Electromagnetics and Differential Forms

Multicovectors and Differential Forms

In today’s lecture we talk about:

- **Multicovectors**

- Exterior product

- Differential p-forms

- First connections to classical vector analysis

- First connections to electromagnetism

Multicovectors

A multicovector η in a point $P \in M$ on a differentiable manifold is defined as an alternating multilinear map over the tangent space $T_P M$.

$$\eta : T_P M \times \cdots \times T_P M \to \mathbb{R}$$

$\mathbb{R}$-linearity in every argument : $\lambda, \mu \in \mathbb{R} \qquad \xi, \xi' \in T_P M$

$$\eta(\ldots, \mu\xi + \lambda\xi', \ldots) = \mu\eta(\ldots, \xi, \ldots) + \lambda\eta(\ldots, \xi', \ldots)$$

Alternating property : $\xi, \xi' \in T_P M$

$$\eta(\ldots, \xi, \ldots, \xi', \ldots) = -\eta(\ldots, \xi', \ldots, \xi, \ldots)$$

Vanishes if $\xi = \xi'$

Space of multicovectors $\bigwedge^p T_P^* M$

The degree p of a multicovector η is the number of arguments it takes.

The space of all multicovectors of degree p at $P \in M$ is denoted as :

$$\bigwedge^p T_P^* M$$

It can be given the structure of a vector space over $\mathbb{R}$ by:

$$(\eta + \eta')(\xi_1, \ldots, \xi_p) := \eta(\xi_1, \ldots, \xi_p) + \eta'(\xi_1, \ldots, \xi_p) \qquad \forall \xi_i \in T_P M$$

$$(\lambda\eta)(\xi_1, \ldots, \xi_p) := \lambda\eta(\xi_1, \ldots, \xi_p) \qquad \forall \lambda \in \mathbb{R} \quad \forall \xi_i \in T_P M$$

Dimension of $\bigwedge^{p} T_P^* M$

Let $T_P M$ be of dimension n and $\eta \in \bigwedge^p T_P^* M$ any multicovector of degree p.

Linearity in each slot $\longrightarrow$ η is completely fixed by action on basis of $T_P M$.

Alternating property $\nearrow$ η is only nonzero if all basis vectors are different.

Alternating property $\searrow$ order of basis vectors only determines sign.

$$\Rightarrow \dim \bigwedge^p T_P^* M = \binom{n}{p}$$

The dimension is the number of options to draw p different basis vectors of $T_P M$, ignoring the order.

Some examples of $\bigwedge^p T_P^* M$

p	$\bigwedge^p T_P^* M$	$\dim \bigwedge^p T_P^* M$
0	$\mathbb{R}$	1
1	$T_P^* M$	n
$> n$	$\{0\}$	0

← By definition

← There is only a finite number of such spaces that are non trivial.

Can we combine these spaces to an overall structure?

$$\bigwedge T_P^* M := \bigwedge^0 T_P^* M \underset{\substack{\uparrow \\ \text{direct sum}}}{\oplus} \bigwedge^1 T_P^* M \oplus \cdots \oplus \bigwedge^n T_P^* M$$

What is the dimension of $\bigwedge T_P^* M$?

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Exterior product - definition

$$\wedge : \bigwedge^p T_P^* M \times \bigwedge^q T_P^* M \to \bigwedge^{(p+q)} T_P^* M$$
$$(\omega, \eta) \mapsto \omega \wedge \eta$$

$$\omega \wedge \eta(\xi_1, \ldots, \xi_{p+q}) = \frac{1}{p!q!} \sum_{\pi \in S^{p+q}} \operatorname{sign}(\pi) \omega(\xi_{\pi(1)}, \ldots, \xi_{\pi(p)}) \eta(\xi_{\pi(p+1)}, \ldots, \xi_{\pi(p+q)})$$

permutations ensure alternating property

$$\xi_i \in T_P M$$

$$S^{p+q} = \{\pi : \{1, \ldots, p+q\} \to \{1, \ldots, p+q\} \mid \pi \text{ is permutation}\}$$

$$\operatorname{sign}(\pi) = \begin{cases} 1 & \text{if } \pi \text{ is even} \\ -1 & \text{if } \pi \text{ is odd} \end{cases}$$

