Asymptotics

Mathematical Biostatistics Boot Camp

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Numerical limits

- Imagine a sequence
  - $a_1 = .9$,
  - $a_2 = .99$,
  - $a_3 = .999$, ...
- Clearly this sequence converges to 1
- Definition of a limit: For any fixed distance we can find a point in the sequence so that the sequence is closer to the limit than that distance from that point on
- $|a_n - 1| = 10^{-n}$

Limits of random variables

- The problem is harder for random variables
- Consider $\bar X_n$ the sample average of the first n of a collection of iid observations
  - Example $\bar X_n$ could be the average of the result of n coin flips (i.e. the sample proportion of heads)
- We say that $\bar X_n$ {\bf converges in probability} to a limit if for any fixed distance the {\em probability} of $\bar X_n$ being closer (further away) than that distance from the limit converges to one (zero)
- $P(|\bar X_n - \mathrm{limit}| < \epsilon) \rightarrow 1$

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The Law of Large Numbers

- Establishing that a random sequence converges to a limit is hard
- Fortunately, we have a theorem that does all the work for us, called the **Law of Large Numbers**
- The law of large numbers states that if $X_1, \ldots X_n$ are iid from a population with mean μ and variance σ^2 then $\bar X_n$ converges in probability to μ
- (There are many variations on the LLN; we are using a particularly lazy one)

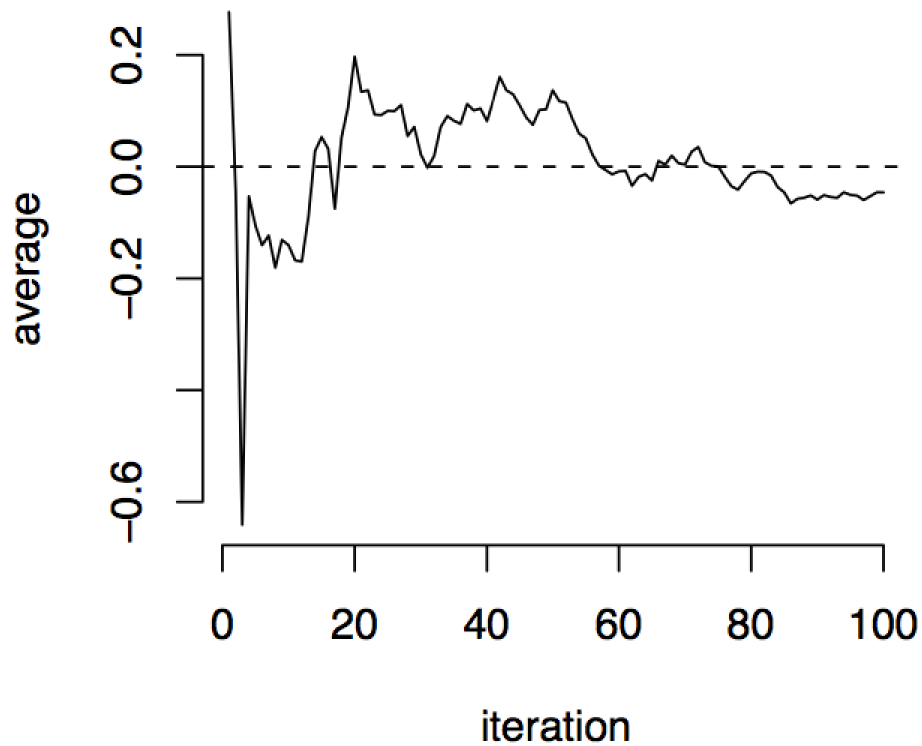
Proof using Chebyshev's inequality

- Recall Chebyshev's inequality states that the probability that a random variable variable is more than k standard deviations from its mean is less than $1/k^2$
- Therefore for the sample mean

$$P\left\{|\bar X_n - \mu| \geq k \ \mathrm{sd}(\bar X_n)\right\} \leq 1/k^2$$

- Pick a distance ϵ and let $k = \epsilon / \mathrm{sd}(\bar X_n)$

$$P(|\bar X_n - \mu| \geq \epsilon) \leq \frac{\mathrm{sd}(\bar X_n)^2}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$


average
0.2
0.0
−0.2
−0.6
0
20
40
60
80
100
iteration

Useful facts

- Functions of convergent random sequences converge to the function evaluated at the limit
- This includes sums, products, differences, ...
- Example $(\bar X_n)^2$ converges to μ^2
- Notice that this is different than $(\sum X_i^2)/n$ which converge to $E[X_i^2] = \sigma^2 + \mu^2$
- We can use this to prove that the sample variance converges to σ^2

