Electromagnetics and Differential Forms

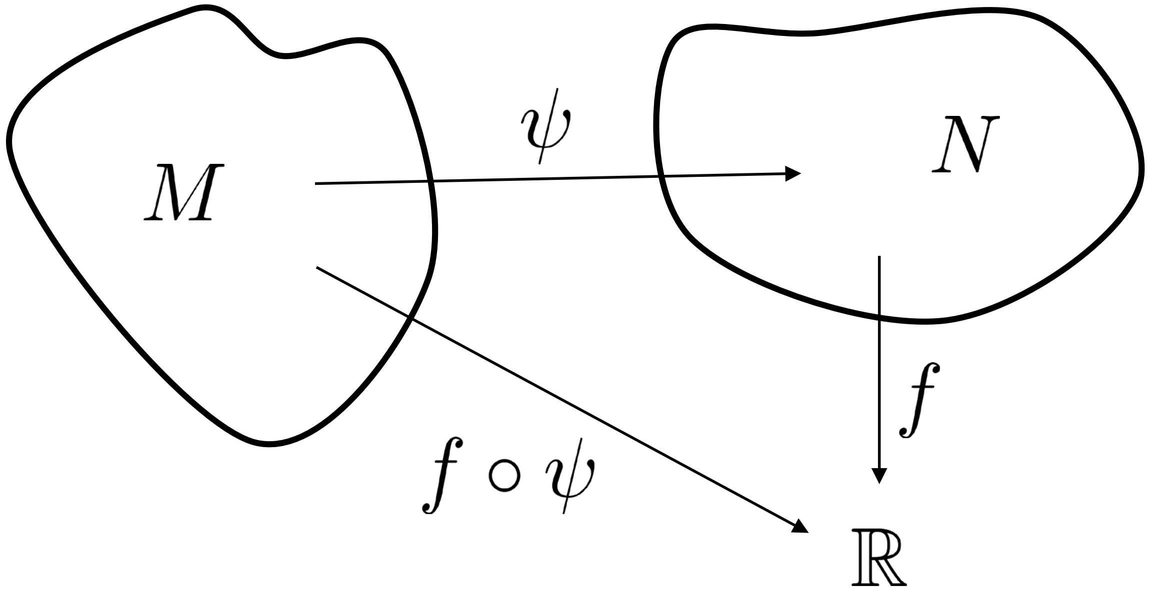
Operations on Differential Forms

In today’s lecture we talk about:

- **Mappings between manifolds**
- Pullback operation
- Trace operator
- Exterior derivative
- Maxwell's equations

Differentiable maps between manifolds

Let M and N be differentiable manifolds.



Consider the maps

$$\psi : M \to N$$

$$f : N \to \mathbb{R} \in \mathcal{C}^{\infty}(N)$$

$$f \circ \psi : M \to \mathbb{R}$$

The map $\psi : M \to N$ is called differentiable if $f \circ \psi : M \to \mathbb{R}$ is differentiable as function over M for any $f \in \mathcal{C}^{\infty}(N)$.

Differentiable maps in coordinates

Let M and N be differentiable manifolds.
Let $\phi_{U,M} : U \subset M \to \mathbb{R}^m$ and $\phi_{V,N} : V \subset N \to \mathbb{R}^n$ be charts.

Differentiability of $\psi : M \to N$ can be formulated in coordinates :

$$\psi : M \to N$$

is a differentiable map between the manifolds M and N.

$$\Leftrightarrow$$

$$\hat{\psi} := \phi_{V,N} \circ \psi \circ \phi_{U,M}^{-1}$$

is a differentiable map for all charts of M and N .

$\hat{\psi} : \mathbb{R}^m \to \mathbb{R}^n$ is the coordinate representation of ψ.

Differentials of maps between manifolds

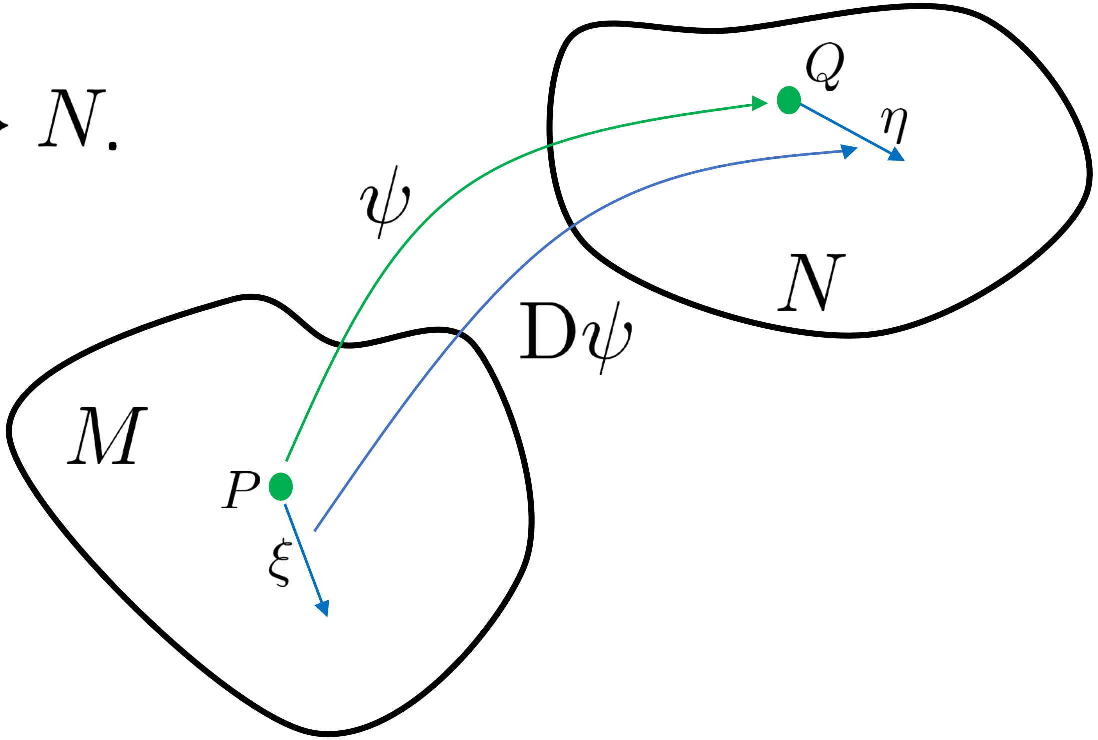
Consider a differentiable map $\psi : M \to N$.
Let $P \in M$ and $Q = \psi(P) \in N$.

