Lagrange Multipliers

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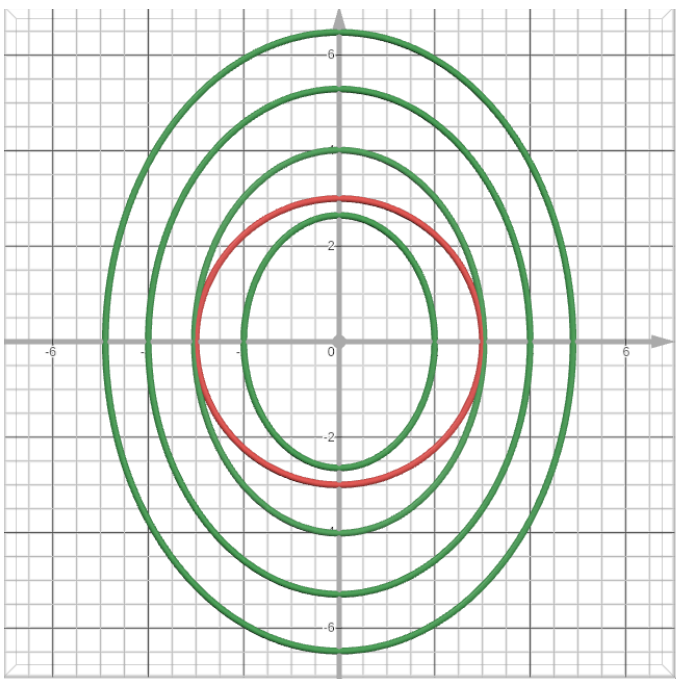
— February 24, 2025 —

Lagrange multipliers is a method which helps to find a maximum or minimum in a function f constrained by some function g in any dimension.

For example: find the maximum point on the parabola $f(x,y) = \frac{1}{4}x^2 + \frac{1}{7}y^2$ *constrained to be in the circle* $x^2 + y^2 = 9$.

If we see the level curves for this example, we can find out that in those points where the circle and the level curve intercept, these lines are parallel



The green curves correspond to $f(x,y)$, the red one is the level curve of $x^2 + y^2 = 9$

Our constraint here is $g(x,y) = x^2 + y^2 - 9$.

If the level curves are parallel, the gradient vectors must be also parallel, we will give an analytic proof of this.

Analytic proof

Let $w = f(x, y, z)$ be a function with a maximum at a point P on the constraint surface $g(x, y, z) = 0$. Now, suppose that $r(t) = \langle x(t), y(t), z(t) \rangle$ is an arbitrary parametric curve that lies entirely on the constraint surface, such that $r(0) = P$. Let $h(t) = f(r(t)) = f(x(t), y(t), z(t))$. Taking the derivative of h:

$$h'(t) = \nabla f_{|_{r(t)}} \cdot r'(t)$$

by the chain rule.

We know that on a constant point, $h'(t)$ must be zero because there's no change in w. Then, using this fact, we know:

$$h'(0) = \nabla f_{|_P} \cdot r'(0) = 0$$

This is, the gradient is perpendicular to the tangent vector $r'(0)$. Because $r(t)$ is an arbitrary curve in the surface, then ∇f is perpendicular to any tangent vector to the surface at P. Thus, ∇f is perpendicular to the constraint surface at P.

Now, applying the same reasoning to ∇g, we get $\nabla g_{|_P} \cdot r'(c) = 0$, so ∇g is normal to the surface at P. this means that ∇f is parallel to ∇g, because they are both perpendicular to the constraint surface.

Two cases

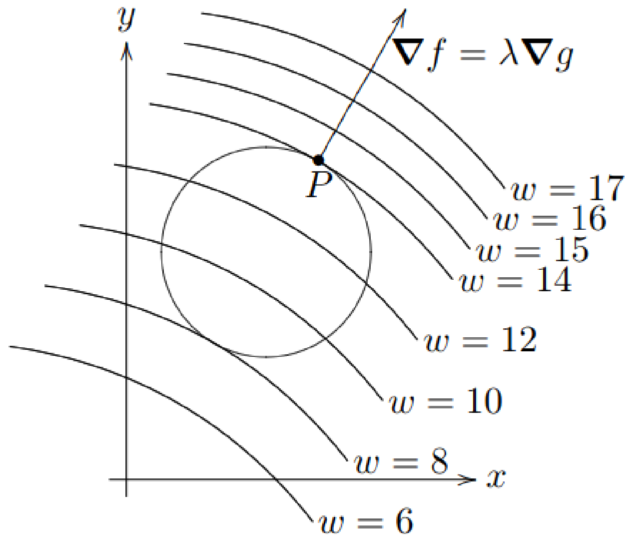
There are two cases, the gradients of f and g can be parallel in the same direction or in different directions.

Lagrange multipliers

The method of Lagrange multipliers consists in: having a function $f(x, y, z)$, find a critical point P constrained by some surface $g(x, y, z) = c$. Applying the reasoning we've proved, we need to find a point where both ∇f and ∇g are parallel, so:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

is the equation we need to solve.



A representation of the equation for some function in $\mathbb{R}^3$, constrained by a circle.

Suppose f and g are functions in $\mathbb{R}^n$. A system of equations can be made

$$\begin{cases} f_{x_1} = \lambda g_{x_1} \\ f_{x_2} = \lambda g_{x_2} \\ \dots \\ f_{x_n} = \lambda g_{x_n} \\ g = c \end{cases}$$

Solving the problem consists of solving this system of equations. A brief example from the book *"Calculus" by Gilbert Strang* says:

EXAMPLE 2 Maximize and minimize $f = x^2 + y^2$ on the ellipse $g = (x-1)^2 + 4y^2 = 4$.

Idea and equations The circles $x^2 + y^2 = c$ grow until they touch the ellipse. The touching point is (x_{min}, y_{min}) and that smallest value of c is f_{min}. As the circles grow they cut through the ellipse. Finally there is a point (x_{max}, y_{max}) where the last circle touches. That largest value of c is f_{max}.

The minimum and maximum are described by the same rule: the circle is tangent to the ellipse (Figure 13.21b). ***The perpendiculars go in the same direction***. Therefore (f_x, f_y) is a multiple of (g_x, g_y), and the unknown multiplier is λ:

$$\begin{aligned} f_x = \lambda g_x: &\qquad 2x = \lambda 2(x-1) \\ f_y = \lambda g_y: &\qquad 2y = \lambda 8y \\ g = k: &\qquad (x-1)^2 + 4y^2 = 4. \end{aligned} \tag{3}$$

To solve this, we first look at the second equation, setting $y = 0$ or $\lambda = 1/4$ solves for that. Then for the last equation we have, if $y = 0$, $(x-1)^2 = 4$, thus $x = 3$ or $x = -1$. Plugging that in the first equation we get $\lambda = 3/2$ or $\lambda = 1/2$. Then, solve for x and y with the different values of λ. This system has four solutions because we have four parameters to set.