Continued

$$
\begin{aligned}
\sum (X_i - \bar X_n)^2/(n-1) &= \frac{\sum X_i^2}{n-1} - \frac{n(\bar X_n)^2}{n-1} \\
&= \frac{n}{n-1} \times \frac{\sum X_i^2}{n} - \frac{n}{n-1} \times (\bar X_n)^2 \\
&\stackrel{p}{\rightarrow} 1 \times (\sigma^2 + \mu^2) - 1 \times \mu^2 \\
&= \sigma^2
\end{aligned}
$$

Hence we also know that the sample standard deviation converges to σ

Discussion

- An estimator is **consistent** if it converges to what you want to estimate
- The LLN basically states that the sample mean is consistent
- We just showed that the sample variance and the sample standard deviation are consistent as well
- Recall also that the sample mean and the sample variance are unbiased as well
- (The sample standard deviation is biased, by the way)

The Central Limit Theorem

- The **Central Limit Theorem** (CLT) is one of the most important theorems in statistics
- For our purposes, the CLT states that the distribution of averages of iid variables, properly normalized, becomes that of a standard normal as the sample size increases
- The CLT applies in an endless variety of settings

The CLT

- Let $X_1, \ldots, X_n$ be a collection of iid random variables with mean μ and variance σ^2
- Let $\bar X_n$ be their sample average
- Then

$$P\left(\frac{\bar X_n - \mu}{\sigma / \sqrt{n}} \leq z\right) \rightarrow \Phi(z)$$

- Notice the form of the normalized quantity

$$\frac{\bar X_n - \mu}{\sigma / \sqrt{n}} = \frac{\mbox{Estimate} - \mbox{Mean of estimate}}{\mbox{Std. Err. of estimate}}.$$

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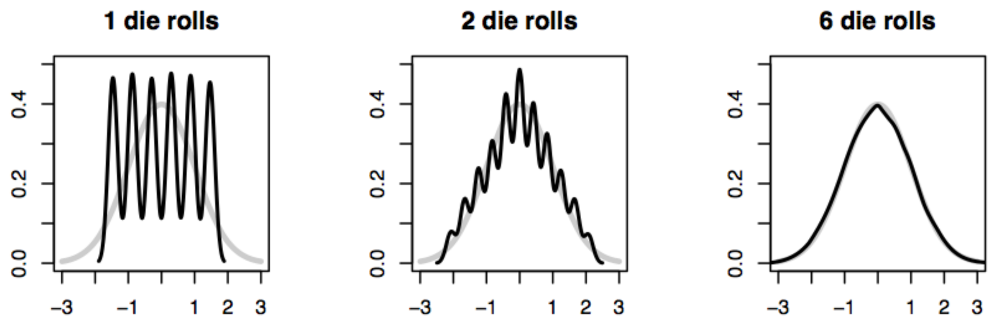
Example

- Simulate a standard normal random variable by rolling n (six sided)
- Let X_i be the outcome for die i
- Then note that $\mu = E[X_i] = 3.5$
- $Var(X_i) = 2.92$
- SE $\sqrt{2.92/n} = 1.71/\sqrt{n}$
- Standardized mean

