

# Linear spaces

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# Chapter 1

## Introduction

### 1.1 Definition of a linear space

Let  $V$  be a non-empty set of objects, called *elements*. The set  $V$  is called linear space if it satisfies the following ten axioms, which are stated in three groups.

#### Axioms of a linear space

Axiom 1. *Closure property of addition*: for every pair of elements  $x, y \in V$ , their sum is written as  $z = x + y$  and  $z \in V$ .

Axiom 2. *Closure property of scalar product*: for any  $x \in V$  and  $a \in \mathbb{R}$ , there is an element  $z = ax \in V$ .

#### Axioms for addition

There are four axioms of addition, we will use a number and a letter to refer to them. However if we are talking of addition properties we will simply use a letter to reference them. The same will be done for the axioms of scalar products.

Axiom 3.a. *Commutative law*: for any  $x, y \in V$ , we have  $x + y = y + x$ .

Axiom 4.b. *Associative law*: for any  $x, y, z \in V$ , we have  $(x+y)+z = x+(y+z)$ .

Axiom 5.c. *Existence of zero as an element*: there is a number in  $V$ , designated as  $O$  (big 'o'), that satisfies

$$x + O = x, \quad \forall x \in V$$

Axiom 6.d. *Opposite elements*: for all  $x \in V$ , the element  $(-1)x$  has the property

$$x + (-1)x = O$$

## Axioms for scalar product

Axiom 7.a. *Associative law*: for all  $x \in V$  and every pair  $a, b \in \mathbb{R}$ , we have

$$a(bx) = (ab)x$$

Axiom 8.b. *Distributive law for addition in  $V$* : for all  $x, y \in V$  and  $a \in \mathbb{R}$ , it is true that

$$a(x + y) = ax + ay$$

Axiom 9.c. *Distributive law for addition in  $\mathbb{R}$* : for any  $x \in V$  and  $a, b \in \mathbb{R}$ , we have

$$(a + b)x = ax + bx$$

Axiom 10.d. *Existence of an identical element*: for all  $x \in V$  there's a unique element  $I$  such that  $Ix = x$  (commonly this element is 1. But, for example, the identical element in matrix spaces is called *identity matrix*, defined as  $I = \text{diag}(1)$ )

## 1.2 Examples of linear spaces

The following examples can be proven to be linear spaces

1. Real numbers
2. The vector space of real numbers  $\mathbb{R}^n$
3. The set of all matrices
4. Polynomials  $P$  with  $\deg P \leq n$  (in this case, if  $\deg P = n$ , we would have a problem with axioms of additions. We can't ensure the sum of two polynomials of degree  $n$  has degree  $n$ ).
5. The set of all polynomials
6. The set of continuous functions on an interval  $[a, b]$ . This space is designated as  $C(a, b)$ .
7. The set of all integrable functions on an interval
8. The set of differentiable functions on an interval
9. A plane in  $\mathbb{R}^3$  with the equation  $ax + by + cz = 0$ . Note that this plane must always go through the origin to be a linear space.

There are plenty of examples for linear spaces. We can “create” a linear space if we define addition and multiplication for that space.

### 1.3 Consequences of the axioms

The following theorems are a consequence of the axioms of linear space.

**Theorem 1.1** (Uniqueness of ‘O’). *In any linear space there is one and only one zero element*

*Proof.* Axiom 5 ensures that there is at least one ‘O’ in  $V$ . Now, suppose there are two zeroes in  $V$ . Let  $x = O_1$  and  $O_2 = O$ , thus  $x + O = x + O_2 = x = O_1$ , but as  $O_1$  is zero,  $O_1 + O_2 = O_2$ , this means that  $O_1 = O_2 = O$   $\square$

**Theorem 1.2** (Uniqueness of opposites). *In any linear space each  $x$  has one and only one opposite  $y$  such that  $x + y = O$*

