Electromagnetics and Differential Forms

Manifolds, Tangent Space & Co.

In today’s lecture we talk about:

- **Differentiable manifolds**
- Tangent spaces and tangent vectors
- Cotangent spaces, covariant vectors and differentials
- Vector fields and 1-forms

Differentiable manifold - Intuitive definition

A differentiable manifold M is a point set that looks locally like $\mathbb{R}^n$, however with a global structure which is not necessarily that of $\mathbb{R}^n$.

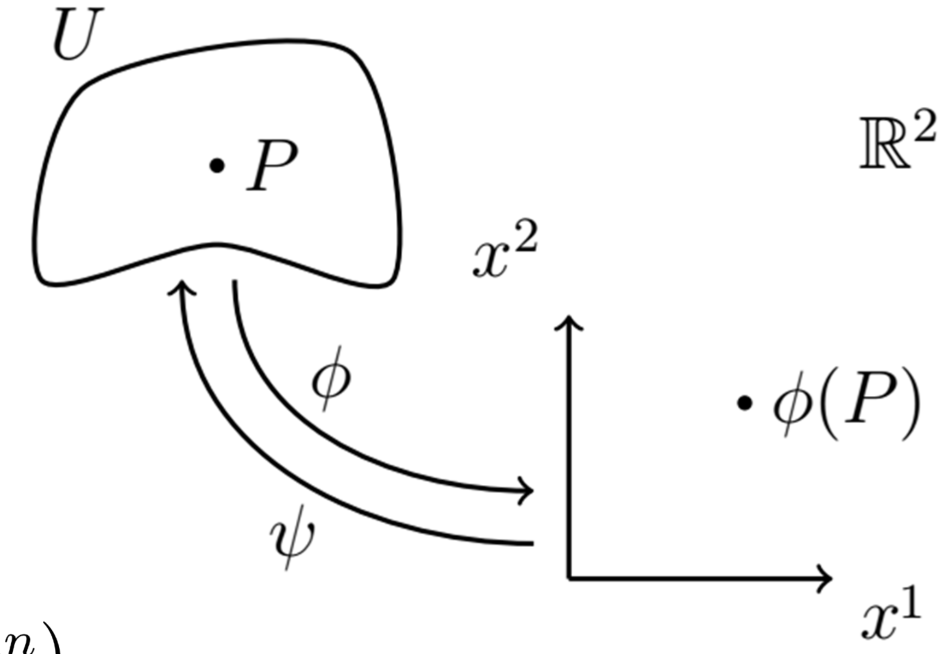
Points and neighborhoods $\Rightarrow$ Topological space (M, τ)

Calculus on manifold $\Rightarrow$ Additional structure needed

Charts

Let $U \subset M$ be an open set on the manifold.

$$\begin{array}{rccc} \phi: & U & \to & \mathbb{R}^n \\ & P & \mapsto & \left(x^1, \ldots, x^n\right) \end{array}$$



ϕ is called a chart, if ϕ is a homeomorphism.
$\left(x^1, \ldots, x^n\right)$ are called local coordinates.

$$\begin{array}{rccc} \phi^{-1}: & \phi(U) \subset \mathbb{R}^n & \to & U \\ & \left(x^1, \ldots, x^n\right) & \mapsto & P\left(x^1, \ldots, x^n\right) \end{array}$$

