Linear spaces

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Chapter 1

Introduction

1.1 Definition of a linear space

Let V be a non-empty set of objects, called *elements*. The set V is called linear space if it satisfies the following ten axioms, which are stated in three groups.

Axioms of a linear space

Axiom 1. *Closure property of addition*: for every pair of elements $x, y \in V$, their sum is written as $z = x + y$ and $z \in V$.

Axiom 2. *Closure property of scalar product*: for any $x \in V$ and $a \in \mathbb{R}$, there is an element $z = ax \in V$.

Axioms for addition

There are four axioms of addition, we will use a number and a letter to refer to them. However if we are talking of addition properties we will simply use a letter to reference them. The same will be done for the axioms of scalar products.

Axiom 3.a. *Commutative law*: for any $x, y \in V$, we have $x + y = y + x$.

Axiom 4.b. *Associative law*: for any $x, y, z \in V$, we have $(x+y)+z = x+(y+z)$.

Axiom 5.c. *Existence of zero as an element*: there is a number in V, designated as O (big 'o'), that satisfies

$$x + O = x, \quad \forall x \in V$$

Axiom 6.d. *Opposite elements*: for all $x \in V$, the element $(-1)x$ has the property

$$x + (-1)x = O$$

Axioms for scalar product

Axiom 7.a. *Associative law*: for all $x \in V$ and every pair $a, b \in \mathbb{R}$, we have

$$a(bx) = (ab)x$$

Axiom 8.b. *Distributive law for addition in V*: for all $x, y \in V$ and $a \in \mathbb{R}$, it is true that

$$a(x + y) = ax + ay$$

Axiom 9.c. *Distributive law for addition in $\mathbb{R}$*: for any $x \in V$ and $a, b \in \mathbb{R}$, we have

$$(a + b)x = ax + bx$$

Axiom 10.d. *Existence of an identical element*: for all $x \in V$ theres an unique element I such that $Ix = x$ (commonly this element is 1. But, for example, the identical element in matrix spaces is called *identity matrix*, defined as $I = \mathrm{diag}(1)$)

1.2 Examples of linear spaces

The following examples can be proven to be linear spaces

1. Real numbers
2. The vector space of real numbers $\mathbb{R}^n$
3. The set of all matrices
4. Polynomials P with $\deg P \leq n$ (in this case, if $\deg P = n$, we would have a problem with axioms of additions. We can't ensure the sum of two polynomials of degree n has degree n).
5. The set of all polynomials
6. The set of continuous functions in an interval $[a, b]$. This space is designated as $C(a, b)$.
7. The set of all integrable functions in an interval
8. The set of differentiable functions in an interval
9. A plane in $\mathbb{R}^3$ with the equation $ax + by + cz = 0$. Note that this plane must always go through the origin to be a linear space.

There are plenty of examples for linear spaces. We can "create" a linear space if we define addition and multiplication for that space.

1.3 Consequences of the axioms

The following theorems are a consequence of the axioms of linear space.

Theorem 1.1 (Uniqueness of 'O')**.** *In any linear space there is one and only one zero element*

Proof. Axiom 5 ensures that there is at least one 'O' in V. Now, suppose there are two zeroes in V. Let $x = O_1$ and $O_2 = O$, thus $x + O = x + O_2 = x = O_1$, but as O_1 is zero, $O_1 + O_2 = O_2$, this means that $O_1 = O_2 = O$ □

Theorem 1.2 (Uniqueness of opposites)**.** *In any linear space each x has one and only one opposite y such that $x + y = O$*

Proof. Axiom 6 ensures there is at least one opposite of x in V. Let $y_1, y_2 \in V$ be two different opposite elements for x. Then $x+y_1 = O$ and $x+y_2 = O$, then

$$(x + y_1) + y_2 = y_2 + O = y_2$$

and

$$y_1 + (x + y_2) = y_1 + O = y_1$$

Thus $y_1 + (x + y_2) = y_1 + (x + y_1) = y_1 + O = O + y_1$, this proves that $y_1 = y_2$. □

1.4 Subspaces of a linear space

Let V be a linear space and let S be a subset of V, if S is also a linear space, then we say that "S is a subspace of V".

