BST 140.751 Midterm exam

Notes:

- You may not use a calculator for this exam.
- Please be neat and write legibly. Use the back of the pages if necessary.
- Good luck!

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printed name

1. Let $X_{11} \ldots, X_{IJ}$ be iid random variables from a distribution with density $f(x) = \frac{1}{2}\exp(-x/2)$ for $x > 0$ for $i = 1, \ldots I$ and $j = 1, \ldots J$. Assume J is a very large number. Let $\bar{X}_j = \frac{1}{I}\sum_{i=1}^{I} X_{ij}$ and $S_j^2 = \sum_{i=1}^{I}(X_{ij} - \bar{X}_j)^2/(I-1)$ Answer the following

   A. Show that the mean and standard deviation associated with f are both 2.

   B. Approximately what number will $\frac{1}{J}\sum_{j=1}^{J}\bar{X}_j$ be? Does your calculation require I to be "large"?

   C. Approximately what number will $\frac{1}{J}\sum_{j=1}^{J} S_j^2$ be? Does your calculation require I to be "large"?

   D. Approximately what number will $\frac{1}{J}\left\{\sum_{j=1}^{J}\left(\bar{X}_j - \frac{1}{J}\sum_{k=1}^{J}\bar{X}_k\right)^2\right\}$ be? Does your calculation require I to be "large"?

   E. Approximately what number will $\frac{1}{J}\sum_{j=1}^{J} Y_j$ be, where $Y_j = I\left(\frac{\sqrt{I}(\bar{X}_j - 2)}{2} > 0\right)$ and $I(A)$ is the indicator that the event A occurred. Does your calculation require I to be "large"?

2. Let f be a continuous density with associated distribution function F. Let $x_0 < x_1$ be points.

$$\tilde{f}(x) = \begin{cases} f(x)/\{F(x_1) - F(x_0)\} & \text{For} \qquad x_0 \leq x \leq x_1 \\ 0 & \text{Otherwise} \end{cases}$$

A. Argue that $\tilde{f}$ is a proper density.

B. What is the distribution function associated with this density?

C. Symbolically write out the α^{th} quantile of this distribution (using only F, F^{-1}, f, x_0 and x_1).

3. Let X follow a shifted and scaled logistic distribution; i.e. having associated density

$$f(x) = \frac{\beta_1 \exp(-\beta_0 - \beta_1 x)}{\{1 + \exp(-\beta_0 - \beta_1 x)\}^2}$$

for $-\infty < x < \infty$ and $\beta_0, \beta_1 > 0$ are constants.

A. Argue that f is a proper density function.

B. Calculate the distribution function associated with f.

C. Let $Y = I(X \leq x_0)$ be an indicator of the event that X is less than x_0. Argue that the log of the odds that $Y = 1$ is $\beta_0 + \beta_1 x_0$.

4. Let $X_1, \ldots, X_n$ be iid from a normal distribution with mean μ and variance σ^2. (I.e. having density $(2\pi\sigma^2)^{-1/2} \exp\{-(x-\mu)^2/2\sigma^2\}$ for $-\infty < x < \infty$.)

   A. Argue that the ML estimate of μ is the sample average.

   B. Argue that the ML estimate of σ is the square root of the biased version of the sample standard variance. (You do not need to demonstrate that it's unbiased.)

   C. Argue that the least squares estimate of μ, i.e. that point which minimizes $\sum_{i=1}^{n}(x_i - \mu)^2$, is also the sample average.