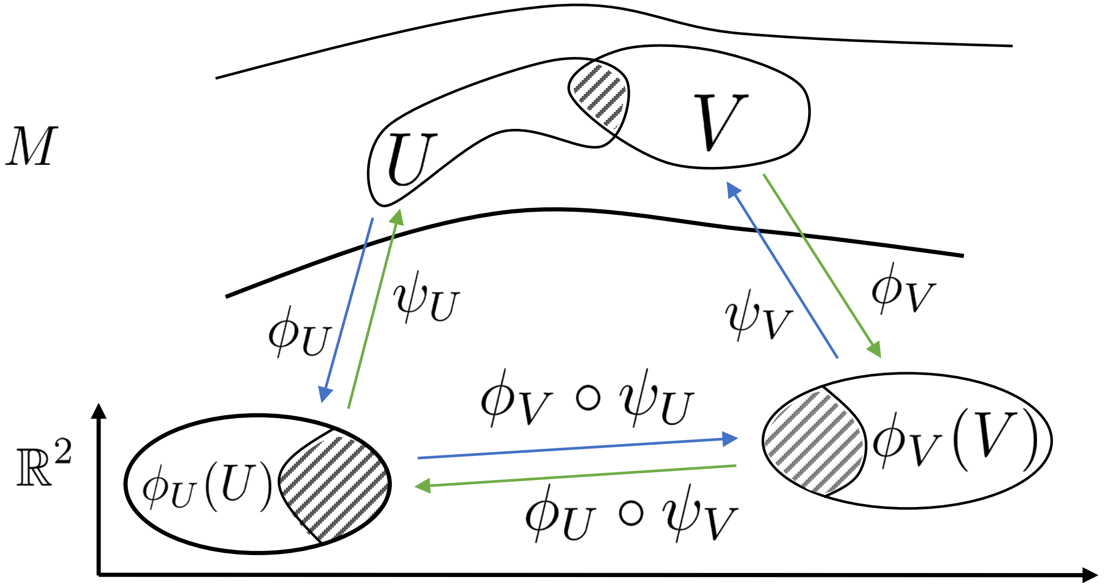
$\psi = \phi^{-1}$ is called a parametrization of U.

Overlapping charts

Let $U, V \subset M$ be open sets on the manifold and ϕ_U, ϕ_V charts for those open sets.

$U \cap V = \emptyset \quad \Rightarrow$ No overlap = No problem

$U \cap V \neq \emptyset \quad \Rightarrow$ Need **smooth** transition map



$$\phi_U \circ \psi_V : \phi_V(U \cap V) \subset \mathbb{R}^n \to \mathbb{R}^n$$

$$\phi_V \circ \psi_U : \phi_U(U \cap V) \subset \mathbb{R}^n \to \mathbb{R}^n$$

Compatibility of charts

Let $U, V \subset M$ be open sets on the manifold and ϕ_U, ϕ_V charts for those open sets.

Two charts ϕ_U, ϕ_V are called $\mathcal{C}^k$**-compatible** if either

$$U \cap V = \emptyset.$$

or

$$\left.\begin{array}{l} \phi_U \circ \phi_V^{-1} : \phi_V\left(U \cap V\right) \subset \mathbb{R}^n \to \mathbb{R}^n \\ \phi_V \circ \phi_U^{-1} : \phi_U\left(U \cap V\right) \subset \mathbb{R}^n \to \mathbb{R}^n \end{array}\right\} \in \mathcal{C}^k(\mathbb{R}^n)$$

are $\mathcal{C}^k$-functions in $\mathbb{R}^n$

Atlas – Collections of charts

An atlas $\mathcal{A}$ of a manifold M is a collection of charts , whose U_i domains cover M.

$$M \subset \bigcup_{i \in I} U_i$$

A $\mathcal{C}^k$-atlas $\mathcal{A}$ is an atlas, where all charts are mutually $\mathcal{C}^k$-compatible.

There exists a maximal $\mathcal{C}^k$-atlas, containing all others as subsets.

Differentiable manifold – Precise definition

$(M, \mathcal{A})$ is called a $\mathcal{C}^k$-differentiable manifold if:

1) M is a second countable topological space;
2) M is a Hausdorff-space;
3) $\mathcal{A}$ is a maximal $\mathcal{C}^k$-differentiable atlas of M.

Remarks :

No loss of generality arises when restricting to manifolds of class $\mathcal{C}^\infty$.
(See Whitney's theorem [4, Remark 3.4])

For practical situations one can restrict to a specific atlas.

