Electromagnetics and Differential Forms

Stokes’ Theorem

In today’s lecture we talk about:

- **Boundary operator**

- Stokes’ theorem

The boundary of a Euclidean simplex

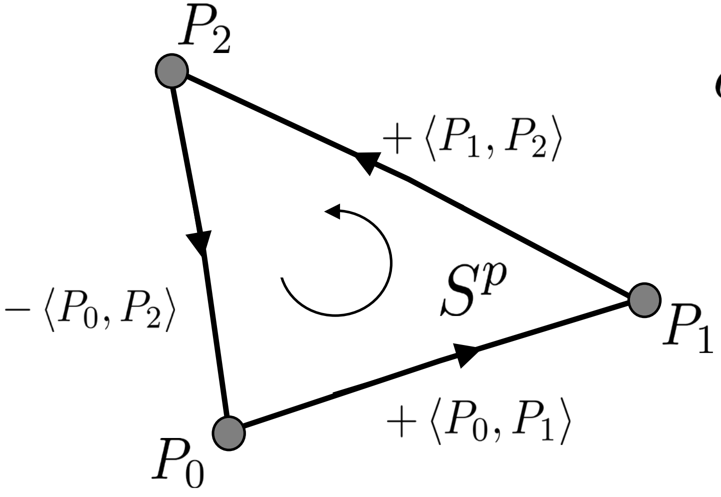
Let $S^p := \langle P_0, \ldots, P_p \rangle$ be a simplex in $\mathbb{R}^n$.

The Euclidean $(p-1)$ - chain

$$\partial S^p := \sum_{k=0}^{p} (-1)^k \langle P_0, \ldots, P_{k-1}, P_{k+1}, \ldots, P_p \rangle$$

is called the **boundary** of S^p.

A consistent orientation for the boundary is induced by the orientation of the simplex.



$$\partial \langle P_0, P_1, P_2 \rangle = +\langle P_1, P_2 \rangle - \langle P_0, P_2 \rangle + \langle P_0, P_1 \rangle$$

The boundary of a simplex on a manifold

Let $\mathcal{S}^p$ be a p-simplex on a n-dimensional manifold M.

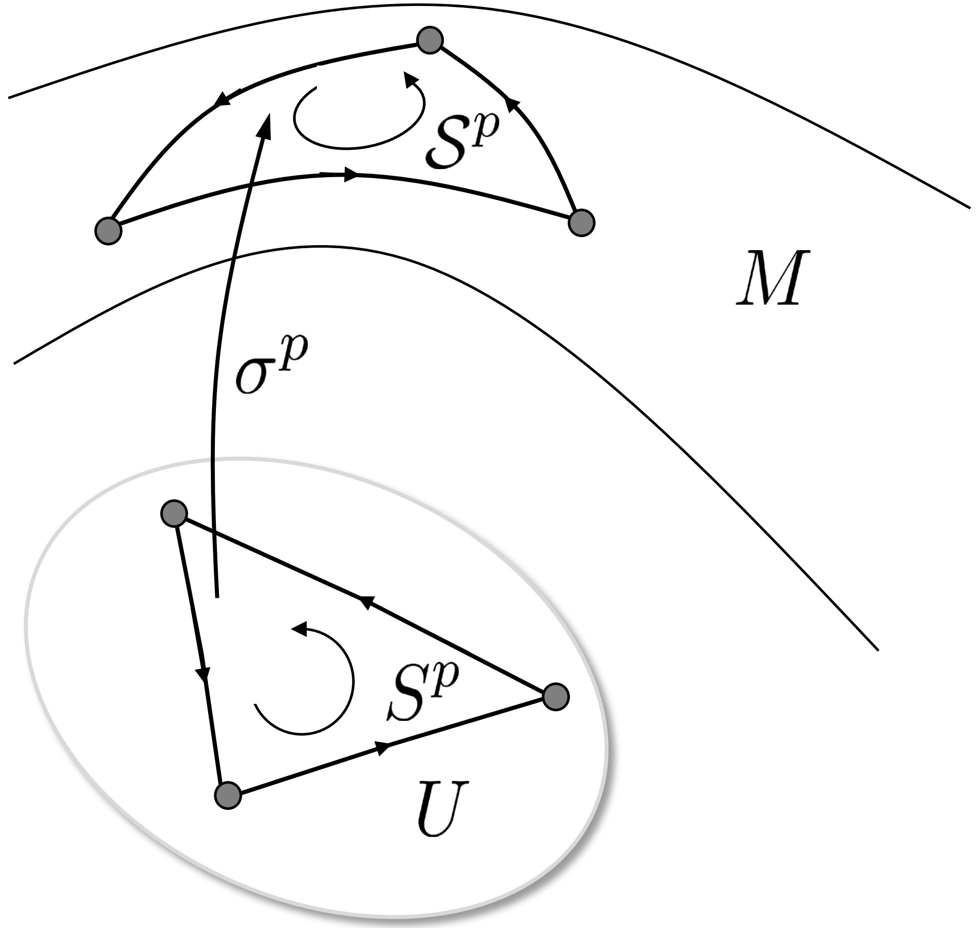
Let (S^p, U, σ^p) be a representation of $\mathcal{S}^p$.

