Some Common Distributions

Mathematical Biostatistics Boot Camp

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The Bernoulli distribution

- The **Bernoulli distribution** arises as the result of a binary outcome
- Bernoulli random variables take (only) the values 1 and 0 with probabilities of (say) p and $1 - p$ respectively
- The PMF for a Bernoulli random variable X is

$$P(X = x) = p^x(1 - p)^{1-x}$$

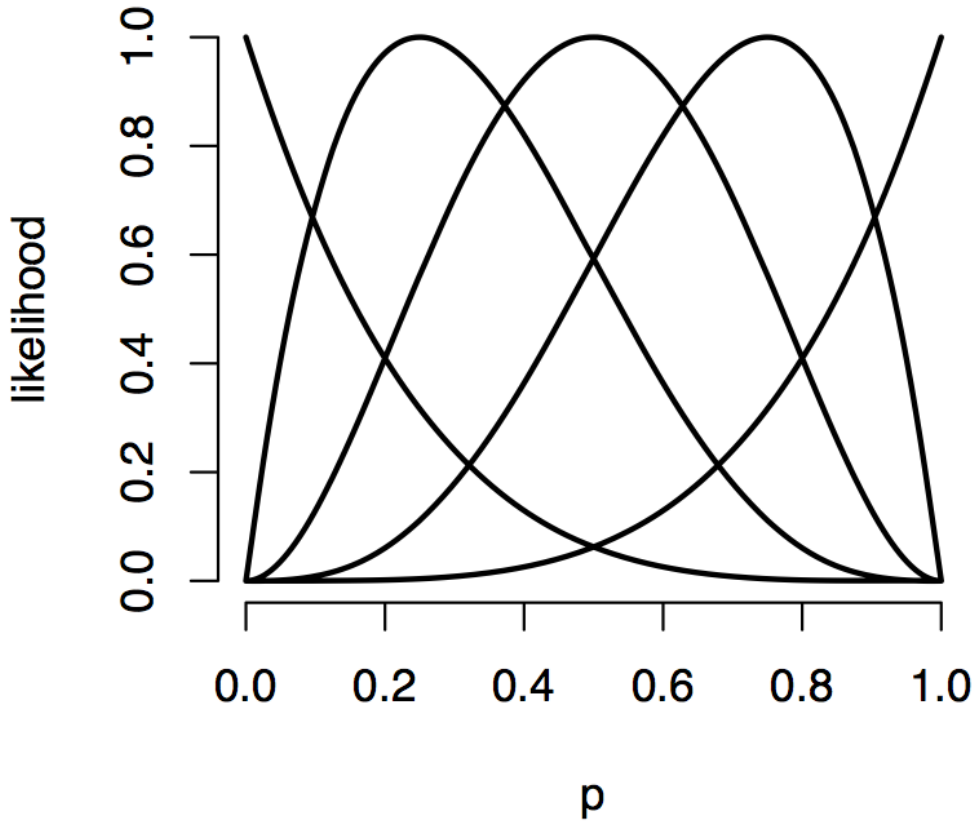
- The mean of a Bernoulli random variable is p and the variance is $p(1 - p)$
- If we let X be a Bernoulli random variable, it is typical to call $X = 1$ as a "success" and $X = 0$ as a "failure"

iid Bernoulli trials

- If several iid Bernoulli observations, say $x_1, \ldots, x_n$, are observed the likelihood is

$$\prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i}$$

- Notice that the likelihood depends only on the sum of the x_i
- Because n is fixed and assumed known, this implies that the sample proportion $\sum_i x_i/n$ contains all of the relevant information about p
- We can maximize the Bernoulli likelihood over p to obtain that $\hat{p} = \sum_i x_i/n$ is the maximum likelihood estimator for p


likelihood
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
p

Binomial trials

- The {\bf binomial random variables} are obtained as the sum of iid Bernoulli trials
- In specific, let $X_1, \ldots, X_n$ be iid Bernoulli(p); then $X = \sum_{i=1}^{n} X_i$ is a binomial random variable
- The binomial mass function is

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, \ldots, n$

- Recall that the notation

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

(read "n choose x") counts the number of ways of selecting x items out of n without replacement disregarding the order of the items

$$\binom{n}{0} = \binom{n}{n} = 1$$

Example justification of the binomial likelihood

- Consider the probability of getting 6 heads out of 10 coin flips from a coin with success probability p
- The probability of getting 6 heads and 4 tails in any specific order is