Differentiable functions on differentiable manifolds

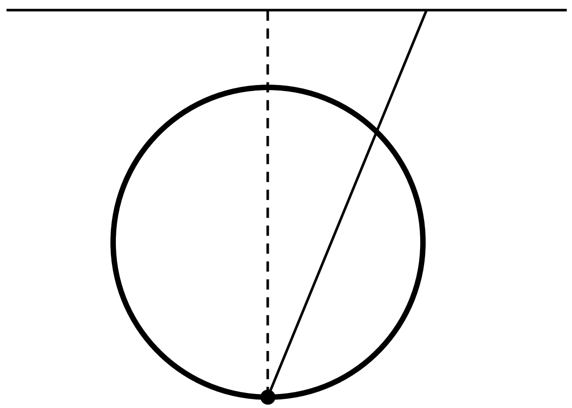
A function $f : M \to \mathbb{R}$ is called differentiable in $x \in M$ if it is continuous and if for a neighborhood U of x and a chart ϕ_U the function $f \circ \phi_U^{-1} : \phi_U(U) \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable in $\phi_U(x)$.

This definition is independent of the chart used, due to the compatibility for the charts.

$\mathcal{C}^k(M)$ denotes the space of functions $f : M \to \mathbb{R}$ which are k-times continuously differentiable on M.

Examples: Manifolds

Sphere S^n:



At least two charts are needed to cover the whole sphere without singularities.

Use e.g. stereographic projections as charts.

$\mathbb{R}^n$:

Only one chart required to cover whole manifold (identity).

Tangent space at every point can be canonically identified with $\mathbb{R}^n$.

$\rightarrow$ flat space

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Tangent space – classical approaches

Tangent vector $\equiv$ n-tupel of numbers with a certain transformation rule under coordinate transformation.

$\rightarrow$ Describes tangent vectors in local coordinates only.
Does not explain what they are.

Equivalence class of smooth curves tangent to each other at a given point of M.

$\rightarrow$ Difficult to introduce the structure of a linear space.

Tangent vectors

Let $\mathcal{C}^\infty(M)$ be the set of all smooth functions $f\,:\,M\,\rightarrow\mathbb{R}$ on the differentiable manifold M.

A map $\xi:\mathcal{C}^\infty(M)\rightarrow\mathbb{R}$ is called a tangent vector to M at the point $P\in M$ if it satisfies :

$\mathbb{R}$- linearity :

$$\xi(\lambda f+\mu g)=\lambda\,\xi(f)+\mu\,\xi(g),\quad \lambda,\mu\in\mathbb{R}\text{ and }f,g\in\mathcal{C}^\infty(M)$$

Local Leibniz rule :

$$\xi(f\cdot g)=f(P)\,\xi(g)+g(P)\,\xi(f),\quad f,g\in\mathcal{C}^\infty(M)$$

Tangent space $\mathbf{T_P M}$

The tangent space $T_P M$ to a point $P \in M$ is the set of all tangent vectors to M at P.

$T_P M$ can be given the structure of a vector space over $\mathbb{R}$ by

$$(\xi + \xi')(f) := \xi(f) + \xi'(f) \qquad \forall \xi, \xi' \in T_P M$$

$$(\lambda\,\xi)(f) := \lambda\xi(f) \qquad \forall \xi \in T_P M \;\; \forall \lambda \in \mathbb{R}$$

In general $T_P M \neq T_{P'} M$.
Tangent vectors are defined locally.

Tangent vector theorem [3]

Let M be a differentiable manifold, $P \in M$ a point on M and $U \subset M$ a neighborhood of P.

Let $\phi : U \to \mathbb{R}^n$ be a chart for U, $\psi = \phi^{-1}$ the parametrization and $(x^1, \ldots, x^n)$ the corresponding local coordinate system for U.

