BST 140.751 final exam

Notes:

- You may not use a calculator for this exam.
- Please be neat and write legibly. Use the back of the pages if necessary.
- Good luck!

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printed name

1. Let $Y = X\beta + \epsilon$ where Y is $n \times p$, X is $n \times p$ of rank $p < n$ and $\epsilon \sim N(0, \sigma^2 I)$. Let $X = UDV'$ where U is $n \times p$ so that $U'U = I$, D is a diagonal matrix $(p \times p)$ and V is a $p \times p$ matrix so that $V'V = I$. Let $\gamma = DV'\beta$.

   A. Argue that $Y = U\gamma + \epsilon$.

   B. Argue that $U'Y = \gamma + \tilde{\epsilon}$ where $\tilde{\epsilon} \sim N(0, \sigma^2 I)$.

   C. Write out and simplify the least squares estimate of γ. What is its distribution?

   D. Suppose that X is orthormal (i.e. $X'X = I$), argue that no matrix inversion is necessary to obtain $\hat{\beta}$ and write out the ML estimate for β.

   E. Let

$$X = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 0 & 1/\sqrt{2} \\ 1/2 & -1/2 & 0 & -1/\sqrt{2} \end{pmatrix}$$

   and $Y = (y_1, y_2, y_3, y_4)$. Derive the ML estimate for β in the terms of the y_i.

2. Let $Y \mid \beta \sim N(X\beta, \sigma^2 I)$

   A. Suppose that σ^2 is known, what is the sufficient statistic for β?

   B. Suppose that $\beta \sim N(\beta_0, \Sigma)$. Derive the posterior distribution for β.

3. Let $\bar{X}$ be the sample average of positive iid variables from a population with mean μ and variance σ^2. Assume σ^2 is known.

   A. Calculate an asymptotic 95% confidence interval for $\log(\mu)$ using the delta method.

   B. Suppose that you were to calculate an ordinary confidence interval for the mean using the logged data, eg with sample mean $\frac{1}{n}\sum_{i=1}^{n}\log(X_i)$. Would this be estimating $\log(\mu)$? If not, what would it be estimating?

   C. Let $\bar{Y}$ be the sample average of an independent collection of positive iid variables from a population with mean δ and known variance τ^2. Calculate a delta method confidence interval for $\log(\mu/\delta)$.

4. Let $X_1 \dots X_n$ be independent Poisson λt_i

   A. Derive the sufficient statistic for λ.

   B. Derive the ML estimate of λ.

   C. Derive the conditional distribution of $X_1 \dots X_n \mid \sum_{i=1}^{n} X_i$? (You may assume that the sum of independent Poissons is Poisson).

   D. Let $t_1 \dots t_n$ be $Gamma(x_i, 1/\lambda)$. Argue that this model is likelihood equivalent to that in A.