Linear spaces

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Chapter 1

Introduction

1.1 Definition of a linear space

Let V be a non-empty set of objects, called *elements*. The set V is called linear space if it satisfies the following ten axioms, which are stated in three groups.

Axioms of a linear space

Axiom 1. *Closure property of addition*: for every pair of elements $x, y \in V$, their sum is written as $z = x + y$ and $z \in V$.

Axiom 2. *Closure property of scalar product*: for any $x \in V$ and $a \in \mathbb{R}$, there is an element $z = ax \in V$.

Axioms for addition

There are four axioms of addition, we will use a number and a letter to refer to them. However if we are talking of addition properties we will simply use a letter to reference them. The same will be done for the axioms of scalar products.

Axiom 3.a. *Commutative law*: for any $x, y \in V$, we have $x + y = y + x$.

Axiom 4.b. *Associative law*: for any $x, y, z \in V$, we have $(x+y)+z = x+(y+z)$.

Axiom 5.c. *Existence of zero as an element*: there is a number in V, designated as O (big 'o'), that satisfies

$$x + O = x, \quad \forall x \in V$$

Axiom 6.d. *Opposite elements*: for all $x \in V$, the element $(-1)x$ has the property

$$x + (-1)x = O$$

Axioms for scalar product

Axiom 7.a. *Associative law*: for all $x \in V$ and every pair $a, b \in \mathbb{R}$, we have

$$a(bx) = (ab)x$$

Axiom 8.b. *Distributive law for addition in V*: for all $x, y \in V$ and $a \in \mathbb{R}$, it is true that

$$a(x + y) = ax + ay$$

Axiom 9.c. *Distributive law for addition in $\mathbb{R}$*: for any $x \in V$ and $a, b \in \mathbb{R}$, we have

$$(a + b)x = ax + bx$$

Axiom 10.d. *Existence of an identical element*: for all $x \in V$ theres an unique element I such that $Ix = x$ (commonly this element is 1. But, for example, the identical element in matrix spaces is called *identity matrix*, defined as $I = \mathrm{diag}(1)$)

1.2 Examples of linear spaces

The following examples can be proven to be linear spaces

1. Real numbers
2. The vector space of real numbers $\mathbb{R}^n$
3. The set of all matrices
4. Polynomials P with $\deg P \leq n$ (in this case, if $\deg P = n$, we would have a problem with axioms of additions. We can't ensure the sum of two polynomials of degree n has degree n).
5. The set of all polynomials
6. The set of continuous functions on an interval $[a, b]$. This space is designated as $C(a, b)$.
7. The set of all integrable functions on an interval
8. The set of differentiable functions on an interval
9. A plane in $\mathbb{R}^3$ with the equation $ax + by + cz = 0$. Note that this plane must always go through the origin to be a linear space.

There are plenty of examples for linear spaces. We can "create" a linear space if we define addition and multiplication for that space.

1.3 Consequences of the axioms

The following theorems are a consequence of the axioms of linear space.

Theorem 1.1 (Uniqueness of 'O'). *In any linear space there is one and only one zero element*

Proof. Axiom 5 ensures that there is at least one 'O' in V. Now, suppose there are two zeroes in V. Let $x = O_1$ and $O_2 = O$, thus $x + O = x + O_2 = x = O_1$, but as O_1 is zero, $O_1 + O_2 = O_2$, this means that $O_1 = O_2 = O$ □

Theorem 1.2 (Uniqueness of opposites). *In any linear space each x has one and only one opposite y such that $x + y = O$*

Proof. Axiom 6 ensures there is at least one opposite of x in V. Let $y_1, y_2 \in V$ be two different opposite elements for x. Then $x+y_1 = O$ and $x+y_2 = O$, then

$$(x + y_1) + y_2 = y_2 + O = y_2$$

and

$$y_1 + (x + y_2) = y_1 + O = y_1$$

Thus $y_1 + (x + y_2) = y_1 + (x + y_1) = y_1 + O = O + y_1$, this proves that $y_1 = y_2$. □

1.4 Subspaces of a linear space

Let V be a linear space and let S be a subset of V, if S is also a linear space, then we say that "S is a subspace of V".

A subset of a linear space if a subspace only if it satisfies the axioms of closure.

Theorem 1.3. *Let V be a linear space, if $S \subset V$ and $S \neq \emptyset$ satisfies the ten axioms of closure then S is a subspace of V.*

The proof for this theorem is easy, and so I discarded it.