Exterior product - properties

Let $\omega \in \bigwedge^p T_P^* M$, $\eta, \eta' \in \bigwedge^q T_P^* M$, $\rho \in \bigwedge^r T_P^* M$ and $\lambda \in \mathbb{R}$ then

$\omega \wedge \eta$ is a multicovector of degree $p + q$ over $T_P M$.

$\omega \wedge \eta = (-1)^{pq} \, \eta \wedge \omega$ Graded commutative law

$(\omega \wedge \eta) \wedge \rho = \omega \wedge (\eta \wedge \rho)$ Associative law

$\omega \wedge (\eta + \lambda \eta') = \omega \wedge \eta + \lambda \, (\omega \wedge \eta')$ Distributive law

Proofs may be done as exercise (easier in coordinate representation)

Exterior algebra (Cartan algebra, Graßmann algebra)

The tuple $\left(\bigwedge T_P^* M, +, \cdot, \wedge\right)$ has the structure of an exterior algebra.

$$\begin{aligned}\bigwedge T_P^* M &:= \bigwedge^0 T_P^* M \oplus \bigwedge^1 T_P^* M \oplus \cdots \oplus \bigwedge^n T_P^* M \\ &= \left\{ (\omega_0, \dots, \omega_n) \,\middle|\, \omega_p \in \bigwedge^p T_P^* M \right\}\end{aligned}$$

$+$ component wise vector addition

$\cdot$ component wise scalar multiplication

$\wedge$ exterior product (generalize definition via distributive law)

This is purely formal and will come naturally when looking at the basis representations $\Rightarrow$ no influence on practical calculations

Exterior product of covectors

Let $\eta^i \in T_P^* M$ and $\xi_i \in T_P M$ with $1 \leq i \leq p \leq n$. Then it holds

(IH) $$\eta^1 \wedge \cdots \wedge \eta^p(\xi_1, \ldots, \xi_p) = \det \begin{pmatrix} \eta^1(\xi_1) & \cdots & \eta^1(\xi_p) \\ \vdots & \ddots & \vdots \\ \eta^p(\xi_1) & \cdots & \eta^p(\xi_p) \end{pmatrix}$$

Proof by induction:

IH … Induction Hypothesis
IB … Induction Base
IS … Induction Step

(IB) : $p = 2$

(IS) : $p \to p + 1$ $\longleftarrow$ Will be based on Laplace theorem for determinants

Exterior product of covariant vectors – IB

Induction base (IB) for $p = 2$:

$$\eta^1 \wedge \eta^2(\xi_1, \xi_2) = \frac{1}{1!1!} \sum_{\pi \in S^2} \mathrm{sign}(\pi) \eta^1(\xi_{\pi(1)}) \eta^2(\xi_{\pi(2)})$$

$$= \eta^1(\xi_1)\eta^2(\xi_2) - \eta^1(\xi_2)\eta^2(\xi_1)$$

$$= \det \begin{pmatrix} \eta^1(\xi_1) & \eta^1(\xi_2) \\ \eta^2(\xi_1) & \eta^2(\xi_2) \end{pmatrix}$$

Exterior product of covariant vectors – IS

Induction step (IS) for $p \to p+1$:

sign change of p-form due to swap of arguments is canceled out

$$(\eta^1 \wedge \cdots \wedge \eta^p) \wedge \eta^{p+1}(\xi_1, \ldots, \xi_p, \xi_{p+1}) =$$

$$= \frac{1}{p!1!} \sum_{\pi \in S^{p+1}} \operatorname{sign}(\pi)\, \eta^1 \wedge \cdots \wedge \eta^p(\xi_{\pi(1)}, \ldots, \xi_{\pi(p)})\eta^{p+1}(\xi_{\pi(p+1)})$$

