{tr}\left\{T_a^{\mathcal{F}} T_d^{\mathcal{F}} T_c^{\mathcal{F}} T_b^{\mathcal{F}}\right\}\right) \left(2m_\pi^2 \; - \; t\right) \\
&+ \; \tfrac{2}{F_\pi^2} \left(\mathrm{tr}\left\{T_a^{\mathcal{F}} T_c^{\mathcal{F}} T_b^{\mathcal{F}} T_d^{\mathcal{F}}\right\} \; + \; \mathrm{tr}\left\{T_a^{\mathcal{F}} T_d^{\mathcal{F}} T_b^{\mathcal{F}} T_c^{\mathcal{F}}\right\}\right) \left(2m_\pi^2 \; - \; s\right) \\
&+ \; \tfrac{2}{F_\pi^2} \left(\mathrm{tr}\left\{T_a^{\mathcal{F}} T_c^{\mathcal{F}} T_d^{\mathcal{F}} T_b^{\mathcal{F}}\right\} \; + \; \mathrm{tr}\left\{T_a^{\mathcal{F}} T_b^{\mathcal{F}} T_d^{\mathcal{F}} T_c^{\mathcal{F}}\right\}\right) \left(2m_\pi^2 \; - \; u\right)
\end{aligned}
$$

Figure E.1: Feynman rules for 4-pion scattering

in the second line. Thus, for fixed indices $(a, b, c, d) \in \{1, \ldots, 9\}^4$ we obtain the following contributions to the 4-point interaction

$$\frac{1}{3F_\pi^2}\left(\left[2\partial_\mu \pi_a \pi_b \partial^\mu \pi_c \pi_d - 2\pi_a \pi_b \partial_\mu \pi_c \partial^\mu \pi_d + m_\pi^2\right] \mathrm{tr}\left\{T_a^{\mathcal{F}} T_b^{\mathcal{F}} T_c^{\mathcal{F}} T_d^{\mathcal{F}}\right\} + \text{permutations of } (a, b, c, d) \ldots\right)$$

All permutations can be obtained by starting with the following six configurations

$$\begin{array}{ccc} t & s & u \\ (a,b,c,d) & (a,c,b,d) & (a,c,d,b) \\ (a,d,c,b) & (a,d,b,c) & (a,b,d,c) \end{array} \tag{E.7}$$

However, under a cyclic permutation the trace $\mathrm{tr}\left\{T_a^{\mathcal{F}}T_b^{\mathcal{F}}T_c^{\mathcal{F}}T_d^{\mathcal{F}}\right\}$ does not change, thus the coefficient stays the same. We evaluate the expression for each of the six configurations and the cyclic permutations separately. If we start with (a,b,c,d) we obtain for the term in square brackets

$$\begin{aligned}
&2\partial_\mu\pi_a\pi_b\partial^\mu\pi_c\pi_d - 2\pi_a\pi_b\partial_\mu\pi_c\partial^\mu\pi_d + m_\pi^2 \quad + \quad \text{cyclic permutations of } (a,b,c,d)\dots \\
&\qquad\qquad\qquad \downarrow \\
&4m_\pi^2 - 2p_a\cdot p_c + 2p_c\cdot p_d - 2p_b\cdot p_d + 2p_d\cdot p_a - 2p_c\cdot p_a + 2p_a\cdot p_b - 2p_d\cdot p_b + 2p_b\cdot p_c \\
&= 4m_\pi^2 - 2(2p_a\cdot p_c + 2p_b\cdot p_d) + 2(p_a+p_c)\cdot(p_b+p_d) \\
&= 4m_\pi^2 - 2((p_a)^2 + (p_b)^2 + (p_c)^2 + (p_d)^2 - 4m_\pi^2 + 2p_a\cdot p_c + 2p_b\cdot p_d) - 2(p_a+p_c)^2 \\
&= 12m_\pi^2 - 2(p_a+p_c)^2 - (p_b+p_d)^2 \\
&= 3(4m_\pi^2 - 2t)
\end{aligned}$$

Repeating this for the other five configurations gives similar results and putting everything together gives exactly the Feynman rule in figure E.1. The result agrees with the expression in [86][13]. For the case of $SU(N_C)$ gauge theories with two fermions this result reduces to the result in [41]. However, the result is also valid for fermions transforming under real and pseudo-real representations of the gauge group. Especially, after explicit evaluation of the trace for the $Sp(4)$ theory in [19], the result agrees with the expression given in the appendix C.1 of [19].

