You must justify your answers.

1) For each of the following graph-theoretical properties, give an example of a real-world network satisfying it (so, for instance, in the first item, you are asked for a directed network and an undirected one):

(1) Undirected *vs* directed

(2) Weighted *vs* unweighted

(3) Multigraph *vs* simple graph

(4) Connected *vs* not necessarily connected

(5) Bipartite *vs* non-bipartite

2) Extend the Handshaking Lemma and its corollary to digraphs and give some real-world application.

3) Why, in the proof of Hall's theorem, the graph G_2 satisfies that $|\mathrm{Neigh}_{G_2}(S)| \geqslant |S|$ for every $S \subsetneq V_1 \setminus V$?

4) We say that a node x is *pivotal* for a pair of nodes u, v when $u \neq x$, $v \neq x$, u and v are connected by some path, and x belongs to every shortest path between them.

(a) For every node in the graph in Fig. 1, find all pairs of nodes for which it is pivotal.

(b) Give an example of a graph where each node is pivotal for some pair of nodes.

(c) Give an example of a graph where each node is pivotal for at least two pairs of nodes.

(d) Give an example of a graph containing a node that is pivotal for every pair of nodes not containing it.

(e) Give an example of a graph with at least 4 nodes with no pivotal node for any pair of nodes.

5) Given a graph $G = (V, E)$, its dual $G^* = (E, E^*)$ is the graph with set of nodes E and, for every $e, e' \in E$, $ee' \in E^*$ iff e and e' are incident in G.

(a) Apply the dual construction to the graph in Fig. 1.

(b) How is the degree of a node $e = uv$ in G^* related to the degrees of u and v in G?

(c) How is the size of G^* related to information (order, size, degrees, ...) of G?

(d) Find a graph G such that the dual of its dual G^* is isomorphic to G, and a graph H such that the dual of its dual H^* is *not* isomorphic to H. (There are small examples of both.)

6) Prove that if a graph G is not connected, then its complement $\overline{G}$ is connected. Provide an example of a connected graph with at least 4 nodes whose complement is also connected.

7) Let $G' = G \setminus e$. Prove that: (a) every independent set in G is also independent in G'; (b) there exists some independent set in G' that is not independent in G'. Are (a) and (b) still true if we replace everywhere "independent" by "maximal independent"?

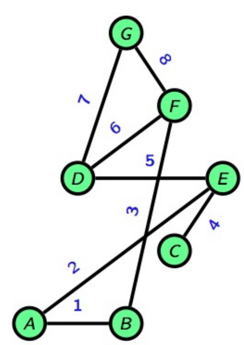

Figure 1.