$$= \sum_{\tilde{\pi} \in \tilde{S}^{p+1}} \operatorname{sign}(\tilde{\pi})\, \eta^1 \wedge \cdots \wedge \eta^p(\xi_{\tilde{\pi}(1)}, \ldots, \xi_{\tilde{\pi}(p)})\eta^{p+1}(\xi_{\tilde{\pi}(p+1)})$$

where

$$\tilde{S}^{p+1} := \{\tilde{\pi} : (1, \ldots, p, p+1) \mapsto (i_1, \ldots, i_p, i_{p+1}) | \, i_1 < \cdots < i_p\}$$

Exterior product of covariant vectors – IS cont.

$$\sum_{\tilde\pi\in\tilde S^{p+1}} \mathrm{sign}(\tilde\pi)\ \eta^1\wedge\cdots\wedge\eta^p(\xi_{\tilde\pi(1)},\ldots,\xi_{\tilde\pi(p)})\eta^{p+1}(\xi_{\tilde\pi(p+1)})$$

$$\begin{array}{cccc}1&2&3&4\\ &j&&\end{array} \xrightarrow[\text{swaps}]{j-1} \begin{array}{cccc}2&1&3&4\end{array} \xrightarrow[\text{swaps}]{(p+1)-1} \begin{array}{cccc}1&3&4&2\end{array}$$

$$= \sum_{j=1}^{p+1}(-1)^{p+1+j}\ \eta^1\wedge\cdots\wedge\eta^p(\xi_1,\ldots,\widehat{\xi_j},\ldots,\xi_{p+1})\,\eta^{p+1}(\xi_j)$$

The hat indicates to skip this argument

IH

$$= \sum_{j=1}^{p+1}(-1)^{p+1+j}\ \det\begin{pmatrix} \eta^1(\xi_1) & \cdots & \widehat{\eta^1(\xi_j)} & \cdots & \eta^1(\xi_{p+1})\\ \vdots & & & & \vdots\\ \eta^p(\xi_1) & \cdots & \widehat{\eta^p(\xi_j)} & \cdots & \eta^p(\xi_{p+1})\end{pmatrix}\eta^{p+1}(\xi_j) \qquad \square$$

Notation - strictly ordered multi-indices

The tuple $J := (j_1 j_2 \dots j_p)$ is called a multi-index.

$$M_{j_1 j_2 \dots j_p} = M_J$$

A multi-index is called strictly ordered if $j_1 < j_2 < \cdots < j_p$ holds.

For the set of all strictly ordered multi-indices, denoted by

$$\mathcal{J}_p^n := \{(j_1 \dots j_p) |\, 1 \leq j_1 < j_2 < \cdots < j_p \leq n\}$$

it holds

$$\#\mathcal{J}_p^n = \left(\begin{array}{c} n \\ p \end{array} \right)$$

Basis of $\bigwedge^p T_P^* M$

Let $(e_i)_{i=1}^n \subset T_P M$ be a basis of the tangent space and $\left(\varepsilon^i\right)_{i=1}^n \subset T_P^* M$ the dual basis.

We define for $J \in \mathcal{J}_p^n$ the multicovectors of degree p :

$$\varepsilon^J = \varepsilon^{j_1 \cdots j_p} := \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_p}$$

The family $\left(\varepsilon^J\right)_{J \in \mathcal{J}_p^n}$ forms a basis of $\bigwedge^p T_P^* M$:

$$\forall \eta \in \bigwedge^p T_P^* M : \qquad \eta = \sum_{J \in \mathcal{J}_p^n} \eta_J \, \varepsilon^J \quad \text{where} \quad \eta_J = \eta(e_{j_1}, \ldots, e_{j_p})$$

Linear independence of the ε^J

Let $(i_1, \ldots, i_p) = I \in \mathcal{J}_p^n$ and $(j_1, \ldots, j_p) = J \in \mathcal{J}_p^n$ be strictly ordered multi-indices.