Define the boundary of $\mathcal{S}^p$ by

$$\partial \mathcal{S}^p := \sigma^p \left(\partial S^p \right)$$

The boundary $\partial \mathcal{S}^p$ is a $(p-1)$- chain on M.

This definition is independent of the representation.



The boundary operator for chains

The boundary operator is defined as

$$\partial : \mathcal{C}_p(M) \to \mathcal{C}_{p-1}(M)$$

$$K = \sum_i \lambda_i \mathcal{S}_i^p \mapsto \partial K = \sum_i \lambda_i \partial \mathcal{S}_i^p$$

The operator is linear

space of p-chains over M.

$$\partial \left(K + \alpha K' \right) = \partial K + \alpha \partial K' \qquad \forall \alpha \in \mathbb{Z} \;\; \forall K, K' \in \mathcal{C}_p(M)$$

The operator commutes with smooth injective maps $\psi : M \to N$

$$\partial \psi \left(\mathcal{S}^p \right) = \partial \psi \left(\sigma^p \left(S^p \right) \right) = \partial \tilde{\sigma}^p (S^p) = \tilde{\sigma}^p (\partial S^p) = \psi \left(\sigma^p (\partial S^p) \right) = \psi(\partial \mathcal{S}^p)$$

Complex property of the boundary operator

The boundary of a boundary is zero!

$$\partial \circ \partial = 0$$

We already saw :

$$\mathrm{d} \circ \mathrm{d} = 0$$

We will see that these statements, viewed from a more abstract perspective, are equivalent.

The proof follows the following idea:

For a Euclidean simplex S^p, the terms

$$\langle P_0, \ldots, P_{j-1}, P_{j+1}, \cdots, P_{k-1}, P_{k+1}, \ldots, P_p \rangle$$

appear twice with opposite signs in the sum. $\Rightarrow \quad \partial \circ \partial \, S^p = 0$

$\Rightarrow$ Generalize result with properties of boundary operator.

In today’s lecture we talk about:

- Boundary operator
- **Stokes’ theorem**

Stokes' theorem for integration on chains [9, p. 109]

Let ω be a $(p-1)$- form on a n-dimensional manifold M and K a p-chain on M.

If ω is continuously differentiable on M, then it holds

One can choose here the smallest open subset of M that contains $\bar{K}$. Such a set always exist due to the definition of simplices on M.

$$\int_K \mathrm{d}\omega = \int_{\partial K} \omega$$

Remark:

It is important for ω to **not** have any singularities on K, even though only the boundary appears in the integral.