E.2 5-pion scattering

We derive the Feynman rule for 5-pion scattering $\pi^a(p_1)\pi^b(p_2)\pi^c(p_3) \to \pi^d(p_4)\pi^e(p_5)$ from the WzW term. The diagram and Rule are summarized figure E.2.
The contributions to 5-pion scattering come from the last line in the Lagrgangian (5.1)

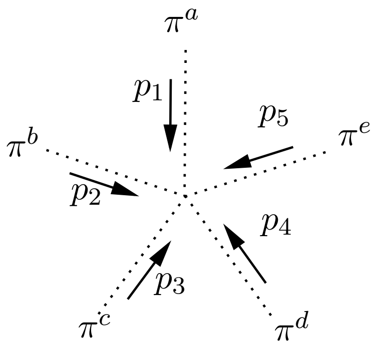
$$\frac{d_{\mathcal{R}}}{15\pi^2 F_\pi^5}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\{\pi\partial_\mu\pi\partial_\nu\pi\partial_\rho\pi\partial_\sigma\pi\} \tag{E.8}$$

For fixed indices $(a,b,c,d,e) \in \{1,\dots,9\}^5$ all terms in the Lagrangian vanish except $a \neq b \neq c \neq d \neq c$, which can be seen by the antisymmetry of the ϵ-tensor (and partial integration if a is equal to any other index). If all indices are different, we can group the contributions by their momentum dependence:

$$\begin{aligned}
p_2,p_3,p_4,p_5 - \text{dependence} &\;\to\; (a,b,c,d,e) \text{ and all permutations of } b,c,d,e \\
p_1,p_3,p_4,p_5 - \text{dependence} &\;\to\; (b,c,d,e,a) \text{ and all permutations of } b,c,d,e \\
p_1,p_2,p_4,p_5 - \text{dependence} &\;\to\; (c,d,e,a,b) \text{ and all permutations of } b,c,d,e \\
p_1,p_2,p_3,p_5 - \text{dependence} &\;\to\; (d,e,a,b,c) \text{ and all permutations of } b,c,d,e \\
p_1,p_2,p_3,p_4 - \text{dependence} &\;\to\; (e,a,b,c,d) \text{ and all permutations of } b,c,d,e
\end{aligned}$$

We collect the five distinct configurations, representing the five types of momentum dependence, into a set

$$I = \{(a,b,c,d,e),(b,c,d,e,a),(c,d,e,a,b),(d,e,a,b,c),(e,a,b,c,d)\} \tag{E.9}$$



$$
\begin{aligned}
&= \quad -i\ M^{5\pi}_{abcde}(p_1,p_2,p_3;p_4,p_5) \\
&= \tfrac{d_{\mathcal{R}}}{15\ \pi^2 F_\pi^2}\ \epsilon^{\mu\nu\rho\lambda} p^2_\mu p^3_\nu p^4_\rho p^5_\lambda\ \mathrm{tr}\left\{T^{\mathcal{F}}_a T^{\mathcal{F}}_{[b} T^{\mathcal{F}}_c T^{\mathcal{F}}_d T^{\mathcal{F}}_{e]}\right\} \\
&+ \tfrac{d_{\mathcal{R}}}{15\ \pi^2 F_\pi^2}\ \epsilon^{\mu\nu\rho\lambda} p^3_\mu p^4_\nu p^5_\rho p^1_\lambda\ \mathrm{tr}\left\{T^{\mathcal{F}}_b T^{\mathcal{F}}_{[c} T^{\mathcal{F}}_d T^{\mathcal{F}}_e T^{\mathcal{F}}_{a]}\right\} \\
&+ \tfrac{d_{\mathcal{R}}}{15\ \pi^2 F_\pi^2}\ \epsilon^{\mu\nu\rho\lambda} p^4_\mu p^5_\nu p^1_\rho p^2_\lambda\ \mathrm{tr}\left\{T^{\mathcal{F}}_c T^{\mathcal{F}}_{[d} T^{\mathcal{F}}_e T^{\mathcal{F}}_a T^{\mathcal{F}}_{b]}\right\} \\
&+ \tfrac{d_{\mathcal{R}}}{15\ \pi^2 F_\pi^2}\ \epsilon^{\mu\nu\rho\lambda} p^5_\mu p^1_\nu p^2_\rho p^3_\lambda\ \mathrm{tr}\left\{T^{\mathcal{F}}_d T^{\mathcal{F}}_{[e} T^{\mathcal{F}}_a T^{\mathcal{F}}_b T^{\mathcal{F}}_{c]}\right\} \\
&+ \tfrac{d_{\mathcal{R}}}{15\ \pi^2 F_\pi^2}\ \epsilon^{\mu\nu\rho\lambda} p^1_\mu p^2_\nu p^3_\rho p^4_\lambda\ \mathrm{tr}\left\{T^{\mathcal{F}}_e T^{\mathcal{F}}_{[a} T^{\mathcal{F}}_b T^{\mathcal{F}}_c T^{\mathcal{F}}_{d]}\right\}
\end{aligned}
$$

Figure E.2: Feynman rules for 5-pion scattering. The square bracket in the trace denotes total antisymmetrization of the indices within the square bracket. The expression for the trace was already introduced in (4.73).