$$\varepsilon^{j_1 \cdots j_p}(e_{i_1}, \ldots, e_{i_p}) \boxed{=} \sum_{\pi \in S^p} \operatorname{sign}(\pi)\, \varepsilon^{j_1}(e_{\pi(i_1)}) \cdots \varepsilon^{j_p}(e_{\pi(i_p)})$$

Leibniz formula for determinants

$$= \sum_{\pi \in S^p} \operatorname{sign}(\pi)\, \delta^{j_1}_{\pi(i_1)} \cdots \delta^{j_p}_{\pi(i_p)}$$

$$= \delta^{j_1}_{i_1} \cdots \delta^{j_p}_{i_p} \quad =: \; \delta^J_I$$

$\Rightarrow \quad \varepsilon^J$ are linear independent.

Spanning property of the $\boldsymbol{\varepsilon^J}$

Let $(e_{j_1}, \ldots, e_{j_p})$ with $(j_1 \ldots j_p) = J \in \mathcal{J}_p^n$ be a test tuple, where the $(e_i)_{i=1}^n \subset T_P M$ are a basis of tangent space.

Let $\eta \in \bigwedge^p T_P^* M$ be any multicovector of degree p.

Due to linearity and the alternating property, η is completely defined by its action on all test tuples.

$$\eta_J = \eta_{j_1 \ldots j_p} := \eta(e_{j_1}, \ldots, e_{j_p})$$

Spanning property of the $\boldsymbol{\varepsilon}^{\boldsymbol{J}}$ - continued

Lets define a multicovector

$$\tilde{\eta} := \sum_{I \in \mathcal{J}_p^n} \eta_I \varepsilon^I$$

but then

$$\begin{aligned} \tilde{\eta}(e_{j_1}, \ldots, e_{j_p}) &:= \sum_{I \in \mathcal{J}_p^n} \eta_I \varepsilon^I (e_{j_1}, \ldots, e_{j_p}) \\ &= \sum_{I \in \mathcal{J}_p^n} \eta_I \delta_J^I \qquad = \qquad \eta_J \qquad = \qquad \eta(e_{j_1}, \ldots, e_{j_p}) \end{aligned}$$

and therefore $\tilde{\eta} = \eta$: hence any η can represented that way.

$\square$

Wrap up: Multicovectors

An alternating multilinear map $\eta : T_P M \times \cdots \times T_P M \to \mathbb{R}$ is called a **multicovector** at $P \in M$ on the manifold M.

Let $\phi(P) = (x^1, \ldots, x^n)$ be local coordinates for a neighborhood $U \ni P$.

$\Rightarrow \quad \varepsilon^J = \mathrm{d}x^{j_1}|_P \wedge \cdots \wedge \mathrm{d}x^{j_p}|_P$ is a basis of $\bigwedge^p T_P^* M$

$\Rightarrow \quad \eta = \sum_{J \in \mathcal{J}_p^n} \eta_{j_1 \ldots j_p} \, \mathrm{d}x^{j_1}|_P \wedge \cdots \wedge \mathrm{d}x^{j_p}|_P$ is a unique representation.
(with respect to a chart)

$\bigwedge^p T_P^* M$ is the space of multicovectors of degree p at $P \in M$.

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(Smooth) Differential p-forms

Smoothness is buried in the target space

A smooth differential p-form ω on M is a local operator

$$\omega : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{p \text{ times}} \to \mathcal{C}^\infty(M)$$

Space of smooth Vector fields

$\mathcal{C}^\infty(M)$-linearity in every argument : $f, g \in \mathcal{C}^\infty(M) \quad v, v' \in \mathcal{X}(M)$

$$\omega(\ldots, fv + gv', \ldots) = f\omega(\ldots, v, \ldots) + g\omega(\ldots, v', \ldots)$$

Alternating property : $v, v' \in \mathcal{X}(M)$

$$\eta(\ldots, v, \ldots, v', \ldots) = -\eta(\ldots, v', \ldots, v, \ldots)$$

Vanishes if $v = v'$

Space of all p-forms

$\mathcal{F}^p(M)$ denotes the space of all smooth differential p-forms on M .

