NOVA SCHOOL OF SCIENCE & TECHNOLOGY

PROBABILIDADES E ESTATÍSTICA D

Ano Lectivo 2023/2024

Exame de Recurso - 8 de Julho de 2024

Grelha de respostas certas

<u>Versão A</u>

Grupo	1			2					3			4		
	a)	b)	c)	a)	b)	c)	d)	e)	a)	b)i.	b)ii.	a)	b)	c)
	D	B	C	F	F	A	D	A	C	A	D	A	A	D

Grupo	5							6	7				
	a)	b)	c)	d)i.	d)ii.	e)i.	e)ii.		a)	b)	c)	d)	e)
	D	C	A	C	C	B	D	A	A	C	D	B	B

<u>Versão B</u>

Grupo	1			2					3			4		
	a)	b)	c)	a)	b)	c)	d)	e)	a)	b)i.	b)ii.	a)	b)	c)
	B	A	A	F	V	D	B	C	D	B	D	C	C	A

Grupo	5							6	7				
	a)	b)	c)	d)i.	d)ii.	e)i.	e)ii.		a)	b)	c)	d)	e)
	A	D	D	B	A	C	A	B	C	C	D	A	A

Resolução abreviada do Exame de Recurso

1. $P(A) > 0,\ P(B) > 0,\ P(A) + P(B) = x,\ P(A \cap B) = y,\ \ x > y$

   (a) $P\left(\overline{A} \cap \overline{B}\right) = P\left(\overline{A \cup B}\right) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] = 1 - x + y$

   (b) $P\left(A \cap \overline{B}\right) + P\left(\overline{A} \cap B\right) + P\left(\overline{A} \cap \overline{B}\right) = P(A) - P(A \cap B) + P(B) - P(A \cap B) + P(A) + 1 - x + y =$
   $= x - 2y + 1 - x + y = 1 - y$

   (c) Se A e B são acontecimentos independentes então $P(A \cap B) = P(A)\,P(B) = P(A)\,[x - P(A)]$

   $$y = P(A)\,[x - P(A)] \Leftrightarrow [P(A)]^2 - xP(A) + y = 0 \Leftrightarrow P(A) = \frac{x \pm \sqrt{x^2 - 4y}}{2}$$

   $$P(A) = \frac{0.6 \pm \sqrt{0.36 - 4 \times 0.0275}}{2} = \frac{0.6 \pm 0.5}{2} = 0.3 \pm 0.25$$

   Logo $P(A) = 0.05$ e $P(B) = 0.6 - 0.05 = 0.55$

2. (a)

$X \setminus Y$	0	1	2	
0	0.1	$0.3 - a$	0	$0.4 - a$
1	0	0.2	0.4	0.6
2	0	a	0	a
	0.1	0.5	0.4	1

$$\sum_{x=0}^{2}\sum_{y=0}^{2} P(X = x; Y = y) = 1 \qquad \text{Condição verdadeira } \forall\, a \in \mathbb{R}$$

$P(X = x; Y = y) \geq 0,\ \ x, y \in \{0, 1, 2\} \Rightarrow 0.3 - a \geq 0, \quad \text{e} \quad a \geq 0 \Rightarrow a \in [0, 0.3]$

$P(X = x) \geq 0,\ \ x \in \{0, 1, 2\} \Rightarrow 0.4 - a \geq 0, \quad \text{e} \quad a \geq 0 \Rightarrow a \in [0, 0.4]$

Então $a \in [0, 0.3]$

Para $a = 0.2$

$X \setminus Y$	0	1	2	
0	0.1	0.1	0	0.2
1	0	0.2	0.4	0.6
2	0	0.2	0	0.2
	0.1	0.5	0.4	1

(b) As v.a.'s X e Y serão independentes se e só se $P(X=x;Y=y)=P(X=x)\,P(Y=y)\,,\ x,y\in\{0,1,2\}$

