Consequences of the Axiom of Completeness

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1 Introduction

The Axiom of Completeness is an assumption made about real numbers, we know that the numbering system is composed of natural, integers and rational numbers. We define the last set to contain both naturals and integers and write $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$. However, a Pythagorean cult member called Hippasus of Metapontum discovered an important fact about numbers.

2 $\sqrt{2}$ is irrational, a hole in $\mathbb{Q}$

Our initial assumption is that $\sqrt{2}$ can be written in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and both have no common factor

$$\left(\frac{p}{q}\right)^2 = 2$$

This implies

$$p^2 = 2q^2$$

From this, we can see that p^2 is even, this means that p is even too (the square of an odd number is odd). We can write p as

$$p = 2r, \qquad r \in \mathbb{Z}$$

Substitute this in the equation above and get

$$q^2 = 2r^2$$

Thus, this implies that q is also even, this is absurd because we have initially assumed them to have no common factor. But as they have, we can rewritte the fraction in lower terms, however, this process would be infinite. Thus $\sqrt{2}$ is not a rational number.

The existence of this proof demonstrates that there is a *hole* in the rational numbers. This set does not contain numbers such as $\sqrt{2}, \sqrt{3}, \pi$, etc. This problem lead mathematicians to build a set that includes irrational numbers. This set is known as the real numbers.

3 Proof: $\sqrt{3}$ is irrational

To prove this, we will first prove the following statement

Theorem 1. *If the square of a number is divisible by a prime, then the number itself is divisible by that prime*

Proof. Let n be any number. Then n has at least a prime factor p, then

$$n^2 = pk \qquad k \in \mathbb{Z}$$

for some k. Now, we know that we can decompose n into prime factors, let $p_1, p_2, \ldots, p_n$ be the prime factors of n, such that

$$n = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$$

Thus $n^2 = p_1^{2e_1} p_2^{2e_2} \cdots p_n^{2e_n}$, from this, we see that it is divisible by p, since $p \in \{p_1, p_2, \ldots, p_n\}$. $\square$

Now, we have to prove that $\sqrt{3}$. Analogous to the proof for $\sqrt{2}$, we let $p, q \in \mathbb{Z}$ be two numbers without any common factor. We try to prove the statement by contradiction, we set

$$\left(\frac{p}{q}\right)^2 = 3$$

Then $p^2 = 3q^2$, but by the theorem 1, we have that if p^2 is divisible by 3, then p must. So $p = 3r$, then

$$3r^2 = q^2$$

This means that q is also divisible by 3. Then p and q have a common prime factor, this violates the initial constraint for p and q to not have any common factor.