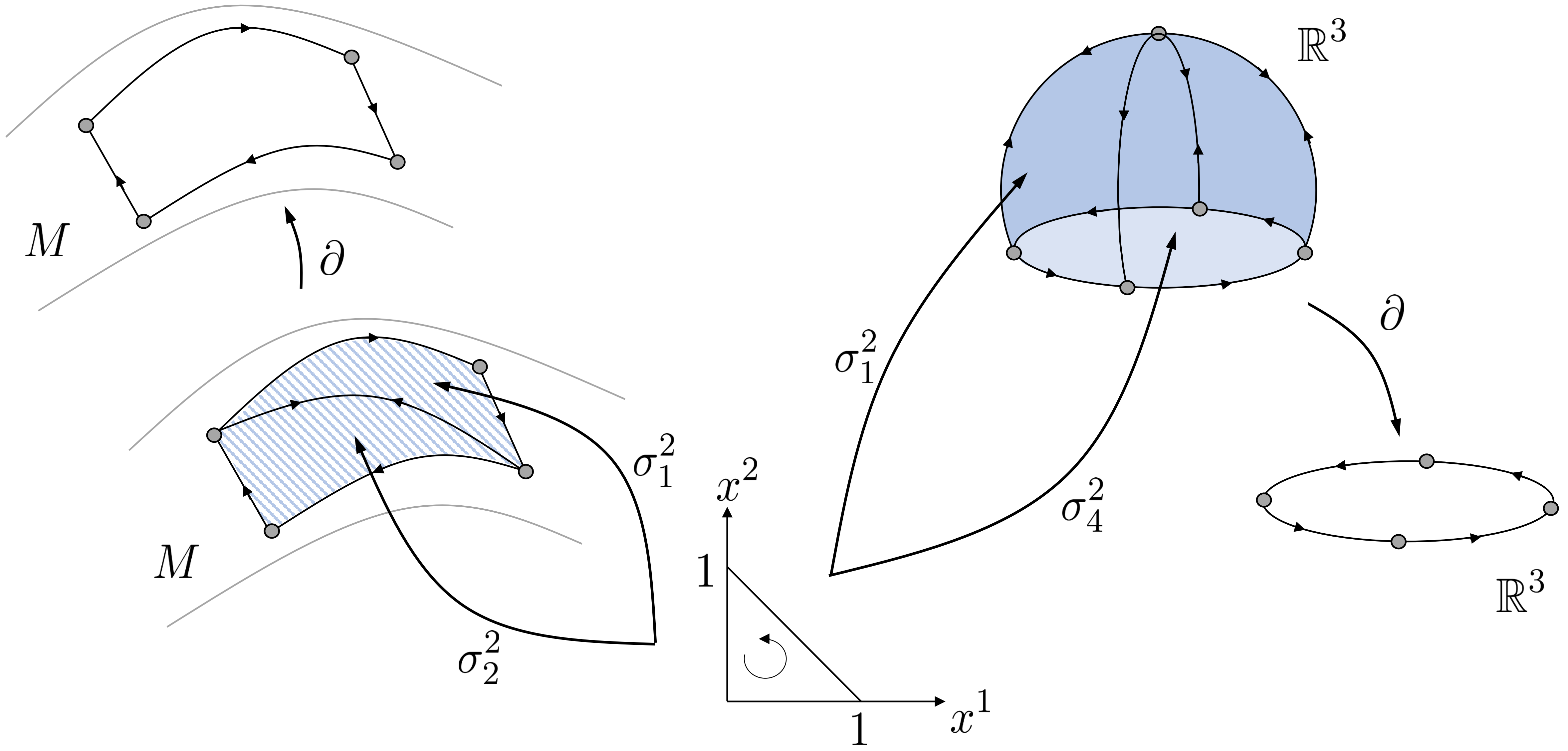
Remarks on Stokes' theorem

1. Problems can occur for infinite chains[1)].
   Here one should additionally assume that ω has a **compact support** and that the chain is **locally finite**[2)].
2. There is a more general version for integration over so-called regular domains on manifolds. See [4, p. 283].
3. Of course one does not need to calculate over all the simplices of the chain explicitly, if the domain and the boundary can be parametrized in a better fashion!
   → **Watch that orientations of K and ∂K are consistent!**

1) Finite chain: consists of finitely many simplices $\mathcal{S}_i$

2) Locally finite chain: every compact subset of M meets only finitely many simplices $\mathcal{S}_i$.
On a compact manifold every locally finite chain is finite.

