T Confidence Intervals

Mathematical Biostatistics Boot Camp

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Independent group t confidence intervals

- Suppose that we want to compare the mean blood pressure between two groups in a randomized trial; those who received the treatment to those who received a placebo

- We cannot use the paired t test because the groups are independent and may have different sample sizes

- We now present methods for comparing independent groups

Notation

- Let $X_1, \ldots, X_{n_x}$ be iid $N(\mu_x, \sigma^2)$

- Let $Y_1, \ldots, Y_{n_y}$ be iid $N(\mu_y, \sigma^2)$

- Let $\bar{X}, \bar{Y}, S_x, S_y$ be the means and standard deviations

- Using the fact that linear combinations of normals are again normal, we know that $\bar{Y} - \bar{X}$ is also normal with mean $\mu_y - \mu_x$ and variance $\sigma^2(\frac{1}{n_x} + \frac{1}{n_y})$

- The pooled variance estimator

$$S_p^2 = \{(n_x - 1)S_x^2 + (n_y - 1)S_y^2\}/(n_x + n_y - 2)$$

is a good estimator of σ^2

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Note

- The pooled estimator is a mixture of the group variances, placing greater weight on whichever has a larger sample size

- If the sample sizes are the same the pooled variance estimate is the average of the group variances

- The pooled estimator is unbiased

$$E[S_p^2] = \frac{(n_x - 1)E[S_x^2] + (n_y - 1)E[S_y^2]}{n_x + n_y - 2}$$

$$= \frac{(n_x - 1)\sigma^2 + (n_y - 1)\sigma^2}{n_x + n_y - 2}$$

- The pooled variance estimate is independent of $\bar{Y} - \bar{X}$ since S_x is independent of $\bar{X}$ and S_y is independent of $\bar{Y}$ and the groups are independent

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Result

- The sum of two independent Chi-squared random variables is Chi-squared with degrees of freedom equal to the sum of the degrees of freedom of the summands

- Therefore

$$(n_x + n_y - 2)S_p^2/\sigma^2 = (n_x - 1)S_x^2/\sigma^2 + (n_y - 1)S_y^2/\sigma^2$$

$$= \chi^2_{n_x - 1} + \chi^2_{n_y - 1}$$

$$= \chi^2_{n_x + n_y - 2}$$

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Putting this all together

- The statistic

$$\frac{\frac{\bar{Y}-\bar{X}-(\mu_y-\mu_x)}{\sigma\left(\frac{1}{n_x}+\frac{1}{n_y}\right)^{1/2}}}{\sqrt{\frac{(n_x+n_y-2)S_p^2}{(n_x+n_y-2)\sigma^2}}} = \frac{\bar{Y}-\bar{X}-(\mu_y-\mu_x)}{S_p\left(\frac{1}{n_x}+\frac{1}{n_y}\right)^{1/2}}$$

  is a standard normal divided by the square root of an independent Chi-squared divided by its degrees of freedom

- Therefore this statistic follows Gosset's t distribution with $n_x + n_y - 2$ degrees of freedom

- Notice the form is (estimator - true value) / SE

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Confidence interval

- Therefore a $(1 - \alpha) \times 100\%$ confidence interval for $\mu_y - \mu_x$ is

$$\bar{Y} - \bar{X} \pm t_{n_x + n_y - 2, 1 - \alpha/2} \, S_p \left(\frac{1}{n_x} + \frac{1}{n_y} \right)^{1/2}$$

- Remember this interval is assuming a constant variance across the two groups

- If there is some doubt, assume a different variance per group, which we will discuss later

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Likelihood method

- Exactly as before,

$$\frac{\bar{Y} - \bar{X}}{S_p \left(\frac{1}{n_x} + \frac{1}{n_y} \right)^{1/2}}$$

follows a non-central t distribution with non-centrality parameter $\dfrac{\mu_y - \mu_x}{\sigma \left(\frac{1}{n_x} + \frac{1}{n_y} \right)^{1/2}}$

- Therefore, we can use this statistic to create a likelihood for $(\mu_y - \mu_x)/\sigma$, a standardized measure of the change in group means

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Example

- Comparing SBP for 8 oral contraceptive users versus 21 controls

- $\bar{X}_{OC} = 132.86$ mmHg with $s_{OC} = 15.34$ mmHg

- $\bar{X}_C = 127.44$ mmHg with $s_C = 18.23$ mmHg

- Pooled variance estimate

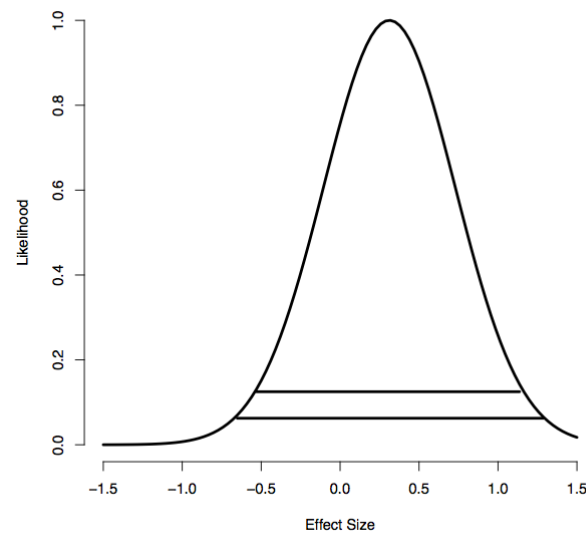$$s_p^2 = \frac{7(15.34)^2 + 20(18.23)^2}{8 + 21 - 2} = 307.8$$

- $t_{27,.975} = 2.052$ (in R, `qt(.975, df = 27)`)

- Interval

$$132.86 - 127.44 \pm 2.052 \left\{ 307.8 \left(\frac{1}{8} + \frac{1}{21} \right)^{1/2} \right\} = [-9.52, 20.36]$$

Likelihood plot for the effect size

Reasonable values for the effect size from the confidence interval

$$[-9.52, 20.36]/sp = [-.54, 1.16]$$

Unequal variances

- Note that under unequal variances

$$\bar{Y} - \bar{X} \sim N\left(\mu_y - \mu_x, \frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}\right)$$

- The statistic

$$\frac{\bar{Y} - \bar{X} - (\mu_y - \mu_x)}{\left(\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}\right)^{1/2}}$$

approximately follows Gosset's t distribution with degrees of freedom equal to

$$\frac{\left(S_x^2/n_x + S_y^2/n_y\right)^2}{\left(\frac{S_x^2}{n_x}\right)^2/(n_x - 1) + \left(\frac{S_y^2}{n_y}\right)^2/(n_y - 1)}$$

Example

- Comparing SBP for 8 oral contraceptive users versus 21 controls

- $\bar{X}_{OC} = 132.86$ mmHg with $s_{OC} = 15.34$ mmHg

- $\bar{X}_C = 127.44$ mmHg with $s_C = 18.23$ mmHg

- $df = 15.04$, $t_{15.04,.975} = 2.13$

- Interval

$$132.86 - 127.44 \pm 2.13 \left(\frac{15.34^2}{8} + \frac{18.23^2}{21} \right)^{1/2} = [-8.91, 19.75]$$

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