and denote by S^4 the group of all permutations $\sigma : \{1,2,3,4\} \to \{1,2,3,4\}$ of four elements. Introducing $\vec{b} = (b_0, b_1, b_2, b_3, b_4)$ we may write all the terms in the Lagrangian, contributing to the vertex as (no summation over the b_i, since summation is already taken into account by the sum in front)

$$
\begin{aligned}
&\frac{d_{\mathcal{R}}}{15\pi^2 F_\pi^5} \sum_{\vec{b}\in I} \sum_{\sigma\in S^4} \epsilon^{\mu\nu\rho\lambda} \pi^{b_0} \partial_\mu \pi^{b_{\sigma(1)}} \partial_\nu \pi^{b_{\sigma(2)}} \partial_\rho \pi^{b_{\sigma(3)}} \partial_\lambda \pi^{b_{\sigma(4)}}\ \mathrm{tr}\Big\{T^{\mathcal{F}}_{b_0} T^{\mathcal{F}}_{b_{\sigma(1)}} T^{\mathcal{F}}_{b_{\sigma(2)}} T^{\mathcal{F}}_{b_{\sigma(3)}} T^{\mathcal{F}}_{b_{\sigma(4)}}\Big\} \\
&= \frac{d_{\mathcal{R}}}{15\pi^2 F_\pi^5} \sum_{\vec{b}\in I} \epsilon^{\mu\nu\rho\lambda} \pi^{b_0} \partial_\mu \pi^{b_1} \partial_\nu \pi^{b_2} \partial_\rho \pi^{b_2} \partial_\lambda \pi^{b_4} \sum_{\sigma\in S^4} \mathrm{sign}(\sigma)\, \mathrm{tr}\Big\{T^{\mathcal{F}}_{b_0} T^{\mathcal{F}}_{b_{\sigma(1)}} T^{\mathcal{F}}_{b_{\sigma(2)}} T^{\mathcal{F}}_{b_{\sigma(3)}} T^{\mathcal{F}}_{b_{\sigma(4)}}\Big\} \\
&= \frac{d_{\mathcal{R}}}{15\pi^2 F_\pi^5} \sum_{\vec{b}\in I} \epsilon^{\mu\nu\rho\lambda} \pi^{b_0} \partial_\mu \pi^{b_1} \partial_\nu \pi^{b_2} \partial_\rho \pi^{b_2} \partial_\lambda \pi^{b_4}\ \mathrm{tr}\Big\{T^{\mathcal{F}}_{b_0} T^{\mathcal{F}}_{[b_1} T^{\mathcal{F}}_{b_2} T^{\mathcal{F}}_{b_3} T^{\mathcal{F}}_{b_4]}\Big\}
\end{aligned}
$$

If we move to momentum space and remove the pion fields, we obtain exactly the expression as in figure E.2. The expression agrees[1] with the result in [86]. In order for this section to be

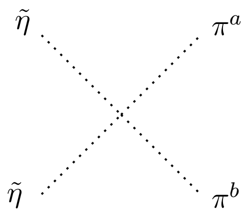
[1] In order to see this one has to note that $\mathrm{tr}\{T^{\mathcal{F}}_b T^{\mathcal{F}}_{[b} T^{\mathcal{F}}_b T^{\mathcal{F}}_b T^{\mathcal{F}}_{b]}\}$ is calculated by adding up all contributions for permutations of b_1, b_2, b_3, b_4 weighted with the signs of the permutations. However, since cyclic permutations of (a, b, c, d, e) all have the same sign (in contrast to cyclic permutations of four elements), the factor $sign(\sigma)$ may also be determined from the sign of the permutation $(b_0, b_{\sigma(1)}, b_{\sigma(2)}, b_{\sigma(3)}, b_{\sigma(4)})$ of (a, b, c, d, e), as is done in [86].

self-contained we add the explicit expression for the totally antisymmetrized trace.

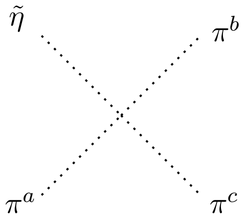
$$
\begin{aligned}
&\mathrm{tr}\Big\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{[b_1}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_4]}\Big\}\\
&=\sum_{\sigma\in S^4}\mathrm{sign}(\sigma)\,\mathrm{tr}\Big\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_{\sigma(1)}}T^{\mathcal{F}}_{b_{\sigma(2)}}T^{\mathcal{F}}_{b_{\sigma(3)}}T^{\mathcal{F}}_{b_{\sigma(4)}}\Big\}\\
&=\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_4}\}-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_1}\}+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_2}\}-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_3}\}\\
&-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_3}\}+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_1}\}-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_2}\}+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_4}\}\\
&-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_4}\}+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_1}\}-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_3}\}+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_2}\}\\
&+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_2}\}-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_1}\}+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_3}\}-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_4}\}\\
&+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_3}\}-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_1}\}+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_4}\}-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_2}\}\\
&-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_2}\}+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_1}\}-\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_3}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_4}\}+\mathrm{tr}\{T^{\mathcal{F}}_{b_0}T^{\mathcal{F}}_{b_2}T^{\mathcal{F}}_{b_1}T^{\mathcal{F}}_{b_4}T^{\mathcal{F}}_{b_3}\}
\end{aligned}
$$