Definition 1.1. *Let $S \subset V$, and $S \neq \emptyset$, where V is a linear space. If $x \in V$ and*

$$x = \sum_{i=1}^{k} c_i x_i$$

where $x_1, x_2, \ldots, x_k \in S$ and $c_1, c_2, \ldots, c_k \in \mathbb{R}$, is called a linear combination of elements in S. The set of linear combinations of the elements of S satisfies the axioms of closure, so it is also a subspace of V. We say that this subspace is generated by S and we call it the linear span of S, designated by $L(S)$. If $S = \emptyset$, we define $L(S) = \{O\}$.

1.5 Dependent and independent subsets of a linear space

In this section we introduce the concept of independence, that is important when working with systems of linear equations, matrices, and other subjects in linear algebra.

Definition 1.2. *Let S be a set of elements of a linear space V. S is dependent if there exists a finite set of distinct elements in $x_1, x_2, \ldots, x_k \in S$, and a set of scalars $c_1, c_2, \ldots, c_k$ where not all of them are zero, that satisfies*

$$\sum_{i=1}^{k} c_i x_i = 0$$

A set is independent if it is not dependent. So the following

$$\sum_{i=1}^{k} c_i x_i = 0, \quad \textit{implies } c_1 = c_2 = \cdots = c_k = 0$$

Independency and dependency are properties of sets of elements. However, we can apply the same concepts to the elements itself. For example, a set of vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ is called independent if there is **not** a linear combination of these vectors that produce the zero vector.

Example 1.1. *Let $u_k(t) = t^k$ for $k = 1, 2, \ldots, n$ and $t \in \mathbb{R}$. The set $V = \{u_1, u_2, \ldots, u_n\}$ is independent except in the subset S where $t = -1$ and n is odd.*

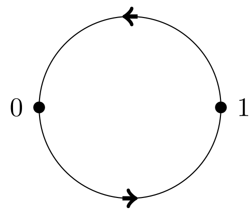
Proof. For S to be independent, there must be $c_1, c_2, \ldots, c_n$, where $c_1 = c_2 = \cdots = c_n = 0$ and

$$\sum_{k=0}^{n} c_k t^k = 0$$

To solve this, we set $c_0 = c_1 = \cdots = c_n$. If we define

$$f(t) = \sum_{k=0}^{n} c_k t^k,$$

note that $f(-1) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ We can draw a picture for this problem. Imagine a circle, in which we have two points. We can travel the circumference counterclockwise starting from 1. We start from 1 because in the case $n = 0$, $f(-1) = 1$.



Now start performing counterclockwise turns, and count how many times you go from 0 to 1, and from 1 to 0.

If $C_{0\to 1}$ and $C_{1\to 0}$ are the counts of going from 0 to 1 and from 1 to 0 respectively, we have that if $C_{0\to 1} = C_{1\to 0}$, then $f(-1) = 1$. Otherwise, we must have $f(-1) = 0$.

But having $C_{0\to 1} = C_{1\to 0}$ and that the total count is $C = C_{0\to 1} + C_{1\to 0}$, means

$$C = 2C_{1\to 0} = 2C_{0\to 1}$$

Hence, C is an even number. To verify that S is dependent, we set $n = 2r-1$ for any integer r and $t = -1$. With the results above, we can see that

$$\sum_{k=0}^{2r-1} c_k(-1)^k = 0$$

if $c_1 = c_2 = \cdots = c_{2r-1}$, but not necessarily zero. This proves that S is dependent. $\square$

Theorem 1.4. *Let $S = \{x_1, x_2, \ldots, x_k\}$ an independent set formed by k elements of a linear space V and let $L(S)$ be the linear span of S. Then, any set of $k+1$ elements from $L(S)$ is dependent.*

What this theorem says is that, taking any set of vectors in the linear span of S, this is, formed by combining elements of S (this vectors can be of any nature), then, if we form a subset of the linear span of S and it has more elements than S itself, the set will be dependent. This is because we are not providing any new "dimension" to the new set. Say $S \in \mathbb{R}^{\nvDash}$, as we are restricted to be in $\mathbb{R}^{\nvDash}$, taking 4 vectors won't make any object in $R^{i>3}$

Proof. Let $T = \{y_1, y_2, \ldots, y_{n+1}\} \subset L(S)$, this means that each y_i is a linear combination of elements in S

$$y_i = \sum_{j=1}^{n} a_{ij}x_j, \quad \text{for } i = 1, 2, \ldots, n+1$$

For T to be dependent, there must be some scalar set $C = \{c_1, c_2, \ldots, c_{n+1}\}$, where not all of them are zero, that satisfies

$$\sum_{i=1}^{n+1} c_i y_i = 0$$

We now want to prove by induction that for $n-1$ elements of T, there is a linear combination that satisfies dependency. Thus, we can try to form an equation that represents T as a linear combination of $n-1$ elements. For this, we are going to take one element of T, multiply it by some scalar and subtract each element of T.