$P(X=0;Y=2)=0$ e $P(X=0)\,P(Y=2)=0.08$, logo X e Y não são v.a.'s independentes

(c) $E(Y)=\displaystyle\sum_{y=0}^{2} y\,P(Y=y)=1.3 \qquad V(Y)=2.1-1.3^2=0.41$

$V(-2Y+1)=(-2)^2\,V(Y)=1.64$

(d) $E(X)=\displaystyle\sum_{x=0}^{2} x\,P(X=x)=1, \quad cov(X,Y)=E(XY)-E(X)\,E(Y)=1.4-1\times 1.3=0.1$

$$\rho(X,Y)=\frac{cov(X,Y)}{\sqrt{V(X)\,V(Y)}}=\frac{0.1}{\sqrt{0.4\times 0.41}}=0.246932399$$

(e) Suporte de M: $S_M=\{0,1,2\}$

$P(M=0)=P(X=0;Y=0)=0.1$
$P(M=1)=P(X=0;Y=1)+P(X=1;Y=0)+P(X=1;Y=1)=0.3$

$$M\left\{\begin{array}{ccc} 0 & 1 & 2 \\ 0.1 & 0.3 & 0.6 \end{array}\right.$$

3. (a)
- $f(x)\geq 0,\ \forall x\in\mathbb{R}\Rightarrow c\geq 0$
- $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx=1\Leftrightarrow c\int_0^{\theta}\frac{x}{\theta^2}\,dx=1\Leftrightarrow\frac{c}{\theta^2}\left[\frac{x^2}{2}\right]_0^{\theta}=1\Leftrightarrow\frac{c}{2}=1\Leftrightarrow c=2$

(b) i. $E(X)=\displaystyle\int_{-\infty}^{+\infty} x\,f_X(x)\,dx=\int_0^2\frac{x^2}{2}dx=\left[\frac{x^3}{6}\right]_0^2=\frac{8}{6}=\frac{4}{3}$

$E\left(X^2\right)=\displaystyle\int_{-\infty}^{+\infty} x^2\,f_X(x)\,dx=\int_0^2\frac{x^3}{2}dx=\left[\frac{x^4}{8}\right]_0^2=\frac{16}{8}=2$

$V(X)=E\left(X^2\right)-E^2(X)=2-\dfrac{16}{9}=\dfrac{2}{9}$

$V(3X+1)=9V(X)=2$

ii. $F_X(x)=\begin{cases} 0, & x<0 \\ x^2/4, & 0\leq x<2 \\ 1, & x\geq 2 \end{cases}$

- $P(|X|>a)=P(X>a)=1-F_X(a)$
- $F_X(1)=1/4$
- $P(X\leq a+h\,|X\leq a)=1$
- $F_X(a)=\dfrac{1}{4}\Leftrightarrow\dfrac{a^2}{4}=\dfrac{1}{4}\Leftrightarrow a=1$

4. X-tempo de uma viagem (em minutos) $\qquad X\sim N(\mu,16)$

(a) $P(X\leq 10)=0.8413\Leftrightarrow P\left(\dfrac{X-\mu}{4}\leq\dfrac{10-\mu}{4}\right)=0.8413\Leftrightarrow P\left(Z\leq\dfrac{10-\mu}{4}\right)=0.8413\Leftrightarrow$

$\Leftrightarrow\dfrac{10-\mu}{4}=\Phi^{-1}(0.8413)\Leftrightarrow\dfrac{10-\mu}{4}=1\Leftrightarrow\mu=6$

(b) Seja $T=\displaystyle\sum_{i=1}^{4}X_i$, sendo X_i o tempo da i-ésima viagem (em minutos). $X_1,\ldots,X_4$ são v.a.'s i.i.d. com distribuição $N(10,16)$. Então $T\sim N(4\times 10,4\times 16)\equiv N(40,64)$

$$P(T>50)=1-P\left(\frac{T-40}{8}\leq\frac{50-40}{8}\right)=1-P(Z\leq 1.25)=1-0.8944=0.1056$$

(c) Seja Y- n.º de viagens com duração inferior a c minutos, de entre $n=40$ viagens $\quad Y\sim B(40,0.1)$
Como $n=40\geq 30$ e $np=40\times 0.1<5$, então $Y\overset{a}{\sim}P(40\times 0.1)\equiv P(4)$

5. $X\sim N\left(\mu,\sigma^2\right)$

(a) Informação populacional: $\sigma=10 \qquad$ Informação amostral: $n=16$, $\overline{x}=194$

- $Z=\sqrt{16}\,\dfrac{\overline{X}-\mu}{10}\sim N(0,1)$
- $1-\alpha=0.92, \quad z_{0.04}=1.75$