The differential of ψ is defined as

$$\begin{aligned} \mathrm{D}\psi|_P : T_P M &\to T_Q N \\ \xi &\mapsto \eta \end{aligned}$$

such that

$$\eta(f) = \xi(f \circ \psi) \qquad \forall f \in \mathcal{C}^\infty(N)$$



In the literature one often finds ψ' instead of $\mathrm{D}\psi$

Coordinate free chain rules

Consider a differentiable map $\psi : M \to N$ and $f : N \to \mathbb{R} \in \mathcal{C}^\infty(N)$
Let $\xi \in T_P M$ be any tangent vector at $P \in M$ and $Q = \psi(P)$.

With the definition of the differential and $\eta = \mathrm{D}\psi|_P\, \xi$ it follows

$$\mathrm{d}(f \circ \psi)|_P(\xi) = \xi(f \circ \psi) = \eta(f) = \mathrm{d}f(\eta)|_Q = \mathrm{d}f|_Q(\mathrm{D}\psi|_P\, \xi)$$

$$\Rightarrow \quad \boxed{\mathrm{d}(f \circ \psi)|_P = \mathrm{d}f|_Q \circ \mathrm{D}\psi|_P}$$

Similar rules hold when it comes to the concatenation of mappings between different manifolds – proof as exercise

Mapping between tangent spaces – in coordinates

$$M \xrightarrow[\psi(P)=Q]{\psi : M \to N} N$$

M		N
$\phi_{U,M} : U \subset M \to \mathbb{R}^m$	charts	$\phi_{V,N} : V \subset N \to \mathbb{R}^n$
$\left(x^1, \ldots, x^m\right)$	local coordinates	$\left(y^1, \ldots, y^n\right)$
$(e_i)_{i=1}^m \subset T_P M$	bases	$(\tilde{e}_i)_{i=1}^n \subset T_Q N$

$$\xi \in T_P M \qquad \xrightarrow[\eta^k = \{\mathrm{D}\psi\}_i^k \, \xi^i]{\mathrm{D}\psi|_P : T_P M \to T_Q N} \qquad \eta = \mathrm{D}\psi|_P \, \xi \in T_Q N$$

$$\xi = \xi^i \, e_i \qquad \qquad \eta = \eta^k \, \tilde{e}_k$$

Differentials of manifold maps – in coordinates

How do we calculate the matrix $\{\mathrm{D}\psi\}_i^k$?

Consider any $f \in \mathcal{C}^\infty(\mathcal{N})$, define $\hat{f} := f \circ \phi_{V,N}^{-1}$ and $\hat{\psi} = \phi_{V,N} \circ \psi \circ \phi_{U,M}^{-1}$

Then

use definitions

$$\boxed{(\mathrm{D}\psi|_P \, e_i)\,(f)} = e_i(f \circ \psi) \quad = \frac{\partial}{\partial x^i}(\hat{f} \circ \hat{\psi})\Big|_{\phi_{U,M}(P)}$$

chain rule

$$= \frac{\partial \hat{f}}{\partial y^k}\Big|_{\phi_{V,N}(Q)} \frac{\partial \hat{\psi}^k}{\partial x^i}\Big|_{\phi_{U,M}(P)} \quad \boxed{= \frac{\partial \hat{\psi}^k}{\partial x^i}\Big|_{\phi_{U,M}(P)} \tilde{e}_k(f)}$$

where $\hat{\psi}^k$ is the k-th component of ψ .

Differentials of manifold maps – in coordinates

We introduce the dual basis $\left(\tilde{\varepsilon}^k\right)_{k=1}^{n} \subset T_Q^* N$ to the $\left(\tilde{e}_i\right)_{i=1}^{n} \subset T_Q N$.

Then

$$\eta^k = \tilde{\varepsilon}^k(\eta) \quad = \tilde{\varepsilon}^k(\mathrm{D}\psi|_P\, \xi) \quad = \tilde{\varepsilon}^k(\mathrm{D}\psi|_P\, e_i)\, \xi^i$$

use result from previous slide.

$$= \left.\frac{\partial \hat{\psi}^j}{\partial x^i}\right|_{\phi_{U,M}(P)} \tilde{\varepsilon}^k(\, \tilde{e}_j)\, \xi^i \qquad = \left.\frac{\partial \hat{\psi}^k}{\partial x^i}\right|_{\phi_{U,M}(P)} \xi^i$$

$$\Rightarrow \quad \{\mathrm{D}\psi\}^k_i \quad = \left.\frac{\partial \hat{\psi}^k}{\partial x^i}\right|_{\phi_{U,M}(P)} = \mathrm{J}\hat{\psi}(P)$$

It is the **Jacobi matrix** of the coordinate representation of ψ ☺

Example – Embedded cylinder in $\mathbb{R}^3$

Local coordinates on cylinder Z :

$$(\varphi, z) \quad e_1 = \frac{\partial}{\partial \varphi} \quad e_2 = \frac{\partial}{\partial z}$$

Cartesian coordinates in $\mathbb{R}^3$:

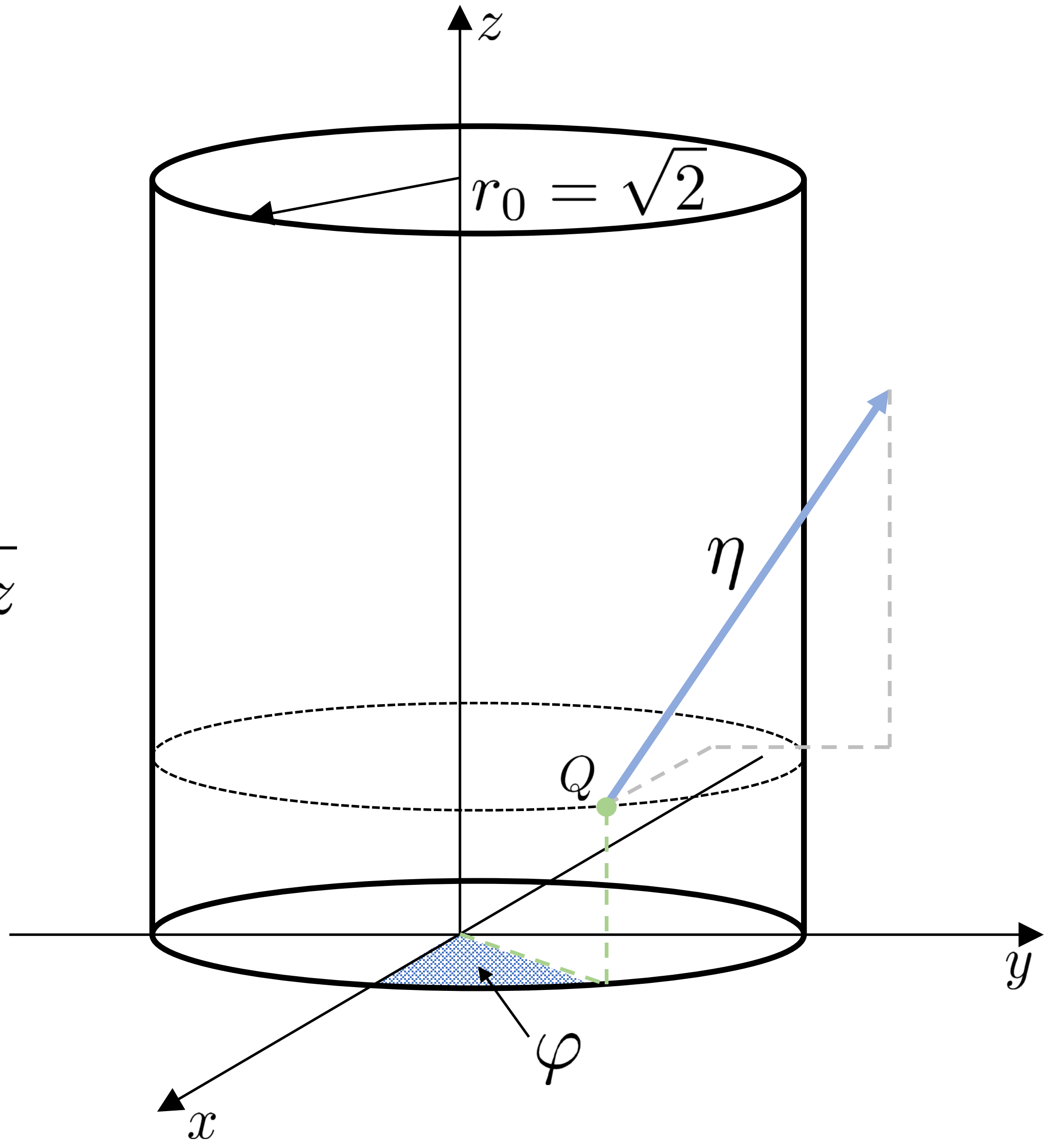
$$(x, y, z) \quad \tilde{e}_1 = \frac{\partial}{\partial x} \quad \tilde{e}_2 = \frac{\partial}{\partial y} \quad \tilde{e}_3 = \frac{\partial}{\partial z}$$

Embedding of cylinder :

$$\psi : Z \hookrightarrow \mathbb{R}^3$$

$$\hat{\psi} = \phi_{V,N} \circ \psi \circ \phi_{U,M}^{-1}$$

$$\hat{\psi}(\varphi, z) = (r_0 \cos\varphi, r_0 \sin\varphi, z)^\top$$



Example – Embedded cylinder in $\mathbb{R}^3$

Let the point $P \in Z$ be represented by the coordinates $\left(\frac{\pi}{4}, 1\right)$ and $Q = \psi(P)$.

$$\boxed{\xi = 1\, e_1 + 2\, e_2} \quad \Rightarrow \quad \xi^1 = 1\,,\ \xi^2 = 2$$

We want to calculate $\eta = \mathrm{D}\psi\, \xi$ with $\eta^k = \{\mathrm{D}\psi\}^k_i\, \xi^i$:

row index

column index

$$\{\mathrm{D}\psi\}^k_i = \left.\frac{\partial \hat{\psi}^k}{\partial x^i}\right|_{\left(\frac{\pi}{4},1\right)} = \left.\begin{pmatrix} -r_0 \sin\varphi & 0 \\ r_0 \cos\varphi & 0 \\ 0 & 1 \end{pmatrix}\right|_{\substack{\left(\frac{\pi}{4},1\right) \\ r_0 = \sqrt{2}}} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \boxed{\eta = -1\, \tilde{e}_1 + 1\, \tilde{e}_2 + 2 \tilde{e}_3}$$

Differential as map between vector fields

Let $\psi : M \to N$ be a differentiable map.
Then

$$\Rightarrow \quad \begin{aligned} \mathrm{D}\psi|_P : T_P M &\to T_Q N \\ \xi &\mapsto \eta \end{aligned}$$

gives a map between the tangent spaces at $P \in M$ and $Q = \psi(P) \in N$.

