Tail recursion
Functional Programming

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Outline

- Stacks
- Stacks in CPUs
  - Exercise 1
- Tail recursion
- Hand-in Tail recursion

Stacks

Stacks are LIFO queues

A stack is a **L**ast **I**n **F**irst **O**ut queue.

CPU and instructions

CPUs executes instructions sequentially

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So how is `x + y * z + u` possible?

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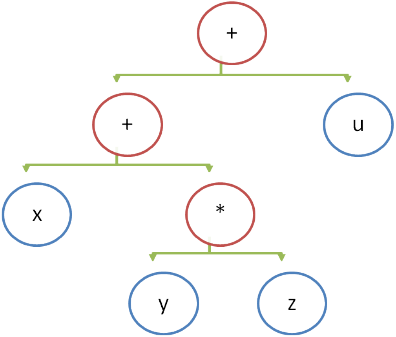

Call stack

Your computer *queues* instructions

Call stack

Your computer *queues* instructions in a **call stack**

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Typically blocks of code

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Typically blocks of code or **procedures / subroutines**

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Stack content

A `call stack` basically
consists of calls into subroutines

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A `call stack` basically consists of calls into subroutines

They are called `stack frames`

Frame content

What does the frame contain? What do we need to know?

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- Space for arguments and variables

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- Space for arguments and variables
- Return address

Frame content

What does the frame contain? What do we need to know?

- Space for arguments and variables
- Return address: Where do we go after we're done?

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Disclaimer: Implementation specific

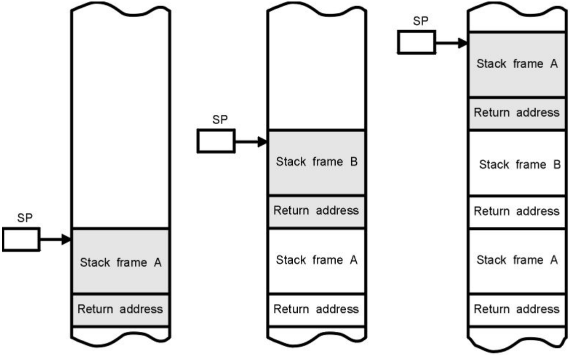
Stack frames

What happens every time we call a function?

Stack frames

What happens every time we call a function?

We create a new stack frame!



Recursion on the stack


```
public void count(int i) {
  if (i > 0) {
    return i + count(i - i);
  } else return i;
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`count(3)`

Recursion on the stack

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```
count(3)
  count(2)
```

Recursion on the stack

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```
count(3)
  count(2)
    count(1)
```

```
public void count(int i) {
  if (i > 0) {
    return i + count(i - i);
  } else return i;
}
```

```
count(3)
  count(2)
    count(1)
      count(0)
```

```java
public void count(int i) {
  if (i > 0) {
    return i + count(i - i);
  } else return i;
}
```

```
count(3)
  count(2)
    count(1)
      count(0)
    // 1 + 0
```

```java
public void count(int i) {
  if (i > 0) {
    return i + count(i - i);
  } else return i;
}
```

```
count(3)
  count(2)
    count(1)
      count(0)
    // 1 + 0
  // 2 + 1
```

Recursion on the stack

```
public void count(int i) {
  if (i > 0) {
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  } else return i;
}
```

```
count(3)
  count(2)
    count(1)
      count(0)
    // 1 + 0
  // 2 + 1
// 3 + 3
```

Small exercise

```java
public long fibonacci(int n) {
  if (n <= 1) return n;
  else return fibonacci(n-1) + fibonacci(n-2);
}
```

```
public long fibonacci(int n) {
  if (n <= 1) return n;
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Small exercise

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public long fibonacci(int n) {
  if (n <= 1) return n;
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```

1. How many stack frames do we create with `fibonacci(2)`?

Small exercise

```java
public long fibonacci(int n) {
    if (n <= 1) return n;
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}
```