- $-1.75 \leq 4\dfrac{\overline{X}-\mu}{10} \leq 1.75 \Leftrightarrow \overline{X} - 1.75\dfrac{10}{4} \leq \mu \leq \overline{X} + 1.75\dfrac{10}{4}$
- $IC_{92\%}\left(\mu\right) \equiv \left[\overline{X} - 1.75\dfrac{10}{4}\ ,\ \overline{X} + 1.75\dfrac{10}{4}\right]$
- $IC_{92\%}\left(\mu\right) = \left[194 - 1.75\dfrac{10}{4}\ ,\ 194 + 1.28\dfrac{10}{4}\right] = [189.625\ ,\ 198.375]$

(b) Informação amostral: $n = 16$, $s^2 = 98$

- $W = (16-1)\dfrac{S^2}{\sigma^2} \sim \chi^2_{16-1}$, ou seja $W = 15\dfrac{S^2}{\sigma^2} \sim \chi^2_{15}$
- $P\left(a \leq W \leq b\right) = 0.95$ e $P\left(W \leq a\right) = P\left(W > b\right) = 0.025$

  $a = \chi^2_{15:0.975} = 6.26$ e $b = \chi^2_{15:0.025} = 27.5$
- $6.26 \leq 15\dfrac{S^2}{\sigma^2} \leq 27.5 \Leftrightarrow \dfrac{15S^2}{27.5} \leq \sigma^2 \leq \dfrac{15S^2}{6.26}$
- $IC_{95\%}\left(\sigma^2\right) \equiv \left[\dfrac{15S^2}{27.5}\ ,\ \dfrac{15S^2}{6.26}\right]$
- Para $s^2 = 98$, a estimativa de σ^2 por IC a 95% de confiança é: $IC_{95\%}\left(\sigma^2\right) = [53.45454545\ ,\ 234.8242812]$

(c) $H_0:\ \mu \geq 210\ \ vs\ \ H_1:\ \mu < 210$

(d) $H_0:\ \mu \geq 200\ \ vs\ \ H_1:\ \mu < 200$

i. $T = \sqrt{16}\dfrac{\overline{X}-200}{S} \underset{\mu=200}{\sim} t_{16-1}$, ou seja $T = 4\dfrac{\overline{X}-200}{S} \underset{\mu=200}{\sim} t_{15}$

ii. $Z = \dfrac{\sqrt{n}}{10}\left(\overline{X}-200\right) \underset{\mu=200}{\sim} N\left(0,1\right)$

$R_{0.2} =]-\infty, z_{1-0.2}[\ = \ ]-\infty, -z_{0.2}[\ = \ ]-\infty, -0.84[$

(e) Informação amostral: $n = 36$, $\hat{p} = \dfrac{27}{36} = 0.75$

i.
- $W = \sqrt{36}\dfrac{\hat{P}-0.8}{\sqrt{0.8\left(1-0.8\right)}} \underset{p=0.8}{\overset{a}{\sim}} N\left(0,1\right)$ ou seja $W = 15\left(\hat{P}-0.8\right) \underset{p=0.8}{\overset{a}{\sim}} N\left(0,1\right)$
- $w_{obs} = 15\left(0.75-0.8\right) = -0.75$

ii.
- $W = 15\left(\hat{P}-0.8\right) \underset{p=0.8}{\overset{a}{\sim}} N\left(0,1\right)$
- $w_{obs} = -0.82$ $\quad R_{obs} =]-\infty, w_{obs}[\ = \ ]-\infty, -0.82[$

  $p-value = P\left(W \in R_{obs}\right) = P\left(W \leq -0.82\right) \approx P\left(Z \leq -0.82\right) =$
  $= 1 - P\left(Z \leq 0.82\right) = 1 - 0.7939 = 0.2061$

6. $W \equiv 9\left(\hat{\Lambda} - \lambda\right) \sim E\left(0,1\right)$

- $W \equiv 9\,\hat{\Lambda} \underset{\lambda=0}{\sim} E\left(0,1\right)$
- $P\left(\text{erro de tipo I}\right) = P\left(\text{Rejeitar } H_0\left|H_0 \text{ verdadeira}\right.\right) = P\left(\left|\hat{\Lambda}\right| > 0.1\left|\lambda = 0\right.\right) = P\left(\left|W\right| > 0.9\right) =$

