

Linear spaces

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Chapter 1

Introduction

1.1 Definition of a linear space

Let V be a non-empty set of objects, called *elements*. The set V is called linear space if it satisfies the following ten axioms, which are stated in three groups.

Axioms of a linear space

Axiom 1. *Closure property of addition*: for every pair of elements $x, y \in V$, their sum is written as $z = x + y$ and $z \in V$.

Axiom 2. *Closure property of scalar product*: for any $x \in V$ and $a \in \mathbb{R}$, there is an element $z = ax \in V$.

Axioms for addition

There are four axioms of addition, we will use a number and a letter to refer to them. However if we are talking of addition properties we will simply use a letter to reference them. The same will be done for the axioms of scalar products.

Axiom 3.a. *Commutative law*: for any $x, y \in V$, we have $x + y = y + x$.

Axiom 4.b. *Associative law*: for any $x, y, z \in V$, we have $(x+y)+z = x+(y+z)$.

Axiom 5.c. *Existence of zero as an element*: there is a number in V , designated as O (big 'o'), that satisfies

$$x + O = x, \quad \forall x \in V$$

Axiom 6.d. *Opposite elements*: for all $x \in V$, the element $(-1)x$ has the property

$$x + (-1)x = O$$

Axioms for scalar product

Axiom 7.a. *Associative law*: for all $x \in V$ and every pair $a, b \in \mathbb{R}$, we have

$$a(bx) = (ab)x$$

Axiom 8.b. *Distributive law for addition in V* : for all $x, y \in V$ and $a \in \mathbb{R}$, it is true that

$$a(x + y) = ax + ay$$

Axiom 9.c. *Distributive law for addition in \mathbb{R}* : for any $x \in V$ and $a, b \in \mathbb{R}$, we have

$$(a + b)x = ax + bx$$

Axiom 10.d. *Existence of an identical element*: for all $x \in V$ there's a unique element I such that $Ix = x$ (commonly this element is 1. But, for example, the identical element in matrix spaces is called *identity matrix*, defined as $I = \text{diag}(1)$)

1.2 Examples of linear spaces

The following examples can be proven to be linear spaces

1. Real numbers
2. The vector space of real numbers \mathbb{R}^n
3. The set of all matrices
4. Polynomials P with $\deg P \leq n$ (in this case, if $\deg P = n$, we would have a problem with axioms of additions. We can't ensure the sum of two polynomials of degree n has degree n).
5. The set of all polynomials
6. The set of continuous functions in an interval $[a, b]$. This space is designated as $C(a, b)$.
7. The set of all integrable functions in an interval
8. The set of differentiable functions in an interval
9. A plane in \mathbb{R}^3 with the equation $ax + by + cz = 0$. Note that this plane must always go through the origin to be a linear space.

There are plenty of examples for linear spaces. We can “create” a linear space if we define addition and multiplication for that space.

1.3 Consequences of the axioms

The following theorems are a consequence of the axioms of linear space.

Theorem 1.1 (Uniqueness of ‘O’). *In any linear space there is one and only one zero element*

Proof. Axiom 5 ensures that there is at least one ‘O’ in V . Now, suppose there are two zeroes in V . Let $x = O_1$ and $O_2 = O$, thus $x + O = x + O_2 = x = O_1$, but as O_1 is zero, $O_1 + O_2 = O_2$, this means that $O_1 = O_2 = O$ \square

Theorem 1.2 (Uniqueness of opposites). *In any linear space each x has one and only one opposite y such that $x + y = O$*

Proof. Axiom 6 ensures there is at least one opposite of x in V . Let $y_1, y_2 \in V$ be two different opposite elements for x . Then $x + y_1 = O$ and $x + y_2 = O$, then

$$(x + y_1) + y_2 = y_2 + O = y_2$$

and

$$y_1 + (x + y_2) = y_1 + O = y_1$$

Thus $y_1 + (x + y_2) = y_1 + (x + y_1) = y_1 + O = O + y_1$, this proves that $y_1 = y_2$. \square

1.4 Subspaces of a linear space

Let V be a linear space and let S be a subset of V , if S is also a linear space, then we say that “ S is a subspace of V ”.

A subset of a linear space is a subspace only if it satisfies the axioms of closure.

Theorem 1.3. *Let V be a linear space, if $S \subset V$ and $S \neq \emptyset$ satisfies the ten axioms of closure then S is a subspace of V .*

The proof for this theorem is easy, and so I discarded it.

Definition 1.1. *Let $S \subset V$, and $S \neq \emptyset$, where V is a linear space. If $x \in V$ and*

$$x = \sum_{i=1}^k c_i x_i$$

where $x_1, x_2, \dots, x_k \in S$ and $c_1, c_2, \dots, c_k \in \mathbb{R}$, is called a linear combination of elements in S . The set of linear combinations of the elements of S satisfies the axioms of closure, so it is also a subspace of V . We say that this subspace is generated by S and we call it the linear span of S , designated by $L(S)$. If $S = \emptyset$, we define $L(S) = \{O\}$.

1.5 Dependent and independent subsets of a linear space

In this section we introduce the concept of independence, that is important when working with systems of linear equations, matrices, and other subjects in linear algebra.

Definition 1.2. Let S be a set of elements of a linear space V . S is dependent if there exists a finite set of distinct elements in $x_1, x_2, \dots, x_k \in S$, and a set of scalars c_1, c_2, \dots, c_k where not all of them are zero, that satisfies

$$\sum_{i=1}^k c_i x_i = 0$$

A set is independent if it is not dependent. So the following

$$\sum_{i=1}^k c_i x_i = 0, \quad \text{implies } c_1 = c_2 = \dots = c_k = 0$$

Independency and dependency are properties of element sets. However, we can apply the same concepts to the elements itself. For example, a set of vectors in $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ is called independent if there is **not** a linear combination of these vectors that produce the zero vector.

Example 1.1. Let $u_k(t) = t^k$ for $k = 1, 2, \dots$ and $t \in \mathbb{R}$. The set $S = u_1, u_2, \dots$ is independent except when $t = -1$.

Proof. For S to be independent, there must be c_1, c_2, \dots, c_n , where $c_1 = c_2 = \dots = c_n = 0$ and

$$\sum_{k=0}^n c_k t^k = 0$$

To solve this, we can set $c_0 = c_1 = \dots = c_n$. If we define

$$f(t) = \sum_{k=0}^n c_k t^k,$$

note that $f(-1) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ We can draw a picture for this prob-

lem. Imagine a circle, in which we have two points. We can travel the circumference counterclockwise