*Proof.* Axiom 6 ensures there is at least one opposite of  $x$  in  $V$ . Let  $y_1, y_2 \in V$  be two different opposite elements for  $x$ . Then  $x + y_1 = O$  and  $x + y_2 = O$ , then

$$(x + y_1) + y_2 = y_2 + O = y_2$$

and

$$y_1 + (x + y_2) = y_1 + O = y_1$$

Thus  $y_1 + (x + y_2) = y_1 + (x + y_1) = y_1 + O = O + y_1$ , this proves that  $y_1 = y_2$ .  $\square$

### 1.4 Subspaces of a linear space

Let  $V$  be a linear space and let  $S$  be a subset of  $V$ , if  $S$  is also a linear space, then we say that “ $S$  is a subspace of  $V$ ”.

A subset of a linear space is a subspace only if it satisfies the axioms of closure.

**Theorem 1.3.** *Let  $V$  be a linear space, if  $S \subset V$  and  $S \neq \emptyset$  satisfies the ten axioms of closure then  $S$  is a subspace of  $V$ .*

The proof for this theorem is easy, and so I discarded it.

**Definition 1.1.** *Let  $S \subset V$ , and  $S \neq \emptyset$ , where  $V$  is a linear space. If  $x \in V$  and*

$$x = \sum_{i=1}^k c_i x_i$$

*where  $x_1, x_2, \dots, x_k \in S$  and  $c_1, c_2, \dots, c_k \in \mathbb{R}$ , is called a linear combination of elements in  $S$ . The set of linear combinations of the elements of  $S$  satisfies the axioms of closure, so it is also a subspace of  $V$ . We say that this subspace is generated by  $S$  and we call it the linear span of  $S$ , designated by  $L(S)$ . If  $S = \emptyset$ , we define  $L(S) = \{O\}$ .*

## 1.5 Dependent and independent subsets of a linear space

In this section we introduce the concept of independence, that is important when working with systems of linear equations, matrices, and other subjects in linear algebra.

**Definition 1.2.** Let  $S$  be a set of elements of a linear space  $V$ .  $S$  is dependent if there exists a finite set of distinct elements in  $x_1, x_2, \dots, x_k \in S$ , and a set of scalars  $c_1, c_2, \dots, c_k$  where not all of them are zero, that satisfies

$$\sum_{i=1}^k c_i x_i = 0$$

A set is independent if it is not dependent. So the following

$$\sum_{i=1}^k c_i x_i = 0, \quad \text{implies } c_1 = c_2 = \dots = c_k = 0$$

Independency and dependency are properties of sets of elements. However, we can apply the same concepts to the elements itself. For example, a set of vectors  $v_1, v_2, \dots, v_n \in \mathbb{R}^n$  is called independent if there is **not** a linear combination of these vectors that produce the zero vector.

**Example 1.1.** Let  $u_k(t) = t^k$  for  $k = 1, 2, \dots, n$  and  $t \in \mathbb{R}$ . The set  $V = \{u_1, u_2, \dots, u_n\}$  is independent except in the subset  $S$  where  $t = -1$  and  $n$  is odd.

*Proof.* For  $S$  to be independent, there must be  $c_1, c_2, \dots, c_n$ , where  $c_1 = c_2 = \dots = c_n = 0$  and

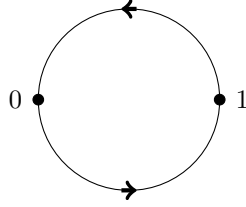
$$\sum_{k=0}^n c_k t^k = 0$$

To solve this, we set  $c_0 = c_1 = \dots = c_n$ . If we define

$$f(t) = \sum_{k=0}^n c_k t^k,$$

note that  $f(-1) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$  We can draw a picture for this prob-

lem. Imagine a circle, in which we have two points. We can travel the circumference counterclockwise starting from 1. We start from 1 because in the case  $n = 0$ ,  $f(-1) = 1$ .



Now start performing counterclockwise turns, and count how many times you go from 0 to 1, and from 1 to 0.

