1 Real Analysis

1.1 De Morgan's Laws: Finite and Infinite Cases

1. Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c \tag{1}$$

for any finite $n \in \mathbb{N}$.

2. Conclude that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c \tag{2}$$

1. Morgan's Laws state that $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$, if we suppose that the initial case is true, suppose that it is true for n and for $n+1$ we have

$$\begin{aligned} ((A_1 \cup \cdots \cup A_n) \cup A_{n+1})^c &= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c \\ &= (A_1^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c \\ &= A_1^c \cap \cdots \cap A_n^c \cap A_{n+1}^c \end{aligned} \tag{3}$$

Which proves the statement.

2. We will prove it for both sides, first we can look at the following logic and apply it to the proof: *If $x \in A$ implies that $x \in B$, then $B \subseteq A$. On the other hand, if $x \in B$ implies that $x \in A$, then $B \supseteq A$.* To put it in words, if x is part of A means that it is *always true* that $x \in B$, then B is contained in A, but in the other way, x being in B does not imply that x is in A. To prove equivalence, both statements should be true, this is, $B \subseteq A$ and $B \supseteq A$.

($\subseteq$) If $x \in \bigcap_{i=1}^{\infty} A_i^c$, then $x \in A_i^c$ for all $i \in \mathbb{N}$. This implies that $x \notin A_i$ for all $i \in \mathbb{N}$, thus $x \notin \bigcup_{i=1}^{\infty} A_i$, which is the same as $x \in (\bigcup_{i=1}^{\infty} A_i)^c$. So we have

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c \tag{4}$$

($\supseteq$) If $x \in (\bigcup_{i=1}^{\infty} A_i)^c$, then $x \notin A_i$ for all $i \in \mathbb{N}$. This is the same as $x \in A_i^c$ for all $i \in \mathbb{N}$. Thus $x \in \bigcap_{i=1}^{\infty} A_i^c$. We conclude that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c \tag{5}$$