$$p^6(1-p)^4$$

- There are

$$\binom{10}{6}$$

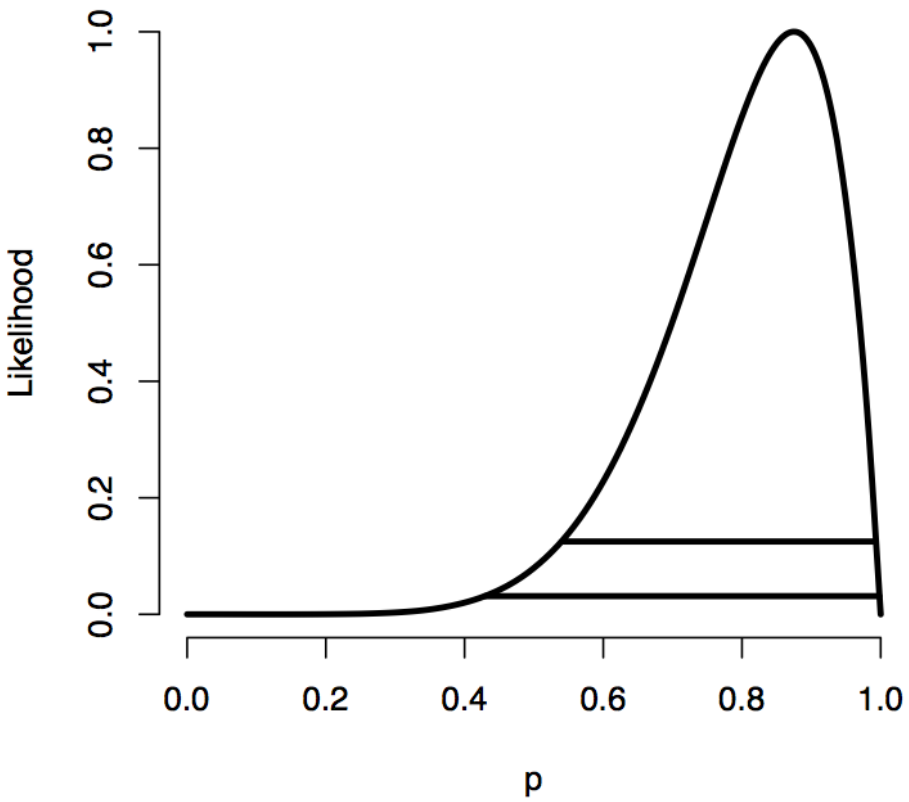
possible orders of 6 heads and 4 tails

Example

- Suppose a friend has 8 children, 7 of which are girls and none are twins
- If each gender has an independent 50% probability for each birth, what's the probability of getting 7 or more girls out of 8 births?

$$\binom{8}{7}.5^7(1-.5)^1 + \binom{8}{8}.5^8(1-.5)^0 \approx 0.04$$

- This calculation is an example of a Pvalue - the probability under a null hypothesis of getting a result as extreme or more extreme than the one actually obtained


Likelihood
0.0
0.2
0.4
0.6
0.8
1.0
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0.8
1.0
p

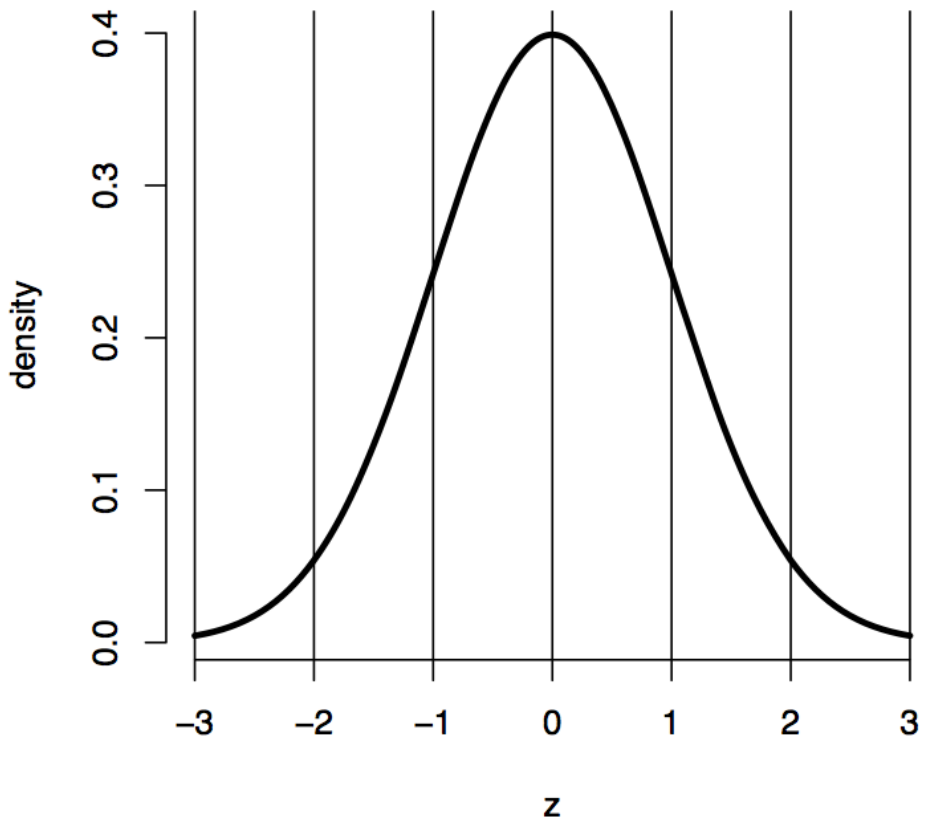
The normal distribution

- A random variable is said to follow a **normal** or **Gaussian** distribution with mean μ and variance σ^2 if the associated density is