If  $C_{0 \rightarrow 1}$  and  $C_{1 \rightarrow 0}$  are the counts of going from 0 to 1 and from 1 to 0 respectively, we have that if  $C_{0 \rightarrow 1} = C_{1 \rightarrow 0}$ , then  $f(-1) = 1$ . Otherwise, we must have  $f(-1) = 0$ .

But having  $C_{0 \rightarrow 1} = C_{1 \rightarrow 0}$  and that the total count is  $C = C_{0 \rightarrow 1} + C_{1 \rightarrow 0}$ , means

$$C = 2C_{1 \rightarrow 0} = 2C_{0 \rightarrow 1}$$

Hence,  $C$  is an even number. To verify that  $S$  is dependent, we set  $n = 2r - 1$  for any integer  $r$  and  $t = -1$ . With the results above, we can see that

$$\sum_{k=0}^{2r-1} c_k (-1)^k = 0$$

if  $c_1 = c_2 = \dots = c_{2r-1}$ , but not necessarily zero. This proves that  $S$  is dependent.  $\square$

**Theorem 1.4.** *Let  $S = \{x_1, x_2, \dots, x_k\}$  an independent set formed by  $k$  elements of a linear space  $V$  and let  $L(S)$  be the linear span of  $S$ . Then, any set of  $k + 1$  elements from  $L(S)$  is dependent.*

What this theorem says is that, taking any set of vectors in the linear span of  $S$ , this is, formed by combining elements of  $S$  (this vectors can be of any nature), then, if we form a subset of the linear span of  $S$  and it has more elements than  $S$  itself, the set will be dependent. This is because we are not providing any new “dimension” to the new set. Say  $S \in \mathbb{R}^{\neq}$ , as we are restricted to be in  $\mathbb{R}^{\neq}$ , taking 4 vectors won’t make any object in  $\mathbb{R}^{i>3}$

*Proof.* Let  $T = \{y_1, y_2, \dots, y_{n+1}\} \subset L(S)$ , this means that each  $y_i$  is a linear combination of elements in  $S$

$$y_i = \sum_{j=1}^n a_{ij} x_j, \quad \text{for } i = 1, 2, \dots, n + 1$$

For  $T$  to be dependent, there must be some scalar set  $C = \{c_1, c_2, \dots, c_{n+1}\}$ , where not all of them are zero, that satisfies

$$\sum_{i=1}^{n+1} c_i y_i = 0$$

We now want to prove by induction that for  $n - 1$  elements of  $T$ , there is a linear combination that satisfies dependency. Thus, we can try to form an equation that represents  $T$  as a linear combination of  $n - 1$  elements. For this, we are going to take one element of  $T$ , multiply it by some scalar and subtract each element of  $T$ .

Take the  $1^{st}$  element in  $T$  and multiply it by  $c_i = \frac{a_{i1}}{a_{11}}$

$$c_i y_1 = a_{i1} x_1 + \sum_{j=2}^n c_i a_{1j} x_j$$

Now subtract  $y_1$

$$\begin{aligned} c_i y_1 - y_i &= a_{i1} x_1 + \sum_{j=2}^n c_i a_{1j} x_j - a_{i1} x_1 - \sum_{j=2}^n a_{ij} x_j \\ &= \sum_{j=2}^n c_i a_{1j} x_j - a_{ij} x_j \\ &= \sum_{j=2}^n (c_i a_{1j} - a_{ij}) x_j \end{aligned} \tag{1.1}$$

Equation (1.1) is indeed a linear combination of  $n - 1$  elements of  $S$ . By induction for  $n$ , we can prove that there are  $n$  scalars  $t_2, t_3, \dots, t_{n+1}$ , that satisfy

$$\sum_{j=2}^{n+1} t_j (c_j y_1 - y_j) = 0 \tag{1.2}$$

As each  $y_i$  is a linear combination of elements of  $S$ , we can write  $y_i$  in terms of  $y_1$ .

Equation (1.2) is solvable, because  $y_i = c_i y_1$ , this is true by the fact that  $T \subset L(S)$ .  $\square$

## 1.6 Basis and dimension