1. How many stack frames do we create with `fibonacci(2)`?
2. How many stack frames do we create with `fibonacci(5)`?

Small exercise

```java
public long fibonacci(int n) {
  if (n <= 1) return n;
  else return fibonacci(n-1) + fibonacci(n-2);
}
```

1. How many stack frames do we create with `fibonacci(2)`?
2. How many stack frames do we create with `fibonacci(5)`?
3. How many stack frames do we create with `fibonacci(10)`?

```
public long fibonacci(int n) {
  if (n <= 1) return n;
  else return fibonacci(n-1) + fibonacci(n-2);
}
```

1. How many stack frames do we create with `fibonacci(2)`?
2. How many stack frames do we create with `fibonacci(5)`?
3. How many stack frames do we create with `fibonacci(10)`?
4. What is the general formula for how many stack frames the `fibonacci` function creates?

Exercise 1

In a new Java project:

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1. Implement a function for factorial using `BigInteger`
   ```
   public static BigInteger factorial(BigInteger i);
   ```

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   `-> VM Options` write ‘`-Xss20m`’

Exercise 1

In a new Java project:

1. Implement a function for factorial using `BigInteger`
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2. Run factorial with `100’000` as input. What happens?
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What is recursion

Think about factorial. What is the input and what is the output?

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Is there a pattern?

What is recursion

Think about factorial. What is the input and what is the output?

Think about fibonacci. What is the input and what is the output?

Is there a pattern?

Yes there is!

```java
public int count(int i) {
  if (i <= 0) return 0;
  else return count(i - 1)
}
```

Stack frame for counting

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public int count(int i) {
  if (i <= 0) return 0;
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}
```

What is in a stack frame?

Stack frame for counting

```java
public int count(int i) {
  if (i <= 0) return 0;
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}
```

What is in a stack frame?

$\boxed{i}$ $\boxed{i - 1}$ $\boxed{count(i - 1)}$

Simplifying stack frames

```java
public int count(int i) {
  if (i <= 0) return 0;
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}
```

```
public int count(int i) {
  if (i <= 0) return 0;
  else return count(i - 1)
}
```

call | i | i - 1 | count(i - 1)

```
public int count(int i) {
  if (i <= 0) return 0;
  else return count(i - 1)
}
```

call	$\boxed{\text{i}}$	$\boxed{\text{i - 1}}$	$\boxed{\text{count(i - 1)}}$
count(3)	$\boxed{3}$	$\boxed{2}$	$\boxed{\text{count(2)}}$

Simplifying stack frames

```
public int count(int i) {
  if (i <= 0) return 0;
  else return count(i - 1)
}
```

call	i	i - 1	count(i - 1)
`count(3)`	3	2	count(2)
`count(2)`	2	1	count(1)

Simplifying stack frames

```
public int count(int i) {
  if (i <= 0) return 0;
  else return count(i - 1)
}
```

call	i	i - 1	count(i - 1)
count(3)	3	2	count(2)
count(2)	2	1	count(1)
count(1)	1	0	count(0)

```
public int count(int i) {
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call	i	i - 1	count(i - 1)
count(3)	3	2	count(2)
count(2)	2	1	count(1)
count(1)	1	0	count(0)
count(0)	0	?	?

cphbusiness COPENHAGEN BUSINESS ACADEMY

```
public int count(int i) {
    if (i <= 0) return 0;
    else return count(i - 1)
}
```

call	i	i - 1	count(i - 1)
count(3)	3	2	count(2)
count(2)	2	1	count(1)
count(1)	1	0	count(0)
count(0)	0	?	?
count(1)	1	0	0

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public int count(int i) {
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count(0)	0	?	?
count(1)	1	0	0
count(2)	2	1	0

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public int count(int i) {
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call	i	i - 1	count(i - 1)
count(3)	3	2	count(2)
count(2)	2	1	count(1)
count(1)	1	0	count(0)
count(0)	0	?	?
count(1)	1	0	0
count(2)	2	1	0
count(3)	3	2	0

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public int count(int i) {
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```

call	i	i - 1	count(i - 1)
`count(3)`	3	2	count(2)