One can generalize that

$$\Rightarrow \quad \begin{aligned} \mathrm{D}\psi : \mathcal{X}(M) &\to \mathcal{X}(N) \\ v &\mapsto w \end{aligned}$$

gives a map between smooth vector fields .

where $\quad w(f) = v(f \circ \psi) \qquad \forall f \in \mathcal{C}^\infty(N)$

Proof as exercise, that these definitions are compatible when seeing a vector field as family of tangent vectors

In today’s lecture we talk about:

- Mappings between manifolds
- **Pullback operation**
- Trace operator
- Exterior derivative
- Maxwell's equations

Pullback map – for differential forms

In the literature one often finds ψ^* instead of $\mathrm{D}\psi^*$

Consider a differentiable map $\psi : M \to N$.

We define the **pullback map** by

$$\mathrm{D}\psi^* : \mathcal{F}^p(N) \to \mathcal{F}^p(M)$$
$$\omega \mapsto \eta$$

where

$$\eta(v_1, \dots, v_p) = \omega(\mathrm{D}\psi\, v_1, \dots, \mathrm{D}\psi\, v_p) \qquad \forall v_i \in \mathcal{X}(M)$$

Note:
The pullback could be defined for multicovectors only, in a similar way. Both definitions are related by identifying a differential form with a family of multicovectors.

Pullback map - properties

$$\begin{tikzcd} \mathcal{F}^p(N) \arrow[r, "\mathrm{D}\psi^*"] \arrow[d, "\mathrm{d}"'] & \mathcal{F}^p(M) \arrow[d, "\mathrm{d}"] \\ \mathcal{F}^{p+1}(N) \arrow[r, "\mathrm{D}\psi^*"'] & \mathcal{F}^{p+1}(M) \end{tikzcd}$$

For the pullback map it holds :

$\mathrm{D}\psi^*$ is a linear map

$$\mathrm{D}\psi^*(\omega + \lambda\omega') = \mathrm{D}\psi^*\,\omega + \lambda\,\mathrm{D}\psi^*\,\omega' \qquad \forall\lambda \in \mathbb{R}, \quad \forall\omega, \omega' \in \mathcal{F}^p(N)$$

$\mathrm{D}\psi^*$ is compatible with the exterior product

$$\mathrm{D}\psi^*(\omega \wedge \eta) = \mathrm{D}\psi^*(\omega) \wedge \mathrm{D}\psi^*(\eta) \qquad \forall\omega \in \mathcal{F}^p(N), \quad \forall\eta \in \mathcal{F}^q(N)$$

$\mathrm{D}\psi^*$ commutes with the exterior derivative

$$\mathrm{D}\psi^*(\mathrm{d}\omega) = \mathrm{d}(\mathrm{D}\psi^*\omega) \qquad \forall\omega \in \mathcal{F}^p(N)$$

The exterior derivative was not introduced yet ... coming soon

Pullback map – in coordinates

M	$\xrightarrow[\psi(P) = Q]{\psi : M \to N}$	N
$\phi_{U,M} : U \subset M \to \mathbb{R}^m$	charts	$\phi_{V,N} : V \subset N \to \mathbb{R}^n$
$\left(x^1, \ldots, x^m\right)$	local coordinates	$\left(y^1, \ldots, y^n\right)$
$(e_i)_{i=1}^m \in \mathcal{X}(M)$	basis	$(\tilde{e}_i)_{i=1}^n \in \mathcal{X}(N)$

$$\mathcal{F}^p(M) \ni \omega = \mathrm{D}\psi\, \tilde{\omega} \xleftarrow{\mathrm{D}\psi^* : \mathcal{F}^p(N) \to \mathcal{F}^p(M)} \tilde{\omega} \in \mathcal{F}^p(N)$$

How does it look like
in coordinates ?

Pullback map – in coordinates

We have to find out how pullback acts on coordinate differentials $\mathrm{d}y^i$

$$(\mathrm{D}\psi^* \mathrm{d}y^i)(e_j) = \mathrm{d}y^i(\mathrm{D}\psi\, e_j) = \{\mathrm{D}\psi\}^k_j\, \mathrm{d}y^i(\tilde{e}_k) = \{\mathrm{D}\psi\}^i_j$$

$$\Rightarrow \quad \boxed{\mathrm{D}\psi^* \mathrm{d}y^i = \{\mathrm{D}\psi\}^i_j\, \mathrm{d}x^j}$$

We again deal here simply with the Jacobian of the coordinate representation of ψ

Pullback distributes over exterior product, hence for any $\tilde{\omega} \in \mathcal{F}^p(N)$

$$\omega = \mathrm{D}\psi^*\, \tilde{\omega} = (\tilde{\omega}_{i_1 \cdots i_p} \circ \psi)\, \underbrace{\{\mathrm{D}\psi\}^{i_1}_{j_1}\, \mathrm{d}x^{j_1}}_{\mathrm{d}(y^{i_1} \circ \psi)} \wedge \cdots \wedge \underbrace{\{\mathrm{D}\psi\}^{i_p}_{j_p}\, \mathrm{d}x^{j_p}}_{\mathrm{d}(y^{i_p} \circ \psi)}$$

Pullback map – in coordinates

We see that using the pullback in coordinates boils down to three simple steps :

$$\omega = \mathrm{D}\psi^* \, \tilde{\omega} = (\tilde{\omega}_{i_1 \cdots i_p} \circ \psi) \, \mathrm{d}(y^{i_1} \circ \psi) \wedge \cdots \wedge \mathrm{d}(y^{i_p} \circ \psi)$$