E.3 $\eta - \pi$ scattering

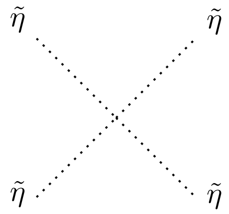
In figure E.3 we depict the diagrams for $\tilde{\eta}(p_1)\tilde{\eta}(p_2) \to \pi^a(p_3)\pi^a(p_4)$ scattering, $\tilde{\eta}(p_1)\pi^a(p_2) \to \pi^b(p_3)\pi^c(p_4)$ scattering and $\tilde{\eta}(p_1)\tilde{\eta}(p_2) \to \tilde{\eta}(p_3)\tilde{\eta}(p_4)$ scattering. The rules presented in figure E.3 can easily be obtained by considering all contributions form the third line in the Lagrangian (5.1), since there is no momentum dependence. Symmetry factors of $N_{\tilde{\eta}}!$ must be added in case $\tilde{\eta}$ appears $N_{\tilde{\eta}}$ times in the expression. The same holds if an identical pion field appears multiple times. This is due to the appearance of all possible field pairings in the path-integral formalism when deriving the Feynman rules.



$$= \quad -i\ M^{2\tilde{\eta}2\pi}_{a} \quad = \quad \frac{m_\pi^2}{2\ F_{\tilde{\eta}}^2}\ \delta_{ab}$$



$$= \quad -\mathrm{i}\ \mathrm{M}^{\tilde{\eta}3\pi}_{\mathrm{abc}} \quad = \quad \frac{\sqrt{2}\ \mathrm{m}_\pi^2}{\mathrm{F}_\pi \mathrm{F}_{\tilde{\eta}}}\ \mathrm{tr}\left\{T^{\mathcal{F}}_a T^{\mathcal{F}}_b T^{\mathcal{F}}_c + T^{\mathcal{F}}_a T^{\mathcal{F}}_c T^{\mathcal{F}}_b\right\}$$

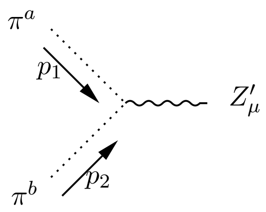


$$= \quad -i\ M^{4\tilde{\eta}} \quad = \quad \frac{m_\pi^2\ F_\pi^2}{2\ F_{\tilde{\eta}}^4}$$

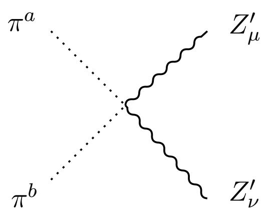
Figure E.3: Feynman rules for $\tilde{\eta}$ scattering off pions and itself.

E.4 $\pi - Z'$ scattering

The Feynman rules for the $Z' - \pi$ vertices can be read off directly from the Lagrangian (4.86). The resulting rules are depicted in figure E.4



$$= 2i\, e_D\, \mathrm{tr}\left\{\left(T_a^{\mathcal{F}} T_b^{\mathcal{F}} - T_b^{\mathcal{F}} T_a^{\mathcal{F}}\right) \mathcal{Q}\right\} (p_1 - p_2)_\mu$$



$$= 4\, e_D^2\, \mathrm{tr}\left\{\left(T_a^{\mathcal{F}} T_b^{\mathcal{F}} + T_b^{\mathcal{F}} T_a^{\mathcal{F}}\right) \mathcal{Q}^2 - 2T_a^{\mathcal{F}} \mathcal{Q} T_b^{\mathcal{F}} \mathcal{Q}\right\} g_{\mu\nu}$$

Figure E.4: Feynman rules for interactions between the dark photon and the dark pions.

E.5 Decay rate $\Gamma(\eta \to 4f)$

This is a very crude estimate in order to estimate the decay rate $\Gamma(\tilde{\eta} \to 4f)$ of $\tilde{\eta}$ into four standard model fermions f. The process is given by the Feynman diagram in figure E.5.

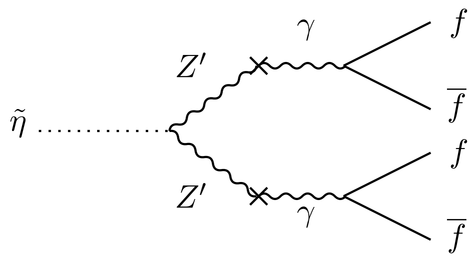


Figure E.5: Feynman diagram for the $\tilde{\eta} \to 4f$ decay.