By defining binary operations $\forall\, \omega, \omega' \in \mathcal{F}^p(M)$ $\forall f \in C^\infty(M)$

$$(\omega + \omega')(v_1, \ldots, v_p) := \omega(v_1, \ldots, v_p) + \omega'(v_1, \ldots, v_p) \quad \forall v_i \in \mathcal{X}(M)$$

$$(f\omega)(\xi_1, \ldots, \xi_p) := f\,\omega(\xi_1, \ldots, \xi_p) \quad \forall v_i \in \mathcal{X}(M)$$

$\mathcal{F}^p(M)$ can be given the structure of a module over $\mathcal{C}^\infty(M)$.

Exterior algebra for p-forms

The exterior product is defined similarly as for multicovectors, but with vector fields as arguments.

$$\wedge : \mathcal{F}^p(M) \times \mathcal{F}^q(M) \to \mathcal{F}^{p+q}(M)$$

It shares the same properties as for multicovectors.

$$\begin{aligned}\mathcal{F}(M) &:= \mathcal{F}^0(M) \oplus \mathcal{F}^1(M) \oplus \cdots \oplus \mathcal{F}^n(M) \\ &= \left\{ (\omega_0, \ldots, \omega_n) \,\middle|\, \omega_p \in \mathcal{F}^p(M) \right\}\end{aligned}$$

is the space of all differential forms on M.

$(\mathcal{F}(M), +, \cdot, \wedge)$ has the structure of an exterior algebra.

p-forms as family of multicovectors

Differential p-form $\rightarrow$ family of multicovectors $(\omega_P)_{P\in M}$.

$$\omega_P \in \bigwedge^p T_P^* M : \omega_P(\xi_1, \ldots, \xi_p) = \omega(v_{\xi_1}, \ldots, v_{\xi_p})\big|_P \quad \forall \xi_i \in T_P M$$

where $\quad v_{\xi_i} \in \mathcal{X}(M) : \; v_{\xi_i}\big|_P = \xi_i$

v_{ξ_i} is a smooth continuation of ξ_i

$\Rightarrow$ When seen from this point of view, p-forms behave like multicovectors when restricting to a point on the manifold.

$\Rightarrow$ Properties of exterior product or basis representation follow directly.

p-forms – Basis representation

Let $\phi(P) = (x^1, \dots, x^n)$ be local coordinates for a open set $U \subset M$.

Any p-form $\omega \in \mathcal{F}^p(M)$ on M can locally be represented by

$$\omega = \sum_{J \in \mathcal{J}_p^n} \omega_J \, \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_p}$$

where $\omega_J = \omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_p}}\right) \in \mathcal{C}^\infty(M)$

This representation is unique (except for the choice of a chart).

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Classical vector analysis

We look at the space $(\mathbb{R}^3\ ,\ \boxed{\cdot}\ ,\ \boxed{\times}\ )$

Euclidean scalar product

cross product

That's a lot of additional structure which goes beyond a pure manifold !

$\Rightarrow$ Is a metric and therefore topological space (induced metric)

$\Rightarrow \mathrm{id} : \mathbb{R}^3 \to \mathbb{R}^3$ is a global chart (flat manifold)

$\Rightarrow \psi : (0,\infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3 \setminus \{0\}$ is another parametrization

$$(r, \varphi, z) \mapsto (r\cos\varphi, r\sin\varphi, z)$$

(cylindrical coordinates)

Translation isomorphisms – coordinate-free [1]

In $\left(\mathbb{R}^3\,,\;\cdot\;,\;\times\;\right)$ we may identify scalar/vector fields with differential forms.
Let $M \subset \mathbb{R}^3$ be an open set in $\mathbb{R}^3$ and $u, v, w \in \mathcal{X}(M)$