$$\frac{\bar X_n - 3.5}{1.71/\sqrt{n}}$$

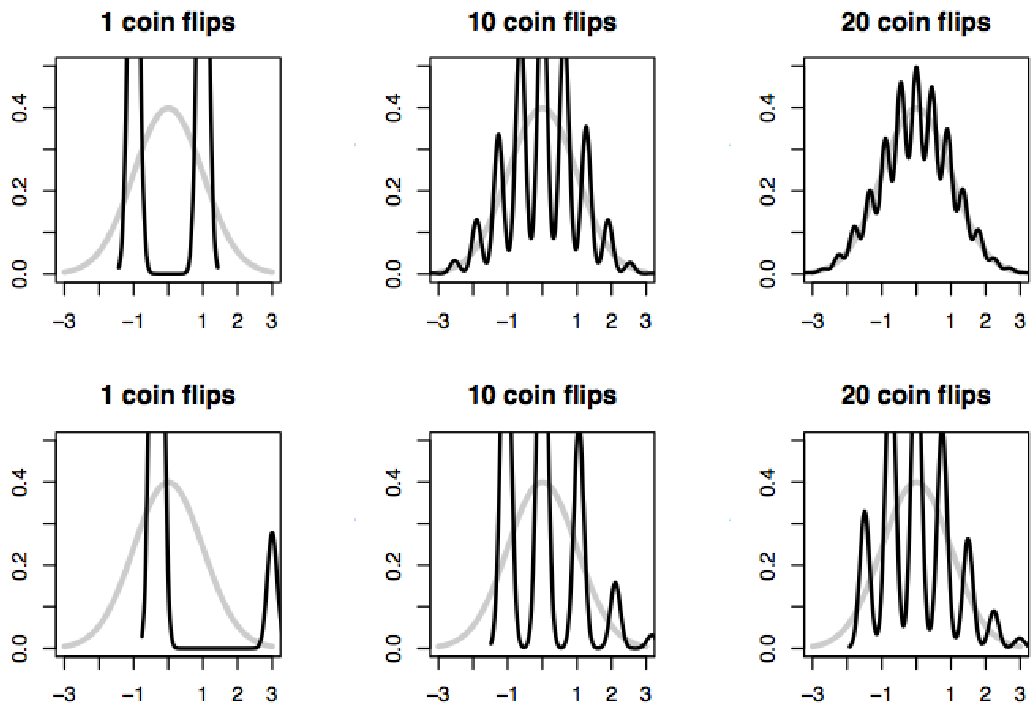

1 die rolls
2 die rolls
6 die rolls
0.0
0.2
0.4
−3
−1
1
2
3

Coin CLT

- Let X_i be the 0 or 1 result of the i^{th} flip of a possibly unfair coin
  - The sample proportion, say $\hat p$, is the average of the coin flips
  - $E[X_i] = p$ and $Var(X_i) = p(1-p)$
  - Standard error of the mean is $\sqrt{p(1-p)/n}$
  - Then

$$\frac{\hat p - p}{\sqrt{p(1-p)/n}}$$

will be approximately normally distributed


1 coin flips
10 coin flips
20 coin flips
1 coin flips
10 coin flips
20 coin flips
0.0
0.2
0.4
–3
–1
1
2
3

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CLT in practice

- In practice the CLT is mostly useful as an approximation

$$P\left(\frac{\bar X_n - \mu}{\sigma / \sqrt{n}} \leq z\right) \approx \Phi(z).$$

- Recall 1.96 is a good approximation to the $.975^{th}$ quantile of the standard normal
- Consider

$$\begin{aligned} .95 &\approx P\left(-1.96 \leq \frac{X_n - \mu}{\sigma / \sqrt{n}} \leq 1.96 \right) \\ &= P\left(\bar X_n + 1.96 \sigma/\sqrt{n} \geq \mu \geq \bar X_n - 1.96\sigma/\sqrt{n}\right), \end{aligned}$$

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Confidence intervals

- Therefore, according to the CLT, the probability that the random interval

$$\bar X_n \pm z_{1-\alpha/2}\sigma/\sqrt{n}$$

contains μ is approximately 95%, where $z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution

- This is called a 95% {\bf confidence interval} for μ
- **Slutsky's theorem**, allows us to replace the unknown σ with s

Sample proportions

- In the event that each X_i is 0 or 1 with common success probability p then $\sigma^2 = p(1-p)$
- The interval takes the form

$$\hat p \pm z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

- Replacing p by $\hat p$ in the standard error results in what is called a Wald confidence interval for p
- Also note that $p(1-p) \leq 1/4$ for $0 \leq p \leq 1$
- Let $\alpha = .05$ so that $z_{1-\alpha/2} = 1.96 \approx 2$ then

$$2\sqrt{\frac{p(1-p)}{n}} \leq 2\sqrt{\frac{1}{4n}} = \frac{1}{\sqrt{n}}$$

- Therefore $\hat p \pm \frac{1}{\sqrt{n}}$ is a quick CI estimate for p