Following [87] the decay rate may be estimated by starting from the anomalous decay of $\tilde{\eta}$ into two off-shell dark photons, given by (4.93), followed by two subsequent decays of the dark off-shell photons into standard model fermions. The rate of the first process is determined by the anomalous contributions to the Lagrangian of the form

$$\mathcal{L}^{\mathrm{IR}}_{\eta Z' Z'} = -\beta \tilde{\eta}^0 \epsilon^{\mu\nu\rho\sigma} F'_{\mu\nu} F'_{\rho\sigma} \tag{E.10}$$

Es explained in [34, Chpt. 22.1], the decay rate is then given by

$$\Gamma(\tilde{\eta} \to Z' Z') \approx \frac{m_{\tilde{\eta}}^3 \beta^2}{\pi} \tag{E.11}$$

Comparing with (4.93) and defining $\alpha_D = e_D / 4\pi$, we deduce

$$\beta = \frac{\alpha_D d_{\mathcal{R}}}{2\pi F_{\tilde{\eta}}} \mathrm{tr}\left\{T_0^F \mathcal{Q}^2\right\} \tag{E.12}$$

According to [87], the $Z' -> f\overline{f}$ decay contributes a suppression by a factor

$$\frac{\alpha}{2\pi}\epsilon^2\left(\frac{m_{\tilde{\eta}}}{2m_{Z'}}\right)^4 \tag{E.13}$$

such that we obtain for the decay rate

$$\Gamma(\tilde{\eta}\to 4f) = \frac{m_{\tilde{\eta}}^3}{\pi}\left(\frac{\alpha_D d_{\mathcal{R}}}{2\pi F_{\tilde{\eta}}}\,\mathrm{tr}\{T_0^F \mathcal{Q}^2\}\right)^2\left(\frac{\alpha}{2\pi}\epsilon^2\left(\frac{m_{\tilde{\eta}}}{2m_{Z'}}\right)^4\right)^2 \tag{E.14}$$

Appendix F

Interpolating scalar meson operators

We present a systematic construction of field bilinears for all the lowest energy states discussed in 3.3. The main theme in the construction is to find an invariant pairing of the fundamental degrees of freedom, which we take to be the Weyl fermions[1] $\psi^{(k)}$ and figure out its behavior under parity and flavor transformations. We start with spin zero operators. Two Weyl fermions may be combined as

$$(\psi^{(k)}, \psi^{(j)}) := \psi^{(k)} E^* S^* \psi^{(j)} \tag{F.1}$$

similar to equation (3.13). This paring is linear in each argument and symmetric under exchange of arguments. The latter can be seen explicitly by transposing the whole expression and then taking into account anti-commutativity of the fermions as well as the properties of E and S. Under a parity transformation (3.34) we observe that

$$\psi^{(k)} E^* S^* \psi^{(j)} \stackrel{\boldsymbol{P}}{\longmapsto} -\delta^{km}\omega_{mn}\delta^{jl}\omega_{li} \left(\psi^{(n)} E^* S^* \psi^{(i)}\right)^* \tag{F.2}$$

Thus, while the paring transforms as a scalar under proper Lorentz transformations [46], the paring does not have a designated behavior under parity. In order to achieve a paring, that transforms as a proper scalar or pseudo-scalar, we need to form linear combinations with the complex conjugate paring. The most general bilinear operator we can build in this fashion is

$$\mathcal{O}_{\mathrm{S/PS}}: \quad O_{kj}\, \psi^{(k)} E^* S^* \psi^{(j)} + \tilde{O}_{kj} \left(\psi^{(k)} E^* S^* \psi^{(j)}\right)^* \tag{F.3}$$

with O_{kj}, $\tilde{O}_{kj}$ being $2N_f^2 = 32$ complex coefficients. $N_f = 4$ denotes the number of Weyl flavors. Because (F.1) is a symmetric bilinear form, actually only $N_f(N_f+1) = 20$ of these coefficients are linear independent. A convenient choice are the symmetric matrices $O_{kj} = \{\omega T_a^{\mathcal{F}}\}_{kj} = \pm\tilde{O}^*_{kj}$ with $T_a^{\mathcal{F}}$ the broken generators (A.1) of the flavor group. Under an $SO(N_f)$ flavor rotation $\psi^{(k)} \mapsto U^k_j \psi^{(j)}$ the coefficients O_{kj}, $\tilde{O}_{kj}$ transform as symmetric covariant 2-tensors. However, due to (A.2) we have

$$O_{kj} \stackrel{U}{\longmapsto} \left\{U^\top \omega T_a^{\mathcal{F}} U\right\}_{kj} = \left\{\omega U^\dagger T_a^{\mathcal{F}} U\right\}_{kj} \tag{F.4}$$

and thus the operators $\mathcal{O}_{\mathrm{S/PS}}$, constructed in this way, transform in the same manner as the conserved currents (3.17) of the broken transformations. Further we have

$$O_{kj}\delta^{km}\omega_{mn}\delta^{jl}\omega_{li} = \left\{\omega\omega T_a^{\mathcal{F}}\omega\right\}_{ni} = \left\{\omega T_a^{\mathcal{F}*}\right\}_{ni} = \pm\tilde{O}_{kj} \tag{F.5}$$

Hence we see that operators with coefficients $O_{kj} = +\tilde{O}^*_{kj}$ behave like pseudo-scalars under a parity transformation (F.2). The operators $\mathcal{O}_{\mathrm{PS}}$ couple to the pNGB states by virtue of (3.52). These are thus interpolating operators for the pNGBs in the theory. In section 3.3