$\begin{aligned} {}^{1}\cdot\;:\;\mathcal{X}(M) &\to \mathcal{F}^1(M) \\ a &\mapsto {}^{1}a \end{aligned}$	${}^{1}a(v) := a \cdot v$
$\begin{aligned} {}^{2}\cdot\;:\;\mathcal{X}(M) &\to \mathcal{F}^2(M) \\ b &\mapsto {}^{2}b \end{aligned}$	${}^{2}b(u, v) := b \cdot (u \times v)$
$\begin{aligned} {}^{3}\cdot\;:\;\mathcal{C}^{\infty}(M) &\to \mathcal{F}^3(M) \\ c &\mapsto {}^{3}c \end{aligned}$	${}^{3}c(u, v, w) := c\, u \cdot (v \times w)$

Cross product as exterior product

Let $a, b \in \mathcal{X}(M)$ be any vector fields.

$$\boxed{{}^2(a \times b) = {}^1a \wedge {}^1b}$$

Proof :

$$\begin{aligned}
{}^2(a \times b)(v_1, v_2) &= (a \times b) \cdot (v_1 \times v_2) \\
&= (a \cdot v_1)(b \cdot v_2) - (a \cdot v_2)(b \cdot v_1) \\
&= \det \begin{pmatrix} a \cdot v_1 & a \cdot v_2 \\ b \cdot v_1 & b \cdot v_2 \end{pmatrix} \\
&= \det \begin{pmatrix} {}^1a(v_1) & {}^1a(v_2) \\ {}^1b(v_1) & {}^1b(v_2) \end{pmatrix} = {}^1a \wedge {}^1b\,(v_1, v_2)
\end{aligned}$$

Example: Vector fields as 1-forms

Since $\left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$ is a Hilbert space the isomorphic property of the translation follows from the Riesz representation theorem.

Let $\left(e_i\right)_{i=1}^3 \subset \mathbb{R}^3$ be a basis and $\left(\varepsilon^i\right)_{i=1}^3$ the dual basis. Then the translation isomorphism is written as

$$^1a = (a \cdot e_1)\varepsilon^1 + (a \cdot e_2)\varepsilon^2 + (a \cdot e_3)\varepsilon^3$$

Proof :

$$^1a(v) = a \cdot v = (a \cdot e_i)v^i = (a \cdot e_i)\varepsilon^i(v)$$

Vector fields as 1-forms – different coordinates

Cartesian coordinates

$$\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}$$

$$a = a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}$$

$$^1 a = a_x \mathrm{d}x + a_y \mathrm{d}y + a_z \mathrm{d}z$$

cylindrical coordinates

use orthonormal dual bases

$$\frac{\partial}{\partial r} \quad \frac{1}{r}\frac{\partial}{\partial \varphi} \quad \frac{\partial}{\partial z}$$

$$a = a_r \frac{\partial}{\partial r} + a_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi} + a_z \frac{\partial}{\partial z}$$

$$^1 a = a_r \mathrm{d}r + a_\varphi r \mathrm{d}\varphi + a_z \mathrm{d}z$$

Example: Vector fields as 2-forms

Let $(e_i)_{i=1}^{3} \subset \mathbb{R}^3$ be a direct orthonormal basis and $\left(\varepsilon^i\right)_{i=1}^{3}$ the dual basis. Then the translation isomorphism is written as

Note it is (3, 1) not (1, 3)

$$^2b = (b \cdot e_1)\varepsilon^2 \wedge \varepsilon^3 + (b \cdot e_2)\varepsilon^3 \wedge \varepsilon^1 + (b \cdot e_3)\varepsilon^1 \wedge \varepsilon^2$$

Proof:

^2b is an alternating multilinear 2-form over a 3D space.