1) substitute coordinates

2) transform differentials

3) simplify by rules of exterior algebra

Example – 1 form in $\mathbb{R}^3$ on cylinder

For the setting see this previous slide

We want to obtain a 1-form on the cylinder surface from a 1-form in $\mathbb{R}^3$, by pullback with the embedding $\psi : Z \hookrightarrow \mathbb{R}^3$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_0 \cos\varphi \\ r_0 \sin\varphi \\ z \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} -r_0 \sin\varphi\ \mathrm{d}\varphi \\ r_0 \cos\varphi\ \mathrm{d}\varphi \\ 1\ \mathrm{d}z \end{pmatrix}$$

$$\omega = \frac{x}{\sqrt{x^2+y^2}}\mathrm{d}x + \frac{y}{\sqrt{x^2+y^2}}\mathrm{d}y + \mathrm{d}z$$

1) $$= \cos\varphi\ \mathrm{d}x + \sin\varphi\ \mathrm{d}y + \mathrm{d}z$$

2) $$= -\cos\varphi\, r_0 \sin\varphi\ \mathrm{d}\varphi + \sin\varphi\, r_0 \cos\varphi\ \mathrm{d}\varphi + \mathrm{d}z$$

3) $$= \mathrm{d}z$$

Coordinate transformations

Coordinate transformations for differential forms can be treated in the framework of pullback maps.

Consider the identity map $\psi = \mathrm{id}_M : M \to M$.

The coordinate representation then corresponds to the chart transition map, i.e., the coordinate transformation

$$\hat{\psi} = \phi_{V,M} \circ \phi_{U,M}^{-1}$$

$\Longrightarrow$ We get the transformation rules for coordinate differentials and for differential forms for free !

Example – Coordinate transformation in $\mathbb{R}^2$

We would like to investigate a coordinate transformation from Cartesian to polar coordinates $(x, y) \to (r, \varphi)$

$$(x, y) = \quad \hat{\psi}(r, \varphi) \quad = \phi_{V,M} \circ \phi_{U,M}^{-1}(r, \varphi) \quad = r\,(\cos\varphi, \sin\varphi)$$

$$\omega = \mathrm{d}x \wedge \mathrm{d}y$$

2)

$$= (\cos\varphi\, \mathrm{d}r - r\sin\varphi\, \mathrm{d}\varphi) \wedge (\sin\varphi\, \mathrm{d}r + r\cos\varphi\, \mathrm{d}\varphi)$$

3)

$$= r\cos^2\!\varphi\ \mathrm{d}r \wedge \mathrm{d}\varphi - r\sin^2\!\varphi\ \mathrm{d}\varphi \wedge \mathrm{d}r$$

$$= r\ \mathrm{d}r \wedge \mathrm{d}\varphi$$

In today’s lecture we talk about:

- Mappings between manifolds
- Pullback operation
- **Trace operator**
- Exterior derivative
- Maxwell's equations

Trace operator

For an example see this [previous slide]

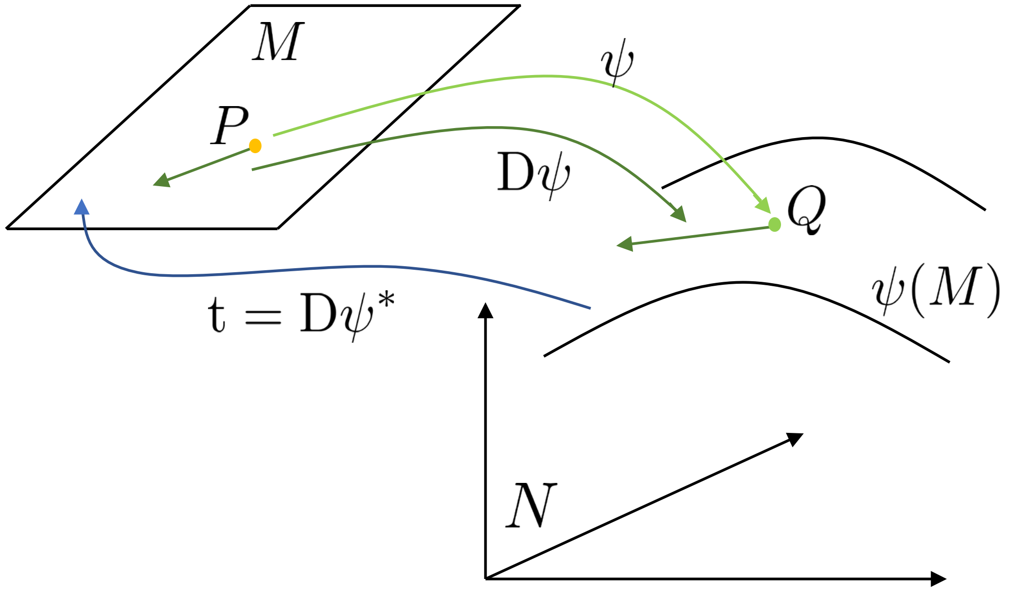
Let N be a differentiable manifold and $M \subset N$ a submanifold.

Let $\psi : M \hookrightarrow N$ be the embedding of M in N.

Then $\mathrm{t} := \mathrm{D}\psi^* : \mathcal{F}^p(N) \to \mathcal{F}^p(M)$ is called **trace operator**

Used for:

1) evaluating p-forms at boundaries.
2) formulation of boundary conditions. (corresponds to Dirichlet conditions)



Decomposition of tangent space

Consider $N = (\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ )$ and $M \subset \mathbb{R}^3$ a sufficiently smooth surface.

Let $\psi : M \hookrightarrow N$ be the embedding of M into N.

$Q \in N$ and $P \in M$ are the same point referred to from different sets and are connected by $Q = \psi(P)$.

