Discrete Random Variables

A random variable is a measurement taken on a defined sample space, or in other words, a random variable associates a real number to all the outcomes of an experiment, e.g. a coin is tossed five times, the sample space consists of all the possible outcomes (heads or tails), a random variable for this experiment can be *the number of tails* in each toss.

Next, all discrete random variables are “distributed” in some way. This distribution is a function called **Probability Mass Function**, and it assigns a probability to each possible value of the random variable. We can see the random variable X (it is denoted with capital letter) as a set of real numbers x (imagine the first example of the coin, then this x is an integer representing the amount of tails that is possible to get, ranging from 0 to 5).

A **PMF** for X is a function $f : X \to \mathbb{R}$, this function sets the probability of each x to a number $n \in [0, 1]$.

To summarize

$$\text{A discrete random variable is a real valued function of the outcome of an experiment, that can take a finite or countably infinite number of values} \tag{1}$$

$$\text{A discrete random variable is associated with a Probability Mass Function (PMF), which gives the probability of each value the random variable takes} \tag{2}$$

$$\text{A function of a random variable gives another random variable whose PMF can be obtained from the PMF of the original random variable} \tag{3}$$

To formally define (1), we say that we have some **experiment** E that has a **sample space** Ω. Here Ω is an **object that contains all the possible outcomes of the experiment**. Ω can be of any nature, for example: we could have a set of tuples consisting of the outcome of throwing two dices, then the sample space is

$$\Omega = \{(x, y) : x, y \in \mathbb{N} \wedge 1 \le x \le 6, 1 \le y \le 6\}$$

or using the Cartesian product $\Omega = \{1, ..., 6\} \times \{1, ..., 6\}$, so this is essentially a discrete plane of points. Then a **random variable** can be $X = \text{"least number rolled"}$, so if we roll (x, y) and $x < y$ then $X = x$, or if $y < x$ then $X = y$.

Now consider a radar that detects planes, this radar has two events that can happen, expressed in set notation, the **sample space** of the radar is

$$\Omega_{radar} = \{\text{"radar succeed"}\}$$
$$\neg\Omega_{radar} = \{\text{"radar failed"}\}$$

then we have that a plane *could fly* or *could not fly* above the radar, so for the plane we have

$$\Omega_{plane} = \{\text{"flew above radar"}\}$$
$$\neg\Omega_{plane} = \{\text{"didn't flew above radar"}\}$$

So, the sample space is $\Omega = \Omega_{plane} \times \Omega_{radar}$, A **random variable** for this event could be the number of times that the plane "flew above radar" but "radar failed", so

$$X = \text{"missing reads of the radar"}$$

Probability Mass Functions

As we defined earlier, a PMF is a function which assigns a probability to all the values that the random variable takes.

The following notation is useful for defining random variables which have a PMF

$$X \sim f_X$$

This means that X is a random variable with Probability Mass Function f_X.

We will now give some examples of $PMFs$

Binomial Distribution

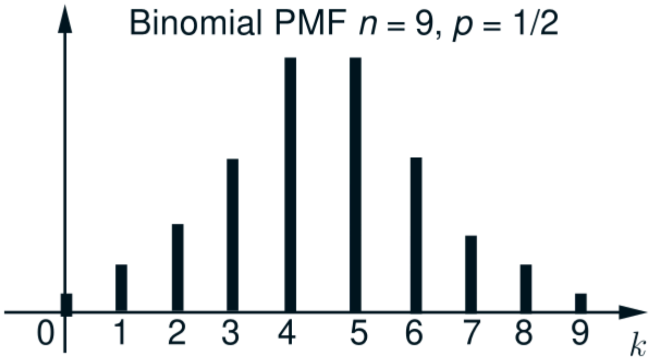
If a six-sided dice is rolled five times, the probability of rolling a 6 is $p = \frac{1}{6}$, otherwise, the probability of not rolling it is $q = 1 - p = \frac{5}{6}$. Let X be the number of heads in the n-toss sequence (instead of tossing five times, we will toss n times), then we refer to X as a **Binomial Random Variable** with parameters n and p, so

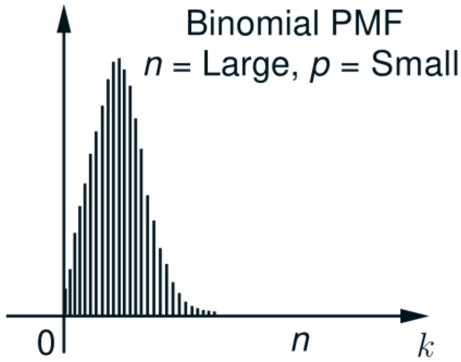
$$X \sim Bin(n, p)$$

The PMF is given by

$$Bin(n, p) : f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \forall x \in \mathbb{Z}^+$$

Here $P(X = x)$ denotes the "probability of the random variable X to take the value x".

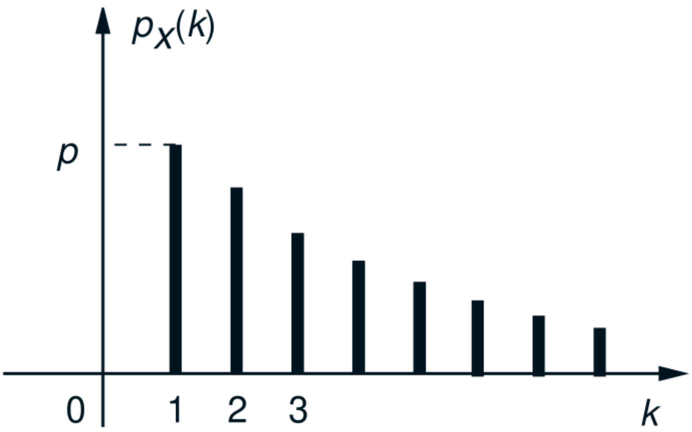




Geometric Distribution

Consider an event that can occur with a probability of $0 < p < 1$. The geometric distribution tells us, given an initial probability p, how likely is the event to happen consecutively. This means that, if you for example toss a coin with $p = 0.5$ to roll tails, then the second probability must be 0.25, and so on. The geometric PMF is given by

$$G(p) : f_X(x) = P(X = x) = (1-p)^{x-1}p \qquad \forall x \in \mathbb{Z}^+$$



Poisson Distribution

Consider the case of a **Binomial Random Variable** which has a very big n (number of samples/experiments) but a very small p. The Poisson distribution serves as an approximation to the Binomial distribution when this happens.

$$P(\lambda) : f_X(x) = P(X = x) = \frac{\lambda^x}{x!}e^{-\lambda} \qquad x \in \mathbb{Z}^+$$

Here λ is a parameter given by $\lambda = np$.
An example of this distribution is, consider a book with
n words, which has probability p of a misspelled word, then $X \sim P(\lambda)$, this because the chance to find one misspelled word is very low.

We will stop here, since we have covered the specific concepts of discrete variables. We can extend the knowledge of conditional probabilities, independence, expectation, convolutions, etc. in a more general way in other PDF, since those do not only apply to discrete cases but also continuous cases.