Conditional Probability

Mathematical Biostatistics Boot Camp

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Conditional probability, motivation

- The probability of getting a one when rolling a (standard) die is usually assumed to be one sixth
- Suppose you were given the extra information that the die roll was an odd number (hence 1, 3 or 5)
- *conditional on this new information*, the probability of a one is now one third

Conditional probability, definition

- Let B be an event so that $P(B) > 0$
- Then the conditional probability of an event A given that B has occurred is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- Notice that if A and B are independent, then

$$P(A \mid B) = \frac{P(A)P(B)}{P(B)} = P(A)$$

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Example

- Consider our die roll example
- $B = \{1, 3, 5\}$
- $A = \{1\}$

$$\begin{aligned} P(\text{one given that roll is odd}) &= P(A \mid B) \\ &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)}{P(B)} \\ &= \frac{1/6}{3/6} = \frac{1}{3} \end{aligned}$$

Conditional densities and mass functions

- Conditional densities or mass functions of one variable conditional on the value of another
- Let $f(x, y)$ be a bivariate density or mass function for random variables X and Y
- Let $f(x)$ and $f(y)$ be the associated marginal mass function or densities disregarding the other variables

$$f(y) = \int f(x, y) dx \quad \text{or} \quad f(y) = \sum_x f(x, y) dx.$$

- Then the **conditional** density or mass function {\em given that $Y = y$} is given by

$$f(x \mid y) = f(x, y)/f(y)$$

Notes

- It is easy to see that, in the discrete case, the definition of conditional probability is exactly as in the definition for conditional events where $A =$ the event that $X = x$ and $B =$ the event that $Y = y$
- The continuous definition is a little harder to motivate, since the events $X = x$ and $Y = y$ each have probability 0
- However, a useful motivation can be performed by taking the appropriate limits as follows
- Define $A = \{X \leq x\}$ while $B = \{Y \in [y, y + \epsilon]\}$

Continued

$$P(X \leq x \mid Y \in [y, y+\epsilon]) = P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(X \leq x, Y \in [y, y+\epsilon])}{P(Y \in [y, y+\epsilon])}$$

$$= \frac{\int_y^{y+\epsilon} \int_{-\infty}^{x} f(x,y)dxdy}{\int_y^{y+\epsilon} f(y)dy}$$

$$= \frac{\epsilon \int_y^{y+\epsilon} \int_{-\infty}^{x} f(x,y)dxdy}{\epsilon \int_y^{y+\epsilon} f(y)dy}$$

Continued

$$= \frac{\frac{\int_{-\infty}^{y+\epsilon} \int_{\infty}^{x} f(x,y)dxdy - \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)dxdy}{\epsilon}}{\frac{\int_{-\infty}^{y+\epsilon} f(y)dy - \int_{-\infty}^{y} f(y)dy}{\epsilon}}$$

$$= \frac{\frac{g_1(y+\epsilon)-g_1(y)}{\epsilon}}{\frac{g_2(y+\epsilon)-g_2(y)}{\epsilon}}$$

where

$$g_1(y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)dxdy \text{ and } g_2(y) = \int_{-\infty}^{y} f(y)dy.$$

- Notice that the limit of the numerator and denominator tends to ${g_1}'$ and ${g_2}'$ as ϵ gets smaller and smaller
- Hence we have that the conditional distribution function is

$$P(X \leq x \mid Y = y) = \frac{\int_{-\infty}^{x} f(x, y) dx}{f(y)}.$$

- Now, taking the derivative with respect to x yields the conditional density

$$f(x \mid y) = \frac{f(x, y)}{f(y)}$$

Geometrically

- Geometrically, the conditional density is obtained by taking the relevant slice of the joint density and appropriately renormalizing it
- This idea extends to any other line, or even non-linear functions

Example

- Let $f(x, y) = ye^{-xy-y}$ for $0 \leq x$ and $0 \leq y$
- Then note

$$f(y) = \int_0^\infty f(x, y) dx = e^{-y} \int_0^\infty ye^{-xy} dx = e^{-y}$$

- Therefore

$$f(x \mid y) = f(x, y)/f(y) = \frac{ye^{-xy-y}}{e^{-y}} = ye^{-xy}$$

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Bayes' rule

- Let $f(x \mid y)$ be the conditional density or mass function for X given that $Y = y$
- Let $f(y)$ be the marginal distribution for y
- Then if y is continuous

$$f(y \mid x) = \frac{f(x \mid y)f(y)}{\int f(x \mid t)f(t)dt}$$

- If y is discrete

$$f(y \mid x) = \frac{f(x \mid y)f(y)}{\sum_t f(x \mid t)f(t)}$$

Notes

- Bayes' rule relates the conditional density of $f(y \mid x)$ to the $f(x \mid y)$ and $f(y)$
- A special case of this kind relationship is for two sets A and B, which yields that