Then, any tangent vector $\xi \in T_P M$ can be represented as

$$\xi(f) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}\bigg|_{\phi(P)} f \circ \psi \qquad \alpha^i \in \mathbb{R}.$$

Tangent vector theorem – Inferences

The $$e_i(f) = \frac{\partial}{\partial x^i}\bigg|_{\phi(P)} f \circ \psi$$ are linearly independent

$\Rightarrow$ The e_i form a basis of the tangent space $T_P M$.

$\Rightarrow$ $T_P M$ is a n-dimensional vector space over $\mathbb{R}$.

The notion of a tangent vector and of differentiation in a given direction coincide.

Contravariant transformation behaviour

Let ϕ_x and ϕ_y be two charts, sharing a common overlapping domain.

$\Rightarrow \psi_x,\ \psi_y$ two local parametrizations.

$\Rightarrow \left(x^1, \ldots, x^n\right),\ \left(y^1, \ldots, y^n\right)$ two local coordinate systems.

Coordinate transformation : $\phi_x \circ \psi_y : \left(y^1, \ldots, y^n\right) \mapsto \left(x^1, \ldots, x^n\right)$

$$\xi(f) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}\bigg|_{\phi_x(P)} f \circ \psi_x \underset{\text{chain rule}}{=} \sum_{k=1}^{n} \underbrace{\sum_{i=1}^{n} \alpha^i \frac{\partial y^k}{\partial x^i}}_{\beta^k :=} \frac{\partial}{\partial y^k}\bigg|_{\phi_y(P)} f \circ \psi_y$$

Contravariant transformation behaviour

$$\xi(f) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}\bigg|_{\phi_x(P)} f \circ \psi_x = \sum_{k=1}^{n} \beta^k \frac{\partial}{\partial y^k}\bigg|_{\phi_y(P)} f \circ \psi_y$$

$$\boxed{\beta^k = \sum_{i=1}^{n} \frac{\partial y^k}{\partial x^i}\, \alpha^i}$$

The α^i, β^k are called contravariant components of the tangent vector.

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Cotangent space $\mathbf{T_P^*M}$

The cotangent space T_P^*M to a point $P \in M$ is the dual space to tangent space.

$$T_P^*M = \mathcal{L}_{\mathbb{R}}\left(T_PM, \mathbb{R}\right)$$

T_P^*M is the set of $\mathbb{R}$-linear maps from T_PM to $\mathbb{R}$.

$$\dim(T_P^*M) = \dim(T_PM)$$

Covariant vectors

The elements $\eta \in T_P^* M$ are called **covariant vectors** or **covectors**.

$\eta \in T_P^* M$ work as “slot machines” :

$$T_P M \ni \xi \longrightarrow \boxed{\eta \in T_P^* M} \longrightarrow \eta(\xi) \in \mathbb{R}$$

Differentials

Let $f \in \mathcal{C}^\infty(M)$ be a smooth function and $P \in M$ a point on M.

The differential $\mathrm{d}f$ at a point P is defined by

$$\mathrm{d}f|_P\,(\xi) = \xi(f) \qquad \forall \xi \in T_P M$$

$\Rightarrow$ Element of cotangent space, $\mathrm{d}f \in T_P^* M$

$\Rightarrow$ Coordinate independent definition

$\Rightarrow$ Not seen as infinitesimally small change

Basis of cotangent space

Let $\phi : U \to \mathbb{R}^n$ be a chart and $(x^1, \ldots, x^n)$ local coordinates.

$\phi^i : P \in M \mapsto x^i$ is the i-th coordinate function.

A basis of $T_P^* M$ is given by

$$\boxed{\varepsilon^i = \mathrm{d}\phi^i|_P}$$

It is the dual basis to the $e_i \in T_P M$

$$\varepsilon^j(e_i) = \mathrm{d}\phi^j|_P(e_i) = e_i(\phi^j) = \left.\frac{\partial}{\partial x^i}\right|_{\phi(P)} (\phi^j \circ \psi) = \left.\frac{\partial}{\partial x^i}\right|_{\phi(P)} x^j = \delta_i^j$$

Shorter (sloppy) notation

Use x^i for the coordinate **and** the coordinate function.