$\Rightarrow$ Calculate the 3 independent components in basis representation

$$^2b(e_2, e_3) = b \cdot (e_2 \times e_3) = b \cdot e_1$$
$$^2b(e_3, e_1) = b \cdot (e_3 \times e_1) = b \cdot e_2$$
$$^2b(e_1, e_2) = b \cdot (e_1 \times e_2) = b \cdot e_3$$

Example: scalar fields as 3-forms

Let $(e_i)_{i=1}^{3} \subset \mathbb{R}^3$ be a direct orthonormal basis and $\left(\varepsilon^i\right)_{i=1}^{3}$ the dual basis. Then the translation isomorphism is written as

$$\boxed{{}^3c = c \;\; \varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3}$$

Proof:

${}^3c$ is an alternating multilinear 3-form over a 3D space.

$\Rightarrow$ Only one independent component in basis representation.

$${}^3c(e_1, e_2, e_3) = c \;\; e_1 \cdot (e_2 \times e_3) = c$$

Translation isomorphisms – In coordinates

In $\left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$ we may identify scalar/vector fields with differential forms.
Let $M \subset \mathbb{R}^3$ be an open set and $(e_i)_{i=1}^3$ a direct orthonormal basis of $\mathbb{R}^3$.

$${}^1. \ :\ \mathcal{X}(M) \to \mathcal{F}^1(M)$$

$$a \mapsto {}^1a = (a \cdot e_1)\varepsilon^1 + (a \cdot e_2)\varepsilon^2 + (a \cdot e_3)\varepsilon^3$$

dual basis

$${}^2. \ :\ \mathcal{X}(M) \to \mathcal{F}^2(M)$$

$$b \mapsto {}^2b = (b \cdot e_1)\varepsilon^2 \wedge \varepsilon^3 + (b \cdot e_2)\varepsilon^3 \wedge \varepsilon^1 + (b \cdot e_3)\varepsilon^1 \wedge \varepsilon^2$$

$${}^3. \ :\ \mathcal{C}^\infty(M) \to \mathcal{F}^3(M)$$

$$c \mapsto {}^3c = c\ \ \varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3$$

Dot product as exterior product

Let $a, b \in \mathcal{X}(M)$ be any vector fields.

$$^3(a \cdot b) = {}^1a \wedge {}^2b$$

Proof :

use Cartesian coordinate representation for a and b.

$$^3(a \cdot b) = (a_x b_x + a_y b_y + a_z b_z)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

$$= (a_x \mathrm{d}x + a_y \mathrm{d}y + a_z \mathrm{d}z) \wedge (b_x \mathrm{d}y \wedge \mathrm{d}z + b_y \mathrm{d}z \wedge \mathrm{d}x + b_z \mathrm{d}x \wedge \mathrm{d}y)$$

$$= {}^1a \wedge {}^2b$$

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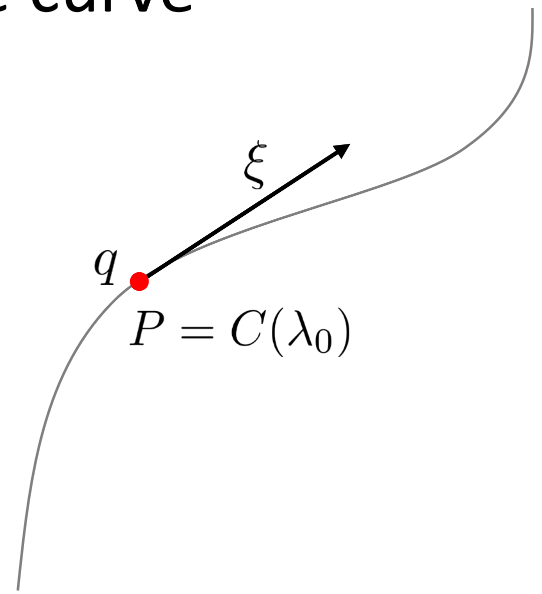
Electric field as 1-form

Consider a charge q moving along a trajectory $C(\lambda)$ parametrized over λ in electric field E.