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid B^c)P(B^c)}.$$

Proof:

- Let X be an indicator that event A has occurred
- Let Y be an indicator that event B has occurred
- Plug into the discrete version of Bayes' rule

Diagnostic tests

- Let $+$ and $-$ be the events that the result of a diagnostic test is positive or negative respectively
- Let D and D^c be the event that the subject of the test has or does not have the disease respectively
- The **sensitivity** is the probability that the test is positive given that the subject actually has the disease, $P(+ \mid D)$
- The **specificity** is the probability that the test is negative given that the subject does not have the disease, $P(- \mid D^c)$

More definitions

- The **positive predictive value** is the probability that the subject has the disease given that the test is positive, $P(D \mid +)$
- The **negative predictive value** is the probability that the subject does not have the disease given that the test is negative, $P(D^c \mid -)$
- The **prevalence of the disease** is the marginal probability of disease, $P(D)$

More definitions

- The **diagnostic likelihood ratio of a positive test**, labeled DLR_+, is $P(+ \mid D)/P(+ \mid D^c)$, which is the

$$sensitivity/(1 - specificity)$$

- The **diagnostic likelihood ratio of a negative test**, labeled DLR_-, is $P(- \mid D)/P(- \mid D^c)$, which is the

$$(1 - sensitivity)/specificity$$

Example

- A study comparing the efficacy of HIV tests, reports on an experiment which concluded that HIV antibody tests have a sensitivity of 99.7% and a specificity of 98.5%
- Suppose that a subject, from a population with a .1% prevalence of HIV, receives a positive test result. What is the probability that this subject has HIV?
- Mathematically, we want $P(D \mid +)$ given the sensitivity, $P(+ \mid D) = .997$, the specificity, $P(- \mid D^c) = .985$, and the prevalence $P(D) = .001$

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Using Bayes' formula

$$
\begin{aligned}
P(D \mid +) &= \frac{P(+ \mid D)P(D)}{P(+ \mid D)P(D) + P(+ \mid D^c)P(D^c)} \\
&= \frac{P(+ \mid D)P(D)}{P(+ \mid D)P(D) + \{1 - P(- \mid D^c)\}\{1 - P(D)\}} \\
&= \frac{.997 \times .001}{.997 \times .001 + .015 \times .999} \\
&= .062
\end{aligned}
$$

- In this population a positive test result only suggests a 6% probability that the subject has the disease
- (The positive predictive value is 6% for this test)

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More on this example

- The low positive predictive value is due to low prevalence of disease and the somewhat modest specificity
- Suppose it was known that the subject was an intravenous drug user and routinely had intercourse with an HIV infected partner
- Notice that the evidence implied by a positive test result does not change because of the prevalence of disease in the subject's population, only our interpretation of that evidence changes

Likelihood ratios

- Using Bayes rule, we have

$$P(D \mid +) = \frac{P(+ \mid D)P(D)}{P(+ \mid D)P(D) + P(+ \mid D^c)P(D^c)}$$

and

$$P(D^c \mid +) = \frac{P(+ \mid D^c)P(D^c)}{P(+ \mid D)P(D) + P(+ \mid D^c)P(D^c)}.$$

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Likelihood ratios

- Therefore

$$\frac{P(D \mid +)}{P(D^c \mid +)} = \frac{P(+ \mid D)}{P(+ \mid D^c)} \times \frac{P(D)}{P(D^c)}$$

  ie

$$\text{post-test odds of } D = DLR_+ \times \text{pre-test odds of } D$$

- Similarly, DLR_- relates the decrease in the odds of the disease after a negative test result to the odds of disease prior to the test.

HIV example revisited

- Suppose a subject has a positive HIV test
- $DLR_+ = .997/(1 - .985) \approx 66$
- The result of the positive test is that the odds of disease is now 66 times the pretest odds
- Or, equivalently, the hypothesis of disease is 66 times more supported by the data than the hypothesis of no disease

HIV example revisited

- Suppose that a subject has a negative test result
- $DLR_- = (1 - .997)/.985 \approx .003$
- Therefore, the post-test odds of disease is now $.3\%$ of the pretest odds given the negative test.
- Or, the hypothesis of disease is supported $.003$ times that of the hypothesis of absence of disease given the negative test result