BST 140.751 Midterm exam
Notes:

- You may not use a calculator for this exam.
- Please be neat and write legibly. Use the back of the pages if necessary.
- Good luck!

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printed name

1. You are playing a game with a friend where you flip a coin and if it comes up heads you give her x dollars and if it comes up tails she gives you cx dollars for some value of c. The coin has probability of a head of p.

   a. What must be true about c for the game to be "fair". That is, to have expected winnings 0 for both players?

   b. Suppose that you play the game n times. What is the mean and variance for your earnings?

   c. How could you make your variance the largest?

2. Let $Y_1, \ldots, Y_I$ be independent random variables so that $Y_i \sim N(x_i\beta, \sigma^2)$ where $\{x_i\}_{i=1}^I$ are known.

   a. Show that the ML estimate of β is $\hat{\beta} = \frac{\sum_{i=1}^I y_i x_i}{\sum x_i^2}$.

   b. Show that this estimate is unbiased.

   c. Derive the variance of $\hat{\beta}$.

   d. Derive the ML estimate of σ^2.

3. Let X_i be independent Poisson(λt_i) for $i = 1, \ldots, I$. The t_i are known. Recall a variable is Poisson(μ) if it has mass function $\frac{\mu^x e^{-\mu}}{x!}$ for $x = 0, 1, \ldots$. The mean and variance of the Poisson mass function is μ.

   a. Argue that the Poisson mass function is a valid mass fuction.

   b. Give a univariate function of the collection $\{X_i\}$ that is sufficient for specifying the likelihood for λ.

   c. Give the maximum likelihood estimate for λ.

   d. Show that the ML estimate of λ is both unbiased and derive its variance.

4. Consider a diagnostic test. Let D be the event that the patient has the disease and T_1 be the event that a first test is positive. Let T_2 be the result that a second test is positive. Assume that the result of the second test is independent of the first.

   a. Symbolically derive the positive predictive value of a positive value on the first test in the terms of its sensitivity and specificity.

   b. Consider the test T_3 defined as positive if either T_1 or T_2 (or both). Derive the sensitivity and specificity of T_3 as a function of the sensitivity and specificity of T_1 and T_2.

   c. Consider the test T_4 defined as positive if both T_1 and T_2. Derive the sensitivity and specificity of T_4 as a function of the sensitivity and specificity of T_1 and T_2.

   d. Say what you can about the ordering ($>$, $\geq$, $<$, $\leq$) between the sensitivities of the four tests. Repeat this for the specificities.