[1] Equivalently we could take the Majorana basis $q_m^{(k)}$ or the $\phi^{(k)}$. The choice of $\psi^{(k)}$ is purely out of convenience in order to pronounce structural features.

another Nambu-Goldstone basis $|\tilde{\pi}^a\rangle$ for the charged states was introduced. The corresponding coefficients matrices O_{kj} are given by using the non-hermitian matrices

$$\begin{array}{llllll} \tilde{T}_1 & = \frac{1}{\sqrt{2}}\left(T_1^{\mathcal{F}} - iT_2^{\mathcal{F}}\right) & \tilde{T}_2 & = \frac{1}{\sqrt{2}}\left(T_1^{\mathcal{F}} + iT_2^{\mathcal{F}}\right) & \tilde{T}_3 & = T_3^{\mathcal{F}} \\ \tilde{T}_4 & = \frac{1}{\sqrt{2}}\left(T_4^{\mathcal{F}} - iT_7^{\mathcal{F}}\right) & \tilde{T}_5 & = \frac{1}{\sqrt{2}}\left(T_5^{\mathcal{F}} - iT_8^{\mathcal{F}}\right) & \tilde{T}_6 & = \frac{1}{\sqrt{2}}\left(T_6^{\mathcal{F}} - iT_9^{\mathcal{F}}\right) \\ \tilde{T}_7 & = \frac{1}{\sqrt{2}}\left(T_4^{\mathcal{F}} + iT_7^{\mathcal{F}}\right) & \tilde{T}_8 & = \frac{1}{\sqrt{2}}\left(T_5^{\mathcal{F}} + iT_8^{\mathcal{F}}\right) & \tilde{T}_9 & = \frac{1}{\sqrt{2}}\left(T_6^{\mathcal{F}} + iT_9^{\mathcal{F}}\right) \end{array} \tag{F.6}$$

instead of the generators $T_a^{\mathcal{F}}$. The pseudo-scalar bilinear operators for pseudo-scalar states are summarized in table F.1 for the case of $N_f = 4$ Weyl flavors. They are given explicitly as parings of the two Dirac fermions in the theory, denoted as $q^{(g)} = u, d$ and their charge conjugates $q_{\mathcal{C}}^{(g)} = u_{\mathcal{C}}, d_{\mathcal{C}}$. The identification may be done by the relations

$$\begin{array}{rll} \overline{q}^{(g)}\gamma^5 q^{(h)} = & \psi^{(2g)}E^*S^*\psi^{(2h-1)} & + \left(\psi^{(2g-1)}E^*S^*\psi^{(2h)}\right)^* \\ \overline{q}^{(g)}\gamma^5 q_{\mathcal{C}}^{(h)} = & \psi^{(2g-1)}E^*S^*\psi^{(2h-1)} & + \left(\psi^{(2g)}E^*S^*\psi^{(2h)}\right)^* \\ \overline{q}_{\mathcal{C}}^{(g)}\gamma^5 q^{(h)} = & \psi^{(2g)}E^*S^*\psi^{(2h)} & + \left(\psi^{(2g-1)}E^*S^*\psi^{(2h-1)}\right)^* \\ \overline{q}_{\mathcal{C}}^{(g)}\gamma^5 q_{\mathcal{C}}^{(h)} = & \psi^{(2g-1)}E^*S^*\psi^{(2h)} & + \left(\psi^{(2g)}E^*S^*\psi^{(2h-1)}\right)^* \end{array}$$

Replacing γ^5 by $\mathbb{1}$ above on the left of the equal sign changes the relative sign to + on the right.