Write always $\big|_P$ **even** if actually $\big|_{\phi(P)}$ is meant.

$$e_i = \left.\frac{\partial}{\partial x^i}\right|_P \quad \longleftrightarrow \quad e_i = \left.\frac{\partial}{\partial x^i}\right|_{\phi(P)} [.] \circ \psi$$

$$\varepsilon^i = \mathrm{d}x^i|_P \quad \longleftrightarrow \quad \varepsilon^i = \mathrm{d}\phi^i|_P$$

Einstein summation convention

Summation signs $\sum$ are omitted.

Summation over indices that occur as pair of lower and upper index.

Summations run from 1 to n.

$$\xi = \alpha^i e_i \quad \longleftrightarrow \quad \xi = \sum_{i=1}^{n} \alpha^i e_i$$

$$\beta^k = T_i^k \alpha^i \quad \longleftrightarrow \quad \beta^k = \sum_{i=1}^{n} T_i^k \alpha^i$$

Basis representation of differentials

$$\xi \in T_P M : \qquad \varepsilon^j(\xi) = \alpha^i \varepsilon^j(e_i) = \alpha^i \delta_i^j = \alpha^j$$

Dual basis extracts components

$$\Rightarrow \qquad \xi = \varepsilon^i(\xi)\, e_i = \mathrm{d}x^i(\xi) \left.\frac{\partial}{\partial x^i}\right|_P$$

$$\Rightarrow \qquad \mathrm{d}f(\xi)|_P = \xi(f) = \mathrm{d}x^i(\xi) \left.\frac{\partial f}{\partial x^i}\right|_P \qquad \forall \xi \in T_P M$$

$$\Rightarrow \qquad \boxed{\mathrm{d}f|_P = \left.\frac{\partial f}{\partial x^i}\right|_P \mathrm{d}x^i}$$

Covariant transformation behaviour

Assume the same situation as in the discussion of contravariant behavior.

$\Rightarrow$ Coordinate transformation : $\phi_x \circ \psi_y : \left(y^1, \ldots, y^n\right) \mapsto \left(x^1, \ldots, x^n\right)$

Two basis sets for $T_P M$

$$e_{x,i} = \left.\frac{\partial}{\partial x^i}\right|_P$$

$$e_{y,i} = \left.\frac{\partial}{\partial y^i}\right|_P$$

$\Rightarrow$

$\eta \in T_P^* M$

$$\eta(e_{x,i}) = a_j \, \mathrm{d}x^j|_P(e_{x,i}) = a_j \, \delta_i^j = a_i$$

$$\eta(e_{y,i}) = b_j \, \mathrm{d}y^j|_P(e_{y,i}) = b_j \, \delta_i^j = b_i$$

Covariant transformation behaviour

$$e_{x,i} = \left.\frac{\partial}{\partial x^i}\right|_P = \sum_{k=1}^{n} \left.\frac{\partial y^k}{\partial x^i}\right|_P \left.\frac{\partial}{\partial y^k}\right|_P = \sum_{k=1}^{n} \left.\frac{\partial y^k}{\partial x^i}\right|_P e_{y,k}$$

$$\Rightarrow \qquad a_i = \eta(e_{x,i}) = \sum_{k=1}^{n} \left.\frac{\partial y^k}{\partial x^i}\right|_P \eta(e_{y,k}) = \sum_{k=1}^{n} \left.\frac{\partial y^k}{\partial x^i}\right|_P b_k$$

$$\eta = a_i \,\mathrm{d}x^i|_P = b_k \,\mathrm{d}y^k|_P \qquad\qquad a_i = \sum_{k=1}^{n} \left.\frac{\partial y^k}{\partial x^i}\right|_P b_k$$

Wrap up: Differentials

They are intrinsically defined by

$$\mathrm{d}f|_P\,(\xi) = \xi(f) \qquad\qquad \forall \xi \in T_P M$$

$\Rightarrow$ No recourse to other differentials or coordinates.