Then we can decompose the tangent space $T_Q N$ into :

$$T_Q N = T_Q N_{\|} \oplus T_Q N_{\perp}$$

where

$$T_Q N_{\|} := \mathrm{D}\psi(T_P M)$$

tangent space parallel to smooth surface M

$$T_Q N_{\perp} := \left(T_Q N_{\|}\right)^{\perp}$$

Example – Boundary conditions for 1-forms

Consider $N = \left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$ and $M \subset \mathbb{R}^3$ a sufficiently smooth surface.
Let $E \in \mathcal{X}(N)$ be a smooth vector field on $\mathbb{R}^3$.

$$\boxed{\mathrm{t}\ {}^1E = 0} \quad \Leftrightarrow \quad \left(\mathrm{t}\ {}^1E\right)(\xi)\Big|_P = 0 \qquad \forall \xi \in T_P M \quad \forall P \in M$$

$$\Leftrightarrow \quad {}^1E(\alpha)\Big|_Q = 0 \qquad \forall \alpha \in T_Q N_{\|}\ \ \forall Q \in \psi(M)$$

$$\Leftrightarrow \quad E|_Q \cdot \alpha\ = 0$$

$$\Leftrightarrow \quad \left(E_{\|}|_Q + E_{\perp}|_Q\right) \cdot \alpha = 0 \quad \Leftrightarrow \quad \boxed{E_{\|} = 0}$$

Example – Boundary conditions for 2-forms

Consider $N = \left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$ and $M \subset \mathbb{R}^3$ a sufficiently smooth surface.
Let $B \in \mathcal{X}(N)$ be a smooth vector field on $\mathbb{R}^3$.

$$\boxed{\mathrm{t}\,{}^2B = 0} \quad \Leftrightarrow \quad \left(\mathrm{t}\,{}^2B\right)(\xi,\xi')\Big|_P = 0 \qquad \forall \xi,\xi' \in T_PM \quad \forall P \in M$$

$$\Leftrightarrow \quad {}^2B(\alpha,\alpha')\Big|_Q = 0 \qquad \forall \alpha,\alpha' \in T_QN_{\|} \quad \forall Q \in \psi(M)$$

$$\Leftrightarrow \quad B|_Q \cdot (\alpha \times \alpha') = 0$$

$$\Leftrightarrow \quad \left(B_{\|}|_Q + B_{\perp}|_Q\right) \cdot \underbrace{(\alpha \times \alpha')}_{\in T_QN_{\perp}} = 0 \quad \Leftrightarrow \quad \boxed{B_{\perp} = 0}$$

Example – Electromagnetic field at interfaces

We can use this to model the conditions for the electromagnetic field at the interface between two different media in $\left(\mathbb{R}^3, \cdot, \times\right)$.

Consider the EM-field $E, B \in \mathcal{X}(\mathbb{R}^3)$ in the first and $\tilde{E}, \tilde{B} \in \mathcal{X}(\mathbb{R}^3)$ in the second medium.

Let $M \subset \mathbb{R}^3$ be a sufficiently smooth surface, representing the interface. Then

$$\boxed{\mathrm{t}\left({}^1E - {}^1\tilde{E} \right) = 0} \quad \Leftrightarrow \quad \boxed{E_{\|} = \tilde{E}_{\|}}$$

$$\boxed{\mathrm{t}\left({}^2B - {}^2\tilde{B} \right) = 0} \quad \Leftrightarrow \quad \boxed{B_{\perp} = \tilde{B}_{\perp}}$$

In today’s lecture we talk about:

- Mappings between manifolds
- Pullback operation
- Trace operator
- **Exterior derivative**
- Maxwell's equations

Differentiation of differential forms - Motivation

Consider $f \in \mathcal{C}^\infty(M) \cong \mathcal{F}^0(M)$, a smooth function on M.

We already considered the differential of such functions.

Intrinsic definition	Coordinate representation
$\mathrm{d}f\,(v) = v(f) \quad \forall v \in \mathcal{X}(M)$	$\mathrm{d}f = \dfrac{\partial f}{\partial x^i}\mathrm{d}x^i$

It turned out that the differential of a **0-form** is a **1-form.**

We want to generalize this concept !

Exterior derivative [2 p. 150]

There exists a unique operator $\mathrm{d} : \mathcal{F}^p(M) \to \mathcal{F}^{p+1}(M)$ which coincides for $p = 0$ with the differential of a smooth function $\mathrm{d} : \mathcal{C}^\infty(M) \to \mathcal{F}^1(M)$ and has the following properties:

$\mathbb{R}$ - linearity

$$\mathrm{d}(\lambda\omega + \mu\eta) = \lambda\,\mathrm{d}\omega + \mu\,\mathrm{d}\eta \qquad \forall \omega, \eta \in \mathcal{F}^p(M) \;\; \forall \lambda, \mu \in \mathbb{R}$$

Graded Leibniz rule

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta \quad \forall \omega \in \mathcal{F}^p(M) \quad \forall \eta \in \mathcal{F}^q(M)$$

Complex property

$$\mathrm{d} \circ \mathrm{d} = 0$$

Exterior derivative – in coordinates

Let $\omega \in \mathcal{F}^p(M)$ be a smooth p-form on a n-dimensional differentiable manifold M.

$$\Rightarrow \quad \omega = \sum_{J \in \mathcal{J}_p^n} \omega_J \varepsilon^J$$

$$\begin{aligned}
\mathrm{d}\varepsilon^J &= \mathrm{d}\left(\mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_p}\right) \\
&= \mathrm{d}\left(\mathrm{d}x^{j_1}\right) \wedge \left(\mathrm{d}x^{j_2} \wedge \cdots \wedge \mathrm{d}x^{j_p}\right) - \mathrm{d}x^{j_1} \wedge \mathrm{d}\left(\mathrm{d}x^{j_2} \wedge \cdots \wedge \mathrm{d}x^{j_p}\right) \\
&= -\mathrm{d}x^{j_1} \wedge \mathrm{d}\left(\mathrm{d}x^{j_2} \wedge \cdots \wedge \mathrm{d}x^{j_p}\right) = \cdots = 0
\end{aligned}$$

$$\Rightarrow \quad \mathrm{d}\omega = \sum_{J_p^n} \mathrm{d}\left(\omega_J \varepsilon^J\right) = \sum_{J_p^n} \mathrm{d}\omega_J \wedge \varepsilon^J + \omega_J \wedge \mathrm{d}\varepsilon^J = \sum_{J \in \mathcal{J}_p^n} \mathrm{d}\omega_J \wedge \varepsilon^J$$