	$\Psi^\dagger\omega EST_a^{\mathcal{F}}\Psi + h.c.$	J^P		$\Psi^\dagger\omega ES\tilde{T}_a\Psi + h.c.$	I_z	$\mathcal{Q}$
π^1	$\frac{1}{\sqrt{2}}\left(\overline{u}\gamma^5 d + \overline{d}\gamma^5 u\right)$	0^-	$\tilde{\pi}^1$	$\overline{u}\gamma^5 d$	1	0
π^2	$\frac{i}{\sqrt{2}}\left(\overline{u}\gamma^5 d - \overline{d}\gamma^5 u\right)$	0^-	$\tilde{\pi}^2$	$\overline{d}\gamma^5 u$	-1	0
π^3	$\frac{1}{\sqrt{2}}\left(\overline{u}\gamma^5 u - \overline{d}\gamma^5 d\right)$	0^-	$\tilde{\pi}^3$	$\frac{1}{\sqrt{2}}\left(\overline{u}\gamma^5 u - \overline{d}\gamma^5 d\right)$	0	0
π^4	$\frac{1}{2}\left(\overline{u_{\mathcal{C}}}\gamma^5 u + \overline{u}\gamma^5 u_{\mathcal{C}}\right)$	0^-	$\tilde{\pi}^4$	$\frac{1}{\sqrt{2}}\overline{u_{\mathcal{C}}}\gamma^5 u$	-1	-2
π^5	$\frac{1}{2}\left(\overline{d_{\mathcal{C}}}\gamma^5 d + \overline{d}\gamma^5 d_{\mathcal{C}}\right)$	0^-	$\tilde{\pi}^5$	$\frac{1}{\sqrt{2}}\overline{d}\gamma^5 d_{\mathcal{C}}$	1	-2
π^6	$\frac{1}{\sqrt{2}}\left(\overline{u}\gamma^5 d_{\mathcal{C}} + \overline{u_{\mathcal{C}}}\gamma^5 d\right)$	0^-	$\tilde{\pi}^6$	$\overline{u}\gamma^5 d_{\mathcal{C}}$	0	-2
π^7	$\frac{i}{2}\left(\overline{u_{\mathcal{C}}}\gamma^5 u - \overline{u}\gamma^5 u_{\mathcal{C}}\right)$	0^-	$\tilde{\pi}^7$	$\frac{1}{\sqrt{2}}\overline{u}\gamma^5 u_{\mathcal{C}}$	1	2
π^8	$\frac{i}{2}\left(\overline{d}\gamma^5 d_{\mathcal{C}} - \overline{d_{\mathcal{C}}}\gamma^5 d\right)$	0^-	$\tilde{\pi}^8$	$\frac{1}{\sqrt{2}}\overline{d}_{\mathcal{C}}\gamma^5 d$	-1	2
π^9	$\frac{i}{\sqrt{2}}\left(\overline{u}\gamma^5 d_{\mathcal{C}} - \overline{u_{\mathcal{C}}}\gamma^5 d\right)$	0^-	$\tilde{\pi}^9$	$\overline{u_{\mathcal{C}}}\gamma^5 d$	0	2
η	$\frac{1}{\sqrt{2}}\left(\overline{u}\gamma^5 u + \overline{d}\gamma^5 d\right)$	0^-	η	$\frac{1}{\sqrt{2}}\left(\overline{u}\gamma^5 u + \overline{d}\gamma^5 d\right)$	0	0

Table F.1: Bilinear interpolation operators for pseudo-scalar states. The general expression (F.3) is evaluated in terms of Dirac fermions $q^{(1)} = u$ and $q^{(2)} = d$.

Appendix G

A non-fundamental example theory: $Sp(4)$ AT2T

For lattice field theory studies of real gauge theory realizations of dark matter, the antisymmetric, traceless, 2-tensor (AT2T) representation of $Sp(4)$ is of interest. $Sp(4)$ is the universal covering group of $SO(5)$. The $Sp(4)$-AT2T representation is locally isomorphic to $SO(5)$-vector. Following the discussion in appendix B it becomes clear how this isomorphism can be made explicit by introduction of an appropriate basis for the representation space V_{AT2T} of the AT2T-representation.
Before discussing the AT2T-representation of general $Sp(2N_C)$ groups, let us first start with the fundamental representation $\mathcal{F}: Sp(2N_C) \to \mathrm{Aut}\left(\mathbb{C}^{2N_C}\right)$. For this we define the matrix[1]

$$\Omega_{2N_C} := \mathbb{1}_{N_C} \otimes i\overline{\sigma}^2 \tag{G.1}$$

Here $\otimes$ denotes the Kronecker product of matrices. In the case of $Sp(4)$ we have

$$\Omega_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{G.2}$$

The group $Sp(2N_C)$ may be defined as the Lie-subgroup of $SU(2N_C)$, whose transformation matrices $U^{\mathcal{F}}$ in the fundamental representation leave invariant the symplectic metric on the representation space $V_{\mathcal{F}} = \mathbb{C}^{2N_C}$, defined by the matrix Ω_{2N_C}. This invariance condition may be formulated as

$$U^{\mathcal{F}\top}\Omega_{2N_C}U^{\mathcal{F}} = \Omega_{2N_C} \tag{G.3}$$

This also defines the representation matrices of the fundamental representation of $Sp(2N_C)$. The invariance condition (G.3) can be used to determine that $Sp(4)$-fundamental transformations may be generated by the matrices $T_4^{\mathcal{F}}, T_5^{\mathcal{F}}, T_6^{\mathcal{F}}, T_7^{\mathcal{F}}, T_8^{\mathcal{F}}, T_9^{\mathcal{F}}, T_{10}^{\mathcal{F}}, T_{11}^{\mathcal{F}}, T_{12}^{\mathcal{F}}$ and $T_{15}^{\mathcal{F}}$, given in (A.1). Now for the AT2T representation we define the representation space V_{AT2T} as the linear space of all antisymmetric two-tensors $Q^{ab} = -Q^{ba}$ satisfying the condition

$$\{\Omega_{2N_C}\}_{ab}\, Q^{ba} = 0 \tag{G.4}$$

This condition of a vanishing symplectic trace can also be formulated as $\mathrm{Tr}_{\mathrm{AT2T}}\{\Omega_{2N_C}Q\} = 0$, if representing the 2-tensor by a matrix. The action of the AT2T representation is now defined as