Take the 1^{st} element in T and multiply it by $c_i = \frac{a_{i1}}{a_{11}}$

$$c_i y_1 = a_{i1}x_1 + \sum_{j=2}^{n} c_i a_{1j} x_j$$

Now subtract y_1

$$\begin{aligned} c_i y_1 - y_i = a_{i1}x_1 + \sum_{j=2}^{n} c_i a_{1j} x_j - a_{i1}x1 + \sum_{j=2}^{n} a_{ij} x_j \\ = \sum_{j=2}^{n} c_i a_{1j} x_j - a_{ij} x_j \qquad (1.1) \\ = \sum_{j=2}^{n} (c_i a_{1j} - a_{ij})\, x_j \end{aligned}$$

Equation (1.1) is indeed a linear combination of $n-1$ elements of S. By induction for n, we can prove that there are n scalars $t_2, t_3, \ldots, t_{n+1}$, that satisfy

$$\sum_{j=2}^{n+1} t_i \left(c_i y_1 - y_i\right) = 0 \qquad (1.2)$$

As each y_i is a linear combination of elements of S, we can write y_i in terms of y_1.

Equation (1.2) is solvable, because $y_i = c_i y_1$, this is true by the fact that $T \subset L(S)$. □

1.6 Basis and dimension

Definition 1.3. *A finite set S of elements of a linear space V is called a finite basis of V is S is independent and spans V. V is of finite dimension if it has a finite basis. Otherwise, V has infinite dimension.*

Theorem 1.5. *Let V be a linear space of finite dimension. Then any finite basis of V has the same number of elements.*

Proof. This theorem can be proved with theorem 1.4, let S and T be two finite bases for V, with k and m elements respectively. If S generates V, then V must have k elements, we know that any set of $k+1$ elements of V is dependent. Thus, T must have $m \geq k$ elements. Applying the same reasoning vice-versa yields that $k = m$. □

This does not mean that a set of $k+1$ elements of V can't span V. It states that, the number of elements for a finite basis of a linear space V of dimension k, must have the same number of elements.

Definition 1.4. *If a linear space V has a finite basis of n elements, we write $n = \dim V$.*

The following theorem will not be proven. However, it has an intuitive explanation.

Theorem 1.6. *Let V be a linear space of finite dimension, with $\dim V = n$. Then*

1. *If S is a finite basis for V, and T is a set of independent elements of V, then $T \subseteq S$.*
2. *Any set of n independent elements of V is a finite basis for V.*

1.6.1 Components

Let V be a linear space with $\dim V = n$, and consider an ordered basis $\{e_1, e_2, \ldots, e_n\}$. This ordered basis is considered as an n-tuple $(e_1, e_2, \ldots, e_n)$.

Definition 1.5. *An ordered basis of a linear space V is a set of elements of V that form a basis and provides information about the order of its elements.*

If $x \in V$, we can express x as a linear combination of elements of the basis

$$x = \sum_{i=1}^{n} c_i e_i \tag{1.3}$$

This ensures that there is only one representation of x, take $x = \sum_{i=1}^{n} c_i e_i$, and $x = \sum_{i=1}^{n} d_i e_i$. Then

$$\sum_{i=1}^{n} c_i e_i = \sum_{i=1}^{n} d_i e_i$$

Then $\sum_{i=1}^{n} (c_i - d_i) e_i = O$, where O is the zero vector/element of V. This means that $c_i = d_i$ for $i = 1, 2, \ldots, n$. So there is only one representation of x in V.

Chapter 2

Euclid spaces, inner products and norms

An euclid space is a linear space with an ordered basis, that satisfies euclid's axioms.

Euclid spaces satisfy the transitive property

$$\text{if } a = b \text{ and } b = c, \text{ then } a = c$$

The sum equality property

$$\text{if } a = b \text{ then } a + c = b + c$$

The sum equality property also holds if we replace $c \to (-1)c$.

The reflective property, which states that two line segments of the same size and direction are the same.

The whole is bigger than the part. Let A be a set and $B \subset A$, then

$$\text{card} > \text{card}B$$

Euclid spaces' satisfy euclidean geometry. They are metric spaces, this means that they have a distance function associated to them.

They can be defined as finite dimensional vector spaces, with inner product, that satisfy the axioms of a linear space (multiplication by scalars and addition of elements).

We define the inner product of vectors to be: *for two vectors $x, y \in \mathbb{R}^{\ltimes}$, the inner product (x, y) is defined as*

$$(x, y) = \sum_{i=1}^{n} x_i y_i \tag{2.1}$$