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Homework 30/03

Apartat 1:

Sigui $\mathcal{B} = (u_1, \dots, u_n)$, aleshores

$$G(\varphi, \mathcal{B}) = \begin{pmatrix} \varphi(u_1, u_1) & \varphi(u_1, u_2) & \cdots & \varphi(u_1, u_n) \\ \varphi(u_2, u_1) & \varphi(u_2, u_2) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(u_n, u_1) & \varphi(u_n, u_2) & \cdots & \varphi(u_n, u_n) \end{pmatrix} \tag{1}$$

Per demostrar la igualtat de l'enunciat, desenvoluparem els dos termes per separat i comprovarem que obtenim el mateix resultat.
Primer de tot, desenvoluparem la $\varphi(u, v)$, on $(u)_{\mathcal{B}} = (\alpha_1, \dots, \alpha_n)$ i $(v)_{\mathcal{B}} = (\beta_1, \dots, \beta_n)$

$$\begin{aligned}\varphi(u, v) &= \varphi(\alpha_1 u_1 + \cdots + \alpha_n u_n, \beta_1 u_1 + \cdots + \beta_n u_n) = \\ &= \varphi(\alpha_1 u_1 + \cdots + \alpha_n u_n, \beta_1 u_1 + \cdots + \beta_n u_n) + \cdots + \varphi(\alpha_1 u_1 + \cdots + \alpha_n u_n, \beta_1 u_1 + \cdots + \beta_n u_n) = \\ &= \sum_{i=1}^{n} \varphi(\alpha_i u_i, \beta_1 u_1 + \cdots + \beta_n u_n) = \sum_{i=1}^{n} \left(\varphi(\alpha_i u_i, \beta_1 u_1) + \cdots + \varphi(\alpha_i u_i, \beta_n u_n)\right) = \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi(\alpha_i u_i, \beta_j u_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \varphi(u_i, \beta_j u_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \varphi(u_i, u_j)\end{aligned}$$

Ara, si aconseguim demostrar la següent igualtat

$$\begin{pmatrix} \alpha_1 & \cdots & \alpha_n \end{pmatrix} G(\varphi, \mathcal{B}) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \varphi(u_i, u_j) \tag{2}$$

haurem demostrat el que voliem. Per demostrar la igualtat (2), anem a fer el producte de matrius.

$$G(\varphi, \mathcal{B}) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} \beta_1 \varphi(u_1, u_1) + \cdots + \beta_n \varphi(u_1, u_n) \\ \vdots \\ \beta_1 \varphi(u_n, u_1) + \cdots + \beta_n \varphi(u_n, u_n) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} \beta_j \varphi(u_1, u_j) \\ \vdots \\ \sum_{j=1}^{n} \beta_j \varphi(u_n, u_j) \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} \alpha_1 & \cdots & \alpha_n \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{n} \beta_j \varphi(u_1, u_j) \\ \vdots \\ \sum_{j=1}^{n} \beta_j \varphi(u_n, u_j) \end{pmatrix} &= \alpha_1 \sum_{j=1}^{n} \beta_j \varphi(u_1, u_j) + \cdots + \alpha_n \sum_{j=1}^{n} \beta_j \varphi(u_n, u_j) = \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \varphi(u_i, u_j) = \varphi(u, v)\end{aligned}$$