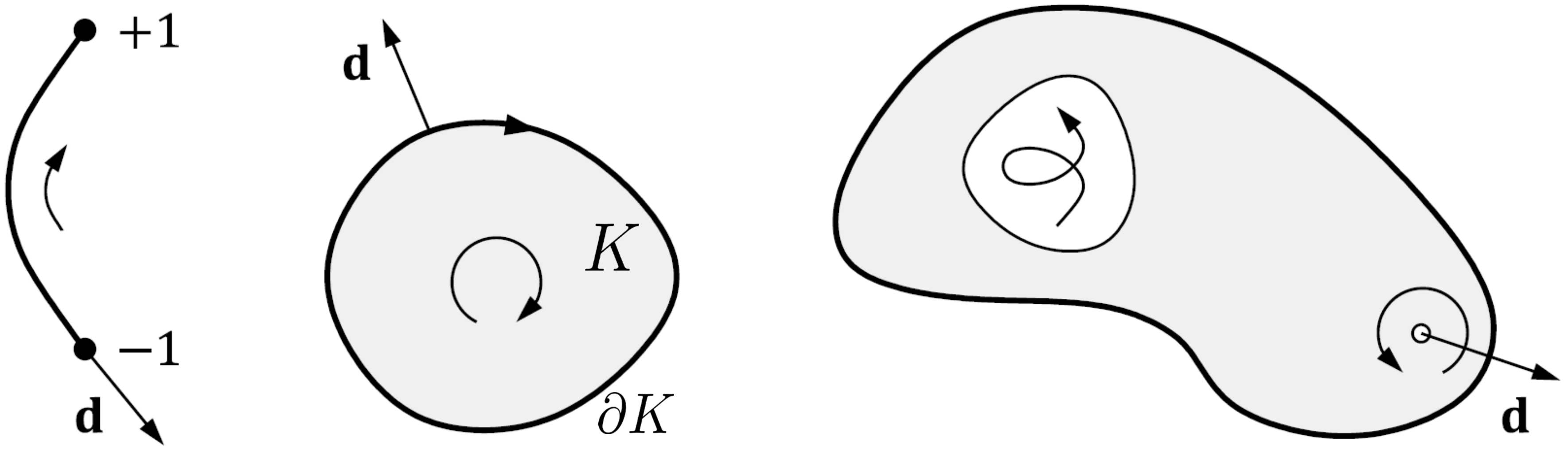
Intuitive representation of integration domains



Intuitive orientation convention

The orientations of K and ∂K are connected by

$$\begin{matrix}\text{Orientation} \\ \text{of } K\end{matrix} \sim \left(\begin{matrix}\text{Transverse direction } \boldsymbol{d} \\ \text{from in} \rightarrow \text{out}\end{matrix}, \begin{matrix}\text{Orientation} \\ \text{of } \partial K\end{matrix}\right)$$



(a) (b) (c)

Example: Fundamental theorem

Let M be a n- dimensional differentiable manifold.
Let $f \in \mathcal{C}^\infty(M)$ be a smooth function on M.
Let C be a path on the manifold starting at $P_1 \in M$ and ending at $P_2 \in M$.

w.l.o.g C can be represented as 1-simplex $\left(\langle 0, 1 \rangle, U, \sigma^1 \right)$.

$$\Rightarrow \quad \partial C = \sigma^1(\partial \langle 0, 1 \rangle) = \sigma^1(\langle 1 \rangle - \langle 0 \rangle) = \sigma^1(\langle 1 \rangle) - \sigma^1(\langle 0 \rangle) = P_2 - P_1$$

$$\Rightarrow \qquad \int_C \mathrm{d}f = \int_{\partial C} f = f(P_2) - f(P_1)$$

Interpret f as 0-form.
Integration of 0-form over 0-chain (point set) is a point evaluation.