$$(2\pi\sigma^2)^{-1/2} e^{-(x-\mu)^2/2\sigma^2}$$

  If X a RV with this density then $E[X] = \mu$ and $Var(X) = \sigma^2$

- We write $X \sim \mathrm{N}(\mu, \sigma^2)$
- When $\mu = 0$ and $\sigma = 1$ the resulting distribution is called **the standard normal distribution**
- The standard normal density function is labeled ϕ
- Standard normal RVs are often labeled Z


density
0.0
0.1
0.2
0.3
0.4
−3
−2
−1
0
1
2
3
z

Facts about the normal density

- If $X \sim \mathrm{N}(\mu, \sigma^2)$ the $Z = \frac{X-\mu}{\sigma}$ is standard normal
- If Z is standard normal

$$X = \mu + \sigma Z \sim \mathrm{N}(\mu, \sigma^2)$$

- The non-standard normal density is

$$\phi\{(x-\mu)/\sigma\}/\sigma$$

More facts about the normal density

1. Approximately 68%, 95% and 99% of the normal density lies within 1, 2 and 3 standard deviations from the mean, respectively
2. -1.28, -1.645, -1.96 and -2.33 are the 10^{th}, 5^{th}, 2.5^{th} and 1^{st} percentiles of the standard normal distribution respectively
3. By symmetry, 1.28, 1.645, 1.96 and 2.33 are the 90^{th}, 95^{th}, 97.5^{th} and 99^{th} percentiles of the standard normal distribution respectively

Question

- What is the 95^{th} percentile of a $N(\mu, \sigma^2)$ distribution?
- We want the point x_0 so that $P(X \leq x_0) = .95$

$$P(X \leq x_0) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x_0 - \mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{x_0 - \mu}{\sigma}\right) = .95$$

- Therefore

$$\frac{x_0 - \mu}{\sigma} = 1.645$$

  or $x_0 = \mu + \sigma 1.645$

- In general $x_0 = \mu + \sigma z_0$ where z_0 is the appropriate standard normal quantile

Question

- What is the probability that a $\mathrm{N}(\mu, \sigma^2)$ RV is 2 standard deviations above the mean?
- We want to know

$$P(X > \mu + 2\sigma) = P\left(\frac{X - \mu}{\sigma} > \frac{\mu + 2\sigma - \mu}{\sigma}\right)$$

$$= P(Z \geq 2)$$

$$\approx 2.5\%$$

Other properties

- The normal distribution is symmetric and peaked about its mean (therefore the mean, median and mode are all equal)
- A constant times a normally distributed random variable is also normally distributed (what is the mean and variance?)
- Sums of normally distributed random variables are again normally distributed even if the variables are dependent (what is the mean and variance?)
- Sample means of normally distributed random variables are again normally distributed (with what mean and variance?)
- The square of a *standard normal* random variable follows what is called **chi-squared** distribution
- The exponent of a normally distributed random variables follows what is called the **log-normal** distribution
- As we will see later, many random variables, properly normalized, *limit* to a normal distribution

Question

If X_i are iid $\mathrm{N}(\mu, \sigma^2)$ with a known variance, what is the likelihood for μ?

$$\mathcal{L}(\mu) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \; exp\left\{-(x_i - \mu)^2/2\sigma^2\right\}$$

$$\propto \exp\left\{ -\sum_{i=1}^{n} (x_i - \mu)^2/2\sigma^2 \right\}$$

$$= \exp\left\{ -\sum_{i=1}^{n} x_i^2/2\sigma^2 + \mu \sum_{i=1}^{n} X_i/\sigma^2 - n\mu^2/2\sigma^2 \right\}$$

$$\propto \exp\left\{\mu n\bar{x}/\sigma^2 - n\mu^2/2\sigma^2\right\}$$

Later we will discuss methods for handling the unknown variance

Question

- If X_i are iid $\mathrm{N}(\mu, \sigma^2)$, with known variance what's the ML estimate of μ?
- We calculated the likelihood for μ on the previous page, the log likelihood is

$$\mu n \bar{x} / \sigma^2 - n\mu^2 / 2\sigma^2$$

- The derivative with respect to μ is

$$n\bar{x}/\sigma^2 - n\mu/\sigma^2 = 0$$

- This yields that $\bar{x}$ is the ml estimate of μ
- Since this doesn't depend on σ it is also the ML estimate with σ unknown

Final thoughts on normal likelihoods

- The maximum likelihood estimate for σ^2 is

$$\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$$

  Which is the biased version of the sample variance

- The ML estimate of σ is simply the square root of this estimate
- To do likelihood inference, the bivariate likelihood of (μ, σ) is difficult to visualize
- Later, we will discuss methods for constructing likelihoods for one parameter at a time