Example – 1-forms and curl of vector fields

Let $\omega \in \mathcal{F}^1(M)$ be a smooth 1-form and M a **3-dimensional** manifold.

$$\Rightarrow \omega = a_1 \mathrm{d}x^1 + a_2 \mathrm{d}x^2 + a_3 \mathrm{d}x^3$$

$$\Rightarrow \mathrm{d}\omega = \left(\frac{\partial a_1}{\partial x^1}\mathrm{d}x^1 + \frac{\partial a_1}{\partial x^2}\mathrm{d}x^2 + \frac{\partial a_1}{\partial x^3}\mathrm{d}x^3\right)\wedge \mathrm{d}x^1 + \left(\frac{\partial a_2}{\partial x^1}\mathrm{d}x^1 + \frac{\partial a_2}{\partial x^2}\mathrm{d}x^2 + \frac{\partial a_2}{\partial x^3}\mathrm{d}x^3\right)\wedge \mathrm{d}x^2 + \ldots$$

$$= \left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3}\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \left(\frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1}\right)\mathrm{d}x^3 \wedge \mathrm{d}x^1 + \left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

If $M = \left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$ and $a \in \mathcal{X}(M)$, $\omega = {}^1a$ it follows

$$\boxed{\mathrm{d}\,{}^1a = {}^2(\mathbf{curl}\, a)}$$

Example – 2-forms and divergence of vector fields

Let $\omega \in \mathcal{F}^2(M)$ be a smooth 2-form and M a **3-dimensional** manifold.

$$\Rightarrow \quad \omega = a_{23}\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + a_{31}\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 + a_{12}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

$$\Rightarrow \quad \mathrm{d}\omega = \left(\frac{\partial a_{23}}{\partial x^1}\mathrm{d}x^1 + \frac{\partial a_{23}}{\partial x^2}\mathrm{d}x^2 + \frac{\partial a_{23}}{\partial x^3}\mathrm{d}x^3 \right) \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \dots$$

$$= \left(\frac{\partial a_{23}}{\partial x^1} + \frac{\partial a_{31}}{\partial x^2} + \frac{\partial a_{12}}{\partial x^3} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

If $M = \left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$ and $a \in \mathcal{X}(M)$, $\omega = {}^2 a$ it follows

$$\boxed{\mathrm{d}\,{}^2 a = {}^3(\mathrm{div}\,a)}$$

Summary – Differential operators in $\left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$

Let $a \in \mathcal{X}(\mathbb{R}^3)$ be a smooth vector field in $\left(\mathbb{R}^3\ ,\ \cdot\ ,\ \times\ \right)$ and $f \in \mathcal{C}^\infty(\mathbb{R}^3)$

Then it holds:

$$\mathrm{d}^{\,0} f = {}^{1}(\mathrm{grad}\, f)$$

$$\mathrm{d}^{\,1} a = {}^{2}(\mathbf{curl}\, a)$$

$$\mathrm{d}^{\,2} a = {}^{3}(\mathrm{div}\, a)$$

These relations are independent of any **coordinate system** or **basis** used !

Divergence in cylindrical coordinates

physical component

normalized basis

$$B = B^r \frac{\partial}{\partial r} + B^\varphi \frac{\partial}{r\partial\varphi} + B^z \frac{\partial}{\partial z}$$

$$^2B = B^r r\,\mathrm{d}\varphi \wedge \mathrm{d}z + B^\varphi \mathrm{d}z \wedge \mathrm{d}r + B^z \mathrm{d}r \wedge r\,\mathrm{d}\varphi$$

translation isomorphism

$$^2\left(\frac{\partial}{\partial r}\right) = r\,\mathrm{d}\varphi \wedge \mathrm{d}z$$

$$^2\left(\frac{\partial}{r\partial\varphi}\right) = \mathrm{d}z \wedge \mathrm{d}r$$

$$^2\left(\frac{\partial}{\partial z}\right) = \mathrm{d}r \wedge r\,\mathrm{d}\varphi$$

$$\begin{aligned}\mathrm{d}\,^2B &= \left(\frac{\partial}{\partial r} rB^r + \frac{\partial}{\partial\varphi} B^\varphi + \frac{\partial}{\partial z} rB^z\right) \mathrm{d}r \wedge \mathrm{d}\varphi \wedge \mathrm{d}z \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r} rB^r + \frac{1}{r}\frac{\partial}{\partial\varphi} B^\varphi + \frac{\partial}{\partial z} B^z\right) \mathrm{d}r \wedge r\,\mathrm{d}\varphi \wedge \mathrm{d}z \quad = \; ^3(\mathrm{div}B)\end{aligned}$$

Properties of differential operators

$$f, g \in \mathcal{C}^\infty(\mathbb{R}^3)$$
$$a, b \in \mathcal{X}(\mathbb{R}^3)$$

From complex property it follows:

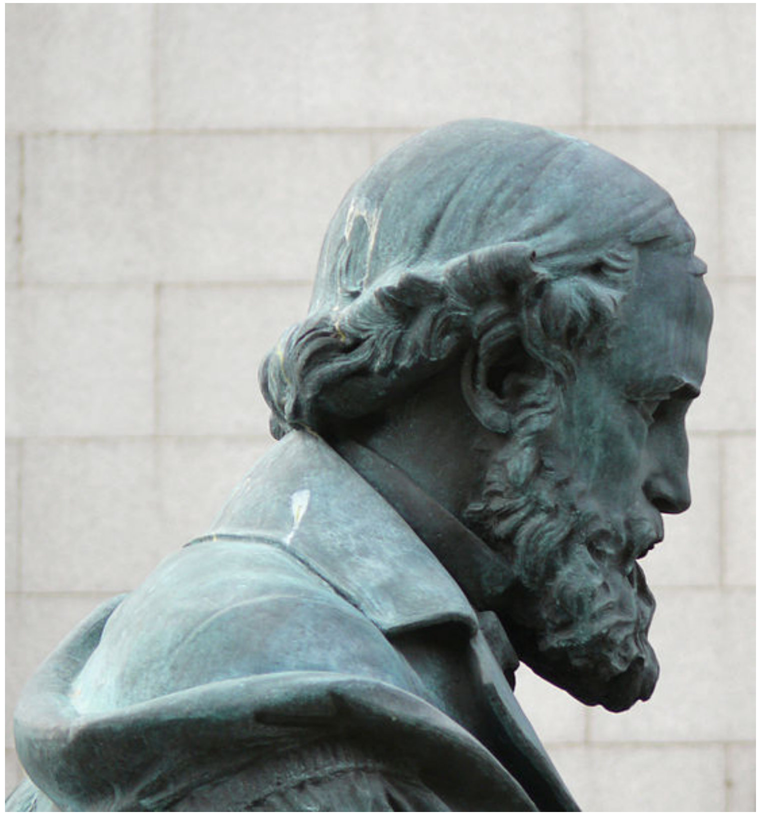
$$\mathrm{d}(\mathrm{d}^{\,0} f) = \mathrm{d}^{\,1}(\mathrm{grad}\ f) = {}^{2}(\mathbf{curl}\,\mathrm{grad}\ f) = 0$$
$$\mathrm{d}(\mathrm{d}^{\,1} a) = \mathrm{d}^{\,2}(\mathbf{curl}\, a) = {}^{3}(\ \mathrm{div}\ \mathbf{curl}\ a) = 0$$

From graded Leibniz rule it follows:

ω	η	product rule in vector analysis
${}^0 f$	${}^0 g$	$\mathrm{grad}\ (f\, g) = f\,\mathrm{grad}\, g + g\,\mathrm{grad}\, f$
${}^0 f$	${}^1 a$	$\mathbf{curl}\ (f\, a) = f\,\mathbf{curl}\, a - a\ \times \mathrm{grad}\, f$
${}^0 f$	${}^2 b$	$\mathrm{div}\ (f\, b) = f\,\mathrm{div}\, b + b\ \cdot \mathrm{grad}\, f$
${}^1 a$	${}^1 b$	$\mathrm{div}\ (a \times b) = b\ \cdot \mathbf{curl}\, a - a\ \cdot \mathbf{curl}\, b$

In today’s lecture we talk about:

- Mappings between manifolds
- Pullback operation
- Trace operator
- Exterior derivative
- **Maxwell's equations**



Statue of James Clerk Maxwell

Source:
https://commons.wikimedia.org/wiki/File:James_Clerk_Maxwell_statue_in_George_Street,_Edinburgh_05.jpg

Maxwell's equations in terms of differential forms

In $\mathbb{R}^3$ we formulate Maxwell's equations with the electromagnetic forms.
Time t is treated as an additional parameter.

$$\mathcal{E} = {}^1E \qquad \mathcal{B} = {}^2B \qquad \mathcal{Q} = {}^3\rho$$

$$\mathcal{D} = {}^2D \qquad \mathcal{H} = {}^1H \qquad \mathcal{J} = {}^2J$$

Faraday's law of induction

$$\mathrm{d}\,\mathcal{E} = -\frac{\partial}{\partial t}\,\mathcal{B}$$

Gauss's law

$$\mathrm{d}\,\mathcal{D} = \mathcal{Q}$$

Ampère's circuital law

$$\mathrm{d}\,\mathcal{H} = \mathcal{J} + \frac{\partial}{\partial t}\,\mathcal{D}$$

Gauss's law for magnetism

$$\mathrm{d}\,\mathcal{B} = 0$$

More relations from EM in terms of forms

We define the form $\mathcal{A} = {}^1 A$, corresponding to the vector potential.
The scalar potential φ is interpreted as 0-form.

Electric field from potentials

$$\mathcal{E} = -\mathrm{d}\varphi - \frac{\partial}{\partial t}\mathcal{A}$$

Magnetic flux density from vector potential

$$\mathcal{B} = \mathrm{d}\,\mathcal{A}$$

Continuity equation

$$\mathrm{d}\,\mathcal{J} + \frac{\partial}{\partial t}\mathcal{Q} = 0$$

Energy densities and Poynting's theorem

We define the Poynting form

$$\mathcal{S} := \mathcal{E} \wedge \mathcal{H} = \ {}^2(E \times H)$$

and the electromagnetic energy densities for <u>linear</u> media

$$\mathcal{W}_{\mathrm{el}} := \frac{1}{2}\mathcal{E} \wedge \mathcal{D} = \frac{{}^3(E \cdot D)}{2} \qquad \mathcal{W}_{\mathrm{mag}} = \frac{1}{2}\mathcal{H} \wedge \mathcal{B} = \frac{{}^3(H \cdot B)}{2}$$

From this, one can prove **Poynting's theorem**

$$\boxed{\mathrm{d}\,\mathcal{S} + \frac{\partial}{\partial t}\left(\mathcal{W}_{\mathrm{el}} + \mathcal{W}_{\mathrm{mag}}\right) = -\mathcal{E} \wedge \mathcal{J}}$$

Prove Poynting's theorem as exercise

Literature

[1] M. Fecko. *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, 2006.

[2] K. Jänich. *Vector Analysis*. Springer, 2001. Link

[3] J. Nestruev. *Smooth manifolds and observables*. Springer, 2003.

[4] C. von Westenholz. *Differential forms in mathematical physics*. North-Holland, 1981.

[5] W. H. Greub. *Multilinear Algebra*. Springer, 1967.

[6] H. J. Dirschmid. *Tensoren und Felder*. Springer, 1996.