Proof of existence of square roots

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Let T be a subset of the real numbers given by

$$T = \left\{ t \in \mathbb{R} : t^2 < 2 \right\}$$

We want to prove that the supremum of this set, which should be irrational, exists. We define $\alpha = \sup T$. We want to get that $\alpha^2 = 2$ by ruling out the posibilities of $\alpha^2 < 2$ and $\alpha^2 > 2$

First, imagine a line of real numbers, we can try to push the supremum by a factor of $1/n$

$$\begin{aligned} \left(\alpha + \frac{1}{n} \right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n} \end{aligned}$$

We now assume that there's some value of n such that

$$\alpha^2 + \frac{2\alpha + 1}{n} < 2$$

After a rearangement we get

$$\frac{2\alpha + 1}{n} > 2 - \alpha^2$$

Now, we can get back to the first inequality. Notice that

$$\left(\alpha + \frac{1}{n} \right)^2 < \alpha^2 + (2 - \alpha^2) = 2$$

This contradicts the fact that $\alpha = \sup T$. Thus, it means that we still can "move towards the right" after α and get a number in T.

Now, we look to a contradiction for $\alpha^2 > 2$. Assuming this is true, we could subtract some amount to α and get a number in T.

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}\end{aligned}$$

Now, analogous to the first part of the Proof

$$\alpha^2 - \frac{2\alpha}{n} > 2$$

Rearranging the inequality we get $2 - \alpha^2 < \frac{-2\alpha}{n}$, this means that

$$\left(\alpha - \frac{1}{n}\right)^2 > \alpha^2 + (2 - \alpha^2) = 2$$

This means that $\alpha^2 > 2$ is false.

And because both $\alpha^2 < 2$ and $\alpha^2 > 2$ are false, this means that $\alpha^2 = 2$, which proves the existence of the square roots in $\mathbb{R}$.