Example: Fundamental theorem in $\mathbb{R}^n$

Let now $M = (\mathbb{R}^n, \cdot\,)$. We may then write

$$\int_C \operatorname{grad} f \cdot \mathrm{d}s = \int_C \mathrm{d}f = \int_{\partial C} f = f(P_2) - f(P_1)$$

The line integral of a gradient field depends only on the end points

Let $M = \mathbb{R}$ and f continuously differentiable such that $\mathrm{d}f = f'\,\mathrm{d}x$
If then $C = [a, b]$ we get the fundamental theorem of real calculus :

$$\int_a^b f'\,\mathrm{d}x = f(b) - f(a)$$

Example: Gauss' theorem in $\left(\mathbb{R}^3, \cdot, \times\right)$

Let $M = \mathbb{R}^3$ and b a smooth vector field on an open set, containing the closure of the volume $V \subset \mathbb{R}^3$.

With ${}^2b \in \mathcal{F}^2(M)$ and $\mathrm{d}^2b = {}^3\mathrm{div}\, b \in \mathcal{F}^3(M)$ we obtain **Gauss' theorem** from classical vector analysis :

$$\boxed{\int_V \mathrm{div}\, b\, \mathrm{d}\tau} = \int_V \mathrm{d}\,{}^2b = \int_{\partial V} {}^2b = \boxed{\int_{\partial V} b \cdot \mathrm{d}A}$$

It is actually sufficient to consider continuously differentiable vector fields.

Example: Stokes' theorem in $(\mathbb{R}^3, \cdot, \times)$

Let $M = \mathbb{R}^3$ and a a smooth vector field on an open set, containing the closure of the surface $A \subset \mathbb{R}^3$.

With ${}^1a \in \mathcal{F}^1(M)$ and $\mathrm{d}\,{}^1a = {}^2(\mathbf{curl}\,a) \in \mathcal{F}^2(M)$ we obtain **Stokes' theorem** from classical vector analysis :

$$\boxed{\int_A \mathbf{curl}\,a \cdot \mathrm{d}A} = \int_A \mathrm{d}\,{}^1a = \int_{\partial A} {}^1a = \boxed{\int_{\partial A} a \cdot \mathrm{d}s}$$

It is actually sufficient to consider continuously differentiable vector fields.

Maxwell's equations in terms of differential forms

In $\mathbb{R}^3$ we formulate Maxwell's equations with the electromagnetic forms.
Time t is treated as an additional parameter.

$$\mathcal{E} = {}^1E \qquad \mathcal{B} = {}^2B \qquad \mathcal{Q} = {}^3\rho$$

$$\mathcal{D} = {}^2D \qquad \mathcal{H} = {}^1H \qquad \mathcal{J} = {}^2J$$

Faraday's law of induction

$$\mathrm{d}\,\mathcal{E} = -\frac{\partial}{\partial t}\,\mathcal{B}$$

$$\mathrm{d}\,\mathcal{D} = \mathcal{Q}$$

Gauss's law

Ampère's circuital law

$$\mathrm{d}\,\mathcal{H} = \mathcal{J} + \frac{\partial}{\partial t}\,\mathcal{D}$$

$$\mathrm{d}\,\mathcal{B} = 0$$

Gauss's law for magnetism

Maxwell's equations in integral form

We can use Stokes' theorem to write Maxwell's equations in an integral form:

$$\mathcal{E} = {}^1E \qquad \mathcal{B} = {}^2B \qquad \mathcal{Q} = {}^3\rho$$

$$\mathcal{D} = {}^2D \qquad \mathcal{H} = {}^1H \qquad \mathcal{J} = {}^2J$$

Faraday's law of induction

$$\int_{\partial A} \mathcal{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathcal{B}$$

Gauss's law

$$\int_{\partial V} \mathcal{D} = \int_V \mathcal{Q}$$

Ampère's circuital law

$$\int_{\partial A} \mathcal{H} = \int_A \mathcal{J} + \frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathcal{D}$$

Gauss's law for magnetism

$$\int_{\partial V} \mathcal{B} = 0$$

Literature

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