A subset of a linear space if a subspace only if it satisfies the axioms of closure.

Theorem 1.3. *Let V be a linear space, if $S \subset V$ and $S \neq \emptyset$ satisfies the ten axioms of closure then S is a subspace of V.*

The proof for this theorem is easy, and so I discarded it.

Definition 1.1. *Let $S \subset V$, and $S \neq \emptyset$, where V is a linear space. If $x \in V$ and*

$$x = \sum_{i=1}^{k} c_i x_i$$

where $x_1, x_2, \ldots, x_k \in S$ and $c_1, c_2, \ldots, c_k \in \mathbb{R}$, is called a linear combination of elements in S. The set of linear combinations of the elements of S satisfies the axioms of closure, so it is also a subspace of V. We say that this subspace is generated by S and we call it the linear span of S, designated by $L(S)$. If $S = \emptyset$, we define $L(S) = \{O\}$.

1.5 Dependent and independent subsets of a linear space

In this section we introduce the concept of independence, that is important when working with systems of linear equations, matrices, and other subjects in linear algebra.

Definition 1.2. *Let S be a set of elements of a linear space V. S is dependent if there exists a finite set of distinct elements in $x_1, x_2, \ldots, x_k \in S$, and a set of scalars $c_1, c_2, \ldots, c_k$ where not all of them are zero, that satisfies*

$$\sum_{i=1}^{k} c_i x_i = 0$$

A set is independent if it is not dependent. So the following

$$\sum_{i=1}^{k} c_i x_i = 0, \quad \textit{implies } c_1 = c_2 = \cdots = c_k = 0$$

Independency and dependency are properties of sets of elements. However, we can apply the same concepts to the elements itself. For example, a set of vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ is called independent if there is **not** a linear combination of these vectors that produce the zero vector.

Example 1.1. *Let $u_k(t) = t^k$ for $k = 1, 2, \ldots, n$ and $t \in \mathbb{R}$. The set $V = \{u_1, u_2, \ldots, u_n\}$ is independent except in the subset S where $t = -1$ and n is odd.*

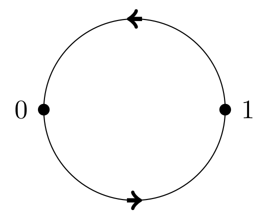
Proof. For S to be independent, there must be $c_1, c_2, \ldots, c_n$, where $c_1 = c_2 = \cdots = c_n = 0$ and

$$\sum_{k=0}^{n} c_k t^k = 0$$

To solve this, we set $c_0 = c_1 = \cdots = c_n$. If we define

$$f(t) = \sum_{k=0}^{n} c_k t^k,$$

note that $f(-1) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ We can draw a picture for this problem. Imagine a circle, in which we have two points. We can travel the circumference counterclockwise starting from 1. We start from 1 because in the case $n = 0$, $f(-1) = 1$.



Now start performing counterclockwise turns, and count how many times you go from 0 to 1, and from 1 to 0.

If $C_{0\to1}$ and $C_{1\to0}$ are the counts of going from 0 to 1 and from 1 to 0 respectively, we have that if $C_{0\to1} = C_{1\to0}$, then $f(-1) = 1$. Otherwise, we must have $f(-1) = 0$.

But having $C_{0\to1} = C_{1\to0}$ and that the total count is $C = C_{0\to1} + C_{1\to0}$, means

$$C = 2C_{1\to0} = 2C_{0\to1}$$

Hence, C is an even number. To verify that S is dependent, we set $n = 2r-1$ for any integer r and $t = -1$. With the results above, we can see that

$$\sum_{k=0}^{2r-1} c_k(-1)^k = 0$$

if $c_1 = c_2 = \cdots = c_{2r-1}$, but not necessarily zero. This proves that S is dependent. $\square$