Simplifying stack frames

```java
public int count(int i) {
  if (i <= 0) return 0;
  else return count(i - 1)
}
```

call	i	i - 1	count(i - 1)
count(2)	2	1	count(1)

```
public int count(int i) {
  if (i <= 0) return 0;
  else return count(i - 1)
}
```

call	$\boxed{i}$	$\boxed{i - 1}$	$\boxed{\text{count}(i - 1)}$
count(1)	$\boxed{1}$	$\boxed{0}$	$\boxed{\text{count}(0)}$

```java
public int count(int i) {
  if (i <= 0) return 0;
  else return count(i - 1)
}
```

call	$\boxed{\text{i}}$	$\boxed{\text{i - 1}}$	$\boxed{\text{count(i - 1)}}$
count(0)	$\boxed{0}$	$\boxed{?}$	$\boxed{?}$

Tail call elimination

This is called **tail call elimination**.

Tail call elimination

This is called **tail call elimination**. Why?

Tail call elimination

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Because there are **no recursive calls**.

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Because there are **no recursive calls**.

We simply update the stack

Java analogy

Using tail call elimination we can go from

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public int count(int i) {
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Using tail call elimination we can go from

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  if (i <= 0) return 0;
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Java analogy

Using tail call elimination we can go from

```
public int count(int i) {
  if (i <= 0) return 0;
  else return count(i - 1)
}
```

to

```
public int count(int i) {
  while(i > 0) {
    i--;
  }
  return i;
}
```

Tail call elimination scope 1/2

Can you think of situations where **tail call elimination** *cannot* be used?

Tail call elimination scope 1/2

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Can this be optimised with tail call elimination?

```
public int count(int i) {
  if (count <= 0) return 0;
  else {
    int newCount = count(i - 1);
    return newCount;
  }
}
```

Tail call elimination scope 1/2

Can you think of situations where **tail call elimination** *cannot* be used?
Can this be optimised with tail call elimination?

```
public int count(int i) {
  if (count <= 0) return 0;
  else {
    int newCount = count(i - 1);
    return newCount;
  }
}
```

No.

Can you think of situations where **tail call elimination** *cannot* be used?
Can this be optimised with tail call elimination?

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public int count(int i) {
  if (count <= 0) return 0;
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    return newCount;
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}
```

No. The recursive call is not the last!

Tail call elimination scope 2/2

Can this be optimised with tail call elimination?

```java
public long sum(long n) {
  if (n <= 1) return n;
  else return n + sum(n - 1);
}
```

Can this be optimised with tail call elimination?

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Let's try that in Java!

Tail recursion

This technique is called **tail recursion**

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- Replaces tail calls with stack frame manipulation

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- Saves stack frames

This technique is called **tail recursion**

- Replaces tail calls with stack frame manipulation
- Saves stack frames
- Note: Java does **not** have it!

Java bytecode example

See the example from `https://github.com/cphbus-functional-programming/java-exercises` in the `tailrecursion` project.

Fibonacci example

Can this be optimised with tail call elimination?

```
public int fibonacci(int i) \{
  if (i < 2) return i;
  else return fibonacci(i - 1) + fibonacci(i - 2)
\}
```

Fibonacci example

Can this be optimised with tail call elimination?

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No.

Fibonacci example

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```

No. There are two recursive calls

Can this be optimised with tail call elimination?

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Can this be optimised with tail call elimination?

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public int fibonacci(int i) \{
  if (i < 2) return i;
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\}
```

No. There are two recursive calls

Can we optimise it to eliminate tail calls?

```
public int fibonacci(int i, int previous, int next) {
  if      (i == 0) return previous;
  else if (i == 1) return next;
  else             return fibonacci(i - 1, next,
                                    (next + previous));
}
```

Hand-in

1. Finish the virtual CPU

Hand-in

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2. Program a tail recursive version of factorial

Hand-in

1. Finish the virtual CPU
2. Program a tail recursive version of factorial

Hint: Use the factorial implementation from Anders on GitHub under **general/ 02 Assignment**