At point $P = C(\lambda_0)$ we have the vector tangent to the curve

$$\xi = \frac{\partial}{\partial \lambda} \in T_P\mathbb{R}^3$$

The notion of a tangent vector and of differentiation in a given direction coincide



We express the work per charge

$$\frac{W(\lambda)}{q} = \frac{W(\lambda_0)}{q} + (\lambda - \lambda_0)\underbrace{E(P)\cdot\xi}_{{}^1E(\xi)} + \mathcal{O}(\lambda^2)$$

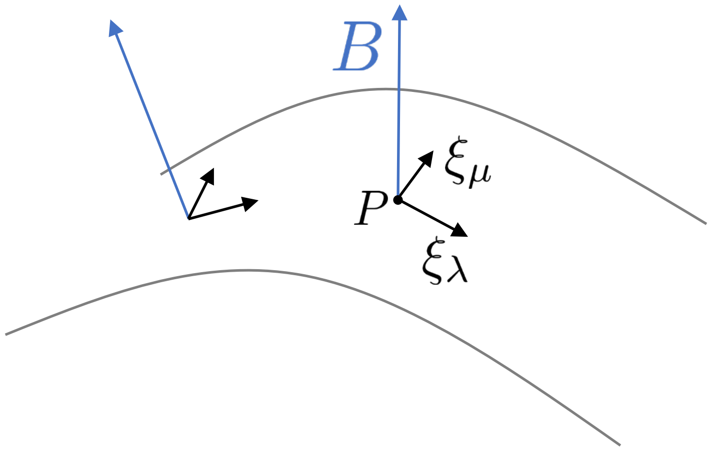
$$\Rightarrow \quad \boxed{\mathcal{E} = {}^1E}$$

model electric field as 1-form in the first place

The equation for the work is invariant under reparametrization

Magnetic flux density as 2-form

Consider a curved surface $S(\lambda, \mu)$, parametrized by λ and μ.
Consider a magnetic flux density B .



At a point $P = S(\lambda_0, \mu_0)$ there are two tangent vectors,

$$\xi_\lambda = \frac{\partial}{\partial\lambda} \in T_P\mathbb{R}^3, \ \xi_\mu = \frac{\partial}{\partial\mu} \in T_P\mathbb{R}^3$$

Magnetic flux through a parallelogram spanned by ξ_λ ,ξ_μ

$$\Phi(\lambda, \mu) = (\lambda - \lambda_0)(\mu - \mu_0)\underbrace{B(P) \cdot (\xi_\lambda \times \xi_\mu)}_{{}^2B(\xi_\lambda, \xi_\mu)} + \mathcal{O}(\lambda^2, \mu^2)$$

model magnetic flux density
as 2-form in the first place

$$\Rightarrow \quad \boxed{\mathcal{B} = {}^2B}$$

Scalar densities as 3-forms

Consider a parallelepiped spanned by 3 vectors $u\,\xi_1, v\,\xi_2, w\,\xi_3 \in \mathbb{R}^3$ at the point P, where $u, v, w \in \mathbb{R}^+$.

One can identify $\xi_1, \xi_2, \xi_3 \in T_P\mathbb{R}^3 \cong \mathbb{R}^3$

The charge within this parallelepiped is given by

Charge density as scalar field

$$Q(u,v,w) = u\,v\,w\,\underbrace{\rho(P)\,(\xi_1 \cdot (\xi_2 \times \xi_3))}_{\text{model charge density as 3-form in the first place}} + \mathcal{O}(u^2, v^2, w^2)$$

model charge density
as 3-form in the first place

$$^3\rho(\xi_1, \xi_2, \xi_3) \;\Rightarrow\; \boxed{\mathcal{Q} = {}^3\rho}$$

Wrap up – electromagnetic quantities as forms

			physical dim.
$\mathcal{E} = {}^1E$	electric field	1 – form	U
$\mathcal{H} = {}^1H$	magnetic field		I
$\mathcal{B} = {}^2B$	magnetic flux density	2 – form	UT
$\mathcal{D} = {}^2D$	electric displacement field		IT
$\mathcal{J} = {}^2J$	electric current density		I
$\mathcal{Q} = {}^3\rho$	electric charge density	3 – form	IT

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