$$\left\{U^{\mathrm{AT2T}}Q\right\}^{ab} = \left\{U^{\mathcal{F}}\right\}^a_c Q^{cd} \left\{U^{\mathcal{F}}\right\}^b_d \tag{G.5}$$

$$U^{\mathrm{AT2T}}Q = U^{\mathcal{F}}QU^{\mathcal{F}\top} \tag{G.6}$$

In order to apply all the formalism in this thesis we need to proof that this representation is real. To do so we use the defining invariance condition (G.3) and define a linear map S

$$SQ := -\Omega_{2N_C}Q\Omega_{2N_C} \tag{G.7}$$

[1]The matrix Ω_{2N} should not be confused with the Maurer-Cartan form (4.12), denoted by the same symbol

and the anti-linear map $JQ = SQ^*$. Now due to $\Omega_{2N_C}^2 = -\mathbb{1}$ and $\Omega_{2N_C}^* = \Omega_{2N_C}$ we obtain $J^2Q = S(SQ^*)^* = \Omega_{2N_C}(\Omega_{2N_C}Q^*\Omega_{2N_C})^*\Omega_{2N_C} = (-1)^2Q$. Since Ω_{2N_C} is unitary, S is also unitary and J is an anti-linear conjugation. Due to the invariance condition (G.3) the map J also is equivariant. Hence, AT2T is a real representation. The map S in (G.7) is the map required to define the augmented charge conjugation (3.8).
In order to charge the dark quarks under the AT2T representation of $Sp(4)$ we organize the color degrees of freedom of the quark in an antisymmetric, traceless matrix.

$$Q^{(g)ab} = \frac{1}{2}\begin{pmatrix} 0 & q^{(g)5} & -q^{(g)3} - iq^{(g)1} & -q^{(g)4} - iq^{(g)2} \\ -q^{(g)5} & 0 & q^{(g)4} - iq^{(g)2} & -q^{(g)3} + iq^{(g)1} \\ q^{(g)3} + iq^{(g)1} & -q^{(g)4} + iq^{(g)2} & 0 & -q^{(g)5} \\ q^{(g)4} + iq^{(g)2} & q^{(g)3} - iq^{(g)1} & q^{(g)5} & 0 \end{pmatrix}^{ab}$$

In this form $Q^{(g)ab} = q^{(g)\alpha}\left\{E_\alpha\right\}_{ab}$, where the index α is the gauge index of the $SO(5)$-vector representation and the basis E_α of V_{AT2T} can be used to connect the two representations. Especially the following, bilinear invariant product

$$(E_\alpha, E_\beta) = \text{Tr}_{\text{AT2T}}\left\{\Omega_{2N_C}E_\alpha\Omega_{2N_C}E_\beta\right\} = \delta_{\alpha\beta} \tag{G.8}$$

on V_{AT2T} takes on a canonical form with respect to this basis. It also holds that $\text{Tr}_{\text{AT2T}}\left\{E_\alpha^\dagger E_\beta\right\} = \delta_{\alpha\beta}$. In the matrix notation $Q^{(g)}$ of the quarks the covariant derivative is given as

$$\text{D}_\mu^{\text{AT2T}}\left[G\right]Q = \partial_\mu Q + G_\mu^{\mathcal{F}}Q + QG_\mu^{\mathcal{F}\top} \tag{G.9}$$

where $G_\mu^F = -igG_\mu^\alpha T_\alpha^{\mathcal{F}}$ and $T_\alpha^{\mathcal{F}}$ are the generators of $Sp(4)$ in the fundamental representation. The Dirac Lagrangian (3.16) of the dark quarks may be formulated as

$$\mathcal{L}_q^{UV} = \sum_g \text{Tr}_{\text{AT2T}}\left\{iQ^{(g)\dagger}\gamma^0\gamma^\mu\text{D}_\mu^{\text{AT2T}}\left[G\right]Q^{(g)} - m_{(g)}Q^{(g)\dagger}\gamma^0Q^{(g)}\right\} \tag{G.10}$$

The trace contracts all the open indices a on the space V_{AT2T}. Adopting the Nambu-Gorkov formalism (3.10), the Dirac spinors may be rewritten as

$$Q^{(g)ab} = \begin{pmatrix} \psi^{(2g-1)ab} \\ -\Omega_{2N_C}E\left(\psi^{(2g)ab}\right)^*\Omega_{2N_C} \end{pmatrix} \tag{G.11}$$

Arranging the fields $\psi^{(k)ab}$ in a vector Ψ as in (3.16) allows to rewrite the dark quark Lagrangian as

$$\mathcal{L}_q^{UV} = \text{Tr}_{\text{AT2T}}\left\{\Psi^\dagger\overline{\sigma}^\mu\text{D}_\mu^{\text{AT2T}}\left[G\right]\Psi - \frac{1}{2}\left(\Omega_{2N_C}\Psi^\dagger ES\Omega_{2N_C}M\Psi^* - \Omega_{2N_C}\Psi^\top E^*\Omega_{2N_C}M\Psi\right)\right\}$$

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