$\Rightarrow$ Coordinate differentials extract the contravariant coordinates of a tangent vector.

$\Rightarrow$ The differentials defined are identical to those under an integral sign.

$\Rightarrow$ The linear relation between differentials follows from their definition.

Example 1D: $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x$

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(Smooth) Vector fields

A smooth vector field v on M is a $\mathbb{R}$-linear operator

$$v : \mathcal{C}^{\infty}(M) \to \mathcal{C}^{\infty}(M)$$

satisfying the Leibniz rule :

$$v(fg) = v(f)g + fv(g) \quad \forall f, g \in \mathcal{C}^{\infty}(M)$$

$\mathcal{X}(M)$ is the space of smooth vector fields on the manifold M.

Vector fields as family of tangent vectors

Vector field $\to$ Family of tangent vectors $(v_P)_{P\in M}$

$$v_P \in T_P M : \quad v_P(f) = v(f)|_P \quad \forall f \in \mathcal{C}^\infty(M)$$

Coordinate representation in local coordinates :

$$v = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i} \quad \text{where} \quad \alpha^i \in \mathcal{C}^\infty(M)$$

(Smooth) Differential 1-forms

$\omega(v)|_P$ depends only on $v|_P$.

A smooth differential 1-form ω on M is a local operator

$$\omega : \mathcal{X}(M) \to \mathcal{C}^\infty(M)$$

satisfying :

$$\omega(fv + gw) = f\omega(v) + g\omega(w) \qquad \begin{array}{l} \forall f, g \in \mathcal{C}^\infty(M) \\ \forall v, w \in \mathcal{X}(M) \end{array}$$

$\mathcal{F}^1(M)$ is the space of smooth differential 1-forms on the manifold M.

$$\mathcal{F}^1(M) := \mathcal{L}_{\mathcal{C}^\infty(M)}(\mathcal{X}(M), \mathcal{C}^\infty(M))$$

Differential 1-forms as family of covectors

Differential 1-form $\rightarrow$ Family of covectors $(\omega_P)_{P \in M}$

$$\omega_P \in T_P^* M : \; \omega_P(\xi) = \omega(v_\xi)|_P \quad \forall \xi \in T_P M$$

where $v_\xi \in \mathcal{X}(M) : \quad v_\xi|_P = \xi$

v_ξ is a smooth continuation of ξ

Coordinate representation in local coordinates:

$$\omega = \sum_{i=1}^{n} a_i \, \mathrm{d}x^i \quad \text{where} \quad a_i \in \mathcal{C}^\infty(M)$$

Algebraic Considerations

$\mathcal{C}^\infty(M)$ is a ring over $\mathbb{R}$.

A (commutative) ring is a field without multiplicative inverse.

$\mathcal{X}(M)$ is a module over $\mathcal{C}^\infty(M)$.

$\mathcal{F}^1(M)$ is a module over $\mathcal{C}^\infty(M)$.

A module over a ring is the generalization of a vector space over a field.

Wrap up: Fields on manifolds

Vector field $\rightarrow$ Assigns a tangent vector to each point on the manifold
v in a smooth way.

$\mathcal{X}(M)$ Space of smooth vector fields.

Differential 1-form $\rightarrow$ Assigns a covector to each point on
ω the manifold in a smooth way.

$\mathcal{F}^1(M)$ Space of smooth differential 1-forms.

Literature

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[3] J. Nestruev. *Smooth manifolds and observables*. Springer, 2003.

[4] C. von Westenholz. *Differential forms in mathematical physics*. North-Holland, 1981.