  $= P\left(W > 0.9\right) = e^{-0.9} = 0.406569659$

7. Informação populacional: $X \sim U\left(2,\theta\right),\ \theta \in]2,+\infty[$ $\qquad E\left(X\right) = \dfrac{2+\theta}{2}$ $\qquad V\left(X\right) = \dfrac{\left(\theta-2\right)^2}{12}$

(a) $E\left(X\right) = \overline{X} \Leftrightarrow \dfrac{2+\theta}{2} = \overline{X} \Leftrightarrow \theta = 2\left(\overline{X}-1\right)$

O estimador dos momentos para o parâmetro θ tem expressão $2\left(\overline{X}-1\right)$

(b) $E\left(\hat{\theta}\right) = \dfrac{n\theta+2}{n+1} \neq \theta$ $\quad \hat{\theta}$ é não centrado $\quad bias\left(\hat{\theta}\right) = E\left(\hat{\theta}\right) - \theta = \dfrac{n\theta+2-n\theta-\theta}{n+1} = \dfrac{2-\theta}{n+1}$

(c) $V\left(\theta^*\right) = V\left(2\overline{X}-2\right) = 4V\left(\overline{X}\right) = 4\dfrac{V\left(X\right)}{n} = \dfrac{\left(\theta-2\right)^2}{3n}$

$$\frac{V\left(\theta^*\right)}{V\left(\ddot{\theta}\right)} = \frac{\dfrac{\left(\theta-2\right)^2}{3n}}{\dfrac{\left(\theta-2\right)^2}{n\left(n+2\right)}} = \frac{n+2}{3} \geq 1$$

$\ddot{\theta}$ é o mais eficiente

(d) $\dfrac{\overline{X}-E\left(\overline{X}\right)}{\sqrt{V\left(\overline{X}\right)}} \overset{a}{\sim} N\left(0,1\right) \qquad E\left(\tilde{\theta}\right)=\theta \qquad V\left(\tilde{\theta}\right)=\dfrac{\left(\theta-2\right)^2}{3n}$

Como $\tilde{\theta}=2\left(\overline{X}-1\right)$ é uma combinação linear de $\overline{X}$, então

$$\frac{\tilde{\theta}-E\left(\tilde{\theta}\right)}{\sqrt{V\left(\tilde{\theta}\right)}}=\frac{\tilde{\theta}-\theta}{\frac{\theta-2}{\sqrt{3n}}}=\sqrt{3n}\frac{\tilde{\theta}-\theta}{\theta-2}\overset{a}{\sim} N\left(0,1\right)$$

(e)

- $W=\sqrt{3n}\dfrac{\tilde{\theta}-\theta}{\tilde{\theta}-2}\overset{a}{\sim} N\left(0,1\right)$
- $P\left(-a\leq W\leq a\right)\approx 1-\alpha \Rightarrow a=z_{\alpha/2}$
- $-z_{\alpha/2}\leq\sqrt{3n}\dfrac{\tilde{\theta}-\theta}{\tilde{\theta}-2}\leq z_{\alpha/2}\Leftrightarrow -\dfrac{z_{\alpha/2}}{\sqrt{3n}}\left(\tilde{\theta}-2\right)\leq\tilde{\theta}-\theta\leq\dfrac{z_{\alpha/2}}{\sqrt{3n}}\left(\tilde{\theta}-2\right)\Leftrightarrow$
  $\Leftrightarrow\tilde{\theta}-\dfrac{z_{\alpha/2}}{\sqrt{3n}}\left(\tilde{\theta}-2\right)\leq\theta\leq\tilde{\theta}+\dfrac{z_{\alpha/2}}{\sqrt{3n}}\left(\tilde{\theta}-2\right)$
- $IC_{(1-\alpha)\times 100\%}(\theta)=\left[\tilde{\theta}-\dfrac{z_{\alpha/2}}{\sqrt{3n}}\left(\tilde{\theta}-2\right),\ \tilde{\theta}+\dfrac{z_{\alpha/2}}{\sqrt{3n}}\left(